Central limit theorem for the sine-β point process at $\beta \leq 2$

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Abstract

The purpose of this paper is to establish the analogue of the Soshnikov Central Limit Theorem for the sine-β process at $\beta \leq 2$. We consider regularized additive functionals, which correspond to 1-Sobolev regular functions $f(x/R)$, in the limit $R \to \infty$. Their convergence to the Gaussian distribution with respect to the Kolmogorov-Smirnov metric at the rate $(\ln R)^{-1/2}$ is established.

The proof is based on the convergence of the circular β-ensemble to the sine-β process, which was shown by Killip and Stoiciu. We find a convenient bound for the Laplace transforms of additive functionals under the circular β-ensemble, which holds under the scaling limit, suggested by Killip and Stoiciu. Further, we show that the additive functionals under the circular β-ensemble with n particles converge to the gaussian distribution as $n \to \infty$ for all 1/2-Sobolev regular functions for $\beta \leq 2$, as was conjectured by Lambert.

Finally, in order to prove the limit theorem for the circular β-ensemble we derive the connection between expectations of multiplicative functionals and the Jack measures, which generalizes the connection between the circular unitary ensemble and the Schur measures given by Gessel's theorem.

1 Introduction

1.1 The circular β-ensemble

The circular β-ensemble is a measure on n-point configurations of the unit circle, defined by the formula

$$d\mathbb{P}^n_\beta(\theta_1, \dots, \theta_n) = Z^{-1}_{n,\beta} \prod_{1 \leq l \leq m \leq n} |e^{i\theta_l} - e^{i\theta_m}|^\beta \prod_{k=1}^n d\theta_k, \quad \theta_j \in (-\pi, \pi),$$

where $Z_{n,\beta}$ is the normalization constant. For $\beta = 2$ the measure reduces to the circular unitary ensemble. As Gessel's theorem states, in this case expectations of multiplicative functionals may be expressed in terms of specializations of Schur polynomials $\{s_\lambda\}$ indexed by partitions, containing at most n non-zero integers $l(\lambda) \leq n$. We refer the reader to Subsection 2.1 for the notation, definitions and basic facts about symmetric functions.

Theorem 1.1 (Gessel [22, 45])**.** *Let f be a continuous function on the unit circle. Then we have the expansion*

$$e^{-n\hat{f}_0} \mathbb{E}^n_2 \left\{ \prod_{j=1}^n e^{f(\theta_j)} \right\} = \sum_{\lambda : l(\lambda) \leq n} s_\lambda(\rho_+) s_\lambda(\rho_-),$$

where $s_\lambda \mapsto s_\lambda(\rho_\pm)$ are homomorphisms from the algebra of symmetric functions $\Lambda \to \mathbb{C}$ defined on the Newton power sums by the formula

$$p_j(\rho_\pm) = j\hat{f}_{\pm j}, \quad p_j(x) = \sum_{l \in \mathbb{N}} x_l^j \in \Lambda, \quad f(\theta) = \sum_{j \in \mathbb{Z}} \hat{f}_j z^{ij\theta}.$$

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The Szegő-Heine Theorem [41, Theorem 1.5.13] states that the expectations described by Theorem 1.1 may be expressed in terms of Toeplitz determinants. Theorem 1.1 then follows from the Cauchy-Binet formula, applied to these determinants, and the Jacobi-Trudi identity for Schur polynomials.

Gessel's expansion establishes the connection between the circular unitary ensemble and the Schur measures (see Subsection 1.3). Our first result is the existence of such connection for general β if one replaces the Schur measures with the Jack measures. Before stating it let us discuss how Theorem 1.1 implies the convergence of additive functionals under $\mathbb{P}_2^n$ in the $n \to \infty$ limit.

A notable consequence of Gessel's theorem is the Strong Szegő Theorem, which states that

$$e^{-n\hat{f}_0}\mathbb{E}_2^n\left\{\prod_{j=1}^n e^{f(\theta_j)}\right\} \to e^{\sum_{k\geq 1} k\hat{f}_k\hat{f}_{-k}}, \qquad \text{as } n \to \infty.$$

Indeed, one obtains the expression for the limit of the considered expectations by applying the homomorphisms $\rho_\pm$ to the Cauchy identity

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \exp\left(\sum_{k\geq 1}\frac{p_k(x)p_k(y)}{k}\right).$$

Finally, the Szegő Theorem implies the convergence of additive functionals $S_f = \sum_{j=1}^n f(\theta_j)$ to the Gaussian distribution without normalization.

The above arguments rely on determinantal structure of the circular unitary ensemble. For general β, the measure $\mathbb{P}_\beta^n$ does not have this property. Recall, however, that Jack [25] introduced a family of symmetric functions $\{J_\lambda^{(\alpha)}\}$, which are parametrized by partitions (see Subsection 2.1). It was further shown by Macdonald [34] that they are orthogonal with respect to the circular β-ensemble

$$\mathbb{E}_{2/\alpha}^n J_\lambda^{(\alpha)}(e^{i\theta_1},\dots,e^{i\theta_n})\overline{J_\mu^{(\alpha)}(e^{i\theta_1},\dots,e^{i\theta_n})} = 0, \qquad \text{for } \lambda \neq \mu. \tag{1.1}$$

For $\alpha = 1$ these functions are Schur polynomials, for which the orthogonality (1.1) follows from the Andréief formula [1]. Together, the orthogonality (1.1) and the Cauchy identity for Jack polynomials

$$\sum_{\lambda\in\mathbb{Y}}\frac{J_\lambda^{(\alpha)}(x)J_\lambda^{(\alpha)}(y)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha} = \exp\left(\frac{1}{\alpha}\sum_{j=1}^\infty\frac{p_j(x)p_j(y)}{j}\right) \tag{1.2}$$

imply for general $\beta = 2/\alpha$

Theorem 1.2 (Gessel-type expansion for the circular β-ensemble)**.** *For any function f on the unit circle, satisfying $\sum_{k\in\mathbb{Z}}|k||\hat{f}_k|^2 < \infty$, and $\alpha \geq 1$ we have*

$$\mathbb{E}_{2/\alpha}^n\left\{\prod_{j=1}^n e^{f(\theta_j)}\right\} = \sum_{\lambda: l(\lambda)\leq n}\frac{J_\lambda^{(\alpha)}(\rho_-)J_\lambda^{(\alpha)}(\rho_+)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\mathcal{A}_\lambda^\alpha(n), \quad \mathcal{A}_\lambda^\alpha(n) = \prod_{(i,j)\in\lambda}\left(1 - \frac{\alpha-1}{n+j\alpha-i}\right),$$

where $(i,j) \in \lambda$ if $i,j \geq 1$ and $j \leq \lambda_i$. The scalar product $\langle\cdot,\cdot\rangle_\alpha$ is defined by the formula (2.1)*. The homomorphisms $\rho_\pm$ are defined by*

$$p_n(\rho_\pm) = \alpha j\hat{f}_{\pm j}.$$

Further, we have that the series

$$\sum_{\lambda\in\mathbb{Y}}\frac{J_\lambda^{(\alpha)}(\rho_-)J_\lambda^{(\alpha)}(\rho_+)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}$$

converge absolutely.

Remark. The requirement $\alpha \geq 1$ ensures the convergence of the expansion. The assertion still holds if one considers the claimed equality to be the equality between formal power series in the Fourier coefficients of f.

Observe the convergence $\mathcal{A}^{\alpha}_{\lambda}(n) \to 1$ as $n \to \infty$. As above we use the β-analogue of the Cauchy identity (1.2) to deduce the generalization of the Szegő Theorem.

Corollary 1.3 (Central Limit Theorem for the circular-β ensemble)**.** *Let f be a function on the unit circle, satisfying $\sum_{k\in\mathbb{Z}}|k||\hat{f}_k|^2 < \infty$. Assume $\beta \leq 2$.*

- *Additive functionals converge to the Gaussian distribution:*

$$e^{-n\hat{f}_0}\mathbb{E}^n_\beta\left\{\prod_{j=1}^{n} e^{f(\theta_j)}\right\} \to e^{\frac{2}{\beta}\sum_{k\geq 1} k\hat{f}_k\hat{f}_{-k}}, \qquad \text{as } n \to \infty.$$

- *If f is real-valued, then the respective additive functional under the circular β-ensemble is subgaussian*

$$e^{-n\hat{f}_0}\mathbb{E}^n_\beta\left\{\prod_{j=1}^{n} e^{f(\theta_j)}\right\} \leq e^{\frac{2}{\beta}\sum_{k\geq 1} k\hat{f}_k\hat{f}_{-k}} \tag{1.3}$$

  uniformly in n.

Theorem 1.2 and Corollary 1.3 are proved in Subsection 2.2.

The idea of using Jack polynomials is not novel. Corollary 1.3 has been established from the orthogonality (1.1) by Jiang and Matsumoto [28] for trigonometric polynomials and general β. For $\beta = 1$ they established the convergence for all 1/2-Sobolev regular functions. For $\beta = 4$ they were able to show the convergence for all functions f, satisfying

$$\sum_{j\in\mathbb{Z}}|j|\ln(1+|j|)|\hat{f}_j|^2 < \infty.$$

Our argument proves the first claim of Corollary 1.3 for arbitrary β if one restricts to functions f, which are holomorphic in a sufficiently large neighborhood of the unit circle. The requirement $\beta \leq 2$ is explained by the coefficients $\mathcal{A}^{\alpha}_{\lambda}(n)$ exponentially growing for large partitions. For $\beta \leq 2$, on the other hand, they may be estimated $\mathcal{A}^{\alpha}_{\lambda}(n) \leq 1$. Using the latter, one may show that Proposition 1 in [28] and, thereby, the convergence to the Gaussian distribution (see proof of Theorem 3 in [28]) hold for arbitrary $\beta \leq 2$.

Another proof of Corollary 1.3 for arbitrary β and sufficiently smooth real-valued function f was given by Lambert [33]. He deduces the convergence from the loop equations, introduced by Johansson [27]. Lambert's method involves analysis of deformations of the circular β-ensemble by multiplicative functionals.

While the condition of 1/2-Sobolev regularity is optimal for $\beta \leq 2$, the question of optimal conditions for $\beta > 2$ is open. Lambert [33] conjectured that 1/2-Sobolev regularity is sufficient if and only if $\beta \leq 2$. Using estimates from Jiang and Matsumoto [28], he gave an example of a function $f \in H_{1/2}(\mathbb{T})$, for which the variance of the respective additive functional $S_f = \sum_{j=1}^{n} f(\theta_j)$ under $\mathbb{P}^n_4$ diverges as $n \to \infty$.

We proceed to the estimate on the rate of the convergence. For a function f on the unit circle denote

$$\|f\|^2_{\dot{H}_p(\mathbb{T})} = \sum_{k\in\mathbb{Z}}|k|^{2p}|\hat{f}_k|^2.$$

Let $H_p(\mathbb{T})$ stand for the Hilbert space of functions on the unit circle endowed with the norm $\sqrt{\|\cdot\|^2_{\dot{H}_p(\mathbb{T})} + \|\cdot\|^2_{L_2(\mathbb{T})}}$ — the space of p-Sobolev regular functions.

Theorem 1.4. *Let $f \in H_1(\mathbb{T})$, assume $\beta \leq 2$. We have*

$$\left|\mathbb{E}^{2n}_\beta\left\{\prod_{j=1}^{n} e^{f(\theta_j)}\right\}\exp\left(-2n\hat{f}_0 - \frac{2}{\beta}\sum_{k\geq 1} k\hat{f}_k\hat{f}_{-k}\right) - 1\right| \leq \frac{8}{\beta^2 n}\exp\left(\frac{1}{\beta}\|f\|^2_{\dot{H}_{1/2}(\mathbb{T})} - \frac{2}{\beta}\Re\sum_{k\geq 1} k\hat{f}_k\hat{f}_{-k}\right)\|f\|^2_{\dot{H}_1(\mathbb{T})}. \tag{1.4}$$

Theorem 1.4 is proved in Subsection 2.3.

Let us again mention the case $\beta = 2$. In this case Theorem 1.1 connects expectations of multiplicative functionals with Schur measures, introduced by Okounkov [38]. The determinantal structure of Schur measures (see [11, 26], also [6] for a review) yields an exact formula [10] for the left-hand side of inequality (1.4) in terms of certain Fredholm determinant. This exact formula implies Theorem 1.4 for $\beta = 2$.

Last, let us say a few more words about related works. The mentioned work of Jiang and Matsumoto [28] has been pursued by Webb [50], who used Stein's method to deduce the rate of the convergence. β-ensembles on the real line were considered in the works [9, 7, 44] for a general class of potentials. Borodin, Gorin and Guionnet established convergence of additive functionals under discrete β-ensembles [8]. We refer the reader to Forrester [21] for a general review on β-ensembles.

1.2 The sine-β process

Let $X_n = \{\theta_1, \dots, \theta_n\}$, $\theta_j \in (-\pi, \pi)$ be a random configuration, distributed according to the circular β-ensemble $X_n \sim \mathbb{P}^n_\beta$. As has been shown by Killip and Stoiciu [30], for any compactly supported continuously differentiable function f on the real line, the random variable

$$\sum_{x \in nX_n} f(x), \quad nX_n = \{n\theta_1, \dots, n\theta_n\}$$

has a limit as $n \to \infty$. These limit distributions define a point process on infinite locally finite subsets of $\mathbb{R}$ — the sine-β process $\mathbb{P}_\beta$. If $X \sim \mathbb{P}_\beta$, then we have

$$\lim_{n\to\infty} \sum_{x \in nX_n} f(x) \overset{\text{distr}}{=} \sum_{x \in X} f(x),$$

which characterizes the sine-β process. The limit measure may be also described by an infinite system of stochastic differential equations. We recall this construction and the precise convergence statement from the paper [30] in Section 4.

Let $\mathrm{Conf}(\mathbb{R})$ stand for the set of locally finite subsets of $\mathbb{R}$ with multiplicities. Both $\mathbb{P}^n_\beta$ and $\mathbb{P}_\beta$ are measures on $\mathrm{Conf}(\mathbb{R})$ with certain σ-algebra (see Subection 4.1). For a bounded Borel compactly supported function additive functional is defined by the formula

$$S_f(X) = \sum_{x \in X} f(x), \quad X \in \mathrm{Conf}(\mathbb{R}).$$

We have that $S_f \in L_1(\mathrm{Conf}(\mathbb{R}), \mathbb{P}_\beta)$. Regularized additive functional is defined by the formula

$$\overline{S}_f(X) = S_f(X) - \mathbb{E}_\beta S_f.$$

As will be shown in Subsection 3.1, its L_2 norm is continuous with respect to the 1/2-Sobolev seminorm of f. Thereby the above definition may be extended to all 1/2-Sobolev regular functions. The convergence $\mathbb{P}^n_\beta \to \mathbb{P}_\beta$ may be stated in terms of these functionals as follows

Proposition 1.5. *For any continuously differentiable compactly supported function f and $\beta \le 2$, we have*

$$\mathbb{E}^n_\beta \exp\left(S_{f(n\cdot)}\right) \to \mathbb{E}_\beta e^{S_f}, \qquad \text{as } n \to \infty.$$

Remark. Proposition 1.5 was proved by Killip and Stoiciu [30] under the restriction $\Re f \le 0$. This condition ensures that

$$\left|e^{S_f}\right| \le 1,$$

and, in particular, that the respective multiplicative functional is not too large. Combining their argument with the uniform subgaussianity stated in Corollary 1.3, we deduce that the claimed convergence holds for all bounded smooth compactly supported functions.

Proposition 1.5 is proved in Section 4.

For a function f on the real line define

$$\|f\|^2_{\dot{H}_p(\mathbb{R})} = \int_{\mathbb{R}} |\lambda|^{2p} |\hat{f}(\lambda)|^2 d\lambda.$$

Let $H_p(\mathbb{R})$ be the Hilbert space of functions on the real line endowed with the norm $\sqrt{\|\cdot\|^2_{\dot{H}_p(\mathbb{R})} + \|\cdot\|^2_{L_2(\mathbb{R})}}$.

Substitution of the functions $f(2n\cdot)$ into the inequalities (1.4), (1.3) and application of Proposition 1.5 give

Theorem 1.6 (Central Limit Theorem for the sine-β process)**.** *Assume $\beta \le 2$. For any $f \in H_1(\mathbb{R})$, we have*

$$\left|\mathbb{E}_\beta e^{\overline{S}_f} \exp\left(-\frac{2}{\beta}\int_0^\infty \lambda \hat{f}(\lambda)\hat{f}(-\lambda)d\lambda\right) - 1\right| \le \frac{16}{\beta^2 n} \exp\left(\frac{1}{\beta}\|f\|^2_{\dot{H}_{1/2}(\mathbb{R})} - \frac{2}{\beta}\Re\int_0^\infty \lambda \hat{f}(\lambda)\hat{f}(-\lambda)d\lambda\right)\|f\|^2_{\dot{H}_1(\mathbb{R})}. \tag{1.5}$$

Further, for any real-valued $f \in H_{1/2}(\mathbb{R})$, we have that the respective additive functional is subgaussian

$$\mathbb{E}_\beta e^{\overline{S}_f} \le \exp\left(\frac{2}{\beta}\int_0^\infty \lambda \hat{f}(\lambda)\hat{f}(-\lambda)d\lambda\right).$$

Remark. Observe that the limit variance is invariant under the dilation of a function:

$$\int_0^\infty \lambda R\hat{f}(R\lambda)R\hat{f}(-R\lambda)d\lambda = \int_0^\infty \lambda \hat{f}(\lambda)\hat{f}(-\lambda)d\lambda.$$

Further, we have $\|f(\cdot/R)\|_{\dot{H}_p} = R^{1/2-p}\|f\|_{\dot{H}_p}$. Therefore, Theorem 1.6 implies the convergence of the additive functionals $\overline{S}_{f(\cdot/R)}$ to the Gaussian distribution as $R \to \infty$.

Theorem 1.6 is proved in Section 3.

We are able to prove only the inequality for multiplicative functionals under the sine-β process. It is interesting if one may derive the scaling limit of the expansion stated in Theorem 1.2 and deduce the connection between the sine-β process and the scaling limits of the Jack measures (see Question 1.9). This was done by Bufetov [4] for $\beta = 2$. The limit of Schur measures in this case is a determinantal point process induced by a product of two Hankel operators.

For a real-valued function f denote

$$F_f(x) = \mathbb{P}_\beta(\overline{S}_f \le x), \quad F_{\mathcal{N}}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{t^2}{2}}dt.$$

Theorem 1.6 implies

Corollary 1.7. *For any real-valued $f \in H_1(\mathbb{R})$ and $\beta \le 2$, there exists a constant C such that for any $R > 0$, the inequality*

$$\sup_x |F_{f(\cdot/R)} - F_{\mathcal{N}}| \le \frac{C}{\sqrt{\ln R}}$$

holds, provided the function f is normalized so that

$$\int_0^\infty \lambda \hat{f}(\lambda)\hat{f}(-\lambda)d\lambda = \frac{\beta}{4}.$$

Corollary 1.7 is proved in Section 5.

The convergence of additive functionals has been shown by Leblé in [32], who used the DLR equations for the sine-β process [18]. Another proof is due to Lambert [33]. Lambert relied on the coupling of the sine-β process with the circular β-ensemble, introduced by Valkó and Virág [49]. Both methods require the function to be compactly supported. This restriction is, however, unnatural, as may be seen with the example of $\beta = 2$, for which the exact formula is available — the scaling limit of the Borodin-Okounkov formula [2, 4]. This formula implies Theorem 1.6 for $\beta = 2$ and, in particular, yields that the convergence of $\overline{S}_{f(\cdot/R)}$ depends not on the decay of the function f itself, but rather on the decay of its Fourier transform. Our method avoids the compact support restriction by deriving an explicit inequality (1.5), which may be extended by continuity to all functions, for which the 1-Sobolev seminorm is finite.

We note that the fact that the variance of additive functionals depends on the seminorm, rather than the norm, is connected, following Ghosh and Peres [24], to the rigidity of the process [23, 24]. In particular, this property allows one to approximate an indicator function by a function with arbitrarily small variance of the respective additive functional. This was used by Chhaibi and Najnudel [14] to show the rigidity of the sine-β process.

The convergence of additive functionals for $\beta = 2$ has first been established by Soshnikov [40]. Corollary 1.7 for $\beta = 2$ is due to Bufetov [5].

1.3 About the Jack measures

In the literature, Jack measures are measures on partitions of a fixed size. They are defined by the formula

$$\mathcal{M}^{\alpha}_{PL,n}(\lambda) = \frac{|\lambda|!\alpha^{|\lambda|}}{H(\lambda,\alpha)H'(\lambda,\alpha)},$$

where

$$H(\lambda,\theta) = \prod_{(i,j)\in\lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + 1), \quad H'(\lambda,\theta) = \prod_{(i,j)\in\lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + \theta), \quad \theta > 0,$$

λ is a partition of n and λ' is its transpose. We will refer to these as to β-Plancherel measures. For $\alpha = 1$ this measure coincides with the Plancherel measure. This measure also appears as a degeneration of the z-measures, introduced by Borodin and Olshanski [12] (the limit $z, z' \to \infty$ in the formula [12, (1.1)]).

A remarkable property of the Plancherel measure is determinantal structure of its poissonisation

$$\mathcal{M}^{\alpha}_{PL,q}(\lambda) = \frac{1}{1-q} q^{|\lambda|} \mathcal{M}^{\alpha}_{PL,|\lambda|}(\lambda)$$

under the embedding

$$\mathbb{Y} \to \mathrm{Conf}(\mathbb{Z}), \quad \lambda \mapsto \{\lambda_j - j\}_{j\in\mathbb{N}} \in \mathrm{Conf}(\mathbb{Z}).$$

The observation that projection of a random diagram after 45 degree turn results in a determinantal point process is due to Okounkov [38]. His result in fact considers a larger class of measures, to which we now proceed.

For arbitrary specializations ρ_1, ρ_2 of the algebra of symmetric functions, Schur measures are defined by the formula

$$\mathcal{M}^{1}_{\rho_1,\rho_2} = Z^{-1} s_\lambda(\rho_1) s_\lambda(\rho_2), \quad Z = \sum_{\lambda\in\mathbb{Y}} s_\lambda(\rho_1) s_\lambda(\rho_2).$$

As has been proved in [38], under the above embedding this measure is determinantal for arbitrary specializations. To obtain the Plancherel measure one specializes the algebra of symmetric functions as follows

$$p_1(\rho_1) = p_1(\rho_2) = \sqrt{q}, \quad p_j(\rho_1) = p_j(\rho_2) = 0, \text{ for } j \geq 2. \tag{1.6}$$

Following Moll [36] and Dimitrov, Gao, Gu, Niedernhofer [15], we define Jack measures for specializations ρ_1, ρ_2 as follows

$$\mathcal{M}^{\alpha}_{\rho_1,\rho_2}(\lambda) = Z^{-1} \frac{J^{(\alpha)}_\lambda(\rho_1) J^{(\alpha)}_\lambda(\rho_2)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda \rangle_\alpha}, \quad Z = \sum_{\lambda\in\mathbb{Y}} \frac{J^{(\alpha)}_\lambda(\rho_1) J^{(\alpha)}_\lambda(\rho_2)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda \rangle_\alpha}. \tag{1.7}$$

For $\alpha = 1$ these coincide with the Schur measures. The β-deformations of the Plancherel measures defined in the beginning may be obtained from the same specialization (1.6) (see [29]).

For a general α consider the embedding

$$\mathbb{Y} \to \mathrm{Conf}(\mathbb{R}), \quad \lambda \mapsto \{\lambda_j - \alpha j\}. \tag{1.8}$$

Strahov [42, 43] showed that the β-Plancherel measures and, more generally, z-measures for parameters $\alpha = 2, 1/2$ are Pfaffian point processes under the embedding (1.8). Recall that for the respective parameters $\beta = 4, 1$, the circulary β-ensemble is also a Pfaffian point process. Theorem 1.2 motivates the following question.

Question 1.8. *Under which specializations ρ_1, ρ_2 the Jack measures* (1.7) *for $\alpha = 2, 1/2$ are Pfaffian point processes under the embedding* (1.8)*?*

Another interesting question is behavior of Jack measure under certain scaling limit. For a smooth compactly supported real-valued function f consider the specializations

$$p_j(\rho_1^n) = \overline{p_j(\rho_2^n)} = \alpha j \widehat{f(n\cdot)}_j,$$

where for large enough n the function $f(n\theta)$ may be considered as a function on the unit circle. Rescale the embedding (1.8) as follows

$$\{\lambda_j - \alpha j\}_{j \in \mathbb{N}} \mapsto \left\{\frac{\lambda_j - \alpha j}{n}\right\}_{j \in \mathbb{N}}. \tag{1.9}$$

Question 1.9. *Do the Jack measures $\mathcal{M}^{\alpha}_{\rho_1^n, \rho_2^n}$ converge to a limit point process under the scaling* (1.9)*?*

In the case $\alpha = 1$, corresponding to Schur measures, the answer is affirmative. This follows from the calculations of Bufetov [4]. The limit process is determinantal; its kernel induces a product of two Hankel operators. Fredholm determinant of the latter describes the convergence of additive functionals under the sine-process.

There are numerous works considering the limit behavior of z-measures and Jack measures. Borodin and Olshanski [12] studied the limit of z-measures on the boundary of the Young graph. Asymptotic shape of a large partition under the β-deformation of the Plancherel measure as well as gaussian fluctuations were investigated by Dolega and Féray [17, 19]. Matsumoto [35] established the connection of large partitions under the Jack measure and the traceless gaussian β-ensemble. Moll [36] has found the limit shape of the large partition under the Jack measure for general specializations. Asymptotic behavior of Jack measures with varying homogeneous specializations were considered by Dimitrov, Gao, Gu, Niedernhofer [15].

1.4 Structure of the paper

The rest of the paper is structured as follows. We first recall the definition of Jack polynomials in Subsection 2.1. Next we prove Theorem 1.2 and Corollary 1.3 in Subsection 2.2. We conclude the proof of Theorem 1.4 in Subsection 2.3. In Section 3 we use the above results and Proposition to prove Theorem 1.6. Section 4 is devoted to the proof of Proposition 1.5. Finally, we show how Corollary 1.7 follows from Theorem 1.6 in Section 5.

1.5 Acknowledgements

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2 Proof of Theorem 1.2, Corollary 1.3 and Theorem 1.4

2.1 Jack polynomials

We refer the reader to Macdonald [34] for basic definitions. A partition $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)}, 0, \ldots)$ is a non-increasing sequence of nonnegative integers such that all but finitely many λ_j are zero. We denote by $l(\lambda)$

the number of non-zero integers and set $|\lambda| = \sum_j \lambda_j$. We denote by $\mathbb{Y}$ the set of all partitions and by $\mathbb{Y}_n$ the set of partitions of n: $\sum_j \lambda_j = n$. The dominance order on $\mathbb{Y}_n$ is defined as follows. We say that $\lambda \geq \mu$ if

$$\sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} \mu_j, \qquad \text{for all } k \geq 0.$$

It is a partial order.

Next we define the algebra of symmetric functions. Let

$$\Lambda_N = \mathbb{C}^{S_n}[x_1, \ldots, x_N] = \bigoplus_{n=0}^{\infty} \Lambda_N^n$$

be the graded by degree algebra of symmetric functions in N variables. Define the projections

$$\pi_N^{N+k} : \Lambda_{N+k}^n \to \Lambda_N^n, \quad \pi_N^{N+k}(x_{N+j}) = 0, j = 1, \ldots k.$$

The algebra of symmetric functions is then defined as

$$\Lambda = \bigoplus_{n=0}^{\infty} \Lambda^n, \quad \Lambda^n = \varprojlim_N \Lambda_N^n.$$

In particular, an element of Λ is a formal power series in infinite variables, which is symmetric under their permutations and whose terms' degrees are bounded.

An example of an element of Λ is a Newton power sum, which is defined by the formula

$$p_k = \sum_{j=1}^{\infty} x_j^k \in \Lambda$$

for $k \geq 1$. For $k = 0$ set $p_0 = 1$.

Theorem 2.1. *We have that p_j form multiplicative basis of Λ:*

$$\Lambda \simeq \mathbb{C}[p_0, p_1, \ldots].$$

For a partition λ define $p_\lambda = \prod_j p_{\lambda_j}$. By Theorem 2.1 for a real $\alpha > 0$ we may define a scalar product on Λ by the formula

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} z_\lambda \alpha^{l(\lambda)}, \quad z_\lambda = \frac{|\lambda|!}{C_\lambda}, \tag{2.1}$$

where C_λ is the number of elements in $S_{|\lambda|}$ with the cycle structure $(\lambda_1, \ldots, \lambda_{l(\lambda)})$.

Another basis is the basis of Schur polynomials. We first define them in finite number of variables

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det(x_i^{\lambda_j + k - j})_{i,j=1}^k}{\det(x_i^{k-j})_{i,j=1}^N} \in \Lambda_N, \quad N \geq l(\lambda).$$

We have that $\deg s_\lambda = |\lambda|$ for any number of variables and that

$$s_\lambda(x_1, \ldots, x_N, 0, \ldots, 0) = s_\lambda(x_1, \ldots, x_N).$$

This yields that they define elements of Λ.

Theorem 2.2. *Schur polynomials $\{s_\lambda\}_{\lambda \in \mathbb{Y}}$ form a basis of Λ.*

Further, for $\alpha = 1$ we have that they are orthonormal with the defined above scalar product

$$\langle s_\lambda, s_\mu \rangle_1 = \delta_{\lambda\mu}.$$

Jack polynomials are counterparts of Schur polynomials for an arbitrary α.

Theorem 2.3. *There exists a unique basis $\{J^{(\alpha)}_\lambda\}_{\lambda\in\mathbb{Y}}$ of Λ, satisfying the following properties*

- $\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\mu\rangle_\alpha = 0$ *for* $\lambda \neq \mu$.
- $J^{(\alpha)}_\lambda = s_\lambda + \sum_{\mu<\lambda} C_{\lambda\mu} s_\mu$ *for some* $C_{\lambda\mu} \in \mathbb{Q}(\alpha)$.

We will refer to these polynomials as to Jack polynomials.

Remark. Usually the second property is given in terms of monomial symmetric functions. Observe, however, that the definition above is equivalent since $J^{(1)}_\lambda = s_\lambda$.

Any symmetric function u in n variables defines a function on n-point configurations $\{z_1, \dots, z_n\} \subset \mathbb{T}$ by a direct substitution of z_j. Define a scalar product on Λ_n by the formula

$$\langle u, v\rangle_{\alpha,n} = \mathbb{E}^n_{2/\alpha} u\bar{v}.$$

Define a constant

$$\mathcal{A}^\alpha_\lambda(n) = \prod_{(i,j)\in\lambda} \left(1 - \frac{\alpha - 1}{n + j\alpha - i}\right),$$

where $(i, j) \in \lambda$ if $i, j \geq 1$ and $j \leq \lambda_i$. Last, let $\pi_n : \Lambda \to \Lambda_n$ be the canonical projection from the projective limit, sending $x_{n+1}, x_{n+2}, \dots$ to zero.

Proposition 2.4 ([34, (10.22), (10.35), (10.37)], [28, Lemma 4.2])**.** *We have*

$$\frac{\langle \pi_n(J^{(\alpha)}_\lambda), \pi_n(J^{(\alpha)}_\mu)\rangle_{\alpha,n}}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha} = \delta_{\lambda,\mu}\delta_{l(\lambda)\leq n}\mathcal{A}^\alpha_\lambda(n).$$

Finally, we recall the Cauchy identity.

Proposition 2.5 ([34, p. 309, p. 377])**.** *We have the equality*

$$\sum_{|\lambda|=k} \frac{J^{(\alpha)}_\lambda(x) J^{(\alpha)}_\lambda(y)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha} = \sum_{|\lambda|=k} \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda \alpha^{l(\lambda)}}.$$

Summing it over all $k \geq 0$ implies the Cauchy identity

$$\sum_{\lambda\in\mathbb{Y}} \frac{J^{(\alpha)}_\lambda(x) J^{(\alpha)}_\lambda(y)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha} = \exp\left(\frac{1}{\alpha}\sum_{j=1}^\infty \frac{p_j(x), p_j(y)}{j}\right).$$

2.2 Proof of Theorem 1.2 and Corollary 1.3

To prove Theorem 1.2 we expand the function $e^{\sum_{j=1}^n f(\theta_j)}$, which is symmetric in the variables $z_j = e^{i\theta_j}$, into Jack polynomials using the Cauchy identity. The Gessel-type expansion then follows from Proposition 2.4, which we use to calculate expectations.

First, factor e^f into a product of functions, which extend analytically into the interior/exterior of the unit circle

$$f(\theta) = \hat{f}_0 + \sum_{k\geq 1} \hat{f}_k z^k + \sum_{k\geq 1} \hat{f}_{-k} z^{-k} = \hat{f}_0 + f_+(\theta) + f_-(\theta), \quad z = e^{i\theta}.$$

Recall that by $\pi_n : \Lambda \to \Lambda_n$ we denoted the canonical projection from the algebra of symmetric functions in infinite number of variables to the algebra of symmetric functions in n variables. We may represent the exponential of additive functionals using the introduced specializations

$$\prod_{j=1}^n e^{f(\theta_j)} = e^{n\hat{f}_0} \exp\left(\alpha^{-1}\sum_{j=1}^\infty \frac{p_j(\rho_+)(\pi_n p_j)(z_1, \dots, z_n)}{j} + \alpha^{-1}\sum_{j=1}^\infty \frac{p_j(\rho_-)(\pi_n p_j)(z_1^{-1}, \dots, z_n^{-1})}{j}\right), \quad z_j = e^{i\theta_j},$$

where $p_j(\rho_\pm) = \alpha j \hat{f}_{\pm j}$ for $j \geq 1$. Since the specializations $\rho_\pm$ and the projections π_n are homomorphisms, we may apply Proposition 2.5 to get the Jack polynomial expansions of factors:

$$\exp\left(\alpha^{-1}\sum_{j=1}^{\infty}\frac{p_j(\rho_\pm)(\pi_n p_j)(z_1^{\pm 1},\ldots,z_n^{\pm 1})}{j}\right) = \sum_{\lambda\in\mathbb{Y}}\frac{J_\lambda^{(\alpha)}(\rho_\pm)(\pi_n J_\lambda^{(\alpha)})(z_1^{\pm 1},\ldots,z_n^{\pm 1})}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}.$$

Jack polynomials are homogeneous: $\deg J_\lambda^{(\alpha)} = |\lambda|$. Therefore, the above series is the power series expansion of $\prod_{j=1}^n e^{f(\theta_j)}$, which is uniformly convergent on $\mathbb{T}^n$ if one assumes that f is holomorphic in some neighborhood of the unit circle. Let us first proceed under this assumption.

Uniform convergence allows us to interchange sums and integration. By Proposition 2.4 we conclude

$$e^{-n\hat{f}_0}\mathbb{E}^n_{2/\alpha}\prod_{j=1}^n e^{f(\theta_j)} = \sum_{\lambda,\mu\in\mathbb{Y}}\frac{J_\lambda^{(\alpha)}(\rho_+)J_\mu^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\frac{\langle \pi_n J_\lambda^{(\alpha)}, \pi_n J_\mu^{(\alpha)}\rangle_{\alpha,n}}{\langle J_\mu^{(\alpha)}, J_\mu^{(\alpha)}\rangle_\alpha} = \sum_{l(\lambda)\leq n}\frac{J_\lambda^{(\alpha)}(\rho_+)J_\mu^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\mathcal{A}^\alpha_\lambda(n), \quad (2.2)$$

which proves the claim for functions, whose Fourier coefficients decay exponentially. We also used that $J_\lambda^{(\alpha)} \in \mathbb{Q}(\alpha)[p_0, p_1, \ldots]$, which implies that $(\pi_n J_\lambda^{(\alpha)})(z_1^{-1},\ldots,z_n^{-1}) = \overline{(\pi_n J_\lambda^{(\alpha)})(z_1,\ldots,z_n)}$ for $z_j \in \mathbb{T}$.

Before proving the first claim for all 1/2-Sobolev regular functions, let us establish the second claim. By the Cauchy-Bunyakovsky-Schwarz inequality we have

$$\sum_{\lambda\in\mathbb{Y}}\left|\frac{J_\lambda^{(\alpha)}(\rho_+)J_\lambda^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\right| \leq \sqrt{\sum_{\lambda\in\mathbb{Y}}\frac{J_\lambda^{(\alpha)}(\rho_+)\overline{J_\lambda^{(\alpha)}(\rho_+)}}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}}\sqrt{\sum_{\lambda\in\mathbb{Y}}\frac{J_\lambda^{(\alpha)}(\rho_-)\overline{J_\lambda^{(\alpha)}(\rho_-)}}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}}.$$

By Proposition 2.5 the left-hand side is finite if

$$\sum_{j\geq 1}\frac{|p_j(\rho_\pm)|^2}{j} = \alpha^2\sum_{j\geq 1} j|\hat{f}_{\pm j}|^2 < +\infty,$$

which holds if $f \in H_{1/2}(\mathbb{T})$.

Let us finally show that the Gessel-type expansion holds for all 1/2-Sobolev regular functions. Consider a function $f \in H_{1/2}(\mathbb{T})$. First, regularize it as follows

$$f_q(\theta) = \hat{f}_0 + \sum_{k\geq 1} q^k \hat{f}_k z^k + \sum_{k\geq 1} q^k \hat{f}_{-k} z^{-k}, \quad z = e^{i\theta}, q \in (0,1).$$

The function f_q is holomorphic in a neighborhood of the unit circle, hence the expansion (2.2) holds for the respective specializations $\rho^q_\pm$. We have that the right-hand side of the equality (2.2) converges as $q \to 1$ to the series for the specializations $\rho_\pm$ by the dominated convergence theorem. Indeed, since the Jack polynomials are homogeneous and $\mathcal{A}^\alpha_\lambda(n) \leq 1$, we have the bound

$$\sum_{l(\lambda)\leq n}\left|\frac{J_\lambda^{(\alpha)}(\rho^q_+)J_\mu^{(\alpha)}(\rho^q_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\mathcal{A}^\alpha_\lambda(n)\right| = \sum_{l(\lambda)\leq n} q^{2|\lambda|}\left|\frac{J_\lambda^{(\alpha)}(\rho_+)J_\mu^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\mathcal{A}^\alpha_\lambda(n)\right| \leq \sum_{l(\lambda)\leq n}\left|\frac{J_\lambda^{(\alpha)}(\rho_+)J_\mu^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}\right|,$$

where the series on the right-hand side converges by the second claim of the theorem. Finally, we show that the convergence

$$\mathbb{E}^n_{2/\alpha}\prod_{j=1}^n e^{f_q(\theta_j)} \to \mathbb{E}^n_{2/\alpha}\prod_{j=1}^n e^{f(\theta_j)}, \quad \text{as } q \to 1$$

holds.

By the above argument the $L_1(\mathbb{P}^n_{2/\alpha}, \mathrm{Conf}(\mathbb{R}))$ norms of the functions $\prod_{j=1}^n e^{f_q(\theta_j)}$ are bounded uniformly in q. Further, the functions $f_q(\theta)$ converge to the function $f(\theta)$ in L_2 as $q \to 1$. Therefore, f_{q_l} converge almost everywhere as $l \to \infty$ for some sequence $\{q_l\}_{l\in\mathbb{N}}$, converging to 1. We conclude that by the Fatou Lemma the multiplicative functional $\prod_{j=1}^n e^{f(\theta_j)}$ is absolutely integrable for any $f \in H_{1/2}(\mathbb{T})$. The bound

$$\mathbb{E}_{2/\alpha}^n \left| \prod_{j=1}^n e^{f(\theta_j)} \right| \leq \mathbb{E}_{2/\alpha}^n \prod_{j=1}^n e^{\Re f(\theta_j)} \leq \limsup_{l\to\infty} \mathbb{E}_{2/\alpha}^n \prod_{j=1}^n e^{\Re f_{q_l}(\theta_j)} \leq e^{\frac{\alpha}{2}\|\Re f\|^2_{\dot{H}_{1/2}(\mathbb{T})}} \tag{2.3}$$

holds. The expression on the right-hand side is the specialized complete Cauchy series from Proposition 2.5. We note that for real-valued f we have $p_k(\rho_+) = \overline{p_k(\rho_-)}$, so the Cauchy series is positive.

To conclude the proof, we apply the Cauchy-Bunyakovsky-Schwarz inequality:

$$\mathbb{E}_{2/\alpha} \left| \prod_{j=1}^n e^{f_q(\theta_j)} - \prod_{j=1}^n e^{f(\theta_j)} \right| \leq \sqrt{\mathbb{E}_{2/\alpha}^n \prod_{j=1}^n e^{2\Re(f_q(\theta_j)+f(\theta_j))}} \sqrt{\mathbb{E}_{2/\alpha}^n \left| \sum_{j=1}^n f_q(\theta_j) - f(\theta_j) \right|^2}.$$

The first factor on the right-hand side is uniformly bounded by the inequality (2.3). The second converges to zero as $q \to 1$, since by Proposition 2.4 we have

$$\left| \sum_{j=1}^n f_q(\theta_j) - f(\theta_j) \right| \leq \sum_{j\in\mathbb{Z}} |\widehat{f}_{q_k} - \widehat{f}_k|^2 \langle p_{|k|}, p_{|k|}\rangle_{\alpha,n} \leq \alpha \sum_{j\in\mathbb{Z}} |\widehat{f}_{q_k} - \widehat{f}_k|^2 |k|,$$

where the second bound follows from substitution of the inequality $\mathcal{A}_\lambda^\alpha(n) \leq 1$ into the expression for $\langle p_k, p_k\rangle_{\alpha,n}$ via Jack polynomials. The right-hand side in the above bound converges to zero as $q \to 1$. Theorem 1.2 is proved.

The second claim of Corollary 1.3 is the inequality (2.3). The first claim follows from the convergence

$$\mathbb{E}_{2/\alpha}^n \prod_{j=1}^n e^{f(\theta_j)} = \sum_{l(\lambda)\leq n} \frac{J_\lambda^{(\alpha)}(\rho_+) J_\mu^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha} \mathcal{A}_\lambda^\alpha(n) \to \sum_\lambda \frac{J_\lambda^{(\alpha)}(\rho_+) J_\mu^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha},$$

as $n \to \infty$, which holds by the monotone convergence Theorem. By Proposition 2.5 the limit is

$$\sum_\lambda \frac{J_\lambda^{(\alpha)}(\rho_+) J_\mu^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha} = \exp\left(\alpha^{-1} \sum_{j\geq 1} \frac{p_j(\rho_+)p_j(\rho_-)}{j}\right) = \exp\left(\alpha \sum_{j\geq 1} j \hat{f}_j \hat{f}_{-j}\right).$$

Corollary 1.3 is proved.

2.3 Proof of Theorem 1.4

For a function $f \in H_{1/2}(\mathbb{T})$ define Jack measure on partitions by the formula

$$\mathcal{M}_f^\alpha(\lambda) = Z^{-1} \frac{J_\lambda^{(\alpha)}(\rho_+) J_\lambda^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}, \quad Z = \sum_{\lambda\in\mathbb{Y}} \frac{J_\lambda^{(\alpha)}(\rho_+) J_\lambda^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha},$$

where $p_k(\rho_\pm) = \alpha j \hat{f}_{\pm k}$. This measure is real-valued if f is real-valued, since $p_k(\rho_+) = \overline{p_k(\rho_-)}$ in this case.

Theorem 1.4 follows from the following lemma.

Lemma 2.6 ([36, Proposition 1.3.1]). *For a real-valued function $f \in H_1(\mathbb{T})$, we have*

$$\mathbb{E}_{\mathcal{M}_f^\alpha} |\lambda| = \frac{\alpha}{2} \|f\|^2_{\dot{H}_1(\mathbb{T})}.$$

Proof. By the Cauchy identity from Proposition 2.5 we have the identity

$$\sum_{\lambda\in\mathbb{Y}} \frac{q^{|\lambda|}p_\lambda(x)p_\lambda(y)}{z_\lambda \alpha^{l(\lambda)}} = \exp\left(\frac{1}{\alpha}\sum_{k\in\mathbb{N}} \frac{q^k p_k(x)p_k(y)}{k}\right).$$

Taking the derivative with respect to q at $q=1$ gives the identity

$$\sum_{\lambda\in\mathbb{Y}} |\lambda| \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda \alpha^{l(\lambda)}} = \frac{1}{\alpha}\exp\left(\frac{1}{\alpha}\sum_{k\in\mathbb{N}} \frac{p_k(x)p_k(y)}{k}\right)\left(\sum_{k\in\mathbb{N}} p_k(x)p_k(y)\right),$$

between formal power series. The assertion of the lemma follows from the substitution of the specializations $\rho_\pm$ into the above power series and application of Proposition 2.5 for the normalization constant Z. □

Denote $\sigma_{\mathbb{T}}(f) = \sum_{j\geq 1} j\hat{f}_j\hat{f}_{-j}$. Observe that $\mathbb{E}^n_{2/\alpha} S_f = n\hat{f}(0)$. By Theorem 1.2 we have

$$\begin{aligned}\left|\frac{\mathbb{E}^{2n}_{2/\alpha} e^{\overline{S}_f}}{e^{\alpha\sigma_{\mathbb{T}}(f)}} - 1\right| \leq e^{-\alpha\Re\sigma_{\mathbb{T}}(f)} &\sum_{l(\lambda)\leq n} \left|\frac{J^{(\alpha)}_\lambda(\rho_+)J^{(\alpha)}_\lambda(\rho_-)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha}(\mathcal{A}^\alpha_\lambda(2n)-1)\right| \\ &+ e^{-\Re\alpha\sigma_{\mathbb{T}}(f)} \sum_{l(\lambda)\geq n+1} \left|\frac{J^{(\alpha)}_\lambda(\rho_+)J^{(\alpha)}_\lambda(\rho_-)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha}\right|.\end{aligned}$$

Estimate for the second term
The second sum may be bounded as follows

$$\begin{aligned}\sum_{l(\lambda)\geq n+1} \left|\frac{J^{(\alpha)}_\lambda(\rho_+)J^{(\alpha)}_\lambda(\rho_-)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha}\right| &\leq \frac{1}{n}\sum_{l(\lambda)\geq n+1} |\lambda| \left|\frac{J^{(\alpha)}_\lambda(\rho_+)J^{(\alpha)}_\lambda(\rho_-)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha}\right| \\ &\leq \frac{1}{n}\sqrt{\sum_{l(\lambda)\geq n+1} |\lambda| \frac{J^{(\alpha)}_\lambda(\rho_+)\overline{J^{(\alpha)}_\lambda(\rho_+)}}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha}}\sqrt{\sum_{l(\lambda)\geq n+1} |\lambda| \frac{J^{(\alpha)}_\lambda(\rho_-)\overline{J^{(\alpha)}_\lambda(\rho_-)}}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha}}.\end{aligned}$$

The sums on the right-hand side may be bounded by sums over all partitions. Further, observe that Lemma 2.6 implies

$$\begin{aligned}\sum_{\lambda\in\mathbb{Y}} |\lambda| \frac{J^{(\alpha)}_\lambda(\rho_\pm)\overline{J^{(\alpha)}_\lambda(\rho_\pm)}}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha} &= e^{\frac{\alpha}{2}\|2\Re f_\pm\|^2_{\dot{H}_{1/2}(\mathbb{T})}} \mathbb{E}_{\mathcal{M}^\alpha_{2\Re f_\pm}} |\lambda| \\ &= \frac{\alpha}{2} e^{\frac{\alpha}{2}\|2\Re f_\pm\|^2_{\dot{H}_{1/2}(\mathbb{T})}} \|2\Re f_\pm\|^2_{\dot{H}_1(\mathbb{T})} \leq 2\alpha e^{\frac{\alpha}{2}\|2\Re f_\pm\|^2_{\dot{H}_{1/2}(\mathbb{T})}} \|f\|^2_{\dot{H}_1(\mathbb{T})}.\end{aligned}$$

We conclude

$$e^{-\Re\alpha\sigma_{\mathbb{T}}(f)} \sum_{l(\lambda)\geq n+1} \left|\frac{J^{(\alpha)}_\lambda(\rho_+)J^{(\alpha)}_\lambda(\rho_-)}{\langle J^{(\alpha)}_\lambda, J^{(\alpha)}_\lambda\rangle_\alpha}\right| \leq \frac{2\alpha}{n}\exp\left(\frac{\alpha}{2}\|f\|^2_{\dot{H}_{1/2}(\mathbb{T})} - \alpha\Re\sigma_{\mathbb{T}}(f)\right)\|f\|^2_{\dot{H}_1(\mathbb{T})}.$$

Estimate for the first term
Observe that for $a_1,\ldots,a_k \in (0,1)$ we have

$$1 - \prod_{j=1}^k (1-a_j) \leq \sum_{j=1}^k a_j.$$

Consequently, if $l(\lambda) \le n$, we have

$$|\mathcal{A}_\lambda^\alpha(2n) - 1| \le (\alpha - 1) \sum_{(m,l)\in\lambda} \frac{1}{2n + l\alpha - m} \le \frac{\alpha - 1}{n}|\lambda|.$$

The resulting estimate by Lemma 2.6 is

$$\begin{aligned}
e^{-\alpha\Re\sigma_{\mathbb{T}}(f)} \sum_{l(\lambda)\le n} \left| \frac{J_\lambda^{(\alpha)}(\rho_+)J_\lambda^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha} (\mathcal{A}_\lambda^\alpha(2n) - 1) \right| &\le e^{-\alpha\Re\sigma_{\mathbb{T}}(f)} \frac{\alpha - 1}{n} \sum_{\lambda\in\mathbb{Y}} |\lambda| \left| \frac{J_\lambda^{(\alpha)}(\rho_+)J_\lambda^{(\alpha)}(\rho_-)}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha} \right| \\
&\le e^{-\alpha\Re\sigma_{\mathbb{T}}(f)} \frac{\alpha - 1}{n} \sqrt{\sum_{\lambda\in\mathbb{Y}} |\lambda| \frac{J_\lambda^{(\alpha)}(\rho_+)\overline{J_\lambda^{(\alpha)}(\rho_+)}}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}} \sqrt{\sum_{\lambda\in\mathbb{Y}} |\lambda| \frac{J_\lambda^{(\alpha)}(\rho_-)\overline{J_\lambda^{(\alpha)}(\rho_-)}}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)}\rangle_\alpha}} \\
&\le \frac{2\alpha(\alpha - 1)}{n} \exp\left(\frac{\alpha}{2}\|f\|^2_{\dot{H}_{1/2}(\mathbb{T})} - \alpha\Re\sigma_{\mathbb{T}}(f)\right) \|f\|^2_{\dot{H}_1(\mathbb{T})}.
\end{aligned}$$

Theorem 1.4 follows by summing these estimates and substituting $\alpha = 2/\beta$.

3 Proof of Theorem 1.6

In this section we explain how Theorem 1.6 follows from Theorem 1.4, Corollary 1.3 and Proposition 1.5. Almost immediately we have the following statement.

Lemma 3.1. *Theorem 1.6 holds for compactly supported smooth functions f on the real line.*

Proof. Substitute a sequence of functions $f(2n\cdot)$ into the inequality (1.4) and take the limit $n \to \infty$. We note that for sufficiently large n the function $f(2n\cdot)$ may be viewed as a function on the unit circle. By Proposition 1.5 we have that

$$\mathbb{E}_\beta^{2n} e^{S_{f(2n\cdot)}} \to \mathbb{E}_\beta e^{S_f}, \qquad \text{as } n \to \infty.$$

To conclude the assertion of the lemma we are left to show that

$$\begin{aligned}
&2n\widehat{f(2n\cdot)}_0 \to \frac{1}{2\pi}\int_{\mathbb{R}} f(x)dx = \mathbb{E}_\beta S_f, \\
&\sum_{j\ge 0} j\widehat{f(2n\cdot)}_j \widehat{f(2n\cdot)}_{-j} \to \int_0^\infty \lambda\hat{f}(\lambda)\hat{f}(-\lambda), \\
&\|f(2n\cdot)\|_{\dot{H}_{1/2}(\mathbb{T})} \to \|f\|_{\dot{H}_{1/2}(\mathbb{R})}, \\
&\frac{1}{\sqrt{n}}\|f(2n\cdot)\|_{\dot{H}_1(\mathbb{T})} \to \sqrt{2}\|f\|_{\dot{H}_1(\mathbb{R})},
\end{aligned}$$

as $n \to \infty$. All of the above limits may be verified directly. The inequality

$$\mathbb{E}_\beta e^{\overline{S}_f} \le e^{\frac{1}{\beta}\|f\|_{\dot{H}_{1/2}(\mathbb{R})}}$$

for a real-valued f is proved similarly. □

To prove Theorem 1.6 for a wider class of functions, we extend it by continuity. Recall that smooth compactly supported functions are dense in $H_1(\mathbb{R})$. The right hand side of the inequality (1.5) is continuous with respect to the norm $\|\cdot\|^2_{H_1(\mathbb{R})} = \|\cdot\|^2_{L_2(\mathbb{R})} + \|\cdot\|^2_{\dot{H}_1(\mathbb{R})}$. The same may be said about the left-hand side once it is established that the expectation $\mathbb{E}_\beta e^{\overline{S}_f}$ is continuous with respect to the Sobolev $\|f\|_{\dot{H}_{1/2}(\mathbb{R})}$ seminorm. Last, we note, that additive functionals are first only defined for absolutely integrable functions $S_f \in L_1(\mathbb{P}_\beta, \mathrm{Conf}(\mathbb{R}))$ for $f \in L_1(\mathbb{R})$. Following Bufetov [3], we extend their definition to a wider class of functions.

3.1 Regularization of additive functionals

For a general point process only compactly supported bounded functions define almost surely finite additive functionals. Observe, that if a process is translationally invariant, then one may extend additive functionals to all absolutely integrable functions, since the first correlation function is constant. However, in this case the sum

$$\sum_{x\in X}\frac{1}{i+x}, \quad X\in \mathrm{Conf}(\mathbb{R})$$

almost surely does not converge absolutely. However, it may be that the limit

$$\lim_{n\to\infty}\sum_{x\in X,|x|\le k_n}\frac{1}{i+x}, \quad X\in \mathrm{Conf}(\mathbb{R})$$

almost surely exists for some sequence k_n (for example, $k_n = n^4$ for the sine-process). Formally, though the additive functional $S_{1/(x+i)}$ is not defined, the limit $S_{\mathbb{I}_{[-n,n]}/(x+i)}$ as $n\to\infty$ exists. Regularization of additive functionals was introduced by Bufetov [3] and then applied to construction of Radon-Nikodym derivatives between Palm measures of determinantal point process.

Proposition 3.2. *There exists a constant C such that for a smooth compactly supported function f we have that*

$$\mathbb{E}_\beta|\overline{S}_f|^2\le C\|f\|^2_{\dot{H}_{1/2}(\mathbb{R})}.$$

Proof. It is sufficient to assume that f is real-valued. In this case the assertion follows from the subgaussianity claimed in Lemma 3.1 (see Proposition 4.16). ☐

We are now able to introduce

Definition 3.3. For a 1/2-Sobolev regular function f the respective regularized additive functional $\overline{S}_f$ is an $L_2(\mathbb{P}_\beta, \mathrm{Conf}(\mathbb{R}))$-limit of a sequence of additive functionals $\overline{S}_{f_k}$, corresponding to smooth compactly supported functions f_k, which converge to f in $H_{1/2}(\mathbb{R})$ as $k\to\infty$.

The following statement concludes the proof of Theorem 1.6.

Lemma 3.4. *We have that the expectation $\mathbb{E}_\beta e^{\overline{S}_f}$ is continuous with respect to the 1/2-Sobolev seminorm $\|f\|_{\dot{H}_{1/2}(\mathbb{R})}$.*

Proof. Indeed, by the Cauchy-Bunyakovsky-Schwarz inequality and Proposition 3.2 we have

$$\left|\mathbb{E}_\beta e^{\overline{S}_f}-\mathbb{E}_\beta e^{\overline{S}_g}\right|\le\sqrt{\mathbb{E}_\beta e^{2S_{\Re f+\Re g}}}\sqrt{\mathbb{E}_\beta|\overline{S}_{f-g}|}\le\sqrt{C}e^{2\|\Re f+\Re g\|_{\dot{H}_{1/2}(\mathbb{R})}}\|f-g\|_{\dot{H}_{1/2}(\mathbb{R})},$$

which implies the desired. ☐

Remark. The rest of the paper is devoted to the proof of Proposition 1.5. If one considers purely imaginary functions f, then Proposition 1.5 has already been established by Killip and Stoiciu in [30]. To avoid using the subgaussianity, one may use the argument of Chhaibi and Najnudel [14] to prove Proposition 3.2. Last, for real-valued f, g we have

$$|\mathbb{E}_\beta e^{i\overline{S}_f}-\mathbb{E}_\beta e^{i\overline{S}_g}|\le\mathbb{E}_\beta|\overline{S}_{f-g}|^2.$$

To conclude, we have established the inequality (1.5) for characteristic functions of real-valued additive functionals. This is sufficient to prove Corollary 1.7.

4 Proof of Proposition 1.5

Let us first present the outline of the proof of Proposition 1.5. Assume we know that for any set of disjoint intervals the distributions of the numbers of particles in these intervals under the point processes $\mathbb{P}_n$ converge. Then expectations of multiplicative functionals Ψ_{1+g} converge for any compactly supported bounded Borel g given the probability of finding large number of particles in any compact subset decays sufficiently rapidly.

Let us explain the condition on the decay. We require that for any $x > 0$, $q > 0$ the series

$$\sum_{j\geq 0} q^j \mathbb{P}_n(N_{[-x,x]} \geq j),$$

where $N_{[-x,x]}$ is the number of particles in $[-x,x]$, converge. We also require that this convergence is uniform in n. Let $\mathcal{I}_j$ be the indicator of the interval $[-x,x]$ having not more than j points. The above condition implies that $\Psi_{1+g}\mathcal{I}_j$ approximates Ψ_{1+g} for large j in L_1 norm. This approximation is uniform in n, so it is sufficient to establish that $\mathbb{E}_n\Psi_{1+g}\mathcal{I}_j$ converges as $n \to \infty$.

Last, we may similarly approximate $\mathbb{E}_n\Psi_{1+g}\mathcal{I}_j$ uniformly in n by approximating Ψ_{1+g} by piecewise constant functions in L_∞ norm. This is possible given g is continuously differentiable. And for piecewise constant functions the convergence follows from the convergence of joint distributions of numbers of particles assumed in the beginning.

The convergence of joint distributions follows from the result of Killip and Stoiciu [30] given the limit point process is simple (see [39, §3.5]). The simplicity of the sine-β process has been established in [18]. However, our calculation will not rely on this simplicity. Instead we work with the construction of Killip and Stoiciu directly, though the idea of the proof is similar to the one explained above.

Recall that Killip and Stoiciu construct the sine-β process (see Subsection 4.5) as a preimage of a random shift of the lattice $\omega + 2\pi\mathbb{Z}$ under a random monotone function ψ, where ω is distributed uniformly in $[0, 2\pi]$, ψ and ω independent. We explain this construction in Subsection 4.1. The circular β-ensemble may be similarly constructed by a sequence of random monotone functions ψ_n (see Subsection 4.6). The result of Killip and Stoiciu states that if one restricts to functions with nonpositive real part, then Proposition 1.5 follows from the convergence $\psi_{n-1}(\cdot/n) \to \psi(\cdot)$ in finite-dimensional distributions and the convergence of L_1 norms of $\psi_{n-1}(x/n)$ to the L_1 norm of $\psi(x)$ for each $x \in \mathbb{R}$.

We replace the decay condition on the number of points by the same decay condition on the differences $\psi(x) - \psi(y)$, $\psi_n(x) - \psi_n(y)$ for all $x, y \in \mathbb{R}$, $x < y$ uniformly in n. This is a purely notational replacement, since the two conditions are equivalent by Definition 4.4. Next, following Killip and Stoiciu, we approximate each point process by a linear interpolation of the respective monotone function (see Subsection 4.2). The decay condition will imply that this approximation is uniform in n (see Lemmata 4.7, 4.9). As above, we also approximate each multiplicative functional by a finite number of particles. This approximation is also uniform in n. Finally, the convergence of the resulting expectation follows from the convergence of the respective random functions in finite-dimensional distributions and from the convergence of L_1 norms of $\psi_{n-1}(x/n)$. See the proof of Lemma 4.10 for the last argument.

The rest of the section is structured as follows. In Subsection 4.1 we present the construction of point processes via random monotone functions. We first prove the general statement — Proposition 4.11 — about convergence of multiplicative functionals in terms of the convergence of random monotone functions. We prove it using the two main arguments above. The linear interpolation argument is given in Subsection 4.2, and the decay argument is presented in Subsection 4.3. The proof of Proposition 4.11 is given in Subsection 4.4. Next we recall the constructions of Killip and Stoiciu of the sine-β process and the circular β-ensemble in terms of the formalism of random functions in Subsections 4.5 and 4.6 respectively. We note that we do not use the explicit constructions, but rather the existence of ones. In particular, to prove Proposition 1.5 we only use Theorem 4.15. Finally, we conclude that the circular β-ensembles meet the requirements of Proposition 4.11 and, consequently, that Proposition 1.5 holds in Subsection 4.7.

4.1 Construction of point processes via random monotone increasing functions

Let $\mathrm{Conf}(\mathbb{R})$ stand for the space of locally finite subsets of $\mathbb{R}$ with multiplicities. That is, any $X \in \mathrm{Conf}(\mathbb{R})$ is an unordered set of points $X = \{x_j\}_{j\in\mathbb{N}}$, $x_j \in \mathbb{R}$, such that only finite number of x_j lies in any compact.

Here and subsequently we denote this number by $X(B) = |X \cap B|$ for any bounded Borel subset $B \subset \mathbb{R}$. We endow the space of configurations with the σ-algebra $\mathfrak{X}$, generated by the subsets

$$C_{k,a,b} = \{X \in \mathrm{Conf}(\mathbb{R}) : X(a,b) = k\}, \quad a, b \in \mathbb{R}, a < b, k \in \mathbb{Z}_{\geq 0}.$$

Definition 4.1. A point process is a measure on $(\mathrm{Conf}(\mathbb{R}), \mathfrak{X})$.

Let $M(\mathbb{R})$ stand for the space of monotone increasing functions on $\mathbb{R}$, endowed with the cylinder σ-algebra. For $u \in M(\mathbb{R})$ denote $c(u) \subset \mathbb{R}$ to be the subset of its critical values — values having more than one preimage. Recall that for a monotone function it is at most countable. Introduce a subset

$$M_{reg} = \{(u, \omega) \in M(\mathbb{R}) \times [0, 2\pi] : c(u) \cap (2\pi\mathbb{Z} + \omega) = \emptyset\} \subset M(\mathbb{R}) \times [0, 2\pi],$$

which we endow with the product σ-algebra $\mathfrak{Y}$. For a measure $\mathcal{P}$ on $M(\mathbb{R})$ denote the product measure

$$\mathcal{P}' = \mathcal{P} \otimes \frac{1}{2\pi}\mathrm{Leb}$$

on $M(\mathbb{R}) \times [0, 2\pi]$.

Proposition 4.2. *For any measure $\mathcal{P}$ on $M(\mathbb{R})$ we have that M_{reg} is a subset of full measure for $\mathcal{P}'$.*

Proof. Let us first show that M_{reg} is measurable. Represent it as an intersection

$$M_{reg} = \cap_{k \in \mathbb{Z}} A_k, \quad A_k = \{(u, \omega) : \omega + 2\pi k \notin c(f)\}.$$

Any A_k is measurable. Indeed, let x be a critical value of $u \in M(\mathbb{R})$. Then u takes the value x in at least two rational points, since u is monotone. Thereby any complement A_k^c is a countable union

$$A_k^c = \cup_{q_1, q_2 \in \mathbb{Q}} B_{q_1, q_2}, \quad B_{q_1, q_2} = \{(u, \omega) : u(q_1) = u(q_2) = \omega + 2\pi k\}.$$

To complete the proof of the measurability it is remaining to observe that any B_{q_1,q_2} is a preimage of the point $(0, 0)$ under the map

$$M(\mathbb{R}) \times [0, 2\pi] \to \mathbb{R}^2, \quad (u, \omega) \mapsto (u(q_1) - \omega - 2\pi k, u(q_2) - \omega - 2\pi k),$$

which is continuous and, consequently, measurable. We note that $\mathfrak{Y}$ is the Borel σ-algebra of the product topology.

To show that M_{reg} is of full measure express the measure of the complement as follows

$$\mathcal{P}'(M_{reg}^c) = \frac{1}{2\pi} \int_{M(\mathbb{R})} \mathrm{Leb}(\{\omega \in [0, 2\pi] : (2\pi\mathbb{Z} + \omega) \cap c(u) \neq \emptyset\}) d\mathcal{P}(u),$$

where the subset under the Lebesgue measure is countable, since $c(u)$ is. This finishes the proof. □

Define a map

$$\Theta : M_{reg} \to \mathrm{Conf}(\mathbb{R}), \quad (u, \omega) \mapsto u^{-1}(\omega + 2\pi\mathbb{Z}),$$

where we include the point $x \in u^{-1}(y)$ to the preimage of the value y if the function u jumps through that value $u_-(x) \leq y \leq u_+(x)$.

Proposition 4.3. *The map $\Theta : (M_{reg}, \mathfrak{Y}) \to (\mathrm{Conf}(\mathbb{R}), \mathfrak{X})$ is measurable.*

Proof. The assertion follows directly from the function

$$\sum_{k \in \mathbb{Z}} \mathbb{I}_{(a,b)}(u^{-1}(\omega + 2\pi k))$$

being measurable for arbitrary $a < b$ on $(M_{reg}, \mathfrak{Y})$. The latter, in turn, follows from each $u^{-1}(\omega + 2\pi k)$ being measurable. Indeed, the representation

$$\{u^{-1}(\omega + 2\pi k) \in (a, b)\} = \cup_{n \in \mathbb{N}} \{(u, \omega) : u(a + 1/n) \leq \omega + 2\pi k \leq u(b - 1/n)\},$$

where each subset in the union is measurable, concludes the proof. □

Definition 4.4. Let $\mathcal{P}$ be a measure on $M(\mathbb{R})$. Then we define the associated point process as the pushforward

$$\mathbb{P}_{\mathcal{P}} := \Theta_* \mathcal{P}'.$$

In particular, finite-dimensional distributions of a process X_- on $\mathbb{R}$ define a point process if $X_a \leq X_b$ for $a \leq b$ almost surely. Slightly abusing notation, for a random monotone function ψ we will denote by $\mathbb{P}_\psi$ both the respective measure on $M(\mathbb{R})$, M_{reg} and the induced point process.

Remark. It is also convenient to define point processes in terms of multiplicative functionals. For a compactly supported bounded Borel function g on $\mathbb{R}$ we define the respective function on configurations by the formula

$$\Psi_{1+g}(X) = \prod_{x \in X} (1 + g(x)), \quad X \in \mathrm{Conf}(\mathbb{R}).$$

It is straightforward that their expectations characterize the point process. In case of Definition 4.4 these are equal to

$$\mathbb{E}_{\mathcal{P}} \Psi_{1+g} = \frac{1}{2\pi} \int_0^{2\pi} \int_{M(\mathbb{R})} \prod_{k \in \mathbb{Z}} (1 + g \circ u^{-1}(\omega + 2\pi k)) d\mathcal{P}(u) d\omega.$$

In [30] Killip and Stoiciu use multiplicative functionals to establish the convergence of the circulary β-ensemble.

4.2 Convergence of point processes by linear interpolation

We proceed to the approximation procedure. We restrict our process from the entire real line to a fixed interval $[-x, x]$. Since we will work with different random monotone functions, it will be convenient to introduce the following notation. Denote by $\{U\}_\psi$ the random variable, induced by a function U on the space of configurations and a point process $\mathbb{P}_\psi$ corresponding to a random monotone function ψ in the sense of Definition 4.4.

Lemma 4.5. *Assume ϕ and ψ are random monotone functions. Let f be a continuously differentiable function supported on $[-x, x]$. Then we have almost surely*

$$\int_0^{2\pi} |\{S_f\}_\psi - \{S_f\}_\phi| d\omega \leq \|f'\|_{L_\infty} \int_{-x}^{x} |\psi(y) - \phi(y)| dy.$$

Proof. Writing the integral by the definition we have

$$\begin{aligned}\int_0^{2\pi} |\{S_f\}_\psi - \{S_f\}_\phi| d\omega &\leq \sum_{j \in \mathbb{Z}} \int_0^{2\pi} |f \circ \psi^{-1}(2\pi j + \omega) - f \circ \phi^{-1}(2\pi j + \omega)| d\omega \\ &= \int_{\mathbb{R}} |f \circ \psi^{-1}(y) - f \circ \phi^{-1}(y)| dy.\end{aligned}$$

Since the function f is continuously differentiable and supported in $[-x, x]$, the integral on the right-hand side is at most

$$\|f'\|_{L_\infty} \int_{\mathbb{R}} |\mathbb{I}_{|\psi^{-1}(y)| \leq x} \psi^{-1}(y) - \mathbb{I}_{|\phi^{-1}(y)| \leq x} \phi^{-1}(y)| dy = \|f'\|_{L_\infty} \int_{-x}^{x} |\psi(y) - \phi(y)| dy,$$

where the equality follows from considering the integral on the left-hand side as the area between the graphs of ψ, ϕ with vertical lines drawn at jumps. $\square$

For a positive integer $k \in \mathbb{N}$ and a monotone function ψ set $[\psi]_k$ be the linear interpolation of ψ in $k+1$ equidistributed points in $[-x, x]$ including the boundary $\{-x, x\}$. The function $[\psi]_k$ is also a monotone function and induces a point process on $[-x, x]$ in the sense of Definition 4.4.

Lemma 4.6. *Let ψ be monotone function. We have that*

$$\int_{-x}^{x} |\psi(y) - [\psi]_k(y)| dy \leq \frac{x}{k}(\psi(x) - \psi(-x)).$$

Proof. Observe that since ψ is monotone and the values of $[\psi]_k$ and ψ coincide in the points of interpolation $y_j = -x + \frac{2x}{k} j$, for $j = 0, 1, \ldots, k$, we may bound the integral between interpolation points by

$$\int_{y_j}^{y_{j+1}} |\psi(y) - [\psi]_k(y)| dy \leq \frac{x}{k}(\psi(y_{j+1}) - \psi(y_j)).$$

Summing up these bounds we deduce the result. □

Lemma 4.7. *Let ψ be a random monotone function. Let f be a continuously differentiable function supported on $[-x, x]$. Then we have*

$$\mathbb{E}_\psi \big| \Psi_{e^f} - \{\Psi_{e^f}\}_{[\psi]_k} \big| \leq \left(\mathbb{E}_\psi \Psi_{e^{4\Re f}} \mathbb{E}_{[\psi]_k} \Psi_{e^{4\Re f}} \right)^{1/4} \sqrt{\frac{x}{k} \|f\|_{L_\infty} \|f'\|_{L_\infty} \mathbb{E}_\psi (\psi(x+1) - \psi(-x-1))^2}.$$

Proof. Two applications of the Cauchy-Bunyakovsky-Schwarz inequality imply the bound

$$\mathbb{E}_\psi \big| \Psi_{e^f} - \{\Psi_{e^f}\}_{[\psi]_k} \big| \leq \left(\mathbb{E}_\psi \Psi_{e^{4\Re f}} \mathbb{E}_{[\psi]_k} \Psi_{e^{4\Re f}} \right)^{1/4} \sqrt{\mathbb{E}_\psi \big| S_f - \{S_f\}_{[\psi]_k} \big|^2}.$$

We have that the number of points in $[-x, x]$ under $\mathbb{P}_\psi$ and $\mathbb{P}_{[\psi]_k}$ does not exceed $\frac{\psi(x+1) - \psi(-x-1)}{2\pi}$ by Definition 4.4. Therefore, the inequality

$$\big| S_f - \{S_f\}_{[\psi]_k} \big| \leq 2\|f\|_{L_\infty} \frac{\psi(x+1) - \psi(-x-1)}{2\pi}$$

holds. Applying it to the L_2 norm we deduce

$$\begin{aligned}
\mathbb{E}_\psi \big| S_f - \{S_f\}_{[\psi]_k} \big|^2 &\leq 2\|f\|_{L_\infty} \mathbb{E}_\psi \frac{\psi(x+1) - \psi(-x-1)}{2\pi} \big| S_f - \{S_f\}_{[\psi]_k} \big| \\
&\leq \frac{x}{k} \|f\|_{L_\infty} \|f'\|_{L_\infty} \mathbb{E}_\psi (\psi(x+1) - \psi(-x-1))^2,
\end{aligned}$$

where the second inequality follows from Lemmata 4.6, 4.7 and the inequality $\psi(x) - \psi(-x) \leq \psi(x+1) - \psi(-x-1)$. This finishes the proof. □

4.3 The decay on the number of points

Let us say that a real-valued random variable X has *subgaussian tails* if there exist real constants C, σ and t_0 such that the inequality

$$\mathbb{P}(|X| \geq t) \leq C e^{-\frac{(t-t_0)^2}{\sigma^2}}$$

holds.

Lemma 4.8.

- *Sums of random variables with subgaussian tails also have subgaussian tails.*
- *A random variable almost surely bounded in absolute value by a random variable with subgaussian tails also has subgaussian tails.*
- *If a sequence of random variables with subgaussian tails with the same constants converges in distribution, then the limiting random variable also has subgaussian tails.*
- *If X has subgaussian tails then the expectation $\mathbb{E}X^2$ may be bounded by a constant depending only on the parameters of the tails.*

Proof. To prove the first claim use the Markov inequality as follows

$$\mathbb{P}(|X+Y| \geq t) = \mathbb{P}(e^{A(X+Y)^2} \geq e^{At^2}) \leq e^{-At^2}\mathbb{E}e^{A(X+Y)^2}, \quad \text{for some } A > 0.$$

The constant A is chosen to be sufficiently small for the function $e^{A(X+Y)^2}$ to be absolutely integrable. Let us show that this is possible. Since we have $e^{A(X+Y)^2} \leq e^{2AX^2}e^{2AY^2}$, so by the Cauchy-Bunyakovsky-Schwarz inequality it is sufficient that e^{4AX^2} and e^{4AY^2} are absolutely integrable. Observe that the bound

$$\mathbb{E}e^{4AX^2} = \sum_{j\geq 0} \mathbb{E}\left(e^{4AX^2} \big| X \in (j, j+1]\right) \mathbb{P}(X \in (j, j+1]) \leq \sum_{j \geq 0} e^{4A(j+1)^2} C e^{\frac{-(j-t_0)^2}{\sigma^2}}$$

holds, where the right-hand side converges absolutely for sufficiently small A.

The second and the third claim follows from the definition. The last claim follows from the inequality

$$\mathbb{E}X^2 = \sum_{j\geq 0} \mathbb{E}\left(X^2 \big| X \in (j, j+1]\right) \mathbb{P}(X \in (j, j+1]) \leq \sum_{j\geq 0} (j+1)^2 C e^{\frac{-(j-t_0)^2}{\sigma^2}}.$$

□

Lemma 4.9. *Let ψ be a random monotone function. Let the random variable $\psi(x+1) - \psi(-x-1)$ have subgaussian tails. Let g be a bounded Borel function supported on $[-x, x]$. Let $\mathcal{I}_j$ be the indicator function of the interval $[-x, x]$ containing at most j points.*

- *The multiplicative functional Ψ_{1+g} is absolutely integrable with respect to the measure $\mathbb{P}_\psi$. Its $L_1(\mathrm{Conf}(\mathbb{R}), \mathbb{P}_\psi)$ norm is bounded by a constant, depending only on the supremum norm of $1+g$ and the subgaussian tail parameters of the difference $\psi(x+1) - \psi(-x-1)$.*
- *We have that $\mathbb{E}_\psi \Psi_{1+g}\mathcal{I}_j$ converges to $\mathbb{E}_\psi \Psi_{1+g}$ as $j \to \infty$. This convergence depends only on the supremum norm of $1+g$ and the subgaussian tail parameters of the difference $\psi(x+1) - \psi(-x-1)$.*

Remark. Let us briefly explain the purpose of Lemma 4.9. Assume we have a family of random monotone functions $\{\psi_k\}_{k\in\mathbb{N}}$ such that $\psi_k(x+1) - \psi_k(-x-1)$ have subgaussian tails with the same constants. Then the expectations $\mathbb{E}_{\psi_k}\Psi_{1+g}$ are bounded by an independent of k constant. Further, the convergence of the expectations $\mathbb{E}_{\psi_k}\Psi_{1+g}\mathcal{I}_j$ as $j \to \infty$ is uniform in k.

Proof. Indeed, let q be the supremum of $|1+g|$. We have that the number of points $N_{[-x,x]}$ in the interval $[-x, x]$ does not exceed $\psi(x+1) - \psi(-x-1)$ by Definition 4.4. The latter has subgaussian tails. Thereby so does the former by the second claim of Lemma 4.8. It is remaining to observe that the expectation is bounded

$$\mathbb{E}_\psi|\Psi_{1+g}| = \sum_{k\geq 0} \mathbb{E}_\psi(|\Psi_{1+g}| \big| N_{[-x,x]} = k)\mathbb{P}_\psi(N_{[-x,x]} = k) \leq C \sum_{k\geq 0} q^k e^{-\frac{(k-t_0)^2}{\sigma^2}}$$

by a converging series. Further, the absolute value of the expectation $\mathbb{E}_\psi \Psi_{1+g}(1 - \mathcal{I}_j)$ is bounded by the series on the right-hand side above for $k \geq j+1$. □

Remark. One may replace the subgaussian decay with any different sufficiently rapid decay.

4.4 Convergence of multiplicative functionals in terms of random monotone functions

Lemma 4.10. *Assume that random monotone functions ψ_n converge in finite-dimensional distributions to a random monotone function ψ as $n \to \infty$. Assume that for each x the convergence*

$$\lim_{n\to\infty} \mathbb{E}_{\psi_n}|\psi_n(x)| = \mathbb{E}_\psi|\psi(x)|.$$

holds. Let f be a continuously differentiable function supported on $[-x, x]$. Assume that random variables $\psi_n(x+1) - \psi_n(-x-1)$ have subgaussian tails with independent of n constants. Then for each k and any compactly supported continuously differentiable f we have the convergence

$$\mathbb{E}_{[\psi_n]_k} \Psi_{e^f} \to \mathbb{E}_{[\psi]_k} \Psi_{e^f}, \qquad \text{as } n \to \infty,$$

where random functions $[\psi_n]_k$ and $[\psi]_k$ linearly interpolate the functions ψ_n, ψ in $k+1$ equidistributed points in $[-x-1, x+1]$.

Proof. By the third claim of Lemma 4.8 the random variable $\psi(x+1) - \psi(-x-1)$ has subgaussian tails. Recall that by $\mathcal{I}_j$ we denoted the indicator function of the subset $[-x, x]$ having at most j points. By Lemma 4.9 (see the remark) we have the convergence

$$\mathbb{E}_{[\psi]_k} \Psi_{e^f} \mathcal{I}_j \to \mathbb{E}_{[\psi]_k} \Psi_{e^f}, \quad \mathbb{E}_{[\psi_n]_k} \Psi_{e^f} \mathcal{I}_j \to \mathbb{E}_{[\psi_n]_k} \Psi_{e^f}, \qquad \text{as } j \to \infty$$

uniformly in n. Therefore it is sufficient to establish that the claimed convergence holds for $\Psi_{e^f} \mathcal{I}_j$ for each j.

We have that the values of ψ_n in the points of interpolation converge in distribution to the values of ψ. By the Skorohod coupling Theorem we may assume they are on a single probability space and the convergence holds almost surely. Further, the convergence of L_1 norms implies the convergence in L_1 norm. From the latter we deduce that the integral

$$\mathbb{E} \int_{-x}^{x} |[\psi_n]_k(y) - [\psi]_k(y)| dy \to 0$$

vanishes as $n \to \infty$. Last, the inequality

$$\mathbb{E}\mathcal{I}_j \left| e^{\{S_f\}_{[\psi_n]_k}} - e^{\{S_f\}_{[\psi]_k}} \right| \leq e^{2j\|f\|_{L_\infty}} \|f'\|_{L_\infty} \mathbb{E} \int_{-x}^{x} |[\psi_n]_k(y) - [\psi]_k(y)| dy$$

holds by Lemma 4.5. This finishes the proof. □

Proposition 4.11. *Assume that random monotone functions ψ_n converge in finite-dimensional distributions to a random monotone function ψ as $n \to \infty$. Assume, further, that for each $x > 0$ the random variables $\psi_n(x) - \psi_n(-x)$ have subgaussian tails for independent of n constant and that the convergence*

$$\lim_{n \to \infty} \mathbb{E}_{\psi_n} |\psi_n(x)| = \mathbb{E}_{\psi} |\psi(x)|$$

holds. Then for any compactly supported continuously differentiable f we have the convergence

$$\mathbb{E}_{\psi_n} \Psi_{e^f} \to \mathbb{E}_{\psi} \Psi_{e^f}, \qquad \text{as } n \to \infty.$$

Proof. First observe that for each $x > 0$ the random variable $\psi(x) - \psi(-x)$ has subgaussian tails by the third claim of Lemma 4.8. Assume that f is supported in $[-x, x]$. Let the functions $[\psi_n]_k$ and $[\psi]_k$ linearly interpolate the functions ψ_n, ψ in $k+1$ equidistributed points in $[-x-1, x+1]$. By Lemmata 4.7, 4.9 and the last claim of Lemma 4.8 the expectations $\mathbb{E}_{[\psi_n]_k} \Psi_{e^f}$, $\mathbb{E}_{[\psi]_k} \Psi_{e^f}$ converge to $\mathbb{E}_{\psi_n} \Psi_{e^f}$, $\mathbb{E}_{\psi} \Psi_{e^f}$ as $k \to \infty$ uniformly in n. The statement then follows from Lemma 4.10. □

4.5 Construction of sine-β process via coupled stochastic differential equations

Let B_1, B_2 be independent standard Brownian motions. For $x \in \mathbb{R}$ introduce the stochastic differential equation

$$dX_x^\beta(t) = x dt + \frac{2}{\sqrt{\beta t}} \Im\{[e^{iX_x^\beta(t)} - 1][dB_1(t) + i dB_2(t)]\}. \tag{4.1}$$

Theorem 4.12 ([30, Proposition 4.5])**.** *For each x there exists a unique strong solution $X_-^{(x)}$ on $(0, +\infty)$, satisfying the following properties*

- *If $x > y$ then $X_x^\beta(t) - X_y^\beta(t)$ is $\mathbb{P}_X$-almost surely nonnegative function of t.*
- *We have*
$$\mathbb{E}X_x^\beta(t) = xt.$$

Definition 4.13 ([30, page 6])**.** Let $\mathcal{P}_\beta$ stand for the finite-dimensional distributions of $X_-^\beta(1)$. The sine-β process is the process associated to $\mathcal{P}_\beta$
$$\mathbb{P}_\beta := \mathbb{P}_{\mathcal{P}_\beta}$$
in the sense of Definition 4.4.

Remark. Let us also mention a different construction of Valkó and Virág [47], who were able to describe the sine-β process as a limit of gaussian β-ensembles. Their construction involves expressing the counting function of the point process as a limit of a system of coupled stochastic differential equations. They were also able to represent this point process as a spectrum of a random operator [49]. The equivalence of two constructions is due to Nakano [37].

4.6 Construction of the circular β-ensemble via random functions

In this subsection we recall a similar construction for the measure $\mathbb{P}_\beta^n$, suggested by Killip and Stoiciu, based on the 5-diagonal model of the circular β-ensemble by Killip and Nenciu [31]. The construction uses the theory of orthogonal polynomials on the unit circle, for which we refer the reader to [41].

A θ_ν distribution is a distribution in the unit disc with the density
$$d\theta_\nu(z) = \frac{\nu-1}{2\pi}(1-|z|^2)^{(\nu-3)/2}\mathbb{I}_{|z|\le 1}d^2z$$
Let $(\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1})$ be independent random variables, distributed according to $\alpha_k \sim \theta_{\beta(k+1)+1}$ for $k = 0, \dots n-2$ and $\alpha_{n-1} \sim e^{i\mu}$, μ is a uniformly distributed $[0, 2\pi]$ random variable. Introduce a sequence of random polynomials on the unit circle, satisfying the Szegö recursion with α_k being the Verblunsky coefficients
$$\begin{aligned}\Phi_{k+1}(z) &= z\Phi_k(z) - \bar{\alpha}_k\Phi_k^*(z),\\ \Phi_{k+1}^*(z) &= \Phi_k^*(z) - \alpha_k z\Phi_k(z),\end{aligned} \quad k = 0, \dots, n-1,$$
where $\Phi_k^*(z) = z^k\overline{\Phi_k(\bar{z}^{-1})}$ are the reciprocal polynomials, $\Phi_0(z) = 1$.

As shown by Cantero, Moral and Velázquez [13], to a sequence of Verbulnsky parameters one may explicitly assign a matrix $\mathcal{C}^{(n)}$ called the CMV matrix. It is connected to the corresponding orthogonal polynomials as follows
$$\det[z - \mathcal{C}^{(n)}] = z\Phi_{n-1}(z) - e^{-i\mu}\Phi_{n-1}^*(z).$$
The result of Killip and Nenciu [31] states that, given the Verblunsky parameters are distributed as above, the spectrum of the corresponding CMV matrix (the set of zeros of the right-hand side of the above expression) is distributed according to $\mathbb{P}_\beta^n$.

Next Killip and Stoiciu define
$$B_k(z) = \frac{z\Phi_k(z)}{\Phi_k^*(z)}, \quad e^{i\psi_k(\theta)} = \frac{B_k(e^{i\theta})}{B_k(1)},$$
where $\psi_k : (-\pi, \pi) \to \mathbb{R}$ is a random function, satisfying $\psi_k(0) = 0$. We have that ψ_k is monotone increasing and defines $\mathbb{P}_\beta^n$ in the sense of Definition 4.4.

Proposition 4.14 ([31, Theorem 1], [30, Proposition 2.2])**.** *The point process $\mathbb{P}_{\psi_{n-1}}$ on* $\mathrm{Conf}((-\pi, \pi))$*, defined in the sense of Definition 4.4, coincides with $\mathbb{P}_\beta^n$.*

When point processes are constructed in the sense of Definition 4.4, it is possible to connect their convergence with the convergence of the corresponding functions. For now we state the latter.

Theorem 4.15 ([30, Proposition 4.4])**.** *We have that finite-dimensional distributions of $\psi_{n-1}(\theta/n)$ converge to finite-dimensional distributions of $X_-^\beta(1)$. Further, we have that*
$$\mathbb{E}|\psi_{n-1}(-/n)| \to \mathbb{E}|X_-^\beta(1)|, \quad \textit{as } n \to \infty.$$

4.7 Uniform estimates on numbers of particles for the circular β-ensemble: proof of Proposition 1.5

First recall the definition of the subgaussian random variable.

Proposition 4.16 ([46, Proposition 2.5.2]). *The following conditions for a real-vlued random variable Z are equivalent*

- *There exists a positive constant C' such that $\mathbb{P}(|Z-\mathbb{E}Z|\geq t)\leq 2e^{-\frac{t^2}{C'}}$ holds.*
- *There exists a positive constant C such that the bound $\mathbb{E}e^{t(Z-\mathbb{E}Z)}\leq e^{Ct^2}$ on the moment generating function holds.*
- *There exists a positive constant C such that the bound $\mathbb{E}|Z-\mathbb{E}Z|^p\leq Cp^{p/2}$ holds for any $p\geq 1$.*

If one of these is satisfied we say that Z is subgaussian with constant C. The constants C, C' are the same up to by multiplication by an absolute constant. We will denote their dependence $C=C(Z)$, $C'=C'(Z)$.

It is obvious that subgaussian random variable has subgaussian tails. In particular, if Z is positive and subgaussian, we have

$$\mathbb{P}(Z\geq t)\leq 2e^{-\frac{(t-\mathbb{E}Z)^2}{C'(Z)}}.$$

The second claim of Theorem 1.3 implies

Lemma 4.17. *For each $x\in\mathbb{R}$ the random variables $\psi_{n-1}(x/n)$ have subgaussian tails with independent of n constants.*

Proof. We have that $\psi_{n-1}(0)=0$. Therefore by Definition 4.4 the random variable $\psi_{n-1}(x/n)$ does not exceed $2\pi(N_{[0,x/n]}+1)$, where $N_{[0,x/n]}$ is a number of points on the interval $[0,x/n]$. Observe that this number is precisely the additive functional $S_{\mathbb{I}_{[0,x]}}(n\cdot)$ corresponding to the indicator function $\mathbb{I}_{[0,x]}$. The indicator function may be bounded by a smooth compactly supported positive function f. The corresponding additive functional $S_{f(n\cdot)}$ bounds the number of points. The second claim of Corollary 1.3 and the convergences established in the proof of Lemma 3.1 imply that the random variables $S_{f(n\cdot)}$ have subgaussian tails with independent of n constants. Therefore so do the random variables $\psi_{n-1}(x/n)$ for each $x\in\mathbb{R}$ by the second claim of Lemma 4.8. The claim then follows from the first assertion of Lemma 4.8. ☐

Proof of Proposition 1.5. The claim follows from Proposition 4.11, applied to the functions $\psi_{n-1}(\cdot/n)$ that induce the circular β-ensemble. They satisfy the requirements of the proposition by Theorem 4.15 and Lemma 4.17. ☐

5 Proof of Corollary 1.7

Corollary 1.7 follows directly from Theorem 1.3 and the Feller smoothing estimate.

Theorem 5.1 ([20, p. 538]). *Assume we are given two real-valued random variables with distribution functions F_1, F_2 and characteristic functions φ_1, φ_2. For any $T>0$ we have that*

$$\sup_{x\in\mathbb{R}}|F_1(x)-F_2(x)|\leq\frac{24}{\sqrt{2\pi^2}T}+\frac{1}{\pi}\int_{-T}^{T}\left|\frac{\varphi_1(y)-\varphi_2(y)}{y}\right|dy.$$

Proof of Theorem 1.7. First observe that

$$\int_0^\infty\lambda R\hat{f}(R\lambda)R\hat{f}(-R\lambda)d\lambda=\int_0^\infty\lambda\hat{f}(\lambda)\hat{f}(-\lambda)d\lambda.,$$

$$\|f(\cdot/R)\|_{\dot{H}_{1/2}(\mathbb{R})}=\|f(\cdot)\|_{\dot{H}_{1/2}(\mathbb{R})},\quad\|f(\cdot/R)\|_{\dot{H}_1(\mathbb{R})}=\frac{1}{\sqrt{R}}\|f(\cdot)\|_{\dot{H}_1(\mathbb{R})}.$$

Denote $\varphi_R(k) = \mathbb{E}_\beta e^{ik\overline{S}_{f(\cdot/R)}}$, $\varphi_{\mathcal{N}}(k) = e^{-k^2/2}$. By Theorem 1.3 we have for some constant K

$$\left|\frac{\varphi_R(k) - \varphi_{\mathcal{N}}(k)}{k}\right| \leq K e^{K|k|^2}|k|,$$

and hence

$$\int_{-T}^{T}\left|\frac{\varphi_R(k) - \varphi_{\mathcal{N}}(k)}{k}\right| dk \leq \frac{2KT^2}{R} e^{KT^2}.$$

Corrolary 1.7 follows from substituting $T = K'\sqrt{\ln R}$ into the above inequality with a sufficiently small constant K' and using Theorem 5.1. □

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