

Oblivious Complexity Classes Revisited: Lower Bounds and Hierarchies

Karthik Gajulapalli*

Zeyong Li[†]

Ilya Volkovich[‡]

Abstract

In this work we study *oblivious* complexity classes. These classes capture the power of interactive proofs where the prover(s) are only given the input size rather than the actual input. In particular, we study the connections between the symmetric polynomial time - S_2P and its oblivious counterpart - O_2P . Among our results:

- For each $k \in \mathbb{N}$, we construct an explicit language $L_k \in O_2P$ that cannot be computed by circuits of size n^k .
- We prove a hierarchy theorem for O_2TIME . In particular, for any time constructible function $t : \mathbb{N} \rightarrow \mathbb{N}$ and any $\varepsilon > 0$ we show that: $O_2TIME[t(n)] \subsetneq O_2TIME[t(n)^{1+\varepsilon}]$.
- We prove new structural results connecting O_2P and S_2P .
- We make partial progress towards the resolution of an open question posed by Goldreich and Meir (TOCT 2015) that relates the complexity of NP to its oblivious counterpart - ONP .
- We identify a natural class of problems in O_2P from computational Ramsey theory, that are not expected to be in P or even BPP .

To the best of our knowledge, these results constitute the first explicit fixed-polynomial lower bound and hierarchy theorem for O_2P . The smallest uniform complexity class for which such lower bounds were previously known was S_2P due to Cai (JCSS 2007). In addition, this is the first uniform hierarchy theorem for a semantic class. All previous results required some non-uniformity. In order to obtain some of the results in the paper, we introduce the notion of *uniformly-sparse extensions* which could be of independent interest.

Our techniques build upon the de-randomization framework of the powerful Range Avoidance problem that has yielded many new interesting explicit circuit lower bounds.

*Georgetown University. Email: kg816@georgetown.edu.

[†]National University of Singapore. Email: li.zeyong@u.nus.edu.

[‡]Boston College. Email: ilya.volkovich@bc.edu.

1 Introduction

Proving circuit lower bounds has been one of the holy grails of theory of computation with strong connections to many fundamental questions in complexity theory. For instance, proving that there exists a function in E^1 that requires exponential-size circuits would entail a strong derandomization: $BPP = P$ and $MA = NP$ [NW94, IW97]. And yet, while by counting arguments (i.e. [Sha49]) the vast majority of Boolean functions/languages do require exponential-size circuits, the best ‘explicit’ lower bounds are still linear! (in fact the best known lower bound for any language in E^{NP} is just linear [LY22]). Indeed, although it is widely *believed* that NP requires super-polynomial-size circuits (i.e. $NP \not\subseteq P/poly$) establishing the statement even for $NEXP$ (i.e. $NEXP \not\subseteq P/poly$), the exponential version of NP , has remained elusive for many years. The best known explicit lower bound is due to a seminal work of Williams [Wil14], where it was shown that $NEXP$ requires super-polynomial-size circuits in a ‘very’ restricted model ($NEXP \not\subseteq ACC^0$).

In the high-end regime, Kannan [Kan82] has shown that the exponential hierarchy requires exponential-size circuits, via diagonalization². More precisely, it was shown that the class $\Sigma_3 E \cap \Pi_3 E$ contains a language that cannot be computed by a circuit family of size $2^n/n$. This result was later improved to $\Delta_3 E = E^{\Sigma_2 P}$ by Miltersen, Vinodchandran and Watanabe [MVW99]. Moreover, it was shown that $\Delta_3 E$ actually requires circuits of ‘maximum possible’ size. Subsequently, the status of the problem remained stagnant for more than two decades until very recently, Chen, Hirahara and Ren [CHR24] and a follow-up work by Li [Li24] improved the result to $S_2 E$ ³. In particular, this result was obtained via solving the Range Avoidance (Avoid) problem with ‘single-valued, symmetric polynomial-time’ algorithm. Indeed, the focus of our work is on ‘oblivious’ symmetric polynomial time and related complexity classes.

1.1 Background

1.1.1 Symmetric Time

Symmetric polynomial time, denoted by $S_2 P$, was introduced independently by Canetti [Can96], and Russell and Sundaram [RS98]. Intuitively speaking, this class captures the interaction between an efficient (polynomial-time) verifier V and two all-powerful provers: the ‘YES’-prover Y and the ‘NO’-prover Z , exhibiting the following behaviour:

- If x is a yes-instance, then the ‘YES’-prover Y can send an irrefutable proof/certificate y to V that will make V *accept*, **regardless** of the communication from Z .
- Likewise, if x is a no-instance, then the ‘NO’-prover can send an irrefutable proof/certificate z to V that will make V *reject*, **regardless** of the communication from Y .

We stress that in both cases the irrefutable certificates can depend on x itself. One can also define $S_2 E$ - the exponential version of $S_2 P$, by allowing the verifier to run in linear-exponential time. For a formal definition see [Definition 2.2](#). A seminal result of [Cai07] provides the best known upper bound $S_2 P \subseteq ZPP^{NP}$. At the same time, $S_2 P$ appears to be a very powerful class as it contains MA and $\Delta_2 P = P^{NP}$.

¹Deterministic time $2^{O(n)}$.

²In fact, this argument could be viewed as solving an instance of the *Range Avoidance* problem. See below.

³Symmetric exponential time. Indeed, $S_2 E \subseteq \Sigma_2 E \cap \Pi_2 E \subseteq \Delta_3 E$. For a formal definition see [Definition 2.2](#).

1.1.2 Oblivious Complexity Classes

The study of oblivious complexity classes was initiated in [CR06] and has subsequently received more attention [Aar07, FSW09, CR11, GM15]. Roughly speaking, let Λ be a complexity class such that in addition to the input x , the machines $M(x, w)$ of Λ also take a witness w (and possibly other inputs). Examples of such classes include: NP , MA , S_2P , etc. The corresponding *oblivious* version of Λ is obtained by stipulating that for every $n \in \mathbb{N}$ there exists a ‘common’ witness w_n for **all** the ‘respective’ inputs of length n . For instance, a language L belongs to ONP – the oblivious version of NP , if there exists a polynomial-time machine $M(x, w)$ such that:

1. $\forall n \in \mathbb{N}$ there exists w_n such that $\forall x \in \{0, 1\}^n : x \in L \implies M(x, w_n) = 1$.
2. $x \notin L \implies \forall w : M(x, w) = 0$.

Thus, in a similar manner, one can define the class O_2P — the oblivious version of S_2P , that is referred to as ‘oblivious symmetric polynomial time’ in the literature. O_2P has the additional requirement that for every $n \in \mathbb{N}$ there exist an irrefutable yes-certificate y^* and an irrefutable no-certificate z^* for all the yes-instances and the no-instances of length n , respectively. For a formal definition, see [Definition 2.4](#).

It is immediate from the definitions that $\text{ONP} \subseteq \text{NP}$, $\text{O}_2\text{P} \subseteq \text{S}_2\text{P}$ and $\text{ONP} \subseteq \text{O}_2\text{P}$. On the other hand, by hard-coding the witnesses/certificates we get that $\text{RP} \subseteq \text{ONP} \subseteq \text{O}_2\text{P} \subseteq \text{P}/\text{poly}$. In addition, it was also observed in [CR06] that O_2P is *self-low*. That is $\text{O}_2\text{P}^{\text{O}_2\text{P}} = \text{O}_2\text{P}$. While the oblivious classes seem to be more restricted than their non-oblivious counterparts, proving any non-trivial upper bounds could still be challenging. In terms of lower bounds, the best known containment of a non-oblivious class is $\text{BPP} \subseteq \text{O}_2\text{P}$ ⁴. For more details and discussion see [CR06, GM15]. Nonetheless, to the best of our knowledge, no “natural” problem for O_2P (or even ONP), known to lie outside of BPP , has been identified in the literature.

1.1.3 Sparsity

A language L is *sparse*, if for every input length $n \in \mathbb{N}$ the number of yes-instances of size n is at most $\text{poly}(n)$. We will denote the class of all sparse languages by SPARSE . Sparse languages have seen many applications in complexity theory. Perhaps, the most fundamental one is known as “Mahaney’s theorem” [Mah82] that implies that a sparse language cannot be NP -hard, unless $\text{P} = \text{NP}$. In [FSW09] and [GM15], sparse languages were also studied in the context of oblivious complexity classes. In particular, they observed that $\text{NP} \cap \text{SPARSE} \subseteq \text{ONP}$. That is, every sparse NP language is also in ONP . The same argument also implies that $\text{NE} = \text{ONE}$, that is, **equality** between the exponential versions of NP and ONP , respectively. Given the former claim we observe that the *Grid Coloring* problem, defined in [AGL23], constitutes a natural ONP (and hence O_2P) problem. For a formal statement, see [Observation 1](#).

Subsequently, Goldreich and Meir [GM15] posed an open question whether a similar relation holds true for coNP and coONP . That is, whether every sparse coNP language is also in coONP ⁵. Motivated by this question, we observe that essentially the same issues arise when one attempts to show that every sparse S_2P language is also in O_2P . While we do not accomplish this task, we make a partial progress by introducing *uniformly-sparse extensions*. The intuition behind this definition

⁴One way to see this is by observing that $\text{BPP} \subseteq \text{RP}^{\text{ONP}}$ and then using the self-lowness of O_2P .

⁵The original (equivalent) formulation of the question in [GM15] was w.r.t NP and co-sparse languages.

is to have a uniform ‘cover’ of the segments of the yes-instances for **all** input lengths. For a formal definition see [Definition 2.19](#). This is our main conceptual contribution. As a corollary, we obtain that $S_2E = O_2E$. Although this might not be a new result, to the best of our knowledge, this result has not appeared in the literature previously.

1.1.4 Range Avoidance

The study of the Range Avoidance problem (**Avoid**) was initiated in [\[KKMP21\]](#). The problem itself takes an input-expanding Boolean Circuit $C : \{0,1\}^n \rightarrow \{0,1\}^{n+1}$ as input and asks to find an element y , outside the range of C . Since its introduction, there has been a steady line of exciting work studying the complexity and applications of **Avoid** [\[Kor22, GLW22, RSW22, CHLR23, GGNS23, ILW23, CGL⁺23, CHR24, Li24, CL24, KP24\]](#).

Informally, **Avoid** algorithmically captures the probabilistic method where the existence of an object with some property follows from a union bound. In particular, Korten [\[Kor22\]](#) showed that solving **Avoid** (deterministically) would result in finding optimal explicit constructions of many important combinatorial objects, including but not limited to Ramsey graphs [\[Rad21\]](#), rigid matrices [\[GLW22, GGNS23\]](#), pseudorandom generators [\[CT22\]](#), two-source extractors [\[CZ19, Li23\]](#), linear codes [\[GLW22\]](#), strings with maximum time-bounded Kolmogorov complexity (K^{poly} -random strings) [\[RSW22\]](#) and truth tables of high circuit complexity [\[Kor22\]](#).

The connection between **Avoid** and hard truth table makes it relevant to the study of circuit lower bounds. It has been observed and pointed out in many prior works (see, e.g. [\[CHR24\]](#)) that proving explicit circuit lower bounds is effectively finding single-valued⁶ constructions of hard truth tables. Indeed, this is the framework adopted for proving circuit lower bounds in [\[CHLR23, CHR24, Li24\]](#): Designing a single-valued algorithm for solving **Avoid**.

1.1.5 Time Hierarchy Theorem

Time Hierarchy theorems are among the most fundamental results in computational complexity theory, which (loosely speaking) assert that computation with more time is strictly more powerful. Time hierarchy theorems are known for deterministic computation (**DTIME**) [\[HS65, HS66\]](#) and non-deterministic computation (**NTIME**) [\[Coo72, SFM78, Žák83\]](#) which are syntactic classes. The situation for semantic classes such as **BPTIME** is much more elusive as it is unclear how to enumerate and simulate all **BPTIME** machines while ensuring that the simulating machine itself remains a proper **BPTIME** machine. In fact, even verifying that a machine is a **BPTIME** machine is itself an undecidable problem. For **BPTIME**, a time hierarchy theorem is only known for its promise version, or when given one bit of advice [\[Bar02, FS04, FST05\]](#). This was further generalized in [\[MP07\]](#), where they show most semantic classes (e.g. **MA**, **S₂TIME**) admit a time hierarchy theorem with one bit of advice.

Along a different line of research, it was shown in [\[LOS21, DPWV22\]](#) that coming up with a pseudo-deterministic algorithm (single-valued randomized algorithms) for estimating the acceptance probability of a circuit would imply a uniform hierarchy theorem for **BPTIME**.

⁶Roughly speaking, a single-valued algorithm on successful executions should output a fixed (canonical) solution given the same input.

1.2 Previous Results

A parallel line of work focused on the ‘low-end’ regime by proving the so-called ‘fixed-polynomial’ circuit lower bounds. That is, the goal is to show that for every $k \in \mathbb{N}$ there is a language L_k (that may depend on k) which cannot be computed by circuits of size n^k . The first result in this sequel — fixed-polynomial lower bounds for the polynomial hierarchy, was obtained by Kannan [Kan82] via diagonalization. In particular, it was shown that for every $k \in \mathbb{N}$ there exists a language $L_k \in \Sigma_4\text{P}$ that cannot be computed by circuits of size n^k . This result was then improved to $\Sigma_2\text{P}$. The key idea behind this and, in fact, the vast majority of subsequent improvements is a ‘win-win’ argument that relies on the *Karp-Lipton* collapse theorem [KL80]: if NP has polynomial-size circuits (i.e. $\text{NP} \subseteq \text{P/poly}$) then the (whole) polynomial hierarchy collapses to $\Sigma_2\text{P}$. More specifically, the argument proceeds by a two-pronged approach:

- Suppose $\text{NP} \not\subseteq \text{P/poly}$. Then the claim follows as $\text{NP} \subseteq \Sigma_2\text{P}$.
- On the other hand, suppose $\text{NP} \subseteq \text{P/poly}$. Then by Karp-Lipton: $\Sigma_4\text{P} = \Sigma_2\text{P}$ and in particular for all $k \in \mathbb{N}$: $L_k \in \Sigma_2\text{P}$.

Indeed, by deepening the collapse, the result was further improved to ZPP^{NP} [KW98, BCG⁺96], P^{prMA} [CR11] and S_2P [Cai07]. By using different versions of the Karp-Lipton theorem, the result has also been extended to PP [Vin05, Aar06] and $\text{MA}/1$ [San09].

Yet, despite the success of the ‘win-win’ argument, the obtained lower bounds are often non-explicit due to the non-constructiveness nature of the argument. Different results [CW04, San09] were required to exhibit explicit ‘hard’ languages in $\Sigma_2\text{P}$, PP and $\text{MA}/1$. Nonetheless, the last word about S_2P is yet to be said. For instance, we know that there is a language in S_2P that requires circuits of size, say, n^2 from such arguments. However, prior to the results of [CHR24, Li24], one could not prove any super-linear lower bound for **any** particular language in S_2P . Another limitation of the ‘win-win’ argument stems from the fact that it only applies to complexity classes which (provably) contain NP . In particular, in [CR06] it was actually shown that if $\text{NP} \subseteq \text{P/poly}$ then the polynomial hierarchy collapses all the way to O_2P ! Unfortunately, this result does not immediately imply fixed-polynomial lower bounds for O_2P ⁷ as it is unknown and, in fact, *unlikely* that O_2P contains NP . Furthermore, such a containment will be ‘self-defeating’. Recall that $\text{O}_2\text{P} \subseteq \text{P/poly}$. Hence, if $\text{NP} \subseteq \text{O}_2\text{P}$ then $\text{NP} \subseteq \text{P/poly}$ which in and of itself already implies the collapse of the whole polynomial hierarchy!

Finally, it is important to mention a result of [FSW09] that for any $k \in \mathbb{N}$, NP has circuits of size n^k iff $\text{ONP}/1$ does. In that sense ONP already nearly captures the hardness of showing fixed-polynomial lower bounds for NP .

1.3 Our Results

In our first result we extend the lower bounds for S_2P and S_2E , to their weaker oblivious counterparts O_2P and O_2E , respectively. This result follows the recent line of research that obtains circuit lower bounds by means of *deterministically* solving (i.e. derandomizing) instances of the Range Avoidance problem [CHLR23, CHR24, Li24].

⁷Indeed, the authors in [CR06] could only obtain fixed-polynomial lower bounds for $\text{NP}^{\text{O}_2\text{P}}$ which was later subsumed by the results of [San09].

Theorem 1. *For all $k \in \mathbb{N}$, $\text{O}_2\text{P} \not\subseteq \text{SIZE}[n^k]$. Moreover, for each k there exists an explicit language $L_k \in \text{O}_2\text{P}$ such that $L_k \notin \text{SIZE}[n^k]$.*

In fact we prove a stronger parameterized version of this statement (see [Theorem 3.2](#), [Corollary 4.2](#), and [Corollary 3.3](#)). We now highlight three main reasons why such a result is fascinating:

1. Our lower bound does not follow the framework of “win-win” style Karp-Lipton collapses. As was mentioned above, since already $\text{O}_2\text{P} \subseteq \text{P}/\text{poly}$ the pre-requisite for proving the bound via the “win-win” argument will be self-defeating.
2. Our proof is constructive and for every $k \in \mathbb{N}$ we define an explicit language $L_k \in \text{O}_2\text{P}$ for which $L_k \not\subseteq \text{SIZE}[n^k]$.
3. O_2P becomes the smallest uniform complexity class known for which fixed-polynomial lower bounds are known. Moreover, after more than 15 years, this class coincides again with the deepest known collapse result of the Karp-Lipton Theorem⁸.

Our second result gives a hierarchy theorem for O_2TIME .

Theorem 2. *For any time constructible function $t : \mathbb{N} \rightarrow \mathbb{N}$ such that $t(n) \geq n$ and any $\varepsilon > 0$ it holds that: $\text{O}_2\text{TIME}[t(n)] \subsetneq \text{O}_2\text{TIME}[t(n)^{1+\varepsilon}]$.*

We remark, that to the best of our knowledge, this is the first known hierarchy theorem for a uniform semantic class (that contains BPTIME). At the same time, we observe that the proof of the non-deterministic time hierarchy theorem (NTIME) (see e.g. [\[Žák83\]](#)) actually extends to the *oblivious* non-deterministic time (ONTIME) since the hard language constructed in their proof is unary and hence is contained in ONTIME . On the other hand, that same language also diagonalizes against **all** NTIME machines which is a superset of all ONTIME machines.

In our time hierarchy theorem for O_2TIME , which goes through a reduction to **Avoid**, one can view **Avoid** as a tool for diagonalization against all circuits of fixed size, which in turn contains all O_2TIME machines with bounded time complexity. This (together with the time hierarchy theorem for ONTIME) might suggest an approach for proving time hierarchy theorem for semantic classes in general: diagonalize against a syntactic class that encompasses the semantic class in consideration.

Finally, we introduce the notion of *uniformly-sparse extensions* (for a formal definition, see [Definition 2.19](#)) to get structural complexity results relating O_2TIME and S_2TIME . This relation provides an alternate method of proving [Theorem 1](#).

Theorem 3. *Let $L \in \text{S}_2\text{P}$. If L has a uniformly-sparse extension then $L \in \text{O}_2\text{P}$.*

While not much was known between the classes O_2TIME and S_2TIME , except that $\text{O}_2\text{TIME} \subseteq \text{S}_2\text{TIME}$, we show new connections between the two classes. In fact, we prove a stronger parameterized version of [Theorem 3](#) that yields as a corollary a proof of the equivalence $\text{S}_2\text{E} = \text{O}_2\text{E}$ (see [Corollary 4.3](#)). Going back to the original motivation, by repeating the same argument, we make a partial progress towards the resolution of the open question posed by Goldreich and Meir in [\[GM15\]](#). See [Lemma 4.1](#) for more details.

⁸Indeed, in the universe of [\[Cai07\]](#) and [\[CR06\]](#) prior to our work, the smallest class has been S_2P , while the deepest known collapse was to O_2P .

Theorem 4. *Let $L \in \text{coNP}$. If L has a uniformly-sparse extension then $L \in \text{coONP}$.*

Finally, we observe that computational Ramsey theory provides some very natural problems in ONP (and hence O_2P). As an example, we re-introduce the grid coloring problem below. While Claim 2.5 in [GM15] suggests a generic way to generate problems via padding arguments⁹, these problems are, however, not very intuitive.

Definition 1.1 (Grid Coloring [AGL23]).

$\text{GC} = \{(1^n 01^m 01^c) \mid \text{the } n \times m \text{ grid can be } c\text{-colored and not have any monochromatic squares.}\}$

Note that Grid Coloring is an example of one of such problems that are in $\text{NP} \cap \text{SPARSE} \subseteq \text{ONP}$, and hence unlikely to be in BPP. Other problems that come from computational Ramsey theory like the *Gallai-Witt* theorem, and the *Van der Waerden's* theorem have a very similar flavor.

Observation 1. $\text{GC} \in \text{ONP} \subseteq \text{O}_2\text{P}$.

Below we make a few remarks. For a further discussion see [Gas10].

- $\text{GC} \in \text{NP}$ since the coloring itself is a witness.
- GC is not known to be in P or even BPP.
- $\text{GC} \in \text{SPARSE}$. In fact, GC has a uniformly-sparse extension.
- Therefore, by the results of [FSW09, GM15], $\text{GC} \in \text{ONP}$.
- On the other hand, by Mahaney's theorem GC is *unlikely* to be NP-complete.

1.4 Proof Overview

Our work builds on the recent line of work on Range Avoidance. [Kor22] provides a reduction of generating hard truth tables from Avoid, and [CHR24, Li24] give a single-valued S_2P time algorithm for Avoid.

Avoid Framework for Circuit Lower bounds Let $\text{TT}_{n,s} : \{0,1\}^{s \log s} \rightarrow \{0,1\}^{2^n}$ be the truth table generator circuit (see Definition 2.13), i.e. $\text{TT}_{n,s}$ take as input an encoding of a n -input s -size circuit and outputs the truth table of the circuit. By construction, $\text{TT}_{n,s}$ maps all circuits of size s (encoded using $s \log s$ bits) to their corresponding truth tables. Then, $\text{Avoid}(\text{TT}_{n,s})$ will output a truth-table not in the range of $\text{TT}_{n,s}$ and hence not decided by any s -sized circuit (a circuit lower bound!!). For correctness we only need to ensure that $s \log s < 2^n$, so the $\text{TT}_{n,s}$ is input-expanding, and hence a valid instance of Avoid.

While the above construction gives us a way of getting explicit exponential lower bounds against even $\text{SIZE}[2^n/n]$, the input to Avoid is also exponential in input length n . As a result, the lower bounds we get are for the exponential class S_2E and not S_2P . Note that one can scale down this lower bound in a black-box manner to get fixed-polynomial lower bounds for S_2P , but will lose explicitness in the process.

⁹The approach is to pick a language L in S_2E that is (conjectured) not in BPE. Then the padded version of L will be in $\text{O}_2\text{P} \setminus \text{BPP}$.

To fix this we modify the above reduction to take as input the prefix truth table generator circuit, $\text{PTT}_{n,s} : \{0,1\}^{s \log s} \rightarrow \{0,1\}^{s \log s + 1}$, where instead of evaluating the input circuit on the whole truth table, $\text{PTT}_{n,s}$ evaluates on the lexicographically first $(s \log s + 1)$ inputs (see [Definition 2.14](#)). Let $f_{n,s} = \text{Avoid}(\text{PTT}_{n,s})$, and define the truth table of the hard language to be $L := f_{n,s} || 0^{2^n - s \log s - 1}$. By construction, L cannot be decided by any n -input s -size circuit. Moreover, when s is polynomial, the size of $\text{PTT}_{n,s}$ is also polynomial¹⁰ ([Lemma 2.15](#)). Hence the single-valued¹¹ algorithm computing $f_{n,s}$ is in S_2P and the explicit fixed-polynomial bounds follow.

To see that the language $L \in \text{O}_2\text{P}$, observe that the S_2P time algorithm is oblivious to x , since for any x of length n , $f_{n,s}$ is the same. One important observation here is that, for the purpose of obtaining circuit lower bound, it suffices to solve Range Avoidance on *one* specific family of circuits (the truth table generating circuit that maps another circuit to its truth table). Hence, while it is unclear whether Range Avoidance can be solved in FO_2P , we could still obtain circuit lower bound for O_2P .

Hierarchy Theorems for O_2TIME To get a hierarchy theorem for O_2TIME , we first get an upper bound on O_2TIME computation via a standard Cook-Levin argument that converts the O_2TIME verifier into a circuit (SAT-formula) for which we can hard code the “YES” and “NO” irrefutable certificates at every input length ([Lemma 3.4](#)). A lower bound follows via the Avoid framework discussed above ([Theorem 3.2](#)). We can now lift the hierarchy theorem on circuit size ([Theorem 2.8](#)) to get a hierarchy on O_2TIME (see [Theorem 3.5](#)).

Sparsity and Lower Bounds We begin by introducing the notion of *uniformly-sparse extensions*. Roughly speaking a sparse language L has a *uniformly-sparse extension* if there is a language $L' \in \text{P}$, such that $L \subseteq L'$ and L' is also sparse (for formal definitions see [Section 2.4](#)).

We show that if a language $L \in \text{S}_2\text{P}$ has a *uniformly-sparse extension*, then $L \in \text{O}_2\text{P}$. Let L' be the *uniformly-sparse extension* of a language $L \in \text{S}_2\text{P}$ and let $X = \{x \in L'\}$. Since $L' \in \text{P}$, we first apply the polynomial time algorithm for L' which let us filter out most inputs, i.e. $x \notin L'$, and hence $x \notin L$. We are now left with deciding membership in L over the set X , where $|X| \leq \text{poly}$.

Let V^* be the polynomial time S_2 -verifier for L , then for every $x \in X$ there exists either an irrefutable YES certificate (y_x) s.t. $V^*(x, y_x, \cdot) = 1$, or an irrefutable NO certificate (z_x) s.t. $V^*(x, \cdot, z_x) = 0$. Let Y be the set of all such y_x 's and Z be the set of all such z_x 's. Now for any $x \in X$, it suffices to find the correct y_x from Y (or z_x from Z) and apply $V^*(x, y_x, z_x)$ to decide x .

In [Lemma 4.1](#) we prove a more efficient parameterized version of this argument. In addition, we are able to apply this in the exponential regime to show the equivalence $\text{O}_2\text{E} = \text{S}_2\text{E}$ (see [Corollary 4.3](#)).

2 Preliminaries

Let $L \subseteq \{0,1\}^*$ be a language. For $n \geq 1$ we define the *n-th slice* of L , $L|_n := L \cap \{0,1\}^n$ as all the strings in L of length n . The characteristic string of $L|_n$, denoted by $\mathcal{X}_{L|_n}$, is the binary string of length 2^n which represents the truth table defined by $L|_n$.

¹⁰In literature the complexity of computing $\text{PTT}_{n,s}$ (Circuit-Eval) is often left as poly, however for our application of getting explicit lower bounds it is crucial to get its fine-grained complexity (see [Lemma 2.10](#) and [Lemma 2.15](#)).

¹¹For the language to be well defined it is essential for the output of our algorithm to be single-valued.

2.1 Complexity Classes

We assume familiarity with complexity theory and notion of non-uniform circuit families (see for e.g. [AB09, Gol08]).

Definition 2.1 (Deterministic Time). Let $t : \mathbb{N} \rightarrow \mathbb{N}$. We say that a language $L \in \text{TIME}[t(n)]$, if there exists a deterministic time multi-tape Turing machine that decides L , in at most $O(t(n))$ steps.

Definition 2.2 (Symmetric Time). Let $t : \mathbb{N} \rightarrow \mathbb{N}$. We say that a language $L \in \text{S}_2\text{TIME}[t(n)]$, if there exists a $O(t(n))$ -time predicate $P(x, y, z)$ that takes $x \in \{0, 1\}^n$ and $y, z \in \{0, 1\}^{t(n)}$ as input, satisfying that:

1. If $x \in L$, then there exists a y such that for all z , $P(x, y, z) = 1$.
2. If $x \notin L$, then there exists a z such that for all y , $P(x, y, z) = 0$.

Moreover, we say $L \in \text{S}_2\text{P}$, if $L \in \text{S}_2\text{TIME}[p(n)]$ for some polynomial $p(n)$, and $L \in \text{S}_2\text{E}$, if $L \in \text{S}_2\text{TIME}[t(n)]$ for $t(n) \leq 2^{O(n)}$.

Definition 2.3 (Single-valued FS_2P algorithm). A single-valued FS_2P algorithm A is specified by a polynomial $\ell(\cdot)$ together with a polynomial-time algorithm $V_A(x, \pi_1, \pi_2)$. On an input $x \in \{0, 1\}^*$, we say that A outputs $y_x \in \{0, 1\}^*$, if the following hold:

1. There exists a $\pi_1 \in \{0, 1\}^{\ell(|x|)}$ such that for every $\pi_2 \in \{0, 1\}^{\ell(|x|)}$, $V_A(x, \pi_1, \pi_2)$ outputs y_x .
2. For every $\pi_1 \in \{0, 1\}^{\ell(|x|)}$ there exists a $\pi_2 \in \{0, 1\}^{\ell(|x|)}$, such that $V_A(x, \pi_1, \pi_2)$ outputs either y_x or $\perp$.

And we say that A solves a search problem Π if on any input x it outputs a string y_x and $y_x \in \Pi_x$, where a search problem Π maps every input $x \in \{0, 1\}^*$ into a solution set $\Pi_x \subseteq \{0, 1\}^*$.

We now formally define O_2TIME - the oblivious version of the class S_2TIME . The key difference is that unlike S_2TIME , where each irrefutable yes/no certificate can depend on the input x itself, in O_2TIME the yes/no certificates can **only** depend on $|x|$, the length of x . In other words, for every input length n , there exist a common YES-certificate $\mathbf{y}^*$ and a common NO-certificate $\mathbf{z}^*$ for checking membership of $x \in L|_n$.

Definition 2.4 (Oblivious Symmetric Time). Let $t : \mathbb{N} \rightarrow \mathbb{N}$. We say that a language $L \in \text{O}_2\text{TIME}[t(n)]$, if there exists a $O(t(n))$ -time predicate $P(x, y, z)$ such that for every $n \in \mathbb{N}$ there exist $\mathbf{y}^*$ and $\mathbf{z}^*$ of length $O(t(n))$ satisfying the following for every input $x \in \{0, 1\}^n$:

1. If $x \in L$, then for all z , $P(x, \mathbf{y}^*, z) = 1$.
2. If $x \notin L$, then for all y , $P(x, y, \mathbf{z}^*) = 0$.

Moreover, we say $L \in \text{O}_2\text{P}$, if $L \in \text{O}_2\text{TIME}[p(n)]$ for some polynomial $p(n)$, and $L \in \text{O}_2\text{E}$, if $L \in \text{O}_2\text{TIME}[t(n)]$ for $t(n) \leq 2^{O(n)}$.

2.2 Nonuniformity

We recall certain circuit properties:

Definition 2.5. A boolean circuit C with n inputs and size s , is a Directed Acyclic Graph (DAG) with $(s + n)$ nodes. There are n source nodes corresponding to the inputs labelled $1, \dots, n$ and one sink node labelled $(n + s)$ corresponding to the output. Each node, labelled $(n + i)$, for $1 \leq i \leq s$ has an in-degree of 2 and corresponds to a gate computing a binary operation over its two incoming edges.

Definition 2.6. Let $s : \mathbb{N} \rightarrow \mathbb{N}$. We say that a language $L \in \text{SIZE}[s(n)]$ if L can be computed by circuit families of size $O(s(n))$ for all sufficiently large input size n .

Definition 2.7. Let $s : \mathbb{N} \rightarrow \mathbb{N}$. We say that a language $L \in i.o.\text{-SIZE}[s(n)]$ if L can be computed by circuit families of size $O(s(n))$ for infinitely many input size n .

By definition, we have $\text{SIZE}[s(n)] \subseteq i.o.\text{-SIZE}[s(n)]$. Hence, circuit lower bounds against $i.o.\text{-SIZE}[s(n)]$ are stronger and sometimes denoted as *almost-everywhere* circuit lower bound in the literature.

We now state the hierarchy theorem for circuit size. The standard proof of this result is existential and goes through a counting argument (see e.g. [AB09]). However, we comment that using the framework of **Avoid**, we can now actually get a constructive size hierarchy theorem, albeit with worse parameters.

Theorem 2.8 (Circuit Size Hierarchy Theorem[AB09]). *For all functions $s : \mathbb{N} \rightarrow \mathbb{N}$ with $n \leq s(n) < o(2^n/n)$:*

$$\text{SIZE}[s(n)] \subsetneq \text{SIZE}[10s(n)] .$$

For our applications, it will be essential to have a tight encoding scheme for circuits. In fact, we will also need the fine-grained complexity of the Turing machine computing Circuit-Eval (i.e. given as input a description of a circuit C and a point x , computes $C(x)$).

Lemma 2.9. *For $n, s \in \mathbb{N}$, and $s \geq n \geq 12$, any n -input, s -size circuit C , there exists an encoding scheme $E_{n,s}$ which encodes C using $5s \log s$ bits.*

Proof. Let C be an n -input, s -size circuit, we now define $E_{n,s}$. Each gate label from $1, \dots, n + s$ can be encoded using $\log(n + s)$ bits. First encode the n inputs using $n \log(n + s)$ bits. Next fix a topological ordering of the remaining gates. For each gate we can encode its two inputs (two previous gates) with $2 \log(n + s)$ bits and the binary operation which requires 4 bits (since there are 16 possible binary operations). So the length of our encoding is $n \log(n + s) + s(2 \log(n + s) + 4) \leq 3s \log(2s) + 4s \leq 5s \log s$ for all $n \geq 12$. $\square$

Lemma 2.10. *For $n, s \in \mathbb{N}$, and $s \geq n$, let $E_{n,s}$ be an encoding of an n -input, s -size circuit C using [Lemma 2.9](#). Then there exists a multi-tape Turing machine M such that $M(E_{n,s}, x) = C(x)$ and it runs in $O(s^2 \log s)$ time.*

Proof. We utilize one tape (memory tape) to store all the intermediate values computed at each gate g_i using $n + s$ cells, and a second tape (evaluation tape) using 6 cells to compute the value at each g_i . We process each gate sequentially as it appears in the encoding scheme, and let g_{i_l} and g_{i_r} be the two gates feeding into g_i . Since [Lemma 2.9](#) encodes the gates in a topological order, we

can assume that when computing g_i , both g_{i_l} and g_{i_r} have already been computed. First copy the value of input bits of x onto the memory tape, and move the head of the input tape to the right by $n \log(n + s)$ steps in $O(s \log s)$ time. Now to compute a gate g_i we write the values of g_{i_l} and g_{i_r} along with the binary operation onto the evaluation tape. We can compute any binary operation with just constant overhead and write its value onto the i th cell of the memory tape. To output the evaluation of the circuit we output the value on the $(n + s)$ th cell of the memory tape. The cost of evaluating each gate is dominated by the 2 read and 1 write operations on the memory tape that take $O(s)$ time. Since the size of the input upper bounds the number of gates we have that the simulation takes $O(s|E_{n,s}|) = O(s^2 \log s)$ time. $\square$

Finally, we recall the famous Cook-Levin theorem that lets us convert a machine $M \in \text{TIME}[t(n)]$ into a circuit $C \in \text{SIZE}[t(n) \log t(n)]$.

Theorem 2.11 (Cook-Levin Theorem [AB09]). *Let $t : \mathbb{N} \rightarrow \mathbb{N}$ be a time constructible function. Then any multi-tape Turing machine running in $\text{TIME}[t(n)]$ time can be simulated by a circuit-family of $\text{SIZE}[t(n) \log t(n)]$.*

2.3 Range Avoidance

Definition 2.12. The Range Avoidance (**Avoid**) problem is defined as follows: given as input the description of a Boolean circuit $C : \{0, 1\}^n \rightarrow \{0, 1\}^m$, for $m > n$, find a $y \in \{0, 1\}^m$ such that $\forall x \in \{0, 1\}^n : C(x) \neq y$.

An important object that connects **Avoid** and circuit lower bound is the truth table generator circuit.

Definition 2.13. [CHR24, Section 2.3] For $n, s \in \mathbb{N}$ where $n \leq s \leq 2^n$, the truth table generator circuit $\text{TT}_{n,s} : \{0, 1\}^{L_{n,s}} \rightarrow \{0, 1\}^{2^n}$ maps a n -input size s circuit using $L_{n,s} = (s + 1)(7 + \log(n + s))$ bits of description¹² into its truth table. Moreover, such circuit can be uniformly constructed in time $\text{poly}(2^n)$.

For the purpose of obtaining fixed polynomial circuit lower bound, we generalise the truth table generator circuit above into one that outputs the prefix of the truth table. We also use a different encoding scheme (with constant factor loss in the parameter) for the convenience of presentation.

Definition 2.14. For $n, s \in \mathbb{N}$ where $12 \leq n \leq s \leq 2^n$ and $|E_{n,s}| = 5s \log s < 2^n$, the prefix truth table generator circuit $\text{PTT}_{n,s} : \{0, 1\}^{|E_{n,s}|} \rightarrow \{0, 1\}^{|E_{n,s}|+1}$ maps a n -input circuit of size s described with $|E_{n,s}|$ bits into the lexicographically first $|E_{n,s}| + 1$ entries of its truth table.

Since we want to prove lower bounds not just in the exponential regime, but also in the polynomial regime for any fixed polynomial, we need a more fine-grained analysis for the running time of uniformly generating $\text{PTT}_{n,s}$.

Lemma 2.15. *The prefix truth table generator circuit $\text{PTT}_{n,s} : \{0, 1\}^{|E_{n,s}|} \rightarrow \{0, 1\}^{|E_{n,s}|+1}$ has size $O(|E_{n,s}|^3)$ and can be uniformly constructed in time $O(|E_{n,s}|^3)$.*

¹²in fact, it maps a stack program of description size $L_{n,s}$ and it is known that every n -input size s circuit has an equivalent stack program of size $L_{n,s}$ [FM05].

Proof. Let M be the multi-tape Turing machine from [Lemma 2.10](#) that takes as input an encoding of a circuit and a bitstring, and evaluates the circuit on that bitstring. Let C be the circuit generated from [Theorem 2.11](#) that simulates M . Then $\text{SIZE}(C) = O(s^2 \log^2 s) = O(|E_{n,s}|^2)$. Making $|E_{n,s}| + 1$ copies of C for each output gate gives a circuit of size $O(|E_{n,s}|^3)$. $\square$

Theorem 2.16 ([\[Li24, CHR24\]](#)). *There exists a single-valued FS_2P algorithm for Avoid. Moreover, on input circuit $C : \{0, 1\}^n \rightarrow \{0, 1\}^{n+1}$, the algorithm runs in time $O(n|C|)^{13}$.*

Theorem 2.17 ([\[Li24, CHR24\]](#)). *There exists an explicit language $L \in \text{S}_2\text{E} \setminus \text{i.o.-SIZE}[2^n/n]$.*

Proof. For any $n \in \mathbb{Z}$, let $\text{TT}_n : \{0, 1\}^{2^n-1} \rightarrow \{0, 1\}^{2^n}$ be the truth table generator circuit. Let $f_n \in \{0, 1\}^{2^n}$ be the canonical solution output by the single-valued algorithm from [Theorem 2.16](#) on input TT_n .

The hard language L is defined as follows: for any $x \in \{0, 1\}^*$, $x \in L$ if and only if the $(x+1)$ -th bit of $f_{|x|} = 1$, treating x as an integer from 0 to $2^n - 1$. $\square$

2.4 Sparse Languages

We define some notions of sparsity below, we first introduce natural definitions of sparsity and *sparse extensions* in the polynomial regime, and then give their generalizations in the fine-grained setting.

Definition 2.18. A language $L \in \text{SPARSE}$ if for all n , $|L \cap \{0, 1\}^n| \leq \text{poly}(n)$. Moreover, L is called *uniformly-sparse* if $L \in \text{P} \cap \text{SPARSE}$.

It is easy to see that $\text{SPARSE} \subseteq \text{P/poly}$. That is, one can identify the yes-instances efficiently, albeit in non-uniform fashion. The purpose of introducing the *uniform-sparsity* is to be able to identify these inputs efficiently in a uniform fashion. Unfortunately, we cannot expect any such language L to lie even in a modestly hard class as, by definition, $L \in \text{P}$. The purpose of the *uniformly-sparse extensions*, on the other hand, is to bridge this gap. One can observe that unlike the *uniformly-sparse* languages, which are contained in P , languages with uniformly-sparse extension can even be undecidable! In particular, any unary language has uniformly-sparse extension in form of 1^* .

Definition 2.19. A language L has a *uniformly-sparse extension*, if there exists a L' s.t. :

1. $L \subseteq L'$
2. L' is *uniformly-sparse*

Generalizing the above definitions in the fine-grained setting, we get:

Definition 2.20. Let $t : \mathbb{N} \rightarrow \mathbb{N}$ be a computable function, then a language L is $t(n)$ -SPARSE if for all n , $|L \cap \{0, 1\}^n| = O(t(n))$. Moreover we say that L is $t(n)$ -uniformly-sparse if $L \in \text{TIME}[t(n)] \cap t(n)$ -SPARSE.

Definition 2.21. L has a $t(n)$ -uniformly-sparse extension, if there exists a L' s.t.:

¹³the running time was implicit in the proof of [\[Li24\]](#), but easy to verify.

1. $L \subseteq L'$
2. L' is $t(n)$ -uniformly-sparse.

Observe that **every** binary language L is 2^n -SPARSE. Furthermore, every such L has a trivial 2^n -uniformly-sparse extension: $\{0, 1\}^*$.

3 Lower Bounds & Hierarchy Theorem

In this section, we first present ([Theorem 3.1](#)) a fine-grained, parameterised version of [Theorem 2.17](#). This allows us to use the Avoid framework and get circuit lower bounds in $S_2\text{TIME}[t(n)]$ instead of S_2E . We then observe that our $S_2\text{TIME}[t(n)]$ witness is oblivious of the input, and hence the lower bounds we get are actually in $O_2\text{TIME}[t(n)]$ as highlighted in [Theorem 3.2](#).

In [Theorem 3.5](#) and [Theorem 3.6](#) we present the first time hierarchy theorem for O_2P . In fact, we note to the best of our knowledge that this is the first known time hierarchy theorem for a semantic class.

Theorem 3.1. *For $n \in \mathbb{N}$, let $t : \mathbb{N} \rightarrow \mathbb{N}$ be a time-constructible function, s.t. $t(n) > n \geq 12$ then*

$$S_2\text{TIME}[t(n)] \not\subseteq i.o.\text{-SIZE} \left[\frac{t(n)^{1/4}}{\log(t(n))} \right].$$

Proof. We construct a language $L_t \in S_2\text{TIME}[t(n)]$ and $L_t \not\subseteq i.o.\text{-SIZE} \left[\frac{t(n)^{1/4}}{\log(t(n))} \right]$.

For any $n \in \mathbb{N}$, let $s = \lfloor \frac{t(n)^{1/4}}{\log(t(n))} \rfloor$ and $|E_{n,s}| = \lceil 5s \log s \rceil$. Let $\text{PTT}_{n,s} : \{0, 1\}^{|E_{n,s}|} \rightarrow \{0, 1\}^{|E_{n,s}|+1}$ be the prefix truth table generator circuit as in [Definition 2.14](#). Let $f_n \in \{0, 1\}^{|E_{n,s}|+1}$ be the canonical solution to $\text{Avoid}(\text{PTT}_{n,s})$ as outputted by the single-valued algorithm from [Theorem 2.16](#).

The hard language L_t is defined as follows: for any $n \in \mathbb{Z}$, the characteristic string of $L_t|_n$ is set to be $\mathcal{X}_{L_t|_n} := f_n || 0^{2^n - |E_{n,s}| - 1}$.

By definition of $\text{PTT}_{n,s}$ and the fact that $f_n \notin \text{Image}(\text{PTT}_{n,s})$, we have that $L_t \notin i.o.\text{-SIZE}[s]$. On the other hand, the single-valued algorithm for finding f_n runs in time $O(|E_{n,s}| \cdot |\text{PTT}_{n,s}|) = O(t(n))$. Hence, $L_t \in S_2\text{TIME}[t(n)]$. $\square$

We make the observation that the witness in the $S_2\text{TIME}$ machine above is oblivious to the actual input x .

Theorem 3.2. *For $n \in \mathbb{N}$, let $t : \mathbb{N} \rightarrow \mathbb{N}$ be a time-constructible function, s.t. $t(n) > n \geq 12$ then*

$$O_2\text{TIME}[t(n)] \not\subseteq i.o.\text{-SIZE} \left[\frac{t(n)^{1/4}}{\log(t(n))} \right].$$

Proof. Consider the same language L_t in the proof of [Theorem 3.1](#). Notice that for any input x of the same length n , the FS_2P algorithm is run on the same circuit $\text{PTT}_{n,s}$ and hence the witness is the same for inputs of the same length. Thus, it follows that $L_t \in O_2\text{TIME}[t(n)]$. $\square$

We now get as a corollary a proof of [Theorem 1](#).

Corollary 3.3. *For all $k \in \mathbb{N}$, there exists an explicit language $L_k \in O_2P$ s.t. $L_k \not\subseteq \text{SIZE}[n^k]$.*

Proof. Fix $t(n) = n^{5k}$. Then there is an explicit hard language L_t as defined in the proof of [Theorem 3.1](#), such that $L_t \not\subseteq \text{SIZE}[n^k]$. Moreover, by [Theorem 3.2](#) we have that $L_t \in \text{O}_2\text{P}$. $\square$

Before proving our hierarchy theorem for O_2TIME , we prove a simple lemma that bounds from above the size of a circuit family computing languages in O_2TIME .

Lemma 3.4. $\text{O}_2\text{TIME}[t(n)] \subseteq \text{SIZE}[t(n) \log(t(n))]$.

Proof. Consider any language $L \in \text{O}_2\text{TIME}[t(n)]$, and let $V(\cdot, \cdot, \cdot)$ be its $t(n)$ -time verifier. For any integer $n \in \mathbb{N}$, let $y_n, z_n \in \{0, 1\}^{t(n)}$ be the irrefutable proofs for input size n . By [Theorem 2.11](#) we can convert $V(\cdot, \cdot, \cdot)$ into a circuit family $\{C_n\} \subseteq \text{SIZE}[t(n) \log(t(n))]$. The values y_n and z_n can be hard-coded into C_n , and hence this circuit will decide L on all inputs of size n . $\square$

Having both an upper bound on the size of circuits simulating an O_2TIME computation, and also a lower bound for O_2TIME against circuits, we can use the circuit size hierarchy ([Theorem 2.8](#)) to define a time hierarchy on O_2TIME .

Theorem 3.5. For $n \in \mathbb{N}$, let $t : \mathbb{N} \rightarrow \mathbb{N}$ be a time constructible function, s.t. $t(n) > n \geq 12$ then: $\text{O}_2\text{TIME}[t(n)] \subsetneq \text{O}_2\text{TIME}[t(n)^4 \log^9(t(n))]$.

Proof. Combining [Theorem 3.2](#), [Lemma 3.4](#), and Circuit Size Hierarchy ([Theorem 2.8](#)) we have:

$$\text{O}_2\text{TIME}[t(n)] \subseteq \text{SIZE}[t(n) \log t(n)] \subsetneq \text{SIZE}[t(n) \log^{\frac{5}{4}} t(n)] ,$$

and

$$\text{O}_2\text{TIME}[t(n)^4 \log^9(t(n))] \not\subseteq \text{SIZE}[t(n) \log^{\frac{5}{4}} t(n)] .$$

$\square$

Theorem 3.6. For $n \in \mathbb{N}$, let $t : \mathbb{N} \rightarrow \mathbb{N}$ be a time constructible function, s.t. $t(n) \geq n$ then: for all $\varepsilon > 0$, $\text{O}_2\text{TIME}[t(n)] \subsetneq \text{O}_2\text{TIME}[t(n)^{1+\varepsilon}]$.

Proof. Let us assume that $\text{O}_2\text{TIME}[t(n)^{1+\varepsilon}] \subseteq \text{O}_2\text{TIME}[t(n)]$, then by translation we have that:

$$\text{O}_2\text{TIME}[t(n)^{(1+\varepsilon)^2}] \subseteq \text{O}_2\text{TIME}[t(n)^{1+\varepsilon}] \subseteq \text{O}_2\text{TIME}[t(n)]$$

Inducting on any $k > 2$, we get that $\text{O}_2\text{TIME}[t(n)^{(1+\varepsilon)^k}] \subseteq \text{O}_2\text{TIME}[t(n)]$. Now setting $k = \lceil 3.1/\varepsilon \rceil$, by Bernoulli's inequality we have that $(1 + \varepsilon)^k \geq (1 + \varepsilon)^{3.1/\varepsilon} \geq 4.1$. Therefore, $\text{O}_2\text{TIME}[t(n)^{4.1}] \subseteq \text{O}_2\text{TIME}[t(n)]$, which contradicts [Theorem 3.5](#). $\square$

Remark 3.7. We note here that while we are able to achieve a better gap in our hierarchy theorem in [Theorem 3.6](#) over [Theorem 3.5](#), there is a trade off. The hierarchy theorem defined in [Theorem 3.5](#) is explicit, that is we have an explicit language not known to be contained in the smaller class. However, when we apply our translation argument to get better parameters in [Theorem 3.6](#) we lose this explicitness.

4 Sparsity

In this section, we use *sparse extensions* to get various structural complexity results. We prove a more fine-grained statement of [Theorem 3](#) which states that any language in $S_2\text{TIME}[t(n)]$ with a *uniformly-sparse extension* is actually in $O_2\text{TIME}[t(n)^2]$. This lets us extract as a corollary another proof of $S_2E = O_2E$. As another application of *sparse extensions*, we are able to recover the fixed polynomial lowerbounds for O_2P from the previous section as stated in [Theorem 1](#). Finally we show connections between *sparse extensions* and open problems posed by [\[GM15\]](#).

Lemma 4.1. *Let $L \in S_2\text{TIME}[t(n)]$. If L has a $t(n)$ -uniformly-sparse extension then $L \in O_2\text{TIME}[t(n)^2]$.*

Proof. For any n , let L' be the $t(n)$ -uniformly-sparse extension of L , and let $\mathcal{F}$ be the $\text{TIME}[t(n)]$ predicate that decides membership in L' . We will now design an $O_2\text{TIME}[t(n)^2]$ verifier V for $L|_n$. Since both L and L' are $t(n)$ -SPARSE, we have that for most $x \in \{0, 1\}^n$: $L|_n(x) = L'|_n(x) = 0$. V will first use $\mathcal{F}$ to efficiently filter out most non-membership in $L'|_n$, and hence $L|_n$ in $\text{TIME}[t(n)]$. Now V only has to decide membership in L over $t(n)$ many inputs $X = \{x \in \{0, 1\}^n : \mathcal{F}(x) = 1\}$. We will use the fact that since $L \in S_2\text{TIME}[t(n)]$, for all $x \in X$, if $x \in L$ there is an irrefutable YES certificate y_x and if $x \notin L$ there is an irrefutable NO certificate z_x and a verifier V^* , running in $\text{TIME}[t(n)]$ s.t.

- if $x \in L$, $\exists y_x, \forall z$ s.t. $V^*(x, y_x, z) = 1$
- if $x \notin L$, $\exists z_x, \forall y$ s.t. $V^*(x, y, z_x) = 0$

$V(x, Y^*, Z^*) :$

- (1) Set output = 1.
- (2) If $\mathcal{F}(x) = 0$, return 0.
- (3) Parse Y^* to get y_x^* .
- (4) For $z_i \in Z^*$, do:
 - (a) output = output $\wedge V^*(x, y_x^*, z_i)$.
- (5) Return output.

Figure 1: $O_2\text{TIME}[t(n)^2]$ Verifier for Language in $S_2\text{TIME}[t(n)]$ with $t(n)$ -uniformly-sparse extension

Consider the string Y^* which encodes a table of YES witnesses y_x^* for every input $x \in X$. When $x \in L$ we set $y_x^* = y_x$, and when $x \notin L$ we will set $y_x^* = 0^{t(n)}$. The size of Y^* is $O(t(n)^2)$, since there are at most $t(n)$ entries in the table each of length $t(n) + n$. For every $x \in X \cap \overline{L}$, let z_x be the irrefutable NO-certificate corresponding to x for V^* . We set Z^* to be the concatenation of all such z_x . The size of Z^* is also at most $t(n)^2$.

We now show that Y^* and Z^* will serve as oblivious irrefutable “YES” and “NO” certificates respectively for V . On input (x, Y^*, Z^*) , V first parses Y^* to find the corresponding y_x^* in time

$\text{TIME}[t(n)^2]$. Then for each $z_i \in Z^*$ we run $V^*(x, y_x^*, z_i)$. If for all z_i , $V^*(x, y_x^*, z_i) = 1$ then V outputs 1, otherwise V will output 0. Since we are making at most $t(n)$ calls that each cost $\text{TIME}[t(n)]$, V runs in $\text{TIME}[t(n)^2]$.

To see correctness, we first analyze the case when $x \in L$, then by construction Y^* includes $y_x^* = y_x$ and V will output 1. On the other hand if $x \notin L$ then there is an irrefutable no-certificate z_x in Z^* so there is no y_i such that $V(x, y_i, z_x) = 1$. Hence V outputs 0. $\square$

By taking $t(n)$ to be a polynomial in [Lemma 4.1](#) we directly get [Corollary 4.2](#) (also [Theorem 3](#)) relating O_2P and S_2P .

Corollary 4.2. *If $L \in \text{S}_2\text{P}$ and L has an uniformly-sparse extension, then $L \in \text{O}_2\text{P}$*

Similarly, one can prove [Theorem 4](#) by showing the same consequence for coNP vs coONP , thus making a partial progress towards the open questions posed by Goldreich and Meir in [\[GM15\]](#). In the exponential regime, since all languages have the trivial 2^n -uniformly-sparse extension we get the equivalence between O_2E and S_2E as seen in [Corollary 4.3](#).

Corollary 4.3. $\text{S}_2\text{E} = \text{O}_2\text{E}$

Proof. As noted in [Section 2.4](#), every language is 2^n -SPARSE, and has the trivial 2^n -uniformly-sparse extension: $\{0, 1\}^*$. When $t(n) = 2^n$, by [Lemma 4.1](#) we get that $\text{S}_2\text{TIME}[2^n] \subseteq \text{O}_2\text{TIME}[2^{2^n}]$. $\square$

In particular, the following lemma shows that the hard language in $\text{S}_2\text{TIME}[t(n)]$ defined in [Theorem 3.1](#) admits a $t(n)$ -uniformly-sparse extension, giving another proof of [Corollary 3.3](#).

Lemma 4.4. *For $n \in \mathbb{N}$, let $t : \mathbb{N} \rightarrow \mathbb{N}$ be a time-constructible function, s.t. $t(n) > n \geq 12$ then, there is an explicit language $L_t \in \text{S}_2\text{TIME}[t(n)]$ s.t. $L_t \notin \text{SIZE}\left[\frac{t(n)^{1/4}}{\log(t(n))}\right]$. Moreover, L_t has a $t(n)$ -uniformly-sparse extension L'_t .*

Proof. Let L_t be the $\text{S}_2\text{TIME}[t(n)]$ language defined in the proof [Theorem 3.1](#) with the characteristic string $\mathcal{X}_{L_t|n} := f_n || 0^{2^n - |E_{n,s}| - 1}$. We now define the language L'_t whose characteristic string $\mathcal{X}_{L'_t|n} := 1^{|E_{n,s}|+1} || 0^{2^n - |E_{n,s}| - 1}$. To see that this L'_t is a $t(n)$ -uniformly-sparse extension of L_t , clearly $L_t \subseteq L'_t$. Moreover membership of $x \in L'_t$ can be decided by checking if the binary value of x is less than or equal to $|E_{n,s}| + 1$ which can be done in $\text{TIME}[n] \subseteq \text{TIME}[t(n)]$. $\square$

Equipped with this lemma we have an alternative proof of fixed polynomial lower bounds for O_2P as stated in [Theorem 1](#).

Corollary 4.5. ([Theorem 1](#)) *For every $k \in \mathbb{N}$, $\text{O}_2\text{P} \not\subseteq \text{SIZE}[n^k]$. Moreover, for every k there is an explicit language L_k in O_2P s.t. $L_k \notin \text{SIZE}[n^k]$.*

Proof. Fix $t(n) = n^{5k}$. Then by [Lemma 4.4](#) there is an explicit language L_k such that $L_k \not\subseteq \text{SIZE}[n^k]$, and L_k has an uniformly-sparse extension. Applying [Lemma 4.1](#) we have that $L_k \in \text{O}_2\text{TIME}[n^{10k}] \subseteq \text{O}_2\text{P}$. $\square$

5 Extension

In this section we show how to extend our result to even, potentially, smaller class $O_2P \cap L_2P$, where L_2P is a complexity classes defined by Korten and Pitassi in [KP24].

Definition 5.1 ([KP24]). A language L belongs to L_2P , if there is a polynomial-time predicate $R(x, y, z)$ and a polynomial p , so that for all $x : R(x, \cdot, \cdot)$ defines a total order on $\{0, 1\}^{p(n)}$ whose minimal element m satisfies $m_1 = L(x)$. That is, for every x , the first bit of the minimal elements denotes whether $x \in L$.

It follows directly from the definition that $L_2P \subseteq S_2P$. Moreover, it has been also shown that L_2P can equivalently be characterized as the class of all languages that are many-to-one reducible, to the so-called *Linear Ordering Principle* (LOP), in polynomial time. At the same time, Avoid also reduces to LOP. Consequently, one can formulate the following analogue of [Theorem 2.16](#).

Theorem 5.2 ([KP24]). *Avoid is many-to-reducible to L_2P , in polynomial time.*

Given that, one can repeat the argument of [Theorem 3.1](#), invoking [Theorem 5.2](#) instead of [Theorem 2.16](#), to obtain the following analogue of [Corollary 3.3](#).

Corollary 5.3. *For all $k \in \mathbb{N}$, there exists an explicit language $A_k \in L_2P$ s.t. $A_k \not\subseteq \text{SIZE}[n^k]$. Moreover, A_k has an uniformly-sparse extension.*

Finally, as $L_2P \subseteq S_2P$ the explicit “hard” languages actually collapse all the way to $L_2P \cap O_2P$.

Corollary 5.4. *For all $k \in \mathbb{N}$, there exists an explicit language $A_k \in L_2P \cap O_2P$ s.t. $A_k \not\subseteq \text{SIZE}[n^k]$.*

Proof. As $L_2P \subseteq S_2P$ and for all $k \in \mathbb{N} : A_k$ has an uniformly-sparse extension, by [Corollary 4.2](#) $A_k \in O_2P$. $\square$

6 Open Problems

We conclude with a few interesting open problems:

- Can we show that every sparse S_2P language is also in O_2P ?
- Can we tighten the gap in the $O_2\text{TIME}$ hierarchy theorem ([Theorem 2](#))?
- Can we show a non-trivial upper bound for O_2P , for example P^{NP} , MA, PP? This would imply explicit fixed-polynomial lower bounds for such classes. On the other hand, we do note that under reasonable derandomization assumptions, $O_2P \subseteq S_2P = P^{\text{NP}}$.
- Can we arrive at something interesting about time hierarchy theorem for semantic classes where fixed-polynomial lower bounds are known e.g. S_2P , ZPP^{NP} , assuming $\text{NP} \not\subseteq P/\text{poly}$? For instance, if $\text{NP} \subseteq P/\text{poly}$, then it follows that $S_2P \subseteq P/\text{poly}$. One could then invoke the circuit size hierarchy theorem ([Theorem 2.8](#)) to establish a hierarchy theorem for $S_2\text{TIME}$, similar to how we obtain the hierarchy theorem for $O_2\text{TIME}$.

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