

Reply to the comments of McMullen et al. (arXiv:2510.04828)

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Abstract McMullen et al. [1] comment that the numerical simulations that *explicitly* include random velocity fluctuations “should exhibit a thermal-fluctuation-dominated range” consistent with the literature, so that our results (J. Fluid Mech. 1008, R2, 2025) [2] “contradict other results in the literature”. First of all, we would give an opposite example against this viewpoint: DNS results (that are badly polluted by numerical noises quickly, as mention in § 2) *implicitly* include random numerical noises, but they also *do not* exhibit a thermal-fluctuation-dominated range. In other words, DNS results in the literature qualitatively agree with ours at this point. In addition, we highly suggest that influences of numerical noises on statistics of turbulent flows given by *all* numerical approaches should be carefully checked, since numerical noises might have huge influences on statistics of chaotic systems (including turbulence), as pointed by Lorenz [3] in 2006. Detailed replies are given below.

Keywords turbulence, chaos, clean numerical simulation

1 Introduction

First of all, we thank all authors (arXiv:2510.04828) [1] for their comments on our paper (J. Fluid Mech. 1008, R2, 2025) [2], since deep discussions and frank exchanges of different viewpoints are helpful for the development of fluid mechanics.

Using a simple model (based on the deterministic Navier-Stokes equations) and clean numerical simulation (CNS) in which artificial numerical noise is negligible over a finite, sufficiently long interval of time, we show evidence that artificial numerical noise in direct numerical simulation (DNS) of Navier-Stokes (NS) turbulence is “approximately equivalent to thermal fluctuation and/or stochastic environmental noise” [2]. Currently, McMullen et al. [1] comment that the numerical simulations that *explicitly* include random velocity fluctuations [2] “should exhibit a thermal-fluctuation-dominated range” consistent with the literature, so that our results [2] “contradict other results in the literature”.

First of all, we would give an opposite example against this viewpoint: DNS results (that are badly polluted by numerical noises quickly, as mention in § 2) *implicitly* include random numerical noises, but they also *do not* exhibit a thermal-fluctuation-dominated range. In other words, DNS results in the literature qualitatively agree with ours at this point.

Our replies are briefly given below .

2 Reliability of numerical simulation of turbulence

In 1890s Poincaré [4] discovered that chaotic system has sensitivity dependence on initial condition, which was named “butterfly-effect” by Lorenz [5] in 1963. More importantly, Lorenz [3] discovered in 2006 that, for a chaotic system, numerical noises have large influences not only on its trajectory but

also on its statistics and characteristics: its maximum Lyapunov exponent invariably alternates between negative and positive values even when the time-step of traditional numerical algorithms becomes rather small. Thus, the basic characteristic of the numerical simulations alternates between chaotic and non-chaotic states, which are fundamentally quite different from each other. A few groups [6–9] confirmed Lorenz’s conclusions and some even pointed out that, “for chaotic systems, numerical convergence *cannot* be guaranteed *forever*” [6], and that “*all* chaotic responses are simply numerical *noises* and have *nothing* to do with differential equations” [7, 8]. These are very negative viewpoints about the reliability of numerical simulations of chaotic systems. They [6–9] illustrated that the *exact* solution of a chaotic system is one thing, but unfortunately its *numerical* simulation might be the other completely different thing, because unavoidable numerical noises increase inexorably to a macro-level due to the butterfly-effect of chaos.

Many researchers reported that Navier-Stokes (NS) turbulence (i.e. turbulence governed by NS equations) are chaotic [10–15]. So, according to the conclusions given by Lorenz [3] and other researchers [6–9], the reliability of numerical simulations (including DNS) of NS turbulence (as a chaotic system) is rather suspect. Thus, we had to answer the following fundamental questions:

- (1) Do numerical noises have huge influences on *spatio-temporal trajectory* of numerical simulations of NS turbulence?
- (2) Are *statistics* of numerical simulations of NS turbulence sensitive to numerical noises?

To gain *reliable* numerical simulation of a chaotic system, Liao [16] proposed the so-called “clean numerical simulation” (CNS) [17–22]. The computational efficiency of CNS has been increased several orders of magnitudes so that some simple NS turbulences can be solved by means of CNS. Briefly speaking, different from traditional numerical algorithms, CNS decreases both of truncation error and round-off error of numerical simulation to such a low level that the numerical noise is much smaller than its “true” solution and thus is negligible in a time-interval $[0, T_c]$ that is long enough for calculating statistics, where T_c is called “critical predictable time”. Note that T_c is a key concept in CNS: results given by CNS are reliable within $t \leq T_c$, but otherwise not. This is essentially different from DNS whose simulation can be *arbitrarily* long if one has enough resources of computation. In fact, one can regard DNS as a special case of CNS, but unfortunately the corresponding T_c of DNS results is too short to calculate statistics. For more details, please see Liao’s book [22].

Using CNS, we can do ‘clean’ numerical experiment of NS turbulence. Comparing CNS results with those given by DNS of the same NS turbulence, one can investigate the influence of numerical noises. It is found [23–25] that

- (A) In all cases, DNS spatio-temporal trajectories of NS turbulence are quickly polluted by numerical noises badly, i.e. numerical noise is mostly at the same order of magnitude as its true solution;
- (B) In most cases, the statistic results of DNS are the same as those given by CNS, indicating the “statistic stability” of these NS turbulences. However, in some cases, numerical noises can lead to large distinctions even in flow type and statistics, indicating that numerical noise sometimes might have large influences in statistics of NS turbulence.

In addition, using CNS to solve a kind of 2D turbulent Kolmogorov flow subject to a specially chosen initial condition that contains micro-level disturbances at different orders of magnitude, Liao and Qin [26] revealed an interesting phenomenon of NS turbulence, called “the noise-expansion cascade”: all micro-level disturbances at different orders of magnitude evolve and grow continuously, step by step, as an

inverse cascade, to reach the macro-level, and each disturbance could greatly change the characteristics of the 2D turbulent Kolmogorov flow[†]. This highly suggests that each disturbance must be considered in the NS turbulence, even if the disturbance is many orders of magnitude smaller than others. Unfortunately, NS turbulence neglects all stochastic disturbances when $t > 0$. This leads to a logic paradox in theory.

Our above-mentioned investigations [23–26] highly suggest that the following three things, i.e.

- (a) exact (or clean) solution of the NS turbulence,
- (b) numerical simulations (that are quickly polluted by numerical noise badly) of the NS turbulence,
- (c) real turbulence in practice,

might be *completely different*. A lot of theoretical, numerical and experimental investigations are needed to reveal the relationships and differences between them. We believe that the same conclusions should hold when the NS turbulence is replaced by other turbulence models, such as fluctuating hydrodynamics (FHD) [27], direct simulation Monte Carlo (DSMC) [28], molecular dynamics (MD) [29], and so on, since turbulence should be chaotic in essence and numerical noises are unavoidable for all numerical approaches. Like DNS, these numerical approaches use double precision and/or low order algorithms so that numerical noise should quickly increase to the same order of magnitude as true solution. Thus, based on our experience [23–26] on NS turbulence, it is *questionable* whether or not these numerical simulations agree with ‘true’ solution of the corresponding mathematical model in statistics. So, we highly suggest that the influences of numerical noises on statistic results given by *all* numerical approaches (including FHD, DSMC, MD and so on) should be checked *very carefully*, before they can be used as “benchmark solution”.

McMullen et al. [1] comment that the numerical simulations that *explicitly* include random velocity fluctuations “should exhibit a thermal-fluctuation-dominated range” consistent with the literature, so that our results [2] “contradict other results in the literature”. Here, we would give an opposite example against this viewpoint: DNS results (that are badly polluted by numerical noises quickly, as mention above) *implicitly* include random numerical noises, but they also *do not* exhibit a thermal-fluctuation-dominated range, as mentioned in the literature. This qualitatively agrees with our results at this point.

Indeed, our results “contradict other results in the literature”. We would like to emphasize here that our results are ‘clean’ and thus reliable, but the influences of numerical noises on the results in the literature have not been checked very carefully. Certainly, more investigations are necessary in future so as to give a sound conclusion.

3 Model for environmental noise and thermal fluctuation

The motivation of our paper [2] is to find some *relationships* between *artificial* numerical noise and *physical* disturbances such as environmental noise and/or thermal fluctuation.

Until now, CNS has been successfully applied *only* to deterministic equations. Thus, in order to gain a ‘clean’ simulation with negligible numerical noises, the deterministic NS turbulence is used in [2], since it as one millennium problem [30] is widely used by turbulence community. Note that the NS turbulence

[†]The related code of CNS of the NS turbulence and some movies can be downloaded via GitHub (<https://github.com/sjtu-liao/2D-Kolmogorov-turbulence>).

itself does not include the stochastic terms about thermal fluctuation and/or environmental noise that can “enter via a random stress tensor that is constrained to satisfy the fluctuation-dissipation relation”. So, we propose in [2] such a *new* model for influence of thermal fluctuation and/or environmental noise: the simulation at $t_{n+1} = (n + 1)\Delta t$ is gained by CNS (with negligible numerical noises) using the NS equations, which is then modified at t_{n+1} by *suddenly* adding a stochastic velocity field, where the random field is taken to be Gaussian white noise with zero mean and standard deviation σ so as to include thermal fluctuations and/or environmental noises. Since the evolution from t_n to t_{n+1} is governed by the deterministic NS turbulence and the stochastic velocity field is *suddenly* added at t_{n+1} , this model is essentially deterministic. Therefore, the numerical algorithms for stochastic differential equations are *unnecessary* for our model adapted in [2].

It is true that the thermal noise strength used in [2] is “many orders of magnitude smaller than what is representative of any physically realizable turbulent flow”. But this is not important and does not change our conclusions reported in [2], since, due to the so-called “noise-expansion cascade” [26], the thermal noise strength quickly increases to an order of magnitude having physical meaning.

Note that NS turbulence neglects all physical stochastic disturbances when $t > 0$. So, it is not an ideal model for thermal fluctuation. Hopefully, one can solve Landau-Lifshitz-Navier-Stokes (LLNS) equations [31] (which consider the influence of thermal fluctuation) by means of CNS in the near future. This is the reason why in [2] we made such a conclusion that numerical noise might be *approximately* equivalent to thermal fluctuation or/and environmental noise.

4 Concluding remarks

Our replies are briefly given below:

- (I) McMullen et al. [1] mentioned that our numerical simulations [2] that *explicitly* include random velocity fluctuations “should exhibit a thermal-fluctuation-dominated range” consistent with the literature. However, as mention in § 2, DNS results *implicitly* include random numerical noises, but they also *do not* exhibit a thermal-fluctuation-dominated range. In other words, DNS results in the literature qualitatively *agrees* with ours in [2] at this point.
- (II) In addition, statistic results in the literature are based on numerical simulations of chaotic systems, which, similar to DNS results, might be badly polluted by numerical noises, too. So, according to our experiences on NS turbulence [23–26], we highly suggest that the influences of numerical noises on statistic results given by *all* approached (including FHD, DSMC, MD and so on) should be checked *very carefully*, before they can be used as ‘benchmark solution’.
- (III) We propose in [2] a *new* simple model for influence of thermal fluctuation and/or environmental noise. This model is *deterministic* in essence, and thus is *unnecessary* to use numerical techniques for stochastic differential equations. Hopefully, LLNS equations can be solved by CNS in future, since it includes thermal fluctuation.

According to our experiences on NS turbulence [23–26], we might artificially add a non-negligible influence on turbulent flows when we investigate them by numerical and/or experimental approaches, because turbulent flows are essentially chaotic. Thus, it is very important for us to know the conditions under which these tiny stochastic disturbances are negligible or non-negligible, the corresponding turbulent flows have statistic stability (or instability), and so on. Hopefully, the above discussions could attract more attentions of turbulence community to investigate *the relationships and differences*

between the mathematical turbulence models, their numerical simulations, and real turbulent flows in practice [25].

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