

THE STRUCTURE OF SEQUENCES WITH ZERO-SUM SUBSEQUENCES OF THE SAME LENGTH ON FINITE ABELIAN GROUPS OF RANK TWO

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ABSTRACT. Let G be an additive finite abelian group, and let $\text{disc}(G)$ denote the smallest positive integer t with the property that every sequence S over G with length $|S| \geq t$ contains two nonempty zero-sum subsequences of distinct lengths. In recent years, Gao et al. established the exact value of $\text{disc}(G)$ for all finite abelian groups of rank 2 and resolved the corresponding inverse problem for the group $C_n \oplus C_n$. In this paper, we characterize the structure of sequences S over $G = C_n \oplus C_{nm}$ (where $m \geq 2$) when $|S| = \text{disc}(G) - 1$ and all nonempty zero-sum subsequences of S have the same length.

1. INTRODUCTION

Throughout this paper, let G be an additive finite abelian group. We denote by C_n the cyclic group of n elements, and denote by C_n^r the direct sum of r copies of C_n .

Let p be a prime number. An old conjecture posed by Graham states that if S is a sequence of length $|S| = p$ over C_p such that all nonempty zero-sum subsequences of S have the same length, then S takes at most two distinct terms. In 1976, P. Erdős and E. Szemerédi [1] showed that Graham's conjecture holds for sufficiently large p . In 2010, W. Gao, Y. Hamidoune and G. Wang [2] proved Graham's conjecture in full generality. Furthermore, they extended this result to all positive integers. Subsequently, D. Gryniewicz [9] provided an alternative proof. In 2012, B. Girard [8] posed the problem of determining the smallest integer t , which is denoted by $\text{disc}(G)$, such that every sequence S over G of length $|S| \geq t$ has two nonempty zero-sum subsequences of distinct lengths. Since then, $\text{disc}(G)$ has been systematically studied by numerous authors, and its exact value has been determined for several classes of finite abelian groups, including the groups of rank at most two, the groups of very large exponent compared to $|G|/\exp(G)$, elementary 2-groups, additional special abelian p -groups and certain groups of rank three (see [4, 5, 6, 10]).

On the other hand, Gao et al. [5] considered the inverse problem associated with $\text{disc}(G)$. In particular, they investigate the set of all positive integers t , denoted by $\mathcal{L}_1(G)$, such that there is a sequence S over G with length $\text{disc}(G) - 1$ and all nonempty zero-sum subsequences of S have the same length t . They conjectured that $|\mathcal{L}_1(G)| = 1$

2020 *Mathematics Subject Classification*. 11B75, 11P70.

Key words and phrases. Finite abelian group; Zero-sum subsequence; Inverse problem.

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for any finite abelian group. In 2020, Gao et al. [4] proved that $\mathcal{L}_1(G) = \{\exp(G)\}$ for the abelian groups of rank at most two, $C_{mp^n} \oplus H$ with m being a positive integer and H being a p -group with $D(H) \leq p^n$, the finite abelian groups of very large exponent compared to $|G|/\exp(G)$. Moreover, they disproved this conjecture by demonstrating that $|\mathcal{L}_1(G)| \geq 2$ for certain abelian p -groups. To gain a deeper understanding of the sequence structures on these groups, they have respectively characterized the structure of sequences of length $\text{disc}(G) - 1$ where all nonempty zero-sum subsequences have the same length on the cyclic group C_n and the group $C_n \oplus C_n$. Recently, X. Li and Q. Yin [10] have successfully extended the scope of application of the conjecture to some groups of rank 3, including the group $C_2 \oplus C_{2m} \oplus C_{2mn}$ and $C_3 \oplus C_{6m} \oplus C_{6m}$, where m and n are positive integers with $m|n$. Currently, research on such inverse zero-sum problems remains insufficient, and existing results are largely confined to specific group structures. To overcome this constraint, it is necessary to employ novel methodologies to characterize the structure of extremal sequences over more finite abelian groups in which all nonempty zero-sum subsequences have the same length.

In this paper, we consider more general finite abelian groups G of rank 2, and we mainly characterize the structure of the sequence S when $|S| = \text{disc}(G) - 1$ and all nonempty zero-sum subsequences of S have the same length.

Our main result is as follows.

Theorem 1.1. *Let $G = C_n \oplus C_{nm}$ with $n, m \geq 2$ be integers. Let S be a sequence over G with length $\text{disc}(G) - 1$ and all nonempty zero-sum subsequences of S have the same length. Then there exists a generating set $\{g_1, g_2\}$ of G with $\text{ord}(g_2) = nm$ such that S has one of the following forms.*

- (1) $S = g_2^{2nm-1} \prod_{i=1}^{n-1} (x_i g_2 + g_1)$, where $\text{ord}(g_1) = n$ and $x_1, \dots, x_{n-1} \in [0, nm - 1]$.
- (2) $S = g_1^{n-2} g_2^{2nm-1} (-(n-1)g_1 + g_2)$.
- (3) $S = g_1^{n-1} g_2^{2nm-1}$.
- (4) $S = g_1^{2nm-1} \prod_{i=1}^{n-1} (-y_i g_1 + g_2)$, where $\text{ord}(g_1) = nm$, and $\sum_{i=1}^{n-1} y_i \in [0, n-1]$.
- (5) $S = g_1^{sn+tn-1} g_2^{2nm+n(1-s)-tn-1}$, where $\text{ord}(g_1) = nm$, $s \in [1, m]$ and $t \in [0, m]$.

The rest of the paper is organized as follows. Section 2 provides some basic notation and preliminaries. Section 3 gives the proof of our main result.

2. PRELIMINARIES

Throughout this paper, our notation and terminology are consistent with [3, 7] and we briefly present some key concepts. Let $\mathbb{Z}$ denote the set of integers, and let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$.

Let G be an abelian group. A family $(e_i)_{i \in I}$ of nonzero elements of G is said to be *independent* if

$$\sum_{i \in I} m_i e_i = 0 \quad \text{implies} \quad m_i e_i = 0 \quad \text{for all } i \in I, \quad \text{where } m_i \in \mathbb{Z}.$$

If $I = [1, r]$ and $(e_1, \dots, e_r)$ is independent, then we simply say that $e_1, \dots, e_r$ are independent elements of G . The tuple $(e_i)_{i \in I}$ is called a *basis* if $(e_i)_{i \in I}$ is independent and $\langle \{e_i : i \in I\} \rangle = G$. If $1 < |G| < \infty$, then we have

$$G \cong C_{n_1} \oplus \dots \oplus C_{n_r},$$

where C_n denotes a cyclic group with n elements, $i \in \mathbb{N}$ and $1 < n_1 \mid \dots \mid n_r$. Then $r = r(G)$ is the *rank* of G and $n_r = \exp(G)$ is the *exponent* of G .

We denote by $\mathcal{F}(G)$ the free (abelian, multiplicative) monoid with basis G . An element $S \in \mathcal{F}(G)$ is called a *sequence* over G and will be written in the form

$$S = g_1 \cdot \dots \cdot g_l = \prod_{g \in G} g^{v_g(S)},$$

where $v_g(S) \geq 0$ is called the *multiplicity* of g in S , and we call

- $\text{supp}(S) = \{g \in G \mid v_g(S) > 0\}$ the *support* of S ,
- $|S| = l = \sum_{g \in G} v_g(S) \in \mathbb{N}_0$ the *length* of S ,
- $\sigma(S) = \sum_{i=1}^l g_i = \sum_{g \in G} v_g(S)g \in G$ the *sum* of S ,
- $\Sigma_k(S) = \{\sum_{i \in I} g_i \mid I \subset [1, l] \text{ with } |I| = k\}$ the *set of k -term subsums* of S , for all $k \in \mathbb{N}$,
- $\Sigma_{\geq k}(S) = \bigcup_{j \geq k} \Sigma_j(S)$,
- $\Sigma(S) = \Sigma_{\geq 1}(S)$ the *set of all subsums* of S ,
- $T = \prod_{g \in G} g^{v_g(T)}$ a *subsequence* of S if $v_g(T) \leq v_g(S)$ for all $g \in G$,
- T a *proper subsequence* of S if T is a subsequence of S and $1 \leq |T| < |S|$,
- $ST^{-1} = \prod_{g \in G} g^{v_g(S) - v_g(T)}$ the subsequence obtained from S by deleting T ,
- S a *zero-sum sequence* if $\sigma(S) = 0$,
- S a *zero-sum free sequence* if there is no nonempty zero-sum subsequence of S ,
- S a *minimal zero-sum sequence* if it is zero-sum and has no proper zero-sum subsequence.

For a finite abelian group G , let $D(G)$ denote the Davenport constant of G , which is defined as the smallest positive integer d such that every sequence over G of length at least d has a nonempty zero-sum subsequence.

We next give several lemmas which will be used in the sequel.

Lemma 2.1. [7, Theorem 5.8.3] *Let $G = C_n \oplus C_{nm}$ with n, m be integers. Then $D(G) = n + nm - 1$.*

Lemma 2.2. [6, Theorem 1.2] *Let G be a finite abelian group with $r(G) \leq 2$. Then $\text{disc}(G) = D(G) + \exp(G)$.*

Lemma 2.3. [4, Theorem 1.4] *Let G be a finite abelian group with $r(G) \leq 2$. Then $\mathcal{L}_1(G) = \{\exp(G)\}$.*

Lemma 2.4. [7, Proposition 5.1.4] *Let G be a finite abelian group and let S be a zero-sum free sequence over G with $|S| = D(G) - 1$. Then $|\Sigma(S)| = |G| - 1$.*

Lemma 2.5. [11, Theorem 3.2] *Let $G = C_{n_1} \oplus C_{n_2}$ be a finite abelian group with $1 < n_1 \mid n_2$ and S be a sequence over G . Then S is a minimal zero-sum sequence of length $|S| = n_1 + n_2 - 1$ if and only if S has one of the following forms:*

(1)

$$S = e_j^{\text{ord}(e_j)-1} \prod_{\nu=1}^{\text{ord}(e_k)} (x_\nu e_j + e_k),$$

where $\{e_1, e_2\}$ is a basis of G with $\text{ord}(e_i) = n_i$ for $i \in [1, 2]$, $\{j, k\} = \{1, 2\}$, $x_1, \dots, x_{\text{ord}(e_k)} \in [0, \text{ord}(e_j) - 1]$ and $x_1 + \dots + x_{\text{ord}(e_k)} \equiv 1 \pmod{\text{ord}(e_j)}$.

(2)

$$S = g_1^{s n_1 - 1} \prod_{\nu=1}^{n_2 + (1-s)n_1} (-x_\nu g_1 + g_2),$$

where $\{g_1, g_2\}$ is a generating set of G with $\text{ord}(g_2) = n_2$, $x_1, \dots, x_{n_2 + (1-s)n_1} \in [0, n_1 - 1]$ and $x_1 + \dots + x_{n_2 + (1-s)n_1} = n_1 - 1$, $s \in [1, n_2/n_1]$ and either $s = 1$ or $n_1 g_1 = n_1 g_2$.

3. PROOF OF THEOREM 1.1

In this section, we present the proof of our main result. To begin with, we establish a crucial lemma.

Lemma 3.1. *Let S be a sequence over a finite abelian group G of length $|S| = \text{disc}(G) - 1$, where all nonempty zero-sum subsequences of S have the same length. Suppose T is a nonempty zero-sum subsequence of S . Then*

$$\text{supp}(T) \cap \Sigma_{\geq 2}(ST^{-1}) = \emptyset.$$

Proof. Assume to the contrary that there exists a subsequence $T' \mid ST^{-1}$ with $|T'| \geq 2$ such that $\sigma(T') = g$, where $g \mid T$. Then $T'Tg^{-1}$ is a zero-sum subsequence of S with length $|T'Tg^{-1}| > |T|$, which is a contradiction. Hence, $\text{supp}(T) \cap \Sigma_{\geq 2}(ST^{-1}) = \emptyset$. $\square$

We are now in position to provide the proof for our main result.

Proof of Theorem 1.1. By Lemmas 2.1 and 2.2, we have $|S| = \text{disc}(G) - 1 = \text{D}(G) + \exp(G) - 1 = n + 2nm - 2$. And it follows from Lemma 2.3 that all nonempty zero-sum subsequences of S have the same length nm .

Since $|S| = n + 2nm - 2 > \text{D}(G) = n + nm - 1$, there exists a zero-sum subsequence T of S with length nm and $0 \nmid S$. Then $|ST^{-1}| = n + nm - 2 = \text{D}(G) - 1$ and ST^{-1} is zero-sum free. It follows from Lemma 2.4 that

$$\Sigma(ST^{-1}) = G \setminus \{0\}.$$

It is easy to see that $ST^{-1}(-\sigma(ST^{-1}))$ is a minimal zero-sum sequence of length $nm + n - 1 = D(G)$. And by Lemma 2.5 we obtain that $ST^{-1}(-\sigma(ST^{-1}))$ has one of the following forms:

$$(3.1) \quad ST^{-1}(-\sigma(ST^{-1})) = e_u^{\text{ord}(e_u)-1} \prod_{i=1}^{\text{ord}(e_v)} (x_i e_u + e_v),$$

where $\{e_1, e_2\}$ is a basis of G with $\text{ord}(e_1) = n$ and $\text{ord}(e_2) = nm$, $\{u, v\} = \{1, 2\}$, $x_1, \dots, x_{\text{ord}(e_v)} \in [0, \text{ord}(e_u) - 1]$ and $x_1 + \dots + x_{\text{ord}(e_v)} \equiv 1 \pmod{\text{ord}(e_u)}$.

$$(3.2) \quad ST^{-1}(-\sigma(ST^{-1})) = g_1^{sn-1} \prod_{i=1}^{nm+(1-s)n} (-y_i g_1 + g_2),$$

where $\{g_1, g_2\}$ is a generating set of G with $\text{ord}(g_2) = nm$, $y_1, \dots, y_{nm+(1-s)n} \in [0, n-1]$ and $y_1 + \dots + y_{nm+(1-s)n} = n-1$, $s \in [1, m]$ and either $s = 1$ or $ng_1 = ng_2$.

We now divide the remaining proof into the following four cases.

Case 1. $ST^{-1}(-\sigma(ST^{-1}))$ is of the form (3.1) with $u = 1$ and $v = 2$. It follows that

$$ST^{-1}(-\sigma(ST^{-1})) = e_1^{n-1} \prod_{i=1}^{nm} (x_i e_1 + e_2),$$

where $x_1, \dots, x_{nm} \in [0, n-1]$ and $x_1 + \dots + x_{nm} \equiv 1 \pmod{n}$.

Subcase 1.1. $ST^{-1} = e_1^{n-1} \prod_{i=1}^{nm-1} (x_i e_1 + e_2)$. If $x_i \neq x_j$ for some $i \neq j \in [1, nm-1]$, then

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, e_1\}.$$

By Lemma 3.1 and $0 \nmid S$, we obtain that T is of the form e_1^{nm} . Therefore, e_1^n is a zero-sum subsequence of S with length $n < nm$, which is a contradiction.

Next we assume that $x_1 = \dots = x_{nm-1}$, i.e. $ST^{-1} = e_1^{n-1} (x_1 e_1 + e_2)^{nm-1}$. Replacing $x_1 e_1 + e_2$ with e_2 , we have that $ST^{-1} = e_1^{n-1} e_2^{nm-1}$ and

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, e_1, e_2\}.$$

If $e_1 \mid T$, then e_1^n is a zero-sum subsequence of S of length $n < nm$, a contradiction. By Lemma 3.1 and $0 \nmid S$, we obtain that T is of the form e_2^{nm} . Therefore,

$$S = e_1^{n-1} e_2^{2nm-1}.$$

Replacing e_1 with g_1 and e_2 with g_2 , $\{g_1, g_2\}$ is a generating set of G where $\text{ord}(g_1) = n$ and $\text{ord}(g_2) = nm$. Thus S is of the form (1).

Subcase 1.2. $ST^{-1} = e_1^{n-2} \prod_{i=1}^{nm} (x_i e_1 + e_2)$ with $\sum_{i=1}^{nm} x_i \equiv 1 \pmod{n}$. It follows that $x_i \neq x_j$ for some $i \neq j \in [1, nm]$. Without loss of generality, we may assume that $0 \leq x_1 \leq \dots \leq x_{nm} \leq n-1$. If x_i, x_j, x_k are pairwise distinct for some $i, j, k \in [1, nm]$ or $x_i - x_j \in [2, n-2]$ for some $i, j \in [1, nm]$, then

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\}.$$

A contradiction with Lemma 3.1 and $0 \nmid S$.

Next we assume that $x_i - x_j \in \{-1, 0, 1\}$ for every $i, j \in [1, nm]$. We may assume that $x_1 = \cdots = x_l = x_{l+1} - 1 = \cdots = x_{nm} - 1$, $l \in [1, nm - 1]$. Thus $\sum_{i=1}^{nm} x_i = nm x_1 + (nm - l) \equiv 1 \pmod{n}$, it deduces that $l = nm - tn - 1$, $t \in [0, m - 1]$. So $ST^{-1} = e_1^{n-2}(x_1 e_1 + e_2)^{nm-tn-1}((x_1 + 1)e_1 + e_2)^{tn+1}$. Replacing $x_1 e_1 + e_2$ with e_2 , we have that $ST^{-1} = e_1^{n-2} e_2^{nm-tn-1} (e_1 + e_2)^{tn+1}$. Then

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, e_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we obtain that T is of the form e_2^{nm} . If $t \geq 1$ and $n \geq 3$, then $e_1^{n-2}(e_1 + e_2)^2 e_2^{nm-2}$ is a zero-sum subsequence of S with length $nm + n - 2 > nm$, which is a contradiction. Therefore $t = 0$ or $n = 2$. If $t = 0$, then

$$S = e_1^{n-2} e_2^{2nm-1} (e_1 + e_2).$$

Replacing e_1 with g_1 and e_2 with g_2 , $\{g_1, g_2\}$ is a generating set of G where $\text{ord}(g_1) = n$ and $\text{ord}(g_2) = nm$. Thus S is of the form (1).

If $n = 2$, then

$$S = e_2^{4m-2t-1} (e_1 + e_2)^{2t+1}.$$

Replacing $e_1 + e_2$ with g_1 and e_2 with g_2 , $\{g_1, g_2\}$ is a generating set of G where $\text{ord}(g_1) = \text{ord}(g_2) = nm$. Thus S is of the form (5).

Case 2. $ST^{-1}(-\sigma(ST^{-1}))$ is of the form (3.1) with $u = 2$ and $v = 1$. It follows that

$$ST^{-1}(-\sigma(ST^{-1})) = e_2^{nm-1} \prod_{i=1}^n (x_i e_2 + e_1),$$

where $x_1, \dots, x_n \in [0, nm - 1]$ and $x_1 + \cdots + x_n \equiv 1 \pmod{nm}$. If $m = 1$, then it reduces to Case 1. Therefore we assume that $m \geq 2$.

Subcase 2.1. $ST^{-1} = e_2^{nm-1} \prod_{i=1}^{n-1} (x_i e_2 + e_1)$. If $x_i \neq x_j$ for some $i \neq j \in [1, n - 1]$, then

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, e_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form e_2^{nm} . Therefore

$$S = e_2^{2nm-1} \prod_{i=1}^{n-1} (x_i e_2 + e_1).$$

Replacing e_1 with g_1 and e_2 with g_2 , $\{g_1, g_2\}$ is a generating set of G where $\text{ord}(g_1) = n$ and $\text{ord}(g_2) = nm$. Thus S is of the form (1).

Next we assume that $x_1 = \cdots = x_{n-1}$, i.e. $ST^{-1} = e_2^{nm-1} (x_1 e_2 + e_1)^{n-1}$. If $x_1 \pmod{m} = 0$, it is easy to see that $\text{ord}(x_1 e_2 + e_1) = n$. By replacing $x_1 e_2 + e_1$ with e_1 , we have $ST^{-1} = e_2^{nm-1} e_1^{n-1}$ and it reduces to Case 1. Next we suppose $x_1 \pmod{m} \in [1, m - 1]$. Then

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, e_2, x_1 e_2 + e_1\}.$$

If $x_1 \pmod{m} \in [2, m - 1]$ and $(x_1 e_2 + e_1) \mid T$, then $(x_1 e_2 + e_1)^n e_2^{nm - (nx_1 \pmod{nm})}$ is a zero-sum subsequence of S with length $nm + n - (nx_1 \pmod{nm}) < nm$, which is a

contradiction. By Lemma 3.1 and $0 \nmid S$, we obtain that T is of the form e_2^{nm} . Therefore

$$S = e_2^{2nm-1}(x_1e_2 + e_1)^{n-1}.$$

Replacing e_1 with g_1 and e_2 with g_2 , $\{g_1, g_2\}$ is a generating set of G where $\text{ord}(g_1) = n$ and $\text{ord}(g_2) = nm$. Thus S is of the form (1).

Next we suppose that $x_1 \pmod{m} = 1$. Replacing $x_1e_1 + e_2$ with $e_1 + e_2$, we have $ST^{-1} = e_2^{nm-1}(e_1 + e_2)^{n-1}$. By Lemma 3.1 and $0 \nmid S$, we obtain that T is of the form $e_2^{nm-tn}(e_1 + e_2)^{tn}$ for $t \in [0, m]$. Therefore

$$S = e_2^{2nm-tn-1}(e_1 + e_2)^{n+tn-1}.$$

Replacing $e_1 + e_2$ with g_1 and e_2 with g_2 , $\{g_1, g_2\}$ forms a generating set of G where $\text{ord}(g_1) = \text{ord}(g_2) = nm$. Thus S is of the form (5).

Subcase 2.2. $ST^{-1} = e_2^{nm-2} \prod_{i=1}^n (x_i e_2 + e_1)$ with $\sum_{i=1}^n x_i \equiv 1 \pmod{nm}$. It follows that $x_i \neq x_j$ for some $i \neq j \in [1, n]$. Without loss of generality, we may assume that $0 \leq x_1 \leq \dots \leq x_n \leq nm - 1$. If x_i, x_j, x_k are pairwise distinct for some $i, j, k \in [1, n]$, or if $x_i - x_j \in [2, nm - 2]$ for some $i, j \in [1, n]$, we infer that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\}.$$

A contradiction with Lemma 3.1 and $0 \nmid S$.

Next we assume that $x_i - x_j \in \{-1, 0, 1\}$ for every $i, j \in [1, n]$. We may assume that $x_1 = \dots = x_l = x_{l+1} - 1 = \dots = x_n - 1$, $l \in [1, n-1]$. Thus $\sum_{i=1}^n x_i = nx_1 + n - l \equiv 1 \pmod{nm}$, it deduces that $l = n - 1$ and $m \mid x_1$. So $ST^{-1} = e_2^{nm-2}(x_1e_2 + e_1)^{n-1}((x_1 + 1)e_2 + e_1)$. By replacing $x_1e_2 + e_1$ with e_1 , we have $ST^{-1} = e_2^{nm-2}e_1^{n-1}(e_2 + e_1)$. Thus it reduces to Case 1 and we are done.

Case 3. $ST^{-1}(-\sigma(ST^{-1}))$ is of the form (3.2) with $ng_1 \neq ng_2$. By (3.2), we have $s = 1$. So we can write

$$ST^{-1}(-\sigma(ST^{-1})) = g_1^{n-1} \prod_{i=1}^{nm} (-y_i g_1 + g_2).$$

Note that $y_1, \dots, y_{nm} \in [0, n-1]$ and $y_1 + \dots + y_{nm} = n - 1$.

Let $\varphi : G \rightarrow G/\langle g_2 \rangle$ denote the canonical epimorphism. Since $\text{ord}(g_2) = nm$ and $\{g_1, g_2\}$ is a generating set of G , we have $n = \text{ord}(\varphi(g_1)) \mid \text{ord}(g_1)$. If $\text{ord}(g_1) = n$, then it reduces to Case 1. So we may assume that $n < \text{ord}(g_1) \leq nm$. Suppose $g_1 = xe_1 + t_0g_2$ for some integer $t_0 \in [0, nm - 1]$ and $x \in [0, n - 1]$, where $\{e_1, g_2\}$ is a basis of G . Then $ng_1 = t_0ng_2$. Since $\text{ord}(g_2) = nm$, we infer that there exists $t' \in [0, m - 1]$ such that

$$ng_1 = t_0ng_2 = t'ng_2.$$

Since $ng_1 \neq ng_2$, we obtain that $t' \in [2, m - 1]$, it deduces that $m \geq 3$.

Subcase 3.1. $ST^{-1} = g_1^{n-2} \prod_{i=1}^{nm} (-y_i g_1 + g_2)$. Without loss of generality, we assume that $n - 1 \geq y_1 \geq \dots \geq y_j > y_{j+1} = \dots = y_{nm} = 0$. Since $y_1 + \dots + y_{nm} = n - 1$, we have $j \in [1, n - 1]$.

If $j \in [2, n - 2]$, we obtain that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\}.$$

A contradiction with Lemma 3.1 and $0 \nmid S$.

If $j = 1$, i.e. $ST^{-1} = g_1^{n-2} g_2^{nm-1} (-(n-1)g_1 + g_2)$, then we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we obtain that T is of the form g_2^{nm} . Therefore

$$S = g_1^{n-2} g_2^{2nm-1} (-(n-1)g_1 + g_2).$$

Thus S is of the form (2).

If $j = n-1$, i.e. $ST^{-1} = g_1^{n-2} g_2^{nm-n+1} (-g_1 + g_2)^{n-1}$, then we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, -g_1 + g_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form $(-g_1 + g_2)^{nm}$. Furthermore, $g_2^{(t'-1)n} (-g_1 + g_2)^n$ is a zero-sum subsequence of S with length $t'n < nm$, which is a contradiction.

Subcase 3.2. $ST^{-1} = g_1^{n-1} \prod_{i=1}^{nm-1} (-y_i g_1 + g_2)$. Without loss of generality, we assume that $n-1 \geq y_1 \geq \dots \geq y_j > y_{j+1} = \dots = y_{nm-1} = 0$. Since $y_1 + \dots + y_{nm} = n-1$, we have $j \in [0, n-1]$.

If $j \in [1, n-1]$, we may first assume that $\sum_{i=1}^j y_i = n-1$, so we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\}.$$

A contradiction with Lemma 3.1 and $0 \nmid S$.

Next we consider that $\sum_{i=1}^j y_i < n-1$, then we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_1\}.$$

Again by Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form g_1^{nm} . Furthermore, $g_1^{nm-n+\sum_{i=1}^j y_i} \prod_{i=1}^{t'n} (-y_i g_1 + g_2)$ is a zero-sum subsequence of S with length $nm + t'n - n + \sum_{i=1}^j y_i > nm$, which is a contradiction.

If $j = 0$, i.e. $ST^{-1} = g_1^{n-1} g_2^{nm-1}$, it follows that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_1, g_2\}.$$

If $g_1 \mid T$, then $g_1^n g_2^{nm-t'n}$ is a zero-sum subsequence of S with length $nm - t'n + n < nm$, which is a contradiction. By Lemma 3.1 and $0 \nmid S$, we obtain that T is of the form g_2^{nm} . Therefore

$$S = g_1^{n-1} g_2^{2nm-1}.$$

Thus S is of the form (3).

Case 4. $ST^{-1}(-\sigma(ST^{-1}))$ is of the form (3.2) with $ng_1 = ng_2$. It follows that

$$ST^{-1}(-\sigma(ST^{-1})) = g_1^{sn-1} \prod_{i=1}^{nm+(1-s)n} (-y_i g_1 + g_2),$$

where $y_1, \dots, y_{nm+(1-s)n} \in [0, n-1]$ and $y_1 + \dots + y_{nm+(1-s)n} = n-1$.

Similar to the proof of Case 3, we have $n \mid \text{ord}(g_1)$. Since $ng_1 = ng_2$ and $\text{ord}(g_2) = nm$, we have $\frac{\text{ord}(g_1)}{n} = \text{ord}(ng_1) = \text{ord}(ng_2) = m$, and so it deduces that $\text{ord}(g_1) = nm$. It is easy to see that $\text{ord}(-g_1 + g_2) = n$.

Subcase 4.1. $ST^{-1} = g_1^{sn-2} \prod_{i=1}^{nm+(1-s)n} (-y_i g_1 + g_2)$. Without loss of generality, we assume that $n-1 \geq y_1 \geq \dots \geq y_j > y_{j+1} = \dots = y_{nm+(1-s)n} = 0$. Since $y_1 + \dots + y_{nm+(1-s)n} = n-1$, we have $j \in [1, n-1]$.

If $j \in [2, n-2]$, we obtain that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\}.$$

A contradiction with Lemma 3.1 and $0 \nmid S$.

Suppose $j = 1$, i.e. $ST^{-1} = g_1^{sn-2} g_2^{nm+n(1-s)-1} (-(n-1)g_1 + g_2)$. If $s \geq 2$, we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\}.$$

A contradiction with Lemma 3.1 and $0 \nmid S$.

If $s = 1$, i.e. $ST^{-1} = g_1^{n-2} g_2^{nm-1} (-(n-1)g_1 + g_2)$. It follows that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we obtain that T is of the form g_2^{nm} . Therefore

$$S = g_1^{n-2} g_2^{2nm-1} (-(n-1)g_1 + g_2).$$

Thus S is of the form (2).

If $j = n-1$, i.e. $ST^{-1} = g_1^{sn-2} g_2^{nm-sn+1} (-g_1 + g_2)^{n-1}$, then by replacing g_1 with e_2 and $-g_1 + g_2$ with e_1 , we see that $\{e_1, e_2\}$ is a basis of G . It follows that $ST^{-1} = e_1^{n-1} e_2^{nm-sn+1} (e_1 + e_2)^{sn-2}$, and this reduces to Case 1.

Subcase 4.2. $ST^{-1} = g_1^{sn-1} \prod_{i=1}^{nm+(1-s)n-1} (-y_i g_1 + g_2)$. Without loss of generality, we assume that $n-1 \geq y_1 \geq \dots \geq y_j > y_{j+1} = \dots = y_{nm+(1-s)n-1} = 0$. Since $y_1 + \dots + y_{nm+(1-s)n} = n-1$, we have $j \in [0, n-1]$.

If $j \in [1, n-2]$, we first assume that $\sum_{i=1}^j y_i = n-1$ and $s \leq m-1$, it then follows that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\}.$$

A contradiction with Lemma 3.1 and $0 \nmid S$.

Next, considering the cases where $\sum_{i=1}^j y_i < n-1$ or $s = m$, we infer that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_1\}.$$

By Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form g_1^{nm} . If $s < m$, then $g_1^{nm-n+\sum_{i=1}^j y_i} \prod_{i=1}^n (-y_i g_1 + g_2)$ is a zero-sum subsequence of S with length $nm + \sum_{i=1}^j y_i > nm$, which is a contradiction. Therefore $s = m$ and

$$S = g_1^{2nm-1} \prod_{i=1}^{n-1} (-y_i g_1 + g_2).$$

Thus S is of the form (4).

If $j = 0$, i.e. $ST^{-1} = g_1^{sn-1} g_2^{nm+n(1-s)-1}$, it follows that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_1, g_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form $g_1^{tn} g_2^{nm-tn}$ for $t \in [0, m]$. Therefore

$$S = g_1^{sn+tn-1} g_2^{2nm+n(1-s)-tn-1}.$$

Thus S is of the form (5).

Suppose $j = n - 1$, i.e. $ST^{-1} = g_1^{sn-1}(-g_1 + g_2)^{n-1} g_2^{nm-sn}$. Replacing g_1 with e_2 and $-g_1 + g_2$ with e_1 , we see that $\{e_1, e_2\}$ is a basis of G . It follows that $ST^{-1} = e_1^{n-1} e_2^{nm-sn} (e_1 + e_2)^{sn-1}$, which reduces to Case 1. $\square$

Acknowledgements. This work was supported by the Sichuan Science and Technology Program (Grant No. 2024NSFSC2051).

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