

Ghost stabilisation for cut finite element exterior calculus

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Abstract

We introduce the cut finite element method in the language of finite element exterior calculus, by formulating a stabilisation – for any form degree – that makes the method robust with respect to the position of the interface relative to the mesh. We prove that the L^2 -norm on the physical domain augmented with this stabilisation is uniformly equivalent to the L^2 -norm on the “active” mesh that contains all the degrees of freedom of the finite element space (including those external to the physical domain). We show how this CutFEEC method can be applied to discretize the Hodge Laplace equations on an unfitted mesh, in any dimension and any topology. A numerical illustration is provided involving a conforming finite element space of H^{curl} posed on a filled torus, with convergence and condition number scaling independent of the position of the boundary with respect to the background mesh.

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1 Introduction

A successful strand of the literature on the numerical approximation of partial differential equations (PDEs) has emphasized the relevance of Hilbert complexes [10] in the design of stable numerical schemes. In this context, differential forms provide a natural unifying language to treat mixed problems set in arbitrary space dimension, possibly on manifolds, and where a combination of differential operators appear [4, 5, 6, 9, 16, 21]. Finite Element Exterior Calculus (FEEC), particularly relevant for the present work, provides a comprehensive view of full and trimmed finite element approximations of the de Rham complex; see the monograph [3] for an introduction. At the same time, the Cut Finite Element Method (CutFEM) has emerged as a versatile technique for handling PDEs on domains with complex geometries, removing the need for the mesh to be compliant with the domain’s boundary; see the recent review [13] and references therein. Through weak imposition of boundary conditions and carefully constructed stabilisation terms over facets close to the boundary, CutFEM leads to stable and well-conditioned discretisations. Moreover, such properties hold independently of the boundary position relative to the unfitted mesh. This work aims at merging these research avenues by proposing a Cut Finite Element Exterior Calculus (CutFEEC) framework that adapts the CutFEM techniques to FEEC. This merger has notably been made possible by the recent extension of the CutFEM technology to mixed problems [17, 18, 19], which are a natural application of FEEC [3].

The prime example of a Hilbert complex is the de Rham complex, which, for an open connected domain $\Omega \subset \mathbb{R}^3$, reads

$$0 \hookrightarrow H^1(\Omega) \xrightarrow{\nabla} \mathbf{H}^{\text{curl}}(\Omega) \xrightarrow{\text{curl}} \mathbf{H}^{\text{div}}(\Omega) \xrightarrow{\text{div}} L^2(\Omega) \rightarrow 0.$$

Using the language of differential forms, the de Rham complex can be extended to a domain Ω of any dimension $n \in \mathbb{N}$ as:

$$0 \hookrightarrow H\Lambda^0\Omega \xrightarrow{d^0} H\Lambda^1\Omega \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} H\Lambda^n\Omega \rightarrow 0, \quad (1.1)$$

where d^k is the exterior derivative and $H\Lambda^k\Omega$ is the space of L^2 -integrable differential k -forms on Ω with L^2 -integrable exterior derivative. The de Rham complex enters the well-posedness analysis of PDEs through its cohomology spaces

$$\mathfrak{H}^k := \text{Ker } d^k / \text{Im } d^{k-1}.$$

These spaces relate to the topology of the domain via their dimension. Preserving these homological structures at the discrete level is crucial for the stability of numerical schemes. A paradigmatic example of PDE problem whose well-posedness hinges on the de Rham complex is the Hodge Laplace equation; see, e.g., [3, Chapter 4]. Its mixed formulation naturally leads to a mixed problem where the unknowns are the codifferential, the (scalar- or vector-) potential, and a Lagrange multiplier to enforce L^2 -orthogonality with respect to harmonic forms.

The Finite Element (FE) approximation of mixed problems became a research topic starting from the late 1970s, when the first conforming approximation of vector-valued spaces in the de Rham complex were developed [7, 24, 26]. Traditional FE methods, such as the ones cited above, require meshes that conform to the geometry of the domain. These can be challenging to generate for intricate geometries or in the presence of evolving interfaces. CutFEM addresses this problem by permitting the interface to cut through the elements of a background mesh, thereby simplifying mesh generation and adaptation [12]. However, ensuring stability and accuracy in such unfitted methods necessitates appropriate stabilisation techniques [11]. Otherwise, when the geometry cuts the mesh in especially nasty ways ($|T \cap \Omega| \ll |T|$ for an element T in the mesh), one observes a significant degradation of the condition number of the associated linear system. Recent works have addressed the application of CutFEM techniques to mixed problems. In [17, 18], the authors introduced a stable CutFEM for Darcy flow problems which, thanks to conformity with respect to the tail end of the de Rham complex, preserves the divergence of the velocity field. Using a similar approach, one can also preserve the divergence condition of Stokes flow in an unfitted setting [19].

In this work, we develop a new CutFEM stabilisation to design unfitted robust discrete L^2 -products for any space in the de Rham complex of differential forms. The stabilisation can be seen as a generalisation of the mixed ghost penalty term introduced in [17, 19, 25]. Orthogonalising against the harmonic forms using an L^2 -product including this stabilisation, we construct an arbitrary-order numerical method which is unfitted but still compatible with the continuous de Rham complex, thus ensuring its natural stability and preservation of relevant constraints (e.g., zero divergence of electric field in Maxwell's equations). We develop facet-based stabilisation terms, but we mention that there exist other equivalent stabilisation terms in the CutFEM literature, though these have not been developed for differential forms, see [13, Section 4.2]. We note that the commonly used assumption of a quasi-uniform mesh [13] is not needed in our construction or analysis. It is sufficient for the relevant arguments to use a local quasi-uniformity, which does not entail particular restrictions on the mesh. In [23], the quasi-uniform assumption is used but a similar remark is made [20, Remark 2.3].

A particularly relevant novelty results from applying this general construction to the special case of 1-forms, which is key to discretize problems such as Maxwell's equations or the H^{curl} -elliptic problem. CutFEM has, indeed, seen little development for problems involving H^{curl} ; see the recent and exhaustive review [13]. There are some previous works which use an unfitted discontinuous Galerkin approach [14, 22]. Otherwise, works which do not use discontinuous/broken finite element spaces are essentially limited to [19, 27, 28][25, Paper D]. Of these, [19] and [25, Paper D] do not contain a stability analysis; [19] considers a vorticity-velocity-pressure formulation for Stokes flow wherein the vorticity is discretized by a H^{curl} -conforming variable, while [25, Paper D] considers numerical experiments of a set of standard formulations of Maxwell's equations made unfitted. The works [27, 28] focus, respectively, on the time-harmonic Maxwell's equations and the H^{curl} -elliptic interface problem. In [28, Remark 3], the authors mention that their proposed method works numerically when using nodal Lagrange elements, but error analysis is provided only for the case of using piecewise discontinuous polynomials. The numerical scheme in [28] uses the lowest order H^{curl} -conforming Nédélec elements of the second kind, but adds diagonal stabilisation terms in the Lagrange multiplier block [27, (4.8)–(4.11)], thus perturbing the saddle point nature of the problem. Similar stabilisation terms also appear in [28]. Moreover, in contrast to our approach, no work mentioned

above considers the presence of non-trivial harmonic forms, which appear when considering domains of general topology.

The remainder of this paper is organised as follows. In Section 2 we discuss a motivating example of mixed problem, the Hodge Laplace problem, whose well-posedness relies on the properties of the de Rham complex, and we summarise some notions of FEEC. In Section 3 we recall the notion of unfitted mesh, and we then introduce in Section 4 the concepts of tangential and normal parts of differential forms, relative to a facet of the mesh. In Section 5 we introduce the novel ghost penalty stabilisation operator, and show the uniform equivalence between the L^2 -norm on the physical domain and the L^2 -norm on the extended domain containing all the degrees of freedom. Finally, in Section 6 we validate our results on a numerical example on the filled torus, demonstrating that the use of the stabilisation drastically reduces the condition number of the discrete system.

2 A motivating example

In this section we briefly discuss the Hodge Laplacian, an archetypal problem whose well-posedness hinges on the properties of the de Rham complex. This example serves as a motivation for the development of a CutFEEC theory, and will be used in Section 6 to numerically demonstrate the efficiency of the ghost penalty stabilisation of Section 5.

2.1 Hodge Laplace equation

Let $B \subset \mathbb{R}^n$ be an open set. We denote by $\Lambda^k B$ the space of smooth differential k -forms on B . We introduce the L^2 -product $(\omega, \zeta)_B := \int_B \omega \wedge \star \zeta$ on $\Lambda^k B$, where $\wedge$ denotes the wedge product and $\star$ the Hodge star operator [3, (6.2)]. The L^2 -norm over B is denoted by $\|\bullet\|_B := \sqrt{(\cdot, \cdot)_B}$. We denote by $L^2 \Lambda^k B$ the space of k -forms over B that are bounded with respect to this norm. With the exterior derivative $d^k : L^2 \Lambda^k B \rightarrow L^2 \Lambda^{k+1} B$, viewed as an unbounded operator, we also define

$$H\Lambda^k B := \left\{ \omega \in L^2 \Lambda^k B : d^k \omega \in L^2 \Lambda^{k+1} B \right\},$$

with norm

$$\|\omega\|_{HB}^2 := \|\omega\|_B^2 + \|d^k \omega\|_B^2.$$

Let now $\Omega \subset \mathbb{R}^n$ be a connected and open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and outward unit normal vector field $\mathbf{n}$. Given $f \in L^2 \Lambda^k \Omega$, the Hodge Laplace equation consists in seeking $\eta \in \Lambda^k \Omega$ such that

$$\begin{aligned} (d^{k-1} \delta^k + \delta^{k+1} d^k) \eta &= f - \pi f && \text{in } \Omega, \\ \pi \eta &= 0 && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where the codifferential $\delta^{k+1} : L^2 \Lambda^{k+1} \Omega \rightarrow L^2 \Lambda^k \Omega$ is the adjoint of d and $\pi := \pi_{\mathcal{H}^k}$ is the orthogonal projection onto the space of harmonic forms

$$\mathcal{H}^k := \left\{ \rho \in L^2 \Lambda^k \Omega : d^k \rho = 0 \text{ and } \delta^k \rho = 0 \right\} = \text{Ker } d^k \cap \text{Ker } \delta^k \cong \text{Ker } d^k / \text{Im } d^{k-1}.$$

Here $\cong$ means vector space isomorphism. The boundary conditions are implicit in the definition of the Hodge Laplace operator $d^{k-1} \delta^k + \delta^{k+1} d^k$, and are revealed upon choosing the domain of the exterior derivative. For ease of presentation, let us in this work choose the domain

$$D(d^k) = H\Lambda^k \Omega,$$

so that the domain of the codifferential becomes (by duality) the space of $L^2(\Omega)$ -integrable forms ω with $\delta \omega \in L^2(\Omega)$ and $\gamma(\star \omega) = 0$. After recalling vector proxies in three space dimensions in Table 2.1, in Table 2.2,

k	Form	Proxy	Sobolev space
0	q	q	$H^1(\Omega)$
1	$v_1 dx^1 + v_2 dx^2 + v_3 dx^3$	$\mathbf{v} = (v_1, v_2, v_3)$	$\mathbf{H}^{\text{curl}}(\Omega)$
2	$w_1 dx^2 \wedge dx^3 - w_2 dx^1 \wedge dx^3 + w_3 dx^1 \wedge dx^2$	$\mathbf{w} = (w_1, w_2, w_3)$	$\mathbf{H}^{\text{div}}(\Omega)$
3	$r dx^1 \wedge dx^2 \wedge dx^3$	r	$L^2(\Omega)$

Table 2.1: Vector proxies in $\mathbb{R}^3$.

$k = 0$	$k = 1$	$k = 2$	$k = 3$
$-\Delta q = f - \tilde{f},$ $\nabla q \cdot \mathbf{n} = 0,$ $\tilde{q} = 0$	$\text{curl curl } \mathbf{v} - \nabla \text{div } \mathbf{v} = \mathbf{f} - \pi \mathbf{f},$ $\mathbf{v} \cdot \mathbf{n} = 0 \text{ and } \text{curl } \mathbf{v} \times \mathbf{n} = 0,$ $\mathbf{v} \perp \mathfrak{H}^1$	$\text{curl curl } \mathbf{w} - \nabla \text{div } \mathbf{w} = \mathbf{f} - \pi \mathbf{f},$ $\mathbf{w} \times \mathbf{n} = 0 \text{ and } \text{div } \mathbf{w} = 0,$ $\mathbf{w} \perp \mathfrak{H}^2$	$-\Delta r = f,$ $r = 0$

Table 2.2: The Hodge Laplace equation in $\mathbb{R}^3$ for different differential form orders. Entries are labeled A , B , C where A are the bulk equation(s) in Ω , B the boundary condition(s) on $\partial\Omega$, and C is the cohomological compatibility condition.

we express problem (2.1) in terms of vector proxies for different values of k ; cf. [5, Section 4.2] for further details.

For convenience of notation, we shall for the remainder omit the index and just write d for the exterior derivative. A variational mixed formulation of (2.1) is (cf. [3, Theorem 4.7]): Find $(\sigma, \eta, \lambda) \in H\Lambda^{k-1}\Omega \times H\Lambda^k\Omega \times \mathfrak{H}^k$ such that

$$\begin{aligned}
(\sigma, \tau)_\Omega - (\eta, d\tau)_\Omega &= 0 & \forall \tau \in H\Lambda^{k-1}\Omega, \\
(d\sigma, \zeta)_\Omega + (d\eta, d\zeta)_\Omega + (\lambda, \zeta)_\Omega &= (f, \zeta)_\Omega & \forall \zeta \in H\Lambda^k\Omega, \\
(\eta, \rho)_\Omega &= 0 & \forall \rho \in \mathfrak{H}^k.
\end{aligned} \tag{2.2}$$

The above formulation enforces in a weak manner $\sigma = \delta\eta$ and $\lambda = \pi f$. The (natural) boundary conditions are $\gamma \star \eta = 0$ and $\gamma \star d\eta = 0$.

Using crucial properties of the de Rham complex such as Hodge decompositions and Poincaré inequalities, it can be proved that this mixed formulation is well-posed [3, Theorems 4.8 and 4.9]. Preserving these properties at the discrete level, as is the case for the compatible discretisations considered here, naturally leads to numerical schemes for which stability can be proved mimicking the arguments for well-posedness of the continuous problem.

2.2 Principles of FEEC

Assume here that Ω is a polytopal domain, and let $\mathcal{T}_h$ be a conforming simplicial mesh of Ω . The Finite Element Exterior Calculus (FEEC) method provides the tools for a discrete formulation of (2.2) which is well-posed [3, (5.3), Section 5.2.3], but non-conforming in general due to the non-inclusion of the discrete space of harmonic forms into the continuous one (these spaces have the same dimension, so the former being included in the latter would actually mean that one could compute harmonic forms exactly). We make a brief summary of the results which pertain to this work.

Let $r \geq 1$ be a polynomial degree. For each $k \in \{0, \dots, n\}$, let $V_h^{k,r} \subset H\Lambda^k\Omega$ be a finite-dimensional space of k -forms that are piecewise polynomial of degree at most r on $\mathcal{T}_h$ and such that $dV_h^{k,r} \subset V_h^{k+1,r-1}$. We adopt the convention that $V_h^{-1,r} := \{0\}$ and $V_h^{n+1,r} := \{0\}$. These spaces define a sub-complex of the de Rham complex:

$$0 \hookrightarrow V_h^{0,r} \xrightarrow{d} V_h^{1,r-1} \xrightarrow{d} \dots \xrightarrow{d} V_h^{n,r-n} \longrightarrow 0. \tag{2.3}$$

It can be shown [3], using, e.g., bounded cochain maps between the continuous and discrete de Rham complexes, that the cohomology spaces of (2.3) are isomorphic to those of (1.1). This diagram (in which the polynomial degree decreases along the sequence) corresponds to the case of full finite element spaces, $V_h^{k,r} := P_r \Lambda^k \Omega_h$. When using trimmed spaces, $P_r^- \Lambda^k \Omega_h \subset P_r \Lambda^k \Omega_h$ in the notation of [3], the degree r remains the same along the diagram. For application of our results to the trimmed spaces, simply replace any occurrence of $V_h^{k,\bullet}$ with $P_r^- \Lambda^k \Omega_h$.

The discrete exterior derivative acting on a discrete space is just the restriction of the exterior derivative from the corresponding continuous space. When relevant, we shall denote it by $d_h := d|_{V_h^{k,r}}$, otherwise by d when there is no confusion. Then, one defines the space of discrete harmonic forms as

$$\mathfrak{H}_h^k := \text{Ker } d_h / dV_h^{k-1,r+1} \cong \left\{ \rho_h \in V_h^{k,r} : d\rho_h = 0 \text{ and } (\rho_h, d\tau_h)_\Omega = 0 \text{ for all } \tau_h \in V_h^{k-1,r+1} \right\}. \quad (2.4)$$

The non-inclusion $\mathfrak{H}_h^k \not\subset \mathfrak{H}^k$ is due to forms in $\mathfrak{H}_h^k$ needing to be orthogonal to all of $dH\Lambda^{k-1}\Omega$, while being in $\mathfrak{H}_h^k$ only guarantees orthogonality to the subspace $dV_h^{k-1,r+1}$.

As mentioned in Section 2.1, the Hodge decomposition of the complex spaces is an essential tool to analyse both the continuous Hodge-Laplace equation and its numerical discretisations. For the complex (2.3), we have the following discrete Hodge decomposition:

$$V_h^{k,r} = (\text{Ker } d_h)^\perp \oplus dV_h^{k-1,r+1} \oplus \mathfrak{H}_h^k. \quad (2.5)$$

Given $f \in L^2 \Lambda^k \Omega$ the FEEC scheme for problem (2.2) reads: Find $(\sigma_h, \eta_h, \lambda_h) \in V_h^{k-1,r+1} \times V_h^{k,r} \times \mathfrak{H}_h^k$ such that

$$\begin{aligned} (\sigma_h, \tau_h)_\Omega - (\eta_h, d\tau_h)_\Omega &= 0 & \forall \tau_h \in V_h^{k-1,r+1}, \\ (d\sigma_h, \zeta_h)_\Omega + (d\eta_h, d\zeta_h)_\Omega + (\lambda_h, \zeta_h)_\Omega &= (f, \zeta_h)_\Omega & \forall \zeta_h \in V_h^{k,r}, \\ (\eta_h, \rho_h)_\Omega &= 0 & \forall \rho_h \in \mathfrak{H}_h^k. \end{aligned}$$

The scheme is well-posed and optimally convergent in the $H\Lambda^k\Omega$ -norm [3, Theorem 5.5].

3 Unfitted mesh

Let $\mathcal{T}_h = \{T\}$ be an *active mesh* such that each (open) simplex $T \in \mathcal{T}_h$ has a nonempty intersection with Ω , and for which

$$\Omega \subset \Omega_h := \left(\bigcup_{T \in \mathcal{T}_h} \overline{T} \right)^\circ.$$

The open set Ω_h is hereafter referred to as the active domain.

Remark 1 (Construction of active mesh). The term active mesh refers to the following construction. Define a background domain $\Omega_0 \supset \Omega$, which is polytopal of dimension n . Consider a conforming triangular mesh $\mathcal{T}_{0,h}$ of Ω_0 . Then, the active mesh is defined as $\mathcal{T}_h := \{T \in \mathcal{T}_{0,h} : T \cap \Omega \neq \emptyset\}$. This construction is illustrated in Figure 3.1.

We say $T \in \mathcal{T}_h$ is *cut* if $T \not\subset \Omega$ and *fully immersed* in Ω otherwise. We denote the set of cut elements by

$$\mathcal{T}_h^{\text{cut}} := \{T \in \mathcal{T}_h : T \not\subset \Omega\}.$$

We denote by $\mathcal{F}_h^\partial$ the set of stabilisation facets [19, Figure 1], which are facets internal to Ω_h for which at least one neighbouring element is intersected by $\partial\Omega$:

$$\mathcal{F}_h^\partial := \left\{ F \in \bigcup_{T \in \mathcal{T}_h^{\text{cut}}} \mathcal{F}_T : F \not\subset \partial\Omega_h \right\},$$

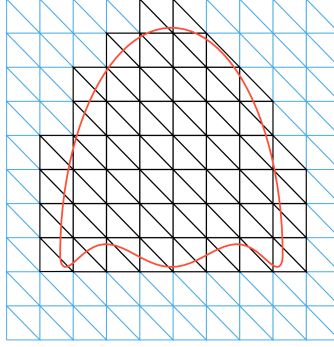


Figure 3.1: Illustration of the active mesh $\mathcal{T}_h$. The boundary $\partial\Omega$ of the domain is in red. The “remainder” of background mesh $\mathcal{T}_{0,h} \setminus \mathcal{T}_h$ is shown in light blue, while the active mesh $\mathcal{T}_h$ is shown in black.

where $\mathcal{F}_T$ gathers the $((n-1)$ -dimensional) facets of $T \in \mathcal{T}_h$. To each facet we associate a unique normal vector $\mathbf{n}$. The context will make it clear which normal is meant, that of a facet or that of $\partial\Omega$.

Assumption 1 (Mesh regularity). For any $T \in \mathcal{T}_h$, let ρ_T and h_T be, respectively, the diameter of the largest ball contained in T and that of T . We assume the following:

- (i) (Shape regularity) For some constant $\kappa > 0$ independent of the mesh size $h := \max_{T \in \mathcal{T}_h} h_T$, we have

$$\frac{h_T}{\rho_T} \leq \kappa \quad \forall T \in \mathcal{T}_h. \quad (3.1)$$

- (ii) (Uniformly bounded cut-to-uncut path) There exists an integer $N > 0$ independent of h such that, for all $T \in \mathcal{T}_h^{\text{cut}}$, there exists a sequence of $N_T \leq N$ elements $\{T_1 = T, T_2, \dots, T_{N_T}\}$ such that T_i and T_{i+1} share a facet for all $1 \leq i \leq N_T - 1$, and $T_{N_T} \in \mathcal{T}_h \setminus \mathcal{T}_h^{\text{cut}}$ is fully immersed in Ω .

The notation $a \approx b$ means that there exist constants $C, C' > 0$ independent of h , but possibly dependent on κ and N such that $a \leq Cb$ and $b \leq C'a$, and similarly we write $a \lesssim b$ if and only if $a \leq Cb$.

Remark 2 (Quasi-uniformity along cut-to-uncut paths). We remark that, by shape regularity and since N is bounded independently of h , all mesh elements along a given cut-to-uncut path have uniformly comparable diameters: if $T_1, \dots, T_{N_T}$ are as in (ii), then $h_{T_i} \approx h_{T_j}$ for all $i, j \in \{1, \dots, N_T\}$.

4 Tangential and normal traces and parts of differential forms

Restrictions as well as tangential and normal traces and parts of differential forms are essential concepts to define the stabilisation in Section 5 below. We consider here an element $T \in \mathcal{T}_h$ and one of its facets $F \subset \partial T$.

For $x \in F$, let $\mathbb{T}_x C$ be the tangent space of $C \in \{F, T\}$ at x . Let

$$\iota : F \hookrightarrow T$$

be the inclusion map of F into T . We note that $\mathbb{T}_x F$ is the vector hyperplane parallel to F in $\mathbb{R}^n$, and that $X \in \mathbb{T}_x F$ can be considered as a vector in $\mathbb{R}^n$; for this reason, we identify $\iota_* X$ with X .

For $\omega \in \Lambda^k T$, by “restriction to the facet F ” we simply mean the restriction of each component of ω to F , i.e., writing $\omega := \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \omega_{j_1 j_2 \dots j_k} dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_k}$, we define

$$\omega|_F := \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} (\omega_{j_1 j_2 \dots j_k}|_F) dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_k}. \quad (4.1)$$

Notice that the result is a form which formally acts on k -tuples of vectors in T , although its components will only be evaluated on F . The reason is that we want $\omega|_F$ as a differential form to also be able to act on vectors normal to F .

The standard trace operator $\gamma := \iota^* : \Lambda^k T \rightarrow \Lambda^k F$ on the facet is defined as the pullback of the inclusion ι : For all $\omega \in \Lambda^k T$,

$$\forall x \in F, \quad (\gamma\omega)_x(X_1, \dots, X_k) := \omega_x(X_1, \dots, X_k) \quad \forall X_1, \dots, X_k \in \mathbb{T}_x F. \quad (4.2)$$

Recall the Hodge star operator $\star : \Lambda^k T \rightarrow \Lambda^{n-k} T$. We can also define a Hodge star $\star_F : \Lambda^k F \rightarrow \Lambda^{n-1-k} F$ on F satisfying $\star_F \star_F \omega = (-1)^{k(n-1-k)} \omega$ for all $\omega \in \Lambda^k F$.

The normal trace operator is defined by “sandwiching” the standard trace between the Hodge star operators on T and on F , see [8, Eqs. (45) and (46)]: $\gamma_n := (-1)^{n(k-1)} \star_F \gamma \star : \Lambda^k T \rightarrow \Lambda^{k-1} F$ is therefore such that, for all $\omega \in \Lambda^k T$,

$$\forall x \in F, \quad (\gamma_n \omega)_x(X_2, \dots, X_k) := \omega_x(n, X_2, \dots, X_k) \quad \forall X_2, \dots, X_k \in \mathbb{T}_x F. \quad (4.3)$$

The latter expression is, by definition, the contraction $n \lrcorner \omega$ of ω . The following identities hold, [8, Eq. (47)]:

$$\star_F \gamma_n = \gamma \star, \quad \gamma_n \star = (-1)^k \star_F \gamma. \quad (4.4)$$

From now on, we shall denote both $\star$ and $\star_F$ by $\star$, the context making it clear which one is meant. The vector proxy interpretation of the trace and normal trace operators are summarized in Table 4.1.

Form degree k	Proxy	Trace (γ)	Normal trace (γ_n)
0	q	q	0
1	v	$n \times (v \times n)$	$v \cdot n$
2	w	$w \cdot n$	$-n \times w$
3	r	0	r

Table 4.1: Trace and normal trace for vector proxies in $\mathbb{R}^3$.

Sometimes, the standard trace is called the tangential trace, in contrast to the normal trace. This nomenclature can be understood comparing the traces with the restriction for sufficiently smooth differential forms. Consider for the remainder of this section a smooth k -form $\omega \in \Lambda^k T$. Using the pullback of the orthogonal projection $\pi_F : T \rightarrow F$, we can define the *tangential part* of ω on F by setting

$$\omega_{\parallel} := (\pi_F^* \circ \gamma) \omega. \quad (4.5)$$

At any $x \in F$, it is also defined as an alternating form in $\text{Alt}^k \mathbb{R}^n$. Setting then

$$\omega_{\perp} := \omega|_F - \omega_{\parallel}, \quad (4.6)$$

we have the following decomposition:

$$\omega|_F = \omega_{\parallel} + \omega_{\perp}. \quad (4.7)$$

Lemma 1 (Characterisation of the tangential and normal parts). *Let $\omega \in \Lambda^k T$ be a smooth k -form. Then the following holds:*

$$\omega_{\parallel} = 0 \iff \gamma \omega = 0, \quad (4.8)$$

$$\omega_{\perp} = 0 \iff \gamma_n \omega = 0. \quad (4.9)$$

Moreover,

$$(\star \omega)_{\parallel} = \star \omega_{\perp} \quad \text{and} \quad (\star \omega)_{\perp} = \star \omega_{\parallel}. \quad (4.10)$$

Proof. 1) *Proof of (4.8).* We start by showing that $\omega_{\parallel} = 0$ if and only if $\gamma\omega = 0$. It is clear by its definition (4.5) that $\omega_{\parallel} = 0$ if $\gamma\omega = 0$. To prove the converse, we just note that, if $(\omega_{\parallel})_x = 0$, then, for all $X_1, \dots, X_k \in \mathbb{T}_x F \subset \mathbb{R}^n$, we have

$$(\gamma\omega)_x(X_1, \dots, X_k) = (\gamma\omega)_{\pi_F(x)}(D\pi_F(X_1), \dots, D\pi_F(X_k)) = (\omega_{\parallel})_x(X_1, \dots, X_k) = 0,$$

the first equality following from the fact that $D\pi_F$ is the orthogonal projection on $\mathbb{T}_x F$.

2) *Proof of (4.9).* Let $Y_1, \dots, Y_k \in \mathbb{R}^n$. For an appropriately chosen coordinate system with respect to F , each vector Y_j can be written as $Y_j = X_j + Y_j^n$, where $X_j = D\pi_F Y_j \in \mathbb{T}_x F$ and Y_j^n is the normal component of Y_j . We then have

$$\begin{aligned} (\omega_{\perp})_x(Y_1, \dots, Y_k) &\stackrel{(4.6), (4.5)}{=} \omega_x(Y_1, \dots, Y_k) - (\pi_F^* \circ \gamma)\omega_x(Y_1, \dots, Y_k) \\ &\stackrel{(4.2)}{=} \omega_x(Y_1, \dots, Y_k) - \omega_x(X_1, \dots, X_k) \\ &= [\omega_x(Y_1^n, X_2, \dots, X_k) + \dots + \omega_x(X_1, \dots, X_{k-1}, Y_k^n) + \omega_x(X_1, \dots, X_k)] - \omega_x(X_1, \dots, X_k) \\ &= \omega_x(Y_1^n, X_2, \dots, X_k) + \dots + \omega_x(X_1, \dots, X_{k-1}, Y_k^n), \end{aligned}$$

where, in the third step, we have used multiple times the fact that, for any $x \in F$, it holds $\omega_x(X_1, \dots, Y_j^n, \dots, Y_k) = \omega_x(X_1, \dots, Y_j^n, \dots, X_k)$ since ω_x is k -linear and alternating. The above relation shows that there exists permutations σ_j , $1 \leq j \leq k$, of the indices $1, \dots, k$ such that, taking c_j such that $Y_j^n = c_j \mathbf{n}$,

$$(\omega_{\perp})_x(Y_1, \dots, Y_k) = \sum_{j=1}^k c_{\sigma_j(k)} (\gamma \mathbf{n} \omega)_x(X_{\sigma_j(1)}, \dots, X_{\sigma_j(k-1)}).$$

This relation shows that $\gamma \mathbf{n} \omega = 0$ implies $\omega_{\perp} = 0$. To prove the converse statement, we observe that, for any $x \in F$ and any vectors $X_2, \dots, X_k \in \mathbb{T}_x F$, since $D\pi_{\mathbb{T}_x F} \mathbf{n} = 0$, we have $(\omega_{\parallel})_x(\mathbf{n}, X_2, \dots, X_k) = 0$ and thus, by (4.6), $(\omega_{\perp})_x(\mathbf{n}, X_2, \dots, X_k) = \omega_x(\mathbf{n}, X_2, \dots, X_k) = (\gamma \mathbf{n} \omega)_x(X_2, \dots, X_k)$. This shows that $\omega_{\perp} = 0$ implies $\gamma \mathbf{n} \omega = 0$. By (4.4), (4.9) also implies that $\gamma(\star \omega) = 0$ if and only if $\omega_{\perp} = 0$, as is stated in [4, Comments after (2.9)].

3) *Proof of (4.10).* Note that $\star \omega = \star \omega_{\parallel} + \star \omega_{\perp} = (\star \omega)_{\parallel} + (\star \omega)_{\perp}$. We have that

$$(\star \omega)_{\parallel} = \star \omega_{\perp}, \quad (\star \omega)_{\perp} = \star \omega_{\parallel},$$

by applying vectors in $\mathbb{R}^n$, respectively tangential and normal to F , to the equation $\star \omega_{\parallel} + \star \omega_{\perp} - (\star \omega)_{\parallel} - (\star \omega)_{\perp} = 0$. $\square$

It can be helpful to look at the decomposition of a k -form $\omega \in \Lambda^k T$ in a coordinate system adapted to the facet F . Fix any $x \in F$. Let $\{\mathbf{n}, e_1, \dots, e_{n-1}\}$ be a basis of $\mathbb{R}^n$ such that $\{e_1, \dots, e_{n-1}\}$ is a basis of $\mathbb{T}_x F$. Let the dual basis be $\{dt, dx^1, \dots, dx^{n-1}\}$, where dt is the dual to $\mathbf{n}$. Define the multi-index set

$$\mathcal{J}_{\ell} := \{(i_1, \dots, i_{\ell}) : 1 \leq i_1 < i_2 < \dots < i_{\ell} \leq n-1\}.$$

Then, for any $x \in F$, we can write

$$(\omega)_x = (\omega_{\perp})_x + (\omega_{\parallel})_x = \sum_{J \in \mathcal{J}_{k-1}} \omega_{t,J}(x) dt \wedge dx^J + \sum_{I \in \mathcal{J}_k} \omega_I(x) dx^I, \quad (4.11)$$

where $\omega_{t,J}$ and ω_I are coefficients with respect to the chosen basis. One can check (upon identifying $\gamma dx^i = dx^i$) that

$$(\gamma \omega)_x = \sum_{I \in \mathcal{J}_k} \omega_I(x) dx^I \quad \text{and} \quad (\gamma \mathbf{n} \omega)_x = \sum_{J \in \mathcal{J}_{k-1}} \omega_{t,J}(x) dx^J. \quad (4.12)$$

These relations shed more light on the relations stated in Lemma 1. They are also useful for the next result.

Lemma 2. *Let $\omega \in \Lambda^k T$ and $\eta \in \Lambda^{n-k} T$. The following relations hold at any $x \in F$:*

$$\omega_{\parallel} \wedge \eta_{\parallel} = 0, \quad \omega_{\perp} \wedge \eta_{\perp} = 0.$$

Proof. Writing the decomposition (4.11) for both ω and η , it is clear that $\omega_\perp \wedge \eta_\perp$ vanishes due to dt being present in both expansions. On the other hand, we have

$$\omega_\parallel \wedge \eta_\parallel = \sum_{I \in \mathcal{J}_k, L \in \mathcal{J}_{n-k}} \omega_I \eta_L dx^I \wedge dx^L,$$

which vanishes since one index at least is common to both I and L , as can be seen by writing (identifying the indices in the families with the sets of these indices):

$$|I \cap L| = |I| + |L| - |I \cup L| = k + (n - k) - |I \cup L| \geq n - (n - 1) = 1. \quad \square$$

5 Ghost penalty

Using, in FEEC discretisations of weak formulations of PDEs, the inner product $(\bullet, \bullet)_{\Omega_h}$ on the active mesh Ω_h in the discrete formulation is possible, but introduces a geometric error that is at least $O(h)$ due to the fact that the mesh is not fitted to the boundary $\partial\Omega$. Avoiding this geometric error requires to only rely on the inner product $(\bullet, \bullet)_\Omega$ in these formulations, which leads to ill-conditioned systems since DOFs attached to elements having a very small intersection with Ω have a very small impact on this inner product. The goal of the ghost penalty stabilisation term designed in this section is precisely to restore robustness while preserving optimal convergence rates. The name “ghost” derives from the finite difference terminology of calling “ghost cells” the elements outside of the physical domain. The main result of this section is Theorem 1, where we prove that the norm associated to the ghost term is equivalent to the $L^2(\Omega_h)$ -norm.

5.1 Jumps of forms

Let $F = \overline{T}_1 \cap \overline{T}_2 \in \mathcal{F}_h^\partial$ with normal $\mathbf{n}$ pointing from T_2 into T_1 . Set, for $\omega \in V_h^{k,r}$ and $i = 1, 2$,

$$\omega_i := \omega|_{T_i},$$

where $\omega|_{T_i}$ denotes the standard component-wise restriction of ω to T_i on the canonical basis of k -alternating multi-linear maps, similar to (4.1). We define the jump over F as

$$\begin{aligned} [\cdot] : \Lambda^k(\overline{T}_1 \cup \overline{T}_2) &\rightarrow C^\infty(F; \text{Alt}^k \mathbb{R}^n), \\ \omega &\mapsto [\omega] := \omega_1|_F - \omega_2|_F. \end{aligned}$$

Remark 3 (Traces and jumps). Since the normal and tangential traces (4.2)–(4.3) depend only on the values of the form in $\text{Alt}^k \mathbb{R}^n$ on the facet F , we can apply these operators to the jump defined above. In other words, $\gamma[\omega]$ and $\gamma_n[\omega]$ make sense.

From the linearity of the jump operator and the relation between the tangential part and the trace operator (cf. Lemma 1), it follows that only the normal part of the form (see Section 4) is involved in the jump.

Proposition 1 (Jump of discrete forms). *For all $\omega \in V_h^{k,r}$, it holds*

$$[\omega] = [\omega_\perp],$$

where $\omega_\perp$ is the normal part of the form defined by (4.6).

Proof. Let $F = \overline{T}_1 \cap \overline{T}_2$. We have that $\omega_i \stackrel{(4.7)}{=} \omega_{i\parallel} + \omega_{i\perp}$ on F for $i = 1, 2$. The discrete k -forms are exactly defined by their degrees of freedom, so that $\gamma\omega$ is continuous across inter-element facets, i.e. $\gamma\omega_1 - \gamma\omega_2 = 0$. Let $\{Y_1, Y_2, \dots, Y_k\}$ be an arbitrary set of k vectors in $\mathbb{R}^n$. By definition (4.5) of the tangential part of ω , it holds

$$(\omega_{i\parallel})_x(Y_1, \dots, Y_k) = (\gamma\omega_i)_x(D\pi_F Y_1, \dots, D\pi_F Y_k) \quad \forall x \in F,$$

where π_F is the orthogonal projection onto F . By linearity of the jump operator, we have:

$$(\omega_{1\parallel} - \omega_{2\parallel})_x(Y_1, \dots, Y_k) = (\gamma\omega_1 - \gamma\omega_2)_x(D\pi_F Y_1, \dots, D\pi_F Y_k) = 0 \quad \forall x \in F. \quad \square$$

The result of Proposition 1 is translated in terms of vector proxies in Table 4.1.

k	Proxy	Jump $[\omega_\perp]$
0	q	0
1	$\mathbf{v}$	$[\mathbf{v} \cdot \mathbf{n}]$
2	$\mathbf{w}$	$[\mathbf{n} \times (\mathbf{w} \times \mathbf{n})]$
3	r	$[r]$

Table 5.1: Jumps $[\omega] = [\omega_\perp]$ for vector proxies in $\mathbb{R}^3$.

5.2 Ghost penalty stabilisation

For an integer $\ell \geq 0$, we define the ℓ -th order directional derivative $\nabla_{\mathbf{n}}^{(\ell)} : V_h^{k,r} \rightarrow L^2 \Lambda^k \Omega_h$ along $\mathbf{n}$ as

$$\nabla_{\mathbf{n}}^{(\ell)} \omega := \sum_{1 \leq j_1 < j_2 < \dots < j_\ell \leq n} (\partial_{\mathbf{n}}^{(\ell)} \omega_{j_1 j_2 \dots j_\ell}) dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_\ell},$$

where $\partial_{\mathbf{n}}^{(\ell)} \omega_{j_1 j_2 \dots j_\ell}$ is the ℓ -th order directional derivatives of its component with respect to a chosen coordinate system. The definition does not depend on the choice of coordinate system, and we also identify $\nabla_{\mathbf{n}}^0 \equiv \text{Id}$.

For $\omega \in V_h^{k,r}$, it is clear that $\nabla_{\mathbf{n}}^{(\ell)} \omega$ is a polynomial form of degree $r - \ell$ and that $\nabla_{\mathbf{n}}^{(\ell)} \omega = 0$ for $\ell > r$. Notice that the jump of the normal derivatives of a discrete k -form ω does not reduce, in general, to the jump of the normal part $[(\nabla_{\mathbf{n}}^{(\ell)} \omega)_\perp]$, in contrast to Proposition 1.

Recall the definitions (4.2) of trace and (4.3) of normal trace, as well as Remark 3 on their applicability to jumps. Letting h_F be the diameter of F , we can then define the (ghost penalty) stabilisation term $s : V_h^{k,r} \times V_h^{k,r} \rightarrow \mathbb{R}$ as

$$s(\omega, \zeta) := \sum_{F \in \mathcal{F}_h^\partial} \sum_{\ell=0}^r \eta h_F^{2\ell+1} \int_F \left(\gamma_{\mathbf{n}}[\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \star \gamma_{\mathbf{n}}[\nabla_{\mathbf{n}}^{(\ell)} \zeta] + \gamma[\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \star \gamma[\nabla_{\mathbf{n}}^{(\ell)} \zeta] \right), \quad (5.1)$$

where $\eta > 0$ is a user-defined penalty parameter. The form s is symmetric and bilinear. Notice that s can be extended to act on smooth enough k -forms and it holds, owing to the single-valuedness of traces,

$$s(\zeta, \omega) = s(\omega, \zeta) = 0 \quad \forall (\zeta, \omega) \in H^{r+1} \Lambda^k \Omega_h \times V_h^{k,r}.$$

We define the ghost inner product and norm respectively as

$$\begin{aligned} (\omega, \zeta)_s &:= (\omega, \zeta)_\Omega + s(\omega, \zeta) & \forall (\omega, \zeta) \in V_h^{k,r} \times V_h^{k,r}, \\ \|\omega\|_s &:= \sqrt{(\omega, \omega)_s} & \forall \omega \in V_h^{k,r}. \end{aligned} \quad (5.2)$$

Remark 4 (Formula (5.1) for vector proxies). Let $n = 3$. The simplifications of the jumps for the case $\ell = 0$ given in Table 5.1 do not extend to the case $\ell \geq 1$. However, the directional derivative of a k -form is still a k -form, so, using the results of Table 4.1, we infer that the integrand in (5.1) is in fact the inner product of the full jumps of the normal derivatives regardless of form order k , see Table 5.2.

The goal of the remainder of this subsection is to show the following equivalence result.

Theorem 1 (Uniform equivalence for L^2 -norms). *The norms $\|\bullet\|_{\Omega_h}$ and $\|\bullet\|_s$ are equivalent on $V_h^{k,r}$, uniformly in h .*

The following technical lemma is what informs the definition of the stabilisation term (5.1).

Lemma 3 (Local control through ghost penalty). *Let $r \geq 1$ be an integer and $\omega \in V_h^{k,r}$. Fix a boundary facet $F \in \mathcal{F}_h^\partial$ shared by the elements T_1 and T_2 of $\mathcal{T}_h$. Then, the following inequality holds:*

$$\|\omega\|_{T_1}^2 \lesssim \|\omega\|_{T_2}^2 + \sum_{\ell=0}^r h_F^{2\ell+1} \int_F \left(\gamma_{\mathbf{n}}[\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \star \gamma_{\mathbf{n}}[\nabla_{\mathbf{n}}^{(\ell)} \omega] + \gamma[\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \star \gamma[\nabla_{\mathbf{n}}^{(\ell)} \omega] \right). \quad (5.3)$$

Form degree k	$s(\bullet, \bullet)$
0 or 3	$\sum_{F \in \mathcal{F}_h^\partial} \sum_{\ell=0}^r \eta h_F^{2\ell+1} \int_F [\partial_{\mathbf{n}}^\ell \bullet] [\partial_{\mathbf{n}}^\ell \bullet]$
1 or 2	$\sum_{F \in \mathcal{F}_h^\partial} \sum_{\ell=0}^r \eta h_F^{2\ell+1} \int_F [\partial_{\mathbf{n}}^\ell \bullet] \cdot [\partial_{\mathbf{n}}^\ell \bullet]$

Table 5.2: Ghost penalty stabilisation (5.1) for vector proxies in $\mathbb{R}^3$, with $\omega, \omega' := \zeta \in V_h^{k,r}$.

Proof. Consider the orthogonal projection onto the hyperplane containing F , which we also denote by π_F . First, we claim that we can assume to start with a simplex T_1 in which the vertex s_F opposite to F is such that $\pi_F s_F \in F$. Indeed, if T_1 is such that $\pi_F s_F \notin F$, then consider the maximal simplex $T'_1 \subseteq T_1$ with facet F and opposite vertex $s'_F \in T_1$ such that $\pi_F s'_F \in F$. By a standard scaling argument as in [15, Lemma 1.25], using norm equivalence for polynomial forms, we have that $\|\omega\|_{T_1} \lesssim \|\omega\|_{T'_1}$.

Consider now a point $x \in T_1$, and its orthogonal projection x_F on F . Let $t \in \mathbb{R}$ be the distance from x to x_F , such that $x = x_F + t\mathbf{n}$. For $i \in \{1, 2\}$, since the restriction $\omega_i := \omega|_{T_i}$ is a polynomial form, there is a canonical extension outside of T_i . The Taylor expansion centered at x_F of ω_1 , or of the extension of ω_2 to T_1 , is given by

$$(\omega_i)_x = \sum_{\ell=0}^r \frac{(\nabla_{\mathbf{n}}^{(\ell)} \omega_i)_{x_F}}{\ell!} t^\ell.$$

Since each ω_i is a polynomial form of degree r , the Taylor expansion is exact. Taking the difference of the two expansions gives

$$(\omega_1)_x = (\omega_2)_x + \sum_{\ell=0}^r \frac{(\nabla_{\mathbf{n}}^{(\ell)} \omega_1)_{x_F} - (\nabla_{\mathbf{n}}^{(\ell)} \omega_2)_{x_F}}{\ell!} t^\ell = (\omega_2)_x + \sum_{\ell=0}^r \frac{[\nabla_{\mathbf{n}}^{(\ell)} \omega]_{x_F}}{\ell!} t^\ell.$$

Taking the squared $L^2(T_1)$ -norm and using triangle inequalities, we get

$$\begin{aligned} \|\omega_1\|_{T_1}^2 &\lesssim \left(\|\omega_2\|_{T_1} + \frac{|t|^\ell}{\ell!} \sum_{\ell=0}^r \|[\nabla_{\mathbf{n}}^{(\ell)} \omega]_{x_F}\|_{T_1} \right)^2 \\ &\lesssim \|\omega_2\|_{T_1}^2 + h_F^{2\ell} \sum_{\ell=0}^r \|[\nabla_{\mathbf{n}}^{(\ell)} \omega]_{x_F}\|_{T_1}^2 \\ &\lesssim \|\omega_2\|_{T_2}^2 + h_F^{2\ell} \sum_{\ell=0}^r \int_{T_1} [\nabla_{\mathbf{n}}^{(\ell)} \omega]_{x_F} \wedge \star [\nabla_{\mathbf{n}}^{(\ell)} \omega]_{x_F}, \end{aligned}$$

where, in the second inequality, we have used the Cauchy–Schwarz inequality on the norm together with $|t| \lesssim h_T \approx h_F$, and the conclusion follows from a polynomial transport argument (similar to the one used in the proof of [15, Lemma 1.25]) to replace $\|\omega_2\|_{T_1}^2$ with $\|\omega_2\|_{T_2}^2$, and the definition of the L^2 -norm to expand the second term.

Let ξ be the k -form on T_1 such that, for all $x \in T_1$, $\xi_x := [\nabla_{\mathbf{n}}^{(\ell)} \omega]_{x_F}$; as above, x_F denotes the projection of x onto F . From (4.7) we can decompose ξ into its normal and tangential parts inside T_1 as

$$\xi = \xi_\perp + \xi_\parallel.$$

By definition, $\xi_x = \xi_{x_F}$ for all $x \in T_1$. Define on T_1 the $(n-k)$ -form $\eta := \star \xi$, for which we also have $\eta_x = \eta_{x_F}$ for all $x \in T_1$. By Lemma 1, we have $\star \xi_\parallel = \eta_\perp$ and $\star \xi_\perp = \eta_\parallel$ which, combined with Lemma 2, gives

$$\xi_\perp \wedge \star \xi_\parallel = \xi_\perp \wedge \eta_\perp = 0 \text{ and } \xi_\parallel \wedge \star \xi_\perp = \xi_\parallel \wedge \eta_\parallel = 0.$$

Hence, at x_F we can decompose the wedge product as

$$\xi \wedge \eta = \xi_{\perp} \wedge \eta_{\parallel} + \xi_{\parallel} \wedge \eta_{\perp}.$$

Decomposing ξ using parallel and normal coordinates to F as in (4.11), we write

$$\xi_x = (\xi_{\perp})_x + (\xi_{\parallel})_x = \sum_{J \in \mathcal{J}_{k-1}} \xi_{t,J}(x) dt \wedge dx^J + \sum_{I \in \mathcal{J}_k} \xi_I(x) dx^I,$$

and similarly for η . By Fubini's theorem, denoting by χ the characteristic function of T_1 and using a coordinate system (t, y) with t the coordinate along $\mathbf{n}$ and y the coordinate along the affine span of F , we get

$$\begin{aligned} \int_{T_1} \xi_{\perp} \wedge \eta_{\parallel} &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi(t, x_F) \sum_{J \in \mathcal{J}_{k-1}, I \in \mathcal{J}_k} \xi_{t,J}(x_F) \eta_I(x_F) dt \wedge dx^J \wedge dx^I \\ &= \int_{\mathbb{R}^{n-1}} \sum_{J \in \mathcal{J}_{k-1}, I \in \mathcal{J}_k} \xi_{t,J}(x_F) \eta_I(x_F) \left(\int_{\mathbb{R}} \chi(t, x_F) dt \right) dx^J \wedge dx^I \\ &= \int_F L_{x_F} \sum_{J \in \mathcal{J}_{k-1}, I \in \mathcal{J}_k} \xi_{t,J}(x_F) \eta_I(x_F) dx^J \wedge dx^I \\ &= \int_F L_{x_F} \sum_{J \in \mathcal{J}_{k-1}} \xi_{t,J} dx^J \wedge \sum_{I \in \mathcal{J}_k} \eta_I dx^I \\ &\stackrel{(4.12)}{=} \int_F L_{x_F} \gamma_{\mathbf{n}} \xi \wedge \gamma \eta, \end{aligned}$$

where, starting from the third equality, L_{x_F} denotes the length of the segment at the vertical of x_F in T_1 . Performing similar computations for $\int_{T_1} \xi_{\parallel} \wedge \eta_{\perp}$ and accounting for the fact that $dx^I \wedge dt = (-1)^k dt \wedge dx^I$, we arrive at

$$\begin{aligned} \int_{T_1} \xi \wedge \eta &= \int_F L_{x_F} \gamma_{\mathbf{n}} \xi \wedge \gamma \eta + (-1)^k \int_F L_{x_F} \gamma \xi \wedge \gamma_{\mathbf{n}} \eta \\ &= \int_F L_{x_F} \gamma_{\mathbf{n}} [\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \gamma (\star [\nabla_{\mathbf{n}}^{(\ell)} \omega]) + (-1)^k \int_F L_{x_F} \gamma [\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \gamma_{\mathbf{n}} (\star [\nabla_{\mathbf{n}}^{(\ell)} \omega]) \\ &\stackrel{(4.4)}{=} \int_F L_{x_F} \gamma_{\mathbf{n}} [\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \star \gamma_{\mathbf{n}} [\nabla_{\mathbf{n}}^{(\ell)} \omega] + \int_F L_{x_F} \gamma [\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \star \gamma [\nabla_{\mathbf{n}}^{(\ell)} \omega] \\ &\lesssim h_F \left(\int_F \gamma_{\mathbf{n}} [\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \star \gamma_{\mathbf{n}} [\nabla_{\mathbf{n}}^{(\ell)} \omega] + \int_F \gamma [\nabla_{\mathbf{n}}^{(\ell)} \omega] \wedge \star \gamma [\nabla_{\mathbf{n}}^{(\ell)} \omega] \right), \end{aligned}$$

where we have expanded ξ and η according to the respective definitions in the second step while, in the conclusion, we have used the fact that $L_{x_F} \lesssim h_F$ and that the integrands are both positive multiples of the volume form on F . $\square$

The next result shows that the ghost penalty term is enough to control the extension of ω to the active mesh.

Lemma 4 (Global control through ghost penalty). *For all $\omega \in V_h^{k,r}$, it holds*

$$\|\omega\|_{\Omega_h}^2 \lesssim \|\omega\|_S^2.$$

Proof. Let $\mathcal{T}_h^{\text{im}} := \mathcal{T}_h \setminus \mathcal{T}_h^{\text{cut}}$ be the set of immersed elements. We note that the norm on fully immersed elements is already controlled by the norm on Ω :

$$\sum_{T \in \mathcal{T}_h^{\text{im}}} \|\omega\|_T^2 \leq \|\omega\|_{\Omega}^2.$$

It remains to evaluate the norm on cut elements.

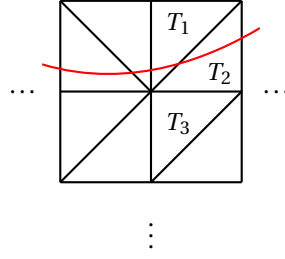


Figure 5.1: Both T_1 and T_2 are cut elements, but T_2 has a fully immersed neighbour T_3 . The boundary $\partial\Omega$ is the red curve. Here $N = 3$.

By Assumption 1-(ii), starting from a cut element, the number of elements we must pass through to reach a fully immersed element is at most N , see Figure 5.1 for an illustration in the case $N = 3$. Let $g : \mathcal{T}_h^{\text{cut}} \rightarrow \mathcal{T}_h^{\text{im}}$ be the map which associates to each cut element $T \in \mathcal{T}_h^{\text{cut}}$ the first fully immersed element $g(T) \in \mathcal{T}_h^{\text{im}}$ which is reached by passing through at most N cut elements. For a given $T' \in \mathcal{T}_h$, any $T \in \mathcal{T}_h$ such that $T' = g(T)$ lies in a chain of elements of cardinality $\leq N$; by Remark 2, all elements along this chain have diameters that are comparable to $h_{T'}$, and therefore lie in a ball of radius $\lesssim Nh_{T'} \lesssim h_{T'}$. As this ball has measure $\approx h_{T'}^n$, and each T in it has measure $\approx h_T^n \approx h_{T'}^n$, we infer that

$$|g^{-1}(T')| \lesssim \frac{h_{T'}^n}{h_{T'}^n} = 1. \quad (5.4)$$

By applying Lemma 3 at most N times, we then have

$$\begin{aligned} \sum_{T \in \mathcal{T}_h^{\text{cut}}} \|\omega\|_T^2 &\lesssim \sum_{T \in \mathcal{T}_h^{\text{cut}}} \|\omega\|_{g(T)}^2 + \sum_{F \in \mathcal{F}_h^\partial} \sum_{\ell=0}^r h_F^{2\ell+1} \int_F \left(\gamma_n[\nabla_n^{(\ell)} \omega] \wedge \star \gamma_n[\nabla_n^{(\ell)} \omega] + \gamma[\nabla_n^{(\ell)} \omega] \wedge \star \gamma[\nabla_n^{(\ell)} \omega] \right) \\ &\stackrel{(5.4)}{\lesssim} \sum_{T' \in \mathcal{T}_h^{\text{im}}} \|\omega\|_{T'}^2 + \sum_{F \in \mathcal{F}_h^\partial} \sum_{\ell=0}^r h_F^{2\ell+1} \int_F \left(\gamma_n[\nabla_n^{(\ell)} \omega] \wedge \star \gamma_n[\nabla_n^{(\ell)} \omega] + \gamma[\nabla_n^{(\ell)} \omega] \wedge \star \gamma[\nabla_n^{(\ell)} \omega] \right). \end{aligned}$$

This completes the proof since $\sum_{T' \in \mathcal{T}_h^{\text{im}}} \|\omega\|_{T'}^2 \leq \|\omega\|_\Omega^2$. $\square$

Proof of Theorem 1. Choose an arbitrary $\omega \in V_h^{k,r}$. In view of Lemma 4, we only have to show that $\|\omega\|_s \lesssim \|\omega\|_{\Omega_h}$. That $\|\omega\|_\Omega \lesssim \|\omega\|_{\Omega_h}$ is clear since $\Omega \subset \Omega_h$. Standard inverse and trace inequalities for polynomials grant

$$\|\nabla_n^{(\ell)} \omega\|_F^2 \lesssim h_F^{-2\ell} \|\omega\|_F^2 \lesssim h_F^{-(2\ell+1)} \|\omega\|_T^2,$$

where T is an element having F as facet (note that $h_F \approx h_T$). Hence, using a triangle inequality followed by the above relation, the norm of the jumps can be estimated as follows: denoting by T_1 and T_2 the two elements on each side of F ,

$$\|[\nabla_n^{(\ell)} \omega]\|_F^2 \lesssim \sum_{T \in \{T_1, T_2\}} h_F^{-(2\ell+1)} \|\omega\|_T^2,$$

from which $\|\omega\|_s \lesssim \|\omega\|_{\Omega_h}$ follows. $\square$

Remark 5 (Macro stabilisation). Notice that the proof of Lemma 4 does not require the stabilisation term to act on all facets in $\mathcal{F}_h^\partial$. It is enough to consider the facets that form a path from each cut element to a fully immersed element, as in the definition of the map g in the proof of Lemma 4. Such a stabilisation term contains fewer summands, and leads naturally to the macro stabilisation procedure outlined in for example [19, Section 6.1].

5.3 Hodge decomposition

Recall the definition (2.4) of the space $\mathfrak{H}_h^k$ of discrete harmonic forms for FEEC, replacing Ω by Ω_h . This space is constructed using orthogonality with respect to the standard inner product of $L^2\Lambda^k\Omega_h$. Since we will use the ghost product $(\bullet, \bullet)_s$ (cf. (5.2)) to define numerical schemes, we need to consistently modify the space of harmonic forms as follows:

$$\mathfrak{H}_s^k := \left\{ \rho_h \in V_h^{k,r} : d\rho_h = 0, (\rho_h, d\tau_h)_s = 0 \quad \forall \tau_h \in V_h^{k-1,r+1} \right\}.$$

Since both $(\bullet, \bullet)_s$ and $(\bullet, \bullet)_{\Omega_h}$ are inner products on $V_h^{k,r}$, and both $\mathfrak{H}_s^k$ and $\mathfrak{H}_h^k$ are orthogonal to the same space, these spaces are isomorphic. We also have

$$\text{Ker } d_h = dV_h^{k-1,r+1} \oplus_s \mathfrak{H}_s^k,$$

where $\oplus_s$ denotes the orthogonal complement with respect to the ghost product.

Correspondingly, we obtain the Hodge decomposition (compare with (2.5)):

$$V_h^{k,r} = (\text{Ker } d_h)^\perp_s \oplus_s dV_h^{k-1,r+1} \oplus_s \mathfrak{H}_s^k.$$

6 Numerical validation

Consider again the Hodge Laplace equation in mixed weak form (2.2). The unfitted method reads: Find $(\sigma_h, \eta_h, \lambda_h) \in V_h^{k-1,r+1} \times V_h^{k,r} \times \mathfrak{H}_s^k$ such that

$$\begin{aligned} (\sigma_h, \tau_h)_s - (\eta_h, d\tau_h)_s &= 0 & \forall \tau_h \in V_h^{k-1,r+1}, \\ (d\sigma_h, \zeta_h)_s + (d\eta_h, d\zeta_h)_s + (\lambda_h, \zeta_h)_s &= (f, \zeta_h)_\Omega & \forall \zeta_h \in V_h^{k,r}, \\ (\eta_h, \rho_h)_s &= 0 & \forall \rho_h \in \mathfrak{H}_s^k. \end{aligned} \tag{6.1}$$

We deliberately take the inner product involving f over the physical domain Ω , rather than the active domain Ω_h , since we do not want to assume an unphysical extension of the data f .

Remark 6 (Divergence constraint for the Poisson equation in mixed form). The proposed method leads to a scheme with particularly interesting properties in the case $k = n$, which corresponds to the Poisson equation in mixed form with homogeneous Dirichlet conditions: Find $(\sigma_h, \eta_h) \in V_h^{n-1,r+1} \times V_h^{n,r}$ such that

$$\begin{aligned} (\sigma_h, \tau_h)_s - (\eta_h, \text{div } \tau_h)_s &= 0 & \forall \tau_h \in V_h^{n-1,r+1}, \\ (\text{div } \sigma_h, \zeta_h)_s &= (f, \zeta_h)_\Omega & \forall \zeta_h \in V_h^{n,r}. \end{aligned}$$

Up to interface conditions and a source term, this is the formulation given in [17, (3.10)-(3.11)]. If $f \in L^2\Lambda^n\Omega$ is regular enough (i.e. has $r+1$ weak normal derivatives in L^2), then $\text{div } \sigma_h$ will be equal, to machine precision, to the L^2 -projection of f with respect to the $(\bullet, \bullet)_s$ -inner product. Applying the same principle for Stokes flow ($f = 0$), the corresponding scheme satisfies the incompressibility condition exactly since $\text{div } \sigma_h = 0$. This observation, which is the content of [19, Theorem 1], can be seen as a consequence of Theorem 1.

6.1 Implementation and results

We validate in this section the scheme (6.1) for $k = 1$. We respectively choose the trimmed spaces $P_1^-\Lambda^0\Omega_h$ (Lagrange) and $P_1^-\Lambda^1\Omega_h$ (Nédélec of the first kind) for σ_h and η_h . This choice means that the exterior derivative d does not reduce the indicated polynomial order, and that the error in the natural stability norm can be

expected to converge with order $O(h)$. The scheme reads as follows. Find $(\sigma_h, \eta_h, \lambda_h) \in P_1^- \Lambda^0 \Omega_h \times P_1^- \Lambda^1 \Omega_h \times \mathfrak{H}_s^1$ such that

$$\begin{aligned} (\sigma_h, \tau_h)_s - (\eta_h, \nabla \tau_h)_s &= 0 & \forall \tau_h \in P_1^- \Lambda^0 \Omega_h, \\ (\nabla \sigma_h, \zeta_h)_s + (\text{curl} \eta_h, \text{curl} \zeta_h)_s + (\lambda_h, \zeta_h)_s &= (f, \zeta_h)_\Omega & \forall \zeta_h \in P_1^- \Lambda^1 \Omega_h, \\ (\eta_h, \rho_h)_s &= 0 & \forall \rho_h \in \mathfrak{H}_s^1, \end{aligned} \quad (6.2)$$

where we use stabilisation parameter $\eta = 1$ in (5.1). Since the trial and test forms are piecewise linear, their second derivatives vanish identically, and since σ_h, τ_h are continuous, we get from Tables 5.1 and 5.2 that

$$\begin{aligned} (\sigma_h, \tau_h)_s &= (\sigma_h, \tau_h)_\Omega + \sum_{F \in \mathcal{F}_h^\partial} h_F^3 ([\nabla \sigma_h \cdot \mathbf{n}], [\nabla \tau_h \cdot \mathbf{n}])_F, \\ (\eta_h, \nabla \tau_h)_s &= (\eta_h, \nabla \tau_h)_\Omega + \sum_{F \in \mathcal{F}_h^\partial} h_F ([\eta_h], [\nabla \tau_h])_F, \\ (\text{curl} \eta_h, \text{curl} \zeta_h)_s &= (\text{curl} \eta_h, \text{curl} \zeta_h)_\Omega + \sum_{F \in \mathcal{F}_h^\partial} h_F ([\text{curl} \eta_h], [\text{curl} \zeta_h])_F, \\ (\lambda_h, \zeta_h)_s &= (\lambda_h, \zeta_h)_\Omega + \sum_{F \in \mathcal{F}_h^\partial} (h_F ([\lambda_h], [\zeta_h])_F + h_F^3 ([\nabla \lambda_h \cdot \mathbf{n}], [\nabla \zeta_h \cdot \mathbf{n}])_F). \end{aligned}$$

We also use the macro parameter $\delta = 0.25$ in the macro stabilisation procedure discussed in Remark 5.

We solve problem (6.2) on the following filled torus embedded in $\mathbb{R}^3$: $\Omega = \{(x, y, z) \in \mathbb{R}^3 : [(\sqrt{x^2 + y^2} - 0.5)^2 + z^2]^{1/2} \leq 0.25\}$, with major radius 0.5 and minor radius 0.25. The mesh is cut via the level set function

$$\phi = \left[\left(\sqrt{x^2 + y^2} - 0.5 \right)^2 + z^2 \right]^{1/2} - 0.25,$$

from a uniform background mesh with mesh size h . The toroidal harmonic forms are scalar multiples of

$$d\theta = \frac{xdy - ydx}{x^2 + y^2} \longleftrightarrow \frac{1}{x^2 + y^2} (-y, x, 0).$$

We pick the right-hand side and the exact solution as

$$f = \left(-\frac{3xy}{(x^2 + y^2)^{\frac{5}{2}}}, \frac{x^2 - 2y^2}{(x^2 + y^2)^{\frac{5}{2}}}, 0 \right), \quad \eta = \left(-\frac{-xy}{(x^2 + y^2)^{\frac{3}{2}}}, \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}, 0 \right).$$

The C++ code used to compute the example can be found in the fork [2] of the CutFEM library [1]. The code was run on a laptop with Intel i7-8565U x 8 CPU and 16GB RAM. To solve the linear system, we use the sparse direct solver MUMPS. Errors and condition numbers for three mesh sizes, $h = \frac{1}{13}, \frac{1}{26}, \frac{1}{52}$, are shown in Figure 6.1. A heat map of the x -component of the solution η is shown in Figure 6.2. We note that the expected rate of convergence $\mathcal{O}(h)$ is indeed achieved for the errors on the curl and gradient, with a slightly better rate for the fields themselves (going possibly towards $\mathcal{O}(h^2)$). The condition number for the scheme based on the ghost stabilisation remains at a reasonable magnitude, while for the unstabilized scheme (using the inner product $(\bullet, \bullet)_\Omega$ instead of $(\bullet, \bullet)_s$) it blows up to a level where solving the linear system becomes infeasible, if not impossible.

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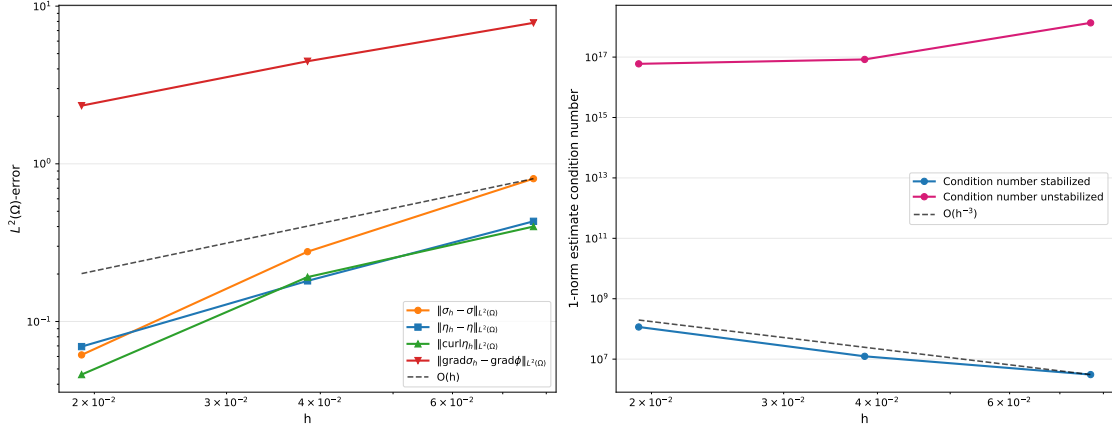


Figure 6.1: Plot of the convergence of the mixed method (6.1) as a function of the mesh size h . The convergence is shown for polynomial order $r = 1$, and the 1-norm estimate of the condition number $\kappa_s := \|\mathcal{A}_s\|_{\text{op}} \|\mathcal{A}_s^{-1}\|_{\text{op}}$ of the system matrix $\mathcal{A}_s$ is also plotted on the right figure.

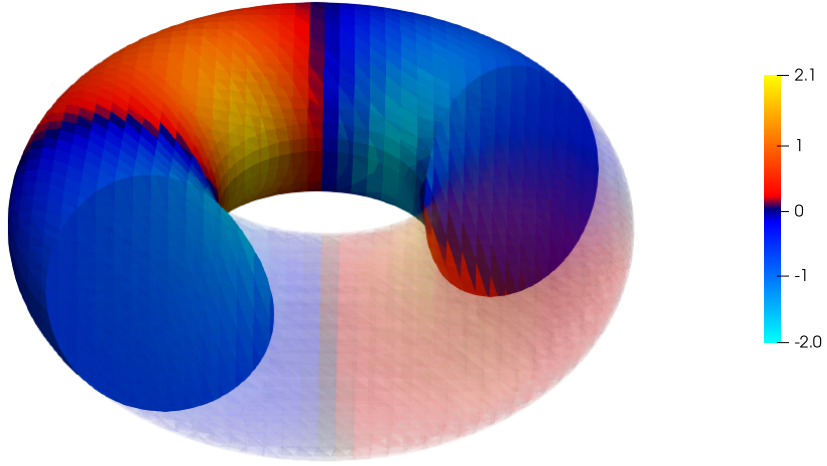


Figure 6.2: Heat map of computed x -component of the solution η_h to the discrete Hodge Laplace equation (6.1) on the filled in torus. The mesh size is $h = 0.019231$ and the polynomial order is $r = 1$.

References

- [1] Cutfem/cutfem-library. URL <https://github.com/CutFEM/CutFEM-Library>. [Online; accessed 2025-09-22].
- [2] tensorfan/cutfem-fork. URL <https://github.com/tensorFan/CutFEM-fork>. [Online; accessed 2025-09-22].
- [3] D. N. Arnold. *Finite Element Exterior Calculus*. Society for Industrial and Applied Mathematics, 2018. doi: 10.1137/1.9781611975543.
- [4] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numerica*, 15:1–155, 2006. doi: 10.1017/S0962492906210018.
- [5] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bulletin of the American Mathematical Society*, 47(2):281–354, 2010. doi: 10.1090/S0273-0979-10-01278-4. URL <http://arxiv.org/abs/0906.4325>.
- [6] F. Bonaldi, D. A. Di Pietro, J. Droniou, and K. Hu. An exterior calculus framework for polytopal methods. *Journal of the European Mathematical Society*, 2025. doi: 10.4171/JEMS/1602. Published online.
- [7] A. Bossavit. A rationale for ‘edge-elements’ in 3-D fields computations. *IEEE Transactions on Magnetics*, 24(1):74–79, 1988. doi: 10.1109/20.43860.
- [8] A. Bossavit. Differential geometry for the student of numerical methods in electromagnetism. 01 1991. URL https://www.researchgate.net/publication/200018385_Differential_Geometry_for_the_student_of_numerical_methods_in_Electromagnetism.
- [9] A. Bossavit. Generating Whitney forms of polynomial degree one and higher. *IEEE Transactions on Magnetics*, 38:341–344, 2002. doi: 10.1109/20.996092.
- [10] J. Brüning and M. Lesch. Hilbert complexes. *J. Funct. Anal.*, 108(1):88–132, 1992. doi: [https://doi.org/10.1016/0022-1236\(92\)90147-B](https://doi.org/10.1016/0022-1236(92)90147-B).
- [11] E. Burman. Ghost penalty. *Comptes Rendus Mathématique*, 348(21):1217–1220, 2010. doi: <https://doi.org/10.1016/j.crma.2010.10.006>.
- [12] E. Burman, S. Claus, P. Hansbo, M. G. Larson, and A. Massing. CutFEM: Discretizing geometry and partial differential equations. *International Journal for Numerical Methods in Engineering*, 104(7):472–501, 2015. doi: 10.1002/nme.4823.
- [13] E. Burman, P. Hansbo, M. G. Larson, and S. Zahedi. Cut finite element methods. *Acta Numerica*, 34:1–121, 2025. doi: 10.1017/S0962492925000017.
- [14] Z. Chen, K. Li, M. Lyu, and X. Xiang. A high order unfitted finite element method for time-harmonic Maxwell interface problems. *International Journal of Numerical Analysis and Modeling*, 21(6):822–849, 2024. doi: <https://doi.org/10.4208/ijnam2024-1033>.
- [15] D. A. Di Pietro and J. Droniou. *The Hybrid High-Order method for polytopal meshes*, volume 19 of *Modeling, Simulation and Application*. Springer International Publishing, 2020. doi: 10.1007/978-3-030-37203-3.
- [16] D. A. Di Pietro, J. Droniou, M.-L. Hanot, and S. Pitassi. Uniform Poincaré inequalities for the discrete de Rham complex of differential forms, 1 2025. URL <http://arxiv.org/abs/2501.16116>.
- [17] T. Frachon, P. Hansbo, E. Nilsson, and S. Zahedi. A divergence preserving cut finite element method for Darcy flow. *SIAM Journal on Scientific Computing*, 46(3):A1793–A1820, 2024. doi: 10.1137/22M149702X.

- [18] T. Frachon, E. Nilsson, and S. Zahedi. Stabilized Lagrange multipliers for Dirichlet boundary conditions in divergence preserving unfitted methods, 2024. URL <http://arxiv.org/abs/2408.10089>.
- [19] T. Frachon, E. Nilsson, and S. Zahedi. Divergence-free cut finite element methods for Stokes flow. *BIT Numerical Mathematics*, 64(4):39, 2024. doi: 10.1007/s10543-024-01040-x.
- [20] C. Gürkan and A. Massing. A stabilized cut discontinuous Galerkin framework for elliptic boundary value and interface problems. *Computer Methods in Applied Mechanics and Engineering*, 348:466–499, 2019. doi: 10.1016/j.cma.2018.12.041.
- [21] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237–339, 2002. doi: 10.1017/S0962492902000041.
- [22] R. Li, Q. Liu, and F. Yang. A reconstructed discontinuous approximation on unfitted meshes to $H(\text{curl})$ and $H(\text{div})$ interface problems. *Computer Methods in Applied Mechanics and Engineering*, 403:115723, 2023. doi: <https://doi.org/10.1016/j.cma.2022.115723>.
- [23] A. Massing, M. G. Larson, A. Logg, and M. E. Rognes. A stabilized Nitsche fictitious domain method for the Stokes problem. *Journal of Scientific Computing*, 61:604–628, 2014. doi: 10.1007/s10915-014-9838-9.
- [24] J.-C. Nédélec. Mixed finite elements in R_3 . *Numerische Mathematik*, 35(3):315–341, 1980. doi: 10.1007/BF01396415.
- [25] E. Nilsson. *Generalized mixed finite element methods: cut elements and virtual elements*. PhD thesis, KTH Royal Institute of Technology, 2024.
- [26] P. A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In I. Galligani and E. Magenes, editors, *Mathematical Aspects of the Finite Element Method*. Springer, 1977. doi: 10.1007/BFb0064470.
- [27] N. Wang, H. Chu, J. Chen, and Y. Cai. Mixed Nitsche extended finite element method for solving three-dimensional $H(\text{curl})$ -elliptic interface problems. *Computers & Mathematics with Applications*, 199: 22–44, 2025. doi: <https://doi.org/10.1016/j.camwa.2025.08.031>.
- [28] F. Yang and X. Xie. An unfitted finite element method with direct extension stabilization for time-harmonic Maxwell problems on smooth domains. *Advances in Computational Mathematics*, 50(3): 51, 2024. doi: 10.1007/s10444-024-10148-1.