

Breaking gyrochronology through the collapse of coronal winds

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ABSTRACT

Gyrochronology, a method for dating aged field stars ($\gtrsim$ a few Gyr) based on their rotation rate, has recently been shown to fail for many stars older than the sun. The explanation most often put forth is that a shutdown or mode change in the stellar dynamo leads to a sharp decrease in angular momentum loss in magnetized coronal winds. In this paper, we explore an alternate possibility, namely a collapse of the wind itself through a reduction of coronal heating. We show that in the low coronal temperature (T_0) limit, even at solar-like low rotation rates (Ω) and coronal magnetic field strength (B_{r0}), magnetocentrifugal effects are important and preclude expression of the mass and angular momentum loss rates as power-laws of T_0 or Ω when T_0 drops below $\simeq 1.5$ MK. Mass loss is found to scale linearly with power input into the wind at all coronal temperatures. Introducing an ad hoc power law relationship $T_0 \propto B_{r0}^\sigma$ while retaining the “standard” dynamo relationship $B_{r0} \propto \Omega$, we show that reproducing the observed break in gyrochronology requires an exponent $\sigma \gtrsim 1.5$, with which is associated a drop by over 3 orders of magnitude in power input into the quiet corona. This appears physically unrealistic, given current observations of chromospheric and coronal non-thermal emission in aged solar-type stars.

1. INTRODUCTION

In a short and epoch-making paper, Skumanich (1972) gathered observations of rotation rates in main-sequence solar-type stars of known ages, and argued that their rotation rate decreases as the inverse square root of their age, a relationship since known as Skumanich’s Law. He also showed that emission in the core of the calcium H and K lines, a known proxy of surface magnetism, also decreased with stellar age following a similar time dependence. This remarkable $t^{-1/2}$ relationship has been harnessed to establish the field of gyrochronology (see, e.g. Barnes 2003; Işık et al. 2023, and reference therein), namely using rotation rate as a proxy for stellar age.

The Skumanich Law has withstood the test of time, and is now considered to hold from ages of ~ 1 –2 Gyr to the solar age ($\simeq 4.5$ Gyr), with the lower bound depending on stellar mass to some extent (Gallet & Bouvier 2015; Lanzafame & Spada 2015, and references therein). It therefore came as a surprise when a number of solar-type field stars slightly older than the sun, when dated through asteroseismology, were found to rotate significantly faster than predicted by Sku-

manich’s Law (van Saders et al. 2016; Metcalfe et al. 2023; Saunders et al. 2024), suggesting a rapid decrease of the angular momentum loss rate ($\dot{J}$) beyond the solar age, as compared to the expectation based on the extrapolation of Skumanich’s Law.

One possible explanation for this sudden drop in $\dot{J}$, and that emphasized thus far (e.g., Metcalfe et al. 2023, and references therein), is a shutdown or mode change in the large-scale dynamo producing the global coronal magnetic field responsible for enhancing angular momentum loss over that characterizing unmagnetized coronal winds. This would imply that the current solar dynamo is running very close to criticality (e.g., Metcalfe et al. 2016). An alternate, related explanation not involving a full shutdown of the large-scale dynamo could be a bifurcation-like switch of its mode of operation, leading to a dominance of higher-order multipoles over the dipolar mode, with a potential large impact on the angular momentum loss rate (Finley & Matt 2018; Garraffo et al. 2018; Metcalfe et al. 2023, and references therein).

In this paper we explore an alternate means of shutting off the angular momentum loss rate, namely a collapse of coronal wind triggered by reduced energy input to coronal heating. The idea that a drop in coronal temperature can lower angular momentum loss in coronal wind through a strong reduction of the asso-

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ciated mass loss rate has already been explored using various empirical and semi-empirical models (see, e.g., [Pantolmos & Matt 2017](#); [Ó Fionnagáin & Vidotto 2018](#)), which have demonstrated the potential viability of this explanatory framework.

The remainder of the paper is organized as follows: we begin (§2) by recalling in some detail the derivation of Skumanich’s Law from wind theory, with emphasis on the various physical assumptions required to arrive at $\Omega \propto t^{-1/2}$. We turn in §3 to the dynamics of winds at coronal base temperatures approaching the hydrostatic corona limit, and on this basis we examine in §4 under which conditions Skumanich’s law can be broken in solar-type stars slightly older than the sun. In §5 we revisit the results of §4 from the point of view of wind energetics and coronal heating, seeking to determine whether scenarios succeeding in breaking Skumanich’s Law are energetically consistent with current ideas and observational constraints. We conclude in §6 by summarizing our main results, and placing them in the context of alternate, dynamo-based explanations for the break of gyrochronology in middle-aged solar-type stars.

2. MAGNETIZED CORONAL WINDS AND THE SKUMANICH $T^{-1/2}$ LAW

Magnetized wind outflows emanating from the corone of solar type stars carry away angular momentum, and thus exert a torque Γ that slows the star’s rotation ([Schatzman 1962](#); [Weber & Davis 1967](#); [Mestel 1968](#)). Assume for now that the stellar moment of inertia (I) remains constant on the main-sequence and that the star rotates as a solid-body at angular velocity Ω ; the stellar angular momentum content is then simply $J = I\Omega$, with $\dot{J} = -\Gamma$ (here and in what follows, the dot indicates time derivative). The specific case $\Gamma \propto \Omega^3$ (see [Durney 1972](#), and §2.2 below) then yields $\dot{\Omega} \propto -\Omega^3$, which integrates to

$$\frac{1}{\Omega^2(t)} - \frac{1}{\Omega_0^2} \propto t - t_0, \quad (1)$$

with initial condition $\Omega_0 \equiv \Omega(t_0)$ on the ZAMS (t_0). In the late phases of spindown, i.e., $t \gg t_0$ and $\Omega(t) \ll \Omega_0$, Skumanich’s Law is recovered:

$$\Omega(t) \propto t^{-1/2}. \quad (2)$$

The missing constants of proportionality in the above expressions, including the $\Gamma \propto \Omega^3$ Ansatz, is of course where physics is hiding; they are determined by stellar structure, setting the stellar moment of inertia, by dynamo theory, which yields the global coronal magnetic field strength and multipole configuration, and by coronal wind speeds and associated mass loss, ultimately set by coronal heating.

Skumanich’s Law was established observationally, but it can also be derived from coronal wind theory under a set of specific physical assumptions. As first shown by [Durney \(1972\)](#), three distinct assumptions lead to Skumanich’s Law:

1. A “dynamo relationship” relating the base coronal magnetic field B_0 linearly to the rotation rate Ω ;
2. A coronal wind that is primarily thermally-driven;
3. A constant stellar moment of inertia (requiring solid-body rotation on the main-sequence).

In the remainder of this section we examine critically the physical underpinnings of these three assumptions.

2.1. The dynamo relationship

The “dynamo relationship” relating a star’s global large-scale surface magnetic field strength to its rotation rate is a crucial input ingredient for any rotational evolution model for solar-type stars. The sun offers an obvious anchor point to establish such a relationship, but even there one runs into major difficulties: at this writing there still exists no consensus model for the solar dynamo, although many acceptable models can be found in the literature ([Charbonneau 2020](#)). While differential rotation is usually considered a key ingredient for the generation of the toroidal component of the sun’s large-scale magnetic field, no such consensus exists with regards to the inductive mechanism amplifying (and reversing) the sun’s large-scale dipole ([Charbonneau & Sokoloff 2023](#)); nor regarding the specific workings of nonlinear backreaction of the magnetic field on the internal flows, responsible for stabilizing the solar cycle amplitude to some more or less stable mean value, or for the mechanism(s) —stochastic or deterministic— responsible for producing the observed cycle-to-cycle amplitude variations ([Charbonneau 2020](#); [Karak 2023](#)).

Even if a consensus did exist on a “solar dynamo model”, extending such a model from the sun to younger or older solar-type stars also faces a number of conceptual difficulties. Even if stellar masses —and thus overall internal structure— remain close to solar, one must specify how large-scale flows such as differential rotation and meridional circulation (the latter also playing an important role in many contemporary solar dynamo models) vary with rotation rate and convective luminosity; also how turbulent induction and diffusion, set by the properties of turbulent convection, vary with rotation and luminosity. Theory and numerical simulations can (and do) provide useful guidance (see, e.g. [Kitchatinov & Rüdiger 1999](#); [Küker et al.](#)

2011; Simard et al. 2016; Varela et al. 2016; Warnecke 2018; Brun et al. 2022, and references therein), but there remains still far too many free parameters and unknowns to allow a physically sound extrapolation from the sun.

Observationally, there is ample indication that stellar magnetic activity increases with increasing rotation rate. Skumanich (1972) already noted that emission in the core of the H and K lines of Calcium, a known proxy of magnetic activity on the sun, also decreased as $t^{-1/2}$, on par with rotation (see also Brown et al. 2022). Observations of stellar coronal X-Ray emission, another proxy of surface magnetism, as well as direct measurements of average surface magnetic fields, are also roughly consistent with a near-linear increase of surface magnetic activity with rotation, holding in slow rotators ($\Omega/\Omega_\odot \lesssim 10$) but flattening out at higher rotation rates (Reiners et al. 2022; Işık et al. 2023, and references therein). In such slow rotators, the data is consistent with a power-law relationship $\langle B_* \rangle \propto \text{Ro}^{-1.25}$, where the Rossby number $\text{Ro} \propto \Omega^{-1}$ at fixed convective turnover time.

With large-scale flows and turbulent induction ultimately driven by the rotational influences on turbulent convection, a dependence of the dynamo-generated global magnetic field strength on rotation is certainly expected; yet the causal chain between the dynamo and CaHK or X-Ray emission is long and complex. In particular, on the sun non-thermal emission is dominated by active regions and smaller photospheric magnetic structures, while what matters for angular momentum loss in coronal winds is the strength of the lowest order magnetic multipoles, which determine the structure of the coronal magnetic field on scales of order of the solar radius and larger (Réville et al. 2015; See et al. 2019, and references therein).

With all these caveat firmly in mind, in everything that follows we retain the (common) dynamo relationship $B \propto \Omega$ as a zeroth-order approximation for middle-age slowly rotating solar-type stars, and proceed.

2.2. The Weber-Davis MHD wind model

We adopt in what follow the magnetohydrodynamical coronal wind model of Weber & Davis (1967) (hereafter WD; see also Belcher & MacGregor 1976 and Sakurai 1985). Formulated in spherical polar coordinates (r, θ, ϕ) , this is defined as an axisymmetric ($\partial/\partial\phi = 0$) equatorial plane ($\theta = \pi/2$) steady-state ($\partial/\partial t = 0$) ideal MHD wind model, in $GM_\odot/r^2$ gravity and with an imposed (split-)monopolar radial field, $B_r(r) \propto 1/r^2$. The latter is actually a fairly realistic representation of the observed solar coronal magnetic field at activity minimum, where the axisymmetric belt

of coronal loops straddling the equator is pulled open radially by the outflowing solar wind a few solar radii above the photosphere (see, e.g., Réville et al. 2015, and references therein). Energy input is effectively achieved by imposing a polytropic relationship $p \propto \rho^\alpha$, with $1 \leq \alpha \leq 5/3$ for an ideal gas. As detailed in §5 below, the polytropic assumption amounts to a specific profile of volumetric heating through the corona and wind; in the absence of a consensus model for coronal heating, the polytropic approximation is in fact commonly used in the majority of extant MHD coronal wind models formulated in two and three spatial dimensions (see, e.g. Washimi & Shibata 1993; Keppens & Goedbloed 1999; Vidotto et al. 2009; Réville et al. 2015; Finley & Matt 2018). The governing equations are solved above a reference radius r_0 ($= 1.15R_\odot$ in all that follows), taken to correspond to the heliocentric radius of peak temperature in the low corona, where density ρ_0 and temperature T_0 are specified.

Its geometrical simplification notwithstanding, the WD model captures in a dynamically consistent manner the magnetocentrifugal driving of the winds, as well as the torque it exerts on the photosphere. Its predictions are well-validated against in situ interplanetary wind measurements at Earth’s orbit (Pizzo et al. 1983), and compare well to numerical simulations using more realistic magnetic field configurations (e.g., Réville et al. 2015). In such 2D or 3D models, as long as coronal arcades are opened by the solar wind at heliocentric radii much smaller than the Alfvén radius, the torque density remains close to the prediction of the WD model. However, the mass loss rate is reduced in proportion to the fraction of the solar surface threaded by magnetic fieldlines closing back onto the photosphere (see Pantolmos & Matt 2017; Ahuir et al. 2020, and references therein).

The steady-state wind solutions are characterized by three invariants, which define conserved quantities along streamlines: the mass flux $r^2\rho u_r$ from the continuity equation, the magnetic flux r^2B_r from the r -component of the induction equation, and a third invariant, $r(u_r B_\phi - u_\phi B_r)$ arising from the ϕ -component of induction equation. These invariants allow to integrate the ϕ -component of the momentum equation to yield a fourth invariant:

$$L = ru_\phi - \frac{rB_\phi B_r}{\mu_0\rho u_r} \quad (3)$$

expressing conservation of angular momentum carried away by the wind, as comprised of two contributions: the specific angular momentum of the outflowing plasma (first term on RHS), and the magnetic torque density (second term on RHS).

Introducing the Alfvén speed $\mathbf{A} = \mathbf{B}/\sqrt{\mu_0\rho}$, and making good use of (3), the aforementioned third invariant can be recast in the form:

$$u_\phi = \Omega r \frac{(u_r^2 L / \Omega r^2) - A_r^2}{u_r^2 - A_r^2} \quad (4)$$

which diverges at the Alfvén radius r_A (where $u_r = A_r$), unless the numerator also vanishes, which then requires:

$$L = \Omega r_A^2 \quad (5)$$

(Weber & Davis 1967; Mestel 1968). This is a truly remarkable result: it is *as if* magnetic stresses force the outflowing plasma to co-rotate with the photosphere out to the Alfvén radius r_A , after which it is released to a Keplerian orbit. With $r_A \gg r_0$ even in a slowly rotating and (relatively) weakly magnetized star as the sun, the loss of angular momentum is vastly enhanced, over the specific angular momentum that would be extracted by an unmagnetized wind of the same mass loss rate.

We follow in this paper the solution procedure described in Belcher & MacGregor (1976); see also Charbonneau (1995), §4, as well as Appendix B herein, for details.

For solar parameters, magnetocentrifugal driving of the wind is weak, contributing only a few percent of the thermal pressure force propelling the wind; however, magnetic torques still completely dominates $\dot{J}$, via the second term on the RHS of eq. (3). The time rate of change of angular momentum is given by

$$\dot{J} = \frac{2}{3} \dot{M} \times L = -\frac{8\pi}{3} \rho_A u_{rA} \Omega r_A^4, \quad (6)$$

where the factor 2/3 results from the projection and integration of the equatorial WD solution over spherical shells, and the second equality from (5) and evaluating the mass flux at the Alfvén radius r_A ; at that location, $u_{rA} = A_{rA} = B_{rA}/\sqrt{\mu_0\rho_A}$; conservation of magnetic flux also imposes $r_0^2 B_{r0} = r_A^2 B_{rA}$, so that eq. (6) can be rewritten as

$$\dot{J} = -\frac{8\pi}{3\mu_0} r_0^4 B_{r0}^2 \Omega A_{rA}^{-1}. \quad (7)$$

At this point, upon assuming a “dynamo relationship” $B_{r0} \propto \Omega$ and a fixed moment of inertia, eq. (7) yields $\dot{\Omega} \propto -\Omega^3$, as required to match Skumanich’s Law, *provided* the Alfvén speed A_{rA} at the Alfvén radius remains constant in the late phases of spin-down.

Figure 1 shows radial profiles of the wind and Alfvén speeds for three WD solutions, all with $\alpha = 1.1$, $T_0 = 1.5 \times 10^6$ K and $B_{r0} \propto \Omega$, differing only in their rotation rates, as color-coded. Plasma velocity and Alfvén speed components are all normalized to the base sound speed

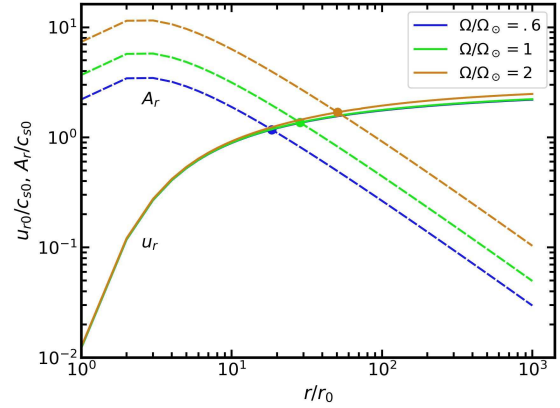


Figure 1. Radial profiles of wind (solid lines) and Alfvén (dashed) speed from WD solutions with rotation rates $\Omega/\Omega_\odot = 0.6$ (blue), 1 (green) and 2 (orange). In all cases $r_0/R_\odot = 1.15$, $T_0 = 1.5 \times 10^6$ K, $\alpha = 1.1$, and $B_{r0} \propto \Omega$.

$c_{s0} = \sqrt{\alpha p_0/\rho_0} = 165$ km s $^{-1}$ here. The wind speed profiles are almost indistinguishable, a direct reflection of the wind being almost entirely thermally-driven. As Ω varies, so does B_{r0} , so that the Alfvén speed profiles shift vertically by an amount directly proportional to B_{r0} —and thus Ω . As a result the Alfvén radius r_A , where solid and dashed curves cross, moves inwards as Ω (and B_{r0}) decreases, with a concomitant decrease of the Alfvén speed A_{rA} at that point. From such wind solutions the variation of A_{rA} with Ω can be measured, and turns out to be well fit by a power law:

$$\frac{A_{rA}(\Omega)}{c_{s0}} = 1.348 \left(\frac{\Omega}{\Omega_\odot} \right)^{0.29}, \quad 0.6 \leq \Omega/\Omega_\odot \leq 2. \quad (8)$$

Again at fixed moment of inertia, Eq. (7) would then “predict” $\Omega \propto t^{-0.58}$, a little steeper than Skumanich’s Law.

Many options are available to recover more precisely Skumanich’s $t^{-1/2}$ Law, for example here assuming a variation of the coronal base density $\rho_0 \propto \Omega^{0.29}$. This type of ad hoc fix is in fact common practice in many semi-empirical angular momentum loss models, in which power-law relationships of the form $\dot{M} \propto \Omega^a$ and/or $B_{r0} \propto \Omega^b$ are introduced, with the exponents a, b adjusted to recover $\dot{J} \propto -\Omega^3$ given other model ingredients (see, e.g., Kawaler 1988; Matt et al. 2012; Ahuir et al. 2020, and Skumanich 2019 for a critical synthesis of these approaches).

2.3. The MacGregor-Brenner spin-down model

The torque exerted by a magnetized wind on the photosphere can be presumed to be efficiently transmitted throughout the convective envelope by turbulent

stresses. Consequently, the wind-mediated torque first spins down the convective envelope, leaving underneath a more rapidly rotating radiative core, a process known as core-envelope decoupling. As the spin-down torque decreases rapidly with the decreasing rotation rate of the envelope, a flux of angular momentum from the core into the overlying convective envelope slows down the former and mitigates the spindown of the latter until, by the solar age, the radiative core spins as a solid body, at the same rate as the average rotation rate of the envelope (Tomczyk et al. 1995).

In a G-type main-sequence star like the Sun, the convective envelope ($0.713 \leq r/R_\odot \leq 1$) accounts for $\simeq 1\%$ of the stellar mass but $\simeq 10\%$ of the stellar moment of inertia. MacGregor & Brenner (1991) have developed a simple two-zone model capturing this process of core-envelope decoupling and later recoupling, involving a single adjustable parameter, namely a coupling timescale τ_c for the exchange of angular momentum between the core and envelope, each rotating as solid bodies but not necessarily at the same angular velocity. For fixed moments of inertia for the core (I_c) and envelope (I_e) on the ZAMS, the governing equations can be expressed as

$$I_c \frac{d\Omega_c}{dt} = -\frac{\Delta J}{\tau_c}, \quad (9)$$

$$I_e \frac{d\Omega_e}{dt} = \frac{\Delta J}{\tau_c} + \dot{J}, \quad (10)$$

where $\dot{J}$ (< 0) is the angular momentum loss rate in the magnetized wind, as given by eq. (7), and

$$\Delta J = \frac{I_e J_c - I_c J_e}{I_c + I_e} = \frac{I_c I_e}{I_c + I_e} (\Omega_c - \Omega_e). \quad (11)$$

The observed rotational evolution of late-type stars in the age range $10^8 - 10^9$ yr offer a relatively stringent constraint on the value of the timescale τ_c , which must be few tens of Myr in order to reproduce the rapid spin down of young solar-type stars in their first $\simeq 10^8$ yr on the main sequence (MacGregor & Brenner 1991; Denissenkov et al. 2010; Gallet & Bouvier 2013). Such a coupling timescale also allows “recoupling” at later times, during which the transfer of angular momentum from the core to the envelope offsets in part loss in the magnetized wind, which provides a natural explanation for the phase of “stalled” spin-down characterizing late-type stars in the 1–2 Gyr age range (Gallet & Bouvier 2013; Curtis et al. 2020; Gordon et al. 2021). A good fit to rotational data does require the coupling timescale τ_c to be a function of stellar mass (see Spada & Lanzafame 2020).

The physical mechanism governing angular momentum exchange between the core and envelope, in prin-

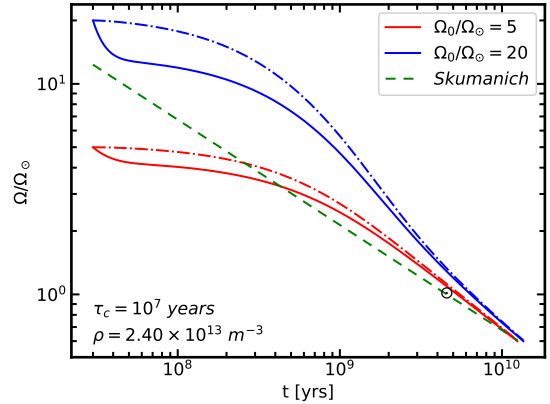


Figure 2. Two representative simulation runs of the MacGregor & Brenner (1991) spindown model, for fixed coupling timescale $\tau_c = 10^7$ yr and initial conditions on the ZAMS $\Omega/\Omega_\odot = 5$ and 25. Solid (dash-dotted) lines give the angular velocities of the convective envelope (radiative core) in each case. The dashed line is the Skumanich $t^{-1/2}$ Law pinned to the present day sun.

ciple setting the value of the coupling timescale τ_c , is not yet identified with confidence. Many plausible options are known and documented, ranging from hydrodynamical shear instabilities (Zahn 1992), gravity waves (Talon & Charbonnel 2005; Denissenkov et al. 2008), turbulence and magnetic stresses driven by MHD instabilities (Balbus & Hawley 1994; Spruit 2002), and/or magnetic stresses from a weak internal fossil magnetic field (Mestel & Weiss 1987; Charbonneau & MacGregor 1993; Ruediger & Kitchatinov 1996).

Figure 2 shows two representative MacGregor & Brenner (1991) rotational evolution simulations, both using a core-envelope coupling timescale $\tau = 10$ Myr and wind torque given by the WD model of the preceding section ($T_0 = 1.5 \times 10^6$ K, $\alpha = 1.1$, $B_{r0} \propto \Omega_e$). The various curves show the time-evolution of the angular velocity for the radiative core Ω_c (dash-dotted lines) and convective envelope Ω_e (solid lines), starting here from solid-body rotation $\Omega_c = \Omega_e$ on the ZAMS, at either 20 (blue) or 5 times (red) the present solar rotation rate $\Omega_\odot = 2.67 \times 10^{-6}$ rad s $^{-1}$. This initial condition is unrealistic, but has no significant influence on the late rotational evolution, on which our subsequent analyses focus.

The rapid early spin-down and “plateau” in the ~ 0.05 – 0.5 Gyr range are both clearly seen, as well as the recoupling at later time and general convergence to the same rotational evolution trend beyond a few Gyr. This is simply a consequence of angular momentum loss in the WD magnetized wind solution increasing sharply with rotation rate, i.e., initially faster rotators spin-

down faster, so that memory of the initial condition is eventually lost (Kawaler 1988). Here, in this asymptotic regime, $\Omega_e \propto t^{-0.58}$, as expected given the rotation rate dependence of the WD torque in the thermally-dominated driving regime, viz. §2.2.

3. MAGNETIZED CORONAL WINDS NEAR THE HYDROSTATIC LIMIT

The mechanism(s) responsible for coronal heating have not yet been identified, but magnetic fields are generally believed to be involved (Judge & Ionson 2024), so it is certainly plausible that a drop in surface magnetism due to decreasing rotation may end up reducing the coronal temperature to the point that a transsonic wind no longer materializes. It is readily shown that in the $1/r^2$ gravity field exterior to a mass $M_\odot$, a hydrostatic polytropic corona of finite radial extent can exist as long as the base coronal temperature satisfies:

$$T_0 \leq \frac{\alpha - 1}{\alpha} \frac{GM_\odot \mu m_p}{k_B r_0}, \quad 1 < \alpha \leq 5/3. \quad (12)$$

where $\mu = 0.5$ is the mean molecular weight of the coronal plasma, α is the polytropic index, and other symbols have their usual meaning (see Appendix A.1). In what follows we refer to this limiting temperature as the *hydrostatic limit*. For $\alpha = 1.1$, this hydrostatic limit is at $T_0 = 0.91 \times 10^6$ K.

We begin by investigating how wind dynamics and associated mass and angular momentum loss rates change as the coronal temperature decreases towards the limiting value given by Eq. (12). A sequence of WD wind solutions is computed for base coronal temperatures in the range $1.125 \leq T_0/10^6 \text{ K} \leq 2.93$, using a fixed polytropic index $\alpha = 1.1$, with rotation rate and average surface field strength fixed at their solar values, $\Omega_\odot = 2.67 \times 10^{-6} \text{ rad s}^{-1}$ and $B_{r0} = 2.79 \times 10^{-4} \text{ T}$ (2.79 G) in all cases. Figure 3 shows the variations of the Alfvén radius r_A , base flow speed u_{r0} , and angular momentum loss rate $\dot{J}$, as a function of coronal base temperature T_0 . Moving from right to left, as T_0 drops from 2 to 1.3×10^6 K, the base wind speed u_{r0} (blue curve), and thus the mass loss rate, drops rapidly; however the Alfvén radius r_A (red curve) increases markedly, a direct consequence of the decreasing wind speed while the Alfvén speed radial profile remains fixed since here both the base field strength and density are held constant (viz. Fig. 1). This more than compensates the drop in mass loss rate, with the consequence that the angular momentum loss rate $\dot{J}$ (green curve, as computed via Eq. (7) *increases*. This behavior is not specific to the WD wind model, and has been noted in geometrically more complex simulations (see, e.g., the 2D

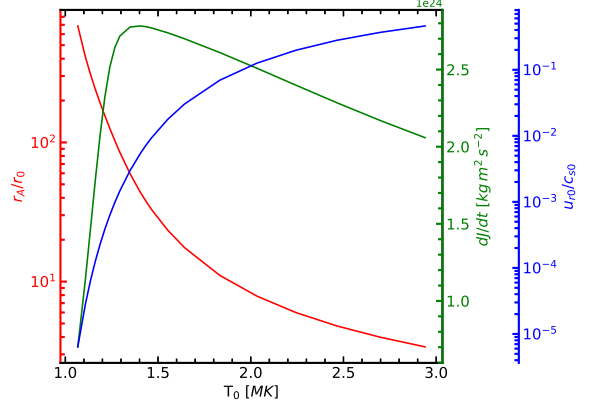


Figure 3. Variations of the Alfvén radius r_A (red), base flow speed u_{r0} (blue), and angular momentum loss rate $\dot{J}$ (green), versus coronal base temperature, as obtained from a sequence of $\alpha = 1.1$ WD wind solutions at fixed solar rotation rate and surface magnetic field strength.

MHD polytropic wind simulations of Pantolmos & Matt 2017). However, once T_0 drops below $\simeq 1.45 \times 10^6$ K the precipitous drop in base flow speed now surpasses the increase in r_A^2 , so that $\dot{J}$ now *decreases* as T_0 is further reduced. The angular momentum loss rate is thus a non-monotonic function of base coronal temperature, peaking here at $T_0 \simeq 1.35 \times 10^6$ K.

The non-monotonic variation of $\dot{J}$ with coronal base temperature reflects a change in dynamical regime taking place in the winds emanating from coronae with decreasing base temperatures. For a slowly rotating, weakly-magnetized solar-type star, once the coronal temperature approaches the hydrostatic limit (12) from above, magnetocentrifugal driving, negligible for a 1.5×10^6 K corona with $\alpha = 1.1$, becomes significant, as it would be in young, rapidly rotating and strongly magnetized stars of solar coronal temperatures. Figure 4 shows a specific example, namely a WD wind solution for a star with solar rotation rate and surface field strength, but a coronal temperature of 1.23×10^6 K and polytropic index $\alpha = 1.125$. The top panel shows radial profile of wind and Alfvén speed components, while the bottom panel depicts the corresponding radial force budget as a function of r .

Because the thermal pressure force is much reduced due to the low coronal temperature, centrifugal and magnetic forces, even if small in absolute terms, dominate the wind dynamics beyond the sonic point (the radius at which $u_r = c_s$, indicated by inverted triangles on Fig. 4), leading to an asymptotic flow speed 858 km s^{-1} , significantly larger than the 162 km s^{-1} characterizing a purely hydrodynamical coronal wind à la Parker

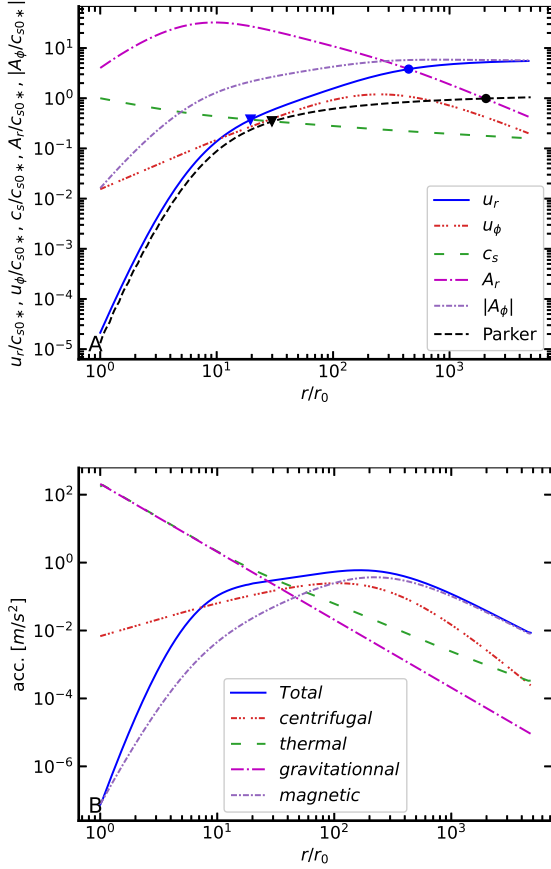


Figure 4. Top panel: Radial profiles of wind variables for a $\alpha = 1.125$ WD solution with solar rotation and magnetic field strength, and a “low” coronal base temperature $T_0 = 1.23 \times 10^6$ K. The black dashed line shows the $u_r(r)$ profile for a non-rotating, unmagnetized coronal wind of the same base temperature and polytropic index. The two solid dot indicates the corresponding Alfvén points. The bottom panel shows the corresponding acceleration terms in the momentum equation: gravity, pressure, centrifugal, and magnetic force, as labeled. The solid blue line gives the resulting Lagrangian acceleration. Note how centrifugal and magnetic forces dominate beyond the sonic point (blue inverted triangle).

(1958) (dashed black line on Fig. 4A) at the same coronal temperature. However, this increase in asymptotic wind speed does not translate into an enhanced mass loss rate, the base flow speed being essentially identical for the WD and Parker wind solutions; at large distances, the WD wind simply has a faster speed but lower density. This is a well known property of coronal winds, independent of the mechanism depositing energy or angular momentum in the wind, over and above thermal driving (Leer & Holzer 1980; Ofman 2010). From the point of view of angular momentum loss, the important

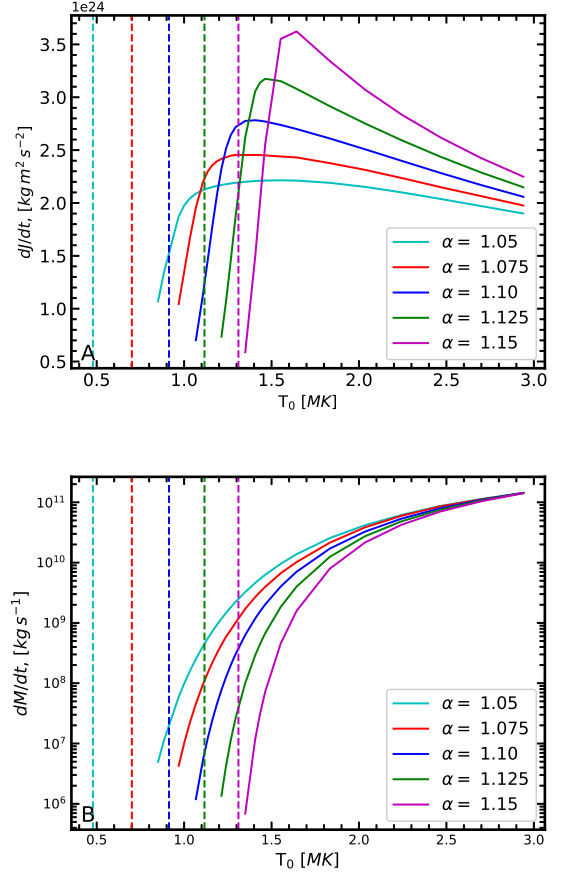


Figure 5. Angular momentum loss rate (top panel) and mass loss rate (bottom panel) as a function of coronal base temperature in WD wind solutions with various values of the polytropic index α , as color-coded. The vertical dashed lines indicate the corresponding hydrostatic corona limiting temperature. In all cases the rotation rate and magnetic field strength are held at their solar values, and the coronal base density is fixed at $\rho_0 = 10^{14} m_p \text{ m}^{-3}$ in calculating the mass loss rate.

take-home message is that for such low base temperature winds, while magnetocentrifugal driving does not greatly affect the base flow speed, it does displace the Alfvén radius significantly inwards, thus further *reducing* the angular momentum loss rate as compared to a naive estimate using the wind profile $u_r(r)$, $\rho(r)$ of the Parker unmagnetized wind model to estimate the location of the Alfvén radius, cf. the two dots on Fig. 4A.

The consequence of this complex dynamical behavior is that, as shown on Figure 5, the angular momentum loss rate ends up varying in a non-monotonic manner with both the base temperature and polytropic index, the latter being another important parameter ultimately related to coronal heating (more on this in §5 below). Note how at the higher base temperatures, the

solutions with polytropic index closer to adiabatic have higher angular momentum loss rates, but wind solutions closer to isothermal ($\alpha \rightarrow 1.0$) retain an approximately constant angular momentum loss rate to much lower base temperatures before wind collapse occurs. Mass loss (Fig. 5B), on the other hand, decreases monotonically with decreasing coronal base temperature T_0 and increasing polytropic index α , the collapse of mass loss taking place at a base temperature decreasing with decreasing α . We note also that the decrease of mass loss with base temperature, although monotonic, cannot be well fit by a power-law in this temperature range, no matter the value of α .

The abruptness seen on Fig. 5 in the collapse of the angular momentum loss rate as coronal temperature drops over the peak in $\dot{J}$ is a direct consequence of magnetocentrifugal driving: as shown on Fig. 4, at low coronal base temperatures it increases the flow speed beyond the slow magnetosonic point, which greatly reduces the outward displacement of the Alfvén radius normally associated with decreasing overall wind speeds (cf. Fig. 3), so that it can no longer compensate the decrease in mass loss rate associated with a lower base flow speed, as it does at higher coronal temperatures. At the dynamical level, magnetocentrifugal driving becomes important at low coronal temperature because, as u_r drops, more winding of the magnetic field can take places, which increases the strength of the azimuthal magnetic component in the wind per unit radial displacement of a plasma volume element. Consequently, both the radial and azimuthal components of the Lorentz force also becomes larger.

Fig. 5 already indicates that there are multiple possible pathways to the shutdown of angular momentum loss. In addition to the (obvious) decrease in base coronal temperature, close examination of Figure 5A reveals that in the coronal temperature range $0.8 \leq T_0 \leq 1.5\text{MK}$, at fixed T_0 a gradual increase in the polytropic index —amounting to reduced power input into the corona, as detailed in §5 below— would generate a moderate rise followed by an extremely swift drop of the angular momentum loss rate.

4. BREAKING SKUMANICH’S LAW

Guided by the modelling results presented in the preceding section, we now investigate the conditions under which Skumanich’s Law can be broken (or not) by collapse of the coronal wind triggered by a gradual reduction of the coronal base temperature as the star spins down. To do so, and following Ó Fionnagáin & Vidotto (2018), we introduce an ad hoc power-law relationship between base coronal temperature and surface magnetic

field B_{r0} which, because of our assumed dynamo relationship $B_{r0} \propto \Omega$ then translates into a power-law relationship with rotation rate:

$$\frac{T_0(\Omega)}{T_\odot} = \left(\frac{\Omega}{\Omega_\odot} \right)^\sigma, \quad \sigma \geq 0. \quad (13)$$

with $T_\odot = 1.5 \times 10^6$ K. Figure 6A shows the rotational evolution resulting from this Ansatz, for varying values of the exponent σ , as labeled. What is plotted is the ratio of the rotation period to that of a reference run with constant $T_0 = 1.5\text{MK}$. All solutions use the dynamo relationship $B_{r0} \propto \Omega$ and $\alpha = 1.1$, and are computed using the MacGregor & Brenner (1991) model. The initial conditions are set so that the angular momentum loss is fixed at the solar value when the model reaches the age of the Sun. In light of Fig. 5A, one might expect that a drop of $\simeq 25\%$ in temperature is required to shut off angular momentum loss, while the Skumanich relation would predict a drop of $\simeq 20\%$ in rotation rate; this would then require $\sigma \simeq 1$ to achieved a significant reduction of angular momentum loss between 5 and 8 Gyr. This expectation is consistent with the results plotted on Fig. 6A, but even at $\sigma = 1.44$ the deviation from Skumanich’s Law at 8 Gyr is modest, with only a $\simeq 10\%$ increase in rotation period.

Figure 6 also reveals that despite a drop by many orders of magnitude in the mass loss rate beyond the solar age for evolutionary sequences with $\sigma \geq 0.5$ (panel C), the corresponding decrease in angular momentum loss rate (panel B) remains quite modest. This is a direct consequence of the peculiar dynamics of magnetized coronal winds at temperatures approaching the hydrostatic corona limit, as detailed in §3. In this regime, mass loss is emphatically not a good predictor of angular momentum loss.

5. WIND ENERGETICS

The energetics of coronal wind acceleration is inseparable from the coronal heating problem (Hansteen & Leer 1995). In regions of the corona overlying the quiet sun, it is estimated that up to 17% of the $\simeq 800 \text{ W m}^{-2}$ energy flux required to maintain the corona in a quasi-steady state is used in accelerating the solar wind; this fraction goes up to $\simeq 90\%$ in coronal holes from which the fast component of the solar wind originates (see discussion in Judge & Ionson 2024, §1.5 and Appendix B)

While a quasi-consensus exists to the effect that magnetic fields are involved at some level, the physical mechanism(s) responsible for coronal heating is yet to be identified with confidence. The collection of review papers introduced by De Moortel & Browning (2015) offer a nice sample of current ideas (see also Judge 2023;

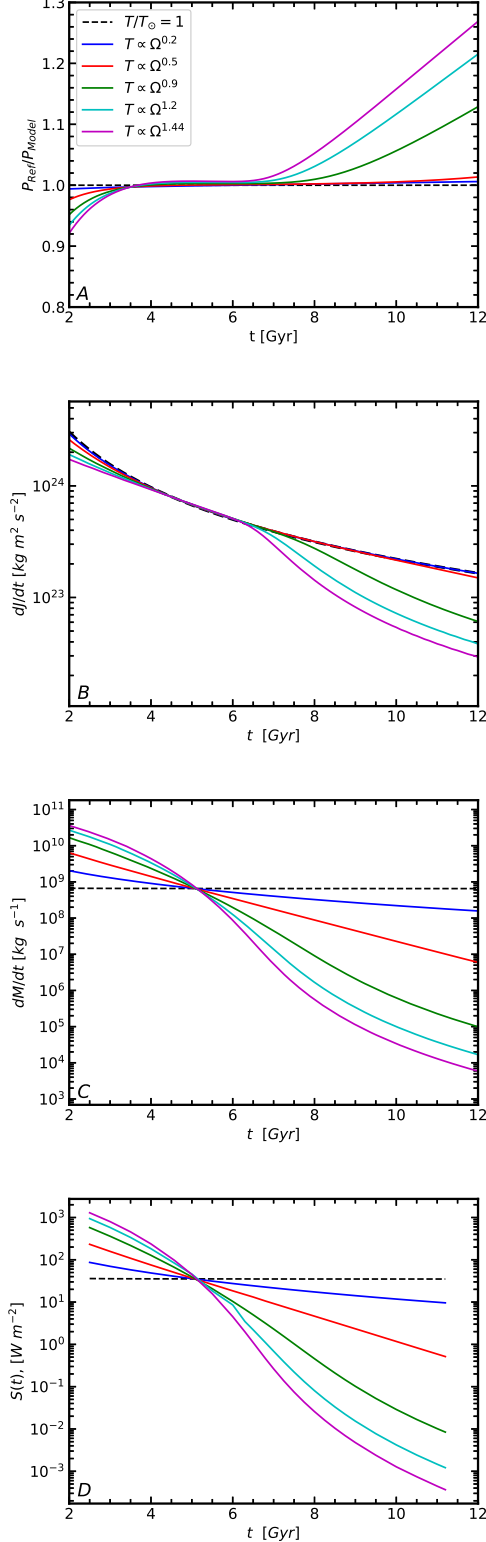


Figure 6. Break of the Skumanich Law from our rotational evolution model using different values of σ in equation 13. The figures show the evolution of (A) the ratio $P_{\text{ref}}/P_{\sigma}$ of rotation periods, (B) $\dot{J}$, (C) $\dot{M}$ and (D) S as a function of time. Here P_{ref} is the rotation period for a model for which temperature is fixed and P_{σ} is the period for models where the temperature evolves as described by equation 13 and S is the integrated volumetric heating rates as described by equation (14). The same colour coding applies to all panels.

Judge et al. 2024). If the observed break of gyrochronology in solar-type stars older than the sun is indeed associated with thermal collapse of coronal winds, then useful constraints regarding coronal heating may perhaps be derived from spindown considerations.

In wind models (including the WD model) or MHD numerical simulations relying on the polytropic approximation $p \propto \rho^{\alpha}$, the coronal base temperature and polytropic index are typically input parameters which can be varied independently; in fact, and as detailed in what follows, the polytropic index embodies a specific form of volumetric heating profile; $\alpha = 1$ corresponds to an isothermal corona, and thus an (unrealistic) infinite energy source, while $\alpha = 5/3$ corresponds to adiabatic expansion, i.e., vanishing volumetric energy input (see, e.g., §4.2 in Lamers & Cassinelli 1999). The coronal base temperature, on the other hand, is a *result* of energy balance between coronal heating, and energy lost to radiation, thermal conduction, and acceleration of the solar wind.

With a polytropic wind solution already computed, it is possible to reconstruct a posteriori the profile of volumetric heating that would result in the same wind solution. One simply needs to write the energy equation, including now explicitly a volumetric source term $s(r)$ on the RHS:

$$\nabla \cdot \left[\rho \mathbf{u} \left(\frac{u^2}{2} + \frac{3p}{2\rho} + \frac{B^2}{2\rho\mu_0} \right) \right] + \nabla \cdot (p\mathbf{u}) + \mathbf{u} \cdot \nabla \Phi - \frac{1}{\mu_0} \mathbf{u} \cdot (\mathbf{B} \cdot \nabla) \mathbf{B} = s(r) \quad (14)$$

Substituting the WD polytropic wind solution $u_r(r)$, $p(r)$, $\rho(r)$, etc. in the LHS then yields $s(r)$ by direct computation. The results of such an exercise are shown on the top panel of Figure 7, for a selection of WD wind solutions computed using distinct combinations of $[\alpha, T_0]$ and otherwise solar parameters. The corresponding wind speed profiles $u_r(r)$ are plotted on the bottom panel.

In all cases the volumetric source term $s(r)$ can be well approximated by a steep power law in heliocentric radius, i.e., $s(r) \propto r^{-\gamma}$, with $\gamma \simeq 4$. It is physically satisfying that the heating profiles extracted in this manner peak at the base of the corona, and decrease rapidly outwards. Yet the relationship to the two heating parameters, the base flow speed T_0 and polytropic index α is intricate; at $T_0 = 2.7$ MK the volumetric heating profiles (in red) are nearly independent of α , but at the lower temperature of 1.4 MK differ by over two orders of magnitudes at all radial positions.

Comparing the flow speed profiles for two $T_0 = 1.4$ MK WD solutions on Fig. 7B (blue curves), note

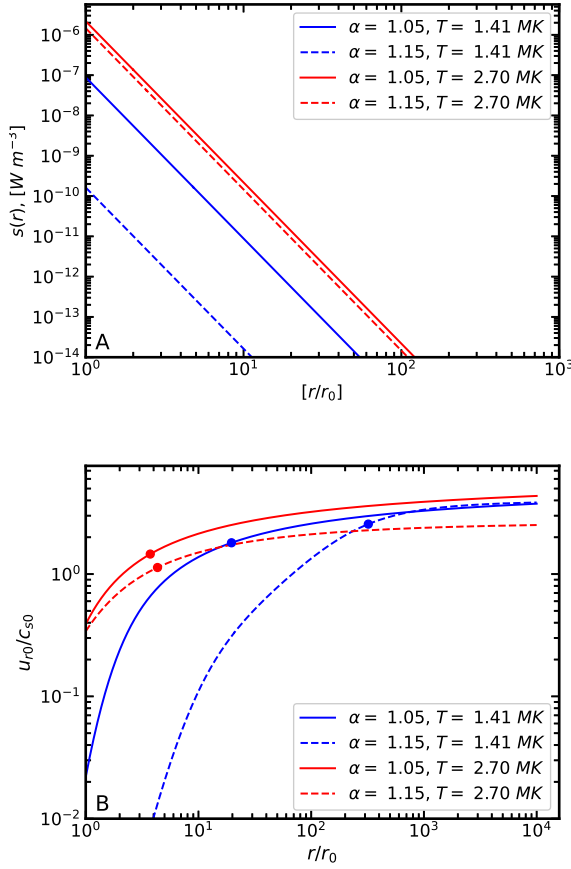


Figure 7. Panel A shows radial profiles of volumetric power input in a sample of polytropic WD wind solutions, reconstructed a posteriori via Eq. (14). Panel B shows the corresponding flow profiles. All solutions are computed for solar rotation and surface magnetic field strength. The solid dots indicate the position of the Alfvén radius r_A in each solution.

how the asymptotic flow speeds are almost identical, while the base flow speed—and thus mass loss rate—is orders of magnitude smaller in the $\alpha = 1.15$ solutions than for $\alpha = 1.05$. Conversely, and despite nearly identical volumetric heating profiles, the two $T_0 = 2.7$ MK wind solutions (in red) have distinctly different acceleration profiles and asymptotic flow speeds, but nearly identical base flow speeds. Perhaps counter-intuitively given the wind profiles on Fig. 7B, all four of these wind solutions have comparable angular momentum loss rates, ranging from $\dot{J} = -1.48 \times 10^{24} \text{ kg m}^2 \text{ s}^{-2}$ for the $(\alpha, T_0) = (1.15, 1.41 \text{ MK})$ solution (dashed blue), up to $\dot{J} = -2.42 \times 10^{24} \text{ kg m}^2 \text{ s}^{-2}$ $(\alpha, T_0) = (1.15, 2.69 \text{ MK})$ (solid blue).

Integrating $s(r)$ from the coronal base out to infinity yields the power required to maintain the coronal temperature and accelerate the wind. This power requirement, expressed as an energy flux at the base of

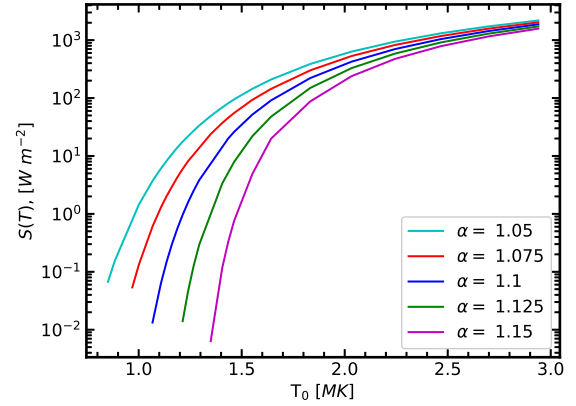


Figure 8. Integrated volumetric heating rates for WD solutions of various α and T_0 , expressed as energy flux at the base of the wind, $r_0/R_\odot = 1.15$. As on Fig. 5, all wind solutions are computed at the solar rotation rate and surface magnetic field strength.

the wind, is plotted on Figure 8, for sequences of WD solutions at fixed polytropic index α and varying coronal base temperature. The total heating rate S drops with decreasing T_0 faster than a power-law, the more swiftly the closer the polytropic index α is to its adiabatic limit $\alpha = 5/3$. Below $\simeq 1.5$ MK, at fixed coronal base temperatures power input decreases rapidly with increasing α , but this sensitivity rapidly disappears as T_0 increases beyond $\simeq 2$ MK. It must be kept in mind that these results pertain to the energy required to propel the wind, and do not include radiative of conductive energy losses. For $\alpha = 1.1$, a $T_0 = 1.5 \times 10^6$ K corona requires $\sim 10^2 \text{ W m}^{-2}$, in line with observational inferences (Judge & Ionson 2024).

Figure 9 shows the correlation between power input and mass (top panel) and angular momentum (bottom panel) loss rates, for a sample of WD wind solutions taken from Figs. 5 and 8. Data points are color-coded in terms of the polytropic index α , as labeled, and base temperatures increase from bottom to top along each colored-dot sequence. Although the mass loss rate clearly cannot be described by a power-law in base temperature (viz. Fig. 5B), a single and very tight linear relationship (power-law of index unity) emerges when correlated against total power input into the wind, characterizing WD wind solutions computed for a wide range of base temperatures and polytropic indices. The power input is clearly a great predictor of the mass loss rate (and vice-versa!). Note again the marked degeneracies characterizing these WD wind solutions, with widely different (α, T_0) pairs associated with identical power input, and thus mass loss rate.

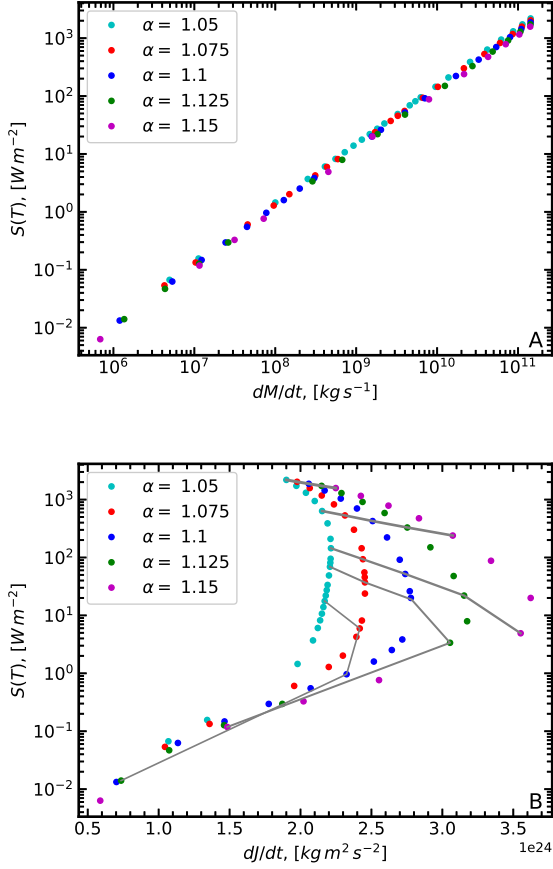


Figure 9. Correlation between integrated volumetric heating rates and mass loss rate (top panel) and angular momentum loss rate (bottom panel) for WD solutions of various α and T_0 . All wind solutions are computed at the solar rotation rate and surface magnetic field strength, and mass loss rates is computed for a base density $\rho_0 = 10^{14} m_p \text{ m}^{-3}$. The base temperature T_0 ranges from 0.85 to 2.94 MK and increases upwards along each sequence of colored dots. The lines on panel B are isotherms for values $T_0 = [1.21, 1.4, 1.55, 2.03, 2.94]$ MK, from bottom to top.

On the other hand, as evidenced on Fig. 9B the angular momentum loss rate shows a more complex relationship to power input, definitely not expressible as a power-law even at fixed α . Note that the horizontal axis spans only a factor of $\simeq 4$ in $\dot{J}$, in contrast to the five orders of magnitude in $\dot{M}$ on panel A; so that in comparison, $\dot{J}$ is almost independent of the power input *despite* the strong dependency of $\dot{M}$.

With these results in hand, we return to the break of Skumanich’s Law in the spin-down simulations of §4. Panel D of Figure 6 shows the power input to the wind calculated a posteriori along each spin-down run of panel A. As one might have anticipated on the basis of Fig. 9A, the drop in power input along the rotational

evolution sequence closely shadows that of the mass loss rate, dropping by some 3 orders of magnitude between 5 and 8 Gyr. Such a large drop, for which no observational evidence currently exists to the best of our knowledge, represents a strong constraint on any explanation for the break in gyrochronology invoking collapse of mass loss in aged solar-type stars.

6. DISCUSSION AND CONCLUSION

Our investigations into the dynamics of polytropic magnetized coronal winds near the hydrostatic limit have revealed a number of somewhat counter-intuitive results, of importance in the context of the spin-down of solar-type stars and observed break of gyrochronology; and raise some warning flags regarding the indiscriminate use of ad hoc power law to relate input wind parameters to mass and/or angular momentum loss rates:

1. At low ($\lesssim 1.2$ MK) coronal base temperatures, magnetocentrifugal driving is important even at the relatively low solar rotation rate (Ω) and coronal magnetic field strength (B_{r0}). It does not greatly affect mass loss, but can lead to a significant decrease of $\dot{J}$ as compared to pure thermal driving.
2. In the magnetohydrodynamical Weber-Davis wind models, at fixed rotation rate and magnetic field strength neither the angular momentum loss rate $\dot{J}$ nor mass loss rate $\dot{M}$ can be described by power-laws in coronal temperature T_0 ; the former is even non-monotonic in T_0 .
3. Again at fixed Ω and B_{r0} , $\dot{M}$ is entirely set by the power input into the wind, itself showing strong degeneracies in (α, T_0) .
4. As coronal base temperature decreases towards the hydrostatic corona limit, the collapse of $\dot{J}$ is very sudden and takes place at significantly larger T_0 than the hydrostatic limit.

We have then attempted to break the Skumanich $t^{-1/2}$ Law for solar-type stars older than the sun, by introducing an ad hoc power-law relationship between T and B_{r0} such that T decreases as the star spins down as $T \propto \Omega^\sigma$. In our $\alpha = 1.1$ magnetized polytropic wind models, a pronounced break in spin-down, as suggested by observations (see, e.g., Fig. 2 in van Saders et al. 2016), requires $\sigma \gtrsim 1.5$. Direct extrapolation of such a steep power law to rapidly rotating young main-sequence solar-type stars would yield unphysically high coronal temperature. Ó Fionnagáin & Vidotto (2018) have bypassed this difficulty by assuming a variation of

coronal temperature with rotation rate described by a double power-law $T \propto \Omega^{1.2}$ for $\Omega < 1.4\Omega_\odot$, switching to $T \propto \Omega^{0.37}$ above that rotation period. Recovering Skumanich's Law using such a temperature dependency would then require the introduction of other ad hoc dependencies in other wind parameters, for example the wind base density.

In polytropic winds models, a class to which the WD model belongs, the energy input is set implicitly by both the assumed coronal base temperature and value for the polytropic index. We demonstrated in §5 that close to the hydrostatic corona limit, the WD wind model exhibits a very tight correlation between mass loss rate and total power input into the wind, so that the ad hoc decreases in coronal temperature required to break the Skumanich $t^{-1/2}$ relationship in §4 implies a drop in power input by over three orders of magnitude as compared to that estimated for the corona of the present-day sun. Observational evidence for variations of chromospheric emission beyond the solar age (e.g., [Lorenzo-Oliveira et al. 2018](#), and references therein) show no evidence for such a drastic drop, although care is warranted considering that the bulk of the observed emission presumably arises in compact, closed magnetic fields overlying active regions, as it does on the sun. To what extent this reflects closely (or not) the temperature of (and power input to) the quiet so-

lar corona remains an open question; between minimum and maximum phases of the solar magnetic activity cycle, chromospheric and coronal emission varies by orders of magnitude, while the temperature of the quiet corona does not. This fact alone indicates that inferring stellar *quiet* coronal temperatures from non-thermal emission measures is a risky proposition, at the very least.

All such observational and physical uncertainties notwithstanding, the modeling results presented in this paper indicate that even under the most favorable conditions, namely a very steep drop of coronal temperature with decreasing global coronal magnetic field and rotation rate, thermal wind collapse alone is unlikely to fully explain the observationally-inferred very rapid drop in angular momentum loss in solar-type stars slightly older than the sun. This implies that additional physical mechanisms, such as a shutdown or topological reconfiguration of the dynamo-generated large-scale coronal magnetic field, must also play a significant role.

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APPENDIX

A. HYDROSTATIC POLYTROPIC CORONAE

We consider a spherically symmetric unmagnetized hydrostatic solar corona under the polytropic approximation

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0} \right)^\alpha . \quad 1 \leq \alpha \leq 5/3 \quad (\text{A1})$$

where a subscript “0” indicates a quantity evaluated at a reference heliocentric radius r_0 corresponding to the base of the corona. Its radial structure is described by the equation of hydrostatic equilibrium in a $1/r^2$ external gravity field, which in view of Eq. (A1) takes the form:

$$c_{s0}^2 \left(\frac{\rho}{\rho_0} \right)^{\alpha-1} \frac{d\rho}{dr} = -\rho \frac{GM_\odot}{r^2} , \quad (\text{A2})$$

where $c_{s0}^2 = \alpha p_0 / \rho_0$ is the base polytropic sound speed. This readily integrates to

$$\frac{\rho(r)}{\rho_0} = \left[1 - \frac{(\alpha-1)GM_\odot}{r_0 c_{s0}^2} \left(1 - \frac{r_0}{r} \right) \right]^{1/(\alpha-1)} , \quad (\text{A3})$$

from which the pressure profile $p(r)/p_0$ follows immediately using Eq. (A1). Both pressure and density vanish in the asymptotic limit $r \rightarrow \infty$ provided

$$c_{s0}^2 \leq (\alpha-1)GM_\odot/r_0 . \quad (\text{A4})$$

in which case the corona extends to a finite radius $r_{\text{top}}/r_0 = (1 - r_0 c_{s0}^2/(\alpha - 1)GM_\odot)^{-1}$. If (A4) is not satisfied, then both pressure and density asymptote to constant values, implying coronal expansion (Parker 1958). For a perfect gas of mean molecular weight μ , the equation of state allows to express c_{s0} in term of the base coronal temperature T_0 as $c_{s0}^2 = \alpha k_B T_0 / \mu m_p$ which, when substituted into Eq. (A4), leads to the constraint (12) on the base temperature setting the hydrostatic limit.

B. NUMERICAL SOLUTION OF THE WEBER-DAVIS WIND EQUATIONS

We follow Belcher & MacGregor (1976) (§II) in reducing the calculation of a Weber-Davis wind solution to a six-dimensional nonlinear root finding problem, from which the full wind solution can be reconstructed. Straightforward (but somewhat tedious) algebraic manipulations allow to combine the governing equations of the WD model —the r - and ϕ -component of the momentum and induction equations, the continuity equation, and the polytropic relationship $p \propto \rho^\alpha$ — into either of the two following forms:

$$\frac{\partial u_r}{\partial r} = \left(\frac{u_r}{r}\right) \frac{(u_r^2 - A_r^2)(2c_s^2 + u_\phi^2 - GM/r) + 2u_r u_\phi A_r A_\phi}{(u_r^2 - A_r^2)(u_r^2 - c_s^2) - u_r^2 A_\phi^2}, \quad (\text{B5})$$

$$\frac{1}{2}(u_r^2 + u_\phi^2) - \frac{GM}{r} + \frac{c_{s0}^2}{\alpha - 1} \left(\frac{\rho}{\rho_0}\right)^{\alpha-1} - \frac{r\Omega A_r A_\phi}{u_r} = E, \quad (\text{B6})$$

The first is essentially a rewriting of the r -component of the momentum equation, while the second is a first integral akin to Bernoulli's equation for this problem, with the constant E corresponding to the total energy per unit mass in a fluid element moving along a flow streamline. The denominator (D) of Eq. (B5) vanishes when the radial flow speed becomes equal to the slow or fast magnetosonic wave speeds. Denoting the radial positions and radial flow speeds at which this takes place as (r_s, u_{rs}) and (r_f, u_{rf}) , respectively, regularity of the solution is enforced by requiring that the numerator (N) also vanishes at these positions; this yields four coupled non-linear algebraic equations:

$$N(r_f, u_{rf}) = 0, \quad D(r_f, u_{rf}) = 0, \quad N(r_s, u_{rs}) = 0, \quad D(r_s, u_{rs}) = 0, \quad (\text{B7})$$

Equation (B6) is then invoked to further require that the energy per unit mass at these points be the same as at the base of the flow, which provides two additional nonlinear algebraic equations:

$$E(r_f, u_{rf}) = E(r_0, u_{r0}), \quad E(r_s, u_{rs}) = E(r_0, u_{r0}). \quad (\text{B8})$$

Solution of these six coupled nonlinear algebraic equations yields the solution vector $[u_{r0}, u_{\phi0}, r_s, u_{rs}, r_f, u_{rf}]$. Any one such solution is defined by a 5-dimensional input vector $[\alpha, c_{s0}, \gamma, \zeta, \beta]$, grouping the polytropic index α ; the base sound speed c_{s0} , measuring coronal temperature; the gravitational escape speed, measuring gravity; the azimuthal speed of the rotating frame, measuring rotation rate; and the Alfvén speed, measuring coronal magnetic field strength. All are evaluated at the base radius r_0 , and the latter three normalized to the base polytropic sound speed c_{s0} .

Strongly nonlinear multidimensional root finding problems of this type are notoriously challenging from a numerical point of view (see, e.g., the insightful discussion in §9.6 of Press et al. 1992, also §4.2 in Charbonneau 1995 for an alternate approach based on nonlinear minimization). Standard gradient-based schemes, such as Newton-Raphson, can be used; however, in the present case, the radius of convergence can be quite small, requiring a careful setting of the step length and a good initial guess to initiate a successful steepest descent to the sought-after solution.

In the context of spin-down calculations in which a large number of WD solutions need to be calculated to compute the angular momentum loss rate $\dot{J}$ along any given spindown sequence, we have opted to pre-compute a grid of WD solutions in a 4-dimensional input parameter space. For the work presented in this paper, this grid covers the (physical) ranges $\alpha \in [1.05, 1.15]$, $T \in [0.81, 4.50]$ MK, $\Omega/\Omega_\odot \in [0.6, 25]$, $B \in [1.67, 69.8]$ G, and gravity fixed at the solar value.

Starting from a solar solution, the grid is filled by increasing sequentially one of the four input parameters, and using the previous member of the sequence as the initial starting guess for the current member. Interpolation is then used to extract a solution for any combination of input parameters. This procedure introduces a small numerical error, but in practice for a tight enough pre-computed grid it remains quite small and insignificant for mass and angular momentum loss computation. Significant interpolation errors are readily detected when reconstructing full radial wind profiles, as they typically manifest themselves as spurious jumps across energetically allowed solution branches, resulting from failure to properly cross a critical points.

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