NAKAJIMA QUIVER VARIETIES IN DIMENSION FOUR

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ABSTRACT. In this paper we classify all four-dimensional Nakajima quiver varieties via a combinatorial approach. For each quiver variety, we describe the symplectic leaves and minimal degenerations between them. Using the resulting Hasse diagrams and secondary hyperplane arrangements, we fully classify the quiver varieties up to isomorphism, a step in the problem of classifying all four-dimensional conical symplectic singularities. As an application, we answer in the negative a question posed by Bellamy, Craw, Rayan, Schedler, and Weiss regarding whether the G_4 quotient singularity (or its projective crepant resolutions) can be realised as a quiver variety.

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1. INTRODUCTION

Let $Q = (Q_0, Q_1, s, t)$ be a finite, connected quiver of rank $\mathsf{rk} := |Q_0|$ and let $\alpha \in \mathbb{Z}_{\geq 0}^{\mathsf{rk}}$ be a dimension vector. The vector space $\mathrm{Rep}(Q, \alpha)$ of complex representations of Q with dimension vector α is an affine variety, and its cotangent bundle admits a Hamiltonian action from the reductive gauge group $G(\alpha) := \prod_{i \in Q_0} \mathrm{GL}(\mathbb{C}, \alpha_i)$, with moment map μ. The Nakajima quiver variety

$$\mathfrak{M}_0(Q, \alpha) := \mu^{-1}(0) /\!/ G(\alpha)$$

is then obtained by Hamiltonian reduction, and is a moduli space parametrising stable representations of the associated preprojective algebra $\Pi(Q)$.

Quiver varieties appear in many different parts of algebraic geometry and representation theory. For example, they give constructions of moduli spaces such as resolutions of Kleinian singularities, their Hilbert schemes of points, and Uhlenbeck and Gieseker instanton moduli spaces.

By work [7] of Bellamy and Schedler, $\mathfrak{M}_0(Q, \alpha)$ is a conical symplectic singularity, admitting projective symplectic (equivalently, crepant) resolutions in precise cases. These singularities are also an important topic in modern representation theory and algebraic geometry. They allow to obtain many properties of deformation quantizations of a variety by studying the resolutions, enhance the search for compact symplectic varieties and possess many fascinating properties, like Białynicki-Birula decomposition or symplectic duality[1], closely related to duality for 3d supersymmetric quantum field theories.

Quiver varieties make up a large part of these singularities, so it is by studying them we begin to classify all conical symplectic singularities in dimension four. This is similar to the work of Nagaoka in [32, Thm. 1.4] on classifying affine hypertoric varieties in this dimension, another class of conical symplectic singularities.

In dimension two quiver varieties are the well-studied Kleinian singularities. There Q has underlying graph $\tilde{\Gamma}$, for Γ a simply laced Dynkin diagram, and α is the associated minimal (*isotropic*) imaginary

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[1]we refer the reader to Kamnitzer's survey [25]

root δ. Combinatorially, the dimension vector is special in the sense that for all vertices i we have $2\delta_i - \sum \delta_j = 0$, the sum taken over the (undirected) neighbours of i. It turns out that the 'expected dimension' of quiver varieties can be determined by generalising this idea.

To see this, let $\Delta = (\Delta_0, \Delta_1)$ be the underlying graph of Q and let i be a vertex with ℓ_i loops. The 'vertex' and 'total' *balances* of a dimension vector $\alpha \in \mathbb{N}^{\mathsf{rk}}$ are

$$(\alpha, \mathsf{e}_i) = 2(1 - \ell_i)\alpha_i - \sum_{i-j} \alpha_j, \qquad (\alpha, \alpha) = \sum_i \alpha_i(\alpha, \mathsf{e}_i),$$

where $(\cdot, \cdot)$ is the bilinear Euler form that associated with the generalised Cartan matrix of Δ. We call a vertex i *balanced* if $(\alpha, \mathsf{e}_i) = 0$ and call α itself balanced every vertex is balanced. In [20], Kac first studied the function

$$p \colon \mathbb{Z}^{\mathsf{rk}} \to \mathbb{Z}, \quad \alpha \mapsto 1 - \tfrac{1}{2}(\alpha, \alpha),$$

to count parameters in indecomposable representations of $M \in \mathrm{Rep}(Q, \alpha)$. Following this, Crawley-Boevey in [13, Thm. 1.1] shows that $\dim \mathfrak{M}_0(Q, \alpha) = 2p(\alpha)$, provided that α is in a combinatorially defined (Definition 2.7) subset Σ_0 of the positive root system R^+ associated with Δ. By [13, Thm. 1.2], Σ_0 consists of dimension vectors for 0-stable (that is, simple) representations of the preprojective algebra $\Pi(Q)$. Roughly speaking geometrically, the functions that define the variety $\mathfrak{M}_0(Q, \alpha)$ will be 'independent' when $\alpha \in \Sigma_0$.

Furthermore, in [14, Thm. 1.1, Proposition 1.2] Crawley-Boevey proves that every $\alpha \in \mathbb{N}\mathsf{R}^+$ decomposes as

$$\alpha = \sum_{k=1}^{r} m_k \beta^{(k)}, \quad \beta^{(k)} \in \Sigma_0 \tag{1.A}$$

known as the *canonical decomposition*. Moreover, if $\alpha = \sum_{k=1}^{r} m_k \beta^{(k)}$ is such a decomposition then

$$\mathfrak{M}_0(\alpha) \simeq \prod_{k=1}^{r} S^{m_k} \mathfrak{M}_0(\beta^{(k)}).$$

$\mathfrak{M}_0(\beta^{(k)})$ is a point for real $\beta^{(k)}$, and if $p(\beta^{(k)}) > 1$ then $m_k = 1$. It follows that to classify all quiver varieties of given dimension $2n$ one can find all the dimension vectors $\alpha \in \Sigma_0$ with $p(\alpha) \leq n$. In this paper we study the case $p(\alpha) = 2$ in order to classify four-dimensional quiver varieties.

1.1. **Main results.** To discover new quiver varieties in higher dimensions, we play the following combinatorial game with dimension vectors. As one might expect, this is just a small modification to the balanced case of Kleinian singularities. We start by finding all dimension vectors α such that

- $p(\alpha) = 2$, i.e. total balance -2,
- $(\alpha, \mathsf{e}_i) \leq 0$, i.e. nonpositive vertex balance, for all $i \in \Delta$.

The second condition (which turns out to be a consequence of $\alpha \in \Sigma_0$, see Lemma 2.10 and [14, Lem. 2.3]) makes the problem finite, as there are only three qualitatively different cases, that we call *types*:

(I) There are only two unbalanced vertices, both of weight 1 and balance -1.
(II) There is a unique unbalanced vertex, of weight 1 and balance -2.
(III) There is a unique unbalanced vertex, of weight 2 and balance -1.

Across three types we give all options for an unbalanced vertex (see local structure in § 3.2), and classify all ways they might be furnished with balanced vertices. Finding every dimension vector α across Types I, II, and III, and removing those that lie outside Σ_0, we obtain our first result.

Theorem 1.1. *The following is a full list of dimension vectors α such that $\alpha \in \Sigma_0$ and $p(\alpha) = 2$. Here Γ is a simply laced Dynkin diagram and $\gamma(\Gamma)$ the highest root in the associated (finite) root system, so that dotted boxes contain the corresponding minimal imaginary root $\delta(\Gamma)$. Unbalanced vertices are underlined, and vertices are assumed balanced otherwise.*

TABLE 1.1. Dimension vectors $\alpha \in \Sigma_0$ with $p(\alpha) = 2$

(Δ, α)		rk, ht
$\mathsf{I}(\Gamma, n)$ $\Gamma \neq \mathsf{E}_8$ $n \geq 0$	$\underline{1} \;\text{---}\; \gamma(\Gamma) \;\text{---}\; \underline{1}$ with $1 \;\ldots\; n$ arcs below	$\mathrm{rk}(\Gamma) + n$ $\mathrm{ht}(\Gamma) + n$
$\mathsf{I}(m)$ $m \geq 4$	$1 - 2 - 3 \overset{1}{\text{---}} \cdots \overset{m-4}{\text{---}} 3 - 2 - 1$ with 1 above the left 3 and 1 above the right 3	$m + 3$ $3m - 1$
I_a	$\underline{1}$ and $\underline{1}$ joined to $3 - 4 - 3 - 2 - 1$, with 2 above 4	8 17
I_b	$\underline{1} - 3$ and $\underline{1} - 3$ joined to $5 - 4 - 3 - 2 - 1$	9 23
I_c	$\underline{1}$ and $\underline{1}$ joined to $3 - 4 - 5 - 6 - 4 - 2$, with 3 above 6	9 29
$\mathsf{II}(\Gamma^{(1)}, \Gamma^{(2)})$	$\gamma(\Gamma^{(1)}) \;\text{---}\; \underline{1} \;\text{---}\; \gamma(\Gamma^{(2)})$	$\mathrm{rk}^{(1)} + \mathrm{rk}^{(2)} + 1$ $\mathrm{ht}^{(1)} + \mathrm{ht}^{(2)} + 1$
$\mathsf{II}(m)$ $m \geq 4$	2 and 2 joined to $4 \overset{1}{\text{---}} \cdots \overset{m-4}{\text{---}} 4 - 3 - 2 - 1$, with $\underline{1}$ above the 4	$m + 3$ $4m - 1$
II_a	$\underline{1} - 4 - 5 - 6 - 4 - 2$, with 2 above 4 and 3 above 6	8 27
II_b	$\underline{1} - 4 - 7 - 10 - 8 - 6 - 4 - 2$, with 5 above 10	9 47
$\mathsf{III}(\Gamma)$ $\Gamma \neq \mathsf{A}$	$\gamma(\Gamma) \;\text{---}\; \underline{2} \;\text{---}\; 1$	$\mathrm{rk}(\Gamma) + 2$ $\mathrm{ht}(\Gamma) + 2$

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TABLE 1.1. (continued)

(Δ, α)		rk, ht
$\mathrm{III}(m,n)$ $m \geq 4$ $n \in \{6,7,8\}$	$\begin{array}{c} 1 \\ \diagdown \\ 2 \overset{1}{\text{---}} \cdots \overset{m-4}{\text{---}} \underline{2} \text{---} 3 \text{---} \cdots \\ \diagup \qquad\qquad \gamma(\mathsf{E}_n) \\ 1 \end{array}$	$m+n-2$ $2m + \{5,11,24\}$
III_a	$\begin{array}{c} \underline{2} \\ \mid \\ 1 \text{---} 2 \text{---} 3 \text{---} 4 \text{---} 5 \text{---} 4 \text{---} 3 \text{---} 2 \text{---} 1 \end{array}$	10 27
III_b	$\begin{array}{c} 4 \\ \mid \\ \underline{2} \text{---} 5 \text{---} 8 \text{---} 7 \text{---} 6 \text{---} 5 \text{---} 4 \text{---} 3 \text{---} 2 \text{---} 1 \end{array}$	11 47

Remark 1.2. We included the height of the dimension vector as it gives the dimension of the associated Coulomb branch.

Remark 1.3. Compare this to [37, Thm. 5.1, 5.3], where authors classified special 'star-shaped' and 'crab-shaped' roots that satisfy $p(\alpha) = 2$. All of these appear in our classification, mostly in the form of borderline cases of infinite families and exceptions, and some of them lie outside Σ_0 and the corresponding quiver varieties are symmetric powers of Kleinian singularities.

Bellamy-Schedler in [7] show that all possible subrefinements of the canonical decomposition classify symplectic leaves $\mathscr{L}$ within quiver varieties and in some cases you can find something about minimal degenerations (Definition 4.5). In our case it works for all minimal degenerations and we classify them based on isotropic decompositions, which are of the form

$$\alpha = \delta(\Gamma) + \gamma(\Gamma') \tag{1.B}$$

with δ the minimal imaginary root in an extended ADE root system of type Γ and γ the highest root in a finite ADE root system of type Γ'. Geometrically, (δ, γ) tells us that the transversal slice to the corresponding two-dimensional leaf $\mathscr{L}$ inside $\mathfrak{M}_0(\alpha)$ is the Kleinian singularity corresponding to γ, and the transversal slice to 0 point inside $\mathscr{L}$ is the Kleinian singularity corresponding to δ. In this way, the data (δ, γ), together with the number of resolutions, allow us to distinguish the quiver varieties up to isomorphism.

Theorem 1.4. *Every dimension vector in Theorem* 1.1 *yields a non-isomorphic conical symplectic singularity. In particular,* $\mathfrak{M}_0(\mathrm{II}(\Gamma^{(1)}, \Gamma^{(2)}))$ *is a product of Kleinian singularities, and we have no examples of symmetric powers arising from dimension vectors in* Σ_0.

Remark 1.5. Note, that in our classification all the roots in Σ_0 produce distinct quiver varieties. However, for higher dimensions this is not necessarily so. In [5, Lem. 8.4] the authors explain that for $\alpha^{(1)}, \alpha^{(2)} \in \Sigma_0$ one can find β in Σ_0 such that $\mathfrak{M}_0(\beta) \simeq \mathfrak{M}_0(\alpha^{(1)}) \times \mathfrak{M}_0(\alpha^{(2)})$ if both $\alpha^{(1)}, \alpha^{(2)}$ have a vertex of weight 1. Namely, one needs to 'glue' the roots by identifying this vertex in both of them, a generalisation of $\mathrm{II}(\Gamma^{(1)}, \Gamma^{(2)})$ outside of Kleinian singularities. In principle, one can do this in several different ways if one of the roots has more than one such vertex and such different graphs will correspond to the same variety. One can see this already for roots with $p(\alpha) = 3$.

Geometrically we thus obtain a full classification of Nakajima quiver varieties in dimension four along with their Namikawa Weyl groups. It so happens that the Namikawa Weyl group associated to each α in our classification is the product of the finite Weyl groups of ADE type that can appear as highest roots in isotropic decompositions (1.B).

Theorem 1.6. *If* $\alpha \in \Sigma_0$ *satisfies* $p(\alpha) = 2$, *then it has one of the following Namikawa Weyl groups:*

TABLE 1.2. Namikawa Weyl groups for 4D quiver varieties

(Δ, α)	Parameter(s)	W_α	$|W_\alpha|$
$\mathsf{I}(\ell, m, n)$	$\ell, m, n \geq 0$	$W(\mathsf{A}_{\ell-1}) \times W(\mathsf{A}_{m-1}) \times W(\mathsf{A}_{n-1})$	$\ell!m!n!$
$\mathsf{I}(\mathsf{D}_m, n)$	$m \geq 4, \ n \geq 0$	$W(\mathsf{A}_{m-1}) \times W(\mathsf{A}_{n-1})$	$m!n!$
$\mathsf{I}(\mathsf{D}_m, n)'$	$m \geq 4, \ n \geq 0$	$W(\mathsf{A}_{n-1}) \times W(\mathsf{D}_{m-1})$	$2^{m-2}(m-1)!n!$
$\mathsf{I}(\mathsf{E}_6, n)$	$n \geq 0$	$W(\mathsf{A}_{n-1}) \times W(\mathsf{D}_5)$	$1920n!$
$\mathsf{I}(\mathsf{E}_7, n)$	$n \geq 0$	$W(\mathsf{A}_{n-1}) \times W(\mathsf{E}_6)$	$51840n!$
$\mathsf{I}(m)$	$m \geq 4$	$W(\mathsf{A}_{m+1})$	$(m+2)!$
I_a		$W(\mathsf{D}_6)$	23040
I_b		$W(\mathsf{D}_7)$	322560
I_c		$W(\mathsf{E}_7)$	2903040
$\mathsf{II}(\Gamma^{(1)}, \Gamma^{(2)})$	$\Gamma^{(1)}, \Gamma^{(2)} \in \{\mathsf{A}, \mathsf{D}, \mathsf{E}\}$	$W(\Gamma^{(1)}) \times W(\Gamma^{(2)})$	$|W(\Gamma^{(1)})||W(\Gamma^{(2)})|$
$\mathsf{II}(m)$	$m \geq 4$	$W(\mathsf{D}_{m+2})$	$2^{m+1}(m+2)!$
II_a		$W(\mathsf{E}_7)$	2903040
II_b		$W(\mathsf{E}_8)$	696729600
$\mathsf{III}(\mathsf{D}_m, i)$	$1 \leq i \leq \lceil \frac{m-3}{2} \rceil$	$W(\mathsf{A}_1) \times W(\mathsf{D}_{i+1}) \times W(\mathsf{D}_{m-i-1})$	$2^{m-1}m!\binom{m}{i+1}$
$\mathsf{III}(\mathsf{E}_7)$		$W(\mathsf{A}_1) \times W(\mathsf{A}_7)$	80640
$\mathsf{III}(\mathsf{E}_8)$		$W(\mathsf{A}_1) \times W(\mathsf{D}_8)$	10321920
$\mathsf{III}(m, 6)$	$m \geq 4$	$W(\mathsf{A}_5) \times W(\mathsf{D}_{m-2})$	$2^m 90(m-2)!$
$\mathsf{III}(m, 7)$	$m \geq 4$	$W(\mathsf{D}_6) \times W(\mathsf{D}_{m-2})$	$2^m 2880(m-2)!$
$\mathsf{III}(m, 8)$	$m \geq 4$	$W(\mathsf{E}_7) \times W(\mathsf{D}_{m-2})$	$2^m 362880(m-2)!$
III_a		$W(\mathsf{A}_9)$	3628800
III_b		$W(\mathsf{D}_{10})$	1857945600

One of the main consequences of our work are the first steps in understanding the $(2, 2)$ case of quiver varieties, the only examples of quiver varieties with divisible dimension vectors that admit projective symplectic resolutions. By [7, Thm. 1.5], these dimension vectors are all of the form $\beta = 2\alpha$, where $\alpha \in \Sigma_0$ satisfies $p(\alpha) = 2$, with the associated quiver varieties being of dimension 10. Hence we have found all $(2, 2)$ dimension vectors. We also connect the classification with conical symplectic singularities arising in other areas, such as nilpotent orbit closures, hypertoric varieties, and finite group quotients. For the latter case, we answer a question posed in [3] regarding the exceptional complex reflection group G_4.

Corollary 1.7. *The G_4 quotient singularity cannot be realised as a quiver variety, nor can either of its projective crepant resolutions.*

In the proof of Theorem 1.4 we show that we may count the number of projective symplectic resolutions admitted by each four-dimensional quiver variety, in particular using the secondary arrangement associated with the Namikawa Weyl group. We count the number in some cases and establish unexpected conjectures about these numbers for the cases which currently exceed available computational capabilities. In particular, we make the claim that every quiver variety with dimension vector in Type II admits a unique projective crepant resolution.

1.2. **Structure of the paper.** In §2 we give a background on quiver varieties, root combinatorics, and the connection between the two. Importantly we summarise the dimension 2 classification and provide balancing results that will aid us in the dimension 4 classification.

The classification of dimension vectors (Theorem 1.1) is given in §3, starting from the 'local structure' of unbalanced vertices and working through all possible balanced terminations. We then rule out any such dimension vectors that lie outside of Σ_0.

In §4 we go into more detail on the geometric structure admitted by quiver varieties, in order to distinguish the varieties $\mathfrak{M}_0(\alpha)$ for all α obtained in §3 . In particular we compute all of their isotropic decompositions, summarised in Table 4.1. As a consequence of this work we may classify all of the Namikawa-Weyl groups that arise for quiver varieties in dimension 4, which is Theorem 1.6.

We complete the classification in § 5, proving Theorem 1.4 using the invariants of minimial degenerations and projective crepant resolutions. This allows us to prove Corollary 1.7 and state several conjectures for the number of such resolutions.

The symplectic leaf closures, and hence full poset of transversal slice singularities, deduced in § 4 are tabulated in § A.

1.3. **Notation.** For $\alpha, \beta \in \mathbb{N}^{\mathsf{rk}}$ we write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i and write $\alpha > \beta$ if further $\alpha \neq \beta$. In general two dimension vectors will be supported on different graphs with their own root systems. We abuse notation and write Σ_0 to be viewed in either. By *weighted graph*, we mean a connected graph with nonnegative integer weights on the vertices. We represent weighted graphs (Δ, α) with weights directly on each vertex and without explicit indexing, in this way using α to infer the Δ on which α is supported. In similar vein, we write $\mathfrak{M}_0(\alpha)$ in place of $\mathfrak{M}_0(Q, \alpha)$ or $\mathfrak{M}_0(\Delta, \alpha)$. In the displayed dimension vectors, vertices are assumed to be balanced (in the sense of Definition 2.2) unless underlined or overlined to denote negative or positive balance, respectively. This allows the adjacency of vertices to be inferred from the compact versions of dimension vectors used going forward. We use some graph theory notions like 'branching' vertex which denotes a vertex connected to more than 2 neighbours. On the contrary, *hanging* vertex is a vertex connected strictly to 1 other vertex.

Our parameter conventions are $m \geq 0$ for A_m and $m \geq 4$ for D_m unless otherwise stated, where A_0 is the empty graph and its extension $\tilde{\mathsf{A}}_0$ consists of a single vertex with a loop. We now give an explanation of the general constructions shown in Table 1.1, using the words *affine* and *extended* interchangeably with respect to ADE Dynkin diagrams.

Notation 1.8. Let Γ be an ADE diagram, choose two vertices of weight 1 in its minimal imaginary root, and let $n > 0$. Then $\mathsf{I}(\Gamma, n)$ is the dimension vector obtained by connecting with a chain of n edges and weight 1 vertices, which gives both distinguished vertices a balance of -1. Specifics for each Γ are as follows:

(A) $\mathsf{I}(\mathsf{A}_m, n)$ can be viewed as a three-parameter family $\mathsf{I}(\ell, m, n)$, two unbalanced vertices of weight 1 connected by three strings of length at least 1. Note that $\mathsf{I}(\mathsf{A}_m, 0)$ coincides with some $\mathsf{II}(\mathsf{A}, \mathsf{A}')$ described below, so we only consider $1 < \ell \leq m \leq n$ for $\mathsf{I}(\ell, m, n)$.

(D) There are two distinct choices here, either to chain the vertices that are not necessarily connected to the same branch vertex, denoted $\mathsf{I}(\Gamma, n)$, or to do this, denoted $\mathsf{I}(\Gamma, n)'$. The former will have a cycle comprising of $m + n - 2$ vertices, whereas for the latter there will be $n + 2$.

(E) There is a unique choice for $\mathsf{I}(\mathsf{E}_6, n)$ and $\mathsf{I}(\mathsf{E}_7, n)$, as in each case $\delta(\Gamma)$ has only two vertices of weight 1. Since we require $\delta(\Gamma)$ to have two vertices of weight 1, the dimension vector $\mathsf{I}(\mathsf{E}_8, n)$ does not exist.

Similarly we may choose two ADE $\Gamma^{(1)}, \Gamma^{(2)}$ and chain together the unique (up to symmetry) weight 1 vertices in each $\delta^{(1)}, \delta^{(2)}$. This is denoted $\mathsf{I}(\Gamma^{(1)}, \Gamma^{(2)}, n)$. If $n = 0$ then the distinguished weight 1 vertices are identified, giving Type II dimension vectors $\mathsf{II}(\Gamma))$ and $\mathsf{II}(\Gamma^{(1)}, \Gamma^{(2)})$, respectively. For example:
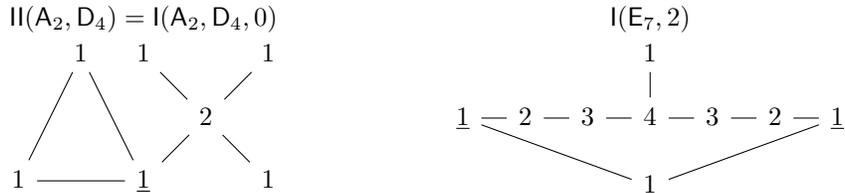
Notation 1.9. Choose a vertex i of weight 2 in $\delta(\Gamma)$ and unbalance it by attaching a weight 1 vertex.

(A) There are no vertices of weight 2. so $\mathsf{III}(\mathsf{A}_m)$ does not exist.

(D) There are $\lceil \frac{m-3}{2} \rceil$ unique choices of a weight 2 vertex up to and including the centre of the chain, as $\mathsf{III}(\mathsf{D}_m, n) = \mathsf{III}(\mathsf{D}_m, m - n - 3)$ as weighted graphs.

(E) For each E_n there is an option of weight 2 vertex that becomes a branch vertex when extended, we denote the resulting dimension vector $\mathsf{III}(\mathsf{E}_n)'$. For E_7 and E_8 there is also a choice that does not create a new branch vertex, which we denote $\mathsf{III}(\mathsf{E}_n)$.

By instead considering $2\delta(\Gamma)$, which has a unique weight 2 vertex that can be unbalanced, we create the dimension vector $\mathrm{III}(2\Gamma)$ for any Γ. For example:
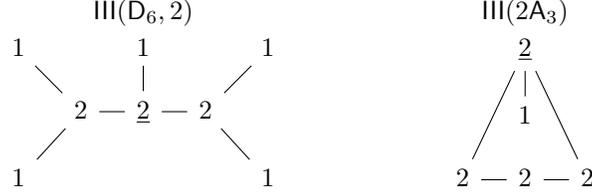
Remark 1.10. Note that $\mathrm{III}(\mathsf{D}_4)$ is the 'star' graph with five vertices and the corresponding variety is a special case ($n = 5$) of the *hyperpolygon space* (in the sense of [3]), which has dimension $2n - 6$ in general.

Notation 1.11. Let $m \geq 4$ and $n \in \{6, 7, 8\}$. The dimension vector $\mathrm{III}(m, n)$ is built by taking $\delta(\mathsf{D}_m$ and identifying one of its branch vertices with the unbalanced vertex in $\gamma(\mathsf{E}_n)$, and then removing the neighbours to the branch vertex. Observe $\mathrm{III}(4, n) = \mathrm{III}(\mathsf{E}_n)'$ for each n, so in Notation 1.9 we only need to consider $\mathrm{III}(\mathsf{E}_7)$ and $\mathrm{III}(\mathsf{E}_8)$.

2. Preliminaries

Following [7], we give the required background on Nakajima quiver varieties, root system combinatorics, and the case of quiver varieties in dimension two. Finally, we start the analysis of four-dimensional quiver varieties by giving balancing results that will underpin the classification of dimension vectors.

2.1. **Quiver varieties.** Take a quiver Q as before, which has vertex set Q_0, arrow set Q_1, and arrow source and target maps $s, t \colon Q_1 \to Q_0$. A (finite-dimensional) *representation M* of Q is an assignment of vector spaces to vertices and corresponding linear maps to arrows. Then $\mathrm{Rep}(Q, \alpha) \coloneqq \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_{s(a)}}, \mathbb{C}^{\alpha_{t(a)}})$ is the space of representations of Q with dimension vector $\alpha \in \mathbb{N}^{\mathrm{rk}}$, and viewed with the Zariski topology becomes an affine variety. The reductive group $G(\alpha) \coloneqq \prod_{i \in \Delta} \mathrm{GL}_{\alpha_i}(\mathbb{C},)$ acts on $\mathrm{Rep}(Q, \alpha)$ by conjugation, and this action is symplectic on the cotangent bundle $T^* \mathrm{Rep}(Q, \alpha)$. Taking $\overline{Q}$ to be the doubled quiver of Q, we may identify $T^* \mathrm{Rep}(Q, \alpha)$ with $\mathrm{Rep}(\overline{Q}, \alpha)$, and consider the moment map $\mu \colon \mathrm{Rep}(\overline{Q}, \alpha) \to \mathrm{Lie}\, G(\alpha)$, identifying the Lie algebra with it dual using the trace pairing. The *Nakajima quiver variety* associated with this data is the affine quotient

$$\mathfrak{M}_\lambda(Q, \alpha) \coloneqq \mu^{-1}(\lambda) /\!/ G(\alpha),$$

a moduli space parametrising semisimple representations of $\Pi^\lambda(Q)$, the *(deformed) preprojective algebra*, see [27]. We restrict to the case $\lambda = 0$, in which case this is a conical symplectic singularity.

Remark 2.1. We note two generalisations of this construction. First of all, one can also take the corresponding GIT quotient $\mathfrak{M}^\theta_\lambda(Q, \alpha) \coloneqq \mu^{-1}(\lambda)^\theta /\!\!/ G(\alpha)$ with stability parameters explained in [27]. This variety also comes with a morphism to $\mathfrak{M}_\lambda(Q, \alpha)$ given by the variation of the GIT. For the classification purposes we study the affine varieties so these do not appear for the most part in this paper. Secondly, Nakajima's construction [33] is more general and involves also an additional vector w of *framing* weights. However, as observed by Crawley-Boevey in [13, §1], every framed quiver variety can be identified with an unframed one. We therefore consider unframed affine quiver varieties.

2.2. **Root combinatorics.** Now let Δ be the underlying graph of Q, so that the data (Δ, α) may be considered as a (nonnegatively) weighted graph. In fact, [16, Lem. 2.2] tells us that $\Pi(Q)$ (and hence $\mathfrak{M}_0(Q, \alpha)$) depends only on Δ, so this is sufficient input data. Write C_Δ for the associated *generalised Cartan matrix*, which has diagonal entries $2(1 - \ell_i)$ and off-diagonal entries $-\Delta_{ij}$, where ℓ_i is the number of loops at vertex i and Δ_{ij} is the number of edges between vertices i and j. This induces the *Euler form* $(\alpha, \beta) \coloneqq \alpha^\top C_\Delta \beta$ which underpins our key combinatorial definitions.

Definition 2.2. Let (Δ, α) be a weighted graph and let $i \in \Delta$. The *vertex balance* and *total balance* are
- $(\alpha, e_i) = 2(1 - \ell_i)\alpha_i - \sum_{j - i} \alpha_j$,
- $(\alpha, \alpha) = \sum_{i \in \Delta} \alpha_i(\alpha_i, e_i)$,

respectively. A vertex is *balanced* if $(\alpha, e_i) = 0$, *positively balanced* if $(\alpha, e_i) > 0$ and *unbalanced* otherwise.

The following balancing condition is elementary, yet fundamental to our analysis.

Lemma 2.3. *If $i, j \in \Delta$ share an edge, then $\alpha_j < \lceil \frac{\alpha_i}{2} \rceil$ implies that $(\alpha, e_j) < 0$. In particular, a balanced vertex necessarily has weight at least half of any neighbouring weight.*

Proof. By assumption either $\alpha_j \leq \frac{\alpha_i}{2} - 1$ (α_i even) or $\alpha_j \leq \frac{\alpha_i}{2} - \frac{1}{2}$ (α_i odd). Then $(\alpha, e_j) \leq 2\alpha_j - \alpha_i$, which is at most -2 in the first case and at most -1 in the second. $\square$

Now, each loopfree ($\ell_i = 0$) vertex i has a corresponding reflection $s_i \alpha \coloneqq \alpha - (\alpha, e_i)e_i$, which generate the *Weyl group* W_Δ. These reflections satisfy $s_i^2 = 1$, $s_i s_j = s_j s_i$ when $\Delta_{ij} = 0$ and $s_i s_j s_i = s_j s_i s_j$ when $\Delta_{ij} = 1$. The group W_Δ is a (generally infinite) Coxeter group. See [22] for more details on root systems.

Definition 2.4. We say that $\alpha \in \mathbb{Z}^{\mathsf{rk}}$ is a *root* if there exists $w \in W_\Delta$ such that $\alpha = w\beta$ for some β in either $\{e_i \mid i \in \Delta \text{ loopfree}\}$ or $\pm F$ where $F \coloneqq \{\beta \in \mathbb{N}^{\mathsf{rk}} \mid \operatorname{supp} \beta \text{ connected}, (\beta, e_i) \leq 0 \text{ for all } i \in \Delta\}$. In the first case we say that α is *real* and in the second case we say that α is *imaginary*.

Denote the set of roots by $\mathsf{R} \subset \mathbb{Z}^{\mathsf{rk}}$, and the positive roots by R^+. Now, as C_Δ is a symmetric matrix with even diagonal entries, the total balance is always even, so we define the function $p(\alpha) \coloneqq 1 - \frac{1}{2}(\alpha, \alpha)$, following [21]. Observe that $\alpha \in \mathsf{R}^+$ is real (respectively imaginary) then $p(\alpha) = 0$ (respectively > 0), however this is not a sufficient condition to be a root. In the latter case call α *isotropic* (imaginary) if $p(\alpha) = 1$, and *anisotropic* (imaginary) otherwise. The *height* of a root α is simply $\mathsf{ht} \coloneqq \sum_i \alpha_i$. In the case of finite root systems, it thus makes sense to speak of a *highest* root.

Notation 2.5. For Γ of type ADE, we write $\gamma(\Gamma)$ for the highest root in its (necessarily finite) root system, and $\delta(\Gamma)$ for the minimal imaginary root in the corresponding affine root system.

TABLE 2.1. Isotropic and highest roots

Γ	$\gamma(\Gamma)$	$\delta(\Gamma)$
A_m $m \geq 0$	$\overline{1} - \cdots\cdots - \overline{1}$	

TABLE 2.1. (continued)

Γ	$\gamma(\Gamma)$	$\delta(\Gamma)$
D_m $m \geq 4$	$1 - \overline{2} - \cdots\cdots - 2 \overset{\diagup 1}{\underset{\diagdown 1}{}}$	$\overset{1}{\diagdown} \underset{1}{\diagup} 2 - \cdots\cdots - 2 \overset{\diagup 1}{\underset{\diagdown 1}{}}$
E_6	$\begin{array}{c} \overline{2} \\ \mid \\ 1 - 2 - 3 - 2 - 1 \end{array}$	$\begin{array}{c} 1 \\ \mid \\ 2 \\ \mid \\ 1 - 2 - 3 - 2 - 1 \end{array}$
E_7	$\begin{array}{c} 2 \\ \mid \\ \overline{2} - 3 - 4 - 3 - 2 - 1 \end{array}$	$\begin{array}{c} 2 \\ \mid \\ 1 - 2 - 3 - 4 - 3 - 2 - 1 \end{array}$
E_8	$\begin{array}{c} 3 \\ \mid \\ 2 - 4 - 6 - 5 - 4 - 3 - \overline{2} \end{array}$	$\begin{array}{c} 3 \\ \mid \\ 2 - 4 - 6 - 5 - 4 - 3 - 2 - 1 \end{array}$

The overlined vertices are positively unbalanced, and are 'extended' with a vertex of weight 1 in each case to form the minimal imaginary root.

Lemma 2.6. *For all $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{\mathsf{rk}}$, we have $p(n\alpha) = n^2(p(\alpha) - 1) + 1$. In particular, if α is isotropic then so is $n\alpha$.*

Proof. Applying bilinearity and the fact that $(\alpha, \alpha) = 2 - 2p(\alpha)$, we obtain

$$p(n\alpha) = 1 - \tfrac{1}{2}(n\alpha, n\alpha) = 1 - \tfrac{1}{2}n^2(\alpha, \alpha) = 1 - \tfrac{1}{2}n^2(2 - 2p(\alpha)) = n^2(p(\alpha) - 1) + 1. \qquad \square$$

If $p(\alpha) = 1$ then the first term vanishes for all n, so the second claim follows.

We now define an important subset of R^+.

Definition 2.7. The set Σ_0 consists of all $\alpha \in \mathsf{R}^+$ such that $p(\alpha) > \sum_{k=1}^r p(\beta^{(k)})$ for every decomposition $\alpha = \sum_{k=1}^r \beta^{(k)}$, with $r \geq 2$ and all $\beta^{(k)} \in \mathsf{R}^+$.

If $\alpha \in \Sigma_0$ is real then $\alpha = e_i$ for some i, i.e. a simple root. If $\alpha \in \Sigma_0$ is isotropic, then it is the minimal imaginary root for some affine Dynkin diagram. See [7, Exm. 1 and 2] for more details. Crawley-Boevey [14, Ppn. 1.2] also shows that for anisotropic root $\beta \in \Sigma_0$, a multiple $n\beta$ is also in Σ_0. By [13, Thm. 1.2] being a root $\alpha \in \Sigma_0$ is equivalent to the existence of a simple representation of the corresponding preprojective algebra (i.e. a point in the quiver variety). The importance of Σ_0 is explained in the following result:

Theorem 2.8 (Thm. 1.1, Ppn. 1.2, [14]). *Every $\alpha \in \mathbb{N}\mathsf{R}^+$ decomposes in terms of Σ_0 as*

$$\alpha = \sum_{k=1}^r m_k \beta^{(k)} \quad \beta^{(k)} \in \Sigma_0. \tag{2.A}$$

This decomposition is canonical in the sense that every other decomposition of α in terms of Σ_0 is a refinement of (2.A). If $\alpha \in \Sigma_0$ then $\dim \mathfrak{M}_0(Q, \alpha) = 2p(\alpha)$. Moreover, if $\alpha = \sum_{k=1}^r m_k \beta^{(k)}$ is the canonical decomposition with distinct $\beta^{(k)}$, then

$$\mathfrak{M}_0(\alpha) \simeq \prod_{k=1}^r S^{m_k} \mathfrak{M}_0(\beta^{(k)}),$$

where S^m denotes the mth symmetric power. $\mathfrak{M}_0(\beta^{(k)})$ is a point for real $\beta^{(k)}$, and if $p(\beta^{(k)}) > 1$ then $m_k = 1$.

Notice that if $\alpha \in \Sigma_0$ then the corresponding canonical decomposition is α itself. Hence the four-dimensional quiver varieties that we aim to classify will have dimension vectors $\alpha \in \Sigma_0$ satisfying $p(\alpha) = 2$ and all the other possible quiver varieties in this dimension are products and symmetric powers of Kleinian singularities.

2.3. **Dimension two.** To motivate what follows, we recall the classification of isotropic roots, that is, the case $p(\alpha) = 1$. This is equivalent to total balance $(\alpha, \alpha) = 0$, so every vertex is balanced.

In [21, Lem. 1.9(d)] Kac shows that

Theorem 2.9. *All totally balanced positive roots are given by $n\delta(\Gamma)$ for some simply laced Dynkin diagram Γ and some $n \in \mathbb{Z}_{\geq 1}$.*

Of these, only the case $n = 1$ belongs to Σ_0. They are connected and belong to the fundamental set of imaginary roots since every vertex has zero balance. Thus quiver varieties in dimension two take the form $\mathfrak{M}_0(\delta(\Gamma))$, that is minimal imaginary roots supported on affine Dynkin diagrams, as shown in Table 2.1. See [7, Exm. 2] for more detail.

2.4. **Balancing lemmas.** Our strategy for finding dimension vectors for 4-dimensional quiver varieties is based on the following statement.

Lemma 2.10 (Thm. 5.8, [13]). *Suppose $\alpha \in \Sigma_0$ is imaginary. Then α has connected support and $(\alpha, e_i) \leq 0$ for all i. In other words, α lies in the fundamental set of positive imaginary roots.*

This is a convenient necessary condition, as we can first classify $p(\alpha) = 2$ dimension vectors with nonpositive local balance, a finite problem.

In the following lemmas we make no assumptions on the balance of α_i (except for the fact that it is unbalanced) so we underline it for full generality. All other vertices are assumed to be balanced. We first investigate restrictions on the number of neighbours a balanced vertex may have.

Lemma 2.11 (Branching). *If (Δ, α) has a weighted subgraph of the form*

$$
\underline{\alpha_i} \;\text{———}\; \alpha_j \begin{cases} \alpha_{j_1} \\ \;\vdots \\ \alpha_{j_n} \end{cases}
$$
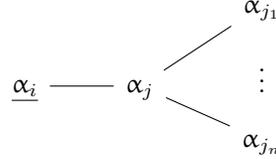
for some $n \in \mathbb{Z}_{\geq 1}$, then $\alpha_i \leq 2\alpha_j - n\lceil\frac{\alpha_j}{2}\rceil$ and $n \leq 3$. In particular $\alpha_j \geq \alpha_i$ if $n = 2$ and $\alpha_j \geq 2\alpha_i$ (or $2\alpha_i + 3$ for α_j odd) if $n = 3$.

Proof. Since α_j is balanced, by definition and Lemma 2.3 we have $2\alpha_j - \alpha_i = \alpha_{j_1} + \cdots + \alpha_{j_n} \geq n\lceil\frac{\alpha_j}{2}\rceil$ and rearranging gives the first claim. If $n \geq 4$, we see that $\alpha_i \leq 2\alpha_j - 4\lceil\frac{\alpha_j}{2}\rceil \leq 2\alpha_j - 2\alpha_j = 0$, which is impossible as $\alpha > 0$. Similarly, if $n \geq 2$ then $\alpha_i \leq 2\alpha_j - 2\lceil\frac{\alpha_j}{2}\rceil \leq 2\alpha_j - \alpha_j = \alpha_j$. $\square$

The second part of Lemma 2.11 implies that in decreasing chains, a sequence of vertices with successively lower weights, branching is impossible. Hence we can fully investigate restrictions for balanced decreasing chains that start with an unbalanced vertex.

Lemma 2.12 (Down-by-d). *Let (Δ, α) have a weighted subgraph of the form*

$$
\underline{\alpha_i} \;\text{———}\; \alpha_j := \alpha_i - d \;\text{———}\; \cdots
$$
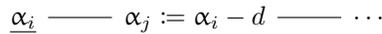
for some $0 < d < \alpha_i$. If this has a balanced continuation (meaning j has at least one other balanced neighbour), then $\alpha_i \geq 3d$ for even d or $\alpha_i \geq 3d + 1$ for odd d. This terminates precisely when d divides α_i.

Proof. By Lemma 2.11 we know that j has at most one other neighbour, as $\alpha_j < \alpha_i$. Since this neighbour is also balanced, it has weight $\alpha_i - 2d \geq \lceil\frac{\alpha_i - d}{2}\rceil$ and the claimed inequalities follow. Assuming a balanced continuation exists, the next weights are $\alpha_i - 2d, \alpha_i - 3d$ and so on. This eventually terminates with

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$\alpha_i - kd$ for some k, but then $2(\alpha_i - kd) = \alpha_i - (k-1)d$ and rearranging yields $\alpha_i = (k+1)d$. Thus d necessarily divides α_i to terminate a decreasing balanced chain. $\qquad\square$

Increasing chains are more subtle, as they can grow unbounded and contain branches at multiple points. We consider the increasing steps relevant to our classification (indeed, the cases in the following lemma are motivated by the structures to be obtained §3), and in each case determine when growing unbounded is the only possibility for the chain. As we are only interested in finite quivers, we can then rule out such behaviour from the classification.

Lemma 2.13 (Up-by-d). *Suppose* (Δ, α) *contains the increasing chain*

$$\underline{\alpha_i} \quad\rule{1.5em}{0.4pt}\quad \alpha_j := \alpha_i + d \quad\rule{1.5em}{0.4pt}\quad \cdots$$

for some $d \geq 0$*. Then for vertex j to admit branching requires*

$(d = 0)$ α_i *is even.*
$(d = 1)$ $\alpha_i \leq 5$ *when odd and* $\alpha_i \leq 2$ *when even.*
$(d = 2)$ $\alpha_i = 1$ *or* $\alpha_i \leq 10$ *when even.*
$(d = 3)$ $\alpha_i \leq 15$ *when odd and* $\alpha_i \leq 6$ *when even.*

Proof. Consider double branching of the form

$$
\underline{\alpha_i} \quad\rule{1.5em}{0.4pt}\quad \alpha_i + d
\begin{array}{c}
\diagup \lceil \tfrac{\alpha_i+d}{2}\rceil + x \\[1em]
\diagdown \lceil \tfrac{\alpha_i+d}{2}\rceil + y
\end{array}
\tag{2.B}
$$

then the balancing condition at j implies $x + y = d$ if $\alpha_i + d$ is even (α_i and d have the same parity) and $x + y = d - 1$ if $\alpha_i + d$ is odd (different parities). Triple branching is even more restrictive so we deal with that on a case-by-case basis.

$(d = 0)$ By Lemma 2.11 the only possible neighbours of $\alpha_j = \alpha_i$ are either a vertex of weight α_i or two vertices of weight $\frac{\alpha_i}{2}$. The latter is only possible if α_i is even and is the only possible balanced termination of such a chain.

$(d = 1)$ By Lemma 2.11, a triple branching is only possible when $\alpha_i + 1 \geq 2\alpha_i$ (α_i odd) or $\alpha_i + 1 \geq 2\alpha_i + 3$ (α_i even), so only with $\alpha_i = 1$ could we induce a triple branch. Now consider (2.B), with $x + y = 0$ if α_i is even and $x + y = 1$ if α_i is odd. In the former case, both new vertices have weight $\frac{\alpha_i + 2}{2}$ ($x = 0, y = 0$), which forces $\alpha_i \leq 2$ by Lemma 2.12 with difference $\frac{\alpha_i}{2}$. In the latter case we may only take $x = 1, y = 0$ and apply Lemma 2.12 with difference $\frac{\alpha_i - 1}{2}$, giving $\alpha_i \leq 5$.

$(d = 2)$ By Lemma 2.11, a triple branching is only possible when $\alpha_i + 2 \geq 2\alpha_i$ (α_i even) or $\alpha_i + 2 \geq \alpha_i + 3$ (α_i odd), so only $\alpha_i = 2$ could induce a triple branch. Now consider (2.B), with $x + y = 2$ if α_i is even and $x + y = 1$ if α_i is odd. If in the even case we have $x = 2, y = 0$ then the top vertex is already balanced, but applying Lemma 2.12 (with difference $\frac{\alpha_i - 2}{2}$) to the lower branch gives $\alpha_i + 2 \geq \frac{3\alpha_i - 6}{2}$, that is $\alpha_i \leq 10$. If $x = y = 1$ then both branches are the same and we apply Lemma 2.12 with difference $\frac{\alpha_i}{2}$ to see $\alpha_i \leq 4$. In the odd case the only option is $x = 1, y = 0$, and Lemma 2.12 with difference $\frac{\alpha_i + 3}{2}$ forces $\alpha_i \leq 1$.

$(d = 3)$ By Lemma 2.11, a triple branching is only possible when $\alpha_i + 3 \geq 2\alpha_i$ (α_i odd) or $\alpha_i + 3 \geq 2\alpha_i + 3$ (α_i even), so only $\alpha_i = 1$ or $\alpha_i = 3$ could induce a triple branch. Now consider (2.B) and suppose α_i is even. Then $x + y = 2$ and we first check $(x, y) = (2, 0)$. If the lower branch is balanced then by Lemma 2.12 we must have $\alpha_i + 3 \geq \frac{3\alpha_i + 6}{2}$, that is, $\alpha_i \leq 0$. The other option is $(x, y) = (1, 1)$, for which both branches are the same. Balancing and Lemma 2.12 again enforces $\alpha_i + 3 \geq \frac{3\alpha_i}{2}$, thus $\alpha_i \leq 6$. If α_i is odd then $\alpha_i + 3$ is even and $x + y = 3$. We first check $(x, y) = (3, 0)$. The lower branch is already balanced, and for the upper branch we use Lemma 2.12 to imply

$\alpha_i + 3 \geq \frac{3\alpha_i - 9}{2}$, so $\alpha_i \leq 15$. If instead $(x, y) = (2, 1)$, then the lower branch balancing implies $\alpha_i + 3 \geq \frac{3\alpha_i + 3}{2}$ by Lemma 2.12. Hence $\alpha_i \leq 3$. $\qquad\square$

Next we look at the types of possible terminations.

Lemma 2.14. *A finite balanced chain of vertices of weight 1 has to either be $\delta(\mathsf{A}_m)$ for some $m \in \mathbb{N}$ or have unbalanced vertices at each end.*

Proof. Suppose we have $\;1 \;\rule[0.3ex]{2em}{0.4pt}\; 1\;$. Since both of them are balanced they can only have neighbours of weight 1. Repeating this procedure, it has to either stop at an unbalanced 1 or become a cycle, as the rank is finite. $\qquad\square$

Lemma 2.15. *A chain starting with $\;\underline{1} \;\rule[0.3ex]{2em}{0.4pt}\; 2 \;\rule[0.3ex]{2em}{0.4pt}\; \cdots\;$ where any vertex connected to 2 on the right is balanced can only end up as some $\delta(\mathsf{D}_m)$ or $\delta(\mathsf{E}_{6,7,8})$.*

Proof. Let (Δ, α) be a weighted graph with fixed unbalanced vertices that are connected in some way. Suppose that $\alpha_i = 1$ has a neighbour satisfying $\alpha_{i+1} = 2$ which needs to be balanced. Suppose $\;\underline{1} \;\rule[0.3ex]{2em}{0.4pt}\; 2 \;\rule[0.3ex]{2em}{0.4pt}\; \cdots\;$ is finished in some way as a chain, i.e. there are no cycles in Δ containing vertex α_{i+1}. Then if we take this balanced chain together with vertex α_i it will form a completely balanced root. We know, however, by Theorem 2.9 that these are given by Γ of types $\tilde{\mathsf{A}}, \tilde{\mathsf{D}}, \tilde{\mathsf{E}}$ and their multiples. Moreover, $\delta(\mathsf{A})$ nor its multiples cannot contain both vertices of weight 1 and 2. $\qquad\square$

3. Dimension vectors

The aim of this section is to prove Theorem 1.1, that is we classify all (Δ, α) such that $\alpha \in \Sigma_0$ and $p(\alpha) = 2$.

3.1. Types of solutions.
We first use Lemma 2.10 and find the $\alpha \in F$ with total balance -2. This restricts the behaviour of our desired dimension vectors to three cases.

Lemma 3.1. *Let $\alpha \in \Sigma_0$ satisfy $p(\alpha) = 2$. Then α has at most two unbalanced vertices, namely:*
  - (I) $i \neq j \in \Delta$ such that $\alpha_i = \alpha_j = 1$ and $(\alpha, \mathrm{e}_i) = (\alpha, \mathrm{e}_j) = -1$,
  - (II) $i \in \Delta_0$ such that $\alpha_i = 1$ and $(\alpha, \mathrm{e}_i) = -2$,
  - (III) $i \in \Delta_0$ such that $\alpha_i = 2$ and $(\alpha, \mathrm{e}_i) = -1$.

Proof. As $\alpha \in \Sigma_0$, by Lemma 2.10 we necessarily have $(\alpha, \mathrm{e}_i) \leq 0$ for all $i \in \Delta_0$. Then $p(\alpha) = 1 - \frac{1}{2}(\alpha, \alpha) = 2$ is equivalent to $(\alpha, \alpha) = -2$, that is

$$\sum_{i \in \Delta_0} \alpha_i(\alpha, \mathrm{e}_i) = -2.$$

As $\alpha \in \mathbb{N}^{\mathrm{rk}}$, it follows that each summand is a nonpositive integer. Hence there are at most two nonzero terms; either they are both -1 or one is -2. $\qquad\square$

Because Lemma 2.10 is not sufficient, some of these dimension vectors obtained in this way may in fact lie outside Σ_0. We deal with this in § 3.4.

3.2. Local unbalanced structure.
Given α in one of the forms in Lemma 3.1, we first determine its unbalanced vertices, and then find all of the ways the remaining vertices can be balanced.

Definition 3.2. The *local structure* of a (possibly unbalanced) vertex i is the weighted subgraph $(\Delta', \alpha') \subseteq (\Delta, \alpha)$ consisting of i and neighbours $i_1, \ldots, i_n$, such that $(\alpha, \mathrm{e}_{i_j})|_{\Delta'} \geq 0$ for all j. In other words, it is still possible for the neighbours of i to be balanced in (Δ, α).

We now classify the local structures that are possible in our case of interest.

Lemma 3.3. *Let $\alpha \in \Sigma_0$ satisfy $p(\alpha) = 2$ and let i be an unbalanced vertex. Then i has one of the following local structures; five in in Type I, thirteen in Type II, and nine in Type III.*

$\mathrm{I}^{(1)}$	$\mathrm{I}^{(2)}$	$\mathrm{I}^{(3)}$
$\underline{1}$ —— 3	$1 =\!= \underline{1}$ —— 1	2 —— $\underline{1}$ —— 1
$\mathrm{I}^{(4)}$	**$\mathrm{I}^{(5)}$**	**$\mathrm{II}^{(1)}$**
$\begin{array}{c} 1 \\ \mid \\ 1 - \underline{1} - 1 \end{array}$	$\circlearrowleft \underline{1}$ —— 1	$\underline{1} =\!= 2$
$\mathrm{II}^{(2)}$	**$\mathrm{II}^{(3)}$**	**$\mathrm{II}^{(4)}$**
$\underline{1}$ —— 4	$1 =\!= \underline{1} =\!= 1$	$1 =\!= \underline{1}$ —— 2
$\mathrm{II}^{(5)}$	**$\mathrm{II}^{(6)}$**	**$\mathrm{II}^{(7)}$**
2 —— $\underline{1}$ —— 2	1 —— $\underline{1}$ —— 3	$\begin{array}{c} 1 \\ \| \| \\ 1 - \underline{1} - 1 \end{array}$
$\mathrm{II}^{(8)}$	**$\mathrm{II}^{(9)}$**	**$\mathrm{II}^{(10)}$**
$\begin{array}{c} 2 \\ \mid \\ 1 - \underline{1} - 1 \end{array}$	$\begin{array}{c} 1 \qquad\quad 1 \\ \diagdown \;\; \diagup \\ 1 - \underline{1} - 1 \end{array}$	$\circlearrowleft \underline{1} =\!= 1$
$\mathrm{II}^{(11)}$	**$\mathrm{II}^{(12)}$**	**$\mathrm{II}^{(13)}$**
$\circlearrowleft \underline{1}$ —— 2	$\begin{array}{c} \frown \\ 1 - \underline{1} - 1 \end{array}$	$\circlearrowleft \underline{1} \circlearrowright$
$\mathrm{III}^{(1)}$	$\mathrm{III}^{(2)}$	$\mathrm{III}^{(3)}$
$\underline{2}$ —— 5	1 —— $\underline{2} =\!= 2$	2 —— $\underline{2}$ —— 3
$\mathrm{III}^{(4)}$	$\mathrm{III}^{(5)}$	$\mathrm{III}^{(6)}$
1 —— $\underline{2}$ —— 4	$\begin{array}{c} 1 \\ \mid \\ 2 - \underline{2} - 2 \end{array}$	$\begin{array}{c} 3 \\ \mid \\ 1 - \underline{2} - 1 \end{array}$
$\mathrm{III}^{(7)}$	$\mathrm{III}^{(8)}$	$\mathrm{III}^{(9)}$
$\begin{array}{c} 1 \qquad\quad 1 \\ \diagdown \;\; \diagup \\ 2 - \underline{2} - 1 \end{array}$	$\begin{array}{c} 1 \quad 1 \quad 1 \\ \diagdown \mid \diagup \\ 1 - \underline{2} - 1 \end{array}$	$\circlearrowleft \underline{2}$ —— 1

Proof. As $\alpha \in \Sigma_0$, we may apply Lemma 3.1. Therefore in each case the weights neighbouring i sum to:

(I) 3, or 1 if $\ell_i = 1$;

(II) 4, or 2 if $\ell_i = 1$, or 0 if $\ell_i = 2$;

(III) 5, or 1 if $\ell_i = 1$.

Recall that we view ℓ_i as i having two neighbours, both of weight α_i. To make sure nothing is missed, we order the possibilities first by the number of loops, then the number of distinct neighbours, then the maximal neighbouring weight present, and then the minimal neighbouring weight present. If $\alpha_i = 2$ then Lemma 2.3 forces at most one edge when connecting i to weight 1 vertices, so we can omit most cases of double edges and higher multiplicities. Similarly for $\alpha_i = 1$ we have at most two edges shared with any neighbour. Hence we see that only the tabulated weighted subgraphs are possible. $\qquad\square$

There is one dimension vector α satisfying $p(\alpha)$ in Type I that doesn't come from a local structure as defined in Definition 3.2. Namely, this is

$$\underline{1} \equiv\equiv\equiv \underline{1} \quad = \mathsf{I}(1,1,1),$$

which consists only of unbalanced vertices.

3.3. **The classification.** We are now ready to find the dimension vectors of interest, proceeding by type to complete the local structures in Table 3.1 with balanced vertices, applying the restrictions in §2.4.

Proposition 3.4. *The dimension vectors of Type I are $\mathsf{I}(\Gamma^{(1)}, \Gamma^{(2)}, n)$, $\mathsf{I}(\Gamma, n)$, $\mathsf{I}(m)$, I_a, I_b, and I_c.*

Proof. For $M, N \in \{1, 2, 3, 4, 5\}$, we attempt to connect $\mathsf{I}^{(M)}$ with $\mathsf{I}^{(N)}$ usng only balanced vertices. Observe that $\mathsf{I}^{(1)}$ can only pair up with itself, as all other local structures have an odd number of balanced vertices with weight 1, and so one would need to connect to the weight 3 vertex, contradicting Lemma 2.14.

(1),(1) Firstly if we look at the local structure of one of the unbalanced vertices. We have the following possible sets of neighbours to balance 3:

$$(1, 2, 2), \quad (1, 4), \quad (2, 3), \quad (5).$$

The first two determine the unbalanced vertices. The first yields

$$\begin{array}{c} \underline{1} \\ 1 \ 2 \ 3 \ 2 \ 1 \\ \underline{1} \end{array} \quad = \mathsf{I}(4),$$

as the decreasing chains are forced by Lemma 2.12. Now take a vertex of weight 4. It requires total weight 5 to be balanced, we may balance with a 2 and a 3, giving.

$$\begin{array}{c} \underline{1} \quad 2 \\ 3 \ 4 \ 3 \ 2 \ 1 \\ \underline{1} \end{array} \quad = \mathsf{I}_a.$$

Otherwise continue the chain with a 5, which must then have neighbours at least 3 that sum to 6. A pair of 3s contradicts Lemma 2.12 as 2 does not divide 5. Continuing with a 6 is our last step by Lemma 2.13 ($d = 1$), so the remaining terminalisation is

$$\begin{array}{c} \underline{1} \quad\quad 3 \\ 3 \ 4 \ 5 \ 6 \ 4 \ 2 \\ \underline{1} \end{array} \quad = \mathsf{I}_c.$$

Now suppose the weight 3 vertices are not identified, to each we must attach weights at least 2 that total 5. If on one copy of $\mathsf{I}^{(1)}$ we attach 2 and 3, the former weight starts a decreasing chain and the latter weight starts a constant balanced chain of 3s. We can only terminate this chain by adding an unbalanced vertex to one of the weight 3 vertices, i.e. by symmetrically attaching the other copy of $\mathsf{I}^{(1)}$ to obtain

$$\begin{array}{c} \underline{1} \quad\quad \underline{1} \\ 1 \ 2 \ 3 \ \cdots \ 3 \ 2 \ 1 \end{array} \quad = \mathsf{I}(m)$$

with $m \geq 5$. Now continue one copy of $\mathsf{I}^{(1)}$ with a weight 5 vertex, so that neighbours must be at least 3 and total 7. There are two options to proceed. One is to take a weight 7 vertex and proceed. This creates a growing chain of consecutive odd weights. By Lemma 2.13(2) we know

this cannot be terminated in a balanced way. Moreover, from the proof of Lemma 2.13(2) one can see that there is no other way to finish it in an unbalanced way with local structure $\mathsf{I}^{(1)}$. We thus proceed with the only remaining option which is to take 4 and 3. The former is a forced terminating downwards chain and the latter necessarily being the other copy of $\mathsf{I}^{(1)}$:

$$\begin{array}{c} \underline{1}\ \ 3 \\ \qquad\quad 5\ \ 4\ \ 3\ \ 2\ \ 1 \qquad = \mathsf{I}_b. \\ \underline{1}\ \ 3 \end{array}$$

(2),(2) By Lemma 2.14, the only possibility is

$$1 = \underline{1}\ \cdots\ \underline{1} = 1 \quad = \mathsf{I}(\mathsf{A}_1, \mathsf{A}_1, n)$$

for some $n \geq 1$.

(2),(3) Similarly to the previous argument, vertices of weight 1 are connected in a chain of 1s. On the other side, we apply Lemma 2.15 to see this must terminate as $\delta(\Gamma)$ for some $\Gamma \neq \mathsf{A}$:

$$\gamma(\Gamma)\ \underline{1}\ \cdots\ \underline{1} = 1 \quad = \mathsf{I}(\Gamma, \mathsf{A}_1, n).$$

(2),(4) Applying Lemma 2.14 again, the only possibility is to have a chain of 1s connecting the unbalanced vertices and to connect another pair of 1s in a cycle. This is

$$\begin{array}{c} 1 \\ 1\ \cdots\ \underline{1}\ \cdots\ \underline{1} = 1 \end{array} \quad = \mathsf{I}(\mathsf{A}_m, \mathsf{A}_1, n).$$

(2),(5) This is another use of Lemma 2.14, giving

$$\overset{\cap}{\underline{1}}\ \cdots\ \underline{1} = 1 \quad = \mathsf{I}(\mathsf{A}_0, \mathsf{A}_1, n)$$

for some $n \geq 1$.

(3),(3) Lemma 2.15 tells us that the only balanced termination of $\underline{1}\ \text{——}\ 2\ \text{——}\ \cdots$ is with some $\delta(\mathsf{D})$ or $\delta(\mathsf{E})$. Suppose we start with one of our unbalanced vertices. However, notice that one of the weight 1 vertices could be unbalanced. If that did not happen this means that each of the unbalanced vertices has its own Γ root, that is

$$\gamma(\Gamma^{(1)})\ \underline{1}\ \cdots\ \underline{1}\ \gamma(\Gamma^{(2)}) \quad = \mathsf{I}(\Gamma^{(1)}, \Gamma^{(2)}, n),$$

for some $n \geq 1$ and $\Gamma^{(1)}, \Gamma^{(2)}$ not of type A. If that did happen, however, we get one of the following:

$$\begin{array}{c} 1 \qquad\qquad 1 \\ \ \ 2\ \cdots\ 2 \\ \underline{1}\underset{\cdots}{\diagdown\diagup}\underline{1} \end{array},\quad \begin{array}{c} 1 \qquad\quad 1 \\ \quad 2\ \cdots\ 2 \\ \underline{1} \qquad\quad \underline{1} \end{array},\quad \begin{array}{c} 1 \\ 2 \\ \underline{1}\ \underline{2}\ 3\ \underline{2}\ 1 \end{array},\quad \begin{array}{c} 2 \\ \underline{1}\ \underline{2}\ 3\ 4\ 3\ \underline{2}\ \underline{1} \end{array} \quad = \mathsf{I}(\Gamma, n).$$

(3),(4) Combining Lemmas 2.14 and 2.15, we see the only possibility is

$$\begin{array}{c} 1 \\ \gamma(\Gamma)\ \underline{1}\ \cdots\ \underline{1}\ \cdots\ 1 \end{array} \quad = \mathsf{I}(\Gamma, \mathsf{A}_m, n),$$

for some $n \geq 1$ and $\Gamma \neq \mathsf{A}$.

(3),(5) As in the above argument, this forces

$$\gamma(\Gamma)\ \underline{1}\ \cdots\ \overset{\cap}{\underline{1}} \quad = \mathsf{I}(\Gamma, \mathsf{A}_0, n)$$

with the same restrictions on n and Γ.

(4),(4) Lemma 2.14 tells us that we have to connect chains of 1s, and there are two ways to do this. If we pair up the hanging 1s between the copies of $\mathsf{I}^{(4)}$, we obtain $\mathsf{I}(\ell, m.n)$, otherwise we have

$$\begin{array}{c} 1 \qquad\qquad 1 \\ 1\ \cdots\ \underline{1}\ \cdots\ \underline{1}\ \cdots\ 1 \end{array} \quad = \mathsf{I}(\mathsf{A}_\ell, \mathsf{A}_m, n)$$

for some $\ell, m \geq 2$ and $n \geq 1$.

(4),(5) Again, by Lemma 2.14 there is only one way to connect chains of 1s,

$$\begin{array}{c} 1 \qquad\qquad\ \cap \\ 1\ \cdots\ \underline{1}\ \cdots\ \underline{1} \end{array} \quad = \mathsf{I}(\mathsf{A}_m, \mathsf{A}_0, n),$$

with the same restrictions on m and n.

(5),(5) This is the limiting case of the previous two arguments, and we get

$$\overset{\cap}{\underline{1}} \ \cdots \ \overset{\cap}{\underline{1}} \quad = \mathsf{I}(\mathsf{A}_0, \mathsf{A}_0, n). \qquad \qquad \square$$

Having finished the most complicated type, we move on to those that have at most one unbalanced vertex.

Proposition 3.5. *The dimension vectors of Type* II *are* $\mathsf{II}(\Gamma^{(1)}, \Gamma^{(2)})$, $\mathsf{II}(\Gamma)$ *for* $\Gamma \neq \mathsf{E}_8$, $\mathsf{II}(m)$, II_a, *and* II_b.

Proof. As before, we proceed by completing local structure in Table 3.1 with balanced vertices.

(1) By Lemma 2.13, the hanging weight 2 can either be completed by a pair of 1s or a 2. The former terminates the process and the latter repeats this same choice. Hence for $m \geq 4$ this is

$$\underline{1} = 2 \ \cdots \ 2 \ \overset{1}{\underset{1}{}} \quad = \mathsf{II}(\mathsf{D}_m)'.$$

(2) The neighbours of 4 must be each at least 2 and must total $2 \times 4 - 1 = 7$. The only options that do not lead to a contradiction are $(3, 4)$, $(2, 2, 3)$, $(2, 5)$, and (7). Choosing $(2, 5)$ forces

$$\begin{matrix} & 2 & & 3 & & & \\ \underline{1} & 4 & 5 & 6 & 4 & 2 \end{matrix} \quad = \mathsf{II}_a.$$

by the proof of Lemma 2.15. Choosing (7) means that the corresponding vertex needs to have neighbours that add up to 10. There are 3 possibilities: (10), $(4, 6)$, and $(5, 5)$. By Lemma 2.12 the last two are impossible to terminate in a balanced way. Proceeding, for (10) the only possible neighbours are $(5, 8)$, $(6, 7)$, and (13). The first one is

$$\begin{matrix} & & 5 & & & & \\ \underline{1} & 4 & 7 & 10 & 8 & 6 & 4 & 2 \end{matrix} \quad = \mathsf{II}_b.$$

The other two options are impossible to terminate by Lemma 2.12 and Lemma 2.13(3), respectively. Finally, choosing $(3, 4)$ has a forced termination from the vertex of weight 3 and, by Lemma 2.13 the chain of 4s terminates as

$$\begin{matrix} 2 & & & \underline{1} & & & \\ & 4 & \cdots & 4 & 3 & 2 & 1 \\ 2 & & & & & & \end{matrix} \quad = \mathsf{II}(m),$$

which has $(2, 2, 3)$ as a special case.

(3) This is already complete, and is $\mathsf{II}(\mathsf{A}_1, \mathsf{A}_1) = \mathsf{I}(\mathsf{A}_1, \mathsf{A}_1, 0)$.

(4) By Lemma 2.15 we obtain

$$\Gamma \ 1 = \underline{1} \quad = \mathsf{II}(\Gamma, \mathsf{A}_1).$$

(5) Completing the weight 2 vertices separately gives rise to

$$\Gamma^{(1)} \ \underline{1} \ \Gamma^{(2)} \quad = \mathsf{II}(\Gamma^{(1)}, \Gamma^{(2)}).$$

for $\Gamma \neq \mathsf{A}$ by Lemma 2.15. We could also terminate by linking the weight 2 vertices in some way, but this is where §2.4 cannot help up. We proceed from one of the ends and go case by case. The possible neighbours to 2 (that don't force a dimension vector without cycles) are $(1, 2)$ and (3). The first one gives rise to a chain of 2 which can only be connected to another 2. However, as the dimension vector has cycles it has to connect to the initial 2 from $\mathsf{I}^{(5)}$, resulting in

$$\begin{matrix} 1 & & & & 1 & \\ & 2 & \cdots & 2 & & \\ & & \underline{1} & & & \end{matrix} \quad = \mathsf{II}(\mathsf{D}_m).$$

Now suppose the next weight is 3, which can have neighbours $(2, 2)$ or (4). The first choice results in a dimension vector without cycles unless one of these 2s was in $\mathsf{I}^{(5)}$. This way we obtain

$$\begin{matrix} 1 & & \\ 2 & & \\ 2 & 3 & 2 \\ \underline{1} & & \end{matrix} \quad = \mathsf{II}(\mathsf{E}_6).$$

Now suppose the next weight is 4. The possible neighbours are $(3, 2)$ or (5). The first terminates the root in a down-by-one chain without cycles unless 2 was in $\mathsf{I}^{(5)}$. A choice of (5) forces 6 and

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the only possible branching by Lemma 2.13(1) is E_8 which does not have a vertex of weight 1 to be unbalanced. Thus the only remaining option is

$$\underset{\underset{\underline{1}}{2\ \ 3\ \ 4\ \ 3\ \ 2}}{\overset{2}{}} \quad = \mathsf{II}(\mathsf{E}_7).$$

(6) Since the hanging 1 does not have another to connect with, this local structure cannot be completed in a balanced way.

(7) By Lemma 2.14, pairing the hanging 1s with an arbitrary chain completes the diagram, for $m \geq 2$, as

$$\underset{1=\underline{1}\ \cdots\ 1}{\overset{1}{}} \quad = \mathsf{II}(\mathsf{A}_1, \mathsf{A}_m).$$

(8) Combining Lemmas 2.14 and 2.15 we obtain

$$\underset{\gamma(\Gamma)\ \ \underline{1}\ \cdots\ 1}{\overset{1}{}} \quad = \mathsf{II}(\Gamma, \mathsf{A}_m)$$

with $\Gamma \neq \mathsf{A}$ and $m \geq 2$.

(9) As before, the only option is to pair up the 1s with arbitrary chains, that is, for $\ell, m \geq 2$,

$$\underset{1\ \cdots\ \underline{1}\ \cdots\ 1}{\overset{1\qquad 1}{}} \quad = \mathsf{II}(\mathsf{A}_\ell, \mathsf{A}_m).$$

(10) This is already complete, and is $\mathsf{II}(\mathsf{A}_0, \mathsf{A}_1)$.

(11) By Lemma 2.15, the hanging weight 2 can only be completed through some $\delta(\mathsf{D})$ or $\delta(\mathsf{E})$:

$$\gamma(\Gamma)\ \overset{\frown}{\underline{1}} \quad = \mathsf{II}(\Gamma, \mathsf{A}_0).$$

(12) Again, by Lemma 2.14 the only way to complete the hanging weight 1 vertices is to connect them with an arbitrary weight 1 chain (for $m \geq 2$).

$$\underset{\underline{1}\ \cdots\ 1}{\overset{\frown\ \ 1}{}} \quad = \mathsf{II}(\mathsf{A}_0, \mathsf{A}_m).$$

(13) This is already complete, and is $\mathsf{II}(\mathsf{A}_0, \mathsf{A}_0)$. □

Finally we consider which dimension vectors arise from terminating the last nine local structures in Table 3.1.

Proposition 3.6. *The dimension vectors in Type* III *are* $\mathsf{III}(\Gamma)$, $\mathsf{III}(2\Gamma)$, $\mathsf{III}(m, n)$, III_a, *and* III_b.

Proof. We proceed in entirely the same way as the previous two results.

(1) The weight 5 vertex must have neighbours each at least 3 that total $2 \times 5 - 2 = 8$, so the options are two $(4, 4)$ or (8). The former has a forced descending termination and the latter can only (by Lemma 2.12) continue by attaching a 4 and a 7. These are

$$\underset{1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 4\ \ 3\ \ 2\ \ 1}{\overset{2}{}} = \mathsf{III}_a, \qquad \underset{\underline{2}\ \ 5\ \ 8\ \ 7\ \ 6\ \ 5\ \ 4\ \ 3\ \ 2\ \ 1}{\overset{4}{}} = \mathsf{III}_b.$$

(2) This diagram is already complete, and is $\mathsf{III}(2\mathsf{A}_1)$.

(3) We must complete the vertex of weight 2 and the vertex of weight 3. By Lemma 2.13(0), the former only terminates with balanced vertices through a chain of 2 similar to D_m. By the proof of Lemma 2.15, the latter terminates with a choice of $\delta(\mathsf{E}_n)$, yielding

$$\underset{\underset{1}{2\cdots\ \underline{2}\ \ 3\ \ \cdots}}{\overset{1}{}} \quad = \mathsf{III}(m, n).$$

(4) We need to complete the vertex of weight 4, with possible neighbours totalling 6. Thus the possibilities are $(3, 3,)$, $(2, 4)$, or (6). By Lemma 2.12, $(3, 3)$ forces

$$\underset{1\ \ 2\ \ 3\ \ 4\ \ 3\ \ 2\ \ 1}{\overset{\overset{1}{2}}{}} \quad = \mathsf{III}(\mathsf{E}_7).$$

Further, by Lemma 2.13 $(2, 4)$ can only end as a chain of 4s that splits at the end.

$$\underset{\underset{2\qquad\qquad 2}{4\ \cdots\ 4}}{\overset{1\ \underline{2}\qquad\qquad 2}{}} \quad = \mathsf{III}(2\mathsf{D}_m).$$

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Next, suppose next vertex is 6. Its possible neighbours are $(4,4), (3,5)$ or (8). The former terminates to $\mathsf{III}(2\mathsf{E}_6)$:

$$
\begin{array}{c}
2 \\
4 \\
1\ \underline{2}\ 4\ 6\ 4\ 2
\end{array}
\quad = \mathsf{III}(2\mathsf{E}_6).
$$

Proceeding with $(3,5)$, this gives rise to

$$
\begin{array}{c}
3 \\
1\ 2\ 3\ 4\ 5\ 6\ 4\ \underline{2}\ 1
\end{array}
\quad = \mathsf{III}(\mathsf{E}_8)
$$

by Lemma 2.12. Lastly, 8 needs to have neighbours of weight 10. By Lemma 2.12 the only options are $(4,6)$ $(8-5=3$ does not divide $8)$, which terminates in

$$
\begin{array}{c}
4 \\
1\ \underline{2}\ 4\ 6\ 8\ 6\ 4\ 2
\end{array}
\quad = \mathsf{III}(2\mathsf{E}_7)
$$

and 10, which, by Lemma 2.12 continues in 12. Here, by Lemma 2.13 we have to branch as further going up is impossible. The only branching is $(6,8)$ which gives

$$
\begin{array}{c}
6 \\
1\ \underline{2}\ 4\ 6\ 8\ 10\ 12\ 8\ 4
\end{array}
\quad = \mathsf{III}(2\mathsf{D}_m).
$$

(5) Both of the remaining vertices with weight 2 must be made balanced. By Lemma 2.13 we may terminate with a pair of weight 1 vertices or we may also connected them with an arbitrary chain of weight 2 vertices. The first option gives rise to

$$
\begin{array}{c}
1 \qquad 1 \qquad 1 \\
2\ \cdots\ \underline{2}\ \cdots\ 2 \\
1 \qquad\qquad 1
\end{array}
\quad = \mathsf{III}(\mathsf{D}_m, i)
$$

for $i \neq 1$. The second option, for $m \geq 2$, is

$$
\begin{array}{c}
2 \\
2\ \cdots\ \underline{2}\ 1
\end{array}
\quad = \mathsf{III}(2\mathsf{A}_m).
$$

(6) By Lemma 2.15 the only possible balanced terminations of the weight 3 vertex chain are

$$
\begin{array}{c}
1 \\
1 \quad 2 \\
\ \ \underline{2}\ 3\ 2\ 1 \\
1
\end{array}
= \mathsf{III}(4,6), \qquad
\begin{array}{c}
1 \qquad 2 \\
\underline{2}\ 3\ 4\ 3\ 2\ 1 \\
1
\end{array}
= \mathsf{III}(4,7), \qquad
\begin{array}{c}
1 \qquad\ \ 3 \\
\underline{2}\ 3\ 4\ 5\ 6\ 4\ 2 \\
1
\end{array}
= \mathsf{III}(4,8).
$$

(7) The only vertex to be made balanced is the remaining 2. By Lemma 2.13 we obtain an arbitrary chain of weight 2 which terminates by splitting into two vertices of weight 1:

$$
\begin{array}{c}
1 \qquad\ 1 \\
1\ \underline{2}\ \cdots\ 2 \\
1 \qquad 1
\end{array}
\quad = \mathsf{III}(\mathsf{D}_m, 1).
$$

(8) This diagram is already complete, and is $\mathsf{III}(\mathsf{D}_4)$.

(9) This diagram is already complete, and is $\mathsf{III}(2\mathsf{A}_0)$. $\qquad\square$

This concludes the list of dimension vectors α with nonpositive local balance and total balance -2, which comprises $Table\,1.1$ along with $\mathsf{III}(2\Gamma)$ and $\mathsf{I}(\Gamma^{(1)}, \Gamma^{(2)}, n > 0)$.

3.4. **Filtering for Σ_0.** We now finish the proof of Theorem 1.1 by removing the $\alpha \notin \Sigma_0$ that satisfy $p(\alpha) = 2$, as they will not give fundamentally (in the sense of the canonical decomposition) new quiver varieties in dimension four. There are two dimension vectors that we can immediately rule out.

Lemma 3.7. *The dimension vectors* $\mathsf{I}(\Gamma^{(1)}, \Gamma^{(2)}, n > 0)$ *and* $\mathsf{III}(2\Gamma)$ *lie outside* Σ_0.

Proof. Simply observe that $\mathsf{I}(\Gamma^{(1)}, \Gamma^{(2)}, n) = \delta(\Gamma^{(1)}) + \delta(\Gamma^{(2)}) + \gamma(\mathsf{A}_{n-1})$ and $\mathsf{III}(2\Gamma) = 2\delta(\Gamma) + e_i$ for some vertex i. Applying p to every term contradicts Σ_0. $\qquad\square$

Apart from the above families, we claim that all of the dimension vectors $\alpha \in \mathsf{R}^+$ with $p(\alpha) = 2$ lie in Σ_0, giving those seen in Table 1.1. We can check this using the following result.

Lemma 3.8. *If* $\alpha \in F \setminus \Sigma_0$ *then either*

(1) $\alpha = n\delta(\Gamma)$ *for some* $n \geq 2$,

(2) *there are two connected components connected by a unique edge, and vertices at each end of this edge both have weight 1, or*

(3) *there are two connected components connected by a unique edge, the vertex at one end is of weight 1 and other is the extending vertex of $n\delta(\Gamma)$ for some $n \geq 2$.*

Proof. This is [13, Thm. 8.1] with $\lambda = 0$ and in characteristic zero. $\qquad\square$

The $p(\alpha) = 2$ dimension vectors of the second form is $\mathsf{I}(\Gamma^{(1)}, \Gamma^{(2)}, n)$ for $n > 0$ and for the third form we get $\mathsf{III}(2\Gamma)$, as we would expect from Lemma 3.7. We now have enough to finish the classification of Σ_0 dimension vectors for out quiver varieties.

Proof of Theorem 1.1. In §3.3 we fully classify all the roots $\alpha \in \mathsf{R}^+$ satisfying $p(\alpha) = 2$ and $(\alpha, \mathrm{e}_i) \leq 0$ at every $i \in \Delta_0$. Now apply Lemma 3.8. Since (1) is $p(\alpha) = 1$ by Theorem 2.9, none of our dimension vectors apply. For (2), the only dimension vector with a weighted subgraph of the form $\gamma(\mathsf{A}_2)$ is $\mathsf{I}(\Gamma, n)$ for $n > 0$, but the two ends of this edge do not lie in disconnected components. Similarly none of the remaining dimension vectors take form (3). $\qquad\square$

4. Symplectic stratification and minimal degenerations

We have obtained all the possible dimension vectors corresponding to 4-dimensional quiver varieties and determined which lie in Σ_0. However, different dimension vectors can correspond to the same variety. As a way to differentiate them, we would like to have some geometric invariants. In this chapter we explain how to compute invariants such as minimal degenerations between the symplectic leaves of quiver varieties, summarise them in the 4-dimensional case using Table 4.1, and then proceed to compute their Namikawa Weyl groups.

4.1. **Symplectic singularities.** We begin by recalling the definition of conical symplectic singularity, which first appeared in [1]. Let X be an affine algebraic variety over $\mathbb{C}$.

Definition 4.1. We say that X has *symplectic singularities* if:

- X is a normal Poisson variety.
- There exists a smooth, dense open subset $U \subset X$ (with complement of codimension 2) on which the Poisson structure comes from a symplectic form ω.
- There exists a resolution of singularities: a birational and projective morphism $Y \to X$, such that the pullback of ω to Y has no poles.

If, in addition to the above conditions, one has a $\mathbb{C}^\times$-action on X which acts on ω with some positive weight and contracts X to the unique fixed point, X is called a *conical symplectic singularity.*

Symplectic singularities possess additional structure, which we explain in the following subsection.

4.2. **Symplectic leaves.** Being Hamiltonian reductions, quiver varieties come equipped with a Poisson structure. Symplectic leaves are maximal connected smooth Poisson submanifolds where the Poisson bracket is non-degenerate. For a better exposition on symplectic leaves, we refer the reader to [30]. By [7, Thm. 1.2], quiver varieties are conical symplectic singularities, so we can use the following theorem (also look at [34, §3.v]):

Theorem 4.2. [24, Thm. 2.3] *A normal symplectic singularity has finitely many symplectic leaves and the latter are algebraic.*

By [13, Cor. 1.4], quiver varieties are irreducible. In particular, this means that there is only one open symplectic leaf. As quiver varieties are *conical* symplectic singularities, there is at most one zero-dimensional leaf. We thus need to study the intermediate leaves, whenever they exist.

The symplectic leaves of quiver varieties can be described using the following combinatorial datum.

4.3. **Representation types.**

Definition 4.3. We say that $\tau = (n_1, \beta^{(1)}; \cdots ; n_k, \beta^{(k)})$ is a *representation type* for α if $\beta^{(i)} \in \Sigma_0$ for all i and $\alpha = \sum_{i=1}^{k} n_i \beta^{(i)}$, $n_i \in \mathbb{N}$.

Recall that quiver variety parametrise representations of the preprojective algebra $\Pi(Q)$. In particular, to each representation type τ we can associate a set $\mathfrak{M}_0(\alpha)_\tau$ consisting of points where the corresponding representation is decomposed precisely as τ. The relation of this object to the geometry of a quiver variety is as follows:

Theorem 4.4. [7, Thm. 1.9] *The symplectic leaves of $\mathfrak{M}_0(\alpha)$ are given precisely by the representation-type strata $\mathfrak{M}_0(\alpha)_\tau$.*

Whenever α belongs to Σ_0, the open symplectic leaf corresponds to the trivial representation type $(\tau = (1, \alpha))$. On the other side, the 0-dimensional leaf has the representation type $\tau = (\alpha_i, e_i)$, a sum of simple roots with coefficients from the dimension vector. Combinatorially, one can think that symplectic leaves are thus subrefinements of the canonical decomposition from Theorem 2.8.

4.4. **Minimal degenerations between leaves and transverse slices.** Symplectic leaves $\mathscr{L}_\tau$ form a poset with respect to the inclusion order on the closures $\overline{\mathscr{L}}_\tau$. The neighbours with respect to this order provide a useful geometric invariant.

Definition 4.5. A *minimal degeneration* is a pair of leaf closures $\overline{\mathscr{L}}_1 \subset \overline{\mathscr{L}}_2$ such that there is no leaf closure $\overline{\mathscr{L}}_3$, which satisfies $\overline{\mathscr{L}}_1 \subset \overline{\mathscr{L}}_3 \subset \overline{\mathscr{L}}_2$.

Remark 4.6. For quiver varieties, in light of Theorem 2.8 and Theorem 4.4, minimal degenerations are 1-step subrefinements of the canonical decomposition.

Given a minimal degeneration $\overline{\mathscr{L}}_1 \subset \overline{\mathscr{L}}_2$ one can take an étale local transverse slice to $\mathscr{L}_1$ inside $\overline{\mathscr{L}}_2$. Following [5, §8], we view a transverse slice (S, o) to a leaf $\mathscr{L}$ of dimension $2k$ as a pointed Poisson variety together with a local isomorphism

$$(\mathfrak{M}_0(\alpha), p) \simeq \left(\mathbb{A}^{2k} \times S, (0, o) \right) \tag{4.A}$$

of Poisson varieties for each $p \in \mathscr{L}$, where $\mathbb{A}^{2k}$ is equipped with the usual symplectic structure. Since the map in (4.A) is Poisson, o will be a symplectic leaf in S so that the leaf in $\mathbb{A}^{2k} \times S$ containing $(0, o)$ is $\mathbb{A}^{2k} \times o$. The slice is étale local by its construction in [15, §4].

By minimality, S will then be an isolated singularity. Kaledin in [24] showed that the stratification by symplectic leaves is the same as stratification by singular loci, so they a give 'local analytic picture'. Thus, in order to distinguish varieties arising from Table 1.1 we will list all their symplectic leaves and minimal degenerations between them. One can present this data in the form of Hasse diagrams, where vertices correspond to leaves and edges to the minimal degenerations between them and the corresponding isolated singularities.

Example 4.7. Let $k \in \mathbb{Z}_{\geq 0}$ and consider the dimension vector

$$\begin{array}{c} k+3 \\ | \\ 1 \text{\textemdash} 2 \text{\textemdash} \cdots\cdots \text{\textemdash} 2k+5 \text{\textemdash} 2k+6 \text{\textemdash} k+4 \text{\textemdash} \underline{2} \end{array}$$

The only unbalanced vertex has balance $-k$ and weight 2, so that the associated quiver variety has dimension $2k + 2$. If $k = 0$ then we see that this is just the Kleinian singularity of type E_8 and taking $k = 1$ recovers the exceptional case III_b from our classification. In the physics literature the case $k = 2$ is studied in [19], which observes that every Hasse diagram in the family has the property of *inversion*. This means that knowing the Hasse diagram for the Coulomb branch could allow to recover the Hasse diagram for the Higgs branch for certain pairs of quiver gauge theories. For more information about such a conjecture in broader sense, see [10, §10.4]. The Hasse diagram of this family, for which the case $k = 1$

is shown in Table A.3, is:

$$\mathscr{L}_{2k+2}$$
$$\Big| \, \mathsf{D}_{2k+8}$$
$$\mathscr{L}_{2k}$$
$$\Big| \, \mathsf{D}_{2(k-1)+8}$$
$$\vdots$$
$$\Big| \, \mathsf{D}_{10}$$
$$\mathscr{L}_2$$
$$\Big| \, \mathsf{E}_8$$
$$\mathscr{L}_0$$

There is work in progress [6] by Bellamy and Schedler to classify the possible isolated singularities that occur as minimal degenerations between symplectic leaf closures in quiver varieties. These include minimal nilpotent orbit and Kleinian singularities, both of which appear for 4-dimensional quiver varieties. In particular, we encounter only three special cases of degenerations between leaves:

- A degeneration between the zero-dimensional leaf $\mathscr{L}_0$ and the open leaf $\mathscr{L}_4$.
- A degeneration between the zero-dimensional leaf $\mathscr{L}_0$ and some two-dimensional leaf $\mathscr{L}$ — *dimension 2 degenerations*.
- A degeneration between a two-dimensional leaf $\mathscr{L}$ and the open leaf $\mathscr{L}_4$ —*codimension 2 degenerations*.

As we will show in §4.5, the first case is uncommon and includes only two varieties, one of which is $\mathbb{C}^4$ and the other one is the minimal nilpotent orbit for $\mathfrak{sl}_3$. For the second case, the transverse slice to the point $\mathscr{L}_0$ in the two-dimensional leaf $\mathscr{L}$ will be its closure $\overline{\mathscr{L}}$. One therefore needs to describe the closures of all two-dimensional leaves. For the third case we need to describe transverse slices to the symplectic leaves of codimension (coincidentally, dimension) 2, and this can also be done combinatorially.

Definition 4.8. [7, Dfn. 1.18] A decomposition $\tau : \alpha = \beta^{(1)} + \ldots + \beta^{(m)} + n_1\gamma^{(1)} + \ldots + n_k\gamma^{(k)}$ is called *isotropic* if

(1) $\beta^{(i)}, \gamma^{(j)} \in \Sigma_0$.
(2) $p(\beta^{(i)}) = 1$
(3) The $\gamma^{(j)}$ are pairwise distinct real roots
(4) If $\overline{Q}''$ is the quiver with $m + k$ vertices without loops and $-\left(\alpha^{(i)}, \alpha^{(j)}\right)$ arrows from vertex i to vertex $j \neq i$ where $\alpha^{(i)}, \alpha^{(j)} \in \{\beta^{(1)}, \ldots, \beta^{(m)}, n_1\gamma^{(1)}, \ldots, n_k\gamma^{(k)}\}$ then Q'', the undoubled quiver, is an affine Dynkin quiver.
(5) The dimension vector $(1_1, \ldots, 1_m, n_1, \ldots, n_k)$ on Q'' gives δ, the minimal imaginary root.

The construction in (4) is known as the *ext-quiver*, due to the number of arrows coinciding with the number of extensions between vertex simples.

4.4.1. *Codimension two degenerations.*

Theorem 4.9. [7, Thm. 1.20] *Let $\alpha \in \Sigma_0$ be imaginary. Then the codimension two strata of $\mathfrak{M}_0(\alpha)$ are in bijection with the isotropic decompositions of α. Moreover, the slice du Val singularity corresponds to the ext-quiver Q''.*

Thus, to understand the codimension-2 leaves and their minimal degenerations, we need to find all isotropic decompositions of our dimension vectors. We proceed to do so in §4.5.

4.4.2. *Dimension two degenerations.* Recall from Theorem 4.4 that symplectic leaves of quiver varieties correspond to representation-type datum.

In [5, Thm. 2.4] the authors, via representation types, construct normalisations of closures of symplectic leaves using products of quotients of 'smaller' quiver varieties. Next they ask whether the leaves themselves are often normal. It turns out that the answer is 'no' in general. However, we *do* have normality for a special type of leaf.

Proposition 4.10. [5, Ppn. 2.8] *If a representation type* $\tau = (1, \delta; n_2, \beta^{(2)}; \ldots; n_r, \beta^{(r)})$ *has* $p(\delta) = 1$ *and* $p(\beta^{(k)}) = 0$ *for all* $k \geq 2$, *then* $\overline{\mathscr{L}}_\tau$ *is normal and isomorphic to the Kleinian singularity associated with* δ.

Notice that these representation types usually are not isotropic. However, they are in the case $p(\alpha) = 2$. In case $p(\alpha)$ every non-trivial representation type of α is of the form $\tau = (1, \delta; n_2, \beta^{(2)}; \cdots; n_k, \beta^{(k)})$ with every $\beta^{(i)}$ a real root, we may apply Proposition 4.10. We thus proceed to classify all the possible representation types for our dimension vectors from Table 1.1.

4.5. **Isotropic decompositions.** We list all isotropic subroots of α with $p(\alpha) = 2$ and subtract, knowing that what remains does not admit another isotropic subroot. Exhaustively listing all isotropic subroots is done by inspection. We check whether any $\delta(\Gamma)$ is a subroot of a given α. To do so we first exclude all possible isotropic $\delta(A)$ roots (so we look for cycles), then all $\delta(D)$ roots (so we look for pairs of branch vertices), and finally all $\delta(E)$ roots. Our routine is demonstrated in the following example:

Example 4.11. Consider

$$\alpha = I(E_7, n) = \underset{1}{\underbrace{1}}\ 2\ 3\ \overset{2}{4}\ 3\ 2\ \underset{n}{\underbrace{1}}$$

Then α contains a cycle and so must have a subroot of the form $\delta(A_n)$ for some n, decomposing as:

$$\underset{1}{\underbrace{1}}\ 2\ 3\ \overset{2}{4}\ 3\ 2\ \underset{n}{\underbrace{1}} = \underset{\delta(A_{n+5})}{\underbrace{1\ 1\ 1\ \overset{0}{1}\ 1\ 1\ 1}} + \underset{\gamma(E_6)}{\underbrace{0\ 1\ 2\ \overset{2}{3}\ 2\ 1\ 0}}$$

Next, α has only one branch vertex, so there are no subroots of type $\delta(D)$. Lastly, one can see that it does not contain $\delta(E_6)$ (not enough branch length) and $\delta(E_8)$ (weight of branch vertex), but does contain $\delta(E_7)$ provided that $n > 1$. This gives the following decomposition:

$$\underset{1}{\underbrace{1}}\ 2\ 3\ \overset{2}{4}\ 3\ 2\ \underset{n}{\underbrace{1}} = \underset{\delta(E_7)}{\underbrace{1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1}} + \underset{\gamma(A_{n-1})}{\underbrace{0\ 0\ 0\ \overset{0}{0}\ 0\ 0\ 0}}$$

We now do this for the remaining dimension vectors in Table 1.1, proceeding by type.

4.5.1. *Type* I. We start with the construction $I(\Gamma, n)$, which by definition contains some $\delta(A)$ and $\delta(\Gamma)$. However it is not possible to subtract both, as $I(\Gamma, n) - \delta(\Gamma)$ is the highest root $\gamma(A_{n-1})$, which is real. For $n \in \{0, 1\}$ the first decomposition in doesn't exist, as for $n = 0$ we no longer have $\delta(\Gamma)$ as a subroot and for $n = 1$ the support is empty after subtracting $\delta(\Gamma)$. If $\Gamma = A_m$, then we have the following decomposition:

$$\underset{I(\ell, m, n)}{\underbrace{\begin{matrix} 1\ \cdots\ 1 \\ \underline{1}\ 1\ \cdots\ 1\ \underline{1} \\ 1\ \cdots\ 1 \end{matrix}}} = \underset{\delta(A_{\ell+m-1})}{\underbrace{\begin{matrix} 1\ \cdots\ 1 \\ 1\ 1\ \cdots\ 1\ 1 \\ 0\ \cdots\ 0 \end{matrix}}} + \underset{\gamma(A_{n-1})}{\underbrace{\begin{matrix} 0\ \cdots\ 0 \\ 0\ 0\ \cdots\ 0\ 0 \\ 1\ \cdots\ 1 \end{matrix}}}$$

if $n > 1$. By symmetry there are two other decompositions with isotropic subroots $\delta(A_{\ell+n-1})$ (whenever $m > 1$) and $\delta(A_{m+n-1})$ (whenever $\ell > 1$). For $\Gamma = D_m$, we have:

$$\underset{I(D_m, n)}{\underbrace{\begin{matrix} 1\qquad\quad 1 \\ 2\ \cdots\ 2 \\ \underline{1}\qquad\quad \underline{1} \\ \underset{1}{\underbrace{1}}\ \cdots\ 1\ \underset{n}{\underbrace{}} \end{matrix}}} = \underset{\delta(A_{m+n-3})}{\underbrace{\begin{matrix} 0\qquad\quad 0 \\ 1\ \cdots\ 1 \\ 1\qquad\quad 1 \\ 1\ \cdots\ 1 \end{matrix}}} + \underset{\gamma(A_{m-1})}{\underbrace{\begin{matrix} 1\qquad\quad 1 \\ 1\ \cdots\ 1 \\ 0\qquad\quad 0 \\ 0\ \cdots\ 0 \end{matrix}}} \underset{n>1}{=} \underset{\delta(D_m)}{\underbrace{\begin{matrix} 1\qquad\quad 1 \\ 2\ \cdots\ 2 \\ 1\qquad\quad 1 \\ 0\ \cdots\ 0 \end{matrix}}} + \underset{\gamma(A_{n-1})}{\underbrace{\begin{matrix} 0\qquad\quad 0 \\ 0\ \cdots\ 0 \\ 0\qquad\quad 0 \\ 1\ \cdots\ 1 \end{matrix}}}$$

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The situation is similar for $I(D_m, n)'$, but the cycle length depends on n only:

$$
\begin{array}{c}
\underset{1}{\overset{1}{}} \\[-2pt]
\underline{1}\ 1 \\
2\ \cdots\ 2 \quad \vdots \\
1 \\
\underline{1}\ 1
\end{array}
\;=\;
\begin{array}{c}
0 \\
1\ 1 \\
0\ \cdots\ 1 \quad \vdots \\
0 \\
1\ 1
\end{array}
\;+\;
\begin{array}{c}
1 \\
0\ 0 \\
2\ \cdots\ 1 \quad \vdots \\
1 \\
0\ 0
\end{array}
\;\underset{n>1}{=}\;
\begin{array}{c}
1 \\
1\ 0 \\
2\ \cdots\ 2 \quad \vdots \\
1 \\
1\ 0
\end{array}
\;+\;
\begin{array}{c}
0 \\
0\ 1 \\
0\ \cdots\ 0 \quad \vdots \\
0 \\
0\ 1
\end{array}
$$

$$\quad I(D_m, n)' \qquad\qquad \delta(A_{n+1}) \qquad\qquad \gamma(D_{m-1}) \qquad\qquad\quad \delta(D_m) \qquad\qquad\quad \gamma(A_{n-1})$$

This leaves $\Gamma = E_6$, which has analogous decompositions:

$$
\begin{array}{c}
1 \\ 2 \\
\underline{1}\ 2\ 3\ 2\ 1 \\
\underset{1}{\overset{}{}}\ 1\ \cdots\ 1
\end{array}
\;=\;
\begin{array}{c}
0 \\ 0 \\
1\ 1\ 1\ 1\ 1 \\
1\ \cdots\ 1
\end{array}
\;+\;
\begin{array}{c}
1 \\ 2 \\
0\ 1\ 2\ 1\ 0 \\
0\ \cdots\ 0
\end{array}
\;\underset{n>1}{=}\;
\begin{array}{c}
1 \\ 2 \\
1\ 2\ 3\ 2\ 1 \\
0\ \cdots\ 0
\end{array}
\;+\;
\begin{array}{c}
0 \\ 0 \\
0\ 0\ 0\ 0\ 0 \\
1\ \cdots\ 1
\end{array}
$$

$$\quad I(E_6, n) \qquad\qquad \delta(A_{n+3}) \qquad\qquad \gamma(D_5) \qquad\qquad\quad \delta(E_6) \qquad\qquad\quad \gamma(A_{n-1})$$

The case $\Gamma = E_7$ was explained in Example 4.11, so this completes $I(\Gamma, n)$. The infinite family $I(m)$ has rank $m + 3$, with $m - 3$ vertices of weight 3 akin to the notational conventions of D_m, which is the only isotropic subroot:

$$
\begin{array}{c}
\underline{1}\qquad \underline{1} \\
1\ 2\ 3\ \cdots\ 3\ 2\ 1
\end{array}
\;=\;
\begin{array}{c}
1\qquad 1 \\
0\ 1\ 2\ \cdots\ 2\ 1\ 0
\end{array}
\;+\;
\begin{array}{c}
0\qquad 0 \\
1\ 1\ 1\ \cdots\ 1\ 1\ 1
\end{array}
$$

$$\qquad I(m) \qquad\qquad\qquad \delta(D_m) \qquad\qquad\qquad \gamma(A_{m+1})$$

The first exception in Type I is I_a, admitting the sole isotropic subroot of type D_5.

$$
\begin{array}{c}
\underline{1}\quad 2 \\
3\ 4\ 3\ 2\ 1 \\
\underline{1}
\end{array}
\;=\;
\begin{array}{c}
1\quad 1 \\
2\ 2\ 1\ 0\ 0 \\
1
\end{array}
\;+\;
\begin{array}{c}
0\quad 1 \\
1\ 2\ 2\ 2\ 1 \\
0
\end{array}
$$

$$\qquad I_a \qquad\qquad\quad \delta(D_5) \qquad\qquad \gamma(D_6)$$

Next we consider the exception I_b which contains only a type E_6 isotropic subroot.

$$
\begin{array}{c}
\underline{1}\ 3 \\
5\ 4\ 3\ 2\ 1 \\
\underline{1}\ 3
\end{array}
\;=\;
\begin{array}{c}
1\ 2 \\
3\ 2\ 1\ 0\ 0 \\
1\ 2
\end{array}
\;+\;
\begin{array}{c}
0\ 1 \\
2\ 2\ 2\ 2\ 1 \\
0\ 1
\end{array}
$$

$$\qquad I_b \qquad\qquad\quad \delta(E_6) \qquad\qquad \gamma(D_7)$$

The final exceptional case in Type I is I_c. We spot the D_7 isotropic subroot, which indeed is the only such subroot we can take.

$$
\begin{array}{c}
\underline{1}\qquad 3 \\
3\ 4\ 5\ 6\ 4\ 2 \\
\underline{1}
\end{array}
\;=\;
\begin{array}{c}
1\qquad 1 \\
2\ 2\ 2\ 2\ 1\ 0 \\
1
\end{array}
\;+\;
\begin{array}{c}
0\qquad 2 \\
1\ 2\ 3\ 4\ 3\ 2 \\
0
\end{array}
$$

$$\qquad I_c \qquad\qquad\quad \delta(D_7) \qquad\qquad \gamma(E_7)$$

4.5.2. Type II.

We start with $II(\Gamma^{(1)}, \Gamma^{(2)})$. The immediate isotropic subroots in this construction are that of $\Gamma^{(1)}$ and $\Gamma^{(2)}$. However notice that $II(\Gamma^{(1)}, \Gamma^{(2)}) = \delta(\Gamma^{(2)}) + \gamma(\Gamma^{(1)})$ and similarly if we subtract $\delta(\Gamma^{(2)})$. To see that there are no other isotropic subroots, consider the fact that every minimal imaginary root only has a weight one vertex at the end of a chain. The construction $II(\Gamma)$ is covered by the $n = 0$ case of $I(\Gamma, n)$ above, so we move to the infinite family $II(m)$. By inspection, the only isotropic subroot is of type D_m.

$$
\begin{array}{c}
2\qquad \underline{1} \\
4\ \cdots\ 4\ 3\ 2\ 1 \\
2
\end{array}
\;=\;
\begin{array}{c}
1\qquad 1 \\
2\ \cdots\ 2\ 1\ 0\ 0 \\
1
\end{array}
\;+\;
\begin{array}{c}
1\qquad 0 \\
2\ \cdots\ 2\ 2\ 2\ 1 \\
1
\end{array}
$$

$$\qquad II(m) \qquad\qquad\quad \delta(D_m) \qquad\qquad \gamma(D_{m+2})$$

The two exceptional cases in Type II admit a single isotropic subroot each, of type D_6 and E_7 respectively.

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$$
\begin{array}{c}
\begin{array}{cc} 2 & 3 \end{array} \\[-2pt]
\underline{1}\ 4\ 5\ 6\ 4\ 2 \\[-2pt]
\mathrm{II}_a
\end{array}
\;=\;
\begin{array}{c}
\begin{array}{cc} 1 & 1 \end{array} \\[-2pt]
1\ 2\ 2\ 2\ 1\ 0 \\[-2pt]
\delta(\mathsf{D}_6)
\end{array}
\;+\;
\begin{array}{c}
\begin{array}{cc} 1 & 2 \end{array} \\[-2pt]
0\ 2\ 3\ 4\ 3\ 2 \\[-2pt]
\gamma(\mathsf{E}_7)
\end{array}
$$

$$
\begin{array}{c}
5 \\[-2pt]
\underline{1}\ 4\ 7\ 10\ 8\ 6\ 4\ 2 \\[-2pt]
\mathrm{II}_b
\end{array}
\;=\;
\begin{array}{c}
2 \\[-2pt]
1\ 2\ 3\ 4\ 3\ 2\ 1\ 0 \\[-2pt]
\delta(\mathsf{E}_7)
\end{array}
\;+\;
\begin{array}{c}
3 \\[-2pt]
0\ 2\ 4\ 6\ 5\ 4\ 3\ 2 \\[-2pt]
\gamma(\mathsf{E}_8)
\end{array}
$$

4.5.3. *Type* III. By construction, subtracting $\delta(\Gamma)$ from $\mathrm{III}(\Gamma)$ results in e_i, where i is the vertex of weight 2 which was unbalanced. As an edge case, $\mathrm{III}(\mathsf{D}_4,1)$ admits five different ways to subtract D_4, each giving a Weyl group factor of type A_1. This is consistent with $\mathsf{A}_1 \times \mathsf{D}_2 \times \mathsf{D}_2 = \mathsf{A}_1^5$.

$$
\begin{array}{c}
\begin{array}{ccc} 1 & & 1 \end{array} \\[-2pt]
2\ 2\ \cdots\ 2\ \underline{2}\ 2\ \cdots\ 2\ 2 \\[-2pt]
\begin{array}{ccc} 1 & & 1 \end{array} \\[-2pt]
\mathrm{III}(\mathsf{D}_m, i)
\end{array}
=
\begin{array}{c}
\begin{array}{ccc} 1 & 0 & 1 \end{array} \\[-2pt]
2\ 2\ \cdots\ 2\ 2\ 2\ \cdots\ 2\ 2 \\[-2pt]
\begin{array}{ccc} 1 & & 1 \end{array} \\[-2pt]
\delta(\mathsf{D}_m)
\end{array}
+
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 0 \end{array} \\[-2pt]
0\ 0\ \cdots\ 0\ 0\ 0\ \cdots\ 0\ 0 \\[-2pt]
\begin{array}{ccc} 0 & & 0 \end{array} \\[-2pt]
\gamma(\mathsf{A}_1)
\end{array}
$$

$$
=
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 1 \end{array} \\[-2pt]
0\ 0\ \cdots\ 1\ 2\ 2\ \cdots\ 2\ 2 \\[-2pt]
\begin{array}{ccc} 0 & & 1 \end{array} \\[-2pt]
\delta(\mathsf{D}_{m-i+1})
\end{array}
+
\begin{array}{c}
\begin{array}{ccc} 1 & 0 & 0 \end{array} \\[-2pt]
2\ 2\ \cdots\ 2\ 2\ 1\ \cdots\ 0\ 0 \\[-2pt]
\begin{array}{ccc} 1 & & 0 \end{array} \\[-2pt]
\gamma(\mathsf{D}_{i+1})
\end{array}
$$

$$
=
\begin{array}{c}
\begin{array}{ccc} 1 & 1 & 0 \end{array} \\[-2pt]
2\ 2\ \cdots\ 2\ 2\ 1\ \cdots\ 0\ 0 \\[-2pt]
\begin{array}{ccc} 1 & & 0 \end{array} \\[-2pt]
\delta(\mathsf{D}_{i+3})
\end{array}
+
\begin{array}{c}
\begin{array}{ccc} 0 & 0 & 1 \end{array} \\[-2pt]
0\ 0\ \cdots\ 0\ 0\ 1\ \cdots\ 2\ 2 \\[-2pt]
\begin{array}{ccc} 0 & & 1 \end{array} \\[-2pt]
\gamma(\mathsf{D}_{m-i-1})
\end{array}
$$

If $\Gamma = \mathsf{E}$, we have the following decompositions:

$$
\begin{array}{c}
1 \\[-2pt] \underline{2} \\[-2pt]
1\ 2\ 3\ 4\ 3\ 2\ 1 \\[-2pt]
\mathrm{III}(\mathsf{E}_7)
\end{array}
=
\begin{array}{c}
0 \\[-2pt] 2 \\[-2pt]
1\ 2\ 3\ 4\ 3\ 2\ 1 \\[-2pt]
\delta(\mathsf{E}_7)
\end{array}
+
\begin{array}{c}
1 \\[-2pt] 0 \\[-2pt]
0\ 0\ 0\ 0\ 0\ 0\ 0 \\[-2pt]
\gamma(\mathsf{A}_1)
\end{array}
$$

$$
=
\begin{array}{c}
1 \\[-2pt] 2 \\[-2pt]
0\ 1\ 2\ 3\ 2\ 1\ 0 \\[-2pt]
\delta(\mathsf{E}_6)
\end{array}
+
\begin{array}{c}
0 \\[-2pt] 0 \\[-2pt]
1\ 1\ 1\ 1\ 1\ 1\ 1 \\[-2pt]
\gamma(\mathsf{A}_7)
\end{array}
$$

$$
\begin{array}{c}
3 \\[-2pt]
1\ 2\ 3\ 4\ 5\ 6\ 4\ \underline{2}\ 1 \\[-2pt]
\mathrm{III}(\mathsf{E}_8)
\end{array}
=
\begin{array}{c}
3 \\[-2pt]
1\ 2\ 3\ 4\ 5\ 6\ 4\ 2\ 0 \\[-2pt]
\delta(\mathsf{E}_8)
\end{array}
+
\begin{array}{c}
0 \\[-2pt]
0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\[-2pt]
\gamma(\mathsf{A}_1)
\end{array}
$$

$$
=
\begin{array}{c}
2 \\[-2pt]
0\ 0\ 1\ 2\ 3\ 4\ 3\ 2\ 1 \\[-2pt]
\delta(\mathsf{E}_7)
\end{array}
+
\begin{array}{c}
1 \\[-2pt]
1\ 2\ 2\ 2\ 2\ 2\ 1\ 0\ 0 \\[-2pt]
\gamma(\mathsf{D}_8)
\end{array}
$$

For the infinite families $\mathrm{III}(m,n)$, (which we recall encompasses $\mathrm{III}(\mathsf{E}_6)$ and $\mathrm{III}(\mathsf{E}_7)'$) we show an extra vertex in the chains, giving more clarity in the second decomposition. For $m = 4$, observe that the second decomposition occurs twice, as there are two ways to subtract $\delta(\mathsf{E}_n)$. However this is included via $\mathsf{D}_2 = \mathsf{A}_1 \times \mathsf{A}_1$.

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$$
\mathrm{III}(m,6):\qquad
\begin{array}{c}
\;\;\;1\qquad\qquad 2\\[-2pt]
1\;\;2\cdots 2\;\underline{2}\;3\;2\;1\\[-2pt]
\;\;\;1
\end{array}
\;=\;
\underbrace{\begin{array}{c}
\;\;\;0\qquad\qquad 1\\[-2pt]
1\;\;2\cdots 2\;2\;2\;1\;0\\[-2pt]
\;\;\;1
\end{array}}_{\delta(\mathsf{D}_{m+1})}
\;+\;
\underbrace{\begin{array}{c}
\;\;\;1\qquad\qquad 1\\[-2pt]
0\;\;0\cdots 0\;0\;1\;1\;1\\[-2pt]
\;\;\;0
\end{array}}_{\gamma(\mathsf{A}_5)}
$$

$$
=\;
\underbrace{\begin{array}{c}
\;\;\;1\qquad\qquad 2\\[-2pt]
0\;\;0\cdots 1\;2\;3\;2\;1\\[-2pt]
\;\;\;0
\end{array}}_{\delta(\mathsf{E}_6)}
\;+\;
\underbrace{\begin{array}{c}
\;\;\;0\qquad\qquad 0\\[-2pt]
1\;\;2\cdots 1\;0\;0\;0\;0\\[-2pt]
\;\;\;1
\end{array}}_{\gamma(\mathsf{D}_{m-2})}
$$

$$
\mathrm{III}(m,7):\qquad
\begin{array}{c}
\;\;\;1\qquad\qquad\quad 2\\[-2pt]
1\;\;2\cdots 2\;\underline{2}\;3\;4\;3\;2\;1\\[-2pt]
\;\;\;1
\end{array}
\;=\;
\underbrace{\begin{array}{c}
\;\;\;1\qquad\qquad\quad 1\\[-2pt]
1\;\;2\cdots 2\;2\;2\;2\;1\;0\;0\\[-2pt]
\;\;\;1
\end{array}}_{\delta(\mathsf{D}_{m+2})}
\;+\;
\underbrace{\begin{array}{c}
\;\;\;0\qquad\qquad\quad 1\\[-2pt]
0\;\;0\cdots 0\;0\;1\;2\;2\;2\;1\\[-2pt]
\;\;\;0
\end{array}}_{\gamma(\mathsf{D}_6)}
$$

$$
=\;
\underbrace{\begin{array}{c}
\;\;\;0\qquad\qquad\quad 2\\[-2pt]
0\;\;0\cdots 1\;2\;3\;4\;3\;2\;1\\[-2pt]
\;\;\;0
\end{array}}_{\delta(\mathsf{E}_7)}
\;+\;
\underbrace{\begin{array}{c}
\;\;\;1\qquad\qquad\quad 0\\[-2pt]
2\;\;2\cdots 1\;0\;0\;0\;0\;0\;0\\[-2pt]
\;\;\;1
\end{array}}_{\gamma(\mathsf{D}_{m-2})}
$$

$$
\mathrm{III}(m,8):\qquad
\begin{array}{c}
\;\;\;1\qquad\qquad\qquad 3\\[-2pt]
1\;\;2\cdots 2\;\underline{2}\;3\;4\;5\;6\;4\;2\\[-2pt]
\;\;\;1
\end{array}
\;=\;
\underbrace{\begin{array}{c}
\;\;\;1\qquad\qquad\qquad 1\\[-2pt]
1\;\;2\cdots 2\;2\;2\;2\;2\;2\;1\;0\\[-2pt]
\;\;\;1
\end{array}}_{\delta(\mathsf{D}_{m+4})}
\;+\;
\underbrace{\begin{array}{c}
\;\;\;0\qquad\qquad\qquad 2\\[-2pt]
0\;\;0\cdots 0\;0\;1\;2\;3\;4\;3\;2\\[-2pt]
\;\;\;0
\end{array}}_{\gamma(\mathsf{E}_7)}
$$

$$
=\;
\underbrace{\begin{array}{c}
\;\;\;0\qquad\qquad\qquad 3\\[-2pt]
0\;\;0\cdots 1\;2\;3\;4\;5\;6\;4\;2\\[-2pt]
\;\;\;0
\end{array}}_{\delta(\mathsf{E}_8)}
\;+\;
\underbrace{\begin{array}{c}
\;\;\;1\qquad\qquad\qquad 0\\[-2pt]
2\;\;2\cdots 1\;0\;0\;0\;0\;0\;0\;0\\[-2pt]
\;\;\;1
\end{array}}_{\gamma(\mathsf{D}_{m-2})}
$$

Finally we have the Type III exceptions. These admit a single type E isotropic subroot each:

$$
\underbrace{\begin{array}{c}
\qquad\underline{2}\\[-2pt]
1\;2\;3\;4\;5\;4\;3\;2\;1
\end{array}}_{\mathrm{III}_a}
\;=\;
\underbrace{\begin{array}{c}
\qquad 2\\[-2pt]
0\;1\;2\;3\;4\;3\;2\;1\;0
\end{array}}_{\delta(\mathsf{E}_7)}
\;+\;
\underbrace{\begin{array}{c}
\qquad 0\\[-2pt]
1\;1\;1\;1\;1\;1\;1\;1\;1\;1
\end{array}}_{\gamma(\mathsf{A}_9)}
$$

$$
\underbrace{\begin{array}{c}
\quad 4\\[-2pt]
\underline{2}\;5\;8\;7\;6\;5\;4\;3\;2\;1
\end{array}}_{\mathrm{III}_b}
\;=\;
\underbrace{\begin{array}{c}
\quad 3\\[-2pt]
2\;4\;6\;5\;4\;3\;2\;1\;0\;0
\end{array}}_{\delta(\mathsf{E}_8)}
\;+\;
\underbrace{\begin{array}{c}
\quad 1\\[-2pt]
0\;1\;2\;2\;2\;2\;2\;2\;2\;1
\end{array}}_{\gamma(\mathsf{D}_{10})}
$$

4.5.4. *Summary.* Noting that the dimension-2 degeneration corresponds to δ and the codimension-2 degeneration corresponds to γ, we tabulate the decomposition data.

TABLE 4.1. Isotropic decompositions for $\alpha \in \Sigma_0$ satisfying $p(\alpha) = 2$

(Δ, α)	Restriction(s)	rk	δ	γ
$\mathsf{I}(\ell,m,n)$	$1 < \ell \le m \le n$ $1 = \ell < m \le n$ $1 = \ell = m < n$ $1 = \ell = m = n$	$\ell + m + n - 1$	$\mathsf{A}_{m+n-1};\ \mathsf{A}_{\ell+n-1};\ \mathsf{A}_{\ell+m-1}$ $\mathsf{A}_{m-1};\ \mathsf{A}_{n-1}$ A_1	$\mathsf{A}_{\ell-1};\ \mathsf{A}_{m-1};\ \mathsf{A}_{n-1}$ $\mathsf{A}_n;\ \mathsf{A}_m$ A_{n-1}
$\mathsf{I}(\mathsf{D}_m,n)$	$n \ge 2$ $n \in \{0,1\}$	$m+n$	$\mathsf{D}_m;\ \mathsf{A}_{m+n-3}$ A_{m+n-3}	$\mathsf{A}_{n-1};\ \mathsf{A}_{m-1}$ A_{m-1}
$\mathsf{I}(\mathsf{D}_m,n)'$	$n \ge 2$ $n \in \{0,1\}$	$m+n$	$\mathsf{D}_m;\ \mathsf{A}_{n+1}$ A_{n+1}	$\mathsf{A}_{n-1};\ \mathsf{D}_{m-1}$ D_{m-1}
$\mathsf{I}(\mathsf{E}_6,n)$	$n \ge 2$ $n \in \{0,1\}$	$n+6$	$\mathsf{E}_6;\ \mathsf{A}_{n+3}$ A_{n+3}	$\mathsf{A}_{n-1};\ \mathsf{D}_5$ D_5

TABLE 4.1. (continued)

(Δ, α)	Restriction(s)	rk	δ	γ
I(E_7, n)	$n \geq 2$ $n \in \{0,1\}$	$n+7$	$E_7;\ A_{n+5}$ A_{n+5}	$A_{n-1};\ E_6$ E_6
I(m)	$m \geq 4$	$m+3$	D_m	A_{m+1}
I$_a$		8	D_5	D_6
I$_b$		9	E_6	D_7
I$_c$		9	D_7	E_7
II($\Gamma^{(1)}, \Gamma^{(2)}$)	$\Gamma^{(1)} \neq A_0 \neq \Gamma^{(2)}$ $\Gamma^{(1)} = A_0 \neq \Gamma^{(2)}$ $\Gamma^{(1)} = A_0 = \Gamma^{(2)}$	$\mathrm{rk}^{(1)} + \mathrm{rk}^{(2)} + 1$	$\Gamma^{(1)};\ \Gamma^{(2)}$ A_0	$\Gamma^{(2)};\ \Gamma^{(1)}$ $\Gamma^{(2)}$
II(m)	$m \geq 4$	$m+3$	D_m	D_{m+2}
II$_a$		8	D_6	E_7
II$_b$		9	E_7	E_8
III(D_m, i)	$2 \leq i \leq \lceil \frac{m-3}{2} \rceil$ $1 = i \leq \lceil \frac{m-3}{2} \rceil$ $m = 4$	$m+2$	$D_m;\ D_{m-i+1};\ D_{i+3}$ $D_m;\ D_m;\ D_m;\ D_4$ $D_4;\ D_4;\ D_4;\ D_4;\ D_4$	$A_1;\ D_{i+1};\ D_{m-i-1}$ $A_1;\ A_1;\ A_1;\ D_{m-2}$ $A_1;\ A_1;\ A_1;\ A_1;\ A_1$
III(E_7)		9	$E_6;\ E_7$	$A_7;\ A_1$
III(E_8)		10	$E_7;\ E_8$	$D_8;\ A_1$
III(m, 6)	$m \geq 5$ $m = 4$	$m+4$	$D_{m+1};\ E_6$ $D_5;\ E_6;\ E_6$	$A_5;\ D_{m-2}$ $A_5;\ A_2;\ A_2$
III(m, 7)	$m \geq 5$ $m = 4$	$m+5$	$D_{m+2};\ E_7$ $D_6;\ E_7;\ E_7$	$D_6;\ D_{m-2}$ $D_6;\ A_2;\ A_2$
III(m, 8)	$m \geq 5$ $m = 4$	$m+6$	$D_{m+4};\ E_8$ $D_8;\ E_8;\ E_8$	$E_7;\ D_{m-2}$ $E_7;\ A_2;\ A_2$
III$_a$		10	E_7	A_9
III$_b$		11	E_8	D_{10}

4.6. **Namikawa's Weyl group.** In [35] Namikawa defined a finite reflection group W, associated to a conical symplectic singularity X. Suppose $Y \to X$ is a $\mathbb{Q}$-factorial terminalisation of X. The group W then acts as a reflection group on $H^2(Y, \mathbb{C})$, which is known to depend on X only.

Let $\mathscr{L}$ be a codimension 2 symplectic leaf and let $x \in \mathscr{L}$. Then the formal neighbourhood of x in X is isomorphic to $\mathbb{C}^{2n-2} \times \mathbb{C}^2/\Gamma$, where $\dim X = 2n$ and $\Gamma \subset SL_2(\mathbb{C})$ is a finite subgroup. Denote by $S_{\mathscr{L}}$ the corresponding Kleinian singularity (of type A, D or E). Via the McKay correspondence we can associate to Γ a Weyl group $W_{\mathscr{L}}$ of types A, D or E and the root space $\hat{\mathfrak{h}}_{\mathscr{L}} \simeq H^2(\widetilde{S}_{\mathscr{L}}, \mathbb{C})$. The fundamental group $\pi_1(\mathscr{L})$ acts on $W_{\mathscr{L}}$ and $\hat{\mathfrak{h}}_{\mathscr{L}}$ by Dynkin diagram automorphisms. Denote by $W'_{\mathscr{L}}$ the centralizer of $\pi_1(\mathscr{L})$ in $W_{\mathscr{L}}$ and denote $\mathfrak{h}_{\mathscr{L}} := \left(\hat{\mathfrak{h}}_{\mathscr{L}} \right)^{\pi_1(\mathscr{L})}$. Taking the product over all leaves we obtain Namikawa's Weyl group

$$W := \prod_{\mathscr{L}} W'_{\mathscr{L}} \tag{4.B}$$

Theorem 4.12. [29, Lem. 2.8] *There is a vector space isomorphism*

$$H^2(Y, \mathbb{C}) \simeq H^2(X^{\mathrm{reg}}, \mathbb{C}) \oplus \bigoplus_{\mathscr{L}} \mathfrak{h}_{\mathscr{L}}$$

We would like to compute the Namikawa Weyl group W when $X := \mathfrak{M}_0(\alpha)$ for $p(\alpha) = 2$. By [4, Cor. 1.3], for generic θ, $Y := \mathfrak{M}_0^\theta(\alpha)$ will be projective symplectic resolution of X.

To compute W we need to find all the codimension 2 leaves $\mathscr{L}$ of a quiver variety and compute the corresponding $W'_{\mathscr{L}}$. The former, as we know by Theorem 4.9, are given by isotropic decompositions and the following theorem shows that the latter step is not required when $p(\alpha) = 2$.

26

Theorem 4.13. *For each symplectic leaf $\mathscr{L}$ of codimension two, corresponding to an isotropic representation type τ with a single imaginary root, we have $W'_{\mathscr{L}} = W_{\mathscr{L}}$.*

Proof. Recall from Definition 4.8 that the *ext-quiver* Q'' for a decomposition $\tau : \alpha = \beta^{(1)} + \ldots + \beta^{(m)} + n_1\gamma^{(1)} + \ldots + n_k\gamma^{(k)}$ is defined as follows. The dualisation $\overline{Q}''$ is the quiver with $m + k$ vertices without loops and $-(\alpha^{(i)}, \alpha^{(j)})$ arrows from vertex i to vertex $j \neq i$ where $\alpha^{(i)}, \alpha^{(j)} \in \{\beta^{(1)}, \ldots, \beta^{(m)}, n_1\gamma^{(1)}, \ldots, n_k\gamma^{(k)}\}$. Moreover, the space $H^2(\widetilde{S}_{\mathscr{L}}, \mathbb{C}) \simeq \hat{\mathfrak{h}}_{\mathscr{L}}$ is the space $\mathbb{C}^{Q''}$.

Following [39, Thm. 3.2, Thm. 3.3, and Ppn. 3.4] (see also [31], [28]), there is a surjective map

$$\kappa : \mathbb{Z}^{Q_0} \to H^2(\mathfrak{M}_0^\theta(\alpha), \mathbb{C})$$

and, in particular, its restriction to each $\mathfrak{h}_{\mathscr{L}} \subseteq \hat{\mathfrak{h}}_{\mathscr{L}}$ can be obtained using [39, Ppn. 3.4] (see also Thm. 1.6 and Rmk. 1.7 in loc. cit.). Explicitly, the restriction of the projection is given by

$$\kappa_{\mathscr{L}} : \chi \longmapsto (\chi \cdot \beta^{(2)}, \ldots, \chi \cdot \beta^{(m)}, \chi \cdot \gamma^{(1)}, \ldots, \chi \cdot \gamma^{(k)}) \in \mathfrak{h}_{\mathscr{L}}$$

One can notice that whenever the representation type τ only has a single imaginary root β, the image of this map is given by $(\chi \cdot \gamma^1, \ldots, \chi \cdot \gamma^k)$, which is a full-rank lattice in $\hat{\mathfrak{h}}_{\mathscr{L}}$, as all the remaining roots are simple. This means that $\hat{\mathfrak{h}}_{\mathscr{L}} = \mathfrak{h}_{\mathscr{L}}$, which entails the claim. $\square$

Corollary 4.14. *For $p(\alpha) = 2$ the* Namikawa Weyl *group of $\mathfrak{M}_0(\alpha)$ is the product of the Weyl groups corresponding to the Kleinian slice singularities across the isotropic decompositions of α.*

Applying this to Table 4.1, we obtain Theorem 1.6.

Remark 4.15. Unfortunately, this simplification does not hold in higher dimensions. Indeed, [39, Exm. 3.6, 3.7] gives counterexamples in dimension 8 and 6, respectively.

5. Arrangements and resolutions

In this section we prove Theorem 1.4, the main classification result. Now that we have computed all the minimal degenerations between the symplectic leaves, we may use this to compare quiver varieties within the Table 1.1 among themselves, and with symmetric powers of Kleinian singularities. To finish the proof we compare the number of projective symplectic resolutions for the remaining cases, but to do this enumeration we need the *secondary arrangement* associated to α.

5.1. **Products and symmetric powers of Kleinian singularities.** Recall that Theorem 2.8 tells us that outwith those induced by Table 1.1, the only possible quiver varieties of dimension 4 are given by the products of lower dimensional quiver varieties (i.e. Kleinian singularities only for $p(\alpha) = 2$) with the symmetric powers of Kleinian singularities. Notice, however, that it does not claim that such varieties are not isomorphic to $\mathfrak{M}_0(\alpha)$ for some $\alpha \in \Sigma_0$.

Remark 5.1. In particular, [5, Lem. 8.4] explains that the quiver variety corresponding to $\amalg(\Gamma^{(1)}, \Gamma^{(2)})$ is the product of Kleinian singularities of type $\Gamma^{(1)}$ and $\Gamma^{(2)}$. We are thus left with symmetric powers to study.

By [14, Thm. 3.4] we know that the symmetric powers of $\mathfrak{M}_0(\gamma)$ have $n\gamma$ as the dimension vector. Simple observation of the minimal imaginary subroots of 2γ yields the following lemma:

Lemma 5.2. *The representation types for 2γ are 2γ, $\gamma + \gamma$, $\gamma + \sum e_i$ and $\sum 2e_i$.*

In particular, as the quiver representations classify points in $S^2(\mathbb{C}^2/\Gamma)$, these representation types label the symplectic leaves, as per Theorem 4.4. Explicitly, $\tau = (1, \gamma; 1, \gamma)$ are the generic points of $\mathscr{L}_4$, as this is the canonical decomposition, $\tau = (2, \gamma)$ is the diagonal leaf, $\tau = (1, \gamma; \sum e_i)$ are points of the type $(x, 0) \simeq (0, x)$ and $\tau = (2, \sum e_i)$ is the $\mathscr{L}_0$ point. From this we observe that the poset of minimal degenerations has the following form:

Theorem 5.3. *Hasse diagrams for $S^2(\mathbb{C}^2/\Gamma)$ have the following form (labeled by τ):*

$$
\begin{array}{ccccc}
 & & \gamma + \gamma & & \\
 & {}^{\Gamma}\diagup & & {}^{A_1}\diagdown & \\
\gamma + \sum e_i & & & & 2\gamma \\
 & {}_{\Gamma}\diagdown & & {}_{\Gamma}\diagup & \\
 & & \sum 2e_i & &
\end{array}
\qquad (5.\mathrm{A})
$$

Proof. The degenerations between the zero point and dimension 2 leaves are given by the closures of the corresponding leaves, which both are $\mathbb{C}^2/\Gamma$. For the upper degenerations we invoke a more general technique. [5, Thm. 8.1] explain that the quiver variety corresponding to the ext-quiver Q'' from Definition 4.8 describes étale local slice to a point in the corresponding symplectic leaf. The ext-quiver for $\tau = 2\gamma$ is the following:

$$2 \supset$$

Which means that the at the points of this leaf, the local structure of the variety variety is $S^2(\mathbb{C}^2) \simeq \mathbb{C}^2 \times \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$. The smooth part is the leaf direction and the étale slice is given by $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$, meaning that the corresponding minimal degeneration is A_1.

Next, we study the etale local slice to the leaf corresponding to $\gamma + \sum e_i$. The corresponding ext-quiver

$$\subset 1 \qquad \delta(\Gamma)$$

is disconnected and consists of two parts. In particular, local structure at points of this leaf is given by $\mathbb{C}^2 \times \mathbb{C}^2/\Gamma$. As before, the minimal degeneration is given by the type Γ singularity. $\square$

Lemma 5.4. *The varieties $S^2\left(\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})\right) \simeq \mathbb{C}^4/\left(S_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2\right)$ and $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$ are not isomorphic.*

Proof. Write $V := \mathbb{C}^4$ and take $V_1, V_2 := V \setminus \mathrm{Sing}$ with Sing being points with a non-trivial stabiliser under the action of the corresponding finite groups $\left(\Gamma_1 := (\mathbb{Z}/2\mathbb{Z})^2 \text{ vs } \Gamma_2 := S_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2\right)$. Both of these groups act symplectically so Sing will have codimension 2 and real codimension 4. This means, by ([18, Thm. 2.3]) that $\pi_1(V \setminus \mathrm{Sing}) \simeq \pi_1(V)$ is trivial in both cases. The fundamental groups of the factors by similar reasons will be isomorphic to $\pi_1\left((V/\Gamma_i) \setminus \mathrm{Sing}\right)$. However, as the action of Γ_1, Γ_2 on $V \setminus \mathrm{Sing}$ is free, the corresponding fundamental groups are $(\mathbb{Z}/2\mathbb{Z})^2 \neq S_2 \ltimes (\mathbb{Z}/2\mathbb{Z})^2$. $\square$

We can now rule out new symmetric powers from the classification of four-dimensional quiver varieties.

Theorem 5.5. *There is no $\alpha \in \Sigma_0$ with $p(\alpha) = 2$ such that $S^2(\Gamma) \simeq \mathfrak{M}_0(\alpha)$ for any type of Kleinian singularity Γ.*

Proof. Table 4.1, as explained in §4.4, contains all the information about minimal degenerations of the roots that are in Σ_0. A simple check shows that the only coinciding Hasse diagrams between those and the ones obtained in Theorem 5.3 are the diagrams for the varieties $S^2\left(\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})\right)$ and $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$. These two are different by Lemma 5.4. $\square$

5.2. **Comparing quiver varieties.** We now use the minimal degenerations and leaf closures of the dimension 2 symplectic leaves to distinguish the quiver varieties in the classification, almost proving Theorem 1.4.

Theorem 5.6. *All of quiver varieties $\mathfrak{M}_0(\alpha)$ from Table 1.2 are non-isomorphic varieties, with the possible exception of $I(1,1,4)$ and $II(D_4)$.*

Proof. As explained in §4.4, Table 4.1 lists geometric invariants of quiver varieties. These distinguish most of the varieties from Table 1.1, as amounts to a simple check. It is clear that, within a given family, the decompositions are all different, so we proceed by the number of decompositions, taking care not to miss any potentially overlapping cases.

(0) Observe that only $I(1,1,1)$ and $II(A_0, A_0)$ only have trivial decompositions.

(1) The cases where $\delta \in \{\mathsf{D}, \mathsf{E}\}$ are all distinct, except from II(5) and I_a which both have $\delta = \mathsf{D}_5$. However the former has $\gamma = \mathsf{D}_7$ and the latter has $\gamma = \mathsf{D}_6$, so these decompositions do not coincide. This leaves the cases where $\delta = \mathsf{A}$, within which only $\mathsf{II}(\mathsf{A}_0, \Gamma^{(2)})$ has $\delta = \mathsf{A}_0$ and only $\mathsf{I}(\mathsf{E}_7, n)$ $(n \in \{0, 1\})$ has $\gamma = \mathsf{E}_6$. We see that $\mathsf{I}(1, 1, 4)$ and $\mathsf{II}(\mathsf{D}_4)$ are distinct dimension vectors with the same decomposition $(\delta, \gamma) = (\mathsf{A}_1, \mathsf{A}_3)$. The data also coincides for $\mathsf{I}(\mathsf{D}_4, n)$ and $\mathsf{I}(\mathsf{D}_4, n)'$ more generally, but this is because the dimension vectors themselves are equal.

(2) We observe that the construction $\mathsf{II}(\Gamma^{(1)}, \Gamma^{(2)})$ is unique in having 'symmetric' decompositions, meaning that $\delta^{(1)} = \gamma^{(2)}$ and $\delta^{(2)} = \gamma^{(1)}$. The families $\mathsf{III}(m, n)$ $(m \geq 5)$ can be separated from the rest by noting that they have one decomposition with $\delta = \mathsf{D}$ and another with $\delta = \mathsf{E}$, all of which are different to each other as well. A similar argument holds for $\mathsf{I}(\mathsf{E}_6, n)$, $\mathsf{I}(\mathsf{E}_7, n)$ and $\mathsf{I}(\mathsf{D}_m, n)$, $\mathsf{I}(\mathsf{D}_m, n)'$, leaving $\mathsf{I}(1, m, n)$ that has unique decompositions.

(3) There are five such dimension vectors: $\mathsf{I}(\ell, m, n)$ $(1 < \ell \leq m \leq n)$ has only A in its decompositions, $\mathsf{III}(\mathsf{D}_m, i)$ $(m \geq 4, i \geq 2)$ has A and D in its decompositions, and the three cases $\mathsf{III}(m, n)$ $(n \in \{6, 7, 8\})$ all have a different E_n in their decompositions.

(4) The dimension vector $\mathsf{III}(\mathsf{D}_m, 1)$ $(m \geq 4)$ is unique in having four decompositions.

(5) Similarly to the previous case, only $\mathsf{III}(\mathsf{D}_4, 1)$ has five decompositions. $\square$

Since we know that $\mathfrak{M}_0(\mathsf{I}(1, 1, 1)) = \mathcal{O}_{min}$ is singular and $\mathfrak{M}_0(\mathsf{II}(\mathsf{A}_0, \mathsf{A}_0)) = \mathbb{C}^4$ is smooth, these quiver varieties are non-isomorphic. For $\mathsf{I}(1, 1, 4)$ and $\mathsf{II}(\mathsf{D}_4)$, we will need an additional invariant to distinguish them geometrically. Recall their dimension vectors and shared Hasse diagram:

$$(5.\mathrm{B})$$
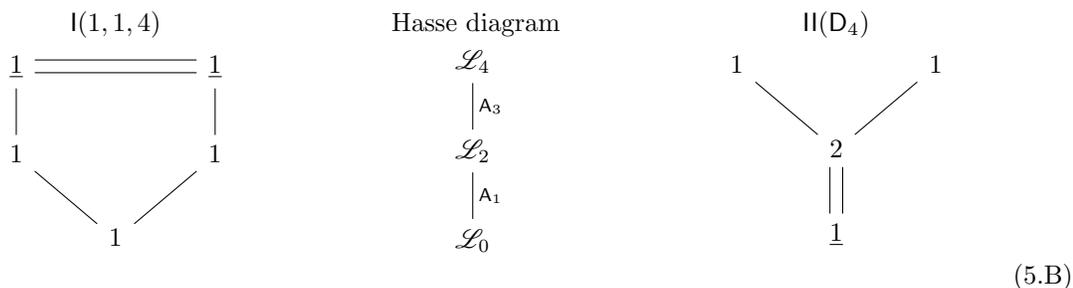
5.3. **Secondary arrangements and the number of resolutions.** In this subsection we explain how to compute the number of projective symplectic resolutions for the quiver varieties from Table 1.1. We then proceed to apply it in particular cases to $\mathsf{II}(\mathsf{D}_4)$ and $\mathsf{I}(1, 1, 4)$, show that they are not isomorphic, and complete the proof of Theorem 1.4.

By [22, Ppn. 5.10], if α is supported on a finite or affine Dynkin diagram and $p(\alpha) \in \{0, 1\}$, then α is a root. For any root $\gamma \in \mathsf{R}^+$ consider the hyperplane $\gamma^\perp := \{\theta \in \mathrm{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Q}) | \theta(\gamma) = 0\}$. We call $\alpha^\perp = \Theta$, the space of stability parameters for GIT quotient quiver varieties.

Definition 5.7 ([4], Dfn. 4.13). The *secondary arrangement* in Θ is given by

$$\mathcal{A}_\alpha := \{\beta^\perp \cap \alpha^\perp \mid \alpha = \beta + (\alpha - \beta) \text{ is a decomposition into two roots in } \mathsf{R}^+\}.$$

Remark 5.8. In general, the secondary arrangement is the hyperplane arrangement defining the Mori fan, which parametrises all crepant resolutions. For quiver varieties the latter has a handy description (above) in terms of root decompositions and these arrangements are also variety invariants that we use to distinguish different quiver varieties. In case of symplectic singularities, symplectic and crepant resolutions are same ([23, Ppn. 3.2], see also [17]). Moreover, all our quiver varieties have a vertex of weight 1, therefore, as shown in [7], they admit projective symplectic resolution, so we henceforth will speak about these. Note that in general quiver varieties also admit proper, nonprojective crepant resolutions. For a detailed discussion, see [26].

In [4, §4.5] (see also Cor. 4.7 and Ppn. 4.12 in loc. cit.) the authors explain that this hyperplane arrangement precisely produces the GIT fan. Moreover, each of the GIT regions presented there is a chamber for the action of the Namikawa Weyl group W. Therefore the number of projective symplectic resolutions can be obtained from this hyperplane arrangement by counting the number of chambers and factoring our the transitive action of W on regions. Denote by $N(\Delta, \alpha)$ the number of projective crepant resolutions of the quiver variety $\mathfrak{M}_0(\Delta, \alpha)$.

Lemma 5.9. [4, §4.5] *The number of projective symplectic resolutions of a quiver variety* $\mathfrak{M}_0(\alpha)$ *for an indivisible* α *is equal to the number of chambers in* $\mathcal{A}_\alpha$ *divided by* $|W|$.

$$N(\alpha) = \frac{|\mathsf{Cham}\mathcal{A}_\alpha|}{|W_\alpha|} \tag{5.C}$$

Remark 5.10. Notice the secondary arrangement contains the Coxeter arrangement associated with the Namikawa Weyl group, which has hyperplanes corresponding to real roots. The question is whether this inclusion is strict — if it is then there are at least two projective symplectic resolutions of $\mathfrak{M}_0(\alpha)$.

We give an explicit example which will be useful later.

Example 5.11. We compute the secondary arrangement for $\mathsf{I}(1,1,3)$ and thus determine the number of projective symplectic resolutions admitted by $\mathfrak{M}_0(\mathsf{I}(1,1,3))$. Using the indexing

$$
\begin{array}{ccc}
\underline{1}_x & =\!=\!= & \underline{1}_w \\
| & & | \\
1_y & —— & 1_z
\end{array}
$$

we have $\mathcal{A}_\alpha \subset \Theta = \alpha^\perp = \{w + x + y + z = 0\}$. Recall from §2.2 that the root system is with respect to the Weyl group of the underlying graph, that is $W_\Delta = \langle s_w, s_x, s_y, s_z \rangle$ with involutive generators, s_w and s_x free, and the rest braiding. The isotropic decomposition $\alpha = \delta(\mathsf{A}_1) + \gamma(\mathsf{A}_2)$ gives the hyperplane $\{w + x = 0\}$, (equivalently, $\{y + z = 0\}$ in Θ). We also have the hyperplane $\{w + z = 0\}$ arising from the decomposition $\alpha = \gamma(\mathsf{A}_2) + \gamma(\mathsf{A}_2)$ into real roots. Similarly there are two decompositions of the form $\alpha = \gamma(\mathsf{A}_3) + \gamma(\mathsf{A}_1)$, giving hyperplanes $\{w = 0\}$ and $\{x = 0\}$. Finally observe that the subroot $\beta := (1 \,—— 1 \,=\!=\!= 1) \notin \Sigma_0$ is imaginary, being the image of the fundamental imaginary subroot $\delta(\mathsf{A}_1)$ under the reflection s_y or s_z. This gives hyperplanes $\{y = 0\}$ and $\{z = 0\}$. Hence $\mathcal{A}_\alpha$ consists of 6 hyperplanes, cutting $\mathbb{R}^3$ into 24 regions[2]. The number of projective symplectic resolutions is then

$$N(\alpha) = \frac{|\mathsf{Cham}\mathcal{A}_\alpha|}{|W_\alpha|} = \frac{24}{6} = 4,$$

using Table 1.2.

This is the last invariant that we need to distinguish all four-dimensional quiver varieties.

Corollary 5.12. *The quiver varieties* $\mathfrak{M}_0(\mathsf{II}(\mathsf{D}_4))$ *and* $\mathfrak{M}_0(\mathsf{I}(1,1,4))$ *are non-isomorphic as affine schemes.*

Proof. We show that the former has a unique projective crepant resolution whilst the latter does not, so they cannot be isomorphic. By Corollary 4.14 and Table 4.1, the two share the Namikawa Weyl group of type A_3, which has six hyperplanes in its associated Coxeter arrangement. We now explicitly describe the hyperplanes in each secondary arrangement by counting the number of decompositions into two positive roots. In the spirit of §4.5, firstly

$$
\begin{array}{ccccccccccccc}
 & 1 & & & 1 & & & 0 & & & 1 & & & 0 & & & 1 & & & 0 \\
\underline{1}=2 & & = & 1=1 & & + & 0=1 & & = & 1=2 & & + & 0=0 & & = & 1=1 & & + & 0=1 & \\
 & 1 & & & 0 & & & 1 & & & 0 & & & 1 & & & 1 & & & 0 \\
\mathsf{II}(\mathsf{D}_4) & & & \beta^{(1)} & & & \gamma(\mathsf{A}_2) & & & \beta^{(2)} & & & \gamma(\mathsf{A}_1) & & & \beta^{(3)} & & & \gamma(\mathsf{A}_1) &
\end{array} \tag{5.D}
$$

the first two decompositions each occurring in two different ways. There is also a single isotropic decomposition since $W_{\mathsf{II}(\mathsf{D}_4)}$ has one factor. The dimension vectors $\beta^{(1)}, \beta^{(2)}, \beta^{(3)}$ are readily checked to be roots, as they all lie in the image of the fundamental imaginary root $\delta(\mathsf{A}_1)$ under simple reflections. This gives at least six hyperplanes in $\mathcal{A}_{\mathsf{II}(\mathsf{D}_4)}$. To see that there are no other permitted decompositions of $\mathsf{II}(\mathsf{D}_4)$ into positive roots, denote the branch vertex by i and the first summand by β. Then as roots necessarily have connected support, i must have at least two neighbours with nonzero weight whenever $\beta_i \neq 0$. In (5.D), the first decomposition gives the cases where $\delta(\mathsf{A}_1) < \beta$ and $\beta_i = 1$. The cases where $\delta(\mathsf{A}_1) < \beta$ and $\beta_i = 2$ are the second equality. The case where every neighbour has nonzero weight and

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[2]interactive version: https://www.desmos.com/3d/m5yp6auo7r

$\beta_i = 1$ is the third equality, and the isotropic decomposition is the case where $\delta(\mathsf{A}_1) \not< \beta$ and $\alpha_i = 1$. This leaves the case where $\delta(\mathsf{A}_1) \not< \beta$ and $\alpha_i = 2$, that is

$$\begin{matrix} & 1 & & & 1 & & & 0 & \\ \underline{1} = 2 & & = & 0 = 1 & & + & 1 = 0 & \\ & 1 & & & 1 & & & 0 & \\ \mathsf{II}(\mathsf{D}_4) & & & & \alpha & & & \gamma(\mathsf{A}_1) & \end{matrix}$$

but a quick computation gives $p(\beta) = -1$ so $\beta \notin \mathsf{R}^+$. Hence the Coxeter arrangement and secondary arrangement associated with $\mathfrak{M}_0(\mathsf{II}(\mathsf{D}_4))$ coincide, so this quiver variety has a unique projective crepant resolution by Lemma 5.9. It now suffices to prove that $|\mathcal{A}_{\mathsf{I}(1,1,4)}| > 6$, as then $N(\mathfrak{M}_0(\mathsf{I}(1,1,4))) > 1$. To this end, consider the decompositions

$$\begin{matrix} & \underline{1} = \underline{1} & & & 1 = 0 & & & 0 = 1 & & & 1 = 0 & & & 0 = 1 & & & 1 = 1 & & & 0 = 0 & \\ 1 & & 1 & 1 & & = & 1 & 1 & 1 & + & 0 & 0 & 0 & = & 1 & 0 & 0 & + & 0 & 1 & 1 & = & 1 & 0 & 0 & + & 0 & 1 & 1 \\ \mathsf{I}(1,1,4) & & & & \gamma(\mathsf{A}_4) & & & & \gamma(\mathsf{A}_1) & & & \gamma(\mathsf{A}_3) & & & \gamma(\mathsf{A}_2) & & & \beta & & & \gamma(\mathsf{A}_1) & \end{matrix}$$

where $\beta = \beta^{(1)} \in \mathsf{R}^+$ from (5.D). Since each decomposition occurs in two different ways, if we include the isotropic decomposition then there is at least 7 hyperplanes in the secondary arrangement. □

We can now prove the main result.

Proof of Theorem 1.4. Combining Theorem 5.6 and Corollary 5.12 we obtain the full proof of Theorem 1.4. Furthermore, adding Theorem 5.5 completes the classification of quiver varieties in dimension 4. □

5.4. **The group quotient G4.** In [3, §1.5], the authors express that it is unknown whether $\mathbb{C}^4/G_4$, a four-dimensional conical symplectic singularity, admits a construction as a quiver variety, as is has been realised as a Hamiltonian reduction in [12, Exm. 5.6]. Applying the classification, we show that this is not the case.

Proof of Corollary 1.7. By [8, §7.2], $\mathbb{C}^4/G_4$ has Namikawa Weyl group S_3, and admits two projective symplectic resolutions. Thus for $\mathbb{C}^4/G_4$ to be a quiver variety it must be isomorphic to either $\mathfrak{M}_0(\mathsf{II}(\mathsf{A}_0,\mathsf{A}_2))$ or $\mathfrak{M}_0(\mathsf{I}(1,1,3))$, as by Theorem 1.4 these are the only four-dimensional quiver varieties with Weyl group S_3. However, the former has a unique projective symplectic resolution (being the direct product of Kleinian singularities), and the latter has four projective symplectic resolutions, as seen in Example 5.11. Hence $\mathbb{C}^4/G_4$ is a conical symplectic singularity in dimension four that cannot be realised as a Nakajima quiver variety.

Moreover, we can show that neither projective symplectic resolution Y of $X := \mathbb{C}^4/G_4$ is of the form $\mathfrak{M}_\theta(Q, \alpha)$. Suppose that Y is isomorphic to $\mathfrak{M}_\theta(Q, \alpha)$ for some θ, α. This isomorphism then gives rise to a map $\pi\colon Y \to \mathfrak{M}_0(Q, \alpha)$. If π is a resolution of singularities, then both X and $\mathfrak{M}_0(Q, \alpha)$ are isomorphic to $\operatorname{Spec}\Gamma(Y, \mathcal{O})$, but this contradicts the above argument. If π is not a resolution of singularities, then by [2, Thm. A.1], $\mathfrak{M}_\theta(Q, \alpha)$ resolves the normalisation of a leaf closure $\overline{\mathscr{L}}$. By [5, Thm. 1.9], the latter is isomorphic to a product of symmetric powers of quiver varieties in lower dimensions. Hence $X \simeq \operatorname{Spec}\Gamma(Y, \mathcal{O}) \simeq \overline{\mathscr{L}}$, which means, in our case, that $X = (\mathbb{C}^2/\Gamma_1 \times \mathbb{C}^2/\Gamma_2) = \mathbb{C}^4/(\Gamma_1 \times \Gamma_2)$ or $X = S^2(\mathbb{C}^2/\Gamma) = \mathbb{C}^4/(S_2 \wr \Gamma)$. But G_4 is an irreducible complex reflection group, so cannot take either of these forms. □

5.5. **Counting resolutions.** To conclude our work we provide computer-aided[3] numerology regarding the number of projective symplectic resolutions admitted by four-dimensional quiver varieties. By [3, Thm. 1.5], all admit at least one projective symplectic resolution, as every α in Table 1.1 is indivisible. Using the computer algebra package `Magma`, [9], we compute the hyperplanes present in the secondary arrangement $\mathcal{A}_\alpha$ by iterating over subroots $\beta < \alpha$, removing those that necessarily cannot be roots (for example $p(\beta) < 0$ or $\operatorname{supp}\beta$ disconnected) and checking the Weyl orbits of the remaining vectors. To compute the number of chambers in $\mathcal{A}_\alpha$ we then pass the hyperplanes to the `julia` package `CountingChambers`, developed in [11]. This process finishes in a reasonable time up to approximately $\dim\Theta = 10$, giving suggestions for the behaviour of $N(\Delta, \alpha)$ for many of the infinite families in our classification.

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[3]code is available from the authors upon request

Conjecture 5.13. *For certain dimension vectors in Type* I *we have the following counts of projective crepant resolutions:*

(1) $N(\mathsf{I}(\mathsf{D}_4, n)) = \frac{n+3}{3}\binom{n+5}{5}$.

(2) $N(\mathsf{I}(\mathsf{D}_m, 1)) = 2^{m-1}$.

(3) $N(\mathsf{I}(\mathsf{D}_m, n)') = \frac{2(m+n-1)}{n!}\prod_{i=1}^{n-1}(2m+i-2)$ *and* $N(\mathsf{I}(\mathsf{D}_m, 0)') = 1$.

(4) $N(\mathsf{I}(\mathsf{E}_{6,7}, n))$ *coincides with* [36, A030648, A030649]*, the dimensions of multiples of the* $\mathsf{E}_{6,7}$ *complex Lie algebra's minimal representation.*

(5) $N(\mathsf{I}(m)) = \prod_{i=0}^{m-1}\frac{(2i)!(m+2i+1)!}{(3i+1)!(m+i)!}$*, which by* [38, Cor. 5.2] *coincides with* [36, A005157]*, the number of totally symmetric plane partitions that fit in an* $m \times m \times m$ *box.*

Our second conjecture is motivated by observing that in each computed case, the number of chambers in the secondary arrangement of α coincided with the size of its Namikawa Weyl group in Table 1.2. This forces $\mathcal{A}_\alpha = \mathcal{B}_\alpha$ and a unique resolution for the quiver variety as in Remark 5.10.

Conjecture 5.14. *If* $\alpha \in \Sigma_0$ *is a dimension vector of Type* II *then the associated quiver variety has a unique projective crepant resolution.*

Clearly this holds for $\mathsf{II}(\alpha^{(1)}, \alpha^{(2)})$, being the product of Kleinian singularities, and similarly for $\mathsf{II}(\mathsf{A}_m)$ which is just $\mathsf{II}(\mathsf{A}_{d-1}, \mathsf{A}_{m-d})$ for some d. However we have also observed a 'trivial' secondary arrangement for $\mathsf{II}(\mathsf{E}_6)$, $\mathsf{II}(\mathsf{E}_7)$, II_a, as well as $\mathsf{II}(\mathsf{D}_m)$, $\mathsf{II}(\mathsf{D}_m)'$, and $\mathsf{II}(m)$ for low $m \geq 4$. We believe that more abstractly it should be possible to rule out the existence of special (that is, not within the Coxeter arrangement) root decompositions for α in Type II.

APPENDIX A. HASSE DIAGRAMS

TABLE A.1. Minimal degeneration poset of $\mathfrak{M}_0(Q, \alpha)$ for $\alpha \in \Sigma_0$ in Type I

(Δ, α)	Hasse diagram(s)

$I(E_6, n)$

$$1$$
$$2$$
$$\underline{1}\ 2\ 3\ 2\ \underline{1}$$

$n \geq 0$

$\mathscr{L}_4$

A_{n-1} D_5

$\mathscr{L}_{21}$ $\mathscr{L}_{22}$

E_6 A_{n+3}

$\mathscr{L}_0$

$n \geq 2$

$\mathscr{L}_4$

D_5

$\mathscr{L}_2$

A_{n+3}

$\mathscr{L}_0$

$n \in \{0, 1\}$

$I(E_7, n)$

$$2$$
$$\underline{1}\ 2\ 3\ 4\ 3\ 2\ \underline{1}$$

$n \geq 0$

$\mathscr{L}_4$

A_{n-1} E_6

$\mathscr{L}_{21}$ $\mathscr{L}_{22}$

E_7 A_{n+5}

$\mathscr{L}_0$

$n \geq 2$

$\mathscr{L}_4$

A_{m-1}

$\mathscr{L}_2$

A_{n+5}

$\mathscr{L}_0$

$n \in \{0, 1\}$

$I(m)$

$$\underline{1}\qquad \underline{1}$$
$$1\ 2\ 3\ \cdots\ 3\ 2\ 1$$

$m \geq 4$

$\mathscr{L}_4$

A_{m+1}

$\mathscr{L}_2$

D_m

$\mathscr{L}_0$

I_a

$$\underline{1}\qquad 2$$
$$3\ 4\ 3\ 2\ 1$$
$$\underline{1}$$

$\mathscr{L}_4$

D_6

$\mathscr{L}_2$

D_5

$\mathscr{L}_0$

I_b

$$\underline{1}\ 3$$
$$5\ 4\ 3\ 2\ 1$$
$$\underline{1}\ 3$$

$\mathscr{L}_4$

D_7

$\mathscr{L}_2$

E_6

$\mathscr{L}_0$

I_c

$$\underline{1}\qquad\quad 3$$
$$3\ 4\ 5\ 6\ 4\ 2$$
$$\underline{1}$$

$\mathscr{L}_4$

E_7

$\mathscr{L}_2$

D_7

$\mathscr{L}_0$

TABLE A.2. Minimal degeneration poset of $\mathfrak{M}_0(Q,\alpha)$ for $\alpha \in \Sigma_0$ in Type II

(Δ, α)	Hasse diagram(s)
$\mathrm{II}(\Gamma^{(1)}, \Gamma^{(2)})$ $\gamma(\Gamma^{(1)}) \perp \gamma(\Gamma^{(2)})$ $\Gamma^{(i)} \in \{\mathsf{A}, \mathsf{D}, \mathsf{E}\}$	**Diagram 1** ($\Gamma^{(1)} \neq \mathsf{A}_0 \neq \Gamma^{(2)}$): $\mathscr{L}_4$ at top with edges $\Gamma^{(1)}$ to $\mathscr{L}_{21}$ and $\Gamma^{(2)}$ to $\mathscr{L}_{22}$; then $\Gamma^{(2)}$ from $\mathscr{L}_{21}$ and $\Gamma^{(1)}$ from $\mathscr{L}_{22}$ down to $\mathscr{L}_0$. **Diagram 2** ($\Gamma^{(1)} = \mathsf{A}_0 \neq \Gamma^{(2)}$): $\mathscr{L}_4 \xrightarrow{\Gamma^{(2)}} \mathscr{L}_2 \xrightarrow{\mathsf{A}_0} \mathscr{L}_0$. **Diagram 3** ($\Gamma^{(1)} = \mathsf{A}_0 = \Gamma^{(2)}$): $\mathscr{L}_4 - \mathscr{L}_0$.
$\mathrm{II}(m)$ $\begin{array}{c} 2 \quad\quad \underline{1} \\ 4 \ \cdots \ 4 \ 3 \ 2 \ 1 \\ 2 \end{array}$ $m \geq 4$	$\mathscr{L}_4 \xrightarrow{\mathsf{D}_{m+2}} \mathscr{L}_2 \xrightarrow{\mathsf{D}_m} \mathscr{L}_0$
II_a $\begin{array}{c} 2 \quad\ 3 \\ \underline{1} \ 4 \ 5 \ 6 \ 4 \ 2 \end{array}$	$\mathscr{L}_4 \xrightarrow{\mathsf{E}_7} \mathscr{L}_2 \xrightarrow{\mathsf{D}_6} \mathscr{L}_0$
II_b $\begin{array}{c} 5 \\ \underline{1} \ 4 \ 7 \ 10 \ 8 \ 6 \ 4 \ 2 \end{array}$	$\mathscr{L}_4 \xrightarrow{\mathsf{E}_8} \mathscr{L}_2 \xrightarrow{\mathsf{E}_7} \mathscr{L}_0$

TABLE A.3. Minimal degeneration poset of $\mathfrak{M}_0(Q,\alpha)$ for $\alpha \in \Sigma_0$ in Type III

(Δ, α)	Hasse diagram(s)
$\mathrm{III}(\mathsf{D}_m, i)$ $\begin{array}{c} 1 \quad\quad 1 \quad\quad 1 \\ 2 \ \cdots \ \underline{2} \ \cdots \ 2 \\ 1 \quad\quad\quad\quad 1 \end{array}$ $1 \leq i \leq \lceil \frac{m-3}{2} \rceil$	**Diagram ($i \neq 1$):** $\mathscr{L}_4$ at top with edges D_{i+1} to $\mathscr{L}_{21}$, A_1 to $\mathscr{L}_{22}$, D_{m-i-1} to $\mathscr{L}_{23}$; then D_{m-i+1}, D_m, D_{i+3} down to $\mathscr{L}_0$. **Diagram ($i = 1$):** $\mathscr{L}_4$ at top with edges A_1 to $\mathscr{L}_{21}$, A_1, A_1 to $\mathscr{L}_{22}$, D_{m-2} to $\mathscr{L}_{23}$, $\mathscr{L}_{24}$; then D_m, D_m, D_4 down to $\mathscr{L}_0$. **Diagram ($m = 4$):** $\mathscr{L}_4$ with A_1 edges to $\mathscr{L}_{21} \cdots \mathscr{L}_{25}$ and D_4 edges down to $\mathscr{L}_0$.
$\mathrm{III}(\mathsf{E}_7)$ $\begin{array}{c} 1 \\ \underline{2} \\ 1 \ 2 \ 3 \ 4 \ 3 \ 2 \ 1 \end{array}$	$\mathscr{L}_4$ at top with edges A_7 to $\mathscr{L}_{21}$ and A_1 to $\mathscr{L}_{22}$; then E_6 from $\mathscr{L}_{21}$ and E_7 from $\mathscr{L}_{22}$ down to $\mathscr{L}_0$.

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(Δ, α)	Hasse diagram(s)
III(E_8) $\begin{smallmatrix} & & & & & & 3 & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 4 & \underline{2} & 1 \end{smallmatrix}$	$\mathscr{L}_4$ at top, with D_8 to $\mathscr{L}_{21}$ and A_1 to $\mathscr{L}_{22}$; $\mathscr{L}_{21}$ with E_7 and $\mathscr{L}_{22}$ with E_8 down to $\mathscr{L}_0$
III($m,6$) $\begin{smallmatrix} & & & & 1 & \\ 1 & & & & 2 & \\ & 2 & \cdots & \underline{2} & 3 & 2 & 1 \\ 1 & & & & & \\ & & m \geq 4 & & & \end{smallmatrix}$	**$m \neq 4$:** $\mathscr{L}_4$ top, A_5 to $\mathscr{L}_{21}$, D_{m-2} to $\mathscr{L}_{22}$; $\mathscr{L}_{21}$ with D_{m+1}, $\mathscr{L}_{22}$ with E_6 to $\mathscr{L}_0$. **$m = 4$:** $\mathscr{L}_4$ top, A_5 to $\mathscr{L}_{21}$, A_2 to $\mathscr{L}_{22}$, A_2 to $\mathscr{L}_{23}$; $\mathscr{L}_{21}$ with D_5, $\mathscr{L}_{22}$ with E_6, $\mathscr{L}_{23}$ with E_6 to $\mathscr{L}_0$.
III($m,7$) $\begin{smallmatrix} 1 & & & & 2 & \\ & 2 & \cdots & \underline{2} & 3 & 4 & 3 & 2 & 1 \\ 1 & & & & & \\ & & m \geq 4 & & & \end{smallmatrix}$	**$m \neq 4$:** $\mathscr{L}_4$ top, D_6 to $\mathscr{L}_{21}$, D_{m-2} to $\mathscr{L}_{22}$; $\mathscr{L}_{21}$ with D_{m+2}, $\mathscr{L}_{22}$ with E_7 to $\mathscr{L}_0$. **$m = 4$:** $\mathscr{L}_4$ top, D_6 to $\mathscr{L}_{21}$, A_2 to $\mathscr{L}_{22}$, A_2 to $\mathscr{L}_{23}$; $\mathscr{L}_{21}$ with D_6, $\mathscr{L}_{22}$ with E_7, $\mathscr{L}_{23}$ with E_7 to $\mathscr{L}_0$.
III($m,8$) $\begin{smallmatrix} 1 & & & & & & 3 & \\ & 2 & \cdots & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \\ 1 & & & & & \\ & & m \geq 4 & & & \end{smallmatrix}$	**$m \neq 4$:** $\mathscr{L}_4$ top, E_7 to $\mathscr{L}_{21}$, D_{m-2} to $\mathscr{L}_{22}$; $\mathscr{L}_{21}$ with D_{m+4}, $\mathscr{L}_{22}$ with E_8 to $\mathscr{L}_0$. **$m = 4$:** $\mathscr{L}_4$ top, E_7 to $\mathscr{L}_{21}$, A_2 to $\mathscr{L}_{22}$, A_2 to $\mathscr{L}_{23}$; $\mathscr{L}_{21}$ with D_8, $\mathscr{L}_{22}$ with E_8, $\mathscr{L}_{23}$ with E_8 to $\mathscr{L}_0$.
III$_a$ $\begin{smallmatrix} & & & & \underline{2} & \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \end{smallmatrix}$	$\mathscr{L}_4 \xrightarrow{\mathsf{A}_9} \mathscr{L}_2 \xrightarrow{\mathsf{E}_7} \mathscr{L}_0$ (vertical chain)
III$_b$ $\begin{smallmatrix} & & 4 & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{smallmatrix}$	$\mathscr{L}_4 \xrightarrow{\mathsf{D}_{10}} \mathscr{L}_2 \xrightarrow{\mathsf{E}_8} \mathscr{L}_0$ (vertical chain)

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