

UNIFORM ASYMPTOTIC APPROXIMATION AND NUMERICAL EVALUATION OF THE REVERSE GENERALIZED BESSEL POLYNOMIAL ZEROS

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Abstract. Uniform asymptotic expansions are derived for the zeros of the reverse generalized Bessel polynomials of large degree n and real parameter a . It is assumed that $-\Delta_1 n + \frac{3}{2} \leq a \leq \Delta_2 n$ for fixed arbitrary $\Delta_1 \in (0, 1)$ and bounded positive Δ_2 . For this parameter range at most one of the zeros is real, with the rest being complex conjugates. The new expansions are uniformly valid for all the zeros, and are shown to be highly accurate for moderate or large values of n . They are consequently used as initial values in a very efficient numerical algorithm designed to obtain the remaining complex zeros using Taylor series.

Key words. Asymptotic expansions, turning point theory, WKB theory, Bessel polynomials, numerical algorithms

AMS subject classifications. 34E05, 33C10, 34M60, 34E20

1. Introduction. The generalized Bessel polynomials are defined by

$$(1.1) \quad y_n(z; a) = \sum_{k=0}^n \binom{n}{k} (n+a-1)_k \left(\frac{1}{2}z\right)^k,$$

where $(\alpha)_k = \Gamma(\alpha+k)/\Gamma(\alpha)$ is Pochhammer's symbol. Their zeros, which are generally complex-valued, arise in a number of applications in applied mathematics [12] and engineering: see, for example, [1, 10, 11, 13]. Previous works addressing the approximation or computation of these zeros include [2, 15]. In addition, [16] presents a general method for computing complex zeros of special functions, one of the cases considered being the zeros of reverse generalized Bessel polynomials. For an investigation into the domains in which the zeros lie, see [3, 4].

In this paper, we derive uniform asymptotic expansions for these zeros as $n \rightarrow \infty$, which are uniformly valid for *all* the zeros, and for $-\Delta_1 n + \frac{3}{2} \leq a \leq \Delta_2 n$ for fixed arbitrary $\Delta_1 \in (0, 1)$ and bounded positive Δ_2 . These expansions are considerably more powerful than existing results, since they are uniformly valid for n large and $|a|$ small or large. These are derived in section 3, employing certain coefficients presented in section 2, and are shown to be highly accurate for moderate or large values of n . Moreover, they can be used as starting values in a new highly efficient numerical algorithm which we develop in section 4, which is designed to obtain the remaining complex zeros using Taylor series.

It is important to mention that the matrix method presented in [15] for calculating the zeros of generalized Bessel polynomials requires good initial approximations and is computationally expensive. In contrast, our algorithm avoids these drawbacks, even when many zeros are required. Moreover, using Taylor series avoids explicit evaluation

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of the function: since we only need the zeros, the overall normalization is irrelevant, simplifying the computation.

In our analysis, following [6, 8, 9], we find it more convenient to consider the reverse Bessel polynomials, which are given by

$$(1.2) \quad \theta_n(z; a) = z^n y_n(z^{-1}; a).$$

Then, from [6, Eqs. (2.3) and (2.4)], define the scaled function

$$(1.3) \quad w_n^{(0)}(z; a) = 2^{-n-a+1} z^{1-n-a/2} e^{-z} \theta_n(z; a),$$

which satisfies the differential equation

$$(1.4) \quad \frac{d^2 w}{dz^2} = \left\{ 1 + \frac{a-2}{z} + \frac{(2n+a)(2n+a-2)}{4z^2} \right\} w.$$

This has a regular singularity at $z = 0$ and an irregular singularity at infinity. The significance of $w_n^{(0)}(z; a)$ is that it is the solution that is recessive at infinity in the right half-plane, since

$$(1.5) \quad w_n^{(0)}(z; a) = 2^{-n-a+1} z^{1-a/2} e^{-z} \{1 + \mathcal{O}(z^{-1})\} \quad (z \rightarrow \infty).$$

As in [8], we also use two numerically satisfactory companion solutions of (1.4), namely

$$(1.6) \quad w_n^{(1)}(z; a) = (-1)^{n+1} z^{n+a/2} e^{-z} V(n+a-1, 2n+a, 2z),$$

and

$$(1.7) \quad w_n^{(-1)}(z; a) = z^{n+a/2} e^{-z} \mathbf{M}(n+a-1, 2n+a, 2z),$$

where $V(a, b, z)$ and $\mathbf{M}(a, b, z)$ are certain confluent hypergeometric functions (see [14, pp. 255–256]). $w_n^{(1)}(z; a)$ and $w_n^{(-1)}(z; a)$ are recessive at $z = \infty$ in the left half-plane $|\arg(-z)| < \frac{1}{2}\pi$, and at $z = 0$, respectively. This follows from the limiting behavior

$$(1.8) \quad w_n^{(1)}(z; a) = 2^{-n-1} z^{(a/2)-1} e^z \{1 + \mathcal{O}(z^{-1})\} \quad (z \rightarrow \infty, |\arg(-z)| \leq \frac{3}{2}\pi - \delta),$$

where δ is an arbitrary small positive constant, and

$$(1.9) \quad w_n^{(-1)}(z; a) = \frac{z^{n+a/2}}{\Gamma(2n+a)} \{1 + \mathcal{O}(z)\} \quad (z \rightarrow 0).$$

2. Liouville-Green coefficients. We define

$$(2.1) \quad u = n + \frac{1}{2}, \quad \alpha = \frac{a-2}{u}.$$

Then the differential equation (1.4) can be rewritten in the form

$$(2.2) \quad \frac{d^2 w}{dz^2} = \{u^2 f(\alpha, z) + g(z)\} w,$$

where

$$(2.3) \quad f(\alpha, z) = \frac{(z + \frac{1}{2}\alpha)^2 + 1 + \alpha}{z^2}, \quad g(z) = -\frac{1}{4z^2}.$$

On factoring we note that

$$(2.4) \quad f(\alpha, z) = \frac{(z - z_1)(z - z_2)}{z^2},$$

where

$$(2.5) \quad z_{1,2}(\alpha) = \pm i\sigma - \frac{1}{2}\alpha,$$

in which

$$(2.6) \quad \sigma = \sqrt{1 + \alpha}.$$

Thus, for large u , (2.2) has turning points at $z = z_{1,2}$. Following [8], we assume

$$(2.7) \quad -1 < -1 + \delta \leq \alpha \leq \alpha_1 < \infty,$$

and, as such, the two turning points are bounded complex conjugates, bounded away from each other and from the pole at $z = 0$.

In our expansions for the zeros, we use a Liouville-Green (LG) variable ξ , along with the Liouville variable ζ , which appears in turning point expansions (see [14, Chaps. 10 and 11]). These are given in the present case by

$$(2.8) \quad \frac{2}{3}\zeta^{3/2} = \xi = \int_{z_1(\alpha)}^z f^{1/2}(\alpha, t) dt \\ = Z - \left(1 + \frac{1}{2}\alpha\right) \ln \left\{ \frac{4Z + 2\alpha(Z + z + 2) + 4 + \alpha^2}{z} \right\} \\ + \frac{1}{2}\alpha \ln(2Z + 2z + \alpha) + \frac{1}{2} \ln(1 + \alpha) + \left(2 + \frac{1}{2}\alpha\right) \ln(2) - \frac{1}{2}(1 + \alpha)\pi i,$$

where

$$(2.9) \quad Z = \{(z - z_1)(z - z_2)\}^{1/2} = \left\{ \left(z + \frac{1}{2}\alpha\right)^2 + 1 + \alpha \right\}^{1/2}.$$

The branch of the square root in (2.9) is chosen so that $Z > 0$ for $z > 0$, $Z < 0$ for $z < 0$, with Z being continuous throughout the upper half of the complex z plane, except along a branch cut connecting $z = 0$ to the turning point z_1 , where the imaginary part of ξ vanishes. This cut traces the so-called anti-Stokes line. With this choice, we have $Z \sim z$ as $z \rightarrow \infty$ in the upper half-plane ($\Im(z) \geq 0$). Furthermore, principal branches are used for the logarithmic terms appearing in (2.8). A detailed description of the associated conformal mapping is provided in [8].

The following coefficients appearing in LG expansions were constructed in [8], and we shall use them here. First, let $\phi \in \mathbb{C}$ be defined by

$$(2.10) \quad \sin(\phi) = \frac{\sigma}{Z},$$

and hence from (2.9),

$$(2.11) \quad \cos(\phi) = \frac{z + \frac{1}{2}\alpha}{Z}.$$

The functions $\sin(\phi)$ and $\cos(\phi)$ are both positive when $z > 0$, and they extend continuously throughout the upper half-plane $\Re(z) \geq 0$, except along the branch cut

extending from $z = 0$ to the turning point z_1 along the anti-Stokes line as described above. In particular, $\cos(\phi)$ is positive for all z in the interval $(-\infty, -\frac{1}{2}\alpha)$, and approaches 1 as $z \rightarrow \infty$ along any ray in the upper half-plane. Also observe that, by combining (2.10) and (2.11), one obtains the relation

$$(2.12) \quad z = \sigma \cot(\phi) - \frac{1}{2}\alpha.$$

With these definitions, the LG coefficients that we shall use are given by

$$(2.13) \quad E_1(\alpha, \phi) = \frac{\sin(\phi) \{5 \cos^2(\phi) - 2\}}{24\sigma} + \frac{\alpha \{\cos(\phi) (5 \cos^2(\phi) - 6) + 1\}}{48(1 + \alpha)},$$

$$(2.14) \quad E_2(\alpha, \phi) = \frac{\alpha \cos(\phi) \sin^3(\phi) \{3 - 5 \cos^2(\phi)\}}{16(1 + \alpha)^{3/2}} \\ + \frac{\sin^2(\phi)}{64(1 + \alpha)^2} \{5(4 - \alpha^2 + 4\alpha) \cos^4(\phi) + (7\alpha^2 - 16\alpha - 16) \cos^2(\phi) - 2\alpha^2\},$$

and for $s = 2, 3, 4, \dots$

$$(2.15) \quad E_{s+1}(\alpha, \phi) = G(\alpha, \phi) \frac{\partial E_s(\alpha, \phi)}{\partial \phi} + \int_0^\phi G(\alpha, \varphi) \sum_{j=1}^{s-1} \frac{\partial E_j(\alpha, \varphi)}{\partial \varphi} \frac{\partial E_{s-j}(\alpha, \varphi)}{\partial \varphi} d\varphi,$$

where

$$(2.16) \quad G(\alpha, \phi) = -\frac{1}{2} \frac{d\phi}{d\xi} = \frac{\cos(\phi) \sin^2(\phi)}{2\sigma} - \frac{\alpha \sin^3(\phi)}{4(1 + \alpha)}.$$

The lower integration limits in (2.15) are chosen for convenience so that $E_s(\alpha, 0) = 0$, which means they vanish as $z \rightarrow \infty$.

Next, let

$$(2.17) \quad a_1 = a_2 = \frac{5}{72}, \quad \tilde{a}_1 = \tilde{a}_2 = -\frac{7}{72},$$

with subsequent terms a_s and $\tilde{a}_s$ satisfying the same recursion formula, viz.

$$(2.18) \quad a_{s+1} = \frac{1}{2}(s+1)a_s + \frac{1}{2} \sum_{j=1}^{s-1} a_j a_{s-j} \quad (s = 2, 3, 4, \dots).$$

Further, define

$$(2.19) \quad \tilde{\mathcal{E}}_s(a, z) = E_s(\alpha, \phi) + (-1)^s \frac{\tilde{a}_s}{s\xi^s},$$

and

$$(2.20) \quad \mathcal{E}_s(a, z) = E_s(\alpha, \phi) + (-1)^s \frac{a_s}{s\xi^s}.$$

Also, let the sequence $d_{2s+1}(\alpha)$ ($s = 0, 1, 2, \dots$) be given by

$$(2.21) \quad \frac{1}{2} \left[u\alpha(\ln(u) - 1) + u(1 + \alpha) \ln(1 + \alpha) \right. \\ \left. + \ln \left\{ \Gamma \left(u + \frac{1}{2} \right) \right\} - \ln \left\{ \Gamma \left(u + u\alpha + \frac{1}{2} \right) \right\} \right] \sim \sum_{s=0}^{\infty} \frac{d_{2s+1}(\alpha)}{u^{2s+1}} \quad (u \rightarrow \infty).$$

The first four terms are given by

$$(2.22) \quad d_1(\alpha) = -\frac{\alpha}{48(1 + \alpha)},$$

$$(2.23) \quad d_3(\alpha) = \frac{7\alpha(3 + 3\alpha + \alpha^2)}{5760(1 + \alpha)^3},$$

$$(2.24) \quad d_5(\alpha) = -\frac{31\alpha(5 + 10\alpha + 10\alpha^2 + 5\alpha^3 + \alpha^4)}{80640(1 + \alpha)^5},$$

and

$$(2.25) \quad d_7(\alpha) = \frac{127\alpha(7 + 21\alpha + 35\alpha^2 + 35\alpha^3 + 21\alpha^4 + 7\alpha^5 + \alpha^6)}{430080(1 + \alpha)^7}.$$

Then from [8, Thm. 3.2]

$$(2.26) \quad \theta_n(uz; a) = \frac{u^{1/6}}{e^{(u+a+2)\pi i/2}} \left\{ \frac{2^a \pi n!}{\Gamma(n + a - 1)} \right\}^{1/2} \left\{ \frac{\zeta}{f(a, z)} \right\}^{1/4} (uz)^{n + \frac{1}{2}a - 1} e^{uz} \\ \times \left\{ \text{Ai} \left(u^{2/3} \zeta \right) A(u, a, z) + \text{Ai}' \left(u^{2/3} \zeta \right) B(u, a, z) \right\},$$

where

$$(2.27) \quad A(u, a, z) \sim \exp \left\{ \sum_{s=1}^{\infty} \frac{\tilde{\mathcal{E}}_{2s}(a, z)}{u^{2s}} \right\} \cosh \left\{ \sum_{s=0}^{\infty} \frac{\tilde{\mathcal{E}}_{2s+1}(a, z) + d_{2s+1}(\alpha)}{u^{2s+1}} \right\},$$

$$(2.28) \quad B(u, a, z) \sim \frac{1}{u^{1/3} \zeta^{1/2}} \exp \left\{ \sum_{s=1}^{\infty} \frac{\mathcal{E}_{2s}(a, z)}{u^{2s}} \right\} \sinh \left\{ \sum_{s=0}^{\infty} \frac{\mathcal{E}_{2s+1}(a, z) + d_{2s+1}(\alpha)}{u^{2s+1}} \right\},$$

as $u \rightarrow \infty$, uniformly for $0 \leq \arg(z) \leq \pi$ with $|z - z_1| \geq \delta > 0$, under the condition (2.4). See [8, Remark 1] on how these expansions can be extended to $|z - z_1| \leq \delta$.

From [8, Lemma 3.1] we have an important result which is required in the next section:

LEMMA 2.1. *Each $(z - z_1)^{1/2} \{E_{2s+1}(\alpha, \phi) + d_{2s+1}(\alpha)\}$ ($s = 0, 1, 2, \dots$), regarded as a function of z , is meromorphic at $z = z_1$.*

3. Uniform asymptotic expansions for the zeros. We derive uniform asymptotic expansions for the complex zeros of $\theta_n(z; a)$ for large n , which are uniformly valid for unrestricted z subject to (2.7). To do so, we use the method of [7]. We begin by defining a function $\mathcal{Z}(u, a, z)$ and coefficient $l_n(a)$ by the pair of equations

$$(3.1) \quad (-1)^n \frac{e^{a\pi i}}{n!} w_n^{(0)}(uz; a) = e^{-\pi i/3} l_n(a) \left\{ \frac{\partial \mathcal{Z}(u, a, z)}{\partial z} \right\}^{-1/2} \text{Ai} \left(u^{2/3} \mathcal{Z}(u, a, z) \right),$$

$$(3.2) \quad \frac{1}{\Gamma(n+a-1)} w_n^{(1)}(uz; a) = l_n(a) \left\{ \frac{\partial \mathcal{Z}(u, a, z)}{\partial z} \right\}^{-1/2} \text{Ai}_1 \left(u^{2/3} \mathcal{Z}(u, a, z) \right).$$

The factor $\{\partial \mathcal{Z}(u, a, z)/\partial z\}^{-1/2}$, along with Airy's equation [5, Eq. 9.2.1], ensures that both functions on the RHS of this pair of equations satisfy a linear second-order differential equation (with independent variable z) that has no first derivative term, which matches the same property of the functions on the LHS of these equations, namely the differential equation (2.2).

Now, from ([5, Eq. 9.2.12]),

$$(3.3) \quad \text{Ai}(z) = e^{\pi i/3} \text{Ai}_1(z) + e^{-\pi i/3} \text{Ai}_{-1}(z),$$

and hence from the connection formula [6, Eq. (2.13)]

$$(3.4) \quad w_n^{(-1)}(z; a) = (-1)^{n+1} \frac{e^{a\pi i}}{n!} w_n^{(0)}(z; a) + \frac{1}{\Gamma(n+a-1)} w_n^{(1)}(z; a),$$

we have

$$(3.5) \quad w_n^{(-1)}(uz; a) = e^{\pi i/3} l_n(a) \left\{ \frac{\partial \mathcal{Z}(u, a, z)}{\partial z} \right\}^{-1/2} \text{Ai}_{-1} \left(u^{2/3} \mathcal{Z}(u, a, z) \right).$$

Moreover, $\zeta \rightarrow \infty$ as $z \rightarrow 0$ or $z \rightarrow \pm\infty$; in these cases $Z \sim \zeta$ (see [7]). The fundamental property of (3.1), (3.2), and (3.4) is that both functions in each identity are recessive at the same singularities, namely $z = -\infty, +\infty, 0$, respectively.

In studying the zeros we do not require the constant $l_n(a)$, but for completeness we note that it can be determined as follows. From (1.3), (1.7), and [5, Eq. 13.2.34]

$$(3.6) \quad \mathcal{W} \left\{ w_n^{(0)}(z; a), w_n^{(-1)}(z; a) \right\} = \frac{2^{1-2n-a}}{\Gamma(n+a-1)}.$$

Thus from (3.1), (3.5), (3.6) and [5, Eq. 9.2.8]

$$(3.7) \quad l_n(a) = \frac{e^{5\pi i/6} e^{(u+a)\pi i/2} u^{1/6}}{2^{n-1}} \left\{ \frac{\pi}{2^n n! \Gamma(n+a-1)} \right\}^{1/2}.$$

We focus on (3.1). Now, from [7, Thm. 2.2], we have

$$(3.8) \quad \mathcal{Z}(u, a, z) \sim \zeta + \sum_{s=1}^{\infty} \frac{\Upsilon_s(a, z)}{u^{2s}} \quad (u \rightarrow \infty),$$

where each $\Upsilon_s(a, z)$ is analytic at the turning point $z = z_1$. This expansion is uniformly valid in an unbounded domain that includes the upper half-plane $0 \leq \arg(z) \leq \pi$. Thus, we shall use it to approximate all the zeros of $\theta_n(z; a)$ with nonnegative imaginary part, with those lying in the lower half-plane simply being the conjugates of these.

The coefficients $\Upsilon_s(a, z)$ are given by [7, Thm. 2.2], with $\hat{E}_{2s+1}(z)$ replaced by $E_{2s+1}(\alpha, \phi) + d_{2s+1}(\alpha)$ ($s = 0, 1, 2, \dots$). The inclusion of the constants $d_{2s+1}(\alpha)$ is required to ensure the required property stated in Lemma 2.1. From [7, Eqs. (2.28), (2.40) - (2.42)] the first four coefficients in the series in (3.8) are given by

$$(3.9) \quad \Upsilon_1 = \frac{3\xi(E_1 + d_1)}{2\zeta^2} - \frac{5}{48\zeta^2},$$

$$(3.10) \quad \Upsilon_2 = -\frac{\Upsilon_1^2}{4\zeta} + \frac{5\Upsilon_1}{32\zeta^3} + \frac{3\xi(E_3 + d_3)}{2\zeta^2} - \frac{1105}{9216\zeta^5},$$

$$(3.11) \quad \Upsilon_3 = -\frac{\Upsilon_1\Upsilon_2}{2\zeta} + \frac{\Upsilon_1^3}{24\zeta^2} - \frac{25\Upsilon_1^2}{128\zeta^4} + \frac{5\Upsilon_2}{32\zeta^3} + \frac{1105\Upsilon_1}{2048\zeta^6} + \frac{3\xi(E_5 + d_5)}{2\zeta^2} - \frac{82825}{98304\zeta^8},$$

and

$$(3.12) \quad \begin{aligned} \Upsilon_4 = & -\frac{\Upsilon_1^4}{64\zeta^3} + \frac{\Upsilon_1^2\Upsilon_2}{8\zeta^2} + \frac{175\Upsilon_1^3}{768\zeta^5} - \frac{\Upsilon_1\Upsilon_3}{2\zeta} - \frac{25\Upsilon_1\Upsilon_2}{64\zeta^4} - \frac{\Upsilon_2^2}{4\zeta} \\ & - \frac{12155\Upsilon_1^2}{8192\zeta^7} + \frac{5\Upsilon_3}{32\zeta^3} + \frac{1105\Upsilon_2}{2048\zeta^6} + \frac{414125\Upsilon_1}{65536\zeta^9} + \frac{3\xi(E_7 + d_7)}{2\zeta^2} - \frac{1282031525}{88080384\zeta^{11}}, \end{aligned}$$

with subsequent ones given by [7, Thm. 2.2].

Next, let $t_m(u, a)$ ($m = 1, 2, 3, \dots, \lfloor (n+1)/2 \rfloor$) be the complex zeros of $\theta_n(t; a)$ in the upper half-plane $\Re(t) \geq 0$, so that

$$(3.13) \quad \theta_n(t_m(u, a); a) = 0 \quad (m = 1, 2, 3, \dots, \lfloor (n+1)/2 \rfloor).$$

From (1.3) and (3.1) they satisfy the implicit equation

$$(3.14) \quad \mathcal{Z}(u, a, u^{-1}t_m(u, a)) = u^{-2/3}a_m \quad (m = 1, 2, 3, \dots),$$

where $x = a_m$ is the m th negative zero of $\text{Ai}(x)$ ordered by increasing absolute values.

From [7, Thm. 3.1] we obtain the uniform asymptotic expansion we seek, namely

$$(3.15) \quad t_m(u, a) \sim u \sum_{s=0}^{\infty} \frac{\tau_{m,s}(\alpha)}{u^{2s}} \quad (u \rightarrow \infty, m = 1, 2, 3, \dots),$$

for coefficients $\tau_{m,s}(\alpha)$ which we determine next.

With the branch cut for Z as described in section 2, and principal branches for the logarithms we have $\xi = -2i|a_m|^{3/2}/(3u)$ for $\zeta = u^{-2/3}a_m$. Thus, on plugging (3.15) into (3.14), using (2.8), (2.9), and (3.8), re-expanding in inverse powers of u , and then equating like powers, we can find in turn the coefficients for each prescribed $m \in \{1, 2, 3, \dots, \lfloor (n+1)/2 \rfloor\}$.

Consequently, the leading term $\tau_{m,0} = \tau_{m,0}(\alpha)$ is given implicitly by

$$(3.16) \quad \begin{aligned} -\frac{2i}{3u}|a_m|^{3/2} = & Z_{m,0} + \left(1 + \frac{1}{2}\alpha\right) \ln \left\{ \frac{\tau_{m,0}}{4Z_{m,0} + 2\alpha(Z_{m,0} + \tau_{m,0} + 2) + 4 + \alpha^2} \right\} \\ & + \frac{1}{2}\alpha \{ \ln(-2Z_{m,0} - 2\tau_{m,0} - \alpha) + \pi i \} + \frac{1}{2} \ln(1 + \alpha) + \left(2 + \frac{1}{2}\alpha\right) \ln(2) - \frac{1}{2}(1 + \alpha)\pi i, \end{aligned}$$

in which $Z_{m,0} = Z(\tau_{m,0})$, where $Z(z)$ is given by (2.9) with the branch as described below that equation. Thus

$$(3.17) \quad Z_{m,0} = Z_{m,0}(\alpha) = -\{(\tau_{m,0} - z_1)(\tau_{m,0} - z_2)\}^{1/2},$$

which is negative for $-\infty < \tau_{m,0} < 0$, since $z = \tau_{m,0}$ lies to the left of the cut along the anti-Stokes line $\Im(\xi) = 0$ from $z = z_1$ to $z = 0$. For numerical purposes, in (3.16) the second logarithm was expressed in a form that ensures the correct branch for these complex values of $\tau_{m,0}$.

Next, let

$$(3.18) \quad \zeta_{m,0} = \zeta(\tau_{m,0}) = u^{-2/3} a_m, \quad \zeta'_{m,0} = \zeta'(\tau_{m,0}), \quad \zeta''_{m,0} = \zeta''(\tau_{m,0}), \quad \dots,$$

and similarly for $s = 1, 2, 3, \dots$ let

$$(3.19) \quad \Upsilon_{m,s} = \Upsilon_s(\tau_{m,0}), \quad \Upsilon'_{m,s} = \Upsilon'_s(\tau_{m,0}), \quad \Upsilon''_{m,s} = \Upsilon''_s(\tau_{m,0}), \quad \dots$$

Then we can apply [7, Thm. 3.1] to find the coefficients in (3.15). Having determined $\tau_{m,0}$, the next four coefficients are of the same form as [7, Eqs.(3.47) - (3.50)] (with the appropriate change of notation). Thus

$$(3.20) \quad \tau_{m,1} = -\frac{\Upsilon_{m,1}}{\zeta'_{m,0}},$$

$$(3.21) \quad \tau_{m,2} = -\frac{1}{2\zeta'_{m,0}} \{ \tau_{m,1}^2 \zeta''_{m,0} + 2\tau_{m,1} \Upsilon'_{m,1} + 2\Upsilon_{m,2} \},$$

$$(3.22) \quad \tau_{m,3} = -\frac{1}{6\zeta'_{m,0}} \{ \tau_{m,1}^3 \zeta'''_{m,0} + 6\tau_{m,1} \tau_{m,2} \zeta''_{m,0} + 3\tau_{m,1}^2 \Upsilon''_{m,1} \\ + 6\tau_{m,2} \Upsilon'_{m,1} + 6\tau_{m,1} \Upsilon'_{m,2} + 6\Upsilon_{m,3} \},$$

and¹

$$(3.23) \quad \tau_{m,4} = -\frac{1}{24\zeta'_{m,0}} \left\{ \tau_{m,1}^4 \zeta^{(4)}_{m,0} + 12\tau_{m,1}^2 \tau_{m,2} \zeta'''_{m,0} + 24\tau_{m,1} \tau_{m,3} \zeta''_{m,0} \right. \\ + 12\tau_{m,2}^2 \zeta''_{m,0} + 4\tau_{m,1}^3 \Upsilon'''_{m,1} + 24\tau_{m,1} \tau_{m,2} \Upsilon''_{m,1} + 12\tau_{m,1}^2 \Upsilon''_{m,2} \\ \left. + 24\tau_{m,3} \Upsilon'_{m,1} + 24\tau_{m,2} \Upsilon'_{m,2} + 24\tau_{m,1} \Upsilon'_{m,3} + 24\Upsilon_{m,4} \right\}.$$

These, of course, result in different coefficients than in the Bessel function case, due to the difference here in the variable ζ as well as the Υ coefficients.

3.1. Numerical examples. We now approximate $t_m(u, a)$ by the series (3.15). In order to do so, we require the derivatives with respect to z of ζ , ξ , ϕ and Υ_s . For the latter, it is convenient to let $\rho = 1/\zeta$. Then from (2.8)

$$(3.24) \quad \rho' = -\frac{3}{2}\rho^4 \xi \xi'.$$

For the other derivatives, we use

$$(3.25) \quad \zeta' = \frac{2\xi' \zeta}{3\xi},$$

$$(3.26) \quad \xi' = f^{1/2}(a, z) = \frac{\sigma}{z \sin(\phi)},$$

¹There is a misprint in [7, Eq. (3.50)]

n, m	t_m	Relative error
15, 1	$-3.1559515225814951808 + 12.586271690843017387i$	1.8×10^{-15}
15, 3	$-6.9360218173803455640 + 8.6292759166638006520i$	6.1×10^{-16}
30, 1	$-4.2425750716206130472 + 27.006358468998877565i$	1.1×10^{-16}
30, 3	$-9.7584463264409865096 + 22.392832031435945931i$	1.3×10^{-16}
30, 10	$-18.102790325129739597 + 9.4722422021510892034i$	7.7×10^{-17}
30, 15	$-19.702854218331257062 + .85611271550820061202i$	2.9×10^{-16}
50, 1	$-5.2055266715795128190 + 46.482961682470093754i$	1.9×10^{-17}
50, 3	$-12.181102558122645217 + 41.239145916888100131i$	7.9×10^{-17}
50, 10	$-24.683402130958153499 + 27.225504025486397962i$	1.4×10^{-16}
50, 15	$-29.379559025204265717 + 18.222895815367965462i$	2.2×10^{-16}
50, 25	$-32.962750529211803345 + .86074820845854851940i$	6.7×10^{-18}

TABLE 1

Approximations to the zeros of the reverse generalized Bessel polynomials for $a = 1.01$ and different values of n and m .

and

$$(3.27) \quad \phi' = -\frac{\sin^2(\phi)}{\sigma},$$

which follow from (2.3), (2.6), (2.8), (2.10), and (2.12).

We compute the first five terms $\tau_{m,s}$ ($s = 0, 1, 2, 3, 4$) in the expansion (3.15). This is achieved by the following steps.

- For each prescribed n and $m \in \{1, 2, 3, \dots, \lfloor n+1 \rfloor\}$ find $\xi = -2i|a_m|^{3/2}/(3u)$ where $u = n + \frac{1}{2}$, and $\zeta = \zeta_{m,0} = a_m u^{-2/3}$. Use these values for ξ and ζ below.
- Use (3.16) to numerically evaluate $\tau_{m,0}$. To obtain the correct root we found it efficient to set $\tau_{m,0} = -0.5 + w$ in the equation and then numerically solve for w . Then use $z = \tau_{m,0}$ in what follows.
- Use (2.6), (2.9)–(2.11), and (3.25)–(3.27) to compute $\zeta'_{m,0}$, $\zeta''_{m,0}$, $\zeta'''_{m,0}$ and $\zeta^{(4)}_{m,0}$.
- From (2.13)–(2.16) and (3.27) for $z = \tau_{m,0}$ compute E_1 and its first three z derivatives, E_3 and its first two z derivatives, E_5 and its z derivative, and E_7 .
- Use (2.19), (2.20), (2.22)–(2.25), (3.9)–(3.12), (3.24), (3.26), and (3.27) and the above values to compute in turn $\Upsilon_{m,1}$, $\Upsilon'_{m,1}$, $\Upsilon''_{m,1}$, $\Upsilon'''_{m,1}$, $\Upsilon_{m,2}$, $\Upsilon'_{m,2}$, $\Upsilon''_{m,2}$, $\Upsilon_{m,3}$, $\Upsilon'_{m,3}$, and $\Upsilon_{m,4}$.
- Evaluate $\tau_{m,s}$ for $s = 1, 2, 3, 4$ in turn from (3.20)–(3.23).

Numerical examples of the approximations t_m obtained using the scheme described above are presented in Tables 1 and 2 for $a = 1.01$, 20.2, respectively, and different values of n and m . The approximations, implemented in Maple², are compared with the values obtained using the numerical algorithm described in [16] (also implemented in Maple). The relative errors from these comparisons are shown in the tables. Relative errors are all close to or better than 10^{-15} .

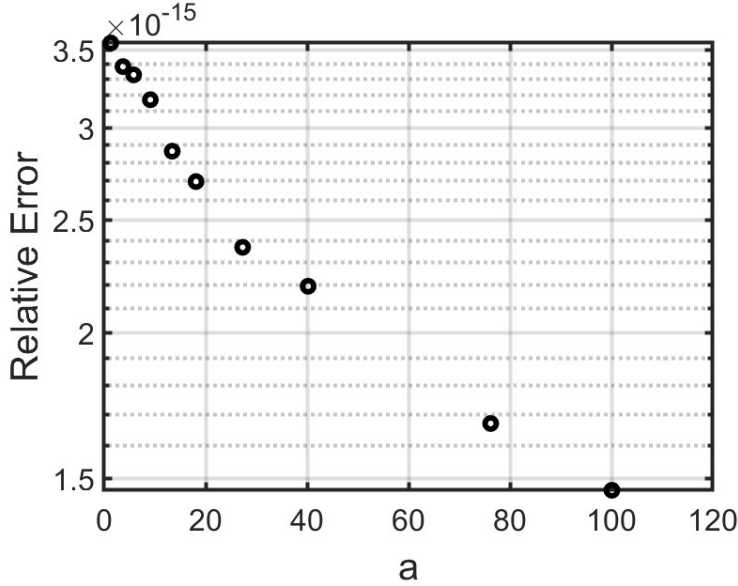
In the implementation of the numerical algorithm, the reverse generalized Bessel

²The file can be obtained from <https://github.com/AmparoGil/AsympZerosRGPs>

n, m	t_m	Relative error
15, 1	$-12.715856054909203812 + 18.788546633810651464i$	1.1×10^{-15}
15, 3	$-16.514653825298059143 + 12.612556755577648289i$	2.6×10^{-15}
30, 1	$-13.800334806578766149 + 34.380365451162645216i$	3.4×10^{-16}
30, 3	$-19.310221900147056579 + 28.210989284732813206i$	2.8×10^{-16}
30, 10	$-27.717880396627235555 + 11.750965665786499280i$	5.4×10^{-17}
30, 15	$-29.339399892921113584 + 1.0590134228243351098i$	2.8×10^{-16}
50, 1	$-14.766307319696546646 + 54.504885286408130512i$	2.9×10^{-16}
50, 3	$-21.724567399352576652 + 48.087744580616150218i$	6.1×10^{-17}
50, 10	$-34.260698846474016613 + 31.438165321383787957i$	1.6×10^{-16}
50, 15	$-38.989834370513922989 + 20.967450446744804559i$	2.3×10^{-16}
50, 25	$-42.605131456252572254 + .98772884689217274567i$	2.6×10^{-18}

TABLE 2

Approximations to the zeros of the reverse generalized Bessel polynomials for $a = 20.2$ and different values of n and m .

FIGURE 1. Relative errors as a function of a (with n fixed at 15 and m at 3).

polynomials are evaluated through their relation with the generalized Laguerre polynomials [5, § 18.34(i)]

$$(3.28) \quad \theta_n(z; a) = \left(-\frac{1}{2}\right)^n n! L_n^{(1-2n-a)}(2z).$$

Furthermore, Figure 1 displays the relative error of the approximations to the zeros for various values of a (with n fixed at 15 and m at 3). It is observed that the relative error decreases as a increases, and the maximum relative error attained is less than 4×10^{-15} .

Notably, equation (3.16) can be solved using Newton's method to obtain the

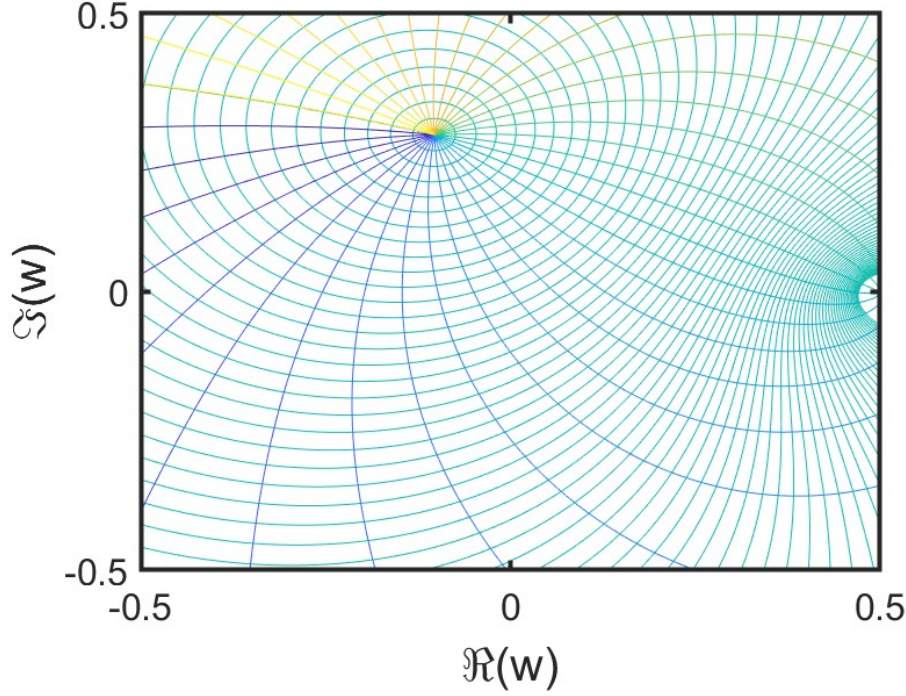


FIGURE 2. Plot of the function $F(w)$ for $a = 1.01$, $m = 10$ and $n = 30$.

corresponding root. By expressing the equation as $F(w) = 0$, Figure 2 illustrates, as an example, a plot of the function $F(w)$ for $a = 1.01$, $m = 10$ and $n = 30$. The root obtained by applying Newton's method is $w = -0.0935299175 + 0.310545771i$, which agrees with the location of the zero of $F(w)$ shown in the figure.

4. A numerical algorithm to compute the zeros of $\theta_n(z; a)$. Given that the asymptotic approximations obtained exhibit high accuracy for moderate to large values of n , they can be employed as initial estimates in the numerical algorithm described in [16] devised to compute all the zeros. Within this algorithm, the fundamental components are an iteration function $T_n(a, z)$ and a step function $H_n(a, z)$:

$$(4.1) \quad T_n(a, z) = z - \frac{1}{\sqrt{\Omega_n(a, z)}} \arctan \left(\frac{\sqrt{\Omega_n(a, z)} w_n^{(0)}(z; a)}{\partial w_n^{(0)}(z; a) / \partial z} \right),$$

$$(4.2) \quad H_n(a, z) = z + \frac{\pi}{\sqrt{\Omega_n(a, z)}},$$

where

$$(4.3) \quad \Omega_n(a, z) = -1 + \frac{2-a}{z} - \left(n + \frac{1}{2}a \right) \left(n + \frac{1}{2}a - 1 \right) \frac{1}{z^2}.$$

The function $w_n^{(0)}(z; a)$ appearing in (4.1) is related to the reverse generalized Bessel polynomials $\theta_n(z; a)$ by (1.3). It is therefore clear that $w_n^{(0)}(z; a)$ can be used

to locate the zeros of $\theta_n(z; a)$. In [9], we discussed the possible use of asymptotic expansions to compute the reverse generalized Bessel polynomials appearing in the iteration function (4.1). However, this is not necessary if accurate approximations are available for the first zero (combined with the use of Taylor series). This is precisely the strategy we discuss next. The resulting algorithm, which avoids the explicit computation of the function in order to locate its zeros, is both accurate and highly efficient.

We use as a starting point for finding the Taylor series the equation satisfied by $w_n = w_n^{(0)}(z; a)$, written in the form

$$(4.4) \quad Pw_n'' + Q_n w_n = 0,$$

where

$$(4.5) \quad P = P(z) = z^2, \quad Q_n = Q_n(a, z) = z^2 \Omega_n(a, z).$$

We have $P^{(m)} = Q_n^{(m)} = 0$ for $m > 2$. Then, we obtain for $w_n = w_n^{(0)}(z; a)$

$$(4.6) \quad \sum_{m=0}^2 \binom{j}{m} \left\{ P^{(m)} w_n^{(j+2-m)} + Q_n^{(m)} w_n^{(j-m)} \right\} = 0,$$

which gives the following recurrence relation for the derivatives

$$(4.7) \quad z^2 w_n^{(k+2)} + 2kz w_n^{(k+1)} + \{Q_n + k(k-1)\} w_n^{(k)} - k(2z + a - 2) w_n^{(k-1)} - k(k-1) w_n^{(k-2)} = 0 \quad (k = 2, 3, 4, \dots).$$

This recurrence relation, which is not ill-conditioned as k becomes large, can be used to compute the derivatives appearing in the Taylor series

$$(4.8) \quad w_n(z_{i+1}) = \sum_{k=0}^N w_n^{(k)}(z_i) \frac{h^k}{k!} + \mathcal{O}(h^{N+1}),$$

$$w_n'(z_{i+1}) = \sum_{k=0}^N w_n^{(k+1)}(z_i) \frac{h^k}{k!} + \mathcal{O}(h^{N+1}).$$

To apply (4.7) in the algorithm step by step, we need $w_n(z_i)$ and $w_n'(z_i)$, which are known from the previous step, and (again suppressing the a -dependence)

$$(4.9) \quad w_n''(z_i) = -\frac{Q_n(z_i)}{P(z_i)} w_n(z_i),$$

$$w_n'''(z_i) = -\frac{Q_n'(z_i)}{P(z_i)} w_n'(z_i) + \left\{ \frac{(2-a)}{z_i^2} - \frac{2(n + \frac{1}{2}a)(n + \frac{1}{2}a - 1)}{z_i^3} \right\} w_n(z_i).$$

The resulting algorithm works as follows: After a zero z_i has been obtained (using an asymptotic approximation for the first zero), the next zero z_{i+1} is obtained by taking the step $H_n(a, z)$ given in (4.2) and then iterating $T_n(a, z)$ given by (4.1) until convergence is reached. Taylor series (4.8) are used to calculate the functions $w_n(z)$ and $w_n'(z)$ needed in the iteration function, and the values $w_n(z_i) = 0$ and $w_n'(z_i) = 1$ are used in this computation. A Matlab implementation of the algorithm

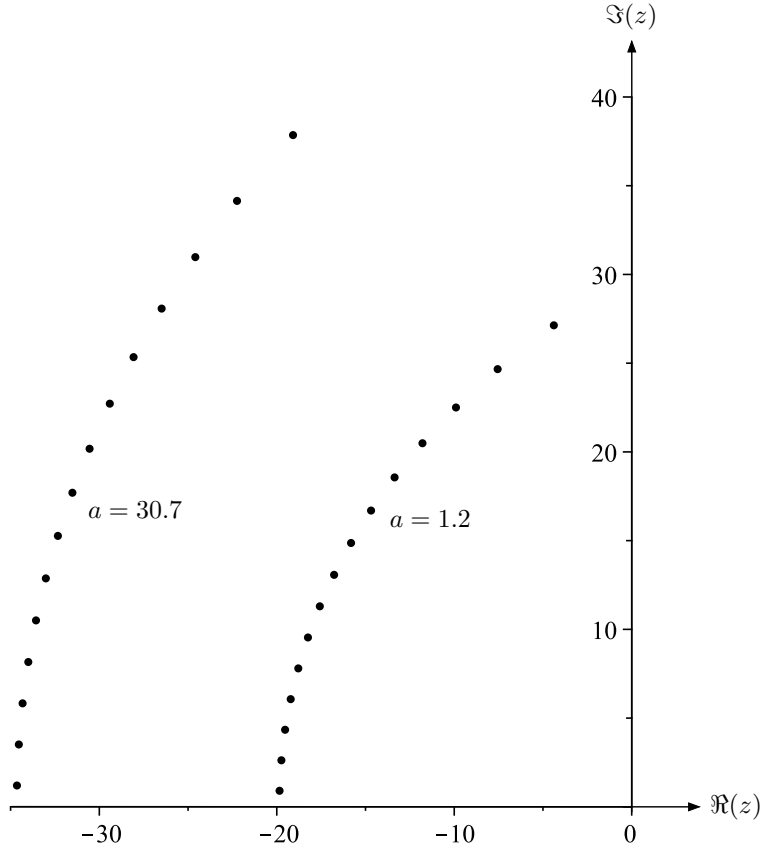


FIGURE 3. Zeros in the second quadrant obtained by the numerical algorithm for $n = 30$ and $a = 1.2, 30.7$.

can be obtained from GitHub³. In the present implementation, and for reasons of computational efficiency, only the first three coefficients of the uniform expansion (3.15) are employed in order to approximate the first zero to be computed (namely, the one with the largest real part in absolute value). The algorithm computes the complex zeros with nonnegative imaginary part (as it was mentioned before, the remaining complex zeros are simply the complex conjugates of those obtained).

Plots of the zeros of $\theta_n(z; a)$ in the second quadrant obtained by the numerical algorithm are shown in Figures 3 and 4 for $n = 30$, $n = 500$, $a = 1.2$ and $a = 30.7$. The accuracy prescribed for the iterations with $T_n(a, z)$ is $\epsilon = 10^{-12}$, although the precision attained for the zeros is higher when the value of n is sufficiently large, as can be observed in Figure 5. The relative errors have been obtained by comparison with the Maple implementation of the numerical algorithm used in subsection 3.1. The number of digits was set to 60 in the Maple implementation in order to ensure the correct evaluation of the quotients appearing in the iteration function. It can be observed that, even when a large number of zeros are computed, the relative errors remain well controlled.

³<https://github.com/AmparoGil/NumerZerosRGBPs>

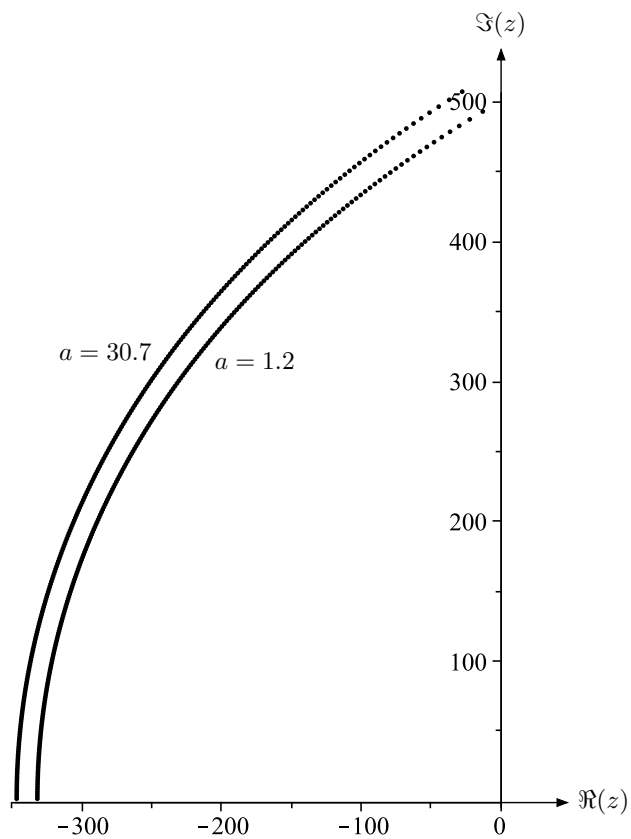


FIGURE 4. Zeros in the second quadrant obtained by the numerical algorithm for $n = 500$ and $a = 1.2, 30.7$.

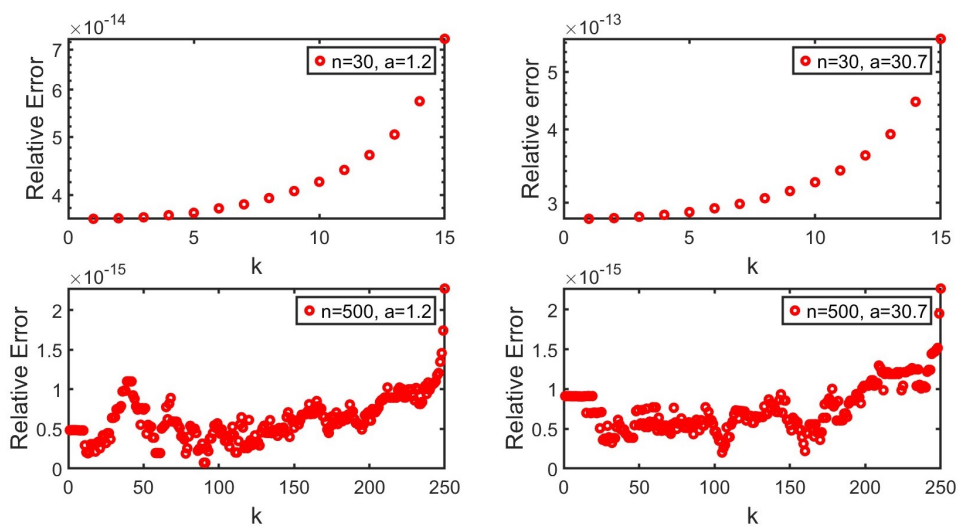


FIGURE 5. Relative errors of the computed zeros obtained using the numerical algorithm.

n	CPU time (s)
30	2.8×10^{-3}
200	5.9×10^{-3}
500	1.1×10^{-2}
1000	1.9×10^{-2}
2000	3.7×10^{-2}

TABLE 3

Representative CPU times obtained from the execution of our numerical algorithm for $a = 2.3$ and several values of n .

Finally, Table 3 presents representative CPU times obtained from the execution of our numerical algorithm for several values of n . CPU times were measured by executing our algorithm in Matlab R2024b on a Dell Latitude 7410 equipped with 16 GB RAM and an Intel Core i5-10210U CPU at 1.6 GHz. Notably, increasing from computing 15 zeros ($n = 30$) to 1000 zeros ($n = 2000$) raises CPU time by only a factor of 10, despite the number of zeros increasing by a far larger factor. This observation provides clear evidence of the efficiency of the algorithm developed for the computation of the zeros.

Acknowledgment. Financial support from Ministerio de Ciencia e Innovación projects PID2021-127252NB-I00 and PID2024-159583NB-I00 (MICIU/ AEI / 10.13039/501100011033 / FEDER, UE) are acknowledged.

Conflict of interest. The authors have no conflict of interest to declare that is relevant to the content of this article.

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