A proof of the $\frac{3}{8}$-conjecture for independent domination in cubic graphs

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Abstract

A set S of vertices in a graph G is a dominating set of G if every vertex not in S is adjacent to a vertex in S. An independent dominating set in G is a dominating set of G with the additional property that it is an independent set. The domination number, $\gamma(G)$, and the independent domination number, $i(G)$, are the minimum cardinalities among all dominating sets and independent dominating sets in G, respectively. By definition, $\gamma(G) \le i(G)$ for all graphs G. Let G be a connected cubic graph of order n. In 1996 Reed [Combin. Probab. Comput. 5 (1996), 277–295] proved a breakthrough result that $\gamma(G) \le \frac{3}{8}n$. We prove the stronger result that if G is different from $K_{3,3}$ and the 5-prism $C_5 \,\square\, K_2$, then $i(G) \le \frac{3}{8}n$. This proves a known conjecture. The bound is tight in the sense that there are infinite families of connected cubic graphs that achieve equality in this bound.

Keywords: Independent domination number; Domination number; Cubic graphs.
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1 Introduction

A set S of vertices in a graph G is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A set is *independent* in G if no two vertices in the set are adjacent. An *independent dominating set*, abbreviated ID-set, of G is a set that is both dominating and independent in G. Equivalently, an ID-set of G is a maximal independent set in G. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an ID-set. An ID-set of cardinality $i(G)$ in G is called an i-set of G. Independent dominating sets have been studied extensively in the literature (see, for example, [8, 10, 15, 16, 18, 19, 21]). A thorough treatise on dominating sets can be found in the so-called "domination books" [11–13]. A survey on independent domination in graphs can be found in [7].

For graph theory notation and terminology, we generally follow [13]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two adjacent vertices in G are *neighbors*. The *open neighborhood* of a vertex v in G is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. We denote the degree of v in G by $\deg_G(v)$, and so $\deg_G(v) = |N_G(v)|$. An *isolated vertex* in G is a vertex of degree 0 in G. For $k \ge 1$ an integer, we let $[k]$ denote the set $\{1, \ldots, k\}$ and we let $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$.

A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. The *complete graph* on n vertices is denoted by K_n, while the *complete bipartite graph* with one partite set of size n and the other of size m is denoted by $K_{n,m}$. A *cubic graph* is graph in which every vertex has degree 3, while a *subcubic graph* is a graph with maximum degree at most 3. If G is a subcubic graph, then we let $V_j(G)$ denote the set of vertices of degree j in G and we let $n_j(G) = |V_j(G)|$ denote the number of vertices of degree j in G for $j \in [3]_0$.

For subsets X and Y of vertices of G, we denote the set of edges with one end in X and the other end in Y by $[X, Y]$. The *girth* of G, denoted $g(G)$, is the length of a shortest cycle in G. A component of a graph G isomorphic to a graph F we call an *F-component* of G. For a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. Further, the subgraph of G obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$; that is, $G - S = G[V(G) \setminus S]$.

2 Motivation and known results

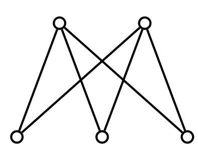
In 1996 Reed [20] proved the following breakthrough result establishing a best possible upper bound on the domination number of a cubic graph.

Theorem 1 ([20]) *If G is a cubic graph of order n, then $\gamma(G) \le \frac{3}{8}n$.*

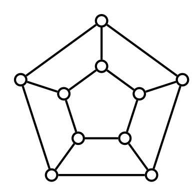
The importance of Reed's paper is reflected in over 100 citations one can find on the MathSciNet. Subsequently, determining best possible bounds on the independent domination number of cubic graphs attracted much interest (see, for example, [1–6, 9, 14]). In 1999 Lam, Shiu, and Sun [17] established the following upper bound on the independent domination number of a connected cubic graph.

Theorem 2 ([17]) *If $G \not\cong K_{3,3}$ is a connected, cubic graph of order n, then $i(G) \le \frac{2}{5}n$.*

The $\frac{2}{5}$-upper bound on the independent domination number in Theorem 2 is best possible in the sense that there exists at least one graph that achieves equality in the bound, namely the 5-prism, $C_5 \,\square\, K_2$, which is the Cartesian product of a 5-cycle with a copy of K_2. The graphs $K_{2,3}$ and $C_5 \,\square\, K_2$ are shown in Figure 1(a) and 1(b), respectively.



(a) $K_{2,3}$



(b) $C_5 \,\square\, K_2$

Figure 1: The graphs $K_{2,3}$ and $C_5 \,\square\, K_2$.

However despite the pleasing result given in Theorem 2, prior to this paper it remained an open problem to characterize the graphs achieving equality in Theorem 2. In their 2013 survey paper on independent domination in graphs, Goddard and Henning [7] conjectured that the 5-prism, $C_5 \,\square\, K_2$, is the only extremal graph achieving equality in the $\frac{2}{5}$-upper bound on the independent domination number $i(G)$ established by Lam, Shiu, and Sun.

The authors in [7] conjectured that Reed's $\frac{3}{8}$-upper bound on the domination number, $\gamma(G)$, also holds for the independent domination, $i(G)$, in a connected, cubic graph, with the exception of the two graphs $K_{3,3}$ and $C_5 \,\square\, K_2$.

Conjecture 1 ([7]) *If $G \notin \{K_{3,3}, C_5 \,\square\, K_2\}$ is a connected cubic graph of order n, then $i(G) \le \frac{3}{8}n$.*

In 2015 Dorbec et al. [6] gave support to Conjecture 1 by proving that the conjecture holds if the cubic graph G contains no subgraph isomorphic to $K_{2,3}$. Recall that in a subcubic graph G, we let $n_j(G)$ denote the number of vertices of degree j in G for $j \in [3]_0$.

Theorem 3 ([6]) *If G is a subcubic graph that does not have a subgraph isomorphic to $K_{2,3}$ and which has no $(C_5 \,\square\, K_2)$-component, then*

$$8i(G) \le 8n_0(G) + 5n_1(G) + 4n_2(G) + 3n_3(G).$$

As a consequence of Theorem 3, we have the main result in [6], noting that if G is a cubic graph of order n, then $n_0(G) = n_1(G) = n_2(G) = 0$ and $n = n_3(G)$.

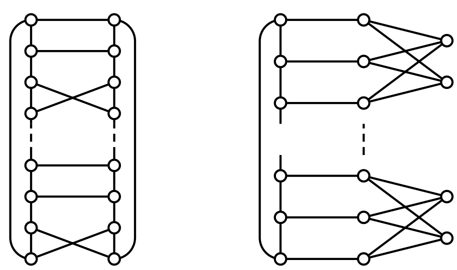
Theorem 4 ([6]) *If $G \neq C_5 \,\square\, K_2$ is a connected cubic graph of order n that does not have a subgraph isomorphic to $K_{2,3}$, then $i(G) \le \frac{3}{8}n$.*

3 Main result

We give a complete proof of Conjecture 1 by adding forbidden subgraphs isomorphic to $K_{2,3}$ in the statement of Theorem 4 back into the mix. We will prove the following result.

Theorem 5 *If $G \notin \{K_{3,3}, C_5 \,\square\, K_2\}$ is a connected cubic graph of order n, then $i(G) \le \frac{3}{8}n$.*

As a consequence of Theorem 5, the 5-prism, $C_5 \,\square\, K_2$, is the unique extremal graph in Theorem 2. The $\frac{3}{8}$-upper bound on the independent domination number given in Theorem 5 is tight. Two infinite families of connected cubic graphs G satisfying $i(G) = \frac{3}{8}n$ are constructed in [7]. These families are also defined in [13, Chapter 6], so we do not redefine them here. However, graphs in these families are illustrated in Figure 2(a) and (b), respectively. Akbari et al. [1] constructed additional families (which we do not define here) of connected cubic graphs G satisfying $i(G) = \frac{3}{8}n$.



(a) (b)

Figure 2: Two infinite families of cubic graphs G of order n satisfying $i(G) = \frac{3}{8}n$

Indeed, the family of extremal graphs achieving the $\frac{3}{8}$-upper bound in Theorem 5 is rich and varied. As a further example of extremal graphs, the graphs G of order n in the infinite family illustrated in Figure 3 also satisfy $i(G) = \frac{3}{8}n$. We remark that the cubic graphs G in the family illustrated in Figure 2(a) do not contain a subgraph isomorphic to $K_{2,3}$ and satisfy $i(G) = \frac{3}{8}n$, while the cubic graphs G in the family illustrated in Figure 2(b) and Figure 3 do contain a subgraph isomorphic to $K_{2,3}$ and satisfy $i(G) = \frac{3}{8}n$.

We proceed as follows. In Section 4 we give an overview of our proof to give the main ideas and direction of the proof. In Section 5, we formally define the (infinite) special families of subgraphs we encounter in our proof, and we prove desirable properties of the families. Thereafter in Section 6, we present a detailed proof of our key result, namely Theorem 6.

4 Overview of proof of Theorem 5

Before presenting a proof of Theorem 5, we give a sketch proof to provide the reader an overview of the detailed proof that follows and the direction of the arguments. In order to prove Theorem 5, we relax the 3-regularity

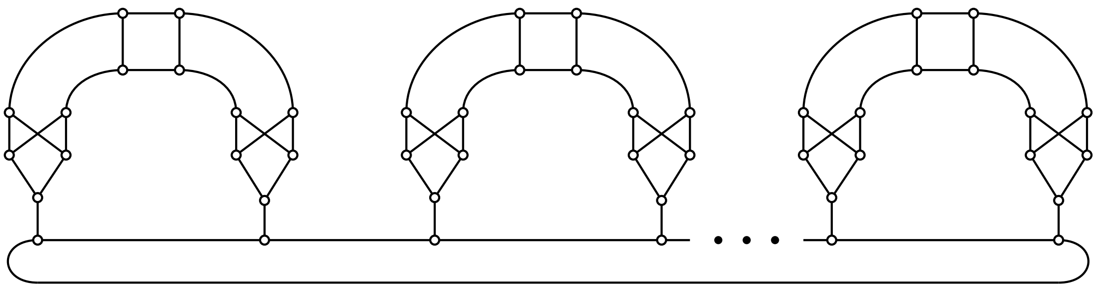


Figure 3: An infinite family of cubic graphs G of order n satisfying $i(G) = \frac{3}{8}n$

condition to a bounded maximum degree 3 condition to make the inductive hypothesis easier to handle, that is, we relax the requirement that $G \notin \{K_{3,3}, C_5 \,\square\, K_2\}$ is a connected cubic graph to the requirement that G is a subcubic graph that contains no $K_{3,3}$-component and no $(C_5 \,\square\, K_2)$-component. We define next what we have coined the vertex weight and the structural weight of such graphs.

4.1 The vertex weight

For a subcubic graph G, we assign weights 8, 5, 4 and 3 to the vertices of G of degrees 0, 1, 2 and 3, respectively. For each vertex v of G, we denote the *weight* of v in G by $\mathrm{w}_G(v)$ (see Table 1).

$\deg_G(v)$	0	1	2	3
$\mathrm{w}_G(v)$	8	5	4	3

Table 1. The weight $\mathrm{w}_G(v)$ of a vertex v in G

For a subset X of vertices in G, we define the *vertex weight* of X in G as the sum of the weights in G of vertices in X; that is,

$$\mathrm{w}_G(X) = \sum_{v \in X} \mathrm{w}_G(v).$$

We define the *vertex weight* of G, denoted by $\mathrm{w}(G)$, as the sum of the weights of vertices in G, that is,

$$\mathrm{w}(G) = 8n_0(G) + 5n_1(G) + 4n_2(G) + 3n_3(G).$$

4.2 The structural weight

We would like to show that $8i(G) \le \mathrm{w}(G)$ since in the special case when G is a cubic graph of order n, we note that $n_0(G) = n_1(G) = n_2(G) = 0$ and $n = n_3(G)$, and so the weight of G simplifies to $\mathrm{w}(G) = 3n$, from which we would infer that $i(G) \le \frac{3}{8}n$, as desired.

However, relaxing the cubic condition results in the so-called "bad" family of subcubic graphs. We construct such a family $\mathcal{B}$ of subcubic graphs in Section 5.2 with the property that if $G \in \mathcal{B}$, then $8i(G) = \mathrm{w}(G) + 2$. We will refer to graphs G in this family $\mathcal{B}$ as *bad graphs* since they satisfy $8i(G) > \mathrm{w}(G)$. We denote by $b(G)$ the number of components of G that belong to the family $\mathcal{B}$.

Moreover relaxing the cubic condition results in so-called "troublesome" configurations of subcubic graphs. In Section 5.3, we define these troublesome configurations, which we call T-*configurations*. We distinguish two types of troublesome configurations, namely T_1-*configurations* and T_2-*configurations*. A T_i-configuration is a troublesome configuration that is joined to vertices not in the subgraph by exactly i edges. We denote by $\mathrm{tc}(G)$ the number of vertex disjoint troublesome configurations in G. We define the Θ-*weight* of G, which we call the *structural weight* of G since it is determined by structural properties of G, by

$$\Theta(G) = 2\mathrm{tc}(G) + 2b(G).$$

4.3 The total weight

We define the *total weight* of G, which we denote by $\Omega(G)$, as the sum of the vertex weight and the structural weight of a subcubic graph G, and so

$$\Omega(G) = \mathrm{w}(G) + \Theta(G).$$

We call the total weight of G the Ω-*weight of* G. Our key result, given in Theorem 6, gives a tight upper bound on the independent domination number of a subcubic graph in terms of its Ω-weight.

Theorem 6 *If G is a subcubic graph of order n that contains no $K_{3,3}$-component and no $(C_5 \,\square\, K_2)$-component, then $8i(G) \le \Omega(G)$.*

In order to prove our main result, namely Theorem 5, we need the above stronger statement given in Theorem 6. Each graph in the family $\mathcal{B}$ of subcubic graphs contains vertices of degree 1 or 2. Moreover, every so-called "troublesome configuration" contains a vertex of degree 2 in G. From these properties, a cubic graph G has no component that belongs to the family $\mathcal{B}$ and contains no troublesome configuration. We therefore infer that if G is a cubic graph, then $\mathrm{tc}(G) = b(G) = 0$, implying that its structural weight is zero, that is, $\Theta(G) = 0$. Further if G is a cubic graph of order n, then $n_0(G) = n_1(G) = n_2(G) = 0$ and $n = n_3(G)$, and so its vertex weight is $3n$, that is, $\mathrm{w}(G) = 3n$. Hence in this case when G is a cubic graph, the inequality in the statement of Theorem 6 simplifies to $8i(G) \le 3n$. Our main result, namely Theorem 5, therefore follows readily from Theorem 6.

5 Special families of graphs and subgraphs

In this section, we define the family $\mathcal{B}$ of so-called 'bad subgraphs' and we define what we have coined 'troublesome configurations' of a subcubic graph G. In order to prove desirable properties of subcubic graphs, we first define 'exit edges' associated with a set X of vertices of a graph G.

5.1 Exit edges of a vertex set

Let X be a proper subset of vertices in a subcubic graph G and let $V = V(G)$. We call an edge an X-*exit edge* in G if it joins a vertex in X to a vertex in $V \setminus X$. We denote the number of X-exit edges in G by $\xi_G(X)$. Those vertices in $G - X$ that are incident in G with at least one X-exit edge have smaller degree in $G - X$ than in G, and therefore have a larger weight in $G - X$ than in G. We refer to the sum of these weight increases as the w-*cost of removing* X from G and denote the weight increase by $\Phi_G(X)$. Thus,

$$\Phi_G(X) = \sum_{v \in V \setminus X} \big(\mathrm{w}_{G-X}(v) - \mathrm{w}_G(v)\big).$$

We prove next the fact that the w-cost of removing a set X of vertices from G is equal to the number of X-exit edges in G plus twice the number of isolated vertices in $G - X$. Consequently, if removing X yields no isolated vertices, then the cost of removing X from G will be precisely the number of X-exit edges in G. For a vertex $v \in V \setminus X$, we denote by $\Phi_G(v)$ the number of X-edges in G that are incident with the vertex v.

Fact 1 *If $X \subset V(G)$, then $\Phi_G(X) = \xi_G(X) + 2n_0(G - X)$.*

Proof. We consider the contribution of a vertex $v \in V \setminus X$ to the cost $\Phi_G(X)$. If the vertex v is incident with no X-exit edge in G, then $\mathrm{w}_{G-X}(v) = \mathrm{w}_G(v)$, and the contribution of v to the sum $\Phi_G(X)$ is zero. Hence, we may assume that v is incident with at least one X-exit edge in G, that is, $\Phi_G(v) \ge 1$. If v is not isolated in $G - X$, then $\mathrm{w}_{G-X}(v) - \mathrm{w}_G(v)$ is precisely the number of exit edges of X incident with v in G, namely $\Phi_G(v)$. If v is isolated in $G - X$, then $\mathrm{w}_{G-X}(v) - \mathrm{w}_G(v)$ is the number of exit edges of X incident with v in G plus an additional 2 noting that in this case $\mathrm{w}_{G-X}(v) = 8$ and $\mathrm{w}_G(v) = 6 - \Phi_G(v)$. $\square$

5.2 The bad family $\mathcal{B}$

In this section, we define the so-called "bad family" $\mathcal{B}$ of graphs. Let B_1 be the graph obtained from $K_{2,3}$ by adding a new vertex r and adding an edge joining r and a vertex of degree 2 in the copy of $K_{2,3}$. The graph B_1, illustrated in Figure 4, we call the *base graph* used in our construction of graphs in the family $\mathcal{B}$.

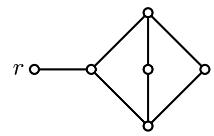


Figure 4: The base graph B_1

Let $\mathcal{B}$ be the family of subcubic graphs that contain the base graph B_1 and is closed under the operation $\mathcal{O}_1$ listed below that extends a graph $G' \in \mathcal{B}$ to a new graph $G \in \mathcal{B}$. As remarked earlier, we refer to graphs in the family $\mathcal{B}$ as *bad graphs*.

- *Operation* $\mathcal{O}_1$: The graph G is obtained from $G' \in \mathcal{B}$ by selecting a vertex v' of degree at most 2 in G', adding a vertex disjoint copy of $K_{2,3}$, and adding an edge joining v' and a vertex of degree 2 in the added copy of $K_{2,3}$. See Figure 5.

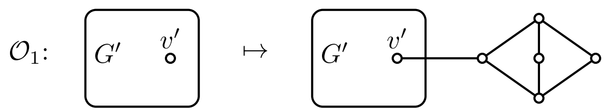


Figure 5: The operation $\mathcal{O}_1$

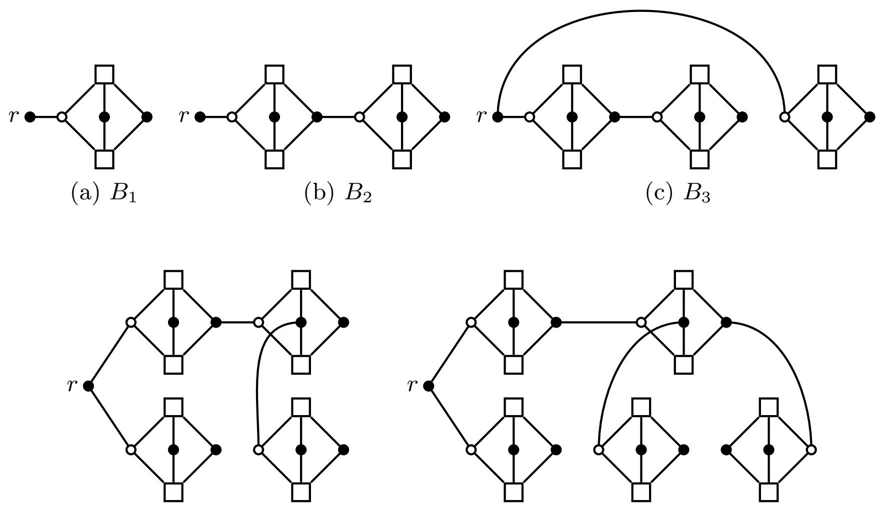
Thus, if $G \in \mathcal{B}$, then either $G = B_1$ or G is constructed from the base graph B_1 by $k-1$ applications of Operation $\mathcal{O}_1$ where $k \ge 2$. Thus, either $G = B_1$ or $k \ge 2$ and $G = B_k$ where $B_1, B_2, \dots, B_k$ is a sequence of graphs in the family $\mathcal{B}$ and the graph B_{i+1} is obtained from the graph B_i by applying Operation $\mathcal{O}_1$ for all $i \in [k-1]$. We call B_{i+1} a *bad-extension* of the graph B_i for $i \in [k-1]$.

If $G \in \mathcal{B}$, then we define the *root* of the graph G as the vertex of G that does not belong to any copy of $K_{2,3}$, equivalently, the root of G is the (unique) vertex that belongs to no cycle. We note that the root has degree 1, 2 or 3 in the graph G. We define $\mathcal{B}_i = \{B \in \mathcal{B} \colon \text{the root of } B \text{ has degree } i \text{ in } B\}$ for $i \in [3]$, and so

$$\mathcal{B} = \bigcup_{i=1}^{3} \mathcal{B}_i.$$

For example, the graph G illustrated in Figure 6(e) belongs to the family $\mathcal{B}$ since it can be constructed from a sequence B_1, B_2, B_3, B_4, B_5 of graphs in $\mathcal{B}$ shown in Figure 6(a)–6(e), where B_{i+1} a bad-extension of B_i for $i \in [4]$ and where $G = B_5$ (in this case, $G = B_k$ where $k = 5$). In this example, the root is the vertex labelled r in Figure 6, and has degree 2 in the resulting graph $G = B_5$, and so the graph G belongs to the subfamily $\mathcal{B}_2$ of $\mathcal{B}$.

We define the *canonical ID-set* of a bad graph G (that belongs to the family $\mathcal{B}$) as follows. Initially if $G = B_1$, we define the canonical ID-set to consist of the root and the two vertices of degree 2 in B_1 (as illustrated by the shaded vertices in Figure 6(a)). Suppose that $k \ge 2$ and let $B_1, B_2, \dots, B_k$ be the sequence of graphs in the family $\mathcal{B}$ used to construct $G \in \mathcal{B}$, and so $G = B_k$ and B_{i+1} a bad-extension of the graph B_i for $i \in [k-1]$. When we apply Operation $\mathcal{O}_1$ to extend the graph B_i to the graph B_{i+1} we add to the current canonical ID-set in B_i the two new vertices of degree 2 in B_{i+1} that belong to the added copy of $K_{2,3}$ for all $i \in [k-1]$. We note that by construction, every vertex of degree 2 in G belongs to the canonical ID-set of G. For example, the canonical ID-set of each of the graphs B_1, B_2, B_3, B_4, B_5 used to construct the graph $G = B_5$ is given by the shaded vertices in Figures 6(a)–6(e).



(a) B_1 (b) B_2 (c) B_3

(d) B_4 (e) $G = B_5$

Figure 6: A graph G in the family $\mathcal{B}_2$ with root vertex r

A vertex of degree 3 in G we call a *large vertex*. Each copy of $K_{2,3}$ in $G \in \mathcal{B}$ contains exactly two vertices that belong to the canonical ID-set of G, and these two vertices have two common large neighbors that belong to the same copy of $K_{2,3}$. We refer to these two large vertices as *non-canonical vertices* of G. Thus each copy of $K_{2,3}$ in $G \in \mathcal{B}$ contains two non-canonical vertices, namely the two vertices of degree 3 in that copy of $K_{2,3}$ that are open twins, where two vertices are open twins if they have the same open neighborhood. We define the *non-canonical independent set* of $G \in \mathcal{B}$ to be the set consisting of all non-canonical vertices of G. Thus if $G \in \mathcal{B}$ is constructed from the base graph B_1 by $k-1$ applications of Operation $\mathcal{O}_1$ where $k \geq 1$, then G contains k vertex disjoint copies of $K_{2,3}$ and the $2k$ non-canonical vertices of G (two from each copy of $K_{2,3}$ in G) form the non-canonical independent set of G. For example, the non-canonical independent set of each of the graphs B_1, B_2, B_3, B_4, B_5 used to construct the graph $G = B_5$ is given by the vertices represented by squares in Figures 6(a)–6(e).

We define the set of vertices in $G \in \mathcal{B}$ that belong to neither the canonical ID-set of G nor the non-canonical independent set of G as the *core independent set* in G, and we refer to the vertices in the core independent set as *core vertices*. Thus if $G \in \mathcal{B}$ is constructed from the base graph B_1 by $k-1$ applications of Operation $\mathcal{O}_1$ where $k \geq 1$, then G contains k core vertices, one from each of the k vertex disjoint copies of $K_{2,3}$. For example, the core independent set of each of the graphs B_1, B_2, B_3, B_4, B_5 used to construct the graph $G = B_5$ is given by the vertices represented by white circled vertices in Figures 6(a)–6(e).

We shall need the following lemmas.

Lemma 1 *If a subcubic graph G is obtained from a subcubic isolate-free graph G' by adding a vertex disjoint copy of $K_{2,3}$, and adding an edge joining a vertex v' of degree at most 2 in G' and a vertex of degree 2 in the added copy of $K_{2,3}$, then the following properties hold.*

(a) $i(G) = i(G') + 2$.

(b) $\mathrm{w}(G) = \mathrm{w}(G') + 16$.

Proof. Let G, G' and v' be as defined in the statement of the lemma. Let F be the copy of $K_{2,3}$ added to G' when constructing the graph G, where v_1, v_2 and v_3 are the three small vertices (of degree 2) in F and u_1 and u_2 are the two large vertices (of degree 3) in F. Further, let v_3 be the vertex of F adjacent to v' in G. Let $X = V(F)$. The construction of G is illustrated in Figure 7.

(a) Every i-set of G' can be extended to an ID-set of G by adding to it the vertices u_1 and u_2, and so $i(G) \leq i(G') + 2$. Conversely, let I be an i-set of G. If $\{u_1, u_2\} \subset I$, then we let $I' = I \setminus \{u_1, u_2\}$. If $\{u_1, u_2\} \not\subset I$, then

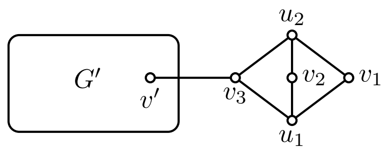


Figure 7: The graphs G' and G in the proof of Lemma 1

$\{v_1, v_2\} \subset I$. Moreover if in this case $v_3 \notin I$, then $v' \in I$ and we let $I' = I \setminus \{v_1, v_2\}$. Suppose that $\{v_1, v_2, v_3\} \subseteq I$. In this case, if a neighbor of v' different from v_3 belongs to the set I, then the set $(I \setminus \{v_1, v_2, v_3\}) \cup \{u_1, u_2\}$ is an ID-set of G of cardinality $|I| - 1$, contradicting the minimality of the set I. Hence, the vertex v_3 is the only neighbor of v' that belongs to the set I. We now let $I' = (I \setminus \{v_1, v_2, v_3\}) \cup \{v'\}$. In all the above cases, the set I' is an ID-set of G' of cardinality $|I| - 2$, and so $i(G') \le |I| - 2 = i(G) - 2$. As observed earlier, $i(G) \le i(G') + 2$. Consequently, $i(G) = i(G') + 2$. This proves part (a).

(b) By supposition, the graph G is a subcubic graph and the graph G' is isolate-free. We infer that the vertex v' has degree 1 or 2 in G' and therefore has degree 2 or 3 in G, respectively. Thus adopting our earlier notation, the edge $v'v_3$ is the only X-exit edge in G. Thus, the number of X-exit edges is $\xi_G(X) = 1$. By Fact 1, the cost of removing X from G is $\Phi_G(X) = \xi_G(X) = 1$ noting that $G' = G - X$ is isolate-free. Thus, $\mathrm{w}(G) = \mathrm{w}(G') + \mathrm{w}_G(X) - \Phi_G(X) = \mathrm{w}(G') + 17 - 1 = \mathrm{w}(G') + 16$, which proves part (b). $\Box$

Suppose that $G \in \mathcal{B}$, and so G is constructed from the base graph B_1 by $k-1$ applications of Operation $\mathcal{O}_1$ where $k \ge 1$. We note that if $k = 1$, then $G = B_1$. Further, we note that $i(B_1) = 3$ and $\mathrm{w}(B_1) = 22$. Hence by repeated applications of Lemma 1, we infer the following properties of graphs in the family $\mathcal{B}$.

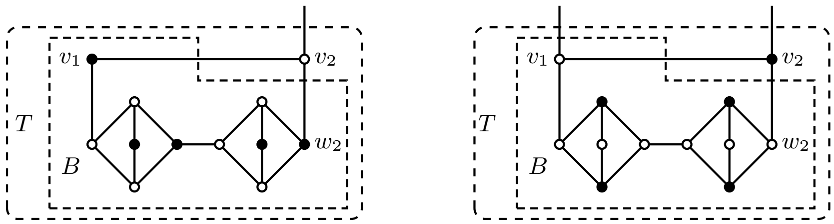
Proposition 1 *If $G \in \mathcal{B}$ is constructed from the base graph B_1 by $k-1$ applications of Operation $\mathcal{O}_1$ where $k \ge 1$, then the following properties hold.*
(a) $i(G) = 2k + 1$, $\mathrm{w}(G) = 16k + 6$, *and* $8i(G) = \mathrm{w}(G) + 2$.
(b) *The canonical ID-set of G is an i-set of G.*
(c) *The non-canonical independent set of G, together with the root vertex of G, is an i-set of G.*
(d) *If v is the root vertex of G or a vertex of degree 2 in G, then $i(G - v) = i(G) - 1$.*
(e) *The core independent set of G is an ID-set of the graph obtained by removing all vertices of degree 2 from G.*
(f) *The graph G contains at most one vertex of degree 1, and no two adjacent vertices of G both have degree 2 in G.*

5.3 Troublesome configurations

In this section, we define what we have coined a 'troublesome configuration' of a subcubic graph G. We define a *troublesome configuration* of G, abbreviated *T-configuration of G*, as a subgraph of G obtained from a bad graph $B \in \mathcal{B}_1$ with root vertex v_1 (of degree 1 in B) by applying the following process.

- adding a new vertex v_2 to B,
- adding an edge joining v_2 to the vertex v_1 and to a vertex w_2 of degree 2 in B different from v_1,
- adding an edge joining v_2 to a vertex not in $V(B) \cup \{v_2\}$, and
- adding at most one edge joining v_1 to a vertex not in $V(B) \cup \{v_2\}$.

If v_1 has degree 2 in G, then we call the troublesome configuration a *type*-1 *troublesome configuration* of G, abbreviated *$T^1_{v_1,v_2}$-configuration of G*. A $T^1_{v_1,v_2}$-configuration is therefore joined to the rest of the graph by exactly one edge, namely an edge that joins v_2 to one vertex not in the subgraph. If v_1 has degree 3 in G, then we call the troublesome configuration a *type*-2 *troublesome configuration* of G, abbreviated *$T^2_{v_1,v_2}$-configuration of G*. A $T^2_{v_1,v_2}$-configuration is therefore joined to the rest of the graph by exactly two edges, namely an edge that joins each of v_1 and v_2 to a vertex not in the subgraph. In a troublesome configuration, we call the vertices v_1 and v_2 the *link vertices* and we call all other vertices the *non-link vertices* of the configuration. Thus the degree of every



(a) A $T^1_{v_1,v_2}$-configuration (b) A $T^2_{v_1,v_2}$-configuration

Figure 8: Type-1 and type-2 troublesome configurations in G

non-link vertex of a troublesome configuration is the same as its degree in G. A $T^1_{v_1,v_2}$-configuration is illustrated in Figure 8(a), and a $T^2_{v_1,v_2}$-configuration is illustrated in Figure 8(b).

The shaded vertices in Figure 8(a) indicate an i-set of $T^1_{v_1,v_2}$ and this set is the canonical ID-set of B. Moreover the shaded vertices in Figure 8(b) indicate an i-set of $T^2_{v_1,v_2}$ that contains the vertex v_2 and the non-canonical vertices of B. More generally, we have the following properties of troublesome configurations of G. These properties follows readily from properties of bad graphs (that belong to the family $\mathcal{B}$) established in Section 5.2.

Proposition 2 *If $T = T^1_{v_1,v_2}$ or $T = T^2_{v_1,v_2}$ is a T_1- or T_2-configuration of G with link vertices v_1 and v_2 that is obtained from a bad graph $B \in \mathcal{B}$ with root vertex v_1, then the following properties hold.*

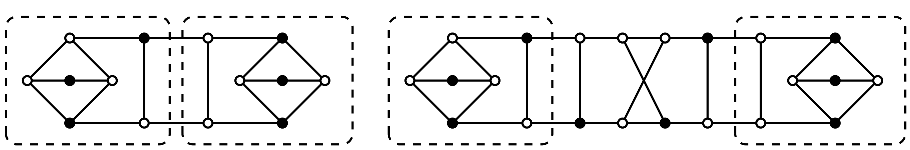
(a) *$i(T) = i(B)$.*
(b) *The canonical ID-set of B is an i-set of T.*
(c) *There exists an i-set of T that contains the link vertex v_2.*
(d) *If S is an arbitrary set of vertices of degree 2 in T that contains neither v_1 nor v_2, then $i(T - S) = i(T) - |S|$. Furthermore, there exists an i-set of $T - S$ that contains neither v_1 nor v_2, and there exists an i-set I_j of $T - S$ that contains v_j for $j \in [2]$.*
(f) *Every vertex in T has degree at least 2, and no two adjacent vertices of T both have degree 2.*

5.4 Examples of graphs achieving equality in Theorem 6

In this section we present a small sample of graphs achieving equality in the upper bound of Theorem 6.

Example 1. If G is the graph shown in Figure 9(a), then $i(G) = 6$ and the shaded vertices indicate an i-set in G. In this example, $\mathrm{w}(G) = 12 \times 3 + 2 \times 4 = 44$ and $\Theta(G) = 2\mathrm{tc}(G) = 2 \times 2 = 4$ (where the two vertex disjoint T_2-configurations are indicated by the subgraphs in the dashed boxes), and so $8i(G) = 48 = 44 + 4 = \mathrm{w}(G) + \Theta(G) = \Omega(G)$.

Moreover if G is the graph shown in Figure 9(b), then $i(G) = 9$ and the shaded vertices indicate an i-set in G. In this example, $\mathrm{w}(G) = 20 \times 3 + 2 \times 4 = 68$ and $\Theta(G) = 2\mathrm{tc}(G) = 2 \times 2 = 4$ (where the two vertex disjoint T_2-configurations are indicated by the subgraphs in the dashed boxes), and so $8i(G) = 72 = 44 + 4 = \mathrm{w}(G) + \Theta(G) = \Omega(G)$.



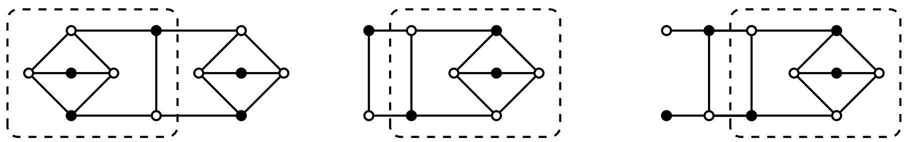
(a) (b)

Figure 9: Examples of graphs achieving equality in the upper bound of Theorem 6

Example 2. If G is the graph shown in Figure 10(a), then $i(G) = 5$ and the shaded vertices indicate an i-set in G. In this example, $\mathrm{w}(G) = 10 \times 3 + 2 \times 4 = 38$ and $\Theta(G) = 2\mathrm{tc}(G) = 2 \times 1 = 2$ (where a T_2-configuration is indicated by the subgraph in the dashed box), and so $8i(G) = 40 = 38 + 2 = \mathrm{w}(G) + \Theta(G) = \Omega(G)$.

If G is the graph shown in Figure 10(b), then $i(G) = 4$ and the shaded vertices indicate an i-set in G. In this example, $\mathrm{w}(G) = 6 \times 3 + 3 \times 4 = 30$ and $\Theta(G) = 2\mathrm{tc}(G) = 2 \times 1 = 2$ (where a T_2-configuration is indicated by the subgraph in the dashed box), and so $8i(G) = 32 = 30 + 2 = \mathrm{w}(G) + \Theta(G) = \Omega(G)$.

If G is the graph shown in Figure 10(c), then $i(G) = 5$ and the shaded vertices indicate an i-set in G. In this example, $\mathrm{w}(G) = 8 \times 3 + 4 + 2 \times 5 = 38$ and $\Theta(G) = 2\mathrm{tc}(G) = 2 \times 1 = 2$ (where a T_2-configuration is indicated by the subgraph in the dashed box), and so $8i(G) = 40 = 38 + 2 = \mathrm{w}(G) + \Theta(G) = \Omega(G)$.



(a) F_1 (b) F_2 (c) F_3

Figure 10: Examples of graphs achieving equality in the upper bound of Theorem 6

Example 3. If G is the graph shown in Figure 11, then $i(G) = 18$ and the shaded vertices indicate an i-set in G. In this example, $\mathrm{w}(G) = 44 \times 3 + 2 \times 4 = 140$ and $\Theta(G) = 2\mathrm{tc}(G) = 2 \times 2 = 4$ (where the T_2-configurations are indicated by the subgraphs in the dashed boxes), and so $8i(G) = 144 = 140 + 4 = \mathrm{w}(G) + \Theta(G) = \Omega(G)$.

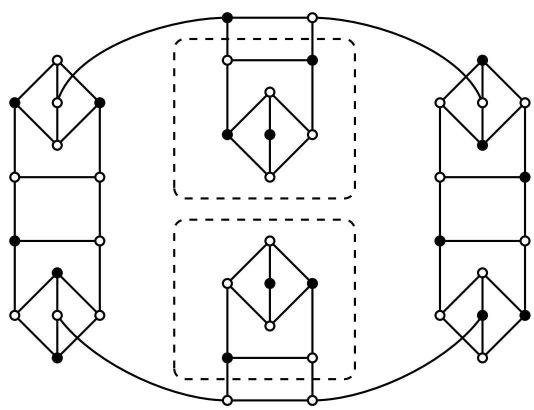


Figure 11: An example of a graph achieving equality in the upper bound of Theorem 6

6 Proof of key result

Recall that the vertex weight of a subcubic graph G is $\mathrm{w}(G) = 8n_0(G) + 5n_1(G) + 4n_2(G) + 3n_3(G)$, and the structural weight of G is $\Theta(G) = 2\mathrm{tc}(G) + 2b(G)$. Moreover the total weight of G, called the Ω-weight of G, is the sum of its vertex weight and structural weight, that is, $\Omega(G) = \mathrm{w}(G) + \Theta(G)$. In this section, we present a proof of our key result, namely Theorem 6. Its statement in terms of the Ω-weight of G is as follows:

Theorem 6. *If G is a subcubic graph of order n that contains no $K_{3,3}$-component and no $(C_5 \,\square\, K_2)$-component, then $8i(G) \le \Omega(G)$.*

Proof. Let G be a subcubic graph of order n that contains no $K_{3,3}$-component and no $(C_5 \,\square\, K_2)$-component, and let $V = V(G)$. We wish to prove that $8i(G) \le \Omega(G)$. Suppose that G is a counterexample to the inequality with minimum order, and let G have order n. Thus, $8i(G) > \Omega(G)$ but every subcubic graph G' of order less than n that contains no $K_{3,3}$-component and no $(C_5 \,\square\, K_2)$-component satisfies $8i(G') \le \Omega(G')$. Since $\Theta(G) \ge 0$, we note that $\mathrm{w}(G) \le \Omega(G)$. Hence, if $8i(G) \le \mathrm{w}(G)$, then $8i(G) \le \Omega(G)$, a contradiction. Therefore, $8i(G) > \mathrm{w}(G)$.

Since our proof of Theorem 6 is long, we break the proof into four parts to enhance clarity and exposition of the proof. The first part of the proof establishes important structural properties of the counterexample G. The second part of the proof considers the case when the minimum degree of G equals 1. The third part of the proof considers the case when the minimum degree of G equals 2. The fourth and final part of the proof considers the case when the graph G is cubic, that is, when G is a 3-regular graph.

6.1 Part 1: Structural properties of G

In this section, namely Part 1 of our proof, we establish structural properties of the counterexample G. A component of a graph that belongs to the family $\mathcal{B}$ is a *bad component* of the graph.

Claim 1 *The subcubic graph G contains at least one subgraph isomorphic to $K_{2,3}$.*

Proof. If G contains no subgraph isomorphic to $K_{2,3}$, then by Theorem 3, we have $8i(G) \le \mathrm{w}(G)$, a contradiction. (□)

By Claim 1, we note that $n \ge 5$. If $n = 5$, then either $G = K_{2,3}$, in which case $8i(G) = 16 < 18 = \mathrm{w}(G)$, or G is obtained from $K_{2,3}$ by adding an edge, in which case $8i(G) = 16 = \mathrm{w}(G)$, contradicting the fact that G is a counterexample. Hence, $n \ge 6$.

Claim 2 *The graph G is connected.*

Proof. If G is not connected, then, by the minimality of $n(G)$, the theorem holds for every component of G, and therefore also for G, a contradiction. (□)

Claim 3 *$G \notin \mathcal{B}$.*

Proof. If $G \in \mathcal{B}$, then $b(G) = 1$ and $\Theta(G) = 2b(G) = 2$, and so $\Omega(G) = \mathrm{w}(G) + 2$. By Proposition 1(a), $8i(G) = \mathrm{w}(G) + 2 = \mathrm{w}(G) + \Theta(G) = \Omega(G)$, a contradiction. (□)

By Claim 3, we have $b(G) = 0$, and so $\Theta(G) = 2\mathrm{tc}(G)$.

Claim 4 *The graph G contains no troublesome configuration.*

Proof. Suppose, to the contrary, that G contains a troublesome configuration, say T. Thus, $T = T_{v_1,v_2}$ is obtained from a bad graph $B \in \mathcal{B}_1$ with root vertex v_1 (of degree 1 in B) by adding a new vertex v_2 to B, adding the edges v_1v_2 and v_2w_2, where w_2 is a vertex of degree 2 in B different from v_1. Furthermore, there are at most two exit edges that join T to vertices in $V(G) \setminus V(T)$ via one edge incident with v_2 and possibly one additional exit edge incident with v_1. Let u be the vertex in B adjacent to v_1 in G.

Let B_u be the subgraph $B - v_1$, and let B_u contain $k \ge 1$ vertex disjoint copies of $K_{2,3}$. The weight of B_u is $\mathrm{w}(B_u) = 16k+2$. Moreover, $i(B_u) = 2k$, which follows by induction and using Lemma 1. Indeed, the non-canonical independent set of B, which we denote by I_u, is an i-set of B_u. Let $G' = G - V(B_u)$. We note that v_1v_2 is an edge of G'. Further, v_1 has degree 1 or 2 in G' and v_2 has degree 2 in G'. Hence every i-set of G' can be extended to an ID-set of G by adding to it the $2k$ vertices from the set I_u, and so $i(G) \le i(G') + |I_u| = i(G') + 2k$. Moreover, $\mathrm{w}(G) = \mathrm{w}(G') + \mathrm{w}(B_u) - 4$ noting that the two edges uv_1 and v_2w_2 joining $V(B_u)$ to $V(G')$ decrease the weights of each of the vertices u, v_1, v_2 and w_2 by 1. Thus, $\mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}(B_u) + 4 = \mathrm{w}(G) - (16k+2) + 4 = \mathrm{w}(G) - 16k + 2$.

We do not create a bad component in G' since such a component would contain the adjacent vertices v_1 and v_2 where we note that both v_1 and v_2 have degree at most 2 in G'. This contradicts the property that at least one of every two adjacent vertices in a bad component is a vertex of degree 3. We therefore infer that $b(G') = 0$.

We note that the troublesome configuration T of G is no longer a troublesome configuration of G'. Further we note that v_1 and v_2 are adjacent vertices in G' where v_1 has degree 1 or 2 and v_2 has degree 2 in G'. Any new troublesome configuration T' of G' that is not a troublesome configuration of G necessarily contains both v_1 and v_2

as its link vertices. However in this case, neither v_1 nor v_2 is adjacent to a vertex not in the T'-configuration noting that both v_1 and v_2 have the same degrees in T' and G'. Hence we do not create a new troublesome configuration. Thus, $\mathrm{tc}(G') = \mathrm{tc}(G) - 1$. As observed earlier, $b(G') = 0$. Thus, $\Theta(G') = 2\mathrm{tc}(G') = 2\mathrm{tc}(G) - 2 = \Theta(G) - 2$, implying that $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le (\mathrm{w}(G) - 16k + 2) + (\Theta(G) - 2) = \Omega(G) - 16k$. Hence, $8i(G) \le 8(i(G') + 2k) \le \Omega(G') + 16k \le \Omega(G)$, a contradiction. (□)

Claim 5 $\Theta(G) = 0$.

Proof. By Claim 3, $b(G) = 0$. By Claim 4, there are no troublesome configurations in G, implying that $\mathrm{tc}(G) = 0$. Hence, $\Theta(G) = b(G) + \mathrm{tc}(G) = 0$. (□)

Claim 6 *The graph G does not contain a $K_{2,3}$-subgraph that contains two vertices of degree 2 in G.*

Proof. Suppose, to the contrary, that G contains a $K_{2,3}$-subgraph, F, that contains two vertices of degree 2 in G. Let u_1 and u_2 be the two vertices of degree 2 in G that belong to F, and let w_1 and w_2 be the two common neighbors of u_1 and u_2. Further, let u_3 be the third common neighbor of w_1 and w_2. We note that the vertex u_3 has degree 2 in F and has degree 3 in G. Let v_1 be the third neighbor of u_3. Let $G' = G - V(F)$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices w_1 and w_2, and so $i(G) \le i(G') + 2$.

We note that $\mathrm{w}(G) = \mathrm{w}(G') + 16$. By Claim 5, $\Theta(G) = 0$ and therefore $\Omega(G) = \mathrm{w}(G) = \mathrm{w}(G') + 16$. We show next that $\Theta(G') = 0$. If G' belongs to $\mathcal{B}$, then so too does the original graph G. Thus, $G' \notin \mathcal{B}$, and so $b(G') = 0$. By Claim 5, the graph G contains no troublesome configuration. Suppose that G' contains a troublesome configuration T'. Necessarily such a subgraph T' contains the vertex v_1. Note that v_1 is not a link vertex of T', since this would imply that T' is troublesome configuration already in G (see Figure 12(a)). Hence v_1 is not a link vertex, see Figure 12(b) as an illustration in this case. But then adding the subgraph F to T', together with the edge u_3v_1, produces a troublesome configuration T of G, and so $\mathrm{tc}(G) \ge 1$, contradicting Claim 5. Hence, $\mathrm{tc}(G') = 0$. Thus, $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \Omega(G) - 16$. As observed earlier, $i(G) \le i(G') + 2$. Hence, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \Omega(G)$, a contradiction. (□)

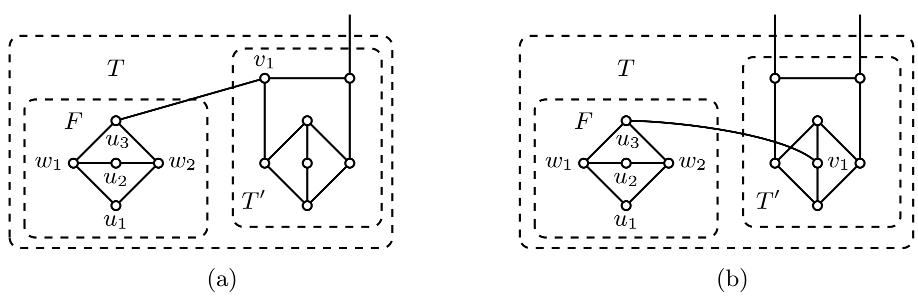


Figure 12: Possible subgraphs in the proof of Claim 6

We shall frequently use the following property that removing an exit edge increases the total weight by at most 3.

Claim 7 *If $X \subset V(G)$ and $G' = G - X$, then removing an X-exit edge when constructing G' increases the total weight by at most* 3.

Proof. Let $X \subset V(G)$ and $G' = G - X$, and consider an exit edge $vv' \in E(G)$ where $v \in X$ and $v' \in V(G')$. If v' is isolated in G', then the vertex v' has degree 1 in G and degree 0 in G', and so $\mathrm{w}_{G'}(v') - \mathrm{w}_G(v') = 8 - 5 = 3$. If $\deg_{G'}(v') \ge 1$, then $\mathrm{w}_{G'}(v') - \mathrm{w}_G(v') = 1$. However, in this case removing the edge vv' may create a bad component or a troublesome configuration, resulting in an increase in the structural weight by 2. Thus the total weight increases by 3 if either we create an isolated vertex - in this case, the vertex weight increases by 3 and the structural weight by 0 - or if we create a bad component or a new troublesome configuration that contains the vertex incident with the exit edge - in this case, the vertex weight increases by 1 and the structural weight increases by 2, resulting in a total increase in the weight by 3. In both cases the removal of the X-exit edge vv' increases the total weight by at most 3. $(\Box)$

By Claim 6, every $K_{2,3}$-subgraph in G contains at most one vertex of degree 2 in G.

Claim 8 *The graph G does not contain a $K_{2,3}$-subgraph that contains a vertex of degree 2 in G.*

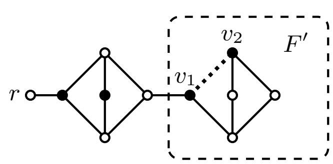
Proof. Suppose, to the contrary, that G contains a $K_{2,3}$-subgraph, F, that contains a vertex, say u, of degree 2 in G. Let w_1 and w_2 be the two neighbors of u in F, and so w_1 and w_2 have degree 3 in F. Let u_1 and u_2 be the common neighbors of w_1 and w_2 in F different from u. Thus, $N_G(w_1) = N_G(w_2) = \{u, u_1, u_2\}$. By Claim 6, both u_1 and u_2 have degree 3 in G. If $u_1u_2 \in E(G)$, then the graph G is determined, and so $n = 5$, a contradiction. Hence, $u_1u_2 \notin E(G)$. Let v_1 and v_2 be the neighbors of u_1 and u_2, respectively, that do not belong to F.

We show firstly that $v_1 \neq v_2$. Suppose, to the contrary, that $v_1 = v_2$. If $\deg_G(v_1) = 2$, then the graph G is determined. In this case, we have $i(G) = 2$, $\mathrm{w}(G) = 20$ and $\Theta(G) = 0$. Thus, $\Omega(G) = 20$, and so $8i(G) < \Omega(G)$, a contradiction. Hence, $\deg_G(v_1) = 3$. Let v be the neighbor of v_1 different from u_1 and u_2. If $\deg_G(v) = 1$, then the graph G is determined, and as before $8i(G) < \Omega(G)$ (noting that $\{u, v_1\}$ is an i-set of G), a contradiction. Hence, $\deg_G(v) \geq 2$. We now let $X = V(F) \cup \{v, v_1\}$ and consider the graph $G' = G - X$. We note that there are at most two X-exit edges in G (and such edges are incident with v). By Claim 7, the removal of an X-exit edge when constructing G' increases the total weight by at most 3. If there are two X-exit edges in G, then $\mathrm{w}_G(X) = 22$, and we infer that $\Omega(G') \leq \Omega(G) - 22 + 6 = \Omega(G) - 16$. On the other hand, if there is only one X-exit edge in G, then $\mathrm{w}_G(X) = 23$, implying that $\Omega(G') \leq \Omega(G) - 23 + 3 = \Omega(G) - 20$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices u and v_1, and so $i(G) \leq i(G') + 2$. Hence, $8i(G) \leq 8i(G') + 16 \leq \Omega(G') + 16 \leq \Omega(G)$, a contradiction. Hence, $v_1 \neq v_2$.

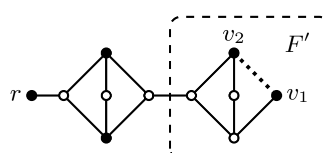
Suppose that $v_1v_2 \in E(G)$. If at least one of v_1 and v_2 has degree 3 in G, then the subgraph of G induced by $V(F) \cup \{v_1, v_2\}$ is a troublesome configuration in G, contradicting Claim 4. Hence, both v_1 and v_2 have degree 2 in G, and so the graph G is determined. In this case, $i(G) = 3$ and $\Omega(G) = \mathrm{w}(G) = 24$, and so $8i(G) = \Omega(G)$, a contradiction. Hence, $v_1v_2 \notin E(G)$.

We now consider the graph G' obtained from $G - V(F)$ by adding the edge $e = v_1v_2$. Let I' be an i-set of G'. If neither v_1 nor v_2 belongs to I', then let $I = I' \cup \{w_1, w_2\}$. If $v_1 \in I'$, then let $I = I' \cup \{u, u_2\}$. If $v_2 \in I'$, then let $I = I' \cup \{u, u_1\}$. In all cases, the set I is an ID-set of G and $|I| = |I'| + 2$. Thus, $i(G) \leq |I| \leq |I'| + 2 = i(G') + 2$. Moreover, $\mathrm{w}(G') = \mathrm{w}(G) - 16 = \Omega(G) - 16$. If $\Theta(G') = 0$, then $\Omega(G') = \mathrm{w}(G') = \Omega(G) - 16$, and so $8i(G) \leq 8(i(G') + 2) \leq \Omega(G') + 16 = \Omega(G)$, a contradiction. Hence, $\Theta(G') > 0$, implying that either $G' \in \mathcal{B}$ (noting that G' is connected) or there is a troublesome configuration in G' that contains the (adjacent) vertices v_1 and v_2. In both cases, we have $\Theta(G') = 2$, and so $\Omega(G') = \mathrm{w}(G') + \Theta(G') = (\Omega(G) - 16) + 2 = \Omega(G) - 14$.

Suppose that $G' \in \mathcal{B}$. We note that G' contains at least one copy of $K_{2,3}$ that contains two vertices of degree 2 in G'. By Claim 6, $G' - v_1v_2$ does not contain a $K_{2,3}$-subgraph that contains two vertices of degree 2 in G. We therefore infer that $G' \in \mathcal{B}_1$ and that G' contains exactly one copy of $K_{2,3}$, say F', that contains two vertices of degree 2 in G'. Moreover, the added edge v_1v_2 belongs to F'. Thus, there are two possibilities that may occur. The first case is that both v_1 and v_2 have degree 3 in G', as illustrated in Figure 13(a), and the second case is that one of v_1 and v_2 has degree 2 in G', as illustrated in Figure 13(b). In both illustrations, we indicate the added edge v_1v_2 by the dotted edge, which recall exists in G' but not in G.



(a)



(b)

Figure 13: Possible graphs G' in the proof of Claim 8

By properties of the bad family $\mathcal{B}$ (see Section 5.2), we infer that there is an i-set, say I', of $G' - v_1v_2$ that contains both v_1 and v_2 and is such that $|I'| = i(G') - 1$ in the first case (as indicated by the shaded vertices in

Figure 13(a)) and $|I'| = i(G')$ in the second case (as indicated by the shaded vertices in Figure 13(b)). In both cases, we have that I' is an ID-set of $G' - v_1v_2$. Moreover, $\{v_1, v_2\} \subset I'$ and $|I'| \le i(G')$. The set I' can be extended to an ID-set of G by adding to it the vertex u. Thus, $i(G) \le i(G') + 1$. We note that $\mathrm{w}(G) = \mathrm{w}(G') + 16$ and $\Theta(G) = \Theta(G') - 2$, and so $\Omega(G) = \mathrm{w}(G) + 16 + \Theta(G) = \mathrm{w}(G') + \Theta(G') - 2 = \Omega(G') + 14$. Therefore, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 < \Omega(G)$, a contradiction. Hence, $G' \notin \mathcal{B}$, and so $b(G') = 0$.

Thus, there is a troublesome configuration, say T', in G' that contains the (adjacent) vertices v_1 and v_2. Let z_1 and z_2 be the two link vertices in T'. Recall that $e = v_1v_2$ is the edge that was added to construct G'. If e does not belong to a copy of $K_{2,3}$ in T' and if $e \ne z_1z_2$, then removing the edge e from T' and adding back the deleted $K_{2,3}$-subgraph F (and the edges u_1v_1 and u_2v_2) produces a troublesome configuration in G, a contradiction. Hence either the edge e belongs to a copy of $K_{2,3}$ in T' or v_1 and v_2 are the two link vertices in T' (that is, $\{v_1, v_2\} = \{z_1, z_2\}$). Let T' be obtained from a bad graph $B' \in \mathcal{B}$, where we may assume (renaming z_1 and z_2 if necessary) that z_1 is the root vertex of B'.

Claim 8.1 *The edge e joins the two link vertices in T', that is, $\{v_1, v_2\} = \{z_1, z_2\}$.*

Proof. Suppose that e belongs to a copy of $K_{2,3}$, say F_e, in T'. Let T' have $k' \ge 1$ vertex disjoint copies of $K_{2,3}$. Let I' be the non-canonical independent set of B', and so $|I'| = 2k'$ (and neither z_1 nor z_2 belong to I'). Moreover, let I'_e be obtained from I' by deleting the two vertices in F_e that belong to I' and replacing them with the vertices v_1 and v_2. We note that $|I'_e| = |I'| = 2k'$.

Suppose firstly that neither v_1 nor v_2 is adjacent to z_1 or z_2. In this case, we let $X^* = V(F) \cup V(B') \setminus \{z_1\}$ and we consider the graph $G^* = G - X^*$. In the case when $k' = 1$, the graph illustrated in Figure 14(a) is an example of a subgraph of G, where the subgraphs F, F_e, T' and G^* are indicated by the dashed boxes. Since $\mathrm{tc}(G) = 0$, any troublesome configuration in G^* necessarily contains the adjacent vertices z_1 and z_2. However, both z_1 and z_2 have degree at most 2 in G^*, and therefore cannot belong to a troublesome configuration in G^*. Moreover, since no bad graph contains two adjacent vertices, both of which has degree at most 2, and since $b(G) = 0$, the connected graph G^* does not belong to the bad family $\mathcal{B}$, that is, $b(G^*) = 0$. Thus, $\Theta(G^*) = 0$ and $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 2 = \mathrm{w}(G) - 16(k' + 1) + 2 = \Omega(G) - 16k' - 14$. We now let $I^* = I'_e \cup \{u\}$. Thus, $|I^*| = 2k' + 1$. In the illustration in Figure 14, the set I^* is indicated by the shaded vertices. Every i-set of G^* can be extended to an ID-set of G by adding to it the set I^*, and so $i(G) \le i(G^*) + |X^*| = i(G^*) + 2k' + 1$. Thus, $8i(G) \le 8(i(G^*) + 2k' + 1) \le \Omega(G^*) + 16k' + 8 = \Omega(G) - 6 < \Omega(G)$, a contradiction.

Hence renaming vertices if necessary, we may assume that $v_2z_2 \in E(G)$. In this case, we let $X^* = N_G[v_1] \cup N_G[u]$ and we consider the graph $G^* = G - X^*$. In the case when $k' = 1$, the graph illustrated in Figure 14(b) is an example of a subgraph of G, where the subgraphs F, F_e, T' and G^* are indicated by the dashed boxes. Since $b(G) = \mathrm{tc}(G) = 0$, we infer from the structure of the graph G^* that $b(G^*) = \mathrm{tc}(G^*) = 0$, and so $\Theta(G^*) = 0$ and $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 5 = \mathrm{w}(G) - 23 + 5 = \Omega(G) - 18$. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertices u and v_1, and so $i(G) \le i(G^*) + 2$. Thus, $8i(G) \le 8(i(G^*) + 2) \le \Omega(G^*) + 16 = \Omega(G) - 2 < \Omega(G)$, a contradiction. (□)

By Claim 8.1, the edge e joins the two link vertices in T', that is, $\{v_1, v_2\} = \{z_1, z_2\}$. Recall that F is an arbitrary $K_{2,3}$-subgraph that contains exactly one vertex of degree 2 in G. Moreover, u_1 and u_2 are the two vertices of degree 2 in F that have degree 3 in G, where v_1 and v_2 are the neighbors of u_1 and u_2, respectively, that do not belong to F. We have shown that $v_1 \ne v_2$ and v_1 and v_2 are not adjacent. Moreover, we have shown that if G' is obtained from $G - V(F)$ by adding the edge $e = v_1v_2$, then G' contains a troublesome configuration, say T', that necessarily contains v_1 and v_2 as the two link vertices of F'. This property, which we call property $(*)$, holds for every $K_{2,3}$-subgraph that contains exactly one vertex of degree 2 in G.

We note that every $K_{2,3}$-subgraph in T' contains only one vertex of degree 2 in G, for otherwise, T' would contain a $K_{2,3}$-subgraph with two vertices of degree 2 in G, contradicting Claim 6. Suppose that T' contains two or more copies of $K_{2,3}$. In this case, we could have chosen the $K_{2,3}$-subgraph F to be adjacent to another $K_{2,3}$-subgraph. With this choice of F, and with the vertices u_1, u_2, v_1, and v_2 as defined earlier (notably, u_1 and u_2 are vertices of degree 3 whose neighbors outside F are v_1 and v_2, respectively), at least one of v_1 or v_2 belongs to a copy of $K_{2,3}$ in T', contradicting property $(*)$ that v_1 and v_2 are necessarily the two link vertices of T' (and noting that the link vertices of T' do not belong to a copy of $K_{2,3}$ in T').

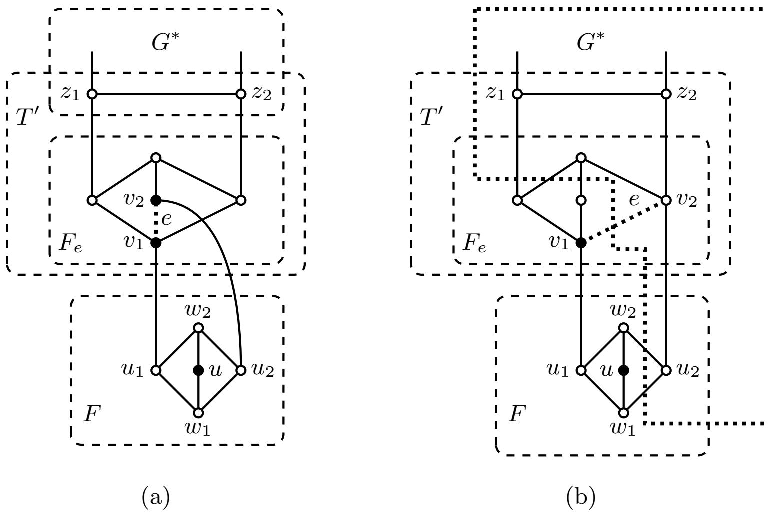


Figure 14: Possible subgraphs in the proof of Claim 8.1

Hence, T' contains exactly one copy of $K_{2,3}$, say F'. Let x be the vertex of F' of degree 2 in G, and let w_3 and w_4 be the two neighbors of x in F'. Let x_1 and x_2 be the common neighbors of w_3 and w_4 in F' different from x. Thus, $N_G(w_3) = N_G(w_4) = \{x, x_1, x_2\}$. Renaming x_1 and x_2 if necessary, we may assume that v_1x_1 and v_2x_2 are edges of G. Thus the graph illustrated in Figure 15 is a subgraph of G, where the added edge e of G' is indicated by the dotted line joining v_1 and v_2.

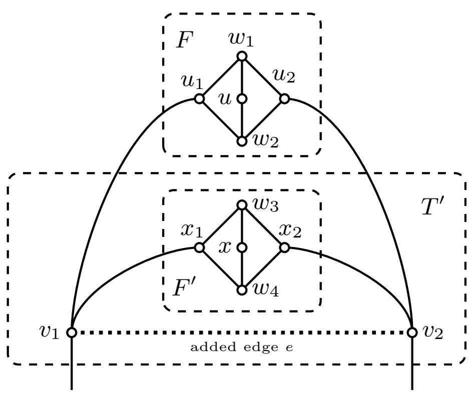


Figure 15: A possible subgraph in the proof of Claim 8

Since T' is a troublesome configuration of G', at least one of v_1 and v_2 has degree 3 in G'. By symmetry, we may assume that $\deg_G(v_1) = 3$ and we let y_1 be the neighbor of v_1 in G different from u_1 and x_1. Suppose that $\deg_G(v_2) = 2$, and so $N_G(v_2) = \{u_2, x_2\}$. If $\deg_G(y_1) = 1$, then the graph G is determined. In this case, $i(G) = |\{u, x, v_1, v_2\}| = 4$ and $\Omega(G) = \mathrm{w}(G) = 16{+}16{+}5{+}4{+}3 = 44$. Thus, $8i(G) < \Omega(G)$, a contradiction. Hence, $\deg_G(y_1) \geq 2$. In this case, we let $X^* = V(F) \cup V(F') \cup \{v_1, v_2, y_1\}$ and we consider the graph $G^* = G - X^*$. We note that every X^*-exit edge is incident with y_1. Moreover, since $\deg_G(y_1) \geq 2$, there is at least one X^*-exit edge. By Claim 7, the removal of an X^*-exit edge when constructing G^* increases the total weight by at most 3. Since there are at most two X^*-exit edges, we have that $\Omega(G^*) \leq \Omega(G) - \mathrm{w}_G(X^*) + 6 \leq \Omega(G) - (16{+}16{+}4{+}3{+}3) + 6 = \Omega(G) - 36$. Every i-set of G^* can be extended to an ID-set of G by adding to it the set $\{u, x, v_1, v_2\}$, and so $i(G) \leq i(G^*) + 4$. Thus, $8i(G) \leq 8(i(G^*) + 4) \leq \Omega(G^*) + 32 = \Omega(G) - 4 < \Omega(G)$, a contradiction. Hence, $\deg_G(v_2) = 3$. Let y_2 be the neighbor of v_2 in G different from u_2 and x_2.

Suppose that $y_1 = y_2$. If $\deg_G(y_1) = 2$, then the graph G is determined, and in this case $i(G) = |\{u, x, v_1, v_2\}| = 4$ and $\Omega(G) = \mathrm{w}(G) = 16{+}16{+}4{+}3{+}3 = 42$. Thus, $8i(G) < \Omega(G)$, a contradiction. Hence, $\deg_G(y_1) = 3$. As before, we let $X^* = V(F) \cup V(F') \cup \{v_1, v_2, y_1\}$ and we consider the graph $G^* = G - X^*$. In this case, there is only one X^*-exit edge, implying analogously as before that $\Omega(G^*) \le \Omega(G) - \mathrm{w}_G(X^*) + 3 = \Omega(G) - (16+16+3+3+3)+3 = \Omega(G) - 38$. As before, $i(G) \le i(G^*) + 4$, yielding the contradiction $8i(G) < \Omega(G)$. Hence, $y_1 \neq y_2$. We now let $X'' = V(F) \cup V(F') \cup \{v_1, v_2, y_1, y_2\}$ and we consider the graph $G'' = G - X''$. Thus the graph illustrated in Figure 16 is a subgraph of G.

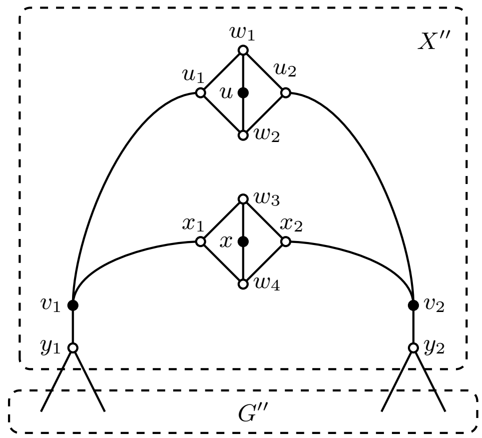


Figure 16: A possible subgraph in the proof of Claim 8

We note that there are at most four X''-exit edges, namely at most two edges incident with y_1 and at most two edges incident with y_2. By Claim 7, removing each X''-exit edge when constructing G'' increases the total weight by at most 3. Thus, $\Omega(G'') \le \Omega(G) - \mathrm{w}_G(X'') + 4 \times 3 = \Omega(G) - (16 + 16 + 12) + 12 = \Omega(G) - 32$. Every i-set of G'' can be extended to an ID-set of G by adding to it the set $\{u, x, v_1, v_2\}$ (as indicated by the shaded vertices in Figure 16), and so $i(G) \le i(G'') + 4$. Thus, $8i(G) \le 8(i(G'') + 4) \le \Omega(G'') + 32 = \Omega(G)$, a contradiction. ($\Box$)

By Claim 8, every vertex that belongs to a $K_{2,3}$-subgraph has degree 3 in G.

Claim 9 *Every $K_{2,3}$-subgraph in G is an induced subgraph in G.*

Proof. Suppose, to the contrary, that G contains a $K_{2,3}$-subgraph, F, that is not an induced subgraph of G. Let the subgraph F have partite sets $\{w_1, w_2\}$ and $\{u_1, u_2, u_3\}$. Renaming vertices if necessary, we may assume that $u_2u_3 \in E(G)$. If $\deg_G(u_1) = 2$, then the graph G is determined, and so $n = 5$, a contradiction. Hence, $\deg_G(u_1) = 3$. We now consider the graph $G' = G - \{u_2, u_3, w_1, w_2\}$. We note that the vertex u_1 has degree 1 in G', and so u_1 does not belong to a troublesome configuration in G'. Moreover, by Claim 6, the (connected) graph $G' \notin \mathcal{B}$. Thus, $\Omega(G') = \Omega(G) - 12 + 2 = \Omega(G) - 10$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex u_2, and so $i(G) \le i(G') + 1$. Thus, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 < \Omega(G)$, a contradiction. ($\Box$)

Claim 10 *$b(G - e) = 0$ for all $e \in E(G)$.*

Proof. Suppose, to the contrary, that the claim is false. Thus there exists an edge e in G whose removal creates a bad component. Let $B \in \mathcal{B}$ be such a bad component in $G - e$. The component B contains at least one copy of $K_{2,3}$, say F, with two (nonadjacent) vertices of degree 2 in B. By Claim 9, F is an induced subgraph of G. Thus the edge e is incident with at most one of the two vertices in F that have degree 2 in B. Therefore, F contains at least one vertex of degree 2 in G, contradicting Claim 8. ($\Box$)

Claim 11 *$b(G - e_1 - e_2) = 0$ for all $e_1, e_2 \in E(G)$.*

Proof. Suppose, to the contrary, that the claim is false. Thus there exists edges e_1 and e_2 in G whose removal creates a bad component. Let $B \in \mathcal{B}$ be such a bad component in $G - e_1 - e_2$, and let v be the root vertex of B.

Let B contain $k \geq 1$ copies of $K_{2,3}$. If $B \in \mathcal{B}_3$, then $k \geq 3$ and there are $k+3 \geq 6$ vertices of degree 2 in B. Thus, B contains at least one copy of $K_{2,3}$ that contains a vertex of degree 2 in G, contradicting Claim 8. Hence, $B \in \mathcal{B}_1 \cup \mathcal{B}_2$.

Suppose that $B \in \mathcal{B}_2$. In this case, $k \geq 2$ and there are $k+2 \geq 4$ vertices of degree 2 in B that belong to $K_{2,3}$-subgraphs. If $k \geq 3$, then there are at least five vertices of degree 2 in B that belong to $K_{2,3}$-subgraphs, implying that B contains at least one copy of $K_{2,3}$ that contains a vertex of degree 2 in G, a contradiction. Hence, $k = 2$. Let F_1 and F_2 be the two copies of $K_{2,3}$ in B, and let u_i and v_i be the two vertices in F_i that have degree 2 in B for $i \in [2]$. By Claims 8 and 9, we infer renaming vertices if necessary that $e_1 = u_1u_2$ and $e_2 = v_1v_2$. However, then the graph G is determined and is illustrated in Figure 17(a), where the edges e_1 and e_2 are dotted. In this case, $i(G) = 3$ (an i-set of G is indicated by the three shaded vertices in Figure 17(a)) and $\Omega(G) = \mathrm{w}(G) = 34$. Thus, $8i(G) < \Omega(G)$, a contradiction.

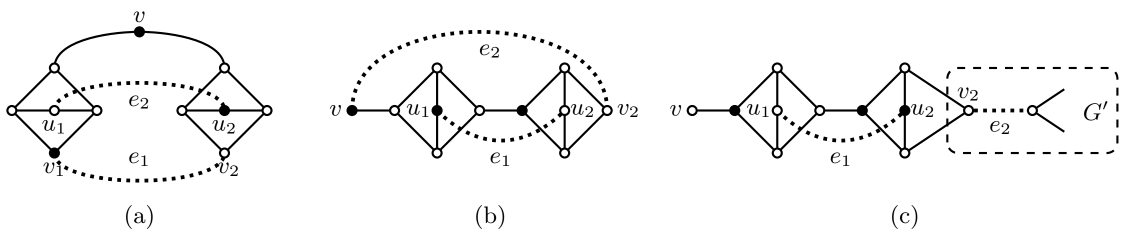


Figure 17: Possible subgraphs in the proof of Claim 11

Hence, $B \in \mathcal{B}_1$, and so $k \geq 1$ and there are $k+1 \geq 2$ vertices of degree 2 in B that belong to $K_{2,3}$-subgraphs. If $k \geq 4$, then there are at least five vertices of degree 2 in B that belong to $K_{2,3}$-subgraphs, implying that B contains at least one copy of $K_{2,3}$ that contains a vertex of degree 2 in G, a contradiction. Therefore, $k \leq 3$.

Suppose that $k = 3$. In this case, there are four vertices of degree 2 in B that belong to $K_{2,3}$-subgraphs. Since no copy of a $K_{2,3}$-subgraph in B contains a vertex of degree 2 in G, we infer that these four vertices of degree 2 in B are incident with the edges e_1 and e_2 in G. The graph G is now determined and there are two non-isomorphic graphs to which G can be isomorphic. By Claim 9, the edge e_i joins vertices that belong to different copies of $K_{2,3}$ in B for $i \in [2]$. The set consisting of the three core vertices in B, together with one end of e_1 and one end of e_2, is an ID-set of G, and so $i(G) \leq 5$. Moreover, $\Omega(G) = \mathrm{w}(G) = 15 \times 3 + 5 = 50$, and so $8i(G) < \Omega(G)$, a contradiction. Therefore, $k \leq 2$.

Suppose that $k = 2$. In this case, there are three vertices of degree 2 in B that belong to $K_{2,3}$-subgraphs. Let F_1 and F_2 be the two copies of $K_{2,3}$ in B, where F_2 is the copy that contains two vertices of degree 2 in B. Let u_1 be the vertex in F_1 of degree 2 in B and let u_2 and v_2 be the two vertices in F_2 of degree 2 in B. Since no copy of a $K_{2,3}$-subgraph in B contains a vertex of degree 2 in G, we infer that the vertices u_1, u_2 and v_2 are incident with the edges e_1 and e_2 in G. Moreover by Claim 9, vertices u_2 and v_2 are not adjacent in G. Renaming u_2 and v_2 if necessary, we may assume that $e_1 = u_1u_2$ and that e_2 is incident with v_2.

Suppose that vertex v is incident with e_2, and so $e_2 = vv_2$. Thus, the graph G is determined as is illustrated in Figure 17(b). In this case, $i(G) = 3$ (an i-set of G is indicated by the three shaded vertices in Figure 17(b)) and $\Omega(G) = \mathrm{w}(G) = 34$. Thus, $8i(G) < \Omega(G)$, a contradiction. Hence, the vertex v is not incident with e_2. We now consider the graph $G' = G - (V(B) \setminus \{v_2\})$ (see Figure 17(c), where the graph G' is indicated by the dashed box). We note that $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 32 + 2 = \Omega(G) - 30$. Every i-set of G' can be extended to an ID-set of G by adding to it the two core vertices of B together with the vertex u_2 (as indicated by the shaded vertices in Figure 17(c). Thus, $i(G) \leq i(G') + 3$, and so $8i(G) < \Omega(G)$, a contradiction.

Hence, $k = 1$. Let F be the copy of $K_{2,3}$ in B, and let u_1 and v_1 be the two vertices of degree 2 in B that belong to F. Since every vertex in F has degree 3 in G, the vertices u_1 and v_1 are incident with the edges e_1 and e_2 in G. Moreover by Claim 9, vertices u_1 and v_1 are not adjacent in G. Renaming edges if necessary, we may assume that the edge e_1 is incident with u_1 and the edge e_2 is incident with v_1. Since $G \neq K_{3,3}$, the vertex v is adjacent to at most one of u_1 or v_1. If v is adjacent to either u_1 or v_1, then v would be a vertex of degree 2 in G that belongs to a copy of $K_{2,3}$, contradicting Claim 8. Hence, v has degree 1 in G. Let u be the neighbor of v. We now consider the graph $G' = G - N[u]$. We note that both u_1 and v_1 have degree 1 in G'. Moreover,

$\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 14 + 4 = \Omega(G) - 10$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex u, and so $i(G) \le i(G') + 1$. Thus, $8i(G) < \Omega(G)$, a contradiction. $_{(\Box)}$

Claim 12 $\mathrm{tc}(G - e) \le 1$ *for all* $e \in E(G)$.

Proof. Suppose, to the contrary, that the claim is false. Thus there exists an edge $e = uv$ in G whose removal creates two troublesome configurations. Let $T_w = T_{w_1,w_2}$ and $T_v = T_{v_1,v_2}$ be the two new troublesome configuration in $G - e$, where w_1 and w_2 are the link vertices of T_w and v_1 and v_2 are the link vertices of T_v. Let T_w be obtained from the bad graph $B_w \in \mathcal{B}_1$ with root vertex w_1 (of degree 1 in B_w), and let x_1 and x_2 be the vertices of B_w adjacent to w_1 and w_2, respectively, in G. Further, let T_v be obtained from the bad graph $B_v \in \mathcal{B}_1$ with root vertex v_1 (of degree 1 in B_v), and let y_1 and y_2 be the vertices of B_v adjacent to v_1 and v_2, respectively, in G. We may assume that $u \in V(B_w)$ and $v \in V(B_v)$. Let u_1 and u_2 be the two neighbors of u in B_w, and let a_1 and a_2 be the two neighbors of v in B_v. By Claim 8, we infer that each of B_w and B_v contain exactly one $K_{2,3}$-subgraph, which we name F_w and F_v, respectively. Thus, $V(F_w) = \{u, u_1, u_2, x_1, x_2\}$ and $V(F_v) = \{v, a_1, a_2, y_1, y_2\}$. If $\deg_G(v_1) = 3$, then let z_1 be the neighbor of v_1 different from y_1 and v_2. Moreover, let z_2 be the neighbor of v_2 different from y_2 and v_1.

Suppose that $\deg_G(v_1) = 3$ and $\deg_G(z_1) = 1$. In this case, we let $X = N_G[u] \cup (V(T_v) \cup \{z_1\})$ and we let $G' = G - X$. Thus, the graph shown in Figure 18 is a subgraph of G, where T_v and T_w are the troublesome configurations indicated in the dashed boxes and where X is the set indicated by the vertices in the dotted region. The removal of the four X-exit edges incident with u_1 and u_2 when constructing G' increases the total weight by 4, noting that the component in G' that contains the vertices w_1 and w_2 contains the path $x_1w_1w_2x_2$, where x_1 and x_2 have degree 1 in G' (and therefore w_1 and w_2 do not belong to a bad component nor a troublesome configurations). Moreover by Claim 7, the removal of the X-exit edge v_2z_2 when constructing G' increases the total weight by at most 3. Therefore, $\Omega(G') \le \mathrm{w}(G) - \mathrm{w}_G(X) + 7 = \Omega(G) - 35 + 7 = \Omega(G) - 28$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices u, v_1 and y_2 (indicated by the shaded vertices in Figure 18), and so $i(G) \le i(G') + 3$. Hence, $8i(G) \le 8(i(G') + 3) \le \Omega(G') + 24 < \Omega(G)$, a contradiction.

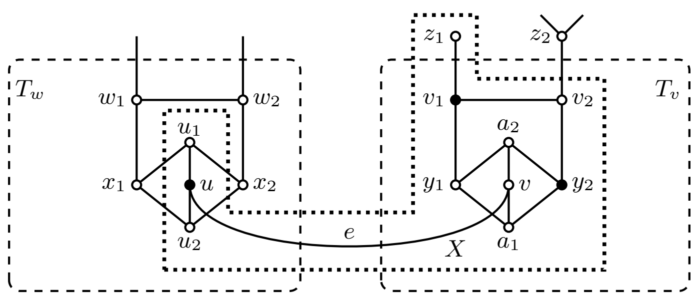


Figure 18: An illustration of a subgraph of G in the proof of Claim 12

Hence either $\deg_G(v_1) = 2$ or $\deg_G(v_1) = 3$ and $\deg_G(z_1) \ge 2$. We now let $X = N_G[u] \cup (V(T_v) \setminus \{v_2, y_2\})$ and let $G' = G - X$. Thus, the graph shown in Figure 19 is a subgraph of G (where in this illustration $\deg_G(v_1) = 3$), where T_v and T_w are the troublesome configurations indicated in the dashed boxes and where X is the set indicated by the vertices in the dotted region. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices u and y_1 (indicated by the shaded vertices in Figure 19), and so $i(G) \le i(G') + 2$. If $\deg_G(v_1) = 2$, then there are seven X-exit edges, and so $\mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 7 = \mathrm{w}(G) - 25 + 7 = \mathrm{w}(G) - 18$. If $\deg_G(v_1) = 3$, then there are eight X-exit edges, and so $\mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 8 = \mathrm{w}(G) - 24 + 8 = \mathrm{w}(G) - 16$. The component in G' that contains the vertices w_1 and w_2 contains at least two vertices of degree 1, namely the vertices x_1 and x_2. Further, we note that the vertex y_2 has degree 1 in G' and its neighbor v_2 has degree 2 in G'. Moreover, if v_1 has degree 3 in G and its neighbor z_1 belongs to a component of G' that contains neither w_1 nor v_2, then such a component is not a bad component (that belongs to $\mathcal{B}$), by Claim 10. We therefore infer that $b(G') = 0$. We also note that the component in G' that contains the vertex w_1 (and w_2) does not contain a troublesome configuration and neither does the component in G' that contains the vertex v_2 (and y_2). Hence either G' has no troublesome configuration or G' has exactly one troublesome configuration and such a troublesome configuration necessarily contains the vertex z_1.

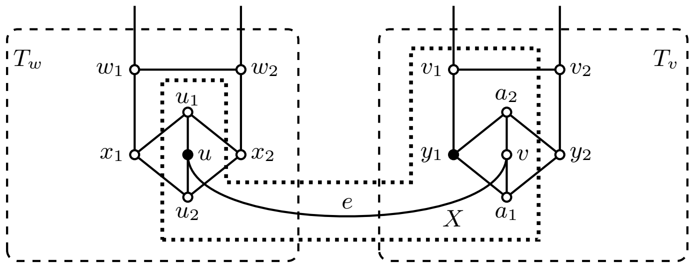


Figure 19: An illustration of a subgraph of G in the proof of Claim 12

Suppose that G' has no troublesome configuration; that is, $\mathrm{tc}(G') = 0$. As observed earlier, $b(G') = 0$, and so $\Theta(G') = 0$. Thus, $\Omega(G') = \mathrm{w}(G') \le \mathrm{w}(G) - 16$. Hence, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 \le \mathrm{w}(G) = \Omega(G)$, a contradiction. Therefore, G' has exactly one troublesome configuration, namely a troublesome configuration, T_{z_1} say, that contains the vertex z_1. In this case, we note that $\deg_G(v_1) = 3$. Interchanging the roles of v_1 and v_2, by analogous arguments (letting $X = N_G[u] \cup (V(T_v) \setminus \{v_1, y_1\})$) we infer that $G - v_2 z_2$ contains a troublesome configuration, T_{z_2} say, that contains the vertex z_2. Thus the graph shown in Figure 20 is a subgraph of G. Let Z be the set of vertices in T_{z_1} different from the two link vertices of T_{z_1} in $G - v_1 z_1$. We note that $G[Z]$ is a copy of $K_{2,3}$. Let I_z be the non-canonical independent set of $G[Z]$, and so I_z consists of the two neighbors of z_1 in T_{z_1}. Let $Y = N_G[u] \cup V(T_v) \cup Z \cup \{z_2\}$. The set Y is indicated by the vertices in the dotted region in Figure 20. We now consider the graph $G'' = G - Y$.

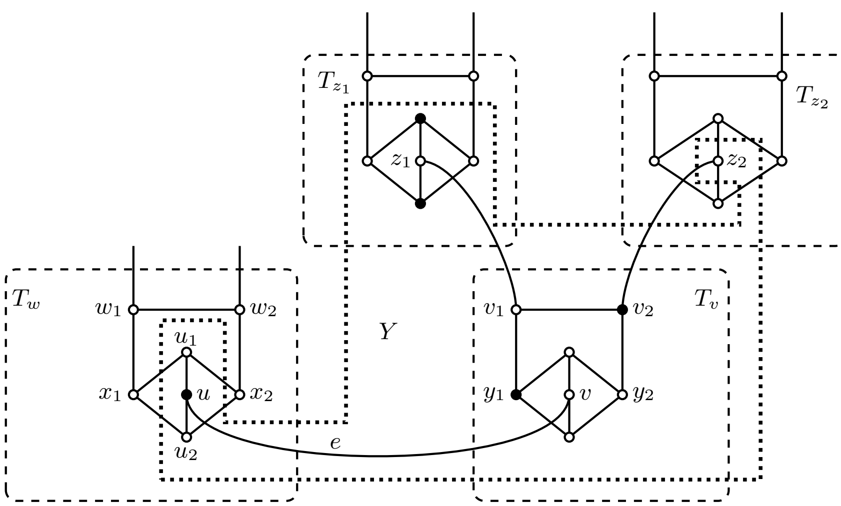


Figure 20: An illustration of a subgraph of G in the proof of Claim 12

Every i-set of G'' can be extended to an ID-set of G by adding to it the set $I_z \cup \{u, v_2, y_1\}$ (as indicated by the five shaded vertices in Figure 20), and so $i(G) \le i(G'') + 5$. We note that $\mathrm{w}_G(Y) = 48$. Since there are eight Y-exit edges, we have $\mathrm{w}(G'') = \mathrm{w}(G) - \mathrm{w}_G(Y) + 8 = \mathrm{w}(G) - 48 + 8 = \mathrm{w}(G) - 40$. Because $\mathrm{tc}(G) = b(G) = 0$ and by the structure of the graph G'' (recalling also the properties of bad components and troublesome configurations in Propositions 1(f) and 2(f)), we infer $\mathrm{tc}(G'') = b(G'') = 0$. Thus, $\Theta(G'') = 0$, and so $\Omega(G'') = \mathrm{w}(G'') = \mathrm{w}(G'') - 40 = \Omega(G) - 40$. Hence, $8i(G) \le 8(i(G'') + 5) \le \Omega(G'') + 40 = \Omega(G)$, a contradiction. (□)

Claim 13 $\mathrm{tc}(G - e) = 0$ *for all* $e \in E(G)$.

Proof. Suppose, to the contrary, that $\mathrm{tc}(G - e) \ge 1$ for some edge $e \in E(G)$. By Claim 12, $\mathrm{tc}(G - e) \le 1$. Thus, $\mathrm{tc}(G - e) = 1$, that is, there exists an edge e in G whose removal creates a troublesome configuration. Let

$T = T_{v_1,v_2}$ be a troublesome configuration in $G - e$ with link vertices v_1 and v_2, where $\deg_G(v_2) = 3$. Thus, T is obtained from a bad graph $B \in \mathcal{B}_1$ with root vertex v_1 (of degree 1 in B). Further, let u_1 and u_2 be the vertices of B adjacent to v_1 and v_2, respectively, in G. If v_1 has degree 3 in G, then let z_1 be the neighbor of v_1 different from u_1 and v_2. Moreover, let z_2 be the neighbor of v_2 different from u_2 and v_1.

Claim 13.1 *The vertex v_1 is not incident with the edge e.*

Proof. Suppose that v_1 is incident with the edge e. In this case, by Claims 8 and 9, the bad graph B contains exactly one copy of $K_{2,3}$, say F, and so $B \in \mathcal{B}_1$. Moreover, the end of the edge e, say u, different from v_1 belongs to F, for otherwise once again G would contain a copy of $K_{2,3}$ that contains a vertex of degree 2 in G, a contradiction. Thus, the graph illustrated in Figure 21 is a subgraph of G, where T is the troublesome configuration indicated in the dashed box and where $e = uv_1$. We now let $X = V(T)$ and we let $G' = G - X$. The removal of the (unique) X-exit edge, namely v_2z_2 when constructing G' increases the total weight by at most 3 by Claim 7. By Claim 10, $b(G') = 0$. Thus, $\Omega(G') \le \mathrm{w}(G) - \mathrm{w}_G(X) + 3 = \Omega(G) - 21 + 3 = \Omega(G) - 18$. Every i-set of G' can be extended to an i-set of G by adding to it the vertices v_1 and u_2, and so $i(G) \le i(G') + 2$. Hence, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \Omega(G) - 2 < \Omega(G)$, a contradiction. $(\Box)$

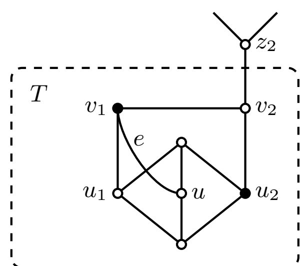


Figure 21: An illustration of a subgraph of G in the proof of Claim 13.1

By Claim 13.1, the vertex v_1 is not incident with the edge e. Let $e = uv$, where $u \in V(B)$.

Claim 13.2 $v \notin V(B)$.

Proof. Suppose that $v \in V(B)$, and so $\{u, v\} \subset V(B)$. By Claims 8 and 9, the bad graph B contains two copies of $K_{2,3}$. Let F_u and F_v be the copies of $K_{2,3}$ that contain u and v, respectively, where we may assume that $u_1 \in V(F_u)$ (and so, $u_2 \in V(F_v)$). Let $X = V(T) \setminus \{u_1, u_2, v_1, v_2\}$ and let $G' = G - X$. Thus, the graph illustrated in Figure 22 is a subgraph of G, where T is the troublesome configuration indicated in the dashed box and the set X is indicated by the vertices in the dotted region. Every i-set of G' can be extended to an ID-set of G by adding to it two vertices (indicated by the two shaded vertices in Figure 22), and so $i(G) \le i(G') + 2$. From the structure of the graph G and since $b(G) = \mathrm{tc}(G) = 0$, we note that $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 24 + 4 = \mathrm{w}(G) - 20 = \Omega(G) - 20$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 < \Omega(G)$, a contradiction. $(\Box)$

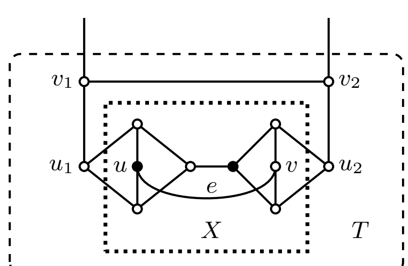


Figure 22: An illustration of a subgraph of G in the proof of Claim 13.2

By Claim 13.2, we have $v \notin V(B)$. By Claim 8 and Claim 9, the bad graph B contains only one copy of $K_{2,3}$, say F_u. Let w_1 and w_2 be the two neighbors of u in B, and so $N_B(w_1) = N_B(w_2) = \{u, u_1, u_2\}$.

Claim 13.3 $\deg_G(v_1) = 3$.

Proof. Suppose that $\deg_G(v_1) = 2$. Let $X = N_G[u_1] = \{u_1, v_1, w_1, w_2\}$ and let $G' = G - X$. Thus, the graph illustrated in Figure 23 is a subgraph of G, where T is the troublesome configuration indicated in the dashed box and the set X is indicated by the vertices in the dotted region. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex u_1 (indicated by the shaded vertes in Figure 23), and so $i(G) \le i(G') + 1$. From the structure of the graph G and since $b(G) = \mathrm{tc}(G) = 0$, we note that $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 13 + 5 = \mathrm{w}(G) - 8 = \Omega(G) - 8$. Thus, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G)$, a contradiction. (□)

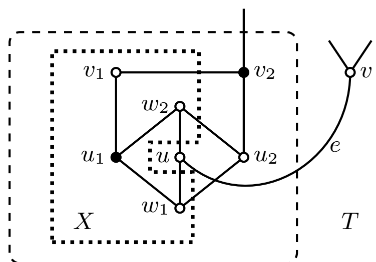


Figure 23: An illustration of a subgraph of G in the proof of Claim 13.3

By Claim 13.3, we have $\deg_G(v_1) = 3$.

Claim 13.4 $z_1 \neq z_2$.

Proof. Suppose that $z_1 = z_2$, and so $\{v_1, v_2\} \subseteq N_G(z_1)$. Let $X = (V(T) \setminus \{u\}) \cup \{z_1\}$ and let $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices u_1 and v_2, and so $i(G) \le i(G') + 2$. If $N_G(z_1) = \{v_1, v_2\}$, then $\deg_G(z_1) = 2$, $\mathrm{w}_G(X) = 22$ and $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 2 = \mathrm{w}(G) - 22 + 2 = \mathrm{w}(G) - 20$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \mathrm{w}(G) - 4 < \Omega(G)$, a contradiction. Hence, $\deg_G(z_1) = 3$. Let w be the neighbor of z_1 different from v_1 and v_2. Thus the graph shown in Figure 24 is a subgraph of G, where T is the troublesome configuration indicated in the dashed box and the set X is indicated by the vertices in the dotted region. As noted earlier, $i(G) \le i(G') + 2$. The removal of the X-exit edge z_1w when constructing G' increases the total weight by at most 3 by Claim 7. Moreover, the removal of the two X-exit edges incident with u when constructing G' increases the total weight by 2 noting that the vertex u does not belong to a bad component or a troublesome configuration in G' but its vertex weight increased by 2. Thus, $\Omega(G') \le \Omega(G) - \mathrm{w}_G(X) + 3 + 2 = \Omega(G) - 21 + 3 + 2 = \Omega(G) - 16$. Hence, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \Omega(G)$, a contradiction. (□)

By Claim 13.4, we have $z_1 \neq z_2$.

Claim 13.5 $\deg_G(z_i) \ge 2$ *for* $i \in [2]$.

Proof. Suppose that $\deg_G(z_1) = 1$ or $\deg_G(z_2) = 1$. By symmetry, we may assume that $\deg_G(z_1) = 1$. Let $X = (V(T) \setminus \{u\}) \cup \{z_1\}$ and let $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices v_1 and u_2, and so $i(G) \le i(G') + 2$. The graph shown in Figure 25 is a subgraph of G, where T is the troublesome configuration indicated in the dashed box and the set X is indicated by the vertices in the dotted region. The removal of the X-exit edge z_2v_2 when constructing G' increases the total weight by at most 3. Moreover, the removal of the two X-exit edges incident with u when constructing G' increases the total weight by 2. Thus, $\Omega(G') \le \Omega(G) - \mathrm{w}_G(X) + 3 + 2 = \Omega(G) - 23 + 3 + 2 = \Omega(G) - 18$. Hence, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 < \Omega(G)$, a contradiction. (□)

By Claim 13.5, we have $\deg_G(z_i) \ge 2$ for $i \in [2]$.

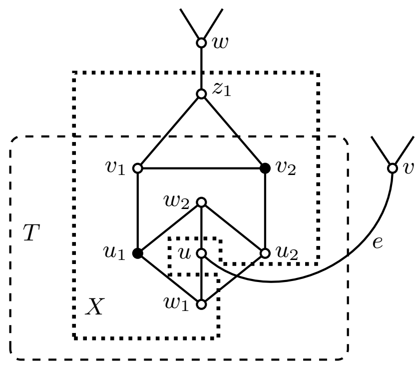


Figure 24: An illustration of a subgraph of G in the proof of Claim 13.4

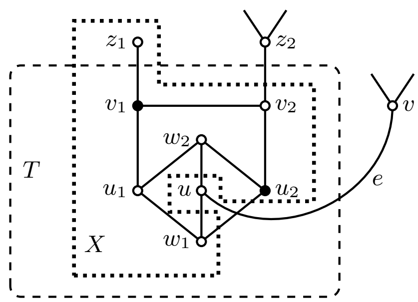


Figure 25: An illustration of a subgraph of G in the proof of Claim 13.5

Claim 13.6 $z_1z_2 \in E(G)$.

Proof. Suppose that $z_1z_2 \notin E(G)$. Let $X = V(T) \setminus \{u\}$ and let G' be obtained from $G - X$ by adding the edge $f = z_1z_2$. Let I' be an i-set of G'. If $I' \cap \{z_1, z_2\} = \emptyset$, then let $I_u = \{u_1, u_2\}$. If $z_1 \in I'$, then let $I_u = \{u_1, v_2\}$. If $z_2 \in I'$, then let $I_u = \{v_1, u_2\}$. In all cases, we have $|I_u| = 2$ and the set I' can be extended to an ID-set of G by adding to it the set I_u. Thus, $i(G) \le |I'| + |I_u| = i(G') + 2$. We note that $\mathrm{w}_G(X) = 18$ and $\mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 2 = \mathrm{w}(G) - 18 + 2 = \mathrm{w}(G) - 16$. If $\Theta(G') = 0$, then $\Omega(G') = \mathrm{w}(G')$, and so $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \mathrm{w}(G') + 16 = \mathrm{w}(G) = \Omega(G)$, a contradiction. Hence, $\Theta(G') > 0$. We therefore infer that the component, G'_f say, of G' that contains the edge f is either a bad component or contains a troublesome configuration.

Claim 13.6.1 $G'_f \notin \mathcal{B}$.

Proof. Suppose that $G'_f \in \mathcal{B}$. Let G'_f have root vertex v_f (of degree 1 in B_f). We note that the degrees of the vertices in G'_f are the same as their degrees in G, except possibly for the vertex u in the case when it belongs to G'_f. By Claims 8 and 9, we infer that $G'_f \in \mathcal{B}_1$, and so the root v_f of G'_f has degree 1 in G. Further, we infer from these claims that G'_f contains exactly one copy of $K_{2,3}$, say H_f. By Claim 13.5, $\deg_G(z_i) \ge 2$ for $i \in [2]$.

Suppose that u belongs to G'_f, implying that u is the root vertex v_f. Since the vertex u is adjacent to neither v_1 nor v_2 in G and since z_i is adjacent to v_i in G for $i \in [2]$, we note that $u \notin \{z_1, z_2\}$. Thus, neither z_1 nor z_2 is the root vertex v_f of G'_f, and so the added edge $f = z_1z_2$ is an edge of H_f. The graph G is now determined and $V(G) = V(T) \cup V(H_f)$. In particular, G has order $n = 12$. There are two possibilities up to isomorphism, depending on whether $v \in \{z_1, z_2\}$ or $v \notin \{z_1, z_2\}$. In both cases, we have $\Omega(G) = \mathrm{w}(G) = 38$ and the set $\{w_1, w_2, z_1, z_2\}$ is an ID-set of G, and so $i(G) \le 4$. Thus, $8i(G) < \Omega(G)$, a contradiction. Hence, $u \notin V(G'_f)$, implying that G' has two components, namely the component G'_f and the component containing the vertices u and v.

As observed earlier, $\deg_G(z_i) \ge 2$ for $i \in [2]$, and so the added edge $f = z_1z_2$ is an edge of H_f. Let $X' =$

$V(G'_f) \cup V(T)$ and let $G'' = G - X'$. Thus the graph shown in Figure 26 is an example of a subgraph of G, where T is the troublesome configuration indicated in the dashed box, $G'_f - f$ is the subgraph indicated in the dashed box, and the set X' is indicated by the vertices in the dotted region. We note that there exists an ID-set, I_f say, of $G'_f - f$ that contains both vertices z_1 and z_2 and is such that $|I_f| \le 3$. Every i-set of G'' can be extended to an ID-set of G by adding to it the set $I_f \cup \{w_1, w_2\}$ (as indicated by the shaded vertices in Figure 26), and so $i(G) \le i(G'')+5$. By Claims 8 and 9, we infer that $b(G'') = 0$ (and note that $\mathrm{tc}(G'') = 0$). Thus, $\Omega(G'') = \mathrm{w}(G'') = \mathrm{w}(G) - \mathrm{w}_G(X) + 1 = \Omega(G) - 43 + 1 = \Omega(G) - 42$. Hence, $8i(G) \le 8(i(G')+5) \le \Omega(G')+40 < \Omega(G)$, a contradiction. (□)

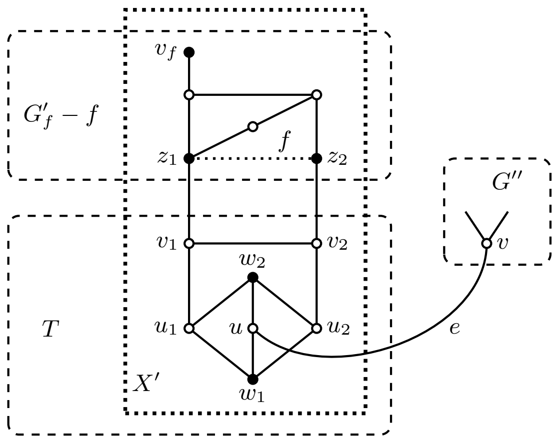


Figure 26: An illustration of a subgraph of G in the proof of Claim 13.6.1

We now return to the proof of Claim 13.6. By Claim 13.6.1, $G'_f \notin \mathcal{B}$, implying that G'_f contains a troublesome configuration, T_f say. Necessarily, T_f contains the added edge f. By Claims 8 and 9, we infer that T_f contains exactly one copy of $K_{2,3}$, and this copy of $K_{2,3}$ contains the added edge $f = z_1z_2$. We may assume, by symmetry, that z_1 has degree 3 in the copy of $K_{2,3}$ in T_f. We now let $X^* = (V(T) \setminus \{u\}) \cup \{z_1\}$ and let $G^* = G - X^*$. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertices v_1 and u_2, and so $i(G) \le i(G^*) + 2$. We note that $\deg_{G^*}(u) = 1$ and the degree of a neighbor of z_1 in G^* is also 1. Thus by the structure of the graph G^* so far determined, we infer that $b(G^*) = 0$ and $\mathrm{tc}(G^*) = 0$. Thus, $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 5 = \Omega(G) - 21 + 5 = \Omega(G) - 16$. Hence, $8i(G) \le 8(i(G^*) + 2) \le \Omega(G^*) + 16 = \Omega(G)$, a contradiction. (□)

By Claim 13.6, $z_1z_2 \in E(G)$.

Claim 13.7 $\deg_G(z_i) = 3$ *for* $i \in [2]$.

Proof. Suppose that $\deg_G(z_i) = 2$ for some $i \in [2]$. Suppose firstly that $\deg_G(z_1) = \deg_G(z_2) = 2$. Thus, $N_G(z_i) = \{v_i, z_{3-i}\}$ for $i \in [2]$. We now let $X = (V(T) \setminus \{u\}) \cup \{z_1, z_2\}$ and let $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{u_1, u_2, z_1\}$, and so $i(G) \le i(G') + 3$. We note that $\Theta(G') = 0$ and $\mathrm{w}_G(X) = 26$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 2 = \Omega(G) - 26 + 2 = \Omega(G) - 24$, and so $8i(G) \le 8(i(G') + 3) \le \Omega(G') + 24 = \Omega(G)$, a contradiction. Hence, at least one of z_1 and z_2 has degree 3 in G. By symmetry, we may assume that $\deg_G(z_2) = 3$, and so, by supposition, $\deg_G(z_1) = 2$. In this case, we let $X = (V(T) \setminus \{u\}) \cup \{z_1\}$ and let $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices v_1 and u_2, and so $i(G) \le i(G') + 2$. We note that $\Theta(G') = 0$ and $\mathrm{w}_G(X) = 22$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 4 = \Omega(G) - 22 + 4 = \Omega(G) - 18$, and so $8i(G) \le 8(i(G')+2) \le \Omega(G')+16 < \Omega(G)$, a contradiction. (□)

By Claim 13.7, $\deg_G(z_i) = 3$ for $i \in [2]$. Let y_i be the neighbor of z_i different from v_i and z_{3-i} for $i \in [2]$.

Claim 13.8 *The following properties hold.*

(a) $\deg_G(y_i) \ge 2$ *for* $i \in [2]$.

(b) $y_1 \neq y_2$.

Proof. (a) Suppose that at least one of y_1 and y_2 has degree 1 in G. By symmetry, we may assume that $\deg_G(y_1) = 1$. In this case, we let $X = N_G[z_1] = \{v_1, y_1, z_1, z_2\}$ and let $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex z_1, and so $i(G) \le i(G') + 1$. The cost of removing the X-exit edge y_2z_2 when constructing G' increases the total weight by at most 3 by Claim 7. Moreover, the removal of the three X-exit edges different from y_2z_2 increases the total weight by 3 since the removal of these three edges does not create a bad component or a troublesome configuration. Thus, $\Omega(G') \le \Omega(G) - \mathrm{w}_G(X) + 3 + 3 = \Omega(G) - 14 + 3 + 3 = \Omega(G) - 8$. Hence, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G)$, a contradiction. This proves part (a).

(b) Suppose that $y_1 = y_2$. In this case, we let $X = (V(T) \setminus \{u\}) \cup \{y_1, z_1, z_2\}$ and let $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{u_1, u_2, z_1\}$, and so $i(G) \le i(G') + 3$. We note that $\Theta(G') = 0$ and $\mathrm{w}_G(X) = 27$, and so $\Omega(G') = \mathrm{w}(G') \le \mathrm{w}(G) - \mathrm{w}_G(X) + 3 = \Omega(G) - 27 + 3 = \Omega(G) - 24$, and so $8i(G) \le 8(i(G') + 3) \le \Omega(G') + 24 = \Omega(G)$, a contradiction. This proves part (b). $(\Box)$

By Claim 13.8, $\deg_G(y_i) \ge 2$ for $i \in [2]$ and $y_1 \neq y_2$. Suppose that y_1 does not belong to a troublesome configuration in $G - y_1z_1$. In this case, we let $X = (V(T) \setminus \{u\}) \cup \{z_1\}$ and let $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices v_1 and u_2, and so $i(G) \le i(G') + 2$. We note that both u and z_2 have degree 1 in G' and do not belong to a bad component or a troublesome configuration in G'. By supposition, y_1 does not belong to a troublesome configuration in G'. Moreover, removing the edge y_1z_1 does not create a bad component containing y_1. Hence, $\Theta(G') = 0$ and $\mathrm{w}_G(X) = 21$, and so $\Omega(G') = \mathrm{w}(G') \le \mathrm{w}(G) - \mathrm{w}_G(X) + 5 = \Omega(G) - 21 + 5 = \Omega(G) - 16$, and so $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \Omega(G)$, a contradiction. Hence, y_1 belongs to a troublesome configuration in $G - y_1z_1$. By symmetry, y_2 belongs to a troublesome configuration in $G - y_2z_2$. Let T_{y_i} be the troublesome configuration in $G - y_iz_i$ that contains the vertex y_i for $i \in [2]$.

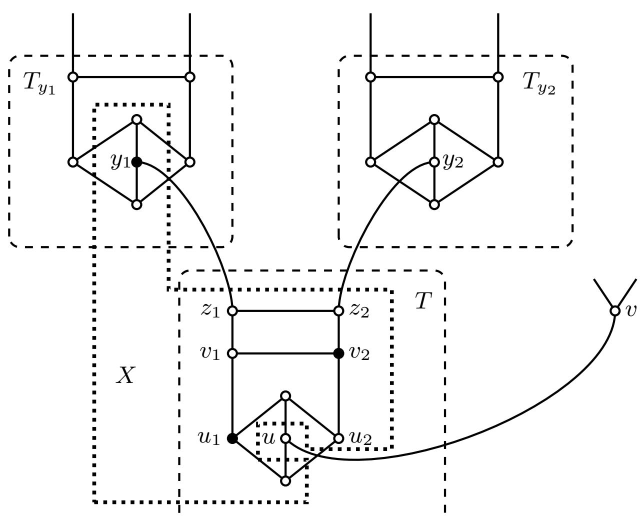


Figure 27: An illustration of a subgraph of G in the proof of Claim 13

We now let $X = (V(T) \setminus \{u\}) \cup \{z_2\} \cup N_G[y_1]$ and let $G' = G - X$. Thus the graph shown in Figure 27 is a subgraph of G, where T, T_{y_1} and T_{y_2} are indicated in the dashed box, and the set X is indicated by the vertices in the dotted region. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{u_1, v_2, y_1\}$ (indicated by the shaded vertices in Figure 27), and so $i(G) \le i(G') + 3$. The cost of removing the X-exit edge y_2z_2 when constructing G' increases the total weight by 3. Moreover, the removal of the six X-exit edges different from y_2z_2 increases the total weight by 6 since the removal of these six edges does not create a bad component or a troublesome configuration. Thus, $\Omega(G') \le \Omega(G) - \mathrm{w}_G(X) + 3 + 6 = \Omega(G) - 33 + 3 + 6 = \Omega(G) - 24$. Hence, $8i(G) \le 8(i(G') + 3) \le \Omega(G') + 24 = \Omega(G)$, a contradiction. This completes the proof of Claim 13. $(\Box)$

We present next two consequences of Claim 13.

Claim 14 $\mathrm{tc}(G - e_1 - e_2) \le 1$ *for all* $e_1, e_2 \in E(G)$.

Proof. Suppose, to the contrary, that the claim is false. Thus there exists edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in G whose removal creates two troublesome configurations. Let $T_w = T_{w_1,w_2}$ and $T_z = T_{z_1,z_2}$ be the two troublesome configuration in $G - e_1 - e_2$, where w_1 and w_2 are the link vertices of T_w and z_1 and z_2 are the link vertices of T_z. By Claims 8, 9, and 13, we infer that the edges e_1 and e_2 both have one end in T_w and one end in T_z. Furthermore, each of T_w and T_z contains exactly two copies of $K_{2,3}$ and each copy of $K_{2,3}$ contains an end of one of the edges e_1 and e_2. We now let $X = (V(T_w) \setminus \{w_1, w_2, x_1, x_2\}) \cup (V(T_z) \setminus \{y_1, y_2, z_1, z_2\})$ and let $G' = G - X$. Renaming vertices if necessary, we may assume that the graph illustrated in Figure 28 is a subgraph of G, where T_w and T_z are the troublesome configurations indicated in the dashed boxes and where X is the set indicated by the vertices in the dotted region. We note that $b(G') = \mathrm{tc}(G') = 0$, and so $\Theta(G') = 0$. Since there are eight X-exit edges, we have $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 8 = \mathrm{w}(G) - 48 + 8 = \Omega(G) - 40$. Every i-set of G' can be extended to an ID-set of G by adding to it four vertices from the set X (indicated, for example, by the shaded vertices in Figure 28), and so $i(G) \le i(G') + 4$. Thus, $8i(G) \le 8(i(G') + 4) \le \Omega(G') + 32 = \Omega(G) - 8 < \Omega(G)$, a contradiction. (□)

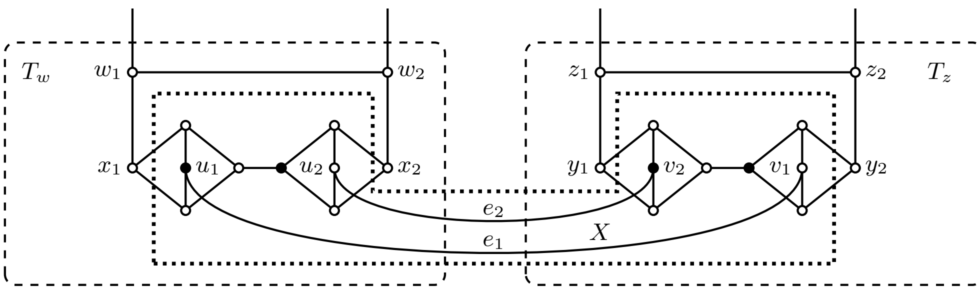


Figure 28: An illustration of a subgraph of G in the proof of Claim 14

Let G_7 be the graph shown in Figure 29. As a special case of Claim 13, we have the following result.

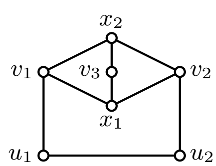


Figure 29: The graph G_7

Claim 15 *The graph* G *does not contain* G_7 *as a subgraph.*

Proof. Suppose, to the contrary, that G contains G_7 as a subgraph. Let F be such a G_7-configuration and let the vertices of F be named as in Figure 29. By Claim 8, there is no $K_{2,3}$-subgraph in G that contains a vertex of degree 2 in G. Hence, $\deg_G(v_3) = 3$. Let v be the neighbor of v_3 different from x_1 and x_2. By Claim 9, either $v \in \{u_1, u_2\}$ or $v \notin V(F)$. We show that at least one of u_1 and u_2 has a neighbor in G that does not belong to $V(F)$. If $v = u_1$ and $\deg_G(u_2) = 2$, then $G = G_7 + u_1v_3$. In this case, $i(G) = 2$ and $\mathrm{w}(G) = 22$, and so $8i(G) < \Omega(G)$, a contradiction. Hence if $v = u_1$, then $\deg_G(u_2) = 3$, and so u_2 has a neighbor in G that does not belong to $V(F)$. By symmetry, if $v = u_2$, then u_1 has a neighbor in G that does not belong to $V(F)$. If $v \notin V(F)$ and $\deg_G(u_1) = \deg_G(u_2) = 2$, then let $X' = V(F) \setminus \{v_3\}$ and let $G' = G - X'$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices v_1 and v_2, and so $i(G) \le i(G') + 2$. We note that $\Theta(G') = 0$ and $\mathrm{w}_G(X') = 20$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 2 = \Omega(G) - 20 + 2 = \Omega(G) - 18$, and so $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 < \Omega(G)$, a contradiction. Hence if $v \notin V(F)$, then at least one of u_1 and u_2 has a neighbor in G that does not belong to $V(F)$. In all cases, at least one of u_1 and u_2 has a neighbor in G that does not belong to $V(F)$. Thus, removing the edge v_3v creates a troublesome configuration, contradicting Claim 13. (□)

6.2 Part 2: The minimum degree of G equals 1

In this section, we consider the case when the minimum degree of G equals 1, that is, when $\delta(G) = 1$. We prove a number of claims that will culminate in a contradiction, thereby showing that this case cannot occur. A *support vertex* in G is a vertex that has at least one neighbor that has degree 1, and we define a *strong support vertex* as a vertex that has at least two neighbors of degree 1 in G.

Claim 16 $\delta(G) \ge 2$.

Proof. Suppose, to the contrary, that G contains a vertex of degree 1.

Claim 16.1 *There is no strong support vertex in G.*

Proof. Suppose, to the contrary, that G contains a strong support vertex v. Let $N_G(v) = \{v_1, v_2, v_3\}$ where $\deg_G(v_1) = \deg_G(v_2) = 1$. Since $n \ge 6$, we note that $\deg_G(v_3) \ge 2$. Suppose that $\deg_G(v_3) = 3$ and the two neighbors of v_3 different from v both have degree 1 in G. In this case, the graph G is determined (and is a double star $S(2, 2)$) and $\mathrm{w}(G) = 26$. Moreover, $i(G) = 3$, and so $8i(G) < \mathrm{w}(G)$, a contradiction. Hence, at most one neighbor of v_3 has degree 1.

Suppose that v_3 has a neighbor, say v_4, of degree 1. Since $n \ge 6$, we note that $\deg_G(v_3) = 3$. Let v_5 be the third neighbor of v_3 different from v and v_4. By our earlier observations, $\deg_G(v_5) \ge 2$. We now let $X = N_G[v] \cup \{v_4\}$ and let $G' = G - X$. By Claim 10, $b(G') = 0$ and by Claim 13, $\mathrm{tc}(G') = 0$. Hence, $\Theta(G') = 0$, and so since there is exactly one X-exit edge, namely v_3v_5, we have $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 1 = \Omega(G) - 21 + 1 = \Omega(G) - 20$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices v and v_4, and so $i(G) \le i(G') + 2$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \Omega(G) - 4 < \Omega(G)$, a contradiction. Hence, no neighbor of v_3 has degree 1 in G.

We now let $X = N_G[v]$ and let $G' = G - X$. By Claim 11, $b(G') = 0$ and by Claim 14, $\mathrm{tc}(G') \le 1$. Hence, $\Theta(G') = 2\mathrm{tc}(G') \le 2$. Since there are two X-exit edges, neither of which is incident to a vertex of degree 1 in G, we have $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le (\mathrm{w}(G) - \mathrm{w}_G(X) + 2) + 2 \le \Omega(G) - 16 + 2 + 2 = \Omega(G) - 12$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v, and so $i(G) \le i(G') + 1$. Thus, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G) - 4 < \Omega(G)$, a contradiction. (□)

By Claim 16.1, every support vertex in G has exactly one leaf neighbor.

Claim 16.2 *No vertex of degree 2 is a support vertex.*

Proof. Suppose, to the contrary, that G contains a support vertex v of degree 2. Let v_1 be the leaf neighbor of v, and let v_2 be the second neighbor of v. Since $n \ge 6$, we note that $\deg_G(v_2) \ge 2$. We now consider the connected subcubic graph $G' = G - \{v, v_1\}$. By Claims 10 and 13, we have $\Theta(G') = 0$. Thus, $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 9 + 1 = \Omega(G) - 8$. Thus, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G)$, a contradiction. (□)

Let $P\colon v_1 \ldots v_k$ be the longest path in G such that vertex v_i is adjacent to a vertex, say u_i, of degree 1 for all $i \in [k]$. Possibly, $k = 1$. By Claim 16.2, $\deg_G(v_i) = 3$ for all $i \in [k]$.

Claim 16.3 $k \le 2$.

Proof. Suppose, to the contrary, that $k \ge 3$. If $v_1v_k \in E(G)$, then the graph G is determined and G is the corona of a cycle of length k, and so $i(G) = k$ and $\mathrm{w}(G) = 8k$, whence $8i(G) = 8k = \mathrm{w}(G)$, a contradiction. Hence, $v_1v_k \notin E(G)$. Let G' be obtained from $G - \{u_2, v_2\}$ by adding the edge v_1v_3. We note that $\mathrm{w}(G) = \mathrm{w}(G') + 8$. If G' contains a troublesome configuration, then such a configuration has a copy of $K_{2,3}$ that contains a vertex of degree 2 in G, contradicting Claim 8. Thus, $\mathrm{tc}(G') = 0$. Moreover since the connected graph G' contains at least two vertices of degree 1, we note that $G' \notin \mathcal{B}$, and so $b(G') = 0$. Thus, $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 8 = \Omega(G) - 8$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex u_2, and so $i(G) \le i(G') + 1$. Hence, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G)$, a contradiction.

By Claim 16.3, we have $k \le 2$. If $k = 1$, then we let $N_G(v_1) = \{a, b, u_1\}$, and if $k = 2$, then we let $N_G(v_1) = \{a, u_1, v_2\}$ and $N_G(v_2) = \{b, u_2, v_1\}$, where possibly $a = b$.

Claim 16.4 *$a \neq b$.*

Proof. Suppose, to the contrary, that $a = b$, implying that $k = 2$. Since $n \ge 6$ (and G contains a copy of $K_{2,3}$), we note that $\deg_G(a) = 3$. Let $N_G(a) = \{a_1, v_1, v_2\}$. [See Figure 30(a).] By the maximality of the path P, the vertex a_1 has degree at least 2 in G. Let $X = \{a, u_1, u_2, v_1, v_2\}$ and let $G' = G - X$. By Claims 10 and 13, we have $\Theta(G') = 0$. Thus, $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 1 = \Omega(G) - 19 + 1 = \Omega(G) - 18$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices v_1 and u_2, and so $i(G) \le i(G') + 2$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \Omega(G) - 2 < \Omega(G)$, a contradiction. $_{(\Box)}$

By Claim 16.4, $a \neq b$. [See Figure 30(b) and 30(c).] Relabeling vertices if necessary, we may assume that $\deg_G(a) \le \deg_G(b)$. By Claim 16.1, we have $2 \le \deg_G(a) \le \deg_G(b)$. If $k = 1$, let $X = \{u_1, v_1\}$ and if $k = 2$, let $X = \{u_1, u_2, v_1, v_2\}$. In both cases, let $X' = X \cup \{a\}$.

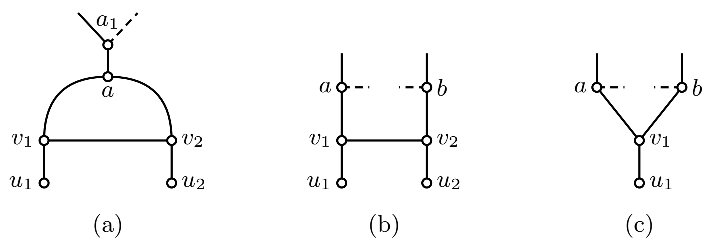


Figure 30: Subgraphs in the proof of Claim 16

Claim 16.5 *The graph $G - X'$ is isolate-free.*

Proof. Suppose, to the contrary, that $G - X'$ has an isolated vertex, z say. By Claim 16.2 and the maximality of the path P, the vertex z has degree at least 2 in G, implying that $\deg_G(z) = 2$ and $N_G(z) = \{a, v_k\}$. Hence, $z = b$. Thus, $2 \le \deg_G(a) \le \deg_G(b) = 2$, and so $\deg_G(a) = \deg_G(b)$ and $ab \in E(G)$. The graph G is therefore determined. However, G does not contain a subgraph isomorphic to $K_{2,3}$, contradicting Claim 1. $_{(\Box)}$

Claim 16.6 *If $k = 2$, then $ab \in E(G)$.*

Proof. Suppose, to the contrary, that $k = 2$ and the vertices a and b are not adjacent. We now consider the graph G' obtained from $G - X$ by adding the edge $e = ab$; that is, $G' = (G - X) + e$. The graph G' is a connected subcubic graph. Recall that $2 \le \deg_G(a) \le \deg_G(b)$. By construction, the degrees of vertices a and b in G' are the same as their degrees in G. Let I' be an i-set of G'. If $a \in I'$, then let $I = I' \cup \{u_1, v_2\}$. If $b \in I'$, then let $I = I' \cup \{v_1, u_2\}$. If neither a nor b belongs to I', then let $I = I' \cup \{u_1, u_2\}$. In all cases, the set I is an ID-set of G, and so $i(G) \le i(G') + 2$.

We show firstly that $b(G') = 0$. Suppose, to the contrary, that $b(G') \ge 1$. Since G' is a connected graph, we infer that $G' \in \mathcal{B}$ and $b(G') = 1$. If G' contains two or more copies of $K_{2,3}$, then at least one of these copies contains a vertex of degree 2 in G, contradicting Claim 8. Hence, $G' \in \mathcal{B}_1$ and G' contains exactly one copy of $K_{2,3}$, say F. If F contains at most one of a and b, then again we contradict Claim 8. Hence, F contains both vertices a and b, and the root vertex of G' has degree 1 in G (and is distinct from a and b). If a or b is adjacent to the root vertex, then we would contradict the maximality of the path P. Hence the graph G is as illustrated in Figure 31, where the added edge e is indicated by the dotted line. Thus, $i(G) = 4$ (the shaded vertices in Figure 31 are an example of an i-set in G) and $\Omega(G) = \mathrm{w}(G) = 38$, and so $8i(G) < \Omega(G)$, a contradiction. Hence, $b(G') = 0$.

We show next that $\mathrm{tc}(G') = 0$. Suppose, to the contrary, that $\mathrm{tc}(G') \ge 1$. Thus, adding the edge $e = ab$ to the graph $G - X$ creates a new troublesome configuration, which we call T_e, that necessarily contains the edge e.

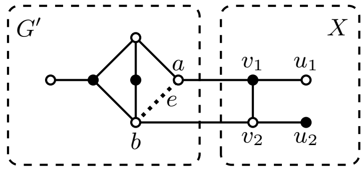


Figure 31: The graph G in the proof of Claim 16.6

Let r_1 and r_2 be the two link vertices of T_e. We may assume that $\deg_G(r_1) \le \deg_G(r_2)$, and so $2 \le \deg_G(r_1) \le \deg_G(r_2) \le 3$. Thus, T_e is obtained from a bad graph $B_e \in \mathcal{B}$ with root vertex r_1. By Claims 8 and 9, the bad graph B_e contains exactly one copy of $K_{2,3}$, say F_e, and so $B_e \in \mathcal{B}_1$. Moreover, F_e contains both vertices a and b, for otherwise F_e would contain a vertex of degree 2 in G, a contradiction.

We now let $X^* = X \cup (V(T_e) \setminus \{r_1, r_2\})$ and we let $G^* = G - X^*$. If $\deg_G(a) = 2$, then the graph illustrated in Figure 32(a) is a subgraph of G, while if $\deg_G(a) = 3$, then the graph illustrated in Figure 32(b) is a subgraph of G where in both cases T_e is the troublesome configuration indicated in the dashed box, the set X^* is indicated in the dashed box, and where $e = ab$ is indicated by the dotted edge. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertices u_1 and v_2 and the neighbor of a in F_e different from b (as indicated by the shaded vertices in Figures 32(a) and 32(b)). Thus, $i(G) \le i(G^*) + 3$. We note that $\Theta(G^*) = 0$, and so $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - 32 + 2 = \Omega(G) - 30$. Hence, $8i(G) \le 8(i(G^*) + 3) \le \Omega(G') + 24 < \Omega(G)$, a contradiction. Hence, $\mathrm{tc}(G') = 0$.

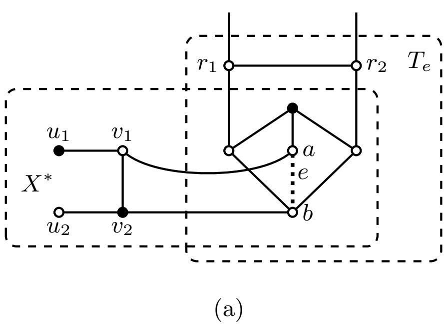


Figure 32: A subgraph of G in the proof of Claim 16.6

We now return to the proof of Claim 16.6. By our earlier observations, $b(G') = 0$ and $\mathrm{tc}(G') = 0$. Hence, $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) = \Omega(G) - 16$. Recall that $i(G) \le i(G') + 2$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \Omega(G)$, a contradiction. $(\Box)$

Claim 16.7 $k = 1$.

Proof. Suppose, to the contrary, that $k = 2$. By Claim 16.6, $ab \in E(G)$. Recall that $2 \le \deg_G(a) \le \deg_G(b)$. If $\deg_G(a) = \deg_G(b) = 2$, then the graph G is determined. In this case, G does not contain a copy of $K_{2,3}$, a contradiction. Hence, $\deg_G(b) = 3$.

We show firstly that $\deg_G(a) = 3$. Suppose, to the contrary, that $\deg_G(a) = 2$. In this case, we let $G' = G - X'$ where recall that $X = \{u_1, u_2, v_1, v_2\}$ and $X' = X \cup \{a\}$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices v_1 and u_2, and so $i(G) \le i(G') + 2$. We note that G' is a connected subcubic graph and $\deg_{G'}(b) = 1$. Moreover, $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 2 = \Omega(G) - 20 + 2 = \Omega(G) - 18$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 < \Omega(G)$, a contradiction. Hence, $\deg_G(a) = 3$. Let $N_G(a) = \{a_1, b, v_1\}$ and $N_G(b) = \{a, b_1, v_2\}$.

We show next that $a_1 \neq b_1$. Suppose, to the contrary, that $a_1 = b_1$. If $\deg_G(a_1) = 2$, then the graph G is determined. In this case, G does not contain a copy of $K_{2,3}$, a contradiction. Hence, $\deg_G(a_1) = 3$. Let a_2

be the neighbor of a_1 different from a and b. If $\deg_G(a_2) = 1$, then the graph G is determined, and again we contradict the supposition that G contain a copy of $K_{2,3}$. Hence, $\deg_G(a_2) \geq 2$. We now let $X^* = X \cup \{a, b, a_1\}$ and consider the connected subcubic graph $G^* = G - X^*$. Every i-set of G^* can be extended to an ID-set of G by adding to it the set $\{b, v_1, u_2\}$, and so $i(G) \leq i(G^*) + 3$. By Claim 10, $b(G^*) = 0$. By Claim 13, $\mathrm{tc}(G^*) = 0$. Thus, $\Theta(G^*) = 0$, and so noting that a_1a_2 is the unique X^*-exit edge and $\deg_G(a_2) \geq 2$, we have $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X) + 1 = \Omega(G) - 25 + 1 = \Omega(G) - 24$. Thus, $8i(G) \leq 8(i(G') + 3) \leq \Omega(G') + 24 = \Omega(G)$, a contradiction. Hence, $a_1 \neq b_1$.

Recall by maximality of the path P that $\deg_G(a_1) \geq 2$ and $\deg_G(b_1) \geq 2$. We now consider the graph $G' = G - X'$, where recall that $X = \{u_1, u_2, v_1, v_2\}$ and $X' = X \cup \{a\}$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices v_1 and u_2, and so $i(G) \leq i(G') + 2$. We note that $\deg_{G'}(b) = 1$, and so there is no troublesome configuration containing b in G'. If the vertex b belongs to a bad component, then such a component has a copy of $K_{2,3}$ that contains a vertex of degree 2 in G, contradicting Claim 8. Hence, b belongs to no bad component in G'. If the removal of the edge aa_1 when constructing G' creates a new troublesome configuration, then by Claim 13 we infer that such a troublesome configuration must contain the three vertices a_1, b and b_1, which is not possible noting that $\deg_{G'}(b) = 1$. Hence, $\mathrm{tc}(G') = 0$. Thus, $\Theta(G') = 0$, and so noting that there are three X'-exit edges and $\deg_G(a_1) \geq 2$, we have $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X') + 3 = \Omega(G) - 19 + 3 = \Omega(G) - 16$. Thus, $8i(G) \leq 8(i(G') + 2) \leq \Omega(G') + 16 = \Omega(G)$, a contradiction. (□)

By Claim 16.7, we have $k = 1$. Recall that $2 \leq \deg_G(a) \leq \deg_G(b)$. Recall that $n \geq 6$. In particular, we note that $\deg_G(b) = 3$.

Claim 16.8 *$ab \notin E(G)$.*

Proof. Suppose, to the contrary, that $ab \in E(G)$. In this case, we let $X^* = \{a, b, u_1, v_1\}$ and consider the graph $G^* = G - X^*$. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertex v_1, and so $i(G) \leq i(G^*) + 1$. Suppose that $\deg_G(a) = 2$, and so there is exactly one X^*-exit edge. By Claim 10, $b(G^*) = 0$. By Claim 13, $\mathrm{tc}(G^*) = 0$. Thus, $\Theta(G^*) = 0$, and so $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X) + 1 = \Omega(G) - 15 + 1 = \Omega(G) - 14$. Thus, $8i(G) \leq 8(i(G^*) + 1) \leq \Omega(G^*) + 8 = \Omega(G)$, a contradiction. Hence, $\deg_G(a) = 3$. Let $N_G(a) = \{a_1, b, v_1\}$ and $N_G(b) = \{a, b_1, v_1\}$. By the maximality of the path P, we note that $\deg_G(a_1) \geq 2$ and $\deg_G(b_1) \geq 2$. If a_1 is isolated in G^*, then the graph G is determined. In this case, $a_1 = b_1$ and $N_G(a_1) = \{a, b\}$. However, then, $n = 5$, a contradiction. Hence, a_1 is not isolated in G^*. By symmetry, b_1 is not isolated in G^*. Thus, $\mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 2 = \mathrm{w}(G) - 14 + 2 = \mathrm{w}(G) - 12$. By Claim 10, we have $b(G^*) = 0$, and by Claim 14, we have $\mathrm{tc}(G^*) \leq 1$. Thus, $\Theta(G^*) \leq 2$, and so $\Omega(G^*) = \mathrm{w}(G^*) + \Theta(G^*) \leq (\mathrm{w}(G) - 12) + 2 = \Omega(G) - 12 + 2 = \Omega(G) - 10$. Thus, $8i(G) \leq 8(i(G^*) + 1) \leq \Omega(G^*) + 8 < \Omega(G)$, a contradiction. (□)

By Claim 16.8, $ab \notin E(G)$.

Claim 16.9 *The graph $G - \{a, b, u_1, v_1\}$ is isolate-free.*

Proof. Suppose, to the contrary, that $G - \{a, b, u_1, v_1\}$ has an isolated vertex. By Claim 16.2 and the maximality of the path P, such a vertex has degree 2 in G with a and b as its two neighbors. If $G - \{a, b, u_1, v_1\}$ contains two isolated vertices, then the graph G is determined and is the graph B_1 shown in Figure 4. However, then $G \in \mathcal{B}$, a contradiction. Hence, $G - \{a, b, u_1, v_1\}$ contains exactly one isolated vertex, say c. As observed earlier, $N_G(c) = \{a, b\}$. Since $n \geq 6$, we note that $\deg_G(b) = 3$. Let $N_G(b) = \{b_1, c, v_1\}$. By the maximality of the path P, we note that $\deg_G(b_1) \geq 2$. We now let $X'' = \{a, b, c, u_1, v_1\}$.

Suppose that $\deg_G(a) = 2$. In this case, we consider the graph $G'' = G - X''$. Every i-set of G'' can be extended to an ID-set of G by adding to it the vertices v_1 and c, and so $i(G) \leq i(G'') + 2$. By Claim 10, $b(G'') = 0$. By Claim 13, $\mathrm{tc}(G'') = 0$. Thus, $\Theta(G'') = 0$, and so $\Omega(G'') = \mathrm{w}(G'') = \mathrm{w}(G) - \mathrm{w}_G(X'') + 1 = \Omega(G) - 19 + 1 = \Omega(G) - 18$. Thus, $8i(G) \leq 8(i(G^*) + 2) \leq \Omega(G^*) + 16 < \Omega(G)$, a contradiction. Hence, $\deg_G(a) = 3$. Let $N_G(a) = \{a_1, c, v_1\}$. By the maximality of the path P, we note that $\deg_G(a_1) \geq 2$. If $a_1 = b_1$, then $G[\{a, a_1, b, c, v_1\}]$ is a copy of $K_{2,3}$ that contains a vertex of degree 2 in G, namely vertex c, contradicting Claim 8. Hence, $a_1 \neq b_1$. Renaming vertices if necessary, we may assume by symmetry that $2 \leq \deg_G(a_1) \leq \deg_G(b_1)$.

We show next that $a_1b_1 \in E(G)$. Suppose, to the contrary, that $a_1b_1 \notin E(G)$. Recall that $X'' = \{a, b, c, u_1, v_1\}$. In this case, we consider the graph G' obtained from $G - X''$ by adding the edge $e = a_1b_1$; that is, $G' = (G - X'') + e$. The graph G' is a connected subcubic graph. Let I'' be an i-set of G'. If $a_1 \in I''$, then let $I = I'' \cup \{b, u_1\}$. If $b_1 \in I''$, then let $I = I'' \cup \{a, u_1\}$. If neither a_1 nor b_1 belongs to I'', then let $I = I'' \cup \{c, v_1\}$. In all cases, the set I is an ID-set of G, and so $i(G) \le i(G') + 2$. By construction, the degrees of vertices a_1 and b_1 in G' are the same as their degrees in G. However adding the edge e may have created a bad graph or a troublesome configuration. Thus, $b(G') + \mathrm{tc}(G') \le 1$, implying that $\Theta(G') \le 2$, and so $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le (\mathrm{w}(G) - \mathrm{w}_G(X'')) + 2 = \mathrm{w}(G) - 18 + 2 = \Omega(G) - 16$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 = \Omega(G)$, a contradiction. Hence, $a_1b_1 \in E(G)$.

In this case, we consider the graph $G'' = G - X''$. Every i-set of G'' can be extended to an ID-set of G by adding to it the vertices v_1 and c, and so $i(G) \le i(G'') + 2$. By Claim 11, $b(G'') = 0$. We note that a_1 and b_1 are adjacent vertices in G'' and both a_1 and b_1 have degree at most 2 in G''. We therefore infer that a_1 and b_1 do not belong to a troublesome configuration in G'', and so $\mathrm{tc}(G'') = 0$. Thus, $\Theta(G'') = 0$, and so $\Omega(G'') = \mathrm{w}(G'') = \mathrm{w}(G) - \mathrm{w}_G(X'') + 2 = \Omega(G) - 18 + 2 = \Omega(G) - 16$. Thus, $8i(G) \le 8(i(G'') + 2) \le \Omega(G'') + 16 = \Omega(G)$, a contradiction. ($\square$)

By Claim 16.9, the graph $G^* = G - X^*$ is isolate-free, where recall that $X^* = \{a, b, u_1, v_1\}$. Further recall that $2 \le \deg_G(a) \le \deg_G(b)$ and that $ab \notin E(G)$. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertex v_1, and so $i(G) \le i(G^*) + 1$. By Claim 10, if there is a bad component in G^*, then such a component is incident with at least three X^*-edges. By Claims 13 and 14, if there is a troublesome configuration in G^*, then such a component is incident with at least two X^*-edges. Thus since there are at most four X^*-edges, namely two edges incident with vertex a and two incident with vertex b, we infer that either $b(G^*) = \mathrm{tc}(G^*) = 0$ or $b(G^*) = 1$ and $\mathrm{tc}(G^*) = 0$ or $b(G^*) = 0$ and $\mathrm{tc}(G^*) \le 2$. In all cases, $b(G^*) + \mathrm{tc}(G^*) \le 2$.

Suppose that $b(G^*) + \mathrm{tc}(G^*) \le 1$, and so $\Theta(G^*) \le 2$ and $\Omega(G^*) \le \mathrm{w}(G^*) + 2$. If $\deg_G(a) = 2$, then $\mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 3 = \mathrm{w}(G) - 15 + 3 = \Omega(G) - 12$, and so $\Omega(G^*) \le \mathrm{w}(G^*) + 2 \le \Omega(G) - 10$. If $\deg_G(a) = 3$, then $\mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 4 = \mathrm{w}(G) - 14 + 4 = \Omega(G) - 10$, and so $\Omega(G^*) \le \mathrm{w}(G^*) + 2 \le \Omega(G) - 8$. In both cases, $\Omega(G^*) + 8 \le \Omega(G)$. Thus, $8i(G) \le 8(i(G^*) + 1) \le \Omega(G^*) + 8 \le \Omega(G)$, a contradiction.

Hence, $b(G^*) + \mathrm{tc}(G^*) = 2$, implying by our earlier observations that $b(G^*) = 0$ and $\mathrm{tc}(G^*) = 2$. Thus, G^* contains two troublesome configurations, say T_w and T_z, each of which contain two neighbors of a or b. In particular, we note that $\deg_G(a) = \deg_G(b) = 3$. Let $N_G(a) = \{a_1, a_2, v_1\}$ and let $N_G(b) = \{b_1, b_2, v_1\}$. Let w_1 and w_2 be the link vertices of T_w and let z_1 and z_2 be the link vertices of T_z. Let T_w be obtained from the bad graph $B_w \in \mathcal{B}_1$ with root vertex w_1, and let T_z be obtained from the bad graph $B_z \in \mathcal{B}_1$ with root vertex z_1. Moreover, let x_1 and x_2 be the vertices of B_w adjacent to w_1 and w_2, respectively, and let y_1 and y_2 be the vertices of B_z adjacent to z_1 and z_2, respectively. By Claims 8 and 9, each of B_w and B_z contains exactly two copies of $K_{2,3}$. Further, each copy of $K_{2,3}$ in B_w and B_z contains a vertex incident with an X^*-exit edge.

We now let $X'' = X^* \cup (V(T_w) \setminus \{x_2, w_2\}) \cup (V(T_z) \setminus \{y_2, z_2\})$ and let $G'' = G - X''$. In the case when $\{a_1, b_1\} \subset V(T_w)$ and $\{a_2, b_2\} \subset V(T_z)$, the graph illustrated in Figure 36 is a subgraph of G, where T_w and T_z are the troublesome configurations indicated in the dashed boxes and where X^* and X'' are the sets indicated by the vertices in the dotted region.

If $\deg_G(w_1) = 3$, then let $e_w = ww_1$ be the edge incident with w_1 not in T_w, and if $\deg_G(z_1) = 3$, then let $e_z = zz_1$ be the edge incident with z_1 not in T_z. The vertices x_2 and y_2 have degree 1 in G'', and the neighbors of x_2 and y_2, namely w_2 and z_2, in G'' have degree 2. Hence, there is no bad component or troublesome configuration in G'' containing these four vertices. By Claim 10, 11, 13 and 14, we have $b(G'') = 0$ and $\mathrm{tc}(G'') \le 1$. Moreover if $\mathrm{tc}(G'') = 1$, then the edges e_w and e_z exist and their removal increases the total weight by $2 + 2 = 4$ (namely, an increase in 2 from the vertex weight and an increase in 2 in the structural weight arising from a troublesome configuration containing w and z). If $\mathrm{tc}(G'') = 0$, then the removal of e_w and e_z, if these edges exist, increases the total weight by at most $2 \times 3 = 6$ in the worst case (which can only occur if w and z are isolated vertices in G''). As observed earlier, the removal of the six X''-exit edges different from e_w and e_z do not increase the structural weight and therefore only increase the vertex weight (by 6). We therefore infer that $\Omega(G'') \le \Omega(G) - \mathrm{w}_G(X'') + 6 + 6 \le \Omega(G) - 74 + 6 + 6 = \Omega(G) - 62$.

Every i-set of G'' can be extended to an ID-set of G'' by adding to it the vertices a, b and u_1, together with two vertices from each of T_w and T_z (as illustrated by the shaded vertices in Figure 36). Thus, $i(G) \le i(G'') + 7$. As observed earlier, $\Omega(G'') + 62 \le \Omega(G)$. Therefore, $8i(G) \le 8(i(G'') + 7) \le \Omega(G'') + 56 < \Omega(G)$, a contradiction. We therefore deduce that our supposition that G contains a vertex of degree 1 is false. Hence, $\delta(G) \ge 2$, completing the proof of Claim 16. ($\square$)

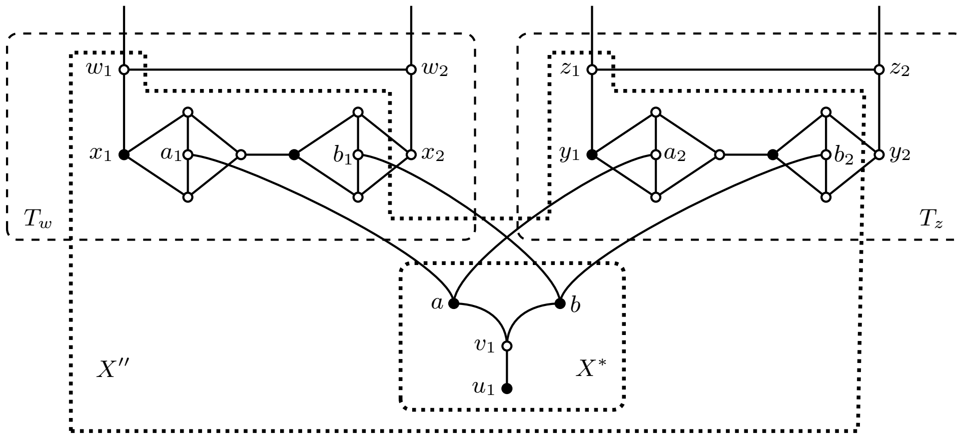


Figure 33: An illustration of a subgraph of G in the proof of Claim 16

6.3 Part 3: The minimum degree of G is at least two

By Claim 16, $\delta(G) \geq 2$. Recall that $n \geq 6$ and that G contains $K_{2,3}$ as a subgraph. By supposition, $G \neq K_{3,3}$. As in the proof of Part 2, in what follows we will frequently use the structural property stated in Claim 8 that there is no $K_{2,3}$-subgraph in G that contains a vertex of degree 2 in G.

We prove next a key structural property of the graph G. For this purpose, we define a *troublesome subgraph* of G as a subgraph T such that there exists a set of edges incident with vertices of T whose removal creates a troublesome configuration in G with vertex set $V(T)$. For example, the subgraph T illustrated in Figure 21 is a troublesome subgraph of G since the removal of the edge e produces a troublesome configuration in $G - e$. As a further example, the subgraph T illustrated in Figure 22 or in Figure 23 is a troublesome subgraph of G since the removal of the edge e (in both figures) produces a troublesome configuration in $G - e$.

Claim 17 *There is no troublesome subgraph in G.*

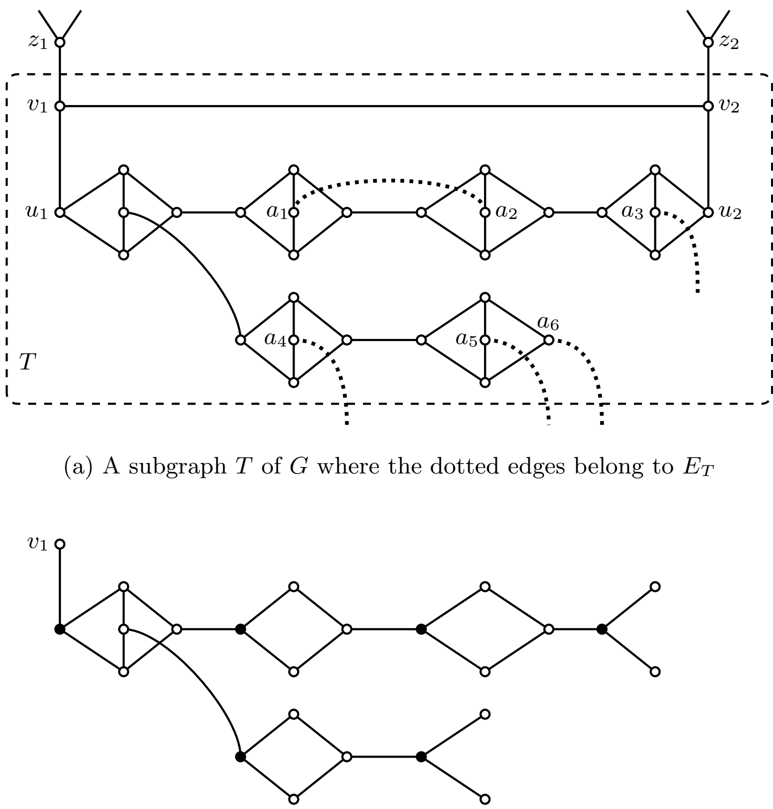
Proof. Suppose, to the contrary, that G contains a troublesome subgraph, say T. Hence there exists a set, say $E_T \subset E(G)$, of edges incident with vertices of T such that $G' = G - E_T$ contains a troublesome configuration, say T', with vertex set $V(T') = V(T)$. By Claim 4, the graph G contains no troublesome configuration, that is, $\mathrm{tc}(G) = 0$. By Claim 13, $\mathrm{tc}(G - e) = 0$ for all $e \in E(G)$. Hence, $|E_T| \geq 2$, that is, at least two edges incident with vertices in T were removed when constructing T'.

Let the troublesome configuration T' in G' have link vertices v_1 and v_2, where $\deg_G(v_2) = 3$. Thus, T' is obtained from the bad graph $B \in \mathcal{B}$ with root vertex v_1 (of degree 1 in B). Further, let u_1 and u_2 be the vertices of B adjacent to v_1 and v_2, respectively, in G. Let z_2 be the neighbor of v_2 different from u_2 and v_1. Let the bad graph B have k copies of $K_{2,3}$ in G'. Since $|E_T| \geq 2$, we note that $k \geq 2$. Let $A = \{a_1, a_2, \dots, a_k\}$ be the set of vertices in B incident with edges in E_T.

Claim 17.1 *If the link vertex v_1 is not incident with an edge of E_T, then $8i(G) \leq \Omega(G)$.*

Proof. Suppose that the link vertex v_1 is not incident with an edge of E_T. If v_1 has degree 3 in G, then let z_1 be the neighbor of v_1 not in T. We now let $X = V(T) \setminus (A \cup \{u_2, v_2\})$ and we consider the graph $G' = G - X$. We note that each vertex in A has degree 1 in G', and belongs to neither a bad component nor a troublesome configuration in G'. Moreover, the vertex u_2 has degree 1 in G' and its neighbor, v_2, in G' has degree 2 in G'. Hence, u_2 and v_2 do not belong to a bad component or a troublesome configuration in G'. If $\deg_G(v_1) = 3$, then we infer by Claims 10 and 13 that the vertex z_1 does not belong to a bad component or a troublesome configuration in G'. Hence, $b(G') = \mathrm{tc}(G') = 0$, and so $\Theta(G') = 0$. We note that there are at most $2k + 4$ X-exit edges. Moreover, $|X| = 4k$ and every vertex of X, except for possibly the vertex v_1, has degree 3 in G. Therefore, $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 2k + 4 \leq \mathrm{w}(G) - 3 \times 4k + 2k + 4 = \mathrm{w}(G) - 10k + 4 = \Omega(G) - 10k + 4$.

Every i-set of G' can be extended to an ID-set of G by adding to it the core independent set (see Section 5.2) of the bad graph B, and so $i(G) \le i(G') + k$. (For example, if the graph shown in Figure 34(a) is a subgraph of G, where the dotted edges are the edges that belong to the set E_T, then $k = 6$ and the subgraph $G[X]$ is illustrated in Figure 34(b). Furthermore, the core independent set of B is indicated by the $k = 6$ shaded vertices in Figure 34(b).) Thus noting that $k \ge 2$, we have $8i(G) \le 8(i(G')+k) \le \Omega(G')+8k \le (\Omega(G)-10k+4)+8k = \Omega(G)-2k+4 \le \Omega(G)$, a contradiction. $(\Box)$



(a) A subgraph T of G where the dotted edges belong to E_T

(b) The subgraph $G[X]$ where $X = V(T) \setminus \{a_1, a_2, a_3, a_4, a_5, a_6, u_2, v_2\}$

Figure 34: An illustration of a subgraph of G in the proof of Claim 17.1

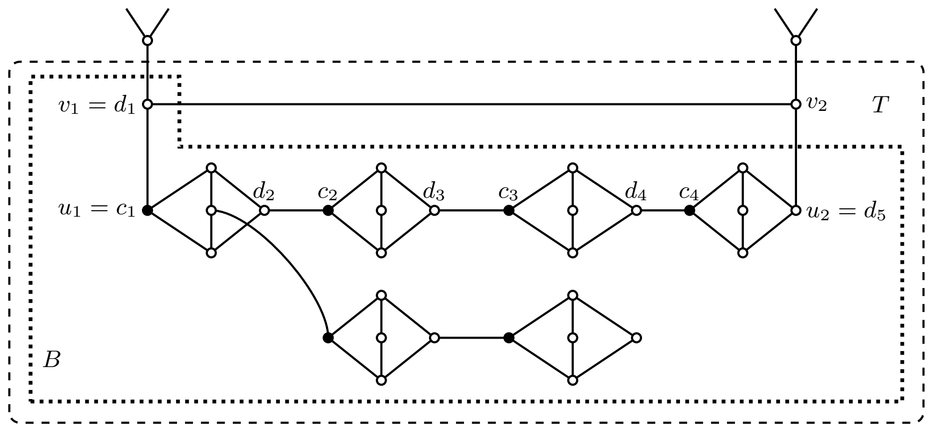
Claim 17.2 *If the link vertex v_1 is incident with an edge of E_T, then $8i(G) < \Omega(G)$.*

Proof. Suppose that the link vertex v_1 is incident with an edge of E_T. Renaming vertices if necessary, we may assume that $e = a_1v_1$. We now let $X = V(T) \setminus (A \setminus \{a_1\})$ and we consider the graph $G' = G - X$. We note that each vertex in $A \setminus \{a_1\}$ has degree 1 in G', and belongs to neither a bad component nor a troublesome configuration in G'. Moreover, the vertex z_2 has degree 1 or 2 in G', and by Claims 10 and 13 we infer that z_2 does not belong to a bad component or a troublesome configuration in G'. Hence, $b(G') = \mathrm{tc}(G') = 0$, and so $\Theta(G') = 0$. Thus, $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 2k - 1 = \mathrm{w}(G) - 3 \times (4k+3) + 2k - 1 = \mathrm{w}(G) - 10k - 10 = \Omega(G) - 10k - 10$.

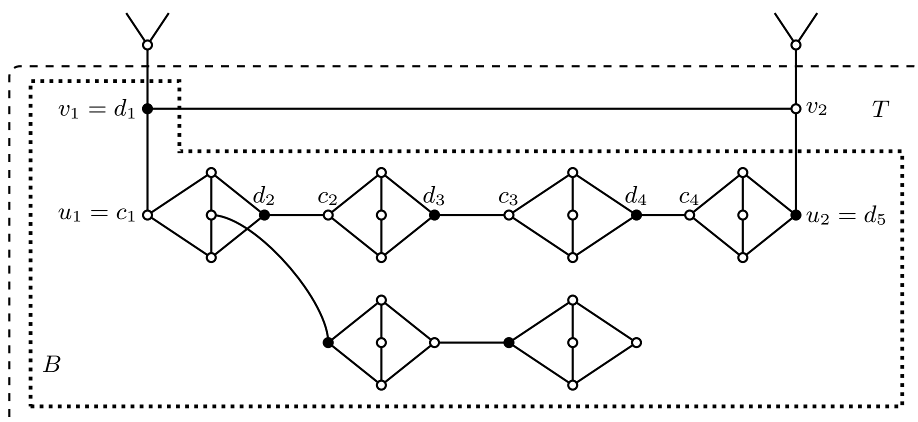
Let I_B be the core independent set (see Section 5.2) of the bad graph B, and so $|I_B| = k$. We define the modified core independent set of B as follows. We say that two copies of $K_{2,3}$ are adjacent if they are joined by an edge. By the structure of the bad graph G, there exists a sequence $F_1, \dots, F_\ell$ of copies of $K_{2,3}$ where $u_1 \in V(F_1)$, $u_2 \in V(F_\ell)$ and if $\ell \ge 2$, then F_{i+1} is adjacent to F_i for all $i \in [\ell - 1]$. Let c_i be the vertex in F_i that belongs to I_B for $i \in [\ell]$. In particular, we note that $c_1 = u_1$. Let $C = \{c_1, \dots, c_\ell\}$. We define $d_1 = v_1$, and if $\ell \ge 2$, then we define d_{i+1} as the vertex of F_i that is adjacent to c_{i+1} in B for $i \in [\ell - 1]$. Moreover, we define $d_{\ell+1} = u_2$, and we let $D = \{d_1, d_2, \dots, d_{\ell+1}\}$. We now define the modified core independent set of B by

$$I_B^* = (I_B \setminus C) \cup D.$$

As an illustration, suppose that T is the troublesome configuration in Figure 35, where B is the associated bad graph used to construct T. In this example, the core independent set I_B is indicated by the shaded vertices in Figure 35(a), and the modified core independent set I_B^* is indicated by the shaded vertices in Figure 35(b). We note that $|I_B| = k$ and $|I_B^*| = k+1$, where here $k = 6$ is the number of copies of $K_{2,3}$ in the bad graph B.



(a) The core independent set, I_B, of B



(b) The modified core independent set, I_B^*, of B

Figure 35: An illustration of a modified core independent set in a bad graph

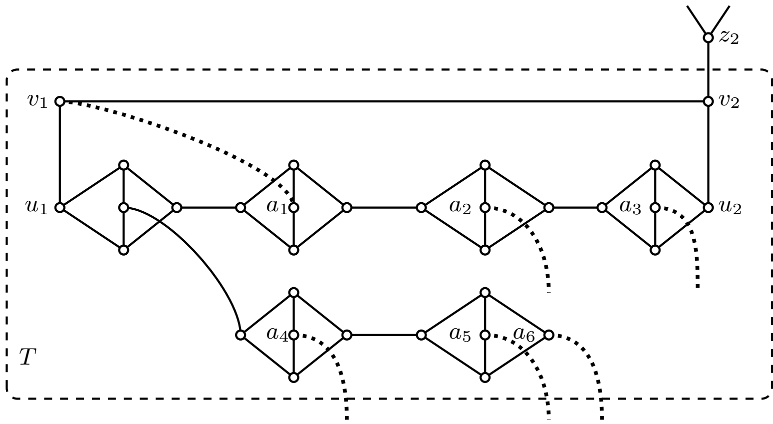
We now return to the proof of Claim 17.2. Every i-set of G' can be extended to an ID-set of G by adding to it the modified core independent set I_B^* of B, and so $i(G) \le i(G') + |I_B^*| = i(G') + k + 1$. (For example, if the graph shown in Figure 36(a) is a subgraph of G, where the dotted edges are the edges that belong to the set E_T, then $k = 6$ and the subgraph $G[X]$ is illustrated in Figure 36(b) where the shaded vertices indicate the vertices of the modified core independent set I_B^* of B.) Recall that $\Omega(G') + 10k + 10 = \Omega(G)$. Thus, $8i(G) \le 8(i(G') + k + 1) \le \Omega(G') + 8k + 8 = \Omega(G) - 2k - 2 < \Omega(G)$, a contradiction. (□)

In both Claims 17.1 and 17.2, we have $8i(G) \le \Omega(G)$, a contradiction. This completes the proof of Claim 17. (□)

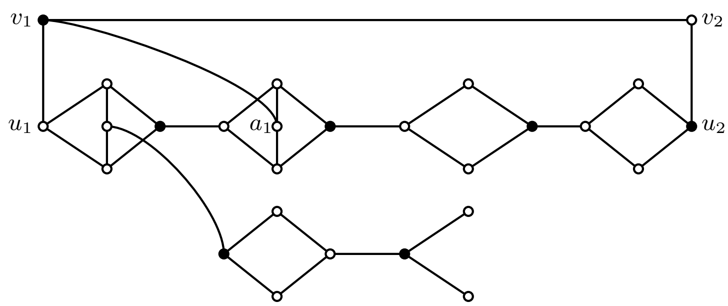
By Claim 17, there is no troublesome subgraph in G. Hence, if G' is an arbitrary induced subgraph of G, then $\mathrm{tc}(G') = 0$, and so $\Theta(G') = 2b(G')$. We state this formally as follows.

Claim 18 *If G' is an induced subgraph of G, then $\Theta(G') = 2b(G')$.*

By Claim 18, the structural weight of an induced subgraph G' of G is determined only by the number of bad components of G', if any, that belong to the bad family $\mathcal{B}$. Recall that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Moreover, every graph in



(a) A subgraph T of G where the dotted edges belong to E_T



(b) The subgraph $G[X]$ where $X = V(T) \setminus \{a_2, a_3, a_4, a_5, a_6\}$

Figure 36: An illustration of a subgraph of G in the proof of Claim 17.2

the family $\mathcal{B}_1$ has one vertex of degree 1 and at least two vertices of degree 2, while every graph in the family $\mathcal{B}_2$ has at least five vertices of degree 2. Furthermore, every graph in the family $\mathcal{B}_3$ has at least six vertices of degree 2. We prove next certain properties that must hold if the graph G contains vertices of degree 2.

Claim 19 *The graph G does not contain a path $v_1v_2v_3$ such that $\deg_G(v_i) = 2$ for all $i \in [3]$.*

Proof. Suppose, to the contrary, that G contains a path $v_1v_2v_3$ such that $\deg_G(v_i) = 2$ for all $i \in [3]$. Let $X = \{v_1, v_2, v_3\}$ and let $G' = G - X$. We note that there are two X-exit edges. If G' contains an isolated vertex, then $G = C_4$, a contradiction. Hence, $\delta(G') \geq 1$ and the graph G' contains at most two vertices of degree 1. By Claim 10, we infer that $b(G') = 0$. Thus, by Claim 18, we have $\Theta(G') = 0$. Thus, $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 2 = \Omega(G) - 12 + 2 = \Omega(G) - 10$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v_2, and so $i(G) \leq i(G') + 1$. Hence, $8i(G) \leq 8(i(G') + 1) \leq \Omega(G') + 8 < \Omega(G)$, a contradiction. (□)

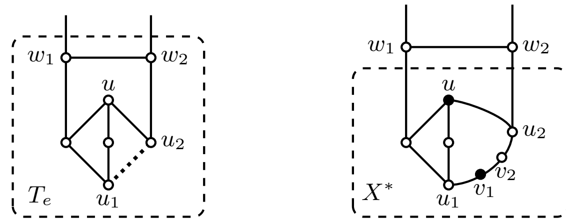
Claim 20 *No two adjacent vertices in G both have degree 2.*

Proof. Suppose, to the contrary, that G contains two adjacent vertices, say v_1 and v_2, of degree 2. Suppose firstly that v_1 and v_2 have a common neighbor v, and so vv_1v_2v is a 3-cycle in G. Let u be the third neighbor of v different from v_1 and v_2. Let $X = \{v, v_1, v_2\}$ and let $G' = G - X$. The graph G' is a connected subcubic graph and all vertices in G' have degree at least 2, except possibly for the vertex u which has degree at least 1. By Claims 10 and 18, we infer that $\Theta(G') = 0$. Thus, $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 1 = \Omega(G) - 11 + 1 = \Omega(G) - 10$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v_1, and so $i(G) \leq i(G') + 1$. Hence, $8i(G) \leq 8(i(G') + 1) \leq \Omega(G') + 8 = \Omega(G)$, a contradiction.

The vertices v_1 and v_2 therefore have no common neighbor. Let u_i be the neighbor of v_i different from v_{3-i} for $i \in [2]$. Since v_1 and v_2 have no common neighbor, $u_1 \neq u_2$. By Claim 19, both vertices u_1 and u_2 have degree 3 in G. We show firstly that u_1u_2 is not an edge in G. Suppose, to the contrary, that $u_1u_2 \in E(G)$. In this case, we let $X = \{v_1, v_2, u_1\}$ and consider the graph $G' = G - X$. We note that G' is isolate-free and the vertex u_2 has degree 1 in G'. By Claim 18, $\mathrm{tc}(G') = 0$. By Claims 10 and 11, we infer that if G' has more than one component, none of them is bad. In addition, if G' is connected, then by Claim 8 we infer $G' \notin \mathcal{B}$. Thus, $\Theta(G') = 0$. Thus, $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 3 = \Omega(G) - 11 + 3 = \Omega(G) - 8$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v_1, and so $i(G) \le i(G') + 1$. Hence, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G)$, a contradiction. Therefore, $u_1u_2 \notin E(G)$.

We now let $X = \{v_1, v_2\}$ and consider the graph G' obtained from $G - X$ by adding the edge $e = u_1u_2$. We note that G' is a connected subcubic graph satisfying $\delta(G') \ge 2$. Furthermore, u_1 and u_2 are adjacent vertices of degree 3 in G'. Let I' be an i-set of G'. If $u_1 \in I'$, then let $I = I' \cup \{v_2\}$. If $u_2 \in I'$ or if neither u_1 nor u_2 belong to I', then let $I = I' \cup \{v_1\}$. In all cases, $|I| = |I'| + 1$ and the set I is an ID-set of G, and so $i(G) \le i(G') + 1$. If $\Theta(G') = 0$, then $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) = \Omega(G) - 8$, and so $8i(G) \le 8(i(G')+1) \le \Omega(G')+8 = \Omega(G)$, a contradiction. Hence, $\Theta(G') \ge 1$, implying that either $G' \in \mathcal{B}$ or the edge u_1u_2 belongs to a troublesome configuration in G'. If $G' \in \mathcal{B}$, then G would contain a copy of $K_{2,3}$ that contains two vertices of degree 2 in G (noting that in this case, the root vertex in G' has degree at least 2), a contradiction. Hence, $G' \notin \mathcal{B}$.

Thus, adding the edge $e = u_1u_2$ to the graph $G - X$ creates a new troublesome configuration, which we call T_e, that necessarily contains the edge e. Let w_1 and w_2 be the two link vertices of T_e. By combining Claim 8 and Claim 17, we infer that T_e contains exactly one copy of $K_{2,3}$, say F_e, and both u_1 and u_2 belong to F_e (for otherwise, F_e would contain a vertex of degree 2 in G, a contradiction). By symmetry and renaming vertices if necessary, we may assume that u_2 is adjacent to the link vertex w_2 of T_e. Let u be the third neighbor of u_2 in T_e different from u_1 and w_2, as illustrated in Figure 37(a). We now let $X^* = V(F_e) \cup X$ and let $G^* = G - X^*$. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertices v_1 and u, as indicated by the shaded vertices in Figure 37(b). Thus, $i(G) \le i(G^*) + 2$. We note that $\Theta(G^*) = \Theta(G) = 0$, and so $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 2 = \Omega(G) - 24 + 2 = \Omega(G) - 22$. Hence, $8i(G) \le 8(i(G^*) + 2) \le \Omega(G^*) + 16 < \Omega(G)$, a contradiction. (□)



(a) A subgraph T_e of G' (b) A subgraph of G

Figure 37: Subgraphs of G' and G in the proof of Claim 20

Claim 21 *A vertex of degree* 3 *in* G *has at most one neighbor of degree* 2*.*

Proof. Suppose, to the contrary, that G contains a vertex, say v, of degree 3 that is adjacent to at least two vertices of degree 2. Let $X = N_G[v]$ and let $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v, and so $i(G) \le i(G') + 1$.

Suppose that all three neighbors of v have degree 2 in G. By Claim 20, the set $N_G(v)$ is therefore an independent set. Suppose that G' contains an isolated vertex, u say. Since $\delta(G) \ge 2$, we note that $\deg_G(u) \ge 2$. If $\deg_G(u) = 3$, then $G = K_{2,3}$, contradicting the fact that $n \ge 6$. If $\deg_G(u) = 2$, then since u is adjacent to two vertices in $N_G(v)$, the graph G contains a path P_3 all of whose vertices have degree 2 in G, contradicting Claim 19. Hence, the graph G' is isolate-free. Since there are three X-exit edges, we note that $\mathrm{w}(G') = (\mathrm{w}(G) - 15) + 3 = \mathrm{w}(G) - 12$. By Claim 11, we infer that $b(G') \le 1$. Thus by Claim 18, we have $\Theta(G') \le 2$, and so $\Omega(G') = \mathrm{w}(G') + \Theta(G') = (\mathrm{w}(G) - 12) + 2 = \Omega(G) - 10$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v, and so $i(G) \le i(G') + 1$. Hence, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 < \Omega(G)$, a contradiction.

Therefore, the vertex v has two neighbors of degree 2 and one of degree 3. Let v_1 and v_2 be the two neighbors of v of degree 2 and let v_3 be the third neighbor of v (of degree 3). Suppose that G' contains an isolated vertex, u say. If $\deg_G(u) = 3$, then G contains a copy of $K_{2,3}$ that contains two vertices of degree 2 in G, contradicting Claim 8. If $\deg_G(u) = 2$, then since u is adjacent to at least one of v_1 and v_2, the graph G contains two adjacent vertices of degree 2, contradicting Claim 20. Hence, the graph G' is isolate-free. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v, and so $i(G) \le i(G')+1$. By Claim 11, we infer that $b(G') \le 1$. Thus by Claim 18, we have $\Theta(G') \le 2$, and so $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le (\mathrm{w}(G) - \mathrm{w}_G(X) + 4) + 2 = \mathrm{w}(G) - 14 + 4 + 2 = \Omega(G) - 8$. Thus, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G)$, a contradiction. $(\Box)$

By Claim 20, both neighbors of a vertex of degree 2 have degree 3 in G. By Claim 21, at most one neighbor of a vertex of degree 3 has degree 2 in G.

Claim 22 *No vertex of degree* 2 *belongs to a triangle.*

Proof. Suppose, to the contrary, that G contains a vertex, say u, of degree 2 that belongs to a triangle, T say. Let v_1 and v_2 be the two neighbors of u, and so $v_1v_2 \in E(G)$ and $V(T) = \{u, v_1, v_2\}$. By our earlier observations, both v_1 and v_2 have degree 3 in G. Let w_i be the neighbor of v_i not in T for $i \in [2]$. Since at most one neighbor of a vertex of degree 3 has degree 2 in G, both w_1 and w_2 have degree 3.

Suppose firstly that $w_1 = w_2$. In this case, we let $X = N_G[v_1]$ and consider the graph $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding the vertex v_1, and so $i(G) \le i(G') + 1$. By Claims 10 and 18, we infer that $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 1 = \mathrm{w}(G) - 13 + 1 = \Omega(G) - 12$. Thus, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 < \Omega(G)$, a contradiction. Hence, $w_1 \ne w_2$.

Suppose that $w_1w_2 \notin E(G)$. In this case, we let $X = N_G[u] = \{u, v_1, v_2\}$ and let G' be obtained from $G - X$ by adding the edge w_1w_2. Let I' be an i-set of G'. If $w_1 \in I'$, then let $I = I \cup \{v_2\}$, while if $w_1 \notin I'$, then let $I = I \cup \{v_1\}$. In both cases, $|I| = |I'| + 1$ and the set I is an ID-set of G, and so $i(G) \le i(G') + 1$. The added edge w_1w_2 increases the structural weight by at most 2 (which occurs if $G' \in \mathcal{B}$ or if the edge w_1w_2 belongs to a troublesome configuration). Hence, $\Theta(G') \le 2$, and so $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le (\mathrm{w}(G) - 10) + 2 = \Omega(G) - 8$. Thus, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G)$, a contradiction.

Hence, $w_1w_2 \in E(G)$. We now let $X = N_G[v_1] = \{u, v_1, v_2, w_1\}$ and consider the graph $G' = G - X$. Every i-set of G' can be extended to an ID-set of G by adding the vertex v_1, and so $i(G) \le i(G')+1$. By combining Claims 8, 10 and 18, we infer that $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - \mathrm{w}_G(X) + 3 = \mathrm{w}(G) - 13 + 3 = \Omega(G) - 10$. Therefore, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 < \Omega(G)$, a contradiction. $(\Box)$

Recall that by Claim 1, the graph G contains at least one subgraph isomorphic to $K_{2,3}$, and by Claim 2, the graph G is connected. By supposition, $G \ne K_{3,3}$ and $G \ne C_5 \,\Box\, K_2$. We proceed further by proving additional structural properties of the graph G. A *diamond* in G is a copy of $K_4 - e$ where e is an arbitrary edge of the complete graph K_4. A graph is *diamond-free* if it does not contain a diamond as an induced subgraph.

Claim 23 *The graph G is diamond-free.*

Proof. Suppose, to the contrary, that G contains a diamond D. Let $V(D) = \{v_1, v_2, v_3, v_4\}$ where v_1v_2 is the missing edge in the diamond. By Claim 22, no vertex of degree 2 belongs to a triangle. Thus, both v_1 and v_2 have degree 3 in G. We now let $X = V(D)$ and we consider the graph $G' = G - X$. Since $n \ge 6$, we note that G' is isolate-free. Every i-set of G' can be extended to an ID-set of G by adding the vertex v_3, and so $i(G) \le i(G')+1$. By Claims 11 and 18, we infer that $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le \mathrm{w}(G) - \mathrm{w}_G(X) + 2 = \mathrm{w}(G) - 12 + 2 = \Omega(G) - 10$. Thus, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 < \Omega(G)$, a contradiction. $(\Box)$

Claim 24 *The graph G does not contain $K_{3,3} - e$ as a subgraph, where e is an edge of the $K_{3,3}$.*

Proof. Suppose, to the contrary, that G contains $K_{3,3} - e$ as a subgraph, where e is an edge of the $K_{3,3}$. Let F be such a subgraph in G, and let F have partite sets $A = \{x_1, x_2, x_3\}$ and $B = \{v_1, v_2, v_3\}$, where $e = x_1v_1$ is the missing edge from the copy of $K_{3,3}$ in F. Since G is diamond-free by Claim 23, the set B is an independent set in G. If $\deg_G(x_1) = 2$ or if $\deg_G(v_1) = 2$, then there is a $K_{2,3}$-subgraph in G that contains a vertex of degree 2 in G, a contradiction. Hence, $\deg_G(x_1) = \deg_G(v_1) = 3$. Let u_1 be the neighbor of v_1 different from x_2 and x_3, and let w_1 be the neighbor of x_1 different from v_2 and v_3. Since $G \neq K_{3,3}$, the vertices u_1 and x_1 are distinct. If $x_1u_1 \in E(G)$, then G contains G_7 as a subgraph (see Figure 29), contradicting Claim 15. Hence, $u_1x_1 \notin E(G)$, and so the vertices u_1 and w_1 are distinct. We now let $X = V(F) \cup \{u_1, w_1\}$ and consider the graph $G' = G - X$.

Suppose that G' contains an isolated vertex, say u. Thus, $N_G(u) = \{u_1, w_1\}$. By Claim 20, $\deg_G(u_1) = \deg_G(w_1) = 3$. By Claim 22, $u_1w_1 \notin E(G)$. In this case, we let $X' = X \cup \{u\}$ and consider the graph $G'' = G - X'$. By Claim 21, the neighbor of u_1 in G'' has degree 3 in G and the neighbor of w_1 in G'' has degree 3 in G. Thus, G'' is isolate-free and there are two X'-exit edges. By Claims 11 and 18, we infer that $\Theta(G'') = 0$, and so $\Omega(G'') = \mathrm{w}(G'') = \mathrm{w}(G) - \mathrm{w}_G(X') + 2 = \mathrm{w}(G) - 28 + 2 = \Omega(G) - 26$. Every i-set of G'' can be extended to an ID-set of G by adding to it the vertices in the set $\{u, v_1, x_1\}$ (indicated by the shaded vertices in Figure 38(a)), and so $i(G) \le i(G'') + 3$. Thus, $8i(G) \le 8(i(G'') + 3) \le \Omega(G'') + 24 < \Omega(G)$, a contradiction.

Hence, G' is isolate-free. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{v_1, x_1\}$ (indicated by the shaded vertices in Figure 38(b)), and so $i(G) \le i(G') + 2$. We note that there are at most four X-exit edges. By Claims 11 and 18, we infer that $b(G') \le 1$ and $\Theta(G') \le 2$, and so $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le (\mathrm{w}(G) - \mathrm{w}_G(X) + 4) + 2 = \mathrm{w}(G) - 24 + 4 + 2 = \Omega(G) - 18$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{v_1, x_1\}$ (indicated by the shaded vertices in Figure 38(b)), and so $i(G) \le i(G') + 2$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 < \Omega(G)$, a contradiction. $_{(\square)}$

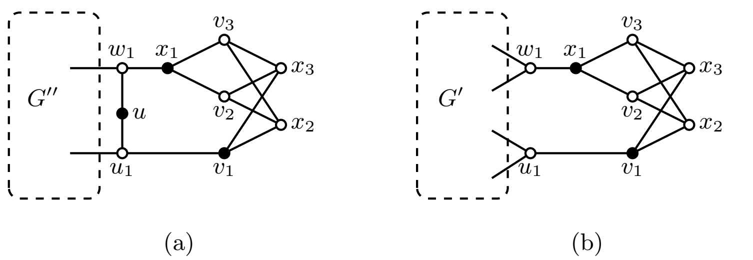


Figure 38: Subgraphs of G in the proof of Claim 24

Claim 25 *Every $K_{2,3}$-subgraph of G belongs to a $G_{8.2}$-configuration that is an induced subgraph in G, where $G_{8.2}$ is the graph in Figure 39(b).*

Proof. Let F be a $K_{2,3}$-subgraph in G with partite sets $A = \{x_1, x_2\}$ and $B = \{v_1, v_2, v_3\}$. Since G is diamond-free by Claim 23, the set B is an independent set in G. Let u_i be the neighbor of v_i different from x_1 and x_2 for $i \in [3]$. By Claim 24, the vertices u_1, u_2 and u_3 are distinct. Let $U = \{u_1, u_2, u_3\}$. Since G_7 is not a subgraph of G, the set C is independent. Thus the subgraph $G[A \cup B \cup C]$ of G induced by the sets $A \cup B \cup C$ (see Figure 39(a)) is an induced subgraph of G. $_{(\square)}$

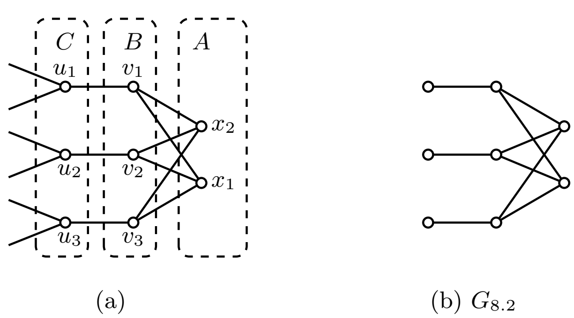


Figure 39: Subgraphs of G in the proof of Claim 25

Claim 26 *The graph G is cubic.*

Proof. Suppose, to the contrary, that $\delta(G) = 2$. Let u be an arbitrary vertex of degree 2 in G. By Claim 20, both neighbors of u have degree 3 in G. Let v be a neighbor of u, and let $N_G(v) = \{u, x, y\}$. By Claim 21, $\deg_G(x) = \deg_G(y) = 3$. We now let $X = N_G[v]$ and we consider the graph $G' = G - X$.

Claim 26.1 *The graph G' is isolate-free.*

Proof. Suppose that G' contains an isolated vertex, say w. Thus, $N_G(w) \subseteq N_G(v)$. If $N_G(w) = N_G(v)$, then $G[\{u, v, w, x, y\}]$ is a copy of $K_{2,3}$ that contains a vertex of degree 2 in G, a contradiction. Hence, w has degree 2 in G and is adjacent to exactly two vertices in $N_G(v)$. Since no two vertices of degree 2 in G are adjacent by Claim 20, we have $N_G(w) = \{x, y\}$. Let x_1 and y_1 be the neighbors of x and y, respectively, different from v and w. Since there is no $K_{2,3}$-subgraph that contains a vertex of degree 2 in G, we note that $x_1 \neq y_1$. By Claim 21, $\deg_G(x_1) = \deg_G(y_1) = 3$. Since $\deg_G(u) = 2$, the vertex u is adjacent to at most one of x_1 and y_1. By symmetry, we may assume that u and x_1 are not adjacent.

We now let $X' = N_G[x] = \{v, w, x, x_1\}$ and consider the graph $G'' = G - X'$. We note that both vertices u and y have degree 1 in G''. If u belongs to a bad component $B_u \in \mathcal{B}$ in G'', then necessarily $B_u = B_1$ where u is the vertex of degree 1 in B_u and where B_u contains exactly one copy of $K_{2,3}$, which we denote by F_u. In this case, the vertex x_1 is adjacent in G to the two vertices of degree 2 in F_u. However, then $G[V(F_u) \cup \{x_1\}]$ is a copy of $K_{3,3}$ with an edge removed, contradicting Claim 24. Hence, u does not belong to a bad component in G''. Analogously, y does not belong to a bad component in G''. By Claim 11, we therefore infer that removing the two X'-exit edges incident with x_1 does not create a bad component. Hence, $b(G'') = 0$, and so by Claim 18 we have $\Theta(G'') = 0$, and so $\Omega(G'') = \mathrm{w}(G'') \leq \mathrm{w}(G) - \mathrm{w}_G(X) + 5 = \mathrm{w}(G) - 13 + 5 = \Omega(G) - 8$. Every i-set of G'' can be extended to an ID-set of G by adding to it the vertex x, and so $i(G) \leq i(G'') + 1$. Thus, $8i(G) \leq 8(i(G'') + 1) \leq \Omega(G'') + 8 = \Omega(G)$, a contradiction. $(\Box)$

Claim 26.2 $xy \notin E(G)$.

Proof. Suppose that $xy \in E(G)$. Thus there are exactly three X-exit edges. By Claim 26.1, the graph G' is isolate-free. By Claim 11, if there is a bad component in G', then all three X-exit edges emanate from such a component and $G' = B_1$. Let r be the root vertex of G'. By Claim 25, the $K_{2,3}$-subgraph in G' belongs to a $G_{8.2}$-configuration (see Figure 39(b)). The graph G is therefore determined (up to isomorphism) and is illustrated in Figure 40. In this case, $i(G) = 3$ (the shaded vertices in Figure 40 indicate an i-set of G) and $\mathrm{w}(G) = 32$, and so $8i(G) < \Omega(G)$, a contradiction. Hence, $b(G') = 0$. By Claim 18, we have $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') \leq \mathrm{w}(G) - \mathrm{w}_G(X) + 3 = \mathrm{w}(G) - 13 + 3 = \Omega(G) - 10$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v, and so $i(G) \leq i(G') + 1$. Thus, $8i(G) \leq 8(i(G') + 1) \leq \Omega(G') + 8 < \Omega(G)$, a contradiction. $(\Box)$

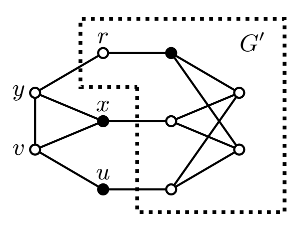


Figure 40: The graph G in the proof of Claim 26.2

By Claim 26.1, the graph G' is isolate-free. By Claim 26.2, $xy \notin E(G)$. Hence, there are at exactly five X-exit edges. By Claim 11, if there is a bad component in G', then at least three X-exit edges emanate from such a component, implying that $b(G') \leq 1$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertex v, and so $i(G) \leq i(G') + 1$. If $b(G') = 0$, then by Claim 18 we have $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') \leq \mathrm{w}(G) - \mathrm{w}_G(X) + 5 = \mathrm{w}(G) - 13 + 5 = \Omega(G) - 8$. Thus, $8i(G) \leq 8(i(G') + 1) \leq \Omega(G') + 8 = \Omega(G)$, a contradiction. Hence, $b(G') = 1$. Let B be the bad component in G', and let r be the root vertex of B.

Claim 26.3 $B \in \mathcal{B}_1$.

Proof. Suppose that $B \in \mathcal{B}_2 \cup \mathcal{B}_3$. If $B \in \mathcal{B}_3$, then at least six X-exit edges emanate from such a component, contradicting the fact that there are five X-exit edges. Hence, $B \in \mathcal{B}_2$. Thus, the root vertex r of B has degree 2 in B and there are either two or three $K_{2,3}$-subgraphs in B. Suppose that there are three $K_{2,3}$-subgraphs in B. In this case the graph G is determined and $V(G) = V(B) \cup X$. The core independent set of B (see Section 5.2) of cardinality 3 can be extended to an ID-set of G by adding to it the set $N_G(v) = \{u, x, y\}$, and so $i(G) \le 6$. (For example, if G is the graph illustrated in Figure 41, then the ID-set of G containing the core independent set of B together with the vertices in $\{u, x, y\}$ is indicated by the six shaded vertices.) Thus, $8i(G) \le 48 < 62 = \Omega(G)$, a contradiction.

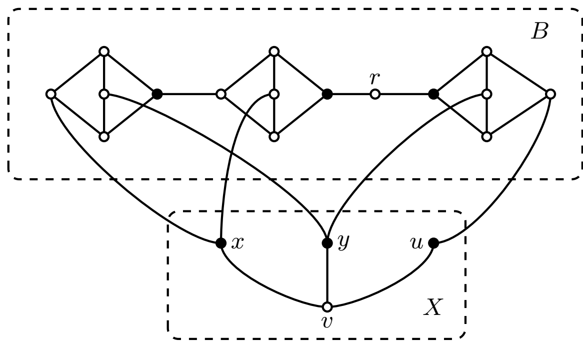


Figure 41: An example of a graph G in the proof of Claim 26.3

Hence, B contains exactly two $K_{2,3}$-subgraphs. If $V(G) = V(B) \cup X$, then the graph G is determined and as before the core independent set (of cardinality 2) of B can be extended to an ID-set of G by adding to it the set $N_G(v) = \{u, x, y\}$, and so $i(G) \le 5$. Moreover, $\Omega(G) = 46$, and so $8i(G) < \Omega(G)$, a contradiction. Hence, $V(G) \ne V(B) \cup X$, implying that there are exactly four X-exit edges that are incident with vertices in B and the root vertex r of B is not incident with any of these four X-exit edges.

Suppose the vertex u is not adjacent to a vertex of B in G. We now let $X^* = V(B) \cup \{v, x, y\}$ and consider the graph $G^* = G - X^*$. In this case, there is exactly one X^*-exit edge and the vertex u has degree 1 in G^*. By Claims 10 and 18, we infer that $b(G^*) = 0$ and $\Theta(G^*) = 0$, and so $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 1 = \mathrm{w}(G) - 43 + 1 = \Omega(G) - 42$. Every i-set of G^* can be extended to an ID-set of G by adding to it the core independent set of B of cardinality 2 and the vertices x and y (as indicated by the four shaded vertices in Figure 42), and so $i(G^*) \le i(G) + 4$. Thus, $8i(G) \le 8(i(G^*) + 4) \le \Omega(G^*) + 32 < \Omega(G)$, a contradiction.

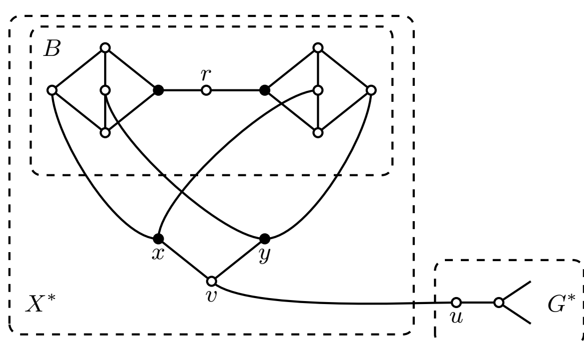


Figure 42: An example of a graph G in the proof of Claim 26.3

Hence, the vertex u is adjacent to a vertex of B in G. Renaming x and y if necessary, we may assume that x is adjacent to exactly one vertex, say x^*, of B in G. In this case, we let $X^* = V(B) \cup X$ and consider the graph

$G^* = G - X^*$. Since there is exactly one X^*-exit edge and the graph G^* is isolate-free, by Claims 10 and 18 we infer that $b(G^*) = 0$ and $\Theta(G^*) = 0$, and so $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 1 = \mathrm{w}(G) - 47 + 1 = \Omega(G) - 46$. Every i-set of G' can be extended to an ID-set of G by adding to it the core independent set of B of cardinality 2, together with the vertices in the set $\{u, x^*, y\}$, and so $i(G^*) \le i(G) + 5$. Thus, $8i(G) \le 8(i(G^*) + 5) \le \Omega(G^*) + 40 < \Omega(G)$, a contradiction. Since both cases produce a contradiction, this completes the proof of Claim 26.3. (□)

By Claim 26.3, we have $B \in \mathcal{B}_1$. Thus, the root vertex r in B has degree 1 in B. Let B have k copies of $K_{2,3}$. We note that $k \le 3$.

Claim 26.4 $k = 1$.

Proof. Suppose that $k \in \{2, 3\}$. If $k = 3$, then the graph G is determined and $V(G) = V(B) \cup X$. In this case, the core independent set of B (see Section 5.2) of cardinality 3 can be extended to an ID-set of G by adding to it the set $N_G(v) = \{u, x, y\}$, and so $i(G) \le 6$. Thus, $8i(G) \le 48 < 62 = \Omega(G)$, a contradiction. Hence, $k = 2$.

Suppose the vertex u is not adjacent to a vertex of B in G. We now let $X^* = V(B) \cup \{v, x, y\}$ and consider the graph $G^* = G - X^*$. In this case, there is exactly one X^*-exit edge and the vertex u has degree 1 in G^*. By Claims 10 and 18, we infer that $b(G^*) = 0$ and $\Theta(G^*) = 0$, and so $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 1 = \mathrm{w}(G) - 43 + 1 = \Omega(G) - 42$. Every i-set of G' can be extended to an ID-set of G by adding to it the core independent set of B of cardinality 2 and the vertices x and y (as indicated by the four shaded vertices in Figure 43), and so $i(G^*) \le i(G) + 4$. Thus, $8i(G) \le 8(i(G^*) + 4) \le \Omega(G^*) + 32 < \Omega(G)$, a contradiction.

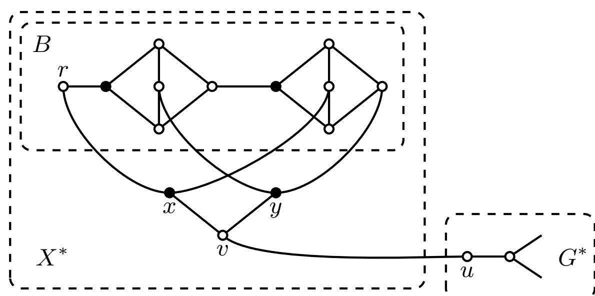


Figure 43: An example of a graph G in the proof of Claim 26.4

Hence, the vertex u is adjacent to a vertex of B in G. Renaming x and y if necessary, we may assume that x is adjacent to exactly one vertex, say x^*, of B in G. In this case, we let $X^* = V(B) \cup X$ and consider the graph $G^* = G - X^*$. Since there is exactly one X^*-exit edge and the graph G^* is isolate-free, by Claims 10 and 18 we infer that $b(G^*) = 0$ and $\Theta(G^*) = 0$, and so $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 1 = \mathrm{w}(G) - 47 + 1 = \Omega(G) - 46$. Every i-set of G' can be extended to an ID-set of G by adding to it the core independent set of B of cardinality 2, together with the vertices in the set $\{u, x^*, y\}$ (as indicated by the five shaded vertices in Figure 44), and so $i(G^*) \le i(G) + 5$. Thus, $8i(G) \le 8(i(G^*) + 5) \le \Omega(G^*) + 40 < \Omega(G)$, a contradiction. Since both cases produce a contradiction, this completes the proof of Claim 26.3. (□)

By Claim 26.4, we have $k = 1$, and so B contains exactly one copy of $K_{2,3}$. Let the copy of $K_{2,3}$ in B have partite sets $\{x_1, x_1\}$ and $\{v_1, v_2, v_3\}$. Renaming vertices if necessary, we may assume that the root vertex of B is adjacent to v_3. By Claim 25, every $K_{2,3}$-subgraph of G belongs to a $G_{8.2}$-configuration that is an induced subgraph in G, where $G_{8.2}$ is the graph in Figure 39(b). Let w_1 and w_2 be the neighbors of v_1 and v_2, respectively, not in B. We note by Claim 25 that $\{r, w_1, w_2\}$ is an independent set. Let w_3 be the third neighbor of v, and so the root vertex r is adjacent to w_3. Thus, $\deg_G(r) = 2$ and $N_G(r) = \{v_3, w_3\}$. We note that $N_G(v) = \{w_1, w_2, w_3\} = \{u, x, y\}$, where recall that $\deg_G(u) = 2$ and $\deg_G(x) = \deg_G(y) = 3$. Since there are no two adjacent vertices of degree 2 in G by Claim 20, we infer that $\deg_G(w_3) = 3$, and so $u \in \{w_1, w_2\}$. Renaming vertices if necessary, we may assume that $u = w_2$, and so $\deg_G(w_2) = 2$ and $N_G(w_2) = \{v, v_2\}$.

We now let $X'' = N_G[v] \cup V(B)$ and we consider the graph $G'' = G - X''$. Suppose that G'' contains an isolated vertex. In this case, such a vertex, say z, has degree 2 in G and is adjacent to w_1 and w_3, implying that the vertex w_3

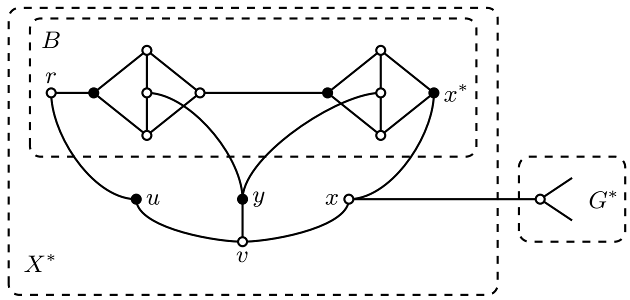


Figure 44: An example of a graph G in the proof of Claim 26.4

has two neighbors of degree 2, namely r and z, contradicting Claim 21. Hence, G'' is isolate-free. Thus, the graph shown in Figure 45 is a subgraph of G, where in this illustration the neighbors of w_1 and w_2 not in G'' are distinct. By Claims 11 and 18 we infer that $b(G'') = 0$ and $\Theta(G'') = 0$, and so $\Omega(G'') = \mathrm{w}(G'') \le \mathrm{w}(G) - \mathrm{w}_G(X'') + 2 = \mathrm{w}(G) - 32 + 2 = \Omega(G) - 30$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{r, v_1, w_2\}$ (as indicated by the three shaded vertices in Figure 45), and so $i(G'') \le i(G) + 3$. Thus, $8i(G) \le 8(i(G'') + 3) \le \Omega(G') + 24 < \Omega(G)$, a contradiction. This completes the proof of Claim 26. (□)

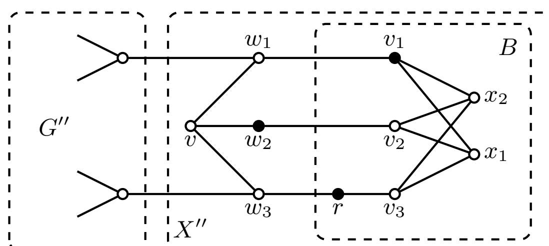


Figure 45: A subgraph of G in the proof of Claim 26

6.4 Part 4: The graph G is cubic

By Claim 26, the graph G is a cubic graph of order n. Recall that $n \ge 6$. By Claim 2, the graph G is connected. By supposition, $G \ne K_{3,3}$ and $G \ne C_5 \,\square\, K_2$. Since G is a cubic graph, we note that $\Theta(G) = 0$, and so $\Omega(G) = \mathrm{w}(G)$. We prove a number of claims that will culminate in a contradiction, thereby proving our main theorem.

Claim 27 *The graph G contains no triangle.*

Proof. Suppose, to the contrary, that G contains a triangle T, where $V(T) = \{v_1, v_2, v_3\}$. Let u_i be the neighbor of v_i not in T for $i \in [3]$. By Claim 23, the graph G is diamond-free, implying that the vertices u_1, u_2 and u_3 are distinct. Let $U = \{u_1, u_2, u_3\}$. If $G[U] = K_3$, then G is the 3-prism $C_3 \,\square\, K_2$. In this case, $i(G) = 2$ and $\mathrm{w}(G) = 18$, and so $8i(G) < \Omega(G)$, a contradiction. Hence renaming vertices if necessary, we may assume that $u_1u_2 \notin E(G)$. Let G' be the graph obtained from $G - V(T)$ by adding the edge u_1u_2. We note that every vertex of G' has degree 3, except for the vertex u_3 which has degree 2 in G'. We therefore infer that $b(G') = 0$.

Suppose that $\mathrm{tc}(G') = 0$. In this case, $\Theta(G') = 0$ and $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 9 + 1 = \Omega(G) - 8$. Let I' be an i-set of G'. If neither u_1 nor u_2 belong to I', then let $I = I' \cup \{v_1\}$. If $u_i \in I'$ for some $i \in [2]$, then let $I = I' \cup \{v_{3-i}\}$. In both cases, $|I| = |I'| + 1 = i(G') + 1$ and the set I is an ID-set of G, and so $i(G) \le i(G') + 1$. Hence, $8i(G) \le 8(i(G') + 1) \le \Omega(G') + 8 = \Omega(G)$, a contradiction.

Hence, $\mathrm{tc}(G') = 1$, implying that there is a troublesome configuration, say T', in G' where T' has exactly one copy of $K_{2,3}$, and this copy contains the vertex u_3 as the vertex of degree 2 in T', and where T' contains the added edge

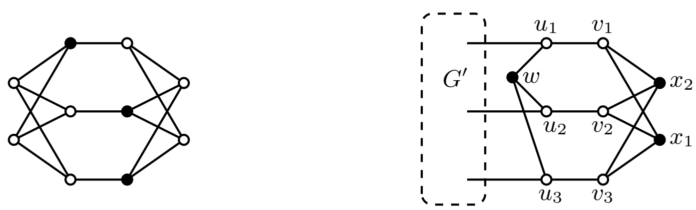
u_1u_2 We now let $X^* = V(T) \cup \{u_3\}$ and $G^* = G - X^*$. In this case, $\mathrm{tc}(G^*) = 0$ and $b(G^*) = 0$, and so $\Theta(G^*) = 0$. Thus, $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - \mathrm{w}_G(X^*) + 4 = \Omega(G) - 12 + 4 = \Omega(G) - 8$. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertex v_3, and so $i(G) \le i(G^*)+1$. Hence, $8i(G) \le 8(i(G^*)+1) \le \Omega(G^*)+8 = \Omega(G)$, a contradiction.

Claim 28 *The graph G does not contain two vertex disjoint copies of $K_{2,3}$ joined by at least one edge.*

Proof. Let F_1 and F_2 be two vertex disjoint copies of $K_{2,3}$, where F_1 has partite sets $X = \{x_1, x_2\}$ and $V_1 = \{v_1, v_2, v_3\}$, and F_2 partite sets $Y = \{y_1, y_2\}$ and $U_1 = \{u_1, u_2, u_3\}$. Suppose, to the contrary, that F_1 and F_2 are joined by at least one edge.

Claim 28.1 *F_1 and F_2 are joined by exactly one edge.*

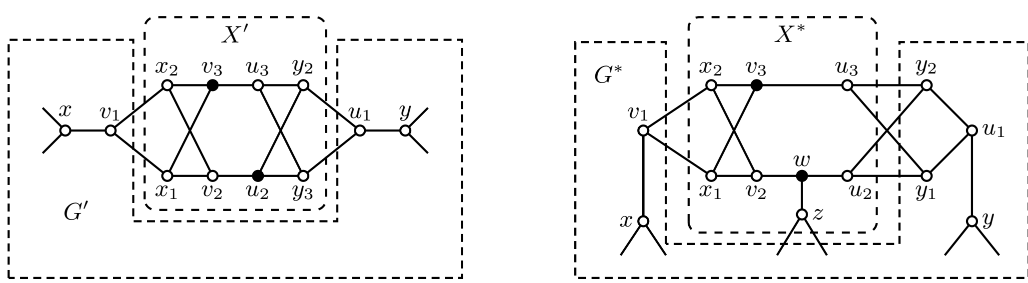
Proof. If F_1 and F_2 are joined by three edges, then $G = G_{10.2}$, where $G_{10.2}$ is the graph illustrated in Figure 46(a). In this case, $i(G) = 3$ (an i-set is indicated by the shaded vertices in Figure 46(a)) and $\mathrm{w}(G) = 30$, and so $8i(G) < \mathrm{w}(G)$, a contradiction.



(a) $G_{10.2}$ (b)

Figure 46: Subgraphs of G in the proof of Claim 28

Suppose next that F_1 and F_2 are joined by exactly two edges. Renaming edges if necessary, we may assume that u_2v_2 and u_3v_3 are edges of G. Let x be the neighbor of v_1 not in X, and let y be the neighbor of u_1 not in Y. Possibly, $x = y$. Let $X' = (V(F_1) \cup V(F_2)) \setminus \{u_1, v_1\}$ and let $G' = G - X'$. Thus the graph in Figure 47(a) is a subgraph of G, where in this illustration $x \neq y$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{u_2, v_3\}$ (indicated by the shaded vertices in Figure 47(a)), and so $i(G) \le i(G') + 2$. Since every vertex of G' has degree 3, except for the vertices u_1 and v_1 both of which have degree 1, we infer that $b(G') = 0$, and so, by Claim 18, we have $\Theta(G') = 0$. Hence, $\Omega(G') = \mathrm{w}(G') \le \mathrm{w}(G) - \mathrm{w}_G(X') + 4 = \Omega(G) - 24 + 4 = \Omega(G) - 20$. Thus, $8i(G) \le 8(i(G') + 2) \le \Omega(G') + 16 < \Omega(G)$, a contradiction. $(\Box)$



(a) (b)

Figure 47: Subgraphs of G in the proof of Claim 28

By Claim 28.1, F_1 and F_2 are joined by exactly one edge. Renaming vertices if necessary, we may assume that u_3v_3 is the edge joining F_1 and F_2.

Claim 28.2 *Neither v_1 nor v_2 has a common neighbor with u_1 or u_2.*

Proof. Suppose that v_1 or v_2 has a common neighbor with u_1 or u_2. By symmetry, we may assume that u_2 and v_2 have a common neighbor, say w. By Claim 25, the vertex w is adjacent to neither u_1 nor v_1. Let z be the neighbor of w different from u_2 and v_2. Let x be the neighbor of v_1 not in X, and let y be the neighbor of u_1 not in Y. Possibly, $x = y$. By Claim 25, the set $\{u_3, x, w\}$ is an independent set and the $\{v_3, y, w\}$ is an independent set. Therefore since w is adjacent to z, we note that $x \neq z$ and $y \neq z$, although possibly $x = y$.

We now let $X^* = (V(F_1) \setminus \{v_1\}) \cup \{u_2, u_3, w, z\}$ and consider the graph $G^* = G - X^*$. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertices v_3 and w (indicated by the shaded vertices in Figure 47(b)), and so $i(G) \leq i(G^*) + 2$. Let z_1 and z_2 be the two neighbors of z different from w. Every vertex of G^* has degree 3 except for three vertices of degree 1 (namely, v_1, y_1 and y_2) and two vertices of degree 2 (namely, z_1 and z_2). Since y_1 and y_2 have a common neighbor in G^* (namely u_1), we note that the only possible component of G^* that belongs to the family $\mathcal{B}$ is a B_1-component containing v_1, z_1 and z_2. However this would imply that z_1 and z_2 are vertices of degree 2 in a copy of $K_{2,3}$ that have a common neighbor, namely z, that does not belong to this copy of $K_{2,3}$, which contradicts Claim 25. Hence, $b(G^*) = 0$, and so, by Claim 18, we have $\Theta(G^*) = 0$. Hence, $\Omega(G^*) = \mathrm{w}(G^*) \leq \mathrm{w}(G) - \mathrm{w}_G(X^*) + 8 = \Omega(G) - 24 + 8 = \Omega(G) - 16$. Thus, $8i(G) \leq 8(i(G^*) + 2) \leq \Omega(G^*) + 16 \leq \Omega(G)$, a contradiction. (□)

We now return to the proof of Claim 28. By Claim 28.2, neither v_1 nor v_2 has a common neighbor with u_1 or u_2. As before, let x be the neighbor of v_1 not in X, and let y be the neighbor of u_1 not in Y. Let w be the neighbor of v_2 not in X, and let z be the neighbor of u_2 not in Y. By our earlier observations, the vertices x, y, w, z are distinct and do not belong to $V(F_1) \cup V(F_2)$. By Claim 25, $xw \notin E(G)$ and $yz \notin E(G)$.

We now let $X^* = \{v_2, v_3, u_3, x_1, x_2, y_1, y_2, w\}$ and we consider the graph $G^* = G - X^*$. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertices v_2 and u_3 (indicated by the shaded vertices in Figure 48), and so $i(G) \leq i(G^*) + 2$. Let w_1 and w_2 be the two neighbors of w different from v_2. Every vertex of G^* has degree 3 except for three vertices of degree 1 (namely, v_1, u_1 and u_2) and two vertices of degree 2 (namely, w_1 and w_2). Analogous arguments as before show that there is no B_1-component containing the vertices w_1 and w_2, for otherwise such a component (of order 6) contains w_1 and w_2 as vertices of degree 2 in a copy of $K_{2,3}$ that have a common neighbor, namely w, that does not belong to this copy of $K_{2,3}$, which contradicts Claim 25. We therefore infer that $b(G^*) = 0$, and so, by Claim 18, we have $\Theta(G^*) = 0$. Hence, $\Omega(G^*) = \mathrm{w}(G^*) \leq \mathrm{w}(G) - \mathrm{w}_G(X^*) + 8 = \Omega(G) - 24 + 8 = \Omega(G) - 16$. Thus, $8i(G) \leq 8(i(G^*) + 2) \leq \Omega(G^*) + 16 = \Omega(G)$, a contradiction. This completes the proof of Claim 28. (□)

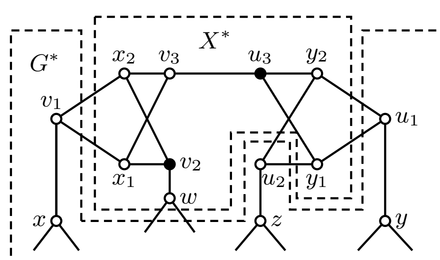
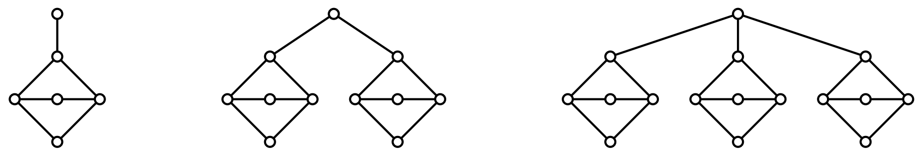


Figure 48: A subgraph of G in the proof of Claim 28

By Claim 28, if the graph G contains two vertex disjoint copies of $K_{2,3}$, then there is no edge joining these two copies of $K_{2,3}$. This yields the following property of the structural weight for an arbitrary induced subgraph G' of G.

Claim 29 *If G' is an arbitrary induced subgraph of G and if B is a component of G' in the family $\mathcal{B}$, then $B \in \mathcal{B}_i$ and B contains exactly i copies of $K_{2,3}$ for some $i \in [3]$ as illustrated in Figure 49.*

By Claim 29, if G' is an arbitrary induced subgraph of G and if B is a component of G' that belongs to $\mathcal{B}$, then either $B \in \mathcal{B}_1$ and B contains one vertex of degree 1 and two vertices of degree 2 (see Figure 49(a)) or $B \in \mathcal{B}_2$



(a) $B \in \mathcal{B}_1$ (b) $B \in \mathcal{B}_2$ (c) $B \in \mathcal{B}_3$

Figure 49: The three possible bad components of G' in Claim 29

and B contains no vertex of degree 1 and five vertices of degree 2 (see Figure 49(b)) or $B \in \mathcal{B}_3$ and B contains no vertex of degree 1 and six vertices of degree 2 (see Figure 49(c)).

By Claim 27, the graph G contains no triangle. Let F be a $K_{2,3}$-subgraph in G with partite sets $A = \{x_1, x_2\}$ and $B = \{v_1, v_2, v_3\}$. Since G is triangle-free, the set B is an independent set in G. Let u_i be the neighbor of v_i different from x_1 and x_2 for $i \in [3]$. By Claim 25, the vertices u_1, u_2 and u_3 are distinct and the set $C = \{u_1, u_2, u_3\}$ is an independent set. We now let $X'' = A \cup B \cup C$ and consider the graph $G'' = G - X''$. Thus the graph illustrated in Figure 50 is an induced subgraph of G.

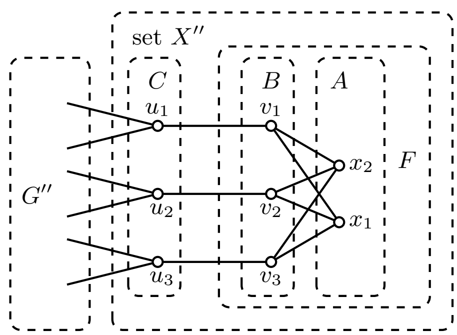


Figure 50: A subgraph of G

Claim 30 *The graph G'' is isolate-free.*

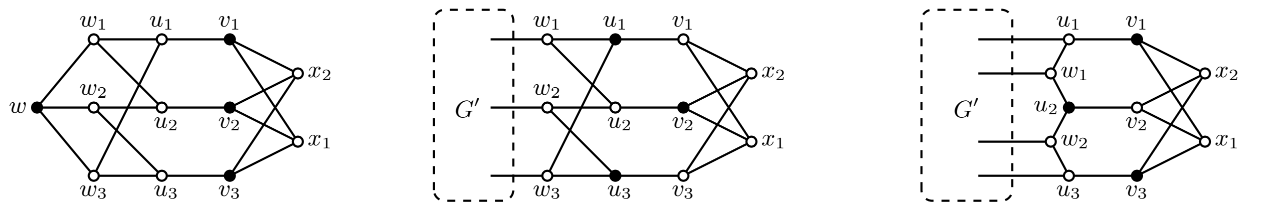
Proof. Suppose, to the contrary, that G'' contains at least one isolated vertex. If G'' contains two isolated vertices, then the graph G is determined and is the graph $G_{10.2}$ shown in Figure 46(a). As noted earlier, in this case $8i(G) < \mathrm{w}(G)$, a contradiction. Hence, G'' contains exactly one isolated vertex, say w. Let $X' = X'' \cup \{w\}$ and let $G' = G - X'$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{w, x_1, x_2\}$ (as indicated by the shaded vertices in Figure 46(b)), and so $i(G) \le i(G') + 3$. Since G' has either one vertex of degree 1 and one vertex of degree 2 or three vertices of degree 2, we infer that $b(G') = 0$, and so by Claim 18, we have $\Theta(G') = 0$. Hence, $\Omega(G') = \mathrm{w}(G') \le \mathrm{w}(G) - \mathrm{w}_G(X') + 3 = \Omega(G) - 27 + 3 = \Omega(G) - 24$. Thus, $8i(G) \le 8(i(G') + 3) \le \Omega(G') + 24 = \Omega(G)$, a contradiction. (□)

By Claim 30, the graph G'' is isolate-free.

Claim 31 *The graph G'' contains at most one vertex of degree* 1.

Proof. Suppose, to the contrary, that G'' contains at least two vertices of degree 1. Let D be the set of vertices of degree 1 in G''. We note that $|D| \le 3$. Suppose firstly that $|D| = 3$. Let $D = \{w_1, w_2, w_3\}$. Let w_i be the common neighbor of u_i and u_{i+1} for $i \in [3]$ where addition is taken modulo 3. If w_1, w_2, w_3 have a common neighbor, say w, then the graph G is determined and is the graph G_{12} shown in Figure 51(a). In this case, $i(G) = 4$ (an i-set is indicated by the shaded vertices in Figure 51(a)) and $\mathrm{w}(G) = 36$, and so $8i(G) < \mathrm{w}(G)$, a contradiction. Hence, w_1, w_2, w_3 do not have a common neighbor.

We now let $X' = X'' \cup D$ and consider the graph $G' = G'' - D = G - X'$. Since w_1, w_2, w_3 have no common neighbor, the graph G' is an isolate-free graph, and so $\mathrm{w}(G') = (\mathrm{w}(G) - 33) + 3 = \mathrm{w}(G) - 30$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{u_1, u_3, v_2\}$ (as indicated by the shaded vertices in Figure 51(b)), and so $i(G) \le i(G') + 3$. Since G' has either one vertex of degree 1 and one vertex of degree 2 or three vertices of degree 2, we infer by Claim 29 that $b(G') = 0$. Thus, by Claim 18, we have $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 30 = \Omega(G) - 30$. Thus, $8i(G) \le 8(i(G') + 3) \le \Omega(G') + 24 < \Omega(G)$, a contradiction.



(a) G_{12} (b) (c)

Figure 51: Subgraphs of G in the proof of Claim 31

Hence, $|D| = 2$, and so G'' contains two vertices of degree 1, say w_1 and w_2. By symmetry, we have two possibilities. First, let w_i be a common neighbor of u_i and u_{i+1} for $i \in [2]$. Thus the graph shown in Figure 51(c) is a subgraph of G. We now let $X' = X'' \cup \{w_1, w_2\}$ and consider the graph $G' = G'' - D = G - X'$. Since the graph G is triangle-free, the graph G' is isolate-free, and so $\mathrm{w}(G') = (\mathrm{w}(G) - 30) + 4 = \mathrm{w}(G) - 26$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{u_2, v_1, v_3\}$ (as indicated by the shaded vertices in Figure 51(c)), and so $i(G) \le i(G') + 3$. Since G' has either two vertices of degree 1 and no vertex of degree 2 or one vertex of degree 1 and two vertices of degree 2 or four vertices of degree 2, we infer that $b(G') \le 1$, and so by Claim 18, we have $\Theta(G') \le 2$. Hence, $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le (\mathrm{w}(G) - 26) + 2 = \mathrm{w}(G) - 24$. Thus, $8i(G) \le 8(i(G') + 3) \le \mathrm{w}(G') + 24 = \mathrm{w}(G)$, a contradiction.

Second, let u_1 and u_2 have two common neighbors w_1 and w_2. Suppose that u_3, w_1 and w_2 have a common neighbor z. Let $X^* = X'' \cup \{w_1, w_2, z\}$. Let $G^* = G - X^*$. By using Claims 10 and 18, we have $\Omega(G^*) = \mathrm{w}(G^*) = (\mathrm{w}(G) - 33) + 1 = \Omega(G) + 32$, since there is one X^*-exit edge. Every i-set of G^* can be extended to an ID-set of G by adding to it the vertices in the set $\{z, v_1, v_2, v_3\}$. Hence, $8i(G) \le 8(i(G^*) + 4) \le \Omega(G^*) + 32 = \Omega(G)$, a contradiction. Thus, u_3, w_1 and w_2 do not have a common neighbor. In this case, let $X' = X'' \cup \{w_1, w_2\}$ and consider the graph $G' = G - X'$. Since u_3, w_1 and w_2 do not have a common neighbor, the graph G' is isolate-free, and so $\mathrm{w}(G') = (\mathrm{w}(G) - 30) + 4 = \mathrm{w}(G) - 26$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices in the set $\{u_2, v_1, v_3\}$ (as indicated by the shaded vertices in Figure 51(c)), and so $i(G) \le i(G') + 3$. Since G' has either two vertices of degree 1 and no vertex of degree 2 or one vertex of degree 1 and two vertices of degree 2 or four vertices of degree 2, we infer that $b(G') \le 1$, and so by Claim 18, we have $\Theta(G') \le 2$. Hence, $\Omega(G') = \mathrm{w}(G') + \Theta(G') \le (\mathrm{w}(G) - 26) + 2 = \mathrm{w}(G) - 24$. Thus, $8i(G) \le 8(i(G') + 3) \le \mathrm{w}(G') + 24 = \mathrm{w}(G)$, a contradiction. (□)

By Claim 31, the graph G'' contains at most one vertex of degree 1. We now define

$$G_{i,j} = G - (N_G[u_i] \cup N_G[v_j])$$

where $i, j \in [3]$ and $i \neq j$.

Claim 32 *At least one of the subgraphs $G_{i,j}$ where $i, j \in [3]$ and $i \neq j$ contains no isolated vertex.*

Proof. Suppose firstly that G'' contains a vertex of degree 1, say w_1. By symmetry, we may assume that w_1 is a common neighbor of u_1 and u_2. Let w be the neighbor of u_1 different from v_1 and w_1. Thus the graph shown in Figure 52(a) is a subgraph of G, where the graph $G_{1,2}$ is the subgraph in the dashed box. If $G_{1,2}$ contains an

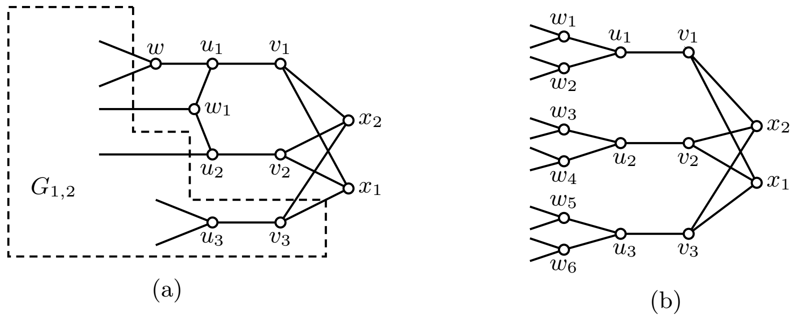


Figure 52: Subgraphs of G in the proof of Claim 32

isolated vertex v, then $N_G(v) = \{w, w_1, u_2\}$. However, then, $G[\{v, u_2, w_1\}] = K_3$, contradicting Claim 27. Hence, $G_{1,2}$ contains no isolated vertex, and desired result of the claim follows.

Suppose next that G'' contains no vertex of degree 1. Thus, every vertex in G'' is adjacent to at most one vertex that belongs to the set $C = \{u_1, u_2, u_3\}$, implying that $N_G[u_i] \cap N_G[u_j] = \emptyset$ for $i, j \in [3]$ and $i \neq j$. Hence the graph shown in Figure 52(b) is a subgraph of G. In particular, we note that the vertices $w_1, w_2, \ldots, w_6$ are distinct. We show that at least one of the subgraphs $G_{i,j}$ where $i, j \in [3]$ and $i \neq j$ contains no isolated vertex. Suppose, to the contrary, that $G_{i,j}$ contains an isolated vertex for all $i, j \in [3]$ and $i \neq j$.

We consider the graph $G_{1,2}$ (illustrated by the subgraph inside the dashed region in Figure 53). By supposition, $G_{1,2}$ contains an isolated vertex. The only possible such isolated vertex is a neighbor of u_2 that is adjacent to both w_1 and w_2. Renaming w_3 and w_4 if necessary, we may assume that w_3 is isolated in $G_{1,2}$, that is, $N_G(w_3) = \{u_2, w_1, w_2\}$. We next consider the graph $G_{1,3}$. By supposition, $G_{1,3}$ contains an isolated vertex. The only possible such isolated vertex is a neighbor of u_3 that is adjacent to both w_1 and w_2. Renaming w_5 and w_6 if necessary, we may assume that w_5 is isolated in $G_{1,3}$, that is, $N_G(w_5) = \{u_3, w_1, w_2\}$. In particular, we note that $C\colon w_1w_3w_2w_5w_1$ is a 4-cycle in G. However, this implies that the graph $G_{2,1}$ does not contain an isolated vertex. Indeed, in $G_{2,1}$, vertices w_1, w_2 and u_3 have degree 1, the neighbors of w_4 in $G_{2,1}$ have degree 2 in $G_{2,1}$, whereas other vertices of $G_{2,1}$ have degree 3. This is a contradiction, proving the claim. (□)

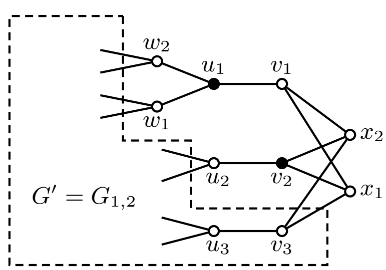


Figure 53: A subgraph of G in the proof of Claim 32

By Claim 32, at least one of the subgraphs $G_{i,j}$ where $i, j \in [3]$ and $i \neq j$ contains no isolated vertex. Renaming vertices if necessary, we may assume that $G_{1,2}$ is isolate-free. For notational convenience, let $G' = G_{1,2}$ and let $X_{1,2} = \{u_1, u_2, v_1, v_2, x_1, x_2, w_1, w_2\}$, and so $G' = G - X_{1,2}$. Every i-set of G' can be extended to an ID-set of G by adding to it the vertices u_1 and v_2 (as indicated by the shaded vertices in Figure 53), and so $i(G) \le i(G') + 2$. If $b(G') = 0$, then by Claim 18, $\Theta(G') = 0$, and so $\Omega(G') = \mathrm{w}(G') = \mathrm{w}(G) - 24 + 8 = \mathrm{w}(G) - 16 = \Omega(G) - 16$. In this case, $8i(G) \le 8(i(G') + 2) \le \mathrm{w}(G') + 16 = \Omega(G)$, a contradiction. Hence, $b(G') \ge 1$. Let B' be a bad component in G', and so $B' \in \mathcal{B}$. We note that $\deg_{G'}(v_3) = 1$, and so G' contains at least one vertex of degree 1.

Claim 33 $B' \notin \mathcal{B}_3$.

Proof. Suppose, to the contrary, that $B' \in \mathcal{B}_3$, and so B' is as illustrated in Figure 49(c). In this case, the six $X_{1,2}$-exit edges from w_1, w_2 and u_2 are all incident with vertices of degree 2 in B'. Let v denote the vertex of degree 3 in B' that does not belong to a copy of $K_{2,3}$. We now let $X^* = V(B') \cup X_{1,2}$ and we let $G^* = G - X^*$. We note that G^* contains one vertex of degree 1, namely v_3, and all other vertices of G^* have degree 3. Hence, $b(G^*) = 0$, and so, by Claim 18, $\Theta(G^*) = 0$. Thus, $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - 24 \times 3 + 2 = \mathrm{w}(G) - 70$. Every i-set of G^* can be extended to an ID-set of G by adding to it the seven vertices in the set $N(v) \cup \{u_2, v_1, w_1, w_2\}$ (as indicated by the shaded vertices in Figure 54), and so $i(G) \le i(G^*) + 7$. Thus, $8i(G) \le 8(i(G^*) + 7) \le \Omega(G^*) + 56 < \Omega(G)$, a contradiction. (□)

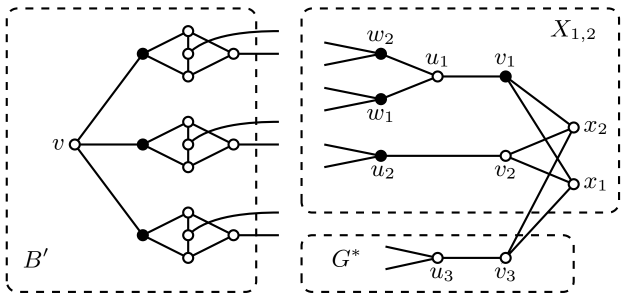


Figure 54: A subgraph of G in the proof of Claim 33

Claim 34 $B' \notin \mathcal{B}_2$.

Proof. Suppose, to the contrary, that $B' \in \mathcal{B}_2$, and so B' is as illustrated in Figure 49(b). In this case, five $X_{1,2}$-exit edges from w_1, w_2 and u_2 are all incident with vertices of degree 2 in B'. Let v denote the vertex of degree 2 in B' that does not belong to a copy of $K_{2,3}$. We now let $X^* = V(B') \cup X_{1,2}$ and we let $G^* = G - X^*$. We note that G^* contains one vertex of degree 1, one vertex of degree 2, and all other vertices of G^* have degree 3. Hence, $b(G^*) = 0$, and so, by Claim 18, we have $\Theta(G^*) = 0$. Thus, $\Omega(G^*) = \mathrm{w}(G^*) = \mathrm{w}(G) - 19 \times 3 + 3 = \mathrm{w}(G) - 54 = \Omega(G) - 54$. We show that every i-set of G^* can be extended to an ID-set of G by adding to it at most six vertices. We will use the property given in Claim 25 that every $K_{2,3}$-subgraph of G belongs to a $G_{8.2}$-configuration that is an induced subgraph in G, where $G_{8.2}$ is the graph in Figure 39(b).

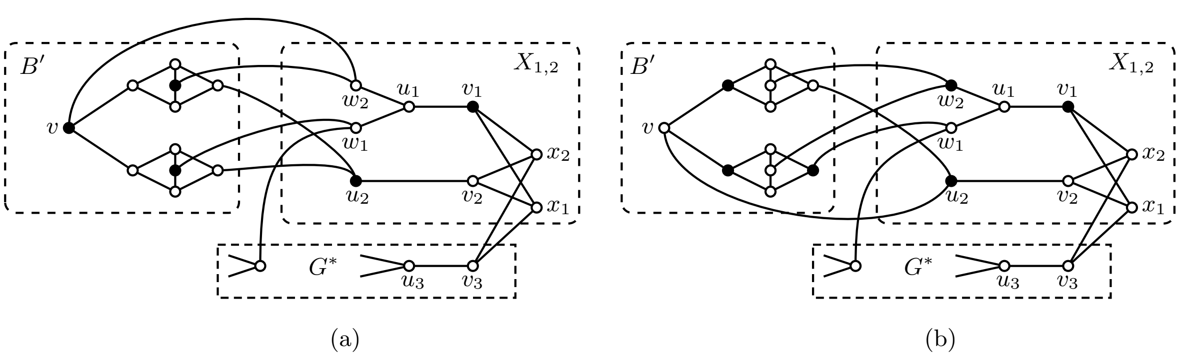


Figure 55: Subgraphs of G in the proof of Claim 34

Suppose firstly that u_2 is adjacent to two vertices in B'. If u_2 is not adjacent to v, then we let I' consist of the three vertices of degree 2 in B' that are not adjacent to u_2, together with the vertices v_1 and u_2 (as indicated by the shaded vertices in Figure 55(a)). If u_2 is adjacent to v, then renaming w_1 and w_2 if necessary, we may assume that w_2 is adjacent to two vertices in B'. In this case, we let I' consist of the two neighbors of v in B', the vertex of degree 2 in B' adjacent to w_1, together with the vertices in the set $\{u_2, v_1, w_2\}$ (as indicated by the shaded vertices in Figure 55(b)). In both cases, every i-set of G^* can be extended to an ID-set of G by adding to the vertices in the set I', and so $i(G) \le i(G^*) + |I'| \le i(G^*) + 6$.

Suppose next that u_2 is adjacent to exactly one vertex in B'. Suppose that u_2 is not adjacent to v. Renaming w_1 and w_2 if necessary, we may assume that w_2 is adjacent to v. We now let I' consist of the vertex v, the vertices of degree 2 in B' adjacent to w_2 and u_2, and the vertices in the set $\{v_1, v_2, w_1\}$ (as indicated by the shaded vertices in Figure 56(a)). If u_2 is adjacent to v, then we let I' consist of the two neighbors of v in B' together with the vertices in the set $\{v_1, v_2, w_1, w_2\}$ (as indicated by the shaded vertices in Figure 56(b)). In both cases, every i-set of G^* can be extended to an ID-set of G by adding to the vertices in the set I', and so $i(G) \le i(G^*) + |I'| = i(G^*) + 6$.

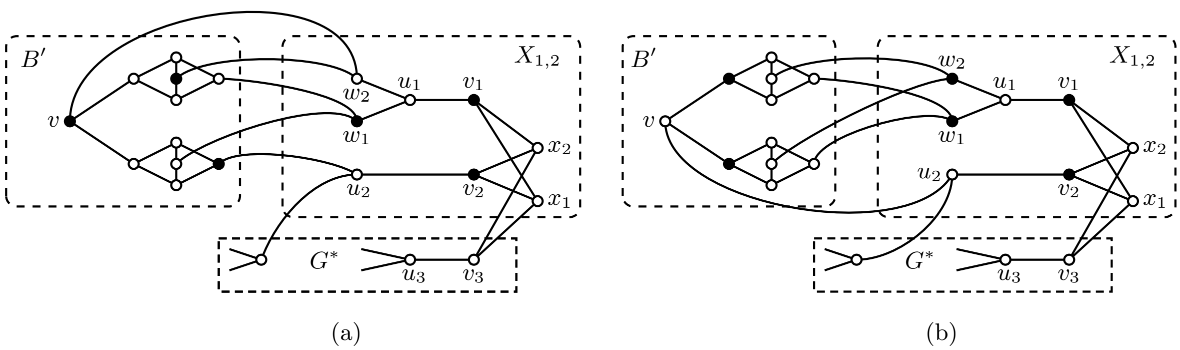


Figure 56: Subgraphs of G in the proof of Claim 34

As observed earlier, $\Omega(G^*) = \Omega(G) - 54$. We have shown that every i-set of G^* can be extended to an ID-set of G by adding to it at most six vertices from X^*, and so $i(G) \le i(G^*) + 6$. Thus, $8i(G) \le 8(i(G^*) + 6) \le \Omega(G^*) + 48 < \Omega(G)$, a contradiction. $_{(\Box)}$

By Claims 33 and 34, we have $B' \in \mathcal{B}_1$. Suppose that $v_3 \in V(B')$. Since v_3 has degree 1 in B', by Claim 29 such a component $B' \in \mathcal{B}_1$ is as illustrated in Figure 49(a). In this case, the graph G contains two vertex disjoint copies of $K_{2,3}$ joined by the edge u_3v_3, contradicting Claim 28. Hence, $v_3 \notin V(B')$.

Let y_1 and y_2 be the two vertices of degree 3 in B' that are not adjacent, and let $N(y_1) = N(y_2) = \{a, b, c\}$. Further, let a' be the vertex of degree 1 in B', where aa' is an edge. Let b' and c' be the neighbors of b and c, respectively, that do not belong to B'. By Claim 25, the subgraph of G induced by the set $V(B') \cup \{b', c'\}$ is a $G_{8.2}$-configuration. In particular, we note that the set $\{a', b', c'\}$ is an independent set. Since $v_3 \notin V(B')$, we note that $a' \neq v_3$. The two neighbors of a' that are not in B' belong to the set $\{u_2, w_1, w_2\}$. Further, the vertices b' and c' belong to the set $\{u_2, w_1, w_2\}$, that is, $\{b', c'\} \subset \{u_2, w_1, w_2\}$. Since the vertex a' is adjacent to two vertices that belong to the set $\{u_2, w_1, w_2\}$, the vertex a' is adjacent to at least one of b' and c', contradicting our earlier observation that $\{a', b', c'\}$ is an independent set. This final contradiction completes our proof of Theorem 6. $\Box$

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