
THE MAKER-BREAKER DIRECTED TRIANGLE GAME

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ABSTRACT

In this work, we investigate Maker-Breaker directed triangle games – a directionally constrained variant of the classical Maker-Breaker triangle game. Our board of interest is a tournament, and the winning sets constitute all directed triangles (i.e. 3-cycles) present in the tournament. We begin by studying the Maker-Breaker directed triangle game played on a specially defined tournament called the parity tournament, and we identify the board size threshold to be $n = 7$, which is to say that if the size (i.e. the number of vertices) of the parity tournament equals n , Breaker has a winning strategy for $3 \leq n < 7$, while Maker can ensure a win for herself for $n \geq 7$. For the $(1 : b)$ biased version of this game, wherein Breaker is allowed to claim b directed edges of the tournament in each of her turns while Maker is allowed to claim only one directed edge in each of hers, we prove that the bias threshold $b^*(n)$ satisfies $\sqrt{(1/12 + o(1))} \, n \leq b^*(n) \leq \sqrt{(8/3 + o(1))} \, n$, which matches the order of magnitude (i.e. $\sqrt{n}$) of the bias threshold for the undirected counterpart of this game. Next, we consider the game on random tournaments $T(n, p)$, wherein the vertices are labeled $1, 2, \dots, n$, and the edge between i and j , for each $i < j$, is directed from i towards j with probability p , independent of all else. We prove that Maker wins this game with probability approaching 1 as $n \rightarrow \infty$ for any fixed $p \in (0, 1)$. Extending the notion of ‘bias’ from undirected games to our directed framework, we introduce the $(1 : \kappa(n))$ flip-biased Maker-Breaker directed triangle game on the parity tournament, where we allow Breaker to strategically flip the directions of a fixed number of edges $\kappa(n)$ of the tournament before the game begins (once again, n indicates the size of the tournament). We show that the flip-bias threshold $\kappa^*(n)$ for this game is asymptotically equal to $n^2/4$ as $n \rightarrow \infty$ (in other words, when $\kappa(n) < \kappa^*(n)$, Maker has a winning strategy, while for $\kappa(n) \geq \kappa^*(n)$, Breaker wins). This work opens up the possibility of studying a variety of directionally constrained Maker-Breaker positional games on tournaments (and more generally, on directed graphs).

1 Introduction

Let X be a finite set which we call the *board*, and let $\mathcal{F}$ be a family of subsets of X referred to as a *family of winning sets* (the elements of $\mathcal{F}$ are referred to as the *winning sets*). A *positional game*, indicated by $(X, \mathcal{F})$, is a *two-player combinatorial game* played on the board X , with the two players taking turns to claim yet-unclaimed elements of X . The game ends when all elements of X have been claimed. When we focus on the special class of games called the *Maker-Breaker positional games*, these two players are called, as the name suggests, *Maker* and *Breaker*. Maker’s objective is to claim all the elements of at least one winning set $F \in \mathcal{F}$, while Breaker tries to prevent her from doing so. Maker wins this game if she is able to accomplish her objective before the end of the game, and Breaker wins otherwise. We shall assume that Maker starts the game by playing the first round. The board X as well as the family $\mathcal{F}$

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of winning sets is revealed to the two players *before* the game begins, making it a *perfect information game*, and the players have complementary goals, thus implying that there can be no draw. We refer the reader to [Bec08],[Hef+14] and [Kri14] for an introduction to the key results and topics in this field.

A special class of such games is the Maker-Breaker positional game *on graphs*, where the underlying board X is the edge-set, E , of a given graph $G = (V, E)$, and the family $\mathcal{F}$ of winning sets comprises the subsets of edges contributing to the formation of a prespecified structure. In other words, Maker’s objective in such a game is to ensure that the subgraph induced by the edges claimed by her contains this prespecified structure. For instance, $\mathcal{F}$ may be the set of all triangles ([CE78], [BS11], [GS22]), all k -cliques for a fixed but arbitrary positive integer k ([Bec08], [BL00]), all spanning trees ([CE78], [Bec82]), all Hamiltonian cycles ([Bec85], [Kri11]) etc.

So far, we have assumed that each player claims a single edge during each of her turns, and such a version of these games is referred to as an *unbiased game*. However, many of these unbiased games happen to be easy wins for Maker. To level the playing field, we may instead allow Breaker to claim b edges during each of her turns, for some pre-decided positive integer b , while allowing Maker to claim a single edge in each of her turns. More generally, we consider the *biased* ($a : b$) Maker-Breaker positional game, where Maker claims exactly a edges in each of her turns, and Breaker claims exactly b edges in each of hers. In particular, $a = b = 1$ represents the unbiased game discussed above. Going forward, we shall assume a game to be unbiased unless stated otherwise. A natural question that arises in such biased games, particularly for the $(1 : b)$ biased game, is as follows: what is the minimum value $b^*(n)$ of the bias b , where n is the size of the graph G (i.e. $|V| = n$), at which the game transitions from being a Maker’s win to a Breaker’s win (i.e. for $b < b^*(n)$, Maker wins, whereas for $b \geq b^*(n)$, Breaker wins)? The value $b^*(n)$ is called the *bias threshold* for that game.

Our work considers a framework with additional *directional constraints* on the winning sets of the traditional Maker-Breaker positional game played on graphs. In particular, we study a ‘directionally constrained’ variant of the usual triangle game ([CE78], [BS11], [GS22]), i.e. we consider directed triangles (or 3-cycles) as our winning sets, and our board is a *tournament* (i.e. a complete directed graph) instead of an undirected graph. For the unbiased version of this directed triangle game on the *parity tournament* $\Pi(n)$ (a special type of tournament with a more ‘regular’ or ‘balanced’ distribution of incoming edges across vertices), we proved that for $n \leq 6$, Breaker has a winning strategy for the game played on $\Pi(n)$, while for all $n > 6$, Maker has a winning strategy. We also identify that the bias threshold for this directed triangle game on the parity tournament is of the order $\sqrt{n}$, matching the order of its undirected counterpart (as seen in [CE78]), and provide an upper and lower bound on $b^*(n)$.

Moving away from the parity tournament, we address the Maker-Breaker directed triangle game played on a random tournament $T(n, p)$ in which, for every pair i, j of vertices with $i < j$, the edge between i and j is oriented, independent of all else, *from i towards j* with probability p , and *from j towards i* with probability $(1 - p)$, where $p \in (0, 1)$ is a given constant. We show that Maker wins with probability tending to 1 as the board size (i.e. the size of the tournament, n) tends to infinity, in this particular game.

Finally, we introduce a new notion of bias, called the *flip-bias*, for the directed triangle game on the parity tournament, whereby Breaker is allowed to preemptively and strategically flip the directions of a fixed number of edges *before* the game begins. These perturbations allow Breaker some additional advantage on the resultant board. For the parity tournament of size n , we define the *flip-bias threshold*, $\kappa^*(n)$, to be the minimum number of edge flips required by Breaker to guarantee a win on the perturbed board (i.e. if the number of edge flips performed by Breaker is less than $\kappa^*(n)$, Maker wins, while if Breaker flips the directions of $\kappa^*(n)$ or more strategically chosen edges, she can ensure a win for herself). We establish that $\kappa^*(n)$ is asymptotically equal to $n^2/4$ as $n \rightarrow \infty$.

1.1 Literature Review

Erdős and Selfridge in [ES73] introduced the notion of Maker-Breaker games, along with the use of potential functions for game analysis, and the formulation of derandomisation techniques to construct explicit winning strategies. The Erdős-Selfridge Theorem from [ES73] is a cornerstone result that provided a criterion for Breaker’s win in unbiased games, which kicked off a cascade of very exciting results in this field of work. Later, Chvátal and Erdős in [CE78] introduced bias into the game framework by allowing Breaker to claim additional moves in each of her turns, thereby proposing a natural question of identifying the bias threshold. Over the past few decades, József Beck, via his many papers ([Bec81], [Bec82], [Bec85], [Bec94]) and an exhaustive monograph ([Bec08]), has proved many insightful results and generalisations of the previous work. These works address a variety of winning substructures that Maker aims to capture, like triangles, Hamiltonian cycles, spanning trees, cliques of fixed sizes, etc. For an overarching introduction to the topic of positional games and its main results, we refer the reader to [Hef+14]. The triangle game

was previously studied by Chvátal and Erdős in [CE78], where they proved that the corresponding bias threshold lies between $\sqrt{2n}$ and $2\sqrt{n}$, where n is the size of the board. Balogh and Samotij in [BS11] improved the upper bound from [CE78] to $1.958\sqrt{n}$ using a randomised strategy. Recently, Glazik and Srivastav in [GS22] further improved this upper bound to $1.633\sqrt{n}$. Tightening these bounds remains a problem that still eludes us.

Our motivation for addressing the Maker-Breaker directed triangle game stems from the intention of introducing directional constraints to the winning sets and analysing similar threshold phenomena and potential differences with their undirected counterparts. Games of a similar flavour have been addressed in [FP21], which analysed the strong connectivity game and the Hamiltonicity game played on a *complete digraph*. Other related games played on undirected boards, but with some directional aspect incorporated into them, include the *tournament game* (studied in [CGL16] and [CM15]), where the board is an undirected complete graph K_n or the Erdős-Rényi random graph $G(n, p)$: whenever Maker claims an undirected edge, she also assigns it a direction, and Maker wins if her set of claimed directed edges contains a prespecified tournament T_k . Another class of games with directional constraints are referred to as the *orientation games* ([BS98], [BKS12], [CL17]), in which players alternately orient the undirected edges (that have not been previously oriented by either player) of K_n to produce a single final tournament: Maker wins if that final tournament has a property P (which is concerned with the presence of a prespecified winning substructure).

1.2 Notation and terminology

Our board of interest for the games discussed in this work are *tournaments*, which are complete graphs endowed with *directed edges*. Given a vertex set $V_n = \{1, 2, \dots, n\}$, we denote by (i, j) the *directed edge* that goes *from* the vertex i to the vertex j , for all distinct $i, j \in V_n$.

Definition 1.1 (Tournament).

A *complete digraph* or a *tournament*, $T(n)$, on V_n is a graph in which precisely one of the two directed edges, (i, j) and (j, i) , is present, for all distinct $i, j \in V_n$, there are no self-loops nor any parallel edges.

We denote by $E(T(n))$ the set of all directed edges present in $T(n)$. For any three distinct vertices $i, j, k \in V_n$, we say that (i, j, k) forms a *directed triangle* if the directed edges (i, j) , (j, k) and (k, i) are all present in $E(T(n))$.

We can now define the *unbiased Maker-Breaker directed triangle game* on a tournament.

Definition 1.2 ((1:1) Maker-Breaker directed triangle game). Let $T(n)$ be a tournament on the vertex set given by V_n . The board on which the Maker-Breaker directed triangle game is played is the edge set $E(T(n))$ of the tournament, and the winning sets are the collection of triples of edges corresponding to all possible directed triangles in $T(n)$. Each winning set comprises the three edges that form the respective directed triangle. We can describe the unbiased (1:1) Maker-Breaker directed triangle game on any such tournaments $T(n)$ as follows:

1. Maker plays the first move and claims one edge from the tournament;
2. Breaker plays next and deletes one edge from the tournament;
3. this goes on alternately until all the edges of the tournament have been exhausted;
4. the winner is:
 - Maker, if she manages to claim a directed triangle by the end of the game,
 - Breaker, if she prevents Maker from claiming such a directed triangle.

We will assume that any game referred to here onward is unbiased, and that Maker is the first player by default, unless stated otherwise.

In particular, we study different *types* of tournaments as our underlying boards for the directed triangle game. To begin with, we first focus on *parity tournaments* $\Pi(n)$ in which the direction of each edge is dictated by a certain parity-based rule.

Definition 1.3 (Parity tournament). In the parity tournament $\Pi(n)$, for any two vertices $i, j \in V_n$ with $i < j$, the direction of the edge between i and j is defined as follows:

- If $(i + j)$ is odd, then $(i, j) \in E(\Pi(n))$ (i.e., the edge is directed from i to j).

- If $(i + j)$ is even, then $(j, i) \in E(\Pi(n))$ (i.e., the edge is directed from j to i).

Another tournament variant of interest for the directed triangle game is the random tournament $T(n, p)$, where the edge between any two distinct $i, j \in V_n$, with $i < j$, will be independently directed from i to j with fixed arbitrary probability p , and from j to i with probability $(1 - p)$, where $p \in (0, 1)$ is a constant.

2 Main Results

In this section, we state our main findings concerning the Maker–Breaker directed triangle game. Our results can naturally divide into three parts. In the first part we consider the game played on the parity tournament, in the second part we study the game on a random tournament with constant edge-orientation probability, and in the third part we introduce the *flip-bias* mechanism, which endows Breaker with additional power by allowing her to reverse the orientation of a fixed number of edges before the game begins.

2.1 Maker–Breaker directed triangle game on the parity tournament

Recall that the *parity tournament* $\Pi(n)$ on n vertices is defined by the rule that for every pair $i, j \in [n]$ with $i < j$,

$$\begin{cases} (i, j) \in E(\Pi(n)) & \text{if } i + j \text{ is odd,} \\ (j, i) \in E(\Pi(n)) & \text{if } i + j \text{ is even.} \end{cases}$$

Our first theorem addresses the size threshold for the board to turn the game into a Maker’s win in the unbiased case and the second one identifies the bounds on the bias threshold for the biased $(1 : b)$ game.

Theorem 2.1. *For the Maker–Breaker directed triangle game played on the parity tournament $\Pi(n)$:*

- If $n \leq 6$, then Breaker has a winning strategy.
- If $n > 6$, then Maker has a winning strategy.

This result identifies $n = 7$ as the threshold; in particular, the tournament $\Pi(7)$ is the smallest instance for which Maker’s winning strategy exists. We use a *cycle hopping strategy*, which will be described in detail when we present the proof, to outline Maker’s winning strategy for $n \geq 7$. For $n \leq 6$, Breaker can use some variants of a pairing strategy to ensure herself a win.

Theorem 2.2. *Let $b^*(n)$ denote the threshold bias for the $(1 : b)$ Maker–Breaker directed triangle game on $\Pi(n)$. Then, for sufficiently large n ,*

$$\sqrt{\left(\frac{1}{12} + o(1)\right)n} \leq b^*(n) \leq \sqrt{\left(\frac{8}{3} + o(1)\right)n}. \quad (1)$$

2.2 Maker’s win on random tournaments

Let $T(n, p)$ be a random tournament on the vertex set V_n , in which, for every pair i, j of vertices with $i < j$, the edge between i and j is oriented from i to j with probability p , and from j to i with probability $(1 - p)$, where $p \in (0, 1)$ is a constant. Our next theorem establishes that Maker wins the directed triangle game on $T(n, p)$ with high probability.

Theorem 2.3. *For any fixed $p \in (0, 1)$, the Maker–Breaker directed triangle game played on the random tournament $T(n, p)$ is a Maker’s win with probability tending to 1 as $n \rightarrow \infty$.*

The proof for this theorem relies on showing that a copy of the parity tournament $\Pi(7)$ is contained within $T(n, p)$ with high probability. We already have a winning strategy for Maker on $\Pi(7)$, and by an extension argument, Maker’s winning strategy on $\Pi(7)$ can then be implemented to devise a winning strategy for her on $T(n, p)$.

Generally, on a tournament $T(n)$, one can define a notion of how "balanced" it is based on the in-degrees and out-degrees of its vertices. Tournaments in which each vertex has equal (when n is odd) or near equal (when n is even) in-degree and out-degree for each vertex are called *regular or near-regular tournaments*. Regular tournaments can be thought of

as being the most balanced. At the other end of the spectrum, we have *transitive tournaments*, which are tournaments where the vertices can be ordered such that every edge is directed from a vertex indexed "higher" in the order to a vertex indexed "lower" in the order. Transitive tournaments are the most imbalanced.

Remark 2.1. The out-degree of a vertex is called its *score*. The scores of vertices in V_n play a crucial role in determining the number of directed triangles (or 3-cycles) in the corresponding tournament. Particularly, as described in [Moo15], we know that a regular or near-regular tournament contains the maximum number of directed triangles for a tournament of that size, whereas the transitive tournament of the same size is acyclic (there is no directed cycle).

2.3 Flip-Bias Threshold

To counteract Maker's advantage in the unbiased game on the parity tournament $\Pi(n)$, we introduce a new mechanism to level the playing field referred to as the *flip-bias*. In this variant of the game, Breaker is allowed to preemptively reverse (or *flip*) the orientation of a number of edges before the (unbiased) game begins. This strategic flipping done by Breaker would decrease the regularity of the tournament, increase the imbalance, and consequently decrease the number of directed triangles (which, recall, form our winning sets) in the tournament. Given enough such flips, Breaker can convert the parity tournament into a transitive tournament with no directed triangles, which will ensure Breaker's victory. We denote by $\kappa^*(n)$ the minimum number of edge flips required on the parity tournament so that the resulting board is a Breaker's win. For ease of calculations, we assume n to be odd for any discussion related to the *flip-bias*, since the parity tournament on odd n is exactly regular and balanced, but we emphasise here to the reader that the argument for even n will be *very similar* modulo some minor tweaks. Our analysis shows the following.

Theorem 2.4. *Assume n is odd. Then, asymptotically,*

$$\kappa^*(n) \sim \frac{n^2}{4}.$$

Taken together, Theorems 2.1–2.4 form the core of our results on the Maker–Breaker directed triangle game. The first couple of results establish the critical size and bias thresholds for the game on the parity tournament, the third shows Maker's win on random tournaments for any fixed $p \in (0, 1)$, and the fourth introduces the flip-bias mechanism and its threshold, thereby quantifying the extra power Breaker must use to counteract Maker's advantage in the unbiased directed triangle game on the parity tournament.

3 Proofs of our results stated in Subsection 2.1

To facilitate the proofs for the theorems stated in Section 2, we first prove some results regarding the structure and count of the directed triangles in a parity tournament.

3.1 Structural characterization of directed triangles – our winning sets

Recall that the parity tournament $\Pi(n)$ is defined on the vertex set V_n . A triple (a, b, c) is said to be a directed triangle in a tournament if $(a, b), (b, c), (c, a)$ are present in $E(T(n))$. Any cyclic rotation of this triple will represent the same underlying directed triangle, and thus we define its canonical representation to be the one in which the smallest element is written first; that is, if $a = \min\{a, b, c\}$, then we take (a, b, c) as the unique canonical representative of the triangle.

Below, we shall establish necessary and sufficient conditions for such a canonical triple to correspond to a directed triangle in the parity tournament. Particularly, we will prove that these conditions force the three vertices to be arranged in ascending order and have alternating parity.

Lemma 3.1. *Let $a, b, c \in V_n$ with $a = \min\{a, b, c\}$. The directed triangle (a, b, c) with edges $(a, b), (b, c), (c, a)$ exists in the parity tournament $\Pi(n)$ (as defined in Definition 1.3) if and only if $a + b$ is odd, $b + c$ is odd, $a + c$ is even and $a < b < c$.*

Proof. In order for (a, b, c) to form a directed triangle in the parity tournament, we require the following (as dictated by the parity rule described in Definition 1.3):

1. Since the edge (a, b) exists and we know that $a < b$, it must be that $a + b$ is odd.

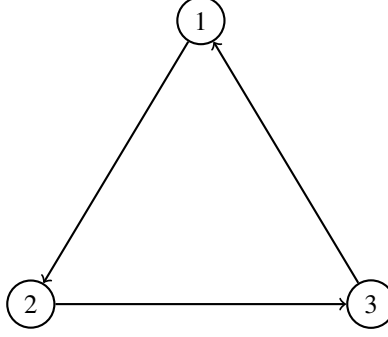


Figure 1: Example of a directed triangle $(1, 2, 3)$ in a parity tournament. Here, $1 + 2 = 3$ (odd), $2 + 3 = 5$ (odd), and $1 + 3 = 4$ (even), satisfying the conditions in Subsection 3.1.

2. Similarly, the edge (c, a) exists where $a < c$, and thus $a + c$ must be even.
3. Now that we know $a + b$ is odd and $a + c$ is even, we can conclude that a and b have opposite parity, while a and c have the same parity. Consequently, b and c have opposite parity, which implies that $b + c$ is odd. Our directed triangle also requires the edge (b, c) to exist, and thus, it must be that $b < c$.

Conversely, let us assume that $a < b < c$ and that $a + b$ is odd, $b + c$ is odd, $a + c$ is even. Then, the parity rule implies the following:

1. Since $a + b$ is odd and $a < b$, the edge between a and b is oriented as (a, b) .
2. Since $b + c$ is odd and $b < c$, the edge between b and c is oriented as (b, c) .
3. Since $a + c$ is even and $a < c$, the edge between a and c is oriented as (c, a) .

These three directed edges together form the directed triangle (a, b, c) . This completes the proof. $\square$

3.2 Counting the winning sets

Having characterised the structure of directed triangles in $\Pi(n)$, it is only natural to count them next. Many criteria determining whether $\Pi(n)$ is a Maker's win or a Breaker's win rely heavily on the count of these winning sets.

Lemma 3.2. *In a parity tournament $\Pi(n)$, the number of directed triangles $w(n)$ is $\sum_{i=1}^{n-2} \left\lceil \frac{i}{2} \right\rceil \cdot \left\lceil \frac{n-(i+1)}{2} \right\rceil$.*

Proof. Fix $b \in \{2, \dots, n-1\}$. In order to come up with triples (a, b, c) that satisfy the conditions stated in Lemma 3.1, we must choose

- a from $\{1, \dots, b-1\}$ with parity opposite to b , and
- c from $\{b+1, \dots, n\}$ with parity opposite to b as well, so that a and c match each other's parity.

The number of integers in $\{1, \dots, b-1\}$ with parity different from b is $\left\lceil \frac{b-1}{2} \right\rceil$. Similarly, the number of integers in $\{b+1, \dots, n\}$ with parity different from b is $\left\lceil \frac{n-b}{2} \right\rceil$. Finally, summing over b we get

$$w(n) = \sum_{b=2}^{n-1} \left\lceil \frac{b-1}{2} \right\rceil \cdot \left\lceil \frac{n-b}{2} \right\rceil = \sum_{i=1}^{n-2} \left\lceil \frac{i}{2} \right\rceil \cdot \left\lceil \frac{n-(i+1)}{2} \right\rceil.$$

$\square$

As expected, this count also aligns with a classical result from [Moo15] concerning 3-cycles in arbitrary tournaments.

Proposition 3.3 (Theorem 4 from [Moo15]). *Let $c_3(T(n))$ denote the number of 3-cycles in the tournament $T(n)$. If $(s_1, s_2, \dots, s_n)$ is the score vector of $T(n)$, where s_i is the number of outgoing edges from vertex i , then*

$$c_3(T(n)) = \binom{n}{3} - \sum_{i=1}^n \binom{s_i}{2} \leq \begin{cases} \frac{1}{24}(n^3 - n), & \text{if } n \text{ is odd,} \\ \frac{1}{24}(n^3 - 4n), & \text{if } n \text{ is even.} \end{cases}$$

Equality holds only for regular tournaments.

Since $\Pi(n)$ is also a special type of regular tournament, we find that $w(n)$, once simplified, is exactly equal to the upper bound from Proposition 3.3.

3.3 Breaker's winning strategy on $\Pi(3)$, $\Pi(4)$, $\Pi(5)$

To begin with, we address the Maker-Breaker directed triangle game played on $\Pi(n)$ for $n = 3, 4, 5$.

Definition 3.1 (Pairing strategy). A pairing strategy for Breaker begins with finding a collection of pairwise disjoint pairs of edges of the board such that each directed triangle contains, fully, one of these pairs. We refer to the two edges, e_1 and e_2 , in any given pair $\{e_1, e_2\}$, as *siblings*, and such a pair is called a *blocking pair*. The strategy prescribes Breaker to, in each of her moves, respond by deleting the sibling of the edge claimed by Maker in the previous turn. In case Maker's previously claimed edge was not a part of any pair, Breaker arbitrarily deletes any edge in the tournament.

Because Breaker always deletes the sibling of whichever edge Maker claims, Maker can never claim both edges of any pair. Recall, from Definition 3.1, that each of our winning sets (i.e. each directed triangle present in our tournament) contains precisely one of these pairs of edges, and one of the siblings belonging to each such pair is always deleted by Breaker. Consequently, Maker cannot claim any of the winning sets entirely, ensuring a win for Breaker.

We refer to the phenomenon where Breaker prevents Maker from claiming a directed triangle, say (a, b, c) , by deleting one of the siblings belonging to the pair, say $\{e_1, e_2\}$, that is contained in (a, b, c) , as " $\{e_1, e_2\}$ blocking the directed triangle (a, b, c) ".

As demonstrated in Figure 3 (here, the paired directed edges are indicated by connecting them using dashed curves), such a pairing strategy can be very useful for determining the outcome of certain games. Another way to represent such pairings, and, more generally, the winning sets in a game, is via a suitable hypergraph. This approach affords us a much easier visualization of the board as well as the progression of the game. For instance, the winning sets arising out of $\Pi(5)$, as well as the pairing illustrated in Figure 3, are more readily seen via the hypergraph representation shown in Figure 2. Particularly, each hyperedge in this hypergraph representation corresponds to the set of 3 directed edges that form a directed triangle in the underlying tournament, and any element in such a hyperedge corresponds to a directed edge from the underlying tournament.

Lemma 3.4. *Breaker has a winning strategy in the Maker-Breaker directed triangle game on $\Pi(n)$, where $n = 3, 4, 5$.*

Proof. To show Breaker's win, it suffices to show that a pairing of edges exists such that conditions mentioned in Definition 3.1 are satisfied. Breaker can then use her pairing strategy to ensure herself a win. Let us consider $n = 3$. According to the parity rule, we have only one directed triangle, which is given by the winning set

$$F_1 = \{(1, 2), (2, 3), (3, 1)\}.$$

We can create an arbitrary pair in this case, say $\{(1, 2), (2, 3)\}$. This pair blocks the only single winning set present, and Breaker wins.

For $n = 4$, the available winning sets in $\Pi(4)$ are

$$F_1 = \{(1, 2), (2, 3), (3, 1)\} \quad \text{and} \quad F_2 = \{(2, 3), (3, 4), (4, 2)\}.$$

For these directed triangles, a possible collection of pairs can be $\{(1, 2), (3, 1)\}$ and $\{(3, 4), (4, 2)\}$, where the former pair blocks F_1 and the latter blocks F_2 .

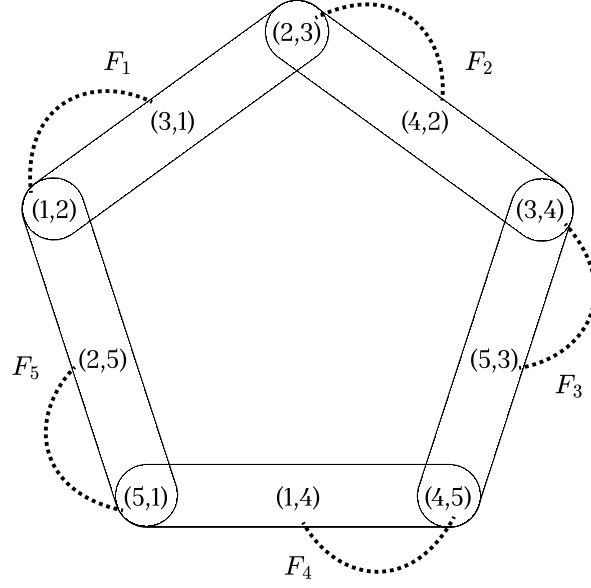


Figure 2: Pairing on $\Pi(5)$'s hypergraph representation

When $n = 5$, there are exactly five winning sets in $\Pi(5)$, which we denote by

$$\begin{aligned} F_1 &= \{(1, 2), (3, 1), (2, 3)\}, \\ F_2 &= \{(2, 3), (4, 2), (3, 4)\}, \\ F_3 &= \{(3, 4), (5, 3), (4, 5)\}, \\ F_4 &= \{(4, 5), (1, 4), (5, 1)\}, \\ F_5 &= \{(5, 1), (2, 5), (1, 2)\}. \end{aligned}$$

Observe that the five winning sets have a cyclic overlap pattern, as shown in Figure 2 (once again, this demonstrates how a hypergraph representation of the game board is visually, and therefore, analytically, helpful). Here, the directed edges are $a_1 = (1, 2)$, $a_2 = (3, 1)$, $a_3 = (2, 3)$, $a_4 = (4, 2)$, $a_5 = (3, 4)$, $a_6 = (5, 3)$, $a_7 = (4, 5)$, $a_8 = (1, 4)$, $a_9 = (5, 1)$, $a_{10} = (2, 5)$, and the winning sets are as follows:

$$\begin{aligned} F_1 &= \{a_1, a_2, a_3\}, \\ F_2 &= \{a_3, a_4, a_5\}, \\ F_3 &= \{a_5, a_6, a_7\}, \\ F_4 &= \{a_7, a_8, a_9\}, \\ F_5 &= \{a_9, a_{10}, a_1\}. \end{aligned}$$

We now come up with the following collection of edge-pairs that satisfies the constraints stated in Definition 3.1 (indicated by the dashed curves in Figure 2):

$$\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}, \{a_7, a_8\}, \{a_9, a_{10}\}.$$

Note that each winning set F_i contains exactly one of the five edge-pairs mentioned above:

$$\begin{aligned} F_1 &= \{a_1, a_2, a_3\} \text{ contains the pair } \{a_1, a_2\}, \\ F_2 &= \{a_3, a_4, a_5\} \text{ contains the pair } \{a_3, a_4\}, \\ F_3 &= \{a_5, a_6, a_7\} \text{ contains the pair } \{a_5, a_6\}, \\ F_4 &= \{a_7, a_8, a_9\} \text{ contains the pair } \{a_7, a_8\}, \\ F_5 &= \{a_9, a_{10}, a_1\} \text{ contains the pair } \{a_9, a_{10}\}. \end{aligned}$$

Hence, Breaker's pairing strategy, as outlined in Definition 3.1, guarantees her a win. $\square$

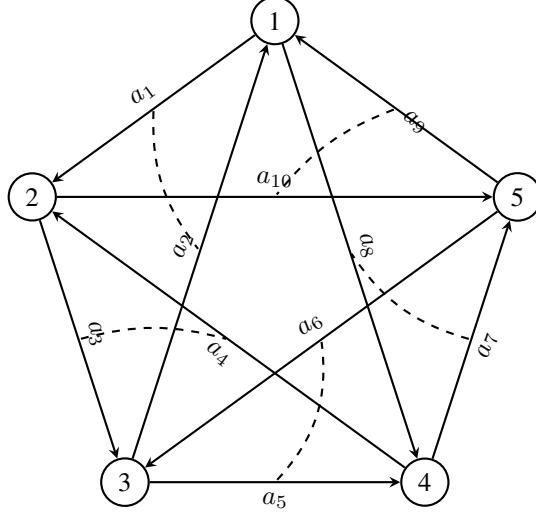


Figure 3: Diagram of $\Pi(5)$ with edges labeled $a_1, \dots, a_{10}$ and dashed arcs showing Breaker's pairing strategy.

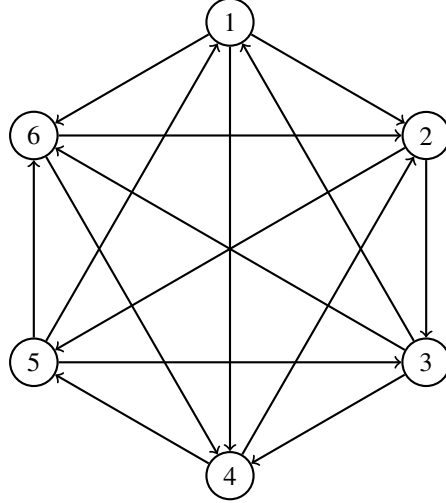


Figure 4: $\Pi(6)$

3.4 Breaker's winning strategy on $\Pi(6)$

Lemma 3.5. *Breaker wins the Maker-Breaker directed triangle game on $\Pi(6)$.*

Proof. In the parity tournament $\Pi(6)$, there are 8 winning sets (i.e. directed triangles), listed as follows:

$$\begin{aligned}
 F_1 &= \{(1, 2), (3, 1), (2, 3)\}, \\
 F_2 &= \{(2, 3), (4, 2), (3, 4)\}, \\
 F_3 &= \{(3, 4), (5, 3), (4, 5)\}, \\
 F_4 &= \{(4, 5), (6, 4), (5, 6)\}, \\
 F_5 &= \{(5, 6), (6, 2), (2, 5)\}, \\
 F_6 &= \{(2, 5), (5, 1), (1, 2)\}, \\
 F_7 &= \{(2, 3), (3, 6), (6, 2)\}, \\
 F_8 &= \{(5, 1), (1, 4), (4, 5)\}.
 \end{aligned}$$

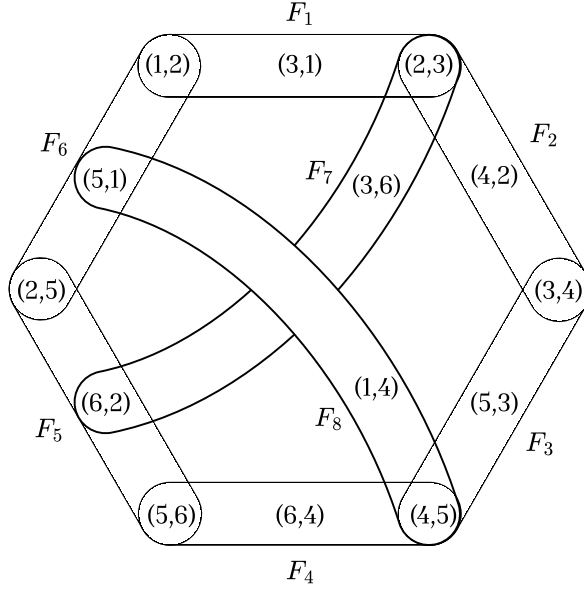


Figure 5: $\Pi(6)$'s hypergraph representation

Once again, we use a pairing strategy, albeit a more nuanced one compared to what we adopted for the parity tournaments of smaller sizes, to demonstrate Breaker's win on $\Pi(6)$. The identity of the player who claims/deletes the edge $(2, 5)$ plays a crucial role in how the pairing is structured, as will be evident in the proof that follows.

Recall that a Maker-Breaker game can be represented as a hypergraph where each hyperedge corresponds to a winning set, which, in our case, is the set of directed edges that form a directed triangle in the underlying parity tournament. Each element in any such hyperedge corresponds to an edge from the parity tournament. A formal definition is provided below.

Definition 3.2 (Hypergraph representation). The Maker-Breaker directed triangle game on any tournament can be represented by a hypergraph $\mathcal{H} = (X, \mathcal{F})$ where:

- X is the set of all directed edges in the tournament;
- $\mathcal{F}$ is the family of hyperedges, with each hyperedge $F \in \mathcal{F}$ consisting of three elements (three directed edges that together form a unique directed triangle in the tournament).

To avoid any confusion, when we say an edge or directed edge, we mean an edge from the underlying tournament, and when we say hyperedge, we are referring to a winning set $F \in \mathcal{F}$ in the hypergraph representation.

A concept that will help us track the effect of Breaker's moves on the underlying hypergraph representation (henceforth denoted by $\mathcal{H}$) of the game is called the *cut* induced by her move.

Definition 3.3 (Cut induced by Breaker's move). Given the hypergraph representation $\mathcal{H} = (X, \mathcal{F})$ of the Maker-Breaker directed triangle game, and a directed edge e from the underlying tournament (recall, from Definition 3.2, that e is an element in X), the *cut* induced by Breaker's move involving the deletion of the edge e is defined as

$$\text{Cut}(\mathcal{H}, e) := (X', \mathcal{F}'),$$

where $\mathcal{F}' = \{F \in \mathcal{F} : e \notin F\}$ is called the family of surviving directed triangles, and $X' = \bigcup_{F \in \mathcal{F}'} F$ is the set of surviving directed edges.

This operation removes all winning sets containing e , reflecting the *blocking* effect of Breaker's move. More generally, we say that a winning set is blocked if it either already contains an edge deleted by Breaker, or contains a blocking pair

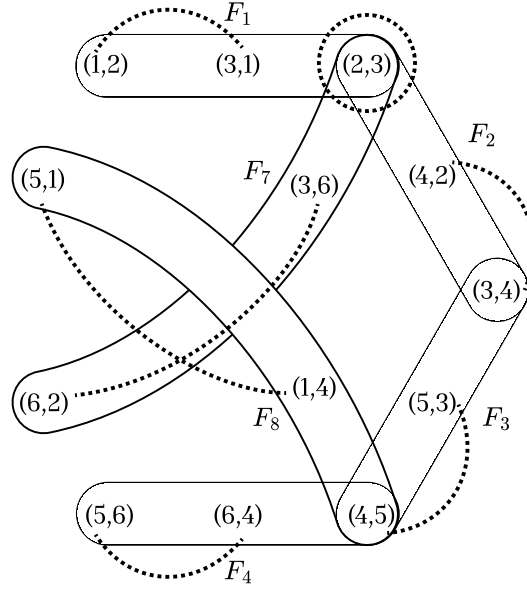


Figure 6: Default pairing on surviving winning sets for Case 1

as defined in Definition 3.1. Furthermore, any element of the hypergraph (which, recall, is a directed edge from the underlying tournament) that is no longer contributing to the formation of any *surviving* winning set is also removed from the hypergraph.

Case 1: Please refer to Figure 5 and Figure 6 for this case's analysis. Assume that Maker's first move, m_1 , is not $(2, 5)$. In the winning strategy that we devise for Breaker, she deletes $(2, 5)$ during her first move. This move by Breaker induces $\text{Cut}(\mathcal{H}, (2, 5))$. As a result, the winning sets F_5 and F_6 are removed, and the surviving winning sets are

$$\mathcal{F}' = \{F_1, F_2, F_3, F_4, F_7, F_8\}.$$

To block these surviving sets, Breaker comes up with the following collection of pairs (recall Definition 3.1), and we refer to this as the *default pairing*:

$$\begin{aligned} \{(1, 2), (3, 1)\}, & \text{ which blocks the winning set } F_1 = \{(1, 2), (3, 1), (2, 3)\}; \\ \{(4, 2), (3, 4)\}, & \text{ which blocks the winning set } F_2 = \{(2, 3), (4, 2), (3, 4)\}; \\ \{(5, 3), (4, 5)\}, & \text{ which blocks the winning set } F_3 = \{(3, 4), (5, 3), (4, 5)\}; \\ \{(6, 4), (5, 6)\}, & \text{ which blocks the winning set } F_4 = \{(4, 5), (6, 4), (5, 6)\}; \\ \{(3, 6), (6, 2)\}, & \text{ which blocks the winning set } F_7 = \{(2, 3), (3, 6), (6, 2)\}; \\ \{(5, 1), (1, 4)\}, & \text{ which blocks the winning set } F_8 = \{(5, 1), (1, 4), (4, 5)\}. \end{aligned}$$

Note that $(2, 3)$ remains unpaired in this default pairing, a fact that will prove useful in the upcoming discussion. Under this default pairing, if Maker subsequently claims one edge from a blocking pair in a directed triangle, Breaker immediately deletes its sibling, ensuring that Maker cannot eventually claim all three edges of any directed triangle. This pairing strategy is outlined in Definition 3.1. However, before using this pairing strategy, Breaker must carefully account for Maker's first move m_1 , and readjust the default pairing in case there is any conflict with Maker's first move. Particularly, a conflict would arise if m_1 was, in fact, one of the directed edges that are already participating in this default pairing. We use a *switch cascade* to resolve this conflict by finding a new pairing (by a minimal perturbation of our default pairing) which does not contain m_1 in any of its pairs. The pairs that participate in the default pairing will be henceforth referred to as the default pairs.

A switch cascade is a finite sequence of local pairing adjustments made by Breaker via which she redefines the pair that blocks each winning set, thereby readjusting the default pairing, in order to account for Maker's first move, which could have been a part of a default pair. The length of the switch cascade is equal to the number of successive pairing adjustments made to finally reach a pairing that blocks all winning sets and does not contain any edges claimed/deleted so far (i.e. edges that are still in play).

Switch Cascade Strategy. Note that Maker's first move, m_1 , may belong to at most one default pair. The switch cascade strategy to find a new pairing, in which none of the pairs contains m_1 , goes as follows:

- If m_1 is an edge that does not appear in any default pair (e.g. if $m_1 = (2, 3)$), then no adjustment is needed to the default pairing (switch cascade is of length 0).
- If m_1 belongs to the default pair that blocks some winning set F_j such that $(2, 3) \in F_j$, then we can adjust the pairing such that, under this new pairing, $(2, 3)$ (which is currently not an element in any default pair) is paired with the remaining directed edge forming F_j (i.e. the directed edge other than m_1 and $(2, 3)$ that forms the winning set F_j). This is a switch cascade of length 1.

For example, if $m_1 = (1, 2)$, then we see that m_1 belongs to the default pair $\{(1, 2), (3, 1)\}$, which blocks the winning set $F_1 = \{(1, 2), (3, 1), (2, 3)\}$, and $(2, 3) \in F_1$. In this case, Breaker redefines the pair that blocks F_1 to be $\{(3, 1), (2, 3)\}$. Under our default pairing, such a switch cascade of length 1 suffices when m_1 is chosen to be any of the directed edges $(1, 2)$, $(3, 1)$, $(4, 2)$, $(3, 4)$, $(3, 6)$, and $(6, 2)$.

- If m_1 belongs to a default pair in a winning set F_k , Breaker is required to adjust the default pairing. Please refer to Figure 6 and Figure 7 for a visual representation. First, we note that $(2, 3)$ is part of no default pair. One can find a path from m_1 to $(2, 3)$ along *adjacent* hyperedges (two hyperedges are said to be adjacent if they have one element in common). We denote this path by $(p_1, p_2, p_3, \dots, p_k)$ such that $p_1 = m_1$ and $p_k = (2, 3)$, where any two consecutive elements, p_i and p_{i+1} , in the path have one common hyperedge that they are both a part of. We also require that this path be such that p_1 and p_2 are a default pair, p_3 and p_4 are a default pair and so on until we reach $p_k = (2, 3)$, which is not part of any default pair. Note that this path is *unique* and has an odd length. Under the default pairing, we are assured that m_1 is paired with p_2 , p_3 is paired with p_4 , and so on, until we reach $(2, 3)$, which remains unpaired. Now, we simply adjust the default pairing by leaving m_1 unpaired, and pairing p_2 with p_3 , p_4 with p_5 , and so on until we pair p_{k-1} with $(2, 3)$. This gives us a new pairing which blocks each surviving winning set (in the same sense as discussed after Definition 3.3) while also accounting for Maker's first move, m_1 . This process of adjustment is called the switch cascade, and this cascade's length is given by $\lfloor \frac{k}{2} \rfloor$.
- If m_1 is $(5, 3)$ or $(4, 5)$ (which appear in F_3 with default pair $\{(5, 3), (4, 5)\}$), then a switch cascade of length 2 is required. For instance, if $m_1 = (5, 3)$, then Breaker first reassigns the blocking pair in F_3 to $\{(3, 4), (4, 5)\}$; if $(3, 4)$ is already used in F_2 by the default pairing, Breaker further adjusts F_2 's pairing to $\{(2, 3), (4, 2)\}$.
- If m_1 is $(6, 4)$, $(5, 6)$, $(5, 1)$, or $(1, 4)$ (which occur in F_4 or F_8), then a switch cascade of length 3 is necessary. In this case, the initial local adjustment forces reassignments in two adjacent winning sets to restore a global pairing.

The fact that $(2, 3)$ is not part of any default pair leaves a gap that gives Breaker ample breathing room to adapt to Maker's first move by employing a switch cascade of the appropriate length. In every instance, the cascade is finite and local, and the final outcome is that every surviving winning set $F \in \mathcal{F}'$ after Breaker's first move is blocked by one of the specified pairs. Hence, Breaker wins in Case 1 by using a pairing strategy (recall Definition 3.1) on the surviving winning sets.

Case 2: In this case, Maker claims $m_1 = (2, 5)$ as her first move. Breaker, according to our prescribed strategy, must then respond immediately with $b_1 = (6, 2)$. We now consider two subcases based on Maker's subsequent second move.

Subcase 2A: Suppose Maker's second move is $m_2 = (5, 1)$. Then the winning set $F_6 = \{(2, 5), (5, 1), (1, 2)\}$ contains two directed edges claimed by Maker namely, $m_1 = (2, 5)$ and $m_2 = (5, 1)$. To block Maker from claiming F_6 , Breaker is forced to delete the edge $(1, 2)$ in her next move, i.e. $b_2 = (1, 2)$. After the first four moves, m_1 , b_1 , m_2 , b_2 , the winning sets F_1 , F_5 , F_6 and F_7 are removed from the hypergraph. Refer to Figure 8. The surviving winning sets after the cut induced by b_1 and b_2 are

$$\mathcal{F}'' = \{F_2, F_3, F_4, F_8\}.$$

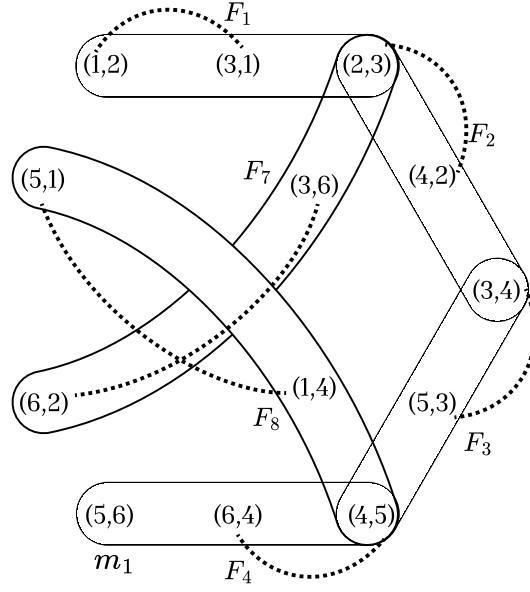


Figure 7: A length 3 switch cascade in case $m_1 = (5, 6)$

We are already aware that $(5, 1)$ is already claimed by Maker as m_2 , and thus it cannot be a part of our pairing. Breaker now decides on the following pairing to block the surviving winning sets:

$$\begin{aligned} \{(2, 3), (4, 2)\}, & \text{ which blocks the winning set } F_2 = ((2, 3), (4, 2), (3, 4)); \\ \{(3, 4), (5, 3)\}, & \text{ which blocks the winning set } F_3 = ((3, 4), (5, 3), (4, 5)); \\ \{(6, 4), (5, 6)\}, & \text{ which blocks the winning set } F_4 = ((4, 5), (6, 4), (5, 6)); \\ \{(1, 4), (4, 5)\}, & \text{ which blocks the winning set } F_8 = ((5, 1), (1, 4), (4, 5)). \end{aligned}$$

Breaker's pairing blocks each remaining winning set. From this point onward, Breaker follows the pairing strategy as outlined in Definition 3.1 and thereby wins the game in Subcase 2A. Since this particular pairing did not require a switch cascade, we call it a *static pairing*.

Subcase 2B: Refer to Figure 9 and Figure 10 for this discussion. In this case, Maker does not choose $(5, 1)$ as her second move (i.e. $m_2 \neq (5, 1)$). Breaker responds by deleting $(5, 1)$ in her second move, i.e. $b_2 = (5, 1)$ (since it remains unclaimed in this case). As a result, the winning sets that contain $(5, 1)$ are blocked, namely F_5, F_6, F_7, F_8 do not survive. After the first four moves, the surviving winning sets are

$$\mathcal{F}''' = \{F_1, F_2, F_3, F_4\}.$$

However, in this particular case, we are yet to account for m_2 . For this, we use a switch cascade strategy similar to the one from Case 1, with the default pairing on $\mathcal{F}'''$ given by:

$$\begin{aligned} \{(1, 2), (3, 1)\}, & \text{ which blocks the winning set } F_1 = \{(1, 2), (3, 1), (2, 3)\}; \\ \{(4, 2), (3, 4)\}, & \text{ which blocks the winning set } F_2 = \{(2, 3), (4, 2), (3, 4)\}; \\ \{(5, 3), (4, 5)\}, & \text{ which blocks the winning set } F_3 = \{(3, 4), (5, 3), (4, 5)\}; \\ \{(6, 4), (5, 6)\}, & \text{ which blocks the winning set } F_4 = \{(4, 5), (6, 4), (5, 6)\}; \end{aligned}$$

Depending on what m_2 was, i.e. what the directed edge claimed by Maker in her second move was, it might conflict with the default pairing mentioned above if m_2 belonged to one of the default pairs. In such a case, Breaker uses her switch cascade strategy (as described in detail for Case 1) to adjust the pairing. For example, if Maker picks $m_2 = (1, 2)$ in F_1 , Breaker reassigns F_1 's blocking pair to be $\{(3, 1), (2, 3)\}$. In other cases, a cascade of local switches of length 1, 2, or 3 may be required, as determined by the specific vertex chosen by Maker. Under the given default pairing, the

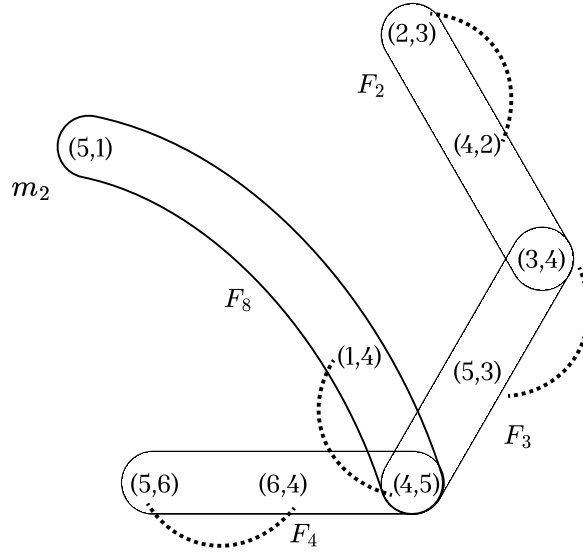


Figure 8: Static pairing on surviving winning sets for Subcase 2A

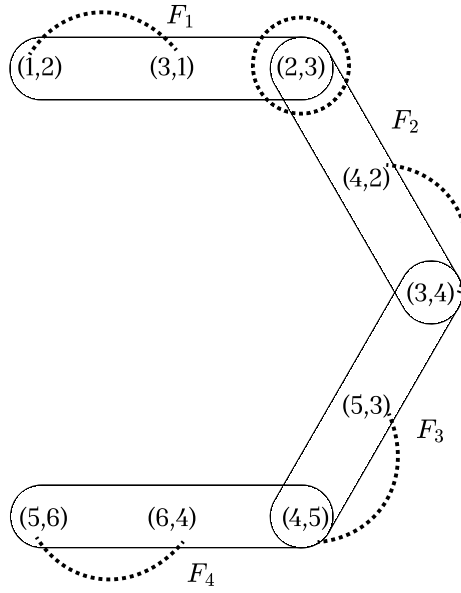


Figure 9: Default pairing on surviving winning sets for Subcase 2B

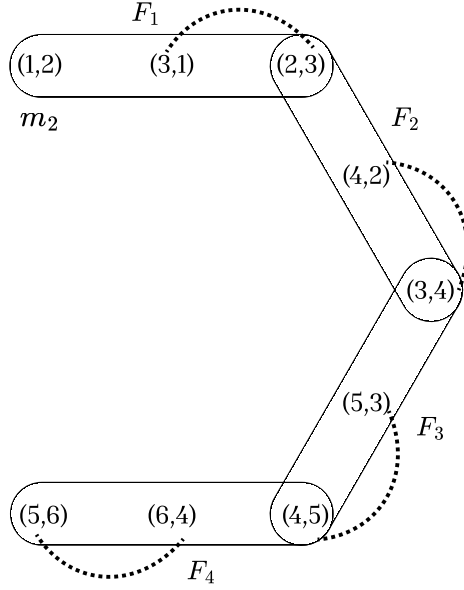


Figure 10: A switch cascade of length 1 in case $m_2 = (1, 2)$

following holds:

$$\begin{aligned}
m_1 = (2, 3) &\implies \text{switch cascade length 0,} \\
m_1 \in \{(1, 2), (3, 1), (4, 2), (3, 4)\} &\implies \text{switch cascade length 1,} \\
m_1 \in \{(5, 3), (4, 5)\} &\implies \text{switch cascade length 2,} \\
m_1 \in \{(6, 4), (5, 6)\} &\implies \text{switch cascade length 3.}
\end{aligned}$$

In every instance, the cascade is finite and local, ensuring that every surviving winning set in $\mathcal{F}'''$ eventually has a blocking pair. As a result, Maker is unable to claim all three edges of any winning set, and Breaker wins in Subcase 2B.

Thus, in both subcases of Case 2, Breaker can successfully block all surviving winning sets, ensuring a Breaker win.

This completes the proof of Lemma 3.5, outlining a winning strategy for Breaker in the Maker-Breaker directed triangle game on $\Pi(6)$. $\square$

3.5 Maker's winning strategy on $\Pi(7)$

Lemma 3.6. *Maker wins the Maker-Breaker directed triangle game on $\Pi(7)$.*

Our approach to proving this statement is twofold. First, we show that regardless of Breaker's first move (modeled via a cut operation as defined in Definition 3.3), there always exists a "cycle" in the surviving hypergraph that starts and ends at the hypergraph-vertex claimed by Maker as m_1 . Second, we demonstrate that Maker can use this cycle to force a win, via a *cycle-hopping strategy*.

Before we proceed, it would help us to define certain structures in the hypergraph representation, $\mathcal{H} = (X, \mathcal{F})$, of the directed triangle game on $\Pi(7)$.

Definition 3.4 (Cycle in a hypergraph). A *cycle*, $\mathcal{C}$, in $\mathcal{H} = (X, \mathcal{F})$ is a sequence of vertices $(v_0, v_1, \dots, v_{m-1}, v_m)$, with $v_0 = v_m$, such that for each $i \in \{1, 2, \dots, m-1\}$ there exists a hyperedge $F_i \in \mathcal{F}$ satisfying $\{v_i, v_{i+1}\} \in F_i$.

Definition 3.5 (Main outer cycle). The cycle $\mathcal{C}$ in $\mathcal{H}$ given by $\mathcal{C} = (v_0, v_1, \dots, v_{14})$, with $v_0 = (1, 2)$, $v_1 = (3, 1)$, $v_2 = (2, 3)$, $v_3 = (4, 2)$, $v_4 = (3, 4)$, $v_5 = (5, 3)$, $v_6 = (4, 5)$, $v_7 = (6, 4)$, $v_8 = (5, 6)$, $v_9 = (7, 5)$, $v_{10} = (6, 7)$, $v_{11} = (1, 6)$, $v_{12} = (7, 1)$, $v_{13} = (2, 7)$, $v_{14} = (1, 2)$, is referred to as the *main outer cycle*.

This main outer cycle is illustrated in Figure 11 by the hyperedges coloured grey. In any cycle, v_i and v_{i+1} belong to a common hyperedge. Note that vertices with even indices, namely v_0, v_2, v_4 , etc, are shared between adjacent winning sets, while odd-indexed vertices $v_1, v_3, \dots, v_{13}$ appear in only one winning set.

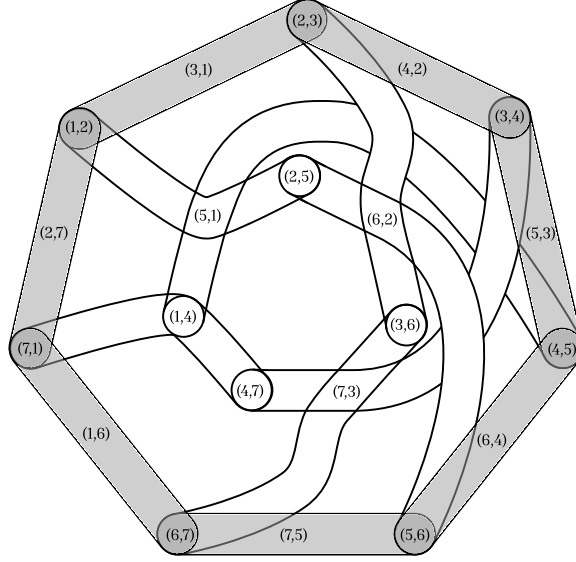


Figure 11: The Main outer cycle C

Definition 3.6 (Internal bridges). In addition to the outer cycle C , the hypergraph contains internal structures, called *bridges*, that connect vertices on C . It is evident from Figure 12 that there are four such bridges:

$$\begin{aligned} B_1 & : ((1, 2), (5, 1), (2, 5), (6, 2), (5, 6)), \\ B_2 & : ((2, 3), (6, 2), (3, 6), (7, 3), (6, 7)), \\ B_3 & : ((3, 4), (7, 3), (4, 7), (1, 4), (7, 1)), \\ B_4 & : ((4, 5), (5, 1), (1, 4), (4, 7), (7, 1)). \end{aligned}$$

In each B_i , each endpoint lies on C and has degree three (i.e. they appear in three winning sets). For example, in B_1 , the endpoints are $(1, 2)$ and $(5, 6)$, each of which are part of three winning sets.

We now prove that, regardless of Breaker's first move, b_1 , there exists a cycle in the surviving hypergraph, $\text{Cut}(\mathcal{H}, b_1)$, containing Maker's first claimed element, m_1 . The lemma stated below assumes, without loss of generality, that $m_1 = (1, 2)$. However, an analogous claim holds for any other choice of m_1 which lies on the main outer cycle and is a part of three winning sets (i.e., $m_1 \in \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 1)\}$). Equivalently, a claim analogous to Lemma 3.7 is true whenever m_1 is an endpoint of an internal bridge.

Lemma 3.7. *If $m_1 = (1, 2)$, the surviving hypergraph $\mathcal{H}' = \text{Cut}(\mathcal{H}, b_1)$, obtained after Breaker deletes the element b_1 during her first move, contains a cycle C of the form stated in Definition 3.4, with $v_0 = v_m = (1, 2)$.*

Proof. We divide the proof into two cases, as follows.

Case 1: For this particular case refer to Figure 13. Suppose b_1 is an internal vertex (i.e., b_1 does not belong to the main outer cycle, $b_1 \notin C$). In this case, the main outer cycle C remains intact in $\mathcal{H}'$ since none of its vertices has been removed, and as $(1, 2) \in C$, our claim stated in Lemma 3.7 is fulfilled by C itself.

Case 2: Suppose b_1 is an element on C (i.e., $b_1 \in C$). We now further divide this case into two subcases, as follows.

- (a) For this subcase, we refer to Figure 14. Suppose b_1 is the endpoint of the internal bridge determined by Maker's first move, m_1 . In our case, $m_1 = (1, 2)$, which lies on Bridge B_1 (as defined in Definition 3.6 and shown in

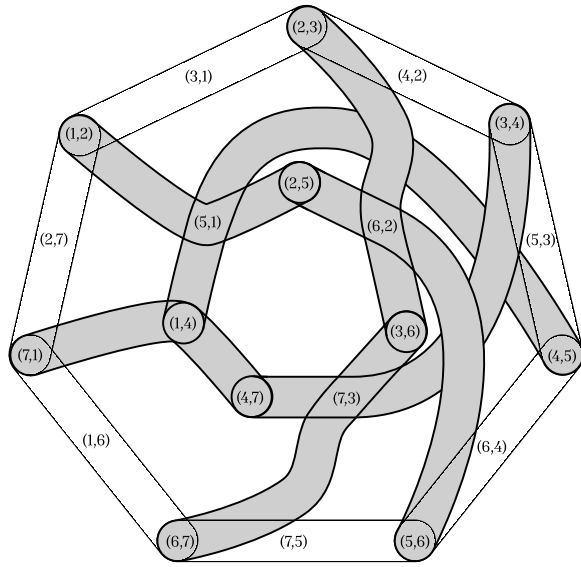


Figure 12: Internal bridges

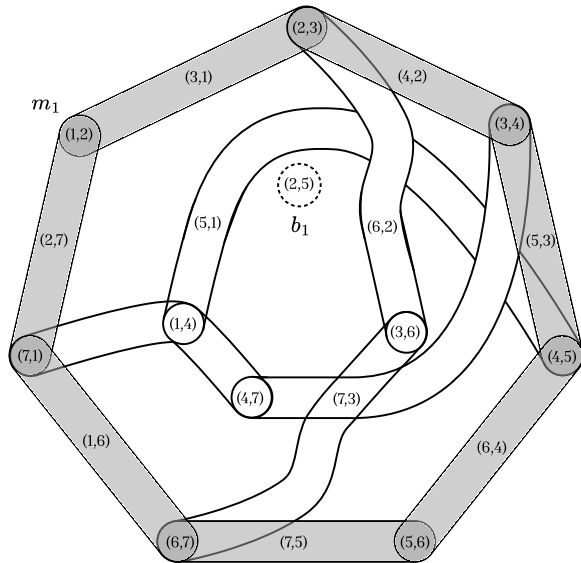


Figure 13: Internal Cut

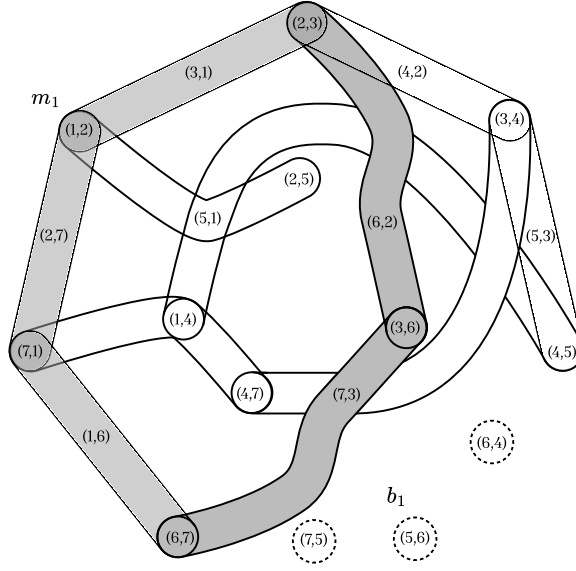


Figure 14: $\text{Cut}(\mathcal{H}, (5, 6))$ when $m_1 = (1, 2)$ and $b_1 = (5, 6)$

Figure 12), with endpoints $(1, 2)$ and $(5, 6)$. Therefore, $b_1 = (5, 6)$ in this case. If Breaker deletes $(5, 6)$ in her first move, then Bridge B_1 is, in a sense, "broken" and the portion of C containing $(5, 6)$ is removed, as modelled by $\text{Cut}(\mathcal{H}, (5, 6))$. However, since the hypergraph contains alternative internal bridges, Maker can still identify cycles which utilise these intact internal bridges and the surviving portion of the main outer cycle. In this particular case where, $m_1 = (1, 2)$ and $b_1 = (5, 6)$, one such surviving cycle is

$$C' = ((1, 2), (3, 1), (2, 3), (6, 2), (3, 6), (7, 3), (6, 7), (1, 6), (7, 1), (2, 7), (1, 2)).$$

Intuitively, C' can be identified by first traversing along C to the endpoint, $(2, 3)$, of the adjacent bridge, B_2 , and then using this internal bridge to bypass the disrupted portion of the hypergraph due to Breaker's first move, b_1 . We refer the reader to Figure 14 for a visual representation of this argument. Similarly, the other intact internal bridges (that is, B_3 and B_4) can be used to identify two more surviving cycles starting and ending at $m_1 = (1, 2)$. Hence, at least one cycle containing $(1, 2)$ exists in $\text{Cut}(\mathcal{H}, (5, 6))$.

- (b) Suppose b_1 is on C but is not the endpoint, $(5, 6)$, of the designated bridge, B_1 . We refer the reader to Figure 15. In this subcase, Maker's designated bridge, determined by $m_1 = (1, 2)$, namely B_1 , remains intact. All eligible possibilities for b_1 in this subcase can be partitioned into two categories, L and R , depending on their relative positions (to the "left" or to the "right") with respect to the internal bridge, B_1 . Referring to Figure 15, we identify

$$\begin{aligned} L &= \{(2, 7), (7, 1), (1, 6), (6, 7), (7, 5)\}, \\ R &= \{(3, 1), (2, 3), (4, 2), (3, 4), (5, 3), (4, 5), (6, 4)\}. \end{aligned}$$

If $b_1 \in L$, we identify the cycle:

$$C'' = ((1, 2), (3, 1), (2, 3), (4, 2), (3, 4), (5, 3), (4, 5), (6, 4), (5, 6), (6, 2), (2, 5), (5, 1), (1, 2))$$

in $\text{Cut}(\mathcal{H}, b_1)$, starting from and ending at $m_1 = (1, 2)$ (refer to Figure 15).

If $b_1 \in R$, we identify the cycle:

$$C''' = ((1, 2), (2, 7), (7, 1), (1, 6), (6, 7), (7, 5), (5, 6), (6, 2), (2, 5), (5, 1), (1, 2))$$

in $\text{Cut}(\mathcal{H}, b_1)$, starting from and ending at $m_1 = (1, 2)$ (refer to Figure 16).

More intuitively, since b_1 lies on C but not B_1 , it would break either the segment of the hypergraph to the left or to the right of the internal bridge, B_1 . Regardless of which of these two segments is broken, the other will

always survive, which is then utilised to identify the desired surviving cycle in $\text{Cut}(\mathcal{H}, b_1)$, starting from and ending at $m_1 = (1, 2)$. $\square$

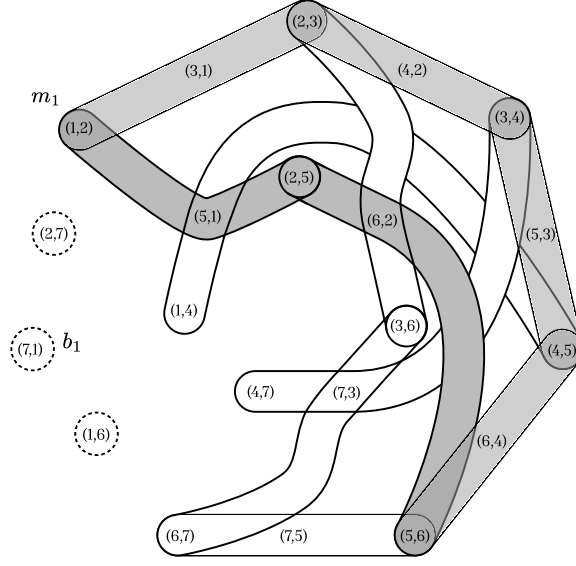


Figure 15: C'' : Cycle to the right of the bridge B_1

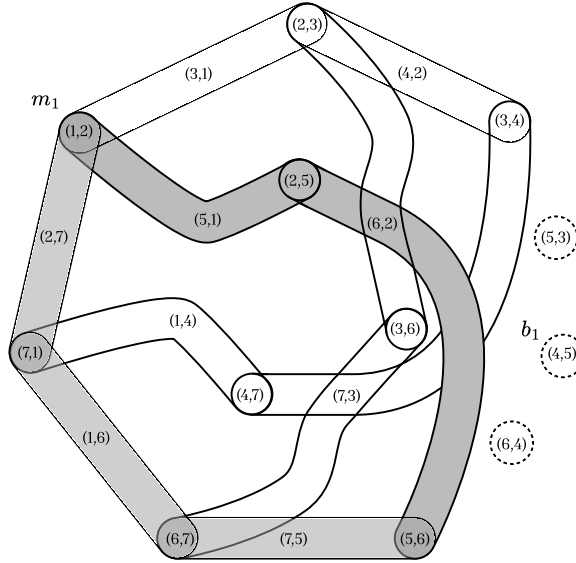


Figure 16: C''' : Cycle to the left of the bridge B_1

3.6 Maker's winning strategy via cycle-hopping

Having established that a cycle $\mathcal{C}$, starting and ending at $(1, 2)$, persists in $\mathcal{H}' = \text{Cut}(\mathcal{H}, b_1)$, regardless of Breaker's first move (refer Lemma 3.7), we now describe how Maker uses this cycle to win the game.

Lemma 3.8. *Provided that Maker claims the element $m_1 = (1, 2)$, she is able to secure a win for herself in the Maker-Breaker directed triangle game on $\Pi(7)$.*

Proof. Maker's strategy is as follows.

1. **Initial Move:** Maker claims $m_1 = (1, 2)$ in her first move.
2. **Identifying a cycle in $\text{Cut}(\mathcal{H}, b_1)$:** As shown in the proof of Lemma 3.7, regardless of what Breaker's first move b_1 is, the hypergraph $\mathcal{H}' = \text{Cut}(\mathcal{H}, b_1)$ has a cycle starting and ending at $m_1 = (1, 2)$. Let this cycle be given by

$$\mathcal{C} = (v_0, v_1, v_2, \dots, v_{k-1}, v_k),$$

where $v_0 = v_k = (1, 2)$. We observe that if such a cycle is composed of t hyperedges, then, since (a) each hyperedge consists of 3 elements, (b) and any two consecutive hyperedges intersect in precisely one element, the number of distinct elements in the cycle will equal $2t$.

3. **Hopping along the cycle (refer to Figure 17 for an example):**

- During her second move, Maker claims v_2 from $\mathcal{C}$, essentially skipping / hopping over v_1 . Thus, $m_1 = v_0$ and $m_2 = v_2$.
 - We note that $\{v_0, v_1, v_2\}$ form a winning set, of which two elements have already been claimed by Maker. This partially occupied winning set is now an immediate threat to Breaker, which she is forced to address by deleting v_1 in her second move. Thus, $b_2 = v_1$.
 - In her third move, Maker now claims $m_3 = v_4$, once again skipping / hopping over v_3 . This creates an immediate threat for Breaker since the winning set $\{v_2, v_3, v_4\}$ is just one element shy of being completely claimed by Maker (i.e., v_3 is the only element of this winning set that is yet to be claimed by Maker). Breaker is thus forced to address this threat by deleting v_3 in her third move, that is $b_3 = v_3$.
 - Maker continues in this manner, sequentially claiming elements on the cycle $\mathcal{C}$ that are shared across hyperedges (i.e., $m_4 = v_6, m_5 = v_8$, and so on). Each time Maker claims such an element, she is one step away from claiming an entire winning set, thus posing an immediate threat for Breaker, and Breaker is compelled to delete the remaining element of this winning set in her next move. This leaves Breaker trailing behind Maker's moves, eliminating each threat left behind by Maker's latest move.
4. **Creation of a double threat:** Following the above-mentioned 'cycle-hopping strategy', Maker will eventually claim v_{k-2} (this follows from the fact that k is even, as justified in 2). Recall that $m_1 = v_0 = v_k$. The move by Maker, claiming v_{k-2} , is crucial as it imposes a 'double threat', since there are now *two* partially occupied winning sets, each of which is only one element shy of being entirely claimed by Maker: these are $\{v_k = v_0, v_{k-1}, v_{k-2}\}$ and $\{v_{k-2}, v_{k-3}, v_{k-4}\}$. Since Breaker is allowed to delete only one element per turn in this unbiased game, she cannot block both of these winning sets simultaneously. Consequently, Maker completes claiming the winning set that remains unblocked after Breaker's latest move, thus winning the game. The complete claim of a hyperedge (which is a winning set) by Maker corresponds to her having claimed all three edges of a directed triangle on $\Pi(7)$. Thus, Maker wins the Maker-Breaker directed triangle game on $\Pi(7)$. $\square$

Lemma 3.7 and Lemma 3.8 together establish Maker's winning strategy on $\Pi(7)$.

3.7 Maker's winning strategy on $\Pi(n)$, for all $n > 7$

Lemma 3.9 (Extension to larger tournaments). *Maker has a winning strategy in the Maker-Breaker directed triangle game on $\Pi(n)$ for all $n \geq 7$.*

Proof. We first note that the parity tournament $\Pi(m)$ is embedded in all parity tournaments $\Pi(n)$, where $n > m$. This implies that each winning set (or, directed triangle) from $\Pi(m)$ is also embedded in $\Pi(n)$. We also know that such a Maker-Breaker game is hypergraph monotone, and the addition of new winning sets (which happens when we consider the larger tournament $\Pi(n)$ compared to the smaller tournament $\Pi(m)$) will, if anything, only serve to provide Maker with greater advantage. In fact, Maker would not even require to deviate from her winning strategy on $\Pi(7)$, when playing the game on $\Pi(n)$ for $n > 7$. A natural winning strategy for Maker on $\Pi(n)$, for $n > 7$, can be described as follows:

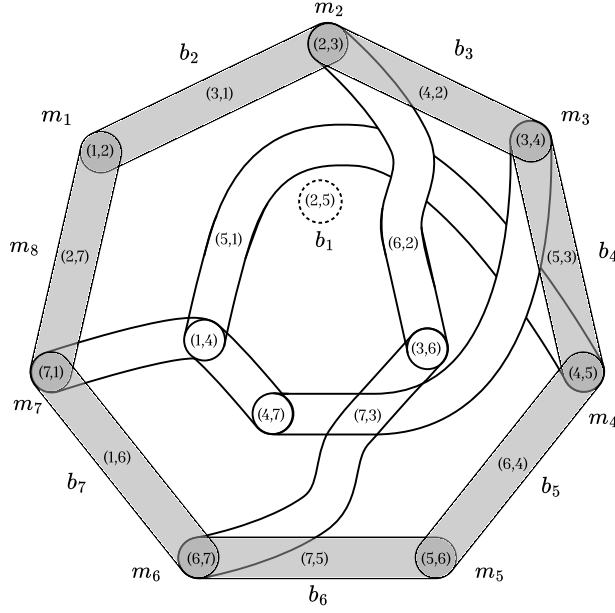


Figure 17: A possible sequence of moves illustrating Maker's cycle hopping strategy when $m_1 = (1, 2)$ and $b_1 = (2, 5)$

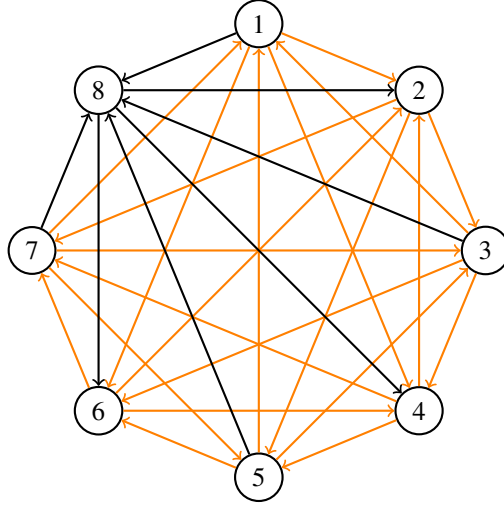


Figure 18: $\Pi(7)$ embedded in $\Pi(8)$

Before the game starts, Maker fixes the vertex set $V_7 = \{1, 2, 3, 4, 5, 6, 7\} \subset V_n$. The sub-tournament induced on this vertex set, V_7 , denoted henceforth by $\Pi[V_7]$, is an exact copy of the parity tournament $\Pi(7)$. Since $\Pi[V_7]$ is isomorphic to $\Pi(7)$, Maker plays according to her winning strategy on $\Pi(7)$ on this induced sub-tournament. If Breaker, in response, during any of her turns, chooses to delete a directed edge from outside the induced tournament $\Pi[V_7]$, Maker ignores this move, and continues choosing directed edges from inside $\Pi[V_7]$ according to the cycle-hopping strategy outlined in Lemma 3.8. If Breaker deviates from blocking the sequential threats imposed by Maker's cycle-hopping strategy by playing outside of $\Pi[V_7]$, then Maker wins in the very next move after this deviation by occupying the winning set that posed a threat to Breaker during her previous move.

It is evident, therefore, that by restricting her play to $\Pi[V_7]$, Maker effectively reduces the game on $\Pi(n)$ to the already-solved game on $\Pi(7)$, and consequently, wins on $\Pi(n)$ for every $n > 7$. $\square$

Lemma 3.7, Lemma 3.8, and Lemma 3.9 together prove Theorem 2.1.

This result addresses the game played on a parity tournament, which is a special type of regular tournament. Intuitively, one could argue that for an arbitrary tournament, which might be less regular, the corresponding game might be easier for Breaker to win, since the number of winning sets would be smaller in the case of more irregular and imbalanced score sequences. Thus, we could guess the board size threshold for an arbitrary tournament to be greater than or equal to 7. However, it remains to find the value of board size thresholds for tournaments with given arbitrary score sequences.

3.8 Bounds on $b^*(n)$

We consider a biased variant of the Maker-Breaker directed triangle game on the parity tournament. Particularly, we address the $(1 : b)$ Maker-Breaker directed triangle game on $\Pi(n)$ with the intention of identifying bounds on the bias threshold, $b^*(n)$, such that for $b \geq b^*(n)$, Breaker has a winning strategy and for $b < b^*(n)$, Maker has a winning strategy.

We use the following criterion for Maker's win, to derive a lower bound on $b^*(n)$:

Theorem 3.10 (From [Bec08]). *Let $\mathcal{F}$ be a family of subsets of a finite set X . Suppose a and b are positive integers, and define*

$$\Delta_2(\mathcal{F}) = \max \left\{ \left| \{A \in \mathcal{F} : \{u, v\} \subset A\} \right| : u, v \in X, u \neq v \right\}.$$

If

$$\sum_{A \in \mathcal{F}} \left(\frac{a+b}{a} \right)^{-|A|} > \frac{a^2 b^2}{(a+b)^3} \Delta_2(\mathcal{F}) |X|, \quad (2)$$

then Maker (as the first player) has a winning strategy for the $(a : b)$ biased Maker-Breaker game on $(X, \mathcal{F})$.

In our biased $(1 : b)$ directed triangle game on $\Pi(n)$, $a = 1$, and each winning set $F \in \mathcal{F}$ has size $|F| = 3$. From the structure of directed triangles, we observe that $\Delta_2(\mathcal{F}) = 1$, since no two edges can appear together in more than one triangle (because specifying two edges in a directed triangle uniquely determines the third). The elements in X correspond to the directed edges of $\Pi(n)$, which is our board. Thus, $|X| = \binom{n}{2}$.

Let $w(n)$ denote the number of directed triangles in $\Pi(n)$. We know from Subsection 3.2 that

$$w(n) = \sum_{i=1}^{n-2} \left\lfloor \frac{i}{2} \right\rfloor \cdot \left\lfloor \frac{n-(i+1)}{2} \right\rfloor.$$

Given these conditions, Inequality 2 from Theorem 3.10 would boil down to

$$\begin{aligned} w(n) (1+b)^{-3} &> \frac{b^2 |X|}{(1+b)^3} \implies w(n) > b^2 |X| \\ \implies b^2 &< \frac{w(n)}{|X|} \implies b < \sqrt{\frac{w(n)}{|X|}}. \end{aligned} \quad (3)$$

Thus, for all b satisfying this inequality, Maker will have a winning strategy in the corresponding $(1 : b)$ Maker-Breaker directed triangle game. Equivalently, the bias threshold $b^*(n)$ must satisfy

$$b^*(n) \geq \sqrt{\frac{w(n)}{|X|}}.$$

We have

$$w(n) = \begin{cases} \frac{n^3 - n}{24}, & \text{if } n \text{ is odd,} \\ \frac{n^3 - 4n}{24}, & \text{if } n \text{ is even,} \end{cases}$$

and

$$|X| = \frac{n(n-1)}{2}.$$

It follows that

$$\sqrt{\frac{w(n)}{|X|}} = \begin{cases} \sqrt{\frac{n+1}{12}} = \sqrt{\left(\frac{1}{12} + o(1)\right)n}, & \text{if } n \text{ is odd,} \\ \sqrt{\frac{(n-2)(n+2)}{12(n-1)}} = \sqrt{\left(\frac{1}{12} + o(1)\right)n}, & \text{if } n \text{ is even.} \end{cases}$$

Thus, for large enough n , Theorem 3.10 implies that Maker can force a win if

$$b < \sqrt{\left(\frac{1}{12} + o(1)\right)n},$$

and consequently, the bias threshold $b^*(n)$ satisfies the lower bound stated in (1) from Theorem 2.2.

For the upper bound, our approach is to compare the directed triangle game to its undirected counterpart, and show that Breaker's win in the undirected triangle game implies Breaker's win in the directed triangle game as well.

First, we let $\mathcal{T}$ be the hypergraph representation for the undirected Maker-Breaker triangle game on the complete graph K_n , that is, the vertex set of $\mathcal{T}$ is the set of edges of K_n and a hyperedge corresponds to a set of edges forming a triangle in K_n . Let $\mathcal{T}'$ be the hypergraph representation for the Maker-Breaker directed triangle game on the parity tournament $\Pi(n)$: recall that, here, the vertex set is the set of directed edges of $\Pi(n)$ and each hyperedge is a 3-set of directed edges forming a directed triangle.

Every edge in $\Pi(n)$, with its direction ignored, corresponds to an edge in K_n . Moreover, every directed triangle in $\Pi(n)$ forms, when orientations are ignored, a triangle in K_n . Therefore, every hyperedge of $\mathcal{T}'$ is also a hyperedge of $\mathcal{T}$. This implies that $\mathcal{T}' \subseteq \mathcal{T}$.

We give below a lemma that allows us to 'borrow' a condition for Breaker's win on the undirected triangle game to the premise of the directed triangle game on the same board, owing to a monotonicity argument common in Maker-Breaker games.

Lemma 3.11. *Let $\mathcal{H} = (X, \mathcal{F})$ and $\mathcal{H}' = (X, \mathcal{F}')$ be hypergraphs on the same vertex set such that $\mathcal{H}' \subseteq \mathcal{H}$. If Breaker has a winning strategy in the $(a : b)$ Maker-Breaker game on $\mathcal{H}$, then Breaker also has a winning strategy in the $(a : b)$ Maker-Breaker game on $\mathcal{H}'$.*

Proof. Assume for contradiction that Breaker has a winning strategy on $\mathcal{H}$, but Maker has one on $\mathcal{H}'$. Then, by following her winning strategy on $\mathcal{H}'$, Maker could ensure the complete claim of some hyperedge $F' \in \mathcal{F}'$. Since $\mathcal{F}' \subseteq \mathcal{F}$, the same set F' also belongs to $\mathcal{F}$, so this strategy would yield a win for Maker on $\mathcal{H}$ as well. This contradicts the assumption that Breaker's strategy on $\mathcal{H}$ is winning. Hence, Maker cannot have a winning strategy on $\mathcal{H}'$, and the claim follows. $\square$

Theorem 3.12 (From [GS22]). *In the biased $(1 : b)$ Maker-Breaker undirected triangle game (with Maker starting first), Breaker has a winning strategy if*

$$b \geq \sqrt{\left(\frac{8}{3} + o(1)\right)n},$$

for sufficiently large n .

This result, along with Lemma 3.11, implies that Breaker would also win the biased $(1 : b)$ Maker-Breaker *directed* triangle game, for n large enough, whenever b satisfies

$$b \geq \sqrt{\left(\frac{8}{3} + o(1)\right)n}. \tag{4}$$

Consequently, the bias threshold $b^*(n)$ satisfies the upper bound in (1) of Theorem 2.2.

4 Proof of our result stated in Subsection 2.2

We have proved two results concerning the parity tournament in Section 3. We shall now direct our attention to the random tournament $T(n, p)$ (recall from Definition 1.3) defined on the vertex set V_n , in which, for every pair i, j of

vertices with $i < j$, the edge between i and j is oriented, independent of all else, from i to j with probability p , and from j to i with probability $(1 - p)$. Particularly, we address the case when p is a constant and $p \in (0, 1)$. We note that the cases of $p = 0$ and $p = 1$ correspond to $T(n, p)$ being transitive. As mentioned previously, transitive tournaments are tournaments where the vertices can be ordered in such a way that every edge is directed from a vertex indexed ‘higher’ in the order to one indexed ‘lower’ in the order. Trivially, transitive tournaments have no directed cycles and consequently no directed triangles. Thus, the Maker-Breaker directed triangle game played on $T(n, p)$ with $p = 0$ or $p = 1$ is trivially won by Breaker.

We have already established in Theorem 2.1 that Maker has a winning strategy on $\Pi(n)$ for $n \geq 7$. Thus, to show that Maker wins on $T(n, p)$ with probability tending to 1 as n tends to infinity, it would suffice to show that a copy of $\Pi(7)$ is embedded in $T(n, p)$ with probability tending to 1 as n tends to infinity. This is the approach we adopt in proving the following lemma, which, in turn, aids us in proving Theorem 2.3.

Lemma 4.1. *Let $T(n, p)$ be a random tournament where p is a constant in $(0, 1)$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(T(n, p) \text{ contains an induced copy of } \Pi(7)\right) = 1.$$

Proof. Recall that $V_n = \{1, 2, 3, \dots, n\}$ is the vertex set on which $T(n, p)$ is defined, and the parity tournament $\Pi(7)$ has been defined on $V_7 = \{1, 2, \dots, 7\}$. Consider any 7-element subset $A \subset V_n$ with $A = \{v_1, v_2, \dots, v_7\}$ such that $v_1 < v_2 < \dots < v_7$. We define the indicator random variable I_A to be equal to 1 if the direction of the edge between v_i and v_j in the tournament induced by $T(n, p)$ on A is the same as the direction of the edge between the corresponding vertices i and j in $\Pi(7)$, for all distinct $i, j \in \{1, 2, \dots, 7\}$. We set I_A to be equal to 0 otherwise.

Let

$$X_n = \sum_{A \subset V_n: |A|=7} I_A,$$

so that X counts the number of copies of $\Pi(7)$ in $T(n, p)$. We shall use the second moment method to show that $\mathbb{P}(X_n > 0) \rightarrow 1$ as $n \rightarrow \infty$.

Calculating the first moment: Each tournament on 7 vertices has $\binom{7}{2} = 21$ edges. By the definition of $\Pi(7)$, for any $A \subset V_n$ with $|A| = 7$, the event that the tournament induced by $T(n, p)$ on A has edges oriented exactly as in $\Pi(7)$, or, in other words, $I_A = 1$, occurs with probability

$$q = \prod_{1 \leq i < j \leq 7} \left[p \mathbb{1}_{\{i+j \text{ odd}\}} + (1-p) \mathbb{1}_{\{i+j \text{ even}\}} \right],$$

and thus, $\mathbb{E}(I_A = 1) = q > 0$. Therefore,

$$\mathbb{E}[X] = \binom{n}{7} q,$$

which is $\Theta(n^7)$ and diverges as $n \rightarrow \infty$.

Calculating the second moment: We have

$$\mathbb{E}[X^2] = \sum_{A, B \subset V_n: |A|=|B|=7} \mathbb{E}[I_A I_B]. \quad (5)$$

For any two 7-subsets, A and B , of V_n , let

$$r = |A \cap B|, \quad 0 \leq r \leq 7.$$

We then partition the sum as

$$\mathbb{E}[X^2] = \sum_{r=0}^7 S_r, \quad \text{where} \quad S_r = \sum_{\substack{A, B \subset V_n: |A|=|B|=7 \\ |A \cap B|=r}} \mathbb{E}[I_A I_B].$$

Case 1: $r = 0$ (Disjoint sets). If A and B are disjoint, which means they have no vertices in common, the edge orientations in A and in B are independently decided in $T(n, p)$, so that

$$\mathbb{E}[I_A I_B] = \mathbb{P}(I_A = 1) \cdot \mathbb{P}(I_B = 1) = q^2.$$

The number of disjoint pairs is

$$N_0 = \binom{n}{7} \binom{n-7}{7} = \Theta(n^{14}),$$

and hence

$$S_0 = N_0 q^2 = \Theta(n^{14}) q^2.$$

Note: We say that $f(n) = \Theta(g(n))$ if there exist positive constants c_1, c_2 and a constant n_0 such that

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \quad \text{for all } n \geq n_0.$$

Case 2: $1 \leq r \leq 7$ (Overlapping sets). Assume that A and B share exactly r vertices. The number of such pairs is

$$N_r = \binom{n}{r} \binom{n-r}{7-r} \binom{n-7}{7-r} = \Theta(n^{14-r}).$$

In this case, the orientations of the $\binom{r}{2}$ shared edges affect both I_A and I_B . Consequently, fewer independent edge realisations (namely, $42 - \binom{r}{2}$ instead of 42) are needed to determine whether the induced tournament on each of A and B matches $\Pi(7)$. Thus, there exists a constant $\rho(r)$ such that

$$\mathbb{E}[I_A I_B] \leq \rho(r) q^2.$$

Remark. The constant $\rho(r)$ captures the effect of which r vertices are common and their positions in the canonical orderings of A and B . Although these details affect the exact probability (i.e., the exact value of $\rho(r)$), they are independent of n and hence only contribute a constant factor. As a result, the overall contribution from such pairs of overlapping sets is

$$S_r = O(n^{14-r}) \rho(r) q^2,$$

and for each fixed $r \geq 1$,

$$\frac{S_r}{(\mathbb{E}[X])^2} = O\left(\frac{n^{14-r}}{n^{14}}\right) = O(n^{-r}),$$

which tends to zero as $n \rightarrow \infty$.

Combining these cases, we obtain

$$\mathbb{E}[X^2] = S_0 + \sum_{r=1}^7 S_r = \Theta(n^{14}) q^2 + o(n^{14}),$$

so that

$$\frac{\mathbb{E}[X^2]}{(\mathbb{E}[X])^2} = 1 + o(1).$$

By the Paley–Zygmund inequality (see [PZ32]), we have

$$\mathbb{P}(X > 0) \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Thus, with high probability, $T(n, p)$ contains an induced copy of $\Pi(7)$. □

Consequently, Maker can use this embedded copy of $\Pi(7)$ in $T(n, p)$ to ensure a win in the Maker–Breaker directed triangle game played on $T(n, p)$. This completes the proof of Theorem 2.3.

A natural direction for future work is to consider a random tournament $T(n, p(n))$, where the edge orientation probability depends on n , and to determine the corresponding threshold probability at which the game transitions from an almost sure Breaker’s win to an almost sure Maker’s win.

5 Proof of our result stated in Subsection 2.3

Traditionally, introducing bias into Maker-Breaker games has been a popular method to even the playing field and provide additional advantage to Breaker. With a similar motivation, we shall now introduce a $(1 : \kappa(n))$ *flip-biased Maker-Breaker directed triangle game* played on the parity tournament $\Pi(n)$. In this variant of the game, Breaker is allowed to strategically and preemptively flip the orientation of $\kappa(n)$ many edges before the game starts, following which the game proceeds via the usual rules for an unbiased directed triangle game as discussed in Definition 1.2. These $\kappa(n)$ edge-flips benefit Breaker in the game as they allow her to reduce the number of directed triangles (i.e. she can make the parity tournament deviate from its regular or balanced structure, and push it towards transitivity). Given a large enough $\kappa(n)$, Breaker can entirely convert the parity tournament $\Pi(n)$, which is a Maker's win for $n \geq 7$, to a transitive tournament $\Lambda(n)$ with *no* directed cycles, which is a trivial win for Breaker.

Note: In this section, we assume that n is odd, since it allows us to study exactly regular tournaments, meaning that each vertex has a score of $\frac{n-1}{2}$, which simplifies the calculations. In contrast, when n is even, the tournaments are near-regular, but not exactly so. The arguments presented below can be adapted for analysis when n is even.

More specifically, Breaker requires exactly $\frac{(n-1)^2}{4}$ edge flips to convert a parity tournament $\Pi(n)$, where n is odd, to a transitive tournament $\Lambda(n)$ on n vertices. This has been established in (8). Naturally, the question remains: what is the minimum number of preemptive edge flips $k^*(n)$ (or, the *flip-bias threshold*) such that for $\kappa(n) \geq k^*(n)$, Breaker has a winning strategy and for $\kappa(n) < k^*(n)$, Maker has a winning strategy in the $(1 : \kappa(n))$ flip-biased Maker-Breaker directed triangle game? Trivially, we have $k^*(n) \leq \frac{(n-1)^2}{4}$ as a consequence of (8).

To prescribe Breaker's edge-flipping strategy, it is crucial for us to quantify the effect of each individual edge-flip on the total number of directed triangles in the resulting perturbed tournament. Breaker's goal is to maximise, via each flip, the reduction in the number of triangles and consequently 'thin' the winning sets enough before Maker's first move.

5.1 Reduction in the number of directed triangles due to an edge flip

Let s_i denote the score (i.e. number of outgoing edges) of vertex i and define the *deviance* as

$$\delta_i = s_i - \frac{n-1}{2}. \quad (6)$$

In the original parity tournament $\Pi(n)$, every vertex i satisfies $s_i = \frac{n-1}{2}$, so that $\delta_i = 0$.

Now consider an edge (j, i) . When this edge is flipped, its direction reverses to (i, j) , and the following changes occur:

1. The score of vertex j decreases by 1, so that its new score is $s'_j = s_j - 1$, and its new deviance becomes $\delta'_j = s'_j - \frac{n-1}{2} = \delta_j - 1$.
2. The score of vertex i increases by 1, so that its new score is $s'_i = s_i + 1$, and its new deviance becomes $\delta'_i = s'_i - \frac{n-1}{2} = \delta_i + 1$.

The count for the number of directed triangles (as a consequence of Proposition 3.3 and as discussed in [Moo15]) in any tournament on the vertex set V_n and a given score sequence is

$$w(n) = \frac{n(n^2 - 1)}{24} - \frac{1}{2} \sum_{v=1}^n \delta_v^2.$$

Assuming that only vertices i and j are affected by the flip, the change in the sum of the squares of the deviances is $\Delta = \delta_j'^2 + \delta_i'^2 - \delta_j^2 - \delta_i^2 = -2\delta_j + 2\delta_i + 2 = 2(1 + \delta_i - \delta_j)$.

The change in the number of directed triangles, given a fixed n , due to flipping the edge (j, i) solely depends on the initial deviance (before the flip was executed) of vertex j, i , and is given by

$$\Delta w(n) = -\frac{1}{2} \Delta = -\frac{1}{2} \cdot 2(1 + \delta_i - \delta_j) = -(1 + \delta_i - \delta_j). \quad (7)$$

5.2 Breaker's flipping strategy

A natural strategy for Breaker is to flip the edges in a sequential manner, such that in each step the imbalance is maximised without undoing the effects of the flips performed in the previous steps. For example, one may proceed as follows:

1. Start at vertex 1 and flip all of its incoming edges (there are $\frac{n-1}{2}$ such edges).
2. Then, at vertex 2, flip all its incoming edges except the edge from vertex 1 (to avoid disturbing the imbalance already created at vertex 1).
3. Continue similarly for vertices 3, 4, $\dots$, always leaving unflipped the edges incoming from the vertices with lower indices.
4. Following this strategy, the final edge that Breaker will flip is $(n, n-2)$, making the tournament transitive.

Under this prescribed strategy, the scores for the lower-indexed vertices will increase while those for the higher-indexed vertices will decrease. Consequently, the number of directed triangles decreases. A more formal description of the strategy is as follows.

Phase 1 (Vertex 1): Breaker begins at vertex 1. In this phase, she flips all edges incoming towards vertex 1 that originate from vertices of the same parity (i.e. odd vertices larger than 1). The exact order and identity of each flip in this phase would not matter much, but as a convention, we will assume that Breaker first flips $(n, 1)$, then $(n-2, 1)$, then $(n-3, 1)$, and so on, until she finally flips $(3, 1)$ and completes Phase 1. The same 'descending' pattern would hold, under our convention, for the next phases. Consequently, the number of edges flipped at vertex 1 is

$$m_1 = \left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n-1}{2}.$$

Phase 2 (Vertex 2): Breaker then moves to vertex 2. Here, she flips all edges incoming from vertices with the same parity as 2 (i.e. even vertices) that have labels larger than 2. The number of edges flipped at vertex 2 is

$$m_2 = \left\lfloor \frac{n-2}{2} \right\rfloor.$$

General Phase i (Vertex i): For an arbitrary vertex i , Breaker flips all incoming edges of the form (j, i) where $j > i$ and i and j have the same parity. In other words, the number of flips at vertex i is given by

$$m_i = \left\lfloor \frac{n-i}{2} \right\rfloor.$$

As a consequence of this strategy, we can calculate the total number of edge-flips performed by Breaker transforming $\Pi(n)$ into $\Lambda(n)$ to be

$$\sum_{i=1}^{n-2} \left\lfloor \frac{n-i}{2} \right\rfloor = \sum_{j=2}^{n-1} \left\lfloor \frac{j}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor = \begin{cases} \frac{n(n-2)}{4}, & n \text{ even,} \\ \frac{(n-1)^2}{4}, & n \text{ odd.} \end{cases} \quad (8)$$

As mentioned earlier, we restrict our analysis to the case where n is odd.

To illustrate and motivate Breaker's flipping strategy, we first discuss the following example concerning a sequence of edge flips that convert a $\Pi(7)$ to a $\Lambda(7)$ (See Table 1). As already mentioned before, we would require $\frac{(n-1)^2}{4} = \frac{(7-1)^2}{4} = 9$ edge flips to make this conversion.

The entries δ_i in Table 1 represent the deviance of vertex i (i.e. $s_i - \frac{n-1}{2}$) **after** that particular edge flip has been completed, and $\Delta w(n)_f$ indicates how many triangles have been removed by that single flip f . Phase i is a collection of the edges flipped that were originally of the form (j, i) , where $j > i$. It also indicates the vertex currently being *fixed*. Summing all $\Delta w(n)_f$ values yields 14, matching the total number of directed triangles in $\Pi(7)$ before any flips. This

Edge Flip Sequence	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	$\Delta w(n)_f$
Parity tournament $\Pi(7)$:	0	0	0	0	0	0	0	
Phase 1 (Vertex 1)								
Flip 1: $7 \rightarrow 1$	1	0	0	0	0	0	-1	-1
Flip 2: $5 \rightarrow 1$	2	0	0	0	-1	0	-1	-2
Flip 3: $3 \rightarrow 1$	3	0	-1	0	-1	0	-1	-3
Phase 2 (Vertex 2)								
Flip 4: $6 \rightarrow 2$	3	1	-1	0	-1	-1	-1	-1
Flip 5: $4 \rightarrow 2$	3	2	-1	-1	-1	-1	-1	-2
Phase 3 (Vertex 3)								
Flip 6: $7 \rightarrow 3$	3	2	0	-1	-1	-1	-2	-1
Flip 7: $5 \rightarrow 3$	3	2	1	-1	-2	-1	-2	-2
Phase 4 (Vertex 4)								
Flip 8: $6 \rightarrow 4$	3	2	1	0	-2	-2	-2	-1
Phase 5 (Vertex 5)								
Flip 9: $7 \rightarrow 5$	3	2	1	0	-1	-2	-3	-1
Transitive tournament $\Lambda(7)$:	3	2	1	0	-1	-2	-3	
Total Triangles Removed:								14

Table 1: A sequence of 9 edge flips, as per Breaker's flipping strategy, transforming the parity tournament $\Pi(7)$ into a transitive tournament $\Lambda(7)$.

implies that these 9 flips reduce the number of triangles from 14 (the maximum value possible for a tournament on 7 vertices) to 0.

A key observation here is the sequence in which the number of directed triangles (as represented by $\Delta w(n)_f$) are reduced by the respective sequence of flips. This phase-wise triangle reduction sequence would look like

$$\underbrace{1, 2, 3}_{\text{Phase 1}}, \underbrace{1, 2}_{\text{Phase 2}}, \underbrace{1, 2}_{\text{Phase 3}}, \underbrace{1}_{\text{Phase 4}}, \underbrace{1}_{\text{Phase 5}} \quad (9)$$

More generally, we already know that the length of a phase i will be given by $m_i = \lfloor \frac{n-i}{2} \rfloor$, and with the above illustration, we can see that as this phase is executed, the sequence of the number of directed triangles reduced in this one phase will be $1, 2, 3, \dots, \lfloor \frac{n-i}{2} \rfloor$. This can intuitively be thought of as Breaker capitalising on the deviance imbalance that she has created with her previously executed flip, which maximises the number of directed triangles reduced by the current flip (as prescribed by the strategy), since this quantity, given by $(1 + \delta_i - \delta_j)$, depends only on the deviance of the endpoints of the edge (j, i) that is flipped. She would continue to double down on the same vertex until all possible flips have been executed, after which, she would move on to the next vertex (next phase). A formal statement explaining this observation follows.

Proposition 5.1. *For a fixed Phase i in Breaker's flipping strategy (as explained in Subsection 5.2), the sequence of the number of directed triangles reduced by each consecutive flip in that particular phase will be $1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor$. Consequently, the total number of directed triangles reduced by the completion of Phase i will be $\sum_{x=1}^z x = \frac{z(z+1)}{2}$, where $z = \lfloor \frac{n-i}{2} \rfloor$.*

Proof. Because of the way Breaker's flipping strategy has been designed during any Phase i , Breaker only concerns herself with edges of the form (j, i) where $j > i$ and i and j have the same parity. Only these edges are then flipped according to the strategy described in Subsection 5.2. Thus, the flips executed during odd phases impact *only* the odd-indexed vertices, and the flips executed during even phases impact *only* the even-indexed vertices. At the beginning of any fixed Phase i , each vertex with index m such that $m \geq i$ and m having the same parity as i , must have equal deviance (as defined in (6)), which is caused by all the phases of the same parity having been executed before the start of Phase i . This deviance will be exactly equal to the number of phases having the same parity as i that have been executed before Phase i , which is $\lfloor \frac{i}{2} \rfloor$, since each of these phases will have exactly one edge flip that would have contributed a -1 to the deviance of vertex m . From the above discussion, we conclude that for each edge to be flipped during Phase i , the vertices at either end of this edge have equal deviance just before Phase i begins. We have already derived (7), which encapsulates the effect of an edge flip, as a function of the deviance of the vertices at either end of that edge, on the number of directed triangles present in the tournament. For the first flip, the change in the number of directed triangles in the tournament equals -1 as $\delta_i = \delta_j = \lfloor \frac{i}{2} \rfloor$ and $-(1 + \delta_i - \delta_j) = -1$. This means the first flip of

Phase i would remove one directed triangle, and would also increase the deviance of vertex i by one, now updating it to $\lfloor \frac{i}{2} \rfloor + 1$. Our next flip in Phase i will then reduce the number of directed triangles by two since $\delta_i = \lfloor \frac{i}{2} \rfloor + 1$, $\delta_k = \lfloor \frac{i}{2} \rfloor$, and $-(1 + \lfloor \frac{i}{2} \rfloor + 1 - \lfloor \frac{i}{2} \rfloor) = -2$, further increasing the deviance of vertex i by one. The next edge flip, which is the third flip in Phase i , would thus change the number of directed triangles by $-(1 + \lfloor \frac{i}{2} \rfloor + 2 - \lfloor \frac{i}{2} \rfloor) = -3$. We repeat this process further for the remaining edge flips in Phase i , and the sequence of the number of triangles reduced by each flip during this phase would be $1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor$. Summing these together would give us the cumulative number of triangles removed by the completion of Phase i , which is $\sum_{x=1}^z x = \frac{z(z+1)}{2}$, where $z = \lfloor \frac{n-i}{2} \rfloor$. $\square$

Now that we have endowed Breaker with a flipping strategy to perturb the board in her favour before the game begins, it remains to identify the threshold number of edge flips $\kappa^*(n)$ required according to this strategy to change the game from Maker's win, as it was on $\Pi(n)$ for $n \geq 7$, to a Breaker's win on the perturbed board.

5.3 An upper bound on the flip-bias threshold

We invoke a criterion for Breaker's win in an unbiased Maker-Breaker game, which is given as follows.

Theorem 5.2 (See [ES73]). *For an unbiased Maker-Breaker game with Maker starting first and with each winning set having size s , if the total number of winning sets $|\mathcal{F}| < 2^{s-1}$, then Breaker has a winning strategy.*

In our case, $s = 3$, and thus, for Breaker to have a winning strategy in the $(1 : \kappa(n))$ flip-biased Maker-Breaker directed triangle game, we want the total number of winning sets left on the board *after* Breaker's edge-flips to obey the above inequality. Formally, the criterion for Breaker's win in the flip-biased Maker-Breaker directed triangle game can now be stated as

$$w(n) - F(k) < 4 \implies F(k) > w(n) - 4, \quad (10)$$

where $w(n)$, recall, is the total number of directed triangles in the parity tournament on n vertices and $F(k)$ is the number of directed triangles reduced by the first k edge-flips performed by Breaker according to her flipping strategy. This is a rather harsh condition, which requires that no more than 3 directed triangles (winning sets) should remain after Breaker is done with her edge flips before the game begins. Note that in our case, $|\mathcal{F}| = w(n) - F(k)$. We can now identify an upper bound $\kappa_{\text{up}}^*(n)$ on the flip-bias threshold as follows:

$$\kappa_{\text{up}}^*(n) = \min\{k \in \mathbb{N} : F(k) > w(n) - 4\}.$$

Assuming that Breaker follows the flipping procedure outlined in Subsection 5.2, what we need to find here is the point, during this procedure, at which Breaker can stop flipping edges and leave Maker with at most 3 directed triangles present in the resulting tournament. Seen from a different perspective, this is the same as trying to find the maximum number of edges that Breaker can afford to keep *unflipped*, beginning from the *end* of the flipping procedure outlined in Subsection 5.2, such that Maker is left with no more than 3 directed triangles.

As explained previously in Subsection 5.2, we can deduce that the last phase, i.e. Phase $(n-2)$, has length $\lfloor \frac{n-(n-2)}{2} \rfloor = 1$, and the singular edge-flip that this phase constitutes would remove one directed triangle. The phase before this one, i.e. Phase $(n-3)$, has length 1 as well, and the corresponding edge-flip would remove one more directed triangle. Phase $(n-4)$ has length 2, of which the first edge-flip would remove a single directed triangle, while the second edge-flip would remove two more directed triangles. To find our upper bound on the flip-bias threshold, we look towards the end of the flipping sequence described in Subsection 5.2, and note that a maximum of 3 edges can be left *unflipped*, starting from the *end* of the flipping sequence. This implies that a minimum of $\sum_{i=1}^{n-2} \lfloor \frac{n-i}{2} \rfloor - 3 = \frac{(n-1)^2}{4} - 3$ edge-flips are required to satisfy the criterion stated in (10). Thus, we have

$$\kappa_{\text{up}}^*(n) = \frac{(n-1)^2}{4} - 3 = \frac{n^2}{4} + O(n). \quad (11)$$

5.4 A lower bound on the flip-bias threshold

To derive the lower bound on $\kappa^*(n)$, we use the following criterion for Maker's win.

Theorem 5.3 (See [Bec81]). *Let $(X, \mathcal{F})$ be a s -uniform hypergraph and define*

$$\Delta_2(\mathcal{F}) = \max \left\{ |\{F \in \mathcal{F} : \{x, y\} \subset F\}| : x, y \in X, x \neq y \right\}.$$

If

$$|\mathcal{F}| > 2^{s-3} \Delta_2(\mathcal{F}) |X|, \quad (12)$$

then Maker has a winning strategy in the Maker–Breaker game on $(X, \mathcal{F})$.

For our particular game, we have $\Delta_2(\mathcal{F}) = 1$, $s = 3$ and $|X| = \binom{n}{2}$. The criterion in (12) thus boils down to

$$w(n) - F(k) > \binom{n}{2}, \quad (13)$$

which would ensure that Maker wins the flip-biased Maker–Breaker directed triangle game. Recall $w(n)$ is the number of directed triangles and $F(k)$ is the number of triangles removed by the first k flips in the flipping sequence from Subsection 5.2. Following an intuition similar to that described in Subsection 5.3, we start from the end of the flipping sequence and find the *minimum* number of edges that can be kept *unflipped* while still satisfying the inequality in (13). This number, viewed from a different perspective, would be exactly equal to the maximum number of edge-flips, counted from the start of our flipping sequence, after which the criterion in (13) would fail, and we can no longer be certain that Maker wins. This number gives us a lower bound $\kappa_{\text{low}}^*(n)$ on $\kappa^*(n)$.

Calculating $\kappa_{\text{low}}^*(n)$ requires us to break the flipping sequence, described in Subsection 5.2, down into blocks, starting from the *end* of the flipping sequence. We have noted above that each of the last two phases, i.e. Phase $(n-2)$ and Phase $(n-3)$, constitutes a single flip, and each of these two flips would remove one directed triangle. Phase $(n-4)$ constitutes 2 flips, of which the first removes one directed triangle while the second removes two more. If we enumerate the flipping sequence described in Subsection 5.2, starting from its end, then we can assign to each edge-flip a value that equals the number of directed triangles that would be removed if that particular edge-flip were to be executed (in the order in which it appears in the reversed flipping sequence of Subsection 5.2). This particular sequence would look like:

$$\underbrace{1}_{\text{Phase } n-2}, \underbrace{1}_{\text{Phase } n-3}, \underbrace{2, 1}_{\text{Phase } n-4}, \underbrace{2, 1}_{\text{Phase } n-5}, \underbrace{3, 2, 1}_{\text{Phase } n-6}, \underbrace{3, 2, 1}_{\text{Phase } n-7}, \underbrace{4, 3, 2, 1}_{\text{Phase } n-8}, \underbrace{4, 3, 2, 1}_{\text{Phase } n-9}, \dots \quad (14)$$

We must now find the minimum index in this sequence beyond which the inequality in (13) is satisfied. Let us denote this index by x . To identify x , we can break our sequence into *blocks* and identify where x falls. This *block partition* is fairly obvious, given our sequence, and would look like:

$$\underbrace{1, 1}_{\text{Block 1}}, \underbrace{2, 1, 2, 1}_{\text{Block 2}}, \underbrace{3, 2, 1, 3, 2, 1}_{\text{Block 3}}, \underbrace{4, 3, 2, 1, 4, 3, 2, 1}_{\text{Block 4}}, \dots \quad (15)$$

More formally, we say that the sequence is partitioned into blocks indexed by $k \geq 1$, where block k constitutes $(k, k-1, \dots, 1)$ written *twice*. The *length* of such a block k , which is the number of flips executed in the block k , is $\text{len}(k) = 2k$. The number of directed triangles removed by the edge-flips in the block k is referred to as the *block sum*, and denoted by $\text{sum}(k) = 2 \sum_{i=1}^k i = k(k+1)$. Our goal is to now make our way down this sequence and identify the block that x falls in. To achieve this, we define the *cumulative block sum* up to and including a block Z as

$$S_Z := \sum_{k=1}^Z \text{sum}(k) = \sum_{k=1}^Z (k^2 + k) = \frac{Z(Z+1)(Z+2)}{3} = \frac{Z^3}{3} + Z^2 + \frac{2Z}{3}, \quad (16)$$

and the *cumulative block length* up to and including Block Z as

$$L_Z := \sum_{k=1}^Z \text{len}(k) = \sum_{k=1}^Z 2k = Z(Z+1). \quad (17)$$

Identifying the block of interest

Let $N := \binom{n}{2} = n(n-1)/2$. As described in (13) and the discussion under it, we are looking for the smallest index x in the sequence where the cumulative sum up until x exceeds N . This index x might not neatly fall at the end of any particular block, and more generally, we are interested in identifying the block K such that

$$S_K \leq N < S_{K+1}. \quad (18)$$

This inequality indicates that the first K many blocks of the sequence have been entirely exhausted in achieving the required cumulative sum, and x either lies exactly at the end of Block K or can be found somewhere within Block $(K + 1)$ to acquire the necessary cumulative sum just exceeding N . Thus, x can be written as:

$$x = L_K + r_{K+1}, \quad \text{where } 0 \leq r_{K+1} < 2(K + 1). \quad (19)$$

Here, r_{K+1} counts the additional flips required from Block $(K + 1)$ and L_K is as defined in (17). From (17) and the bounds on $r_{(K+1)}$ from (19), we have

$$x = L_K + O(K) = K^2 + O(K). \quad (20)$$

Asymptotics of K

From (18) and (16), we have

$$\frac{K(K+1)(K+2)}{3} \leq N < \frac{(K+1)(K+2)(K+3)}{3} \implies N = \frac{K^3}{3} + O(K^2).$$

This can be rewritten as

$$3N = K^3 (1 + O(K^{-1})). \quad (21)$$

Taking cube roots, we obtain

$$(3N)^{1/3} = K (1 + O(K^{-1}))^{1/3}. \quad (22)$$

By the binomial expansion, for $|u| < 1$ we have

$$(1 + u)^{1/3} = 1 + \frac{1}{3}u + O(u^2). \quad (23)$$

Substituting $u = O(K^{-1})$ into (23) gives

$$(1 + u)^{1/3} = 1 + O(K^{-1}). \quad (24)$$

Hence, combining (22) and (24),

$$(3N)^{1/3} = K (1 + O(K^{-1})) = K + O(1). \quad (25)$$

Recall that $N = \binom{n}{2} = \frac{n(n-1)}{2}$. Thus, from (25), we have

$$K = (3N)^{1/3} + O(1) = \left(\frac{3}{2}\right)^{1/3} n^{2/3} + O(1). \quad (26)$$

Using (20) and (26), we get

$$x = \left(\frac{3}{2}\right)^{2/3} n^{4/3} + O(n^{2/3}) \quad (27)$$

Consequently, we have that the maximum number of edges that can be flipped by Breaker while still maintaining Maker's win is $(n-1)^2/4 - x$, which is equal to our lower bound, $\kappa_{\text{low}}^*(n)$, for the flip-bias threshold.

$$\kappa_{\text{low}}^*(n) = \frac{(n-1)^2}{4} - x = \frac{n^2}{4} - \frac{n}{2} + \frac{1}{4} - \left(\frac{3}{2}\right)^{2/3} n^{4/3} + O(n^{2/3}) = \frac{n^2}{4} - \left(\frac{3}{2}\right)^{2/3} n^{4/3} + O(n) \quad (28)$$

Thus, from (11) and (28) and this completes the proof of Theorem 2.4.

It would be interesting to extend this framework to other classical Maker–Breaker games, now played on tournaments instead, with directionally constrained winning sets, and to study their corresponding threshold phenomena and their relation to tournament score sequences.

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