Single-shot antidistinguishability of unitary operations

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The notion of antidistinguishability captures the possibility of ruling out certain alternatives in a quantum experiment without identifying the actual outcome. Although extensively studied for quantum states, the antidistinguishability of quantum channels remains largely unexplored. In this work, we investigate the single-shot antidistinguishability of unitary operations. We analyse two scenarios: antidistinguishability with single-system probes and with entangled probes. For sets of three unitaries, we first prove that all maximally entangled states are equivalent in their performance as probe. In the qubit case, we further establish that maximally entangled probes are always sufficient: if a set of three qubit unitaries is antidistinguishable with either a single-system or non-maximally entangled probe, then it is also antidistinguishable with a maximally entangled one. However, in higher dimension, this equivalence fails. In *dimension 3*, there exists a set of unitaries that are antidistinguishable with non-maximally entangled probe or single-system probe but not with maximally entangled probe. We also establish that union of two antidistinguishable sets of three qubit unitaries also forms a set of antidistinguishable unitaries. Lastly, we provide methods to construct antidistinguishable unitaries from non-antidistinguishable ones.

I. INTRODUCTION

The concept of distinguishability of physical processes plays a crucial role in understanding quantum theory and, by extension, the physical world. Alongside distinguishability, there exists a weaker notion known as antidistinguishability. While distinguishability allows one to identify which process occurred from a set of known processes based on an outcome, antidistinguishability enables the negative identification of certain processes. In addition to distinguishability [1–5], the notion of antidistinguisahbility has significant implications for discussions on the reality of quantum states [6, 7]. Beyond foundational insights, distinguishability and antidistinguishability find broad applications in quantum information and communication [8–10].

Extensive research on these tasks has focused primarily on quantum states, though the antidistinguishability of quantum states [11–18] has received comparatively less attention than distinguishability [19–29]. Despite being fewer in number, there is a considerable body of work addressing the distinguishability of quantum channels [30–43]. However, the study of antidistinguishability for quantum channels remains nearly unexplored. This disparity arises from the inherent complexity of the problem: antidistinguishability requires consideration of at least three different processes, as the notions of distinguishability and antidistinguishability coincide in the context of two processes. One of the earliest treatments of antidistinguishability for channels is presented in [41], where the channels under consideration are quantum measurements. In a similar vein, this work focusses on the antidistinguishability of unitary operations, since dealing with general quantum channels is often highly intricate.

Unitary operators represent one of the most fundamental classes of quantum channels and have a significant importance in quantum communication [44, 45], information processing [46], error correction [47], cryptography [48], and computation[49]. While there exists a body of research on the discrimination of unitary operators [34, 40] in various contexts, the problem of antidiscrimination remains unexplored. The present study aims to provide a foundational perspective on the antidistinguishability of unitaries, analogous to the well-studied discrimination problem.

This work formulates the problem of antidistinguishing a priori known sets of unitary operations sampled from an equal probability distribution. Before addressing the main problem, a brief overview of the antidistinguishability of quantum states is provided. The notion of antidistinguishability for unitary operations is then defined within the single-shot scenario, demonstrating that the antidistinguishability of unitaries ultimately reduces to the antidistinguishability of the corresponding evolved states for an optimally chosen initial probe state. Based on the nature of the initial probe, scenarios are categorized into two types: antidistinguishability with a single-system probe and antidistinguishability with an entangled-system probe. The study primarily is directed on evaluating the antidistinguishability of three unitaries. Subsequently, it is shown that all maximally entangled states perform equivalently in the antidistinguishability task for any three unitaries. The analysis then progresses to ranking different probes in this context. The results reveal that for any three qubit unitaries, if they are antidistinguishable using a non-maximally entangled probe or a single-system probe, they are also antidistinguishable using a maximally entangled state. This implies that the maximally entangled state serves as the optimal probe for antidistinguishing three qubit unitaries. However, this property does not hold when the unitary dimension increases to three. In higher dimensions, there exists a set of three unitaries that are antidistinguishable using single-system or non-maximally entangled probes but not with maximally entangled probes. After that, we move into the results regarding the features of a set of antidistinguishable unitaries. We showed that the union of two antidistinguishable sets of three qubit unitaries also forms a set of antidistinguishable unitaries. Additionally, some methods are proposed for constructing antidistinguishable sets from those that are not antidistinguishable. These findings draw direct parallels to previously established results on the antidistinguishability of quantum states.

The structure of the paper is as follows: the subsequent section provides an overview of the antidistinguishability of quantum states. Following this, the single-shot antidistinguishing task for unitary operations is formulated. Section IV presents the main results. The conclusion summarizes the key findings and discusses open problems and possible directions for future research.

II. ANTIDISTINGUISHABILITY OF QUANTUM STATES

Assume, we are given n a priori known quantum states $\{\rho_k\}_{k=1}^n$ sampled from the probability distribution $\{q_k\}_{k=1}^n$. Antidistinguishability of n quantum states $\{\rho_k\}_{k=1}^n$ is a linear function [41] which is defined as,

$$\mathcal{A}[\{\rho_k\}_k, \{q_k\}_k] = \max_{\{M\}} \left\{ \sum_{k,a} q_k p(a \neq k|\rho_k, M) \right\}. \quad (1)$$

As $\sum_a (p(a \neq k|\rho_k, M) + p(a = k|\rho_k, M)) = 1$ and $\sum_k q_k = 1$, the above expression becomes,

$$\mathcal{A}[\{\rho_k\}_k, \{q_k\}_k] = 1 - \min_{\{M\}} \left\{ \sum_k q_k \operatorname{Tr}(\rho_k M_k) \right\}. \quad (2)$$

The necessary and sufficient conditions [11] for perfect antidistinguishability of three pairwise nonorthogonal pure quantum states, i.e.,

$$\mathcal{A}[\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}] = 1, \quad (3)$$

are as follows:

$$x_1 + x_2 + x_3 < 1 \quad (4a)$$

$$(x_1 + x_2 + x_3 - 1)^2 \geqslant 4x_1x_2x_3, \quad (4b)$$

where $x_1 = |\langle\psi_1|\psi_2\rangle|^2, x_2 = |\langle\psi_1|\psi_3\rangle|^2, x_3 = |\langle\psi_2|\psi_3\rangle|^2$.

If we increase the number of the states in the set, ref. [13] gives some necessary and sufficient (not simultaneously) conditions for the antidistinguishability and non-antidistinguishability of the given set depending on the modulus of pairwise inner products of the states.

III. ANTIDISTINGUISHABILITY OF UNITARY OPERATIONS

Unitary operator (U) is a linear operator $U : H \to H$ on a Hilbert space H that satisfies $U^\dagger U = UU^\dagger = \mathbb{1}$.

We consider a priori known set of r unitaries $\{U_x\}_x$ acting on a d-dimensional quantum state and $x \in \{1, \cdots, r\}$. We assume that all the unitaries are sampled from a probability distribution $\{p_x\}_x$, i.e., $p_x > 0$ and $\sum_x p_x = 1$. In this paper, we consider the unitaries which are not distinguishable as distinguishable unitaries are trivially antidistinguishable. To antidistinguish the unitaries, the unitary device is fed with a known quantum state, single or entangled and the device carries out one of r unitary operations. After this process, the device gives an evolved state as the output. Therefore, one can perform any measurement on the evolved state and this measurement can be optimized such that the antidistinguishability of these evolved states will be maximum. Let us describe this optimal measurement by a set of POVM elements $\{N_b\}_b$, where $b \in \{1, \cdots, r\}$ and $\sum_b N_b = \mathbb{1}$. The protocol is successful in antidistinguishing the unitaries if b is not equal to x. Any classical post-processing of outcome b can be included in the measurement $\{N_b\}_b$. Depending on the initial probing state, we can formulate two situations which are described in detail in the next subsection.

A. With single system probe

At first, the unitary device is given a single quantum state ρ. After the device applying any of the unitaries from the set $\{U_x\}_{x=1}^r$, the output state will be $U_x \rho U_x^\dagger$. After performing the measurement $\{N_b\}_{b=1}^r$, the antidistinguishability of the set of unitaries becomes the antidistinguishability of the set of evolved states. So antidistinguishability in this scenario, denoted by $\mathcal{A}_S$, is defined as,

$$\begin{aligned}
\mathcal{A}_S\left[\{U_x\}_x, \{p_x\}_x\right] &= \max_\rho \sum_x p_x p(b \neq x|x) \\
&= 1 - \max_\rho \sum_x p_x p(b = x|x) \\
&= 1 - \min_{\rho, N_b} \sum_x p_x \operatorname{Tr}\left(U_x \rho U_x^\dagger N_{b=x}\right) \\
&= \max_\rho \mathcal{A}\left[\left\{U_x \rho U_x^\dagger\right\}_x, \{p_x\}_x\right].
\end{aligned}$$

$\mathcal{A}\left[\{\rho_k\}_k, \{q_k\}_k\right]$ denotes the antidistinguishability of the set of states $\{\rho_k\}_k$, sampled from the probability distribution $\{q_k\}_k$. As of now, we are interested to antidistinguish three unitary operations U_1, U_2, U_3. If a pure state $\rho = |\psi\rangle\langle\psi|$ acts as the probe, we need to compute the square of inner product terms $\left|\langle\psi|U_i^\dagger U_j|\psi\rangle\right|^2$, where $i, j \in \{1, 2, 3\}$ and $i \neq j$, to find the antidistinguishability of three unitaries by employing (4). As $U_i^\dagger U_j$ is an unitary, we can write the spectral decomposition of this operator as, $U_i^\dagger U_j = \sum_{l=1}^{d} e^{\mathrm{i}\theta_{l_{ij}}} |\psi_{l_{ij}}\rangle\langle\psi_{l_{ij}}|$, where $e^{\mathrm{i}\theta_{l_{ij}}}$ are the eigenvalues of $U_i^\dagger U_j$ and $|\psi_{l_{ij}}\rangle$ is the eigenvector corresponding to l'th eigenvalue. Therefore $|\psi\rangle$ can be decomposed into the linear combination of the eigenstates $|\psi_{l_{ij}}\rangle$, i.e., $|\psi\rangle = \sum_{l=1}^{d} \alpha_{l_{ij}} |\psi_{l_{ij}}\rangle$. So,

$$\begin{aligned} \left|\langle\psi|U_i^\dagger U_j|\psi\rangle\right|^2 &= \left|\sum_{l=1}^{d}] \left|\alpha_{l_{ij}}\right|^2 e^{\mathrm{i}\theta_{l_{ij}}}\right|^2 \\ &= \left|con\{e^{\mathrm{i}\theta_{l_{ij}}}\}\right|^2, \end{aligned} \tag{5}$$

where $con\{e^{\mathrm{i}\theta_{l_{ij}}}\}$ denotes the set of complex numbers that can be written as the convex combinations of $\{e^{\mathrm{i}\theta_{l_{ij}}}\}$.

B. With entangled probe

Now, the unitary device is fed with a $d \otimes d'$ entangled state ρ_{AB} and the device applies one of the unitaries from the set $\{U_x\}_{x=1}^{r}$ on the part A of the entangled state. The evolved state will be $(U_x \otimes \mathbb{1})\rho_{AB}(U_x^\dagger \otimes \mathbb{1})$. Then one can perform the optimal measurement of dimension dd'. Similarly, the antidistinguishability of the unitaries in this scenario, denoted as $\mathcal{A}_E$, can be written as,

$$\begin{aligned} &\mathcal{A}_E\left[\{U_x\}_x, \{p_x\}_x\right] \\ =\ &\max_{\rho_{AB}} \sum_x p_x p(b \neq x|x) \\ =\ &1 - \min_{\rho_{AB}, N_b} \sum_x p_x \mathrm{Tr}\left[(U_x \otimes \mathbb{1})\rho_{AB}(U_x^\dagger \otimes \mathbb{1})N_{b=x}\right] \\ =\ &\max_{\rho_{AB}} \mathcal{A}\left[\left\{(U_x \otimes \mathbb{1})\rho_{AB}(U_x^\dagger \otimes \mathbb{1})\right\}_x, \{p_x\}_x\right]. \end{aligned} \tag{6}$$

This expression suggests that if the input state is product state, this expression will reduce to (5), i.e., the distinguishability with single system scenario. So, in the context of probing state, we use single state and product state alternatively.

Lemma 1. *For antidistinguishability in entanglement assisted scenario, the sufficient initial entangled state is of $d \otimes d$ dimension for d dimensional unitaries.*

Proof. [40] showed that entangled state of $d \otimes d$ dimension is sufficient for the distinguishability of two d-dimensional unitaries. For the antidistinguishability, we eventually deal with $\left|\langle\psi|U_i^\dagger U_j|\psi\rangle\right|^2$, similar to the case of antidistinguishability. So the proof of sufficiency of $d \otimes d$ dimensional entangled probe remains same. □

For more detailed analysis, we need to compute the square of inner product terms $\left|\langle\psi_{AB}|(U_i^\dagger \otimes \mathbb{1})(U_j \otimes \mathbb{1})|\psi_{AB}\rangle\right|^2$, where $i, j \in \{1,2,3\}$ and $|\psi_{AB}\rangle = \sum_{w=1}^{d} C_w |\eta_w\rangle_A |\chi_w\rangle_B$.

$$\begin{aligned} &\left|\langle\psi_{AB}|(U_i^\dagger \otimes \mathbb{1})(U_j \otimes \mathbb{1})|\psi_{AB}\rangle\right|^2 \\ &= \left|\sum_{w=1}^{d} |C_w|^2 \langle\eta_w|U_i^\dagger U_j|\eta_w\rangle\right|^2 \\ &= \left|\sum_{w=1}^{d} |C_w|^2 \sum_{l=1}^{d} \left|\beta_{l_{ij}}^{w}\right|^2 e^{\mathrm{i}\theta_{l_{ij}}}\right|^2 \\ &= \left|\sum_{l=1}^{d}\sum_{w=1}^{d} |C_w|^2 \left|\beta_{l_{ij}}^{w}\right|^2 e^{\mathrm{i}\theta_{l_{ij}}}\right|^2 \\ &= \left|con\{e^{\mathrm{i}\theta_{l_{ij}}}\}\right|^2. \end{aligned} \tag{7}$$

One can check $\sum_{l=1}^{d}\sum_{w=1}^{d} |C_w|^2 |\beta_{l_{ij}}^{w}|^2 = 1$, that means these coefficients indeed make a convex combinations of the eigenvalues $e^{\mathrm{i}\theta_{l_{ij}}}$. Third line comes from the fact that $U_i^\dagger U_j = \sum_{l=1}^{d} e^{\mathrm{i}\theta_{l_{ij}}} |\psi_{l_{ij}}\rangle\langle\psi_{l_{ij}}|$ and $|\eta_w\rangle = \sum_{l=1}^{d} \beta_{l_{ij}}^{w} |\psi_{l_{ij}}\rangle$.

IV. RESULTS

In this section, we present most of the results regarding the antidistinguishability of three unitaries and all the unitaries are sampled from equal probability distribution. Our first theorem reflects the usefulness of maximally entangled state as the probe in any dimension.

Theorem 1. *All the maximally entangled states are equivalent for antidistinguishability of three unitaries.*

Proof. Let us take the maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |kk\rangle$ as the initial input state. We need to compute $\left|\langle\Phi^+|U_i^\dagger U_j|\Phi^+\rangle\right|^2$, where $i, j \in \{1,2,3\}$ and $i \neq j$. So,

$$\begin{aligned} \left|\langle\Phi^+|U_i^\dagger U_j|\Phi^+\rangle\right|^2 &= \frac{1}{d^2}\left|\sum_{k=1}^{d} \langle k|U_i^\dagger U_j|k\rangle\right|^2 \\ &= \frac{1}{d^2}\left|\mathrm{Tr}(U_i^\dagger U_j)\right|^2 \\ &= \frac{1}{d^2}\left|\sum_{l=1}^{d} e^{\mathrm{i}\theta_{l_{ij}}}\right|^2. \end{aligned} \tag{8}$$

As trace is basis independent property, it does not depend on the initial maximally entangled state and that gives our desired result. □

Now we are interested between the hierarchy of entangled probe. For the simplest case, we take any three qubit unitaries and show that the qubit unitaries antidistinguishable by non-maximally entangled probe are always antidistinguishable with maximally entangled probe.

Theorem 2. *If any three qubit unitaries are antidistinguishable with non-maximally entangled probe, they will be antidistinguishable with maximally entangled probe.*

Proof. Suppose three qubit unitaries A_1, A_2, A_3 are antidistinguishable with non-maximally entangled probe $|\tau\rangle$. Let us denote the following quantities:

$$\begin{aligned} g_1 &= \left|\langle\tau|A_1^\dagger A_2|\tau\rangle\right|^2 \\ &= \left|con\{e^{\mathbb{i}\Theta_{12}^1}, e^{\mathbb{i}\Theta_{12}^2}\}\right|^2, \\ g_2 &= \left|\langle\tau|A_2^\dagger A_3|\tau\rangle\right|^2 \\ &= \left|con\{e^{\mathbb{i}\Theta_{23}^1}, e^{\mathbb{i}\Theta_{23}^2}\}\right|^2, \\ g_3 &= \left|\langle\tau|A_3^\dagger A_1|\tau\rangle\right|^2 \\ &= \left|con\{e^{\mathbb{i}\Theta_{31}^1}, e^{\mathbb{i}\Theta_{31}^2}\}\right|^2. \end{aligned} \tag{9}$$

$e^{\mathbb{i}\Theta_{ij}^1}$ and $e^{\mathbb{i}\Theta_{ij}^2}$ are the eigenvalues of $U_i^\dagger U_j$. For non-maximally entangled state, the convex combinations of eigenvalues are of unequal probability weights, whereas maximally entangled probe gives convex combinations of equal weights. Now we will calculate the quantities of (9) for both the non-maximally and maximally entangled probe. For non-maximally entangled probe,

$$\begin{aligned} g_{1_{NM}} &= \left|te^{\mathbb{i}\Theta_{12}^1} + (1-t)e^{\mathbb{i}\Theta_{12}^2}\right|^2 \\ &= t^2 + (1-t)^2 + 2t(1-t)\cos(\Theta_{12}^1 - \Theta_{12}^2) \\ &= \frac{1+\cos(\Theta_{12}^1 - \Theta_{12}^2)}{2} \\ &\quad +2(t-\frac{1}{2})^2(1-\cos(\Theta_{12}^1 - \Theta_{12}^2)). \end{aligned} \tag{10}$$

Similarly, one can calculate $g_{2_{NM}}$ and $g_{3_{NM}}$. Now, for maximally entangled probe,

$$\begin{aligned} g_{1_M} &= \frac{1}{4}\left|e^{\mathbb{i}\Theta_{12}^1} + e^{\mathbb{i}\Theta_{12}^2}\right|^2 \\ &= \frac{1+\cos(\Theta_{12}^1 - \Theta_{12}^2)}{2}. \end{aligned} \tag{11}$$

It is easy to notice that, for any values of Θ_{12}^1 and Θ_{12}^2, $g_{1_M} \leqslant g_{1_{NM}}$ as $(t-\frac{1}{2})^2 \geqslant 0$ and $(1-\cos(\Theta_{12}^1 - \Theta_{12}^2)) \geqslant 0$. $g_{1_M} = g_{1_{NM}}$ when $t = \frac{1}{2}$. Similarly, $g_{2_M} \leqslant g_{2_{NM}}$ and $g_{3_M} \leqslant g_{3_{NM}}$. As per the statement of the theorem, the unitaries are antidistinguishable with non-maximally entangled state. From (4), we can write,

$$\begin{aligned} & g_{1_{NM}} + g_{2_{NM}} + g_{3_{NM}} < 1, \\ & (g_{1_{NM}} + g_{2_{NM}} + g_{3_{NM}} - 1)^2 \geqslant 4g_{1_{NM}}g_{2_{NM}}g_{3_{NM}}. \end{aligned} \tag{12}$$

We know, $g_{1_M} + g_{2_M} + g_{3_M} \leqslant g_{1_{NM}} + g_{2_{NM}} + g_{3_{NM}}$, which is less than 1 from (12). So first condition is satisfied for maximally entangled probe. Now for the second condition, $(g_{1_M} + g_{2_M} + g_{3_M} - 1)^2 \geqslant (g_{1_{NM}} + g_{2_{NM}} + g_{3_{NM}} - 1)^2 \geqslant 4g_{1_{NM}}g_{2_{NM}}g_{3_{NM}} \geqslant 4g_{1_M}g_{2_M}g_{3_M}$. So g_{1_M}, g_{2_M} and g_{3_M} satisfied the necessary and sufficient conditions of antidistinguishability. That achieves our claim. □

But if we increase the dimension of the unitaries to 3, it is very easy to construct three unitaries which are antidistinguishable with non-maximally entangled state but not antidistinguishable with maximally entangled state.

Theorem 3. *Consider the following three unitaries:*

$$\begin{aligned} V_1 &= |\omega_1\rangle\langle 1| + |\omega_2\rangle\langle 2| + |\omega_3\rangle\langle 3| + \cdots + |\omega_d\rangle\langle d|, \\ V_2 &= \left(\frac{1}{2}|\omega_1\rangle + \frac{\sqrt{3}}{2}|\omega_2\rangle\right)\langle 1| + \left(-\frac{\sqrt{3}}{2}|\omega_1\rangle + \frac{1}{2}|\omega_2\rangle\right)\langle 2| + |\omega_3\rangle\langle 3| + \cdots + |\omega_d\rangle\langle d|, \\ V_3 &= \left(\frac{1}{2}|\omega_1\rangle + \frac{\sqrt{3}}{2}|\omega_3\rangle\right)\langle 1| + |\omega_2\rangle\langle 2| + \left(-\frac{\sqrt{3}}{2}|\omega_1\rangle + \frac{1}{2}|\omega_3\rangle\right)\langle 3| + \cdots + |\omega_d\rangle\langle d|. \end{aligned} \tag{13}$$

These unitaries are antidistinguishable with non-maximally entangled state but not antidistinguishable with maximally entangled state for $d \geqslant 3$.

Proof. For non-maximally entangled state, we select the probe $(\sqrt{p}|11\rangle + \sqrt{1-p}|22\rangle)$ with $p \in (\frac{14-2\sqrt{7}}{21}, 1]$, $p = 1$ corresponds to single system. With this probe, the evolved states will be $\{\sqrt{p}|\omega_1\rangle|1\rangle + \sqrt{1-p}|\omega_2\rangle|2\rangle, \quad \sqrt{p}\left(\frac{1}{2}|\omega_1\rangle + \frac{\sqrt{3}}{2}|\omega_2\rangle\right)|1\rangle + \sqrt{1-p}\left(-\frac{\sqrt{3}}{2}|\omega_1\rangle + \frac{1}{2}|\omega_2\rangle\right)|2\rangle,$
$\sqrt{p}\left(\frac{1}{2}|\omega_1\rangle + \frac{\sqrt{3}}{2}|\omega_3\rangle\right)|1\rangle + \sqrt{1-p}|\omega_2\rangle|2\rangle\}$, which are antidistinguishable with the given values of p.

For maximally entangled probe, we have to check the following quantities:

$$\begin{aligned} y_1 &= \frac{1}{d^2}\left|\mathrm{Tr}(V_1^\dagger V_2)\right|^2 = (1-\frac{1}{d})^2, \\ y_2 &= \frac{1}{d^2}\left|\mathrm{Tr}(V_2^\dagger V_3)\right|^2 = (1-\frac{7}{4d})^2, \\ y_3 &= \frac{1}{d^2}\left|\mathrm{Tr}(V_3^\dagger V_1)\right|^2 = (1-\frac{1}{d})^2. \end{aligned} \tag{14}$$

For $d \geqslant 3$, $y_1 + y_2 + y_3 > 1$. From (8) and (4), we can infer V_1, V_2, V_3 are not antidistinguishable with maximally entangled state. □

Now we extend this hierarchy checking between single system and entangled system.

Theorem 4. *If any three qubit unitaries are antidistinguishable with single system, they must be antidistinguishable with maximally entangled probe.*

Proof. If three unitaries B_1, B_2, B_3 are antidistinguishable with single system $|\Delta\rangle$, we can write

$$\begin{aligned} g_{1_s} &= \left|\langle\Delta|B_1^\dagger B_2|\Delta\rangle\right|^2 \\ &= \left|s e^{\mathbb{i}\Omega_{12}^1} + (1-s)e^{\mathbb{i}\Omega_{12}^2}\right|^2, \end{aligned} \tag{15}$$

where $s \in (0,1)$ and $e^{\mathbb{i}\Omega_{12}^1}, e^{\mathbb{i}\Omega_{12}^2}$ are the eigenvalues of $B_1^\dagger B_2$. Maximally entangled probe yields convex combination of the eigenvalues with equal weights.

$$g_{1_M} = \frac{1}{4}\left|e^{\mathbb{i}\Omega_{12}^1} + e^{\mathbb{i}\Omega_{12}^2}\right|^2. \tag{16}$$

We already proved in Theorem 2 that $g_{1_M} \leqslant g_{1_s}$. A similar argument holds for the remaining terms, exactly as in the proof of Theorem 2. As $g_{1_s}, g_{2_s}, g_{3_s}$ satisfy the necessary and sufficient conditions of (4), $g_{1_M}, g_{2_M}, g_{3_M}$ also satisfy the same conditions. This completes the proof. □

From Theorem 2 and Theorem 4, we can conclude that for the antidistinguishability of three qubit unitaries, the maximally entangled state is a sufficient probe. But the increase in the dimension of the unitaries does not imply the same result. For the dimension $\geqslant 3$, we find an example of three unitaries which are antidistinguishable with both the single system and non-maximally entangled system but not antidistinguishable with maximally entangled system.

Theorem 5. *The unitaries described at (13) are antidistinguishable with single system and non-maximally entangled system but not antidistinguishable with maximally entangled system for $d \geqslant 3$.*

Proof. With the single system probe $|1\rangle$, the evolved states after unitaries being operated are $\{|\omega_1\rangle, \frac{1}{2}|\omega_1\rangle + \frac{\sqrt{3}}{2}|\omega_2\rangle, \frac{1}{2}|\omega_1\rangle + \frac{\sqrt{3}}{2}|\omega_3\rangle\}$. These three states are antidistinguishable as these states satisfy the conditions (4).

Theorem 3 already proved that the unitaries of (13) are antidistinguishable with non-maximally entangled state and not antidistinguishable with maximally entangled state. □

At this point, we diverted the next results to the characteristics of a set of antidistinguishable unitaries. Our next theorem depicts the closure property of antidistinguishable sets of three qubit unitaries.

Theorem 6. *If $\mathbb{S}_1$ and $\mathbb{S}_2$ are the two sets, each consists of three antidistinguishable qubit unitaries, the unitaries of the set $\mathbb{S}_1 \bigcup \mathbb{S}_2$ are also antidistinguishable.*

Proof. From Theorems 2 and 4, we can assert that maximally entangled state is the sufficient probe to antidistinguish three qubit unitaries. Theorem 1 tells that any maximally entangled state is equivalent for the task. So that makes possible to find a common probe, i.e., any maximally entangled probe which can antidistinguish both the sets $\mathbb{S}_1$ and $\mathbb{S}_2$. After using this probe, we can denote the corresponding two sets of evolved states as $\mathbb{S}'_1$ and $\mathbb{S}'_2$, which are the sets of antidistinguishable states. From proposition 2 of [12], we know that $\mathbb{S}'_1 \bigcup \mathbb{S}'_2$ is also a set of antidistinguishable states. That facilitates that the unitaries of the set $\mathbb{S}_1 \bigcup \mathbb{S}_2$ are antidistinguishable. □

Our last two results tell the procedure of transforming a set of non-antidistinguishable unitaries to antidistinguishable unitaries.

Theorem 7. *Any finite set of qubit unitaries are either antidistinguishable or the set can be made antidistinguishable by adding one unitary.*

Proof. For any single qubit state $|\chi\rangle$ as a probing state, the antidistinguishability of any finite set of unitaries, i.e., $\{\mathcal{U}_1, \cdots, \mathcal{U}_n\}$ eventually reduces to the antidistinguishability of the set of evolved states $\mathbb{S} = \{\mathcal{U}_1|\chi\rangle, \cdots, \mathcal{U}_n|\chi\rangle\}$. Proposition 7 of [12] tells that any set of qubit pure states are either antidistinguishable or it can be made antidistinguishable by adding one pure qubit state. Similarly we can say either $\mathbb{S}$ is a set of antidistinguishable unitaries or we can add one more unitary to make the set antidistinguishable without changing the probing state. □

While Theorem 7 ensures that any finite set of qubit unitaries can be extended to an antidistinguishable set, below we show that, having tensor product of unitaries does not, in general, preserve their non-antidistinguishability.

Theorem 8. *There exist two not antidistinguishable qubit unitaries* $W_1 = |0\rangle\langle 0| + |1\rangle\langle 1|, W_2 = |+\rangle\langle 0| - |-\rangle\langle 1|$ *such that the set* $\{W_i \otimes W_j\}_{i,j=1}^{2}$ *are antidistinguishable.*

Proof. Eigenvalues of $W_1^\dagger W_2$ are $(\frac{1+\mathbb{i}}{\sqrt{2}}, \frac{1-\mathbb{i}}{\sqrt{2}})$. So none of the convex combination of these two eigenvalues can give zero, that means W_1 and W_2 are not distinguishable [40], which implies they are not antidistinguishable.
Now we take the following $2 \otimes 2$ four unitaries $\{W_1 \otimes W_1, W_1 \otimes W_2, W_2 \otimes W_1, W_2 \otimes W_2\}$. With the product probe $|00\rangle$, the four evolved states are $\{|00\rangle, |0+\rangle, |+0\rangle, |++\rangle\}$. The ref. [6] showed that these four states are antidistinguishable with an entangled measurement. □

Since indistinguishability and non-antidistinguishability are equivalent notions for a pair of unitaries, it is instructive to identify a set of three unitaries that are individually non-antidistinguishable yet whose tensor products constitute an antidistinguishable set.

Theorem 9. *There exist three not antidistinguishable qubit unitaries* $Q_1 = |0\rangle\langle 0| + |1\rangle\langle 1|, Q_2 = |\eta\rangle\langle 0| - |\eta^\perp\rangle\langle 1|$ *and* $Q_3 = |\zeta\rangle\langle 0| - |\zeta^\perp\rangle\langle 1|$ *such that the set* $\{Q_i \otimes Q_j\}_{i,j=1}^{3}$ *are antidistinguishable, where* $|\eta\rangle = \cos(5\pi/18)|0\rangle + \sin(5\pi/18)|1\rangle$ *and* $|\zeta\rangle = \cos(19\pi/60)|0\rangle + e^{\mathbb{i}2\pi/3}\sin(19\pi/60)|1\rangle$.

Proof. We have already established that any maximally entangled state serves as a sufficient probe for verifying the antidistinguishability of three qubit unitaries (Theorems 1, 2, and 4). Using the Bell state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ as the probe, the three unitaries $\{Q_i\}_{i=1}^{3}$ generate the evolved states $\left\{|\phi^+\rangle,\ |\phi'\rangle = \frac{1}{\sqrt{2}}(|\eta\rangle|0\rangle - |\eta^\perp\rangle|1\rangle),\ |\bar{\phi}\rangle = \frac{1}{\sqrt{2}}(|\zeta\rangle|0\rangle - |\zeta^\perp\rangle|1\rangle)\right\}$. A direct calculation shows that $|\langle\phi^+|\phi'\rangle|^2 + |\langle\phi'|\bar{\phi}\rangle|^2 + |\langle\phi^+|\bar{\phi}\rangle|^2 < 4\,|\langle\phi^+|\phi'\rangle|^2\,|\langle\phi'|\bar{\phi}\rangle|^2\,|\langle\phi^+|\bar{\phi}\rangle|^2$, which is precisely the opposite of the second condition in (4). This indicates that the three states are not antidistinguishable, and consequently, the corresponding unitaries are not antidistinguishable either.

Next, we consider the extended set of unitaries $\{Q_i \otimes Q_j\}_{i,j=1}^{3}$. Choosing the probe state $|00\rangle$, we obtain the nine evolved states: $\{|00\rangle,\ |\eta 0\rangle,\ |\zeta 0\rangle,\ |\eta 0\rangle,\ |\eta\eta\rangle,\ |\eta\zeta\rangle,\ |\zeta 0\rangle,\ |\zeta\eta\rangle,\ |\zeta\zeta\rangle\}$. A semidefinite program confirms that these states are perfectly antidistinguishable [9]. This completes the proof. □

V. CONCLUSION

In this work, we initiated a systematic study of the single-shot antidistinguishability of unitary operations. By formulating the task in analogy with the well-studied case of quantum states, we demonstrated that the problem ultimately reduces to the antidistinguishability of appropriately evolved states for optimally chosen probe states. Our analysis revealed several key structural insights. For three unitaries, all the maximally entangled probes work equivalently in antidistinguishaing task. For three qubit unitaries, we established that maximally entangled probes are always sufficient. However, we also showed that this hierarchy breaks down in higher dimensions, where there exist examples of unitaries that are antidistinguishable only with single-system and non-maximally entangled probes but not with maximally entangled probes. Beyond these structural results, we proved closure properties of antidistinguishable sets of three qubit unitaries and identified procedures for constructing antidistinguishable sets from those that are not. These findings parallel and extend known results in the antidistinguishability of quantum states, highlighting a deeper connection between the two frameworks.

Our results leave several open questions. Most notably, the characterization of antidistinguishable sets of higher-dimensional unitaries remains unexplored, and the role of entanglement in such settings warrants further investigation. Another intriguing direction is the exploration of multi-shot scenarios and the potential resource-theoretic formulation of antidistinguishability for quantum channels. We hope that this work lays the foundation for a broader understanding of the operational and structural aspects of antidistinguishability in quantum theory.

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