

A Unified Representation and Transformation of Electromagnetic Configurations Based on Generalized Hertz Potentials

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Abstract

We present a unified framework that fully represents electromagnetic potentials, fields, and sources in vacuum, based on a reinterpretation of the classical Hertz-potential formalism. In this construction, ϕ , $\mathbf{A}$, $\mathbf{E}$, $\mathbf{B}$, ρ , and $\mathbf{J}$ are systematically derived from a single vector wavefield $\mathbf{\Gamma}(\mathbf{x}, t)$ (called the Γ -potential), which is structurally aligned with the classical electric Hertz potential but of broader scope. A surjective mapping is established from such wavefields to all electromagnetic configurations in vacuum (that are sufficiently regular). This mapping induces a well-defined algebraic correspondence between the solution space of Maxwell's equations and the linear space of $C_t^3 C_x^3$ vector wavefields (modulo the relevant symmetries), thereby enabling a framework for structural analysis of electromagnetic fields via their associated wavefields. Gauge freedom and the Lorenz gauge are naturally preserved; charge conservation and Maxwell's equations are inherently encoded in this representation.

Building on this framework, we also introduce a transformation that provides a systematic method for generating new electromagnetic solutions from known ones. This transformation, called the Γ -transformation, generalizes classical gauge transformations and may facilitate the exploration of hidden structures and symmetries in the solution space of Maxwell's equations.

Keywords: Maxwell Equations, Gamma Potential, Gamma Transformation, Gauge symmetry, Electromagnetic Solution Space, Hertz Potentials

1. Introduction

Mathematically, the physical phenomena of electromagnetism can be described by six quantities as functions of spacetime: charge density ρ , current density $\mathbf{J}$, electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, scalar potential ϕ , and vector potential $\mathbf{A}$.

These six quantities can be naturally grouped into three conceptual pairs. ρ and $\mathbf{J}$ form the "source pair", representing the electric charge distribution and its flow. The second pair, containing $\mathbf{E}$ and $\mathbf{B}$, is the "field pair", representing the physical electromagnetic fields, measurable through their effects. The third is the

"potential pair"— φ and A —often regarded as mathematical constructs for expressing the fields, though their possible physical significance is suggested by the Aharonov–Bohm effect [1]. Due to gauge freedom, the potentials are not uniquely defined; however, they are determined by the fields up to a gauge transformation.

These three pairs describe electromagnetic phenomena at three distinct but interconnected levels. They are not independent. Within and across these pairs, comprehensive mathematical relationships exist and are explicitly described by a set of equations [2, 3]. Charges and currents act as the sources of the fields and, in principle, determine the full electromagnetic configuration. Fields describe the observable effects that form the basis of measurement and practical applications. Potentials, beyond their possible physical significance, provide a mathematical representation of the fields.

This interconnection and interdependence suggest an underlying structural redundancy. As a manifestation of this, Maxwell’s equations—describing the relation among fields and sources—are formally overdetermined. Such redundancy indicates that the electromagnetic configuration, as captured by these pairs, may admit a more compact and structurally unified formulation.

In this regard, the Hertz potential [3–5] may already provide a meaningful clue. Although it is often introduced as a technical tool for solving polarization and radiation problems in homogeneous media, the Hertz potential can generate both scalar and vector potentials—and hence the electric and magnetic fields—from a single vector function. This construction hints at the possibility of extending it to represent the full electromagnetic configuration, at a deeper structural level.

However, this structural perspective on the Hertz formalism remains largely underexplored in the literature. The present work is motivated by this observation, and seeks to reinterpret and extend the Hertz framework into a general representation capable of describing all electromagnetic configurations in vacuum—including those involving arbitrary sources—with full structural clarity. This representation, which we call the **Γ -representation**, forms the theoretical foundation of this work.

As is well known, the interrelations among the electromagnetic quantities are governed by several fundamental equations. For completeness and consistency of notation and format, we list them below.

Within the source pair, charge conservation is expressed by the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (1)$$

This indicates that electric charge is a conserved quantity, and current density represents its motion.

For the field pair, the dynamics and coupling to sources are described by Maxwell’s equations, which in SI units (with $c = 1/\sqrt{\epsilon_0\mu_0}$) can be written as:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (4)$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}. \quad (5)$$

These equations are internally consistent and compatible with the continuity equation, illustrating the redundancy in the description.

Within the potential pair, a gauge condition imposes a constraint between φ and $\mathbf{A}$ without altering the physical fields. In this paper, the Lorenz gauge condition is adopted:

$$\frac{1}{c^2} \frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \quad (6)$$

The fields can be expressed in terms of the potentials via:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (7)$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi. \quad (8)$$

Finally, under the Lorenz gauge, the relation between potentials and sources satisfies the following wave equations (which can be derived from Eqs. (2)–(8)):

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \frac{\rho}{\epsilon_0}, \quad (9)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \quad (10)$$

In this paper, we refer to the collection of electromagnetic potentials, fields, and sources as an **electromagnetic configuration** (or **EM configuration** in short). This term encompasses the scalar and vector potentials φ and $\mathbf{A}$, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, and the associated charge-current sources ρ and $\mathbf{J}$.

Unless otherwise specified, potentials are assumed to satisfy the Lorenz gauge condition in this paper. In contexts involving the solution space of Maxwell's equations, the term **EM solution** will also be used interchangeably with EM configuration, when no ambiguity arises.

To facilitate notation, we compactly denote an EM configuration (with its six quantities collectively satisfying Eqs. (1)–(8)) as

$$[\text{EM}] \triangleq [\varphi, \mathbf{A}, \mathbf{E}, \mathbf{B}, \rho, \mathbf{J}].$$

Accordingly, we write $[\text{EM}_1], [\text{EM}_2], \dots$ to label distinct configurations, and write $[\text{EM}(\mathbf{\Gamma})] \triangleq [\varphi_{\mathbf{\Gamma}}, \mathbf{A}_{\mathbf{\Gamma}}, \mathbf{E}_{\mathbf{\Gamma}}, \mathbf{B}_{\mathbf{\Gamma}}, \rho_{\mathbf{\Gamma}}, \mathbf{J}_{\mathbf{\Gamma}}]$ to denote the configuration associated with a given function $\mathbf{\Gamma}$.

The structure of this paper is as follows: Section 2 introduces the main construction and presents the general theorem and its converse. Section 3 discusses the symmetries of the EM solutions. Section 4 introduces the Γ -transformation. Section 5 further explores structure of the solution space of Maxwell's equations. Section 6 applies the proposed representation to electromagnetic fields in media and analyzes its relation to Hertz potentials. Section 7 offers a summary and outlook on further research.

2. Γ -Representation: The Main Theorems and Proofs

We now introduce the main mathematical structure of the proposed representation—a general construction that derives electromagnetic potentials, fields, and sources from a single vector wavefield.

Theorem 1 (Γ -Representation Theorem).

Let $\mathbf{\Gamma}(\mathbf{x}, t) \in C_t^3 C_x^3(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$ be a vector field satisfying the vector wave equation:

$$\square \mathbf{\Gamma}(\mathbf{x}, t) \triangleq \frac{1}{c^2} \frac{\partial^2 \mathbf{\Gamma}(\mathbf{x}, t)}{\partial t^2} - \nabla^2 \mathbf{\Gamma}(\mathbf{x}, t) = \mathbf{G}(\mathbf{x}, t). \quad (11)$$

Define:

$$\varphi_\Gamma = -\nabla \cdot \mathbf{\Gamma}, \quad (12)$$

$$\mathbf{A}_\Gamma = \frac{1}{c^2} \frac{\partial \mathbf{\Gamma}}{\partial t}, \quad (13)$$

$$\mathbf{E}_\Gamma = -\frac{1}{c^2} \frac{\partial^2 \mathbf{\Gamma}}{\partial t^2} + \nabla(\nabla \cdot \mathbf{\Gamma}), \quad (14)$$

$$\text{or an alternative form (shown in the proof): } \mathbf{E}_\Gamma = \nabla \times (\nabla \times \mathbf{\Gamma}) - \mathbf{G}, \quad (14a)$$

$$\mathbf{B}_\Gamma = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{\Gamma}), \quad (15)$$

$$\rho_\Gamma = -\varepsilon_0 \nabla \cdot \mathbf{G}, \quad (16)$$

$$\mathbf{J}_\Gamma = \varepsilon_0 \frac{\partial \mathbf{G}}{\partial t}. \quad (17)$$

Then:

1. φ_Γ and $\mathbf{A}_\Gamma$, regarded as the scalar and vector potentials, satisfy the Lorenz gauge condition.
2. $\mathbf{E}_\Gamma$ and $\mathbf{B}_\Gamma$, considered as the electric and magnetic fields, are related to φ_Γ and $\mathbf{A}_\Gamma$ via the standard field–potential relations of classical electrodynamics (i.e., Eqs. (7) and (8)).
3. ρ_Γ and $\mathbf{J}_\Gamma$, interpreted as the charge and current densities, together with $\mathbf{E}_\Gamma$ and $\mathbf{B}_\Gamma$, satisfy Maxwell's equations in vacuum, along with the continuity equation for charge conservation.

Proof.

The assumption $\mathbf{\Gamma} \in C_t^3 C_x^3$ ensures that all derivatives appearing in Eqs. (11)–(17) and in Maxwell's equations exist as classical derivatives and are continuous; mixed derivatives commute.

Eq. (6), the Lorenz gauge condition, follows from Eqs. (12) and (13):

$$\frac{1}{c^2} \frac{\partial \varphi_\Gamma}{\partial t} + \nabla \cdot \mathbf{A}_\Gamma = -\frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{\Gamma}) + \nabla \cdot \left(\frac{1}{c^2} \frac{\partial \mathbf{\Gamma}}{\partial t} \right) = 0.$$

The two expressions for the electric field, Eqs. (14) and (14a), are equivalent. This can be confirmed by applying the vector identity $\nabla^2 \mathbf{\Gamma} = \nabla(\nabla \cdot \mathbf{\Gamma}) - \nabla \times (\nabla \times \mathbf{\Gamma})$ to Eq. (11) and reorganizing terms.

Eq. (7), which relates the vector potential to the magnetic field, follows directly from Eqs. (13) and (15).

Eq. (8), which expresses the electric field in terms of the potentials, follows by inserting Eqs. (12) and (13) into Eq. (14):

$$\mathbf{E}_r = -\frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}}{\partial t} \right) + \nabla(\nabla \cdot \mathbf{r}) = -\frac{\partial \mathbf{A}_r}{\partial t} - \nabla \phi_r.$$

To verify Maxwell's equations, Eqs. (2)–(5):

- Eq. (2) (Gauss's law) follows by taking divergence of Eq. (14a), using Eq. (16), and noting that the divergence of a curl vanishes:

$$\nabla \cdot \mathbf{E}_r = \nabla \cdot (\nabla \times (\nabla \times \mathbf{r})) - \nabla \cdot \mathbf{G} = \frac{\rho_r}{\varepsilon_0}.$$

- Eq. (3) (Gauss's law for magnetism) holds since $\mathbf{B} = \nabla \times \mathbf{A}$ (Eq. (7)) and the divergence of any curl is zero.

- Eq. (4) (Faraday's law of induction) follows by taking the curl of Eq. (14):

$$\nabla \times \mathbf{E}_r = -\frac{1}{c^2} \frac{\partial^2 (\nabla \times \mathbf{r})}{\partial t^2} + \nabla \times (\nabla(\nabla \cdot \mathbf{r})) = -\frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{r}) \right) = -\frac{\partial \mathbf{B}_r}{\partial t}.$$

- Eq. (5) (Ampère–Maxwell law) follows from rewriting Eq. (14a) as $\nabla \times (\nabla \times \mathbf{r}) = \mathbf{E}_r + \mathbf{G}$ and substituting it into the curl of Eq. (15), and then using Eq. (17):

$$\nabla \times \mathbf{B}_r = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times (\nabla \times \mathbf{r})) = \frac{1}{c^2} \frac{\partial \mathbf{E}_r}{\partial t} + \frac{1}{c^2} \frac{\partial \mathbf{G}}{\partial t} = \frac{1}{c^2} \frac{\partial \mathbf{E}_r}{\partial t} + \mu_0 \mathbf{J}_r.$$

Finally, Eq. (1), the continuity equation, follows from Eqs. (16)–(17):

$$\frac{\partial \rho_r}{\partial t} + \nabla \cdot \mathbf{J}_r = -\varepsilon_0 \frac{\partial (\nabla \cdot \mathbf{G})}{\partial t} + \varepsilon_0 \nabla \cdot \left(\frac{\partial \mathbf{G}}{\partial t} \right) = 0.$$

Q.E.D.

For Maxwell's equations and other identities in Eqs. (1)–(10) to hold pointwise, a minimal regularity threshold is required. In particular, it is necessary (and sufficient) that $\rho \in C_t^1 C_x^0$ and $\mathbf{J} \in C_t^0 C_x^1$ for the continuity equation (Eq. (1)) to hold in the strong (pointwise) sense. For Eqs. (9)–(10) with source $(\rho, \mathbf{J})$ to hold pointwise, it is required that $\mathbf{A} \in C_t^2 C_x^3$ and $\phi \in C_t^3 C_x^2$. These, in turn, imply $\mathbf{B} \in C_t^2 C_x^2$ via Eq. (7) and $\mathbf{E} \in C_t^1 C_x^1$ via Eq. (8).

We refer to vacuum configurations meeting these regularity requirements as **regular electromagnetic configurations**. Cases with spatial or temporal jumps/impulses (e.g., line/sheet charges or time-impulse currents) fall below this continuity threshold; some identities then hold only in the distributional (weak) sense. These “non-regular” configurations are excluded from the discussion in this paper.

The assumption in Theorem 1, $\mathbf{r} \in C_t^3 C_x^3$, is sufficient and essentially minimal (within this framework) to ensure that the induced potentials, fields, and sources form a regular electromagnetic configuration.

Applying the notation introduced in Section 1, Theorem 1 essentially states that a vector wavefield $\mathbf{r}$, together with its associated source field $\mathbf{G}$, gives rise to an EM configuration $[\text{EM}(\mathbf{r})] =$

$[\varphi, \mathbf{A}, \mathbf{E}, \mathbf{B}, \rho, \mathbf{J}]$ through Eqs. (12)–(17). In other words, the full electromagnetic configuration—including the fields, potentials, and sources—along with Maxwell’s equations and other interrelations, is fully encoded in the single wavefield $\mathbf{F}$.

Such configurations will be referred to as **Γ -representations**, where the wavefield $\mathbf{F}$ is called the **Γ -potential**, and the corresponding source field $\mathbf{G}$ is termed the **Γ -source**.

According to Theorem 1, a spacetime vector field $\mathbf{F}$ qualifies as a valid Γ -potential if it is of class $C_t^3 C_x^3$, ensuring that $\square \mathbf{F}$ and its first derivatives are well defined and the induced mappings to $(\varphi, \mathbf{A}, \mathbf{E}, \mathbf{B}, \rho, \mathbf{J})$ are classical (i.e., valid pointwise).

This mapping, $\mathbf{F} \mapsto [\text{EM}(\mathbf{F})]$, however, is not injective: if two Γ -potentials differ by a constant vector field, they yield the same EM configuration. This nontrivial degree of freedom will be further discussed in Section 3.

A natural question then arises: is this mapping surjective—that is, can every regular electromagnetic configuration in vacuum be realized as a Γ -representation? The answer is affirmative. Every regular EM configuration (i.e., satisfying the continuity threshold stated above) admits at least one corresponding Γ -potential, though not necessarily unique. This is formalized in the following theorem.

Theorem 2 (Converse Γ -Representation Theorem).

Let $[\text{EM}] = [\varphi, \mathbf{A}, \mathbf{E}, \mathbf{B}, \rho, \mathbf{J}]$ be a regular vacuum electromagnetic configuration. Then there exists a spacetime vector field $\mathbf{F}$ that serves as the Γ -potential of $[\text{EM}]$.

In other words, Theorem 2 asserts that every vacuum solution of Maxwell’s equations that meets the continuity threshold (i.e., the “regular” class) can be fully represented by a single $C_t^3 C_x^3$ vector wavefield (as the Γ -potential). To prove this, we directly construct such a Γ -potential from an arbitrary regular EM configuration $[\varphi, \mathbf{A}, \mathbf{E}, \mathbf{B}, \rho, \mathbf{J}]$.

Proof.

As scalar and vector potentials of a regular EM configuration, we have $\varphi \in C_t^3 C_x^2$ and $\mathbf{A} \in C_t^2 C_x^3$. Fix a reference time t_0 and choose a spatial vector field $\mathbf{F}_{t_0}(\mathbf{x}) \in C_x^3$ such that

$$\nabla \cdot \mathbf{F}_{t_0}(\mathbf{x}) = -\varphi(\mathbf{x}, t_0).$$

Define the spacetime vector field:

$$\mathbf{F}(\mathbf{x}, t) = \mathbf{F}_{t_0}(\mathbf{x}) + \int_{t_0}^t c^2 \mathbf{A}(\mathbf{x}, \tau) d\tau. \quad (18)$$

Then $\mathbf{F} \in C_t^3 C_x^3$, so it qualifies as a Γ -potential. Such regularity also guarantees the existence of a vector field $\mathbf{G} \in C_t^1 C_x^1$ satisfying the wave equation $\square \mathbf{F}(\mathbf{x}, t) = \mathbf{G}$.

We now verify that such a vector field $\mathbf{F}$, as a Γ -potential, reproduces the given EM configuration $[\varphi, \mathbf{A}, \mathbf{E}, \mathbf{B}, \rho, \mathbf{J}]$ via Eqs. (12)–(17). In other words, with $\mathbf{F}$ given by Eq. (18), the relations in Eqs. (12)–(17) hold identically and recover the prescribed fields, potentials, and sources in $[\text{EM}]$.

Eq. (12) follows by taking the divergence of $\mathbf{F}$ in Eq. (18) and applying the Lorenz gauge condition:

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}_{t_0} + \int_{t_0}^t c^2 (\nabla \cdot \mathbf{A}) d\tau = -\varphi(\mathbf{x}, t_0) - \int_{t_0}^t c^2 \left(\frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right) d\tau = -\varphi(\mathbf{x}, t).$$

Eq. (13) arises from the time derivative of $\mathbf{F}$ in Eq. (18) and the time-independence of $\mathbf{F}_{t_0}(\mathbf{x})$:

$$\frac{\partial}{\partial t} \mathbf{F}(\mathbf{x}, t) = \frac{\partial}{\partial t} \mathbf{F}_{t_0}(\mathbf{x}) + \frac{\partial}{\partial t} \int_{t_0}^t c^2 \mathbf{A}(\mathbf{x}, \tau) d\tau = c^2 \mathbf{A}(\mathbf{x}, t),$$

Eq. (15) can be obtained by inserting Eq. (13) into Eq. (7):

$$\mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t) = \nabla \times \left(\frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{F}(\mathbf{x}, t) \right) = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{F}(\mathbf{x}, t)).$$

Eq. (14) is yielded by inserting φ (Eq. (12)) and $\mathbf{A}$ (Eq. (13)) into $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi$ (Eq. (8)):

$$\mathbf{E} = -\frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{F} \right) - \nabla(-\nabla \cdot \mathbf{F}) = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{F} + \nabla(\nabla \cdot \mathbf{F}).$$

Eq. (14a) is then recovered applying the wave equation $\square \mathbf{F} = \mathbf{G}$ to Eq. (14):

$$\mathbf{E} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{F} + \nabla(\nabla \cdot \mathbf{F}) = \left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{F} + \nabla^2 \mathbf{F} \right) + \nabla \times (\nabla \times \mathbf{F}) = -\mathbf{G} + \nabla \times (\nabla \times \mathbf{F}).$$

Eq. (16) is a consequence of taking divergence of Eq. (14a) and applying Eq. (2):

$$\nabla \cdot \mathbf{E} = \nabla \cdot (\nabla \times (\nabla \times \mathbf{F})) - \nabla \cdot \mathbf{G} = -\nabla \cdot \mathbf{G} = \frac{\rho}{\epsilon_0}.$$

Finally, for Eq. (17), substituting Eqs. (14) and (15) into Eq. (5) yields:

$$\nabla \times \left(\frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{F}) \right) = \frac{1}{c^2} \frac{\partial}{\partial t} \left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{F}(\mathbf{x}, t) + \nabla(\nabla \cdot \mathbf{F}) \right) + \mu_0 \mathbf{J},$$

which simplifies to:

$$\frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times (\nabla \times \mathbf{F})) = \frac{1}{c^2} \frac{\partial}{\partial t} \left(-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{F}(\mathbf{x}, t) + \nabla(\nabla \cdot \mathbf{F}) \right) + \mu_0 \mathbf{J}.$$

Reorganizing terms and applying the wave equation $\square \mathbf{F} = \mathbf{G}$, we obtain:

$$\frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{F}(\mathbf{x}, t) - \nabla(\nabla \cdot \mathbf{F}) + \nabla \times (\nabla \times \mathbf{F}) \right) = \frac{1}{c^2} \frac{\partial}{\partial t} (\square \mathbf{F}) = \frac{1}{c^2} \frac{\partial \mathbf{G}}{\partial t} = \mu_0 \mathbf{J}.$$

Hence $\mathbf{J} = \frac{1}{\mu_0 c^2} \frac{\partial \mathbf{G}}{\partial t} = \epsilon_0 \frac{\partial \mathbf{G}}{\partial t}$, which is Eq. (17).

Q.E.D.

The above construction of $\mathbf{F}$ from a given electromagnetic configuration further illustrates the role of Γ -potential. As defined in Eq. (18), $\mathbf{F}$ is essentially the time integral of vector potential $\mathbf{A}$, with an initial condition $\mathbf{F}_{t_0}(\mathbf{x})$ which is effectively the “spatial integral” or “inverse divergence” of scalar potential φ at the initial time t_0 .

More generally, the significance of the Γ -potential can be understood through Eqs. (12) and (13). Specifically, $\boldsymbol{\Gamma}$ may be viewed either as the preimage of scalar potential φ under the divergence operator (Eq. (12)), or as the preimage of vector potential $\boldsymbol{A}$ under the time derivative (Eq. (13)). The Lorenz gauge condition bridges these two preimages, enabling the single function $\boldsymbol{\Gamma}$ to represent both φ and $\boldsymbol{A}$, and thereby represent the full electromagnetic configuration—including fields $\boldsymbol{E}$ and $\boldsymbol{B}$, and sources ρ and $\boldsymbol{J}$.

If the three canonical pairs in electrodynamics—namely, the source pair $(\rho, \boldsymbol{J})$, the field pair $(\boldsymbol{E}, \boldsymbol{B})$, and the potential pair $(\varphi, \boldsymbol{A})$ —are understood as describing electromagnetic phenomena at three successive levels, then the Γ -potential may be interpreted as extending this hierarchy by adding a fourth layer beyond the potentials.

As can be seen from its mathematical construction, the concept of Γ -potential bears a strong resemblance to the classical Hertz potentials—particularly the electric Hertz potential [3–5]. At the heart of the Hertz formalism lies a simple yet profound idea: using a single vector function to generate both the scalar and vector potentials, and thereby the complete electromagnetic fields. The Γ -potential inherits this spirit and attempts to generalize it into a symmetric and comprehensive framework that applies to all regular electromagnetic configurations in vacuum, including those involving charge and current sources.

The relationship between the Hertz potentials and the Γ -potential will be further discussed in Section 6.

3. Symmetry in Γ -Representation

From the perspective of their interrelations, the four hierarchical levels of electromagnetic quantities—from sources and fields to potentials and the Γ -potential—are connected through successive applications of differential operators. At each level, the quantities can be obtained from those at one level below via operations such as gradient, divergence, curl, or time derivative. Specifically, the sources ρ and $\boldsymbol{J}$, as expressed in Eqs. (2) and (5), can be derived from combinations of divergence, curl, and time derivative of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$; the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, in turn, can be obtained from the scalar and vector potentials via gradient, curl, and time derivative operations, as shown in Eqs. (7) and (8). Finally, as shown in Eqs. (12) and (13), the potentials φ and $\boldsymbol{A}$ can be derived from the Γ -potential through divergence and time differentiation.

As a natural consequence of this nested differential structure, each level admits certain degrees of freedom—analogueous to "integration constants"—that leave all higher-level qualities unchanged. These structural freedoms encompass and generalize the traditional gauge symmetry, extending its principle across all four levels of electromagnetic description. Within the Γ -representation framework, such freedoms manifest as structural invariances, offering a unified and deeper expression of symmetry that may reveal the intrinsic mathematical nature of electromagnetic configurations.

The following theorem characterizes these invariances in a systematic and hierarchical manner.

Theorem 3 (Γ -Representation Invariance Theorem).

Let $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ be two Γ -potentials corresponding to electromagnetic configurations $[\text{EM}_1] = [\varphi_1, \boldsymbol{A}_1, \boldsymbol{E}_1, \boldsymbol{B}_1, \rho_1, \boldsymbol{J}_1]$ and $[\text{EM}_2] = [\varphi_2, \boldsymbol{A}_2, \boldsymbol{E}_2, \boldsymbol{B}_2, \rho_2, \boldsymbol{J}_2]$, respectively. Define $\boldsymbol{\gamma} = \boldsymbol{\Gamma}_2 - \boldsymbol{\Gamma}_1$. Then:

- (a) If $\nabla \cdot \boldsymbol{\gamma} = 0$ and $\frac{\partial \boldsymbol{\gamma}}{\partial t} = 0$, then $[\text{EM}_2] = [\text{EM}_1]$.

(b) If $\square \boldsymbol{\gamma} = 0$ and $\nabla \times (\nabla \times \boldsymbol{\gamma}) = 0$, and assuming standard decay at infinity for $\mathbf{B}_2 - \mathbf{B}_1$, then $\mathbf{E}_2 = \mathbf{E}_1$, $\mathbf{B}_2 = \mathbf{B}_1$, $\rho_2 = \rho_1$, and $\mathbf{J}_2 = \mathbf{J}_1$; moreover, the potentials $(\varphi_2, \mathbf{A}_2)$ and $(\varphi_1, \mathbf{A}_1)$ are related by a gauge transformation that preserves the Lorenz gauge condition.

(c) If $\square \boldsymbol{\gamma} = 0$, then $\rho_2 = \rho_1$ and $\mathbf{J}_2 = \mathbf{J}_1$.

Proof.

As assumed, $\boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_1 + \boldsymbol{\gamma}$. Substituting into Eqs. (12) and (13), we obtain:

$$\varphi_2 = \varphi_1 - \nabla \cdot \boldsymbol{\gamma}, \quad \mathbf{A}_2 = \mathbf{A}_1 + \frac{1}{c^2} \frac{\partial \boldsymbol{\gamma}}{\partial t}. \quad (19)$$

(a) If $\nabla \cdot \boldsymbol{\gamma} = 0$ and $\frac{\partial \boldsymbol{\gamma}}{\partial t} = 0$, then Eq. (19) yields $\varphi_2 = \varphi_1$ and $\mathbf{A}_2 = \mathbf{A}_1$. Since both scalar and vector potentials are equal, the corresponding fields and sources derived from them are also equal. Thus, $[\text{EM}_2] = [\text{EM}_1]$.

(b) From $\square \boldsymbol{\gamma} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{\gamma}}{\partial t^2} - \nabla^2 \boldsymbol{\gamma} = 0$ and $\nabla \times (\nabla \times \boldsymbol{\gamma}) = \nabla(\nabla \cdot \boldsymbol{\gamma}) - \nabla^2 \boldsymbol{\gamma} = 0$, we obtain

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{\gamma}}{\partial t^2} = \nabla(\nabla \cdot \boldsymbol{\gamma}). \quad (20)$$

Define a scalar $\chi(\mathbf{x}, t)$ by prescribing its time derivative,

$$\frac{\partial \chi}{\partial t} = \nabla \cdot \boldsymbol{\gamma}. \quad (21)$$

Taking the gradient of Eq. (21) and using Eq. (20) gives

$$\nabla \frac{\partial \chi}{\partial t} = \frac{\partial}{\partial t} (\nabla \chi) = \nabla(\nabla \cdot \boldsymbol{\gamma}) = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{\gamma}}{\partial t^2},$$

hence

$$\frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \boldsymbol{\gamma}}{\partial t} - \nabla \chi \right) = 0. \quad (22)$$

That is, $\frac{1}{c^2} \frac{\partial \boldsymbol{\gamma}}{\partial t} - \nabla \chi$ is time-independent, and so is its curl:

$$\mathbf{K} \triangleq \nabla \times \left(\frac{1}{c^2} \frac{\partial \boldsymbol{\gamma}}{\partial t} - \nabla \chi \right) = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \boldsymbol{\gamma}). \quad (23)$$

Besides being time-independent, $\mathbf{K}$ also satisfies

$$\nabla \cdot \mathbf{K} = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \cdot (\nabla \times \boldsymbol{\gamma})) = 0, \quad \nabla \times \mathbf{K} = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times (\nabla \times \boldsymbol{\gamma})) = 0.$$

Substituting these into the vector identity $\Delta \mathbf{K} = \nabla(\nabla \cdot \mathbf{K}) - \nabla \times (\nabla \times \mathbf{K})$ gives $\Delta \mathbf{K} = 0$; i.e., $\mathbf{K}$ is a harmonic vector field. Moreover, by Eq. (15), we actually have

$$\mathbf{B}_2 - \mathbf{B}_1 = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \boldsymbol{\gamma}) = \mathbf{K}.$$

The assumed standard decay at infinity for $\mathbf{B}_2 - \mathbf{B}_1$ implies that $\mathbf{K}(\mathbf{x}) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$. Liouville/Helmholtz uniqueness for harmonic vector fields then ensures $\mathbf{K} \equiv 0$.

With Eq. (22) and $\mathbf{K} \equiv 0$ in Eq. (23), the vector $\frac{1}{c^2} \frac{\partial \boldsymbol{\gamma}}{\partial t} - \nabla \chi$ is time-independent and curl-free. Hence, on a simply connected domain (or in $\mathbb{R}^3$ with decay), we can write

$$\frac{1}{c^2} \frac{\partial \boldsymbol{\gamma}}{\partial t} - \nabla \chi = \nabla f(x). \quad (24)$$

for some scalar $f(x)$. Replacing χ by $\chi + f(x)$, Eq. (21) still stands and Eq. (24) gives

$$\nabla \chi = \frac{1}{c^2} \frac{\partial \boldsymbol{\gamma}}{\partial t}. \quad (25)$$

Substituting Eq. (25) and (21) into Eq. (19), we find

$$\varphi_2 = \varphi_1 - \frac{\partial \chi}{\partial t}, \quad \mathbf{A}_2 = \mathbf{A}_1 + \nabla \chi.$$

Thus $(\varphi_2, \mathbf{A}_2)$ and $(\varphi_1, \mathbf{A}_1)$ are related by a gauge transformation, hence the corresponding fields and sources remain unchanged: $\mathbf{E}_2 = \mathbf{E}_1$, $\mathbf{B}_2 = \mathbf{B}_1$, $\rho_2 = \rho_1$, and $\mathbf{J}_2 = \mathbf{J}_1$.

Finally, since the gauge function χ satisfies

$$\square \chi = \frac{1}{c^2} \frac{\partial^2 \chi}{\partial t^2} - \nabla^2 \chi = \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\partial \chi}{\partial t} \right) - \nabla \cdot (\nabla \chi) = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \cdot \boldsymbol{\gamma}) - \nabla \cdot \left(\frac{1}{c^2} \frac{\partial \boldsymbol{\gamma}}{\partial t} \right) = 0,$$

the Lorenz gauge condition is preserved.

(c) If $\square \boldsymbol{\gamma} = 0$, then the corresponding Γ -sources coincide:

$$\mathbf{G}_2 = \square \boldsymbol{\Gamma}_2 = \square (\boldsymbol{\Gamma}_1 + \boldsymbol{\gamma}) = \square \boldsymbol{\Gamma}_1 + \square \boldsymbol{\gamma} = \square \boldsymbol{\Gamma}_1 = \mathbf{G}_1.$$

By Eqs. (16) and (17), equal Γ -sources imply equal charge and current densities: $\rho_2 = \rho_1$, $\mathbf{J}_2 = \mathbf{J}_1$. However, unless $\nabla \times (\nabla \times \boldsymbol{\gamma}) = 0$, the difference in Γ -potentials, $\boldsymbol{\gamma}$, leads to differences in the fields $\mathbf{E}$ and $\mathbf{B}$, as seen from Eqs. (14a) and (15).

Q.E.D.

As a representation of electromagnetic configurations, the Γ -potential inherits and elucidates the intrinsic symmetry structure embedded in classical electrodynamics, as formalized in Theorem 3. These invariances arise from the layered derivative relationships among potentials, fields, and sources. The Γ -representation provides a unified framework in which such structural freedoms are not only preserved but also rendered more transparent and systematically classified:

- When two Γ -potentials differ by a static and divergence-free vector field, they represent the same electromagnetic configuration (Theorem 3(a)).
- When the difference between the two Γ -potentials, $\boldsymbol{\gamma}$, is a solution to the homogeneous wave equation and has vanishing double curl, the scalar and vector potentials are related by a Lorenz-gauge-preserving gauge transformation, and the corresponding fields and sources remain identical (provided $\mathbf{B}_2 - \mathbf{B}_1$ decays at infinity; otherwise $\mathbf{B}$ may differ by a time-independent harmonic field; see Theorem 3(b)).

- When the difference satisfies the homogeneous wave equation but possesses a nonzero double curl, the Γ -sources remain unchanged, implying that charge and current densities remain the same, while the fields $\mathbf{E}$ and $\mathbf{B}$ may differ (Theorem 3(c)).
- Extending beyond Theorem 3, if the difference $\boldsymbol{\gamma}$ does not satisfy the homogeneous wave equation—i.e., if it corresponds to a nonzero Γ -source $\square\boldsymbol{\gamma} = \mathbf{g} \neq 0$ —then the two Γ -potentials describe genuinely distinct electromagnetic configurations, with differences in all associated quantities including charge density ρ and current density $\mathbf{J}$.

In this way, while providing a unified description of potentials, fields, and sources, the Γ -representation also helps to reveal the layered symmetries of electromagnetic theory.

4. Γ -Transformations

As is well known, the governing relations of classical electrodynamics—including Maxwell’s equations, the charge continuity equation, the Lorenz gauge condition, the potential-field relations, and the associated wave equations (essentially Eqs. (1)–(10))—are all linear. This linearity naturally extends to the Γ -potential framework. The relations that link electromagnetic quantities to the Γ -potential (Eqs. (11)–(17)) are likewise linear. This implies that the Γ -representation is additive and thus enables the construction of new EM solutions by linear superposition.

To proceed with a more concrete discussion, we first define basic operations on electromagnetic configurations, treating them formally as algebraic objects.

Let $[\text{EM}_1] = [\varphi_1, \mathbf{A}_1, \mathbf{E}_1, \mathbf{B}_1, \rho_1, \mathbf{J}_1]$ and $[\text{EM}_2] = [\varphi_2, \mathbf{A}_2, \mathbf{E}_2, \mathbf{B}_2, \rho_2, \mathbf{J}_2]$ be two EM solutions, and $\alpha \in \mathbb{R}$. We define

- the **sum** of $[\text{EM}_1]$ and $[\text{EM}_2]$ as:

$$[\text{EM}_1] + [\text{EM}_2] = [\varphi_1 + \varphi_2, \mathbf{A}_1 + \mathbf{A}_2, \mathbf{E}_1 + \mathbf{E}_2, \mathbf{B}_1 + \mathbf{B}_2, \rho_1 + \rho_2, \mathbf{J}_1 + \mathbf{J}_2],$$

- the **scalar multiplication** of α and an EM solution $[\text{EM}] = [\varphi, \mathbf{A}, \mathbf{E}, \mathbf{B}, \rho, \mathbf{J}]$ as:

$$\alpha[\text{EM}] = [\alpha\varphi, \alpha\mathbf{A}, \alpha\mathbf{E}, \alpha\mathbf{B}, \alpha\rho, \alpha\mathbf{J}].$$

Under these operations, any linear combination of EM solutions remains a valid EM solution. Moreover, such combinations correspond directly to linear combinations of their associated Γ -potentials, as formalized in the following theorem.

Theorem 4 (Γ -Potential Linearity Theorem).

Let Γ_1 and Γ_2 be two Γ -potentials corresponding to EM solutions $[\text{EM}(\Gamma_1)] = [\varphi_1, \mathbf{A}_1, \mathbf{E}_1, \mathbf{B}_1, \rho_1, \mathbf{J}_1]$ and $[\text{EM}(\Gamma_2)] = [\varphi_2, \mathbf{A}_2, \mathbf{E}_2, \mathbf{B}_2, \rho_2, \mathbf{J}_2]$, and let $\alpha, \beta \in \mathbb{R}$. Then:

$$[\text{EM}(\alpha\Gamma_1 + \beta\Gamma_2)] = \alpha[\text{EM}(\Gamma_1)] + \beta[\text{EM}(\Gamma_2)].$$

The proof follows directly from the linearity of all involved equations (and is omitted here).

Theorem 4 enables us to construct new EM solutions by linearly combining existing ones through their Γ -potentials. In particular, for any given EM solution, we can generate a new solution by adding to it an auxiliary one constructed from a vector wavefield $\boldsymbol{\Gamma}$ as a Γ -potential. That is,

$$[\text{EM}'] = [\text{EM}_0] + [\text{EM}(\boldsymbol{\Gamma})].$$

In this case, we refer to $[EM']$ as the Γ -**transformation** of $[EM_0]$ by Γ .

According to Theorem 1, a spacetime vector field qualifies as a Γ -potential if it is $C_t^3 C_x^3$. Any such vector field can thus be used to transform an EM solution into another. Because all electromagnetic quantities—fields, potentials, and sources—are explicitly constructed from the Γ -potential via Eqs. (12)–(17), this framework enables a direct and flexible method for modifying specific aspects of an EM solution. In principle, we can selectively adjust a particular property of an EM solution by transforming it with a Γ -potential that induces a targeted modification of that property.

As a simple illustration, assuming we wish to cancel the magnetic field of a configuration $[EM(\Gamma_1)]$, we can choose a vector field Γ_2 such that $\nabla \times \Gamma_2 = -\nabla \times \Gamma_1$ and use it to transform $[EM(\Gamma_1)]$; the resulting solution $[EM'] = EM(\Gamma_1) + EM(\Gamma_2)$ then has vanishing magnetic field (by Eq. (15)).

Interestingly, the vector quantities in a given EM solution—such as $\mathbf{E}$, $\mathbf{B}$, or $\mathbf{A}$ —can themselves be used as Γ -potentials (provided sufficient regularity). This opens the door to recursive or self-referential transformations. For example, taking the electric field $\mathbf{E}$ from a solution $[EM]$ and using $\mathbf{E}$ (or a vector function of $\mathbf{E}$) as a Γ -potential yields a transformed solution:

$$[EM'] = [EM] + [EM(\mathbf{E})]$$

Here, the new configuration may incorporate feedback-like effects, potentially enhancing or reorienting the original electric field. If the original $\mathbf{E}$ is time-dependent and spatially localized, the transformed field may exhibit intensified radiation characteristics or constructive interference in specific directions.

Similarly, we can use the magnetic field $\mathbf{B}$ (or a vector function of $\mathbf{B}$) as a Γ -potential:

$$[EM'] = [EM] + [EM(\mathbf{B})]$$

Such a transformation can be interpreted as embedding magnetic topology into the evolving field configuration. In particular, if $\mathbf{B}$ carries nontrivial structure (e.g., braided or knotted field lines), this operation may lead to new solutions with enhanced magnetic helicity or topological stability.

Another example is to use the vector potential $\mathbf{A}$ (or a vector function of $\mathbf{A}$):

$$[EM'] = [EM] + [EM(\mathbf{A})]$$

This may correspond to augmenting the system with a self-referenced gauge structure. Since $\mathbf{A}$ directly contributes to both electric and magnetic fields, this transformation may shift the overall energy distribution and potentially induce novel polarization patterns.

These transformations can also be combined or iterated systematically. For instance, applying successive transformations with a fixed Γ -potential,

$$[EM^{(n)}] = [EM^{(n-1)}] + [EM(\Gamma)]$$

generates a chain of solutions $[EM^{(0)}], [EM^{(1)}], \dots, [EM^{(n)}]$. The cumulative effect may include progressive buildup of energy, helicity, or spatial complexity.

In summary, these Γ -transformations provide a tool for constructing structured or topologically nontrivial electromagnetic fields and systematically generating new configurations from known ones. In particular, choosing a knotted magnetic field [6, 7] as the Γ -potential may yield configurations with

interlinked electric and magnetic structures, with potential applications in topological optics, structured light, and plasma confinement.

Moreover, from a broader perspective, the Γ -potential framework also enables further ways to map EM solutions to new ones. Since the Γ -potential determines the entire EM configuration, applying to it any operator on vector fields that preserves $C_t^3 C_x^3$ regularity yields another admissible Γ -potential—and thus a new EM solution. In this sense, transforming a Γ -potential induces a general transformation of the associated EM solution.

Scalar multiplication and linear combination with another Γ -potential are special cases of such operators and recover the Γ -transformations discussed earlier in this section. More generally, owing to the linearity of the defining relations and of the vector wave equation, linear operators can carry structured relations from Γ -potentials through to the EM quantities, as formalized below.

Theorem 5 (Operator-Induced Γ -Transformation Theorem).

Let $\mathbf{\Gamma}(\mathbf{x}, t) \in C_t^3 C_x^3(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$ be a Γ -potential associated with $[\text{EM}(\mathbf{\Gamma})] = [\varphi, \mathbf{A}, \mathbf{E}, \mathbf{B}, \rho, \mathbf{J}]$, and $\square \mathbf{\Gamma} = \mathbf{G}$. Let $\mathcal{L}$ be a linear operator on vector fields such that, for this $\mathbf{\Gamma}$, $\mathcal{L}[\mathbf{\Gamma}] \in C_t^3 C_x^3$. Assume that $\mathcal{L}$ commutes with the basic derivatives and the wave operator:

$$[\mathcal{L}, \partial_t] = [\mathcal{L}, \nabla \cdot] = [\mathcal{L}, \nabla \times] = [\mathcal{L}, \square] = 0, \text{ (where } [A, B] = AB - BA\text{).}$$

If there exists a linear scalar companion $\mathcal{L}_s$ such that, for all sufficiently smooth vector fields $\mathbf{X}$ and scalar functions ψ ,

$$\nabla \cdot (\mathcal{L}[\mathbf{X}]) = \mathcal{L}_s[\nabla \cdot \mathbf{X}], \quad \mathcal{L}[\nabla \psi] = \nabla(\mathcal{L}_s[\psi]),$$

Then the EM configuration associated with $\mathcal{L}[\mathbf{\Gamma}]$, denoted $[\text{EM}(\mathcal{L}[\mathbf{\Gamma}])] = [\varphi', \mathbf{A}', \mathbf{E}', \mathbf{B}', \rho', \mathbf{J}']$, satisfies

$$\varphi' = \mathcal{L}_s[\varphi], \mathbf{A}' = \mathcal{L}[\mathbf{A}], \mathbf{E}' = \mathcal{L}[\mathbf{E}], \mathbf{B}' = \mathcal{L}[\mathbf{B}], \rho' = \mathcal{L}_s[\rho], \mathbf{J}' = \mathcal{L}[\mathbf{J}],$$

and the Γ -source transforms as $\mathbf{G}' = \mathcal{L}[\mathbf{G}]$.

The proof follows directly from linearity and the stated commutation/compatibility properties and is omitted here.

Depending on the particular operator $\mathcal{L}$, $\mathbf{\Gamma}$ may need higher regularity to ensure $\mathcal{L}[\mathbf{\Gamma}] \in C_t^3 C_x^3$ (e.g., for a spacetime operator of order (m_t, m_x) , it suffices that $\mathbf{\Gamma} \in C_t^{3+m_t} C_x^{3+m_x}$). Under the structural assumptions on $\mathcal{L}$ and $\mathcal{L}_s$, the scalar outputs $\mathcal{L}_s[\varphi]$ and $\mathcal{L}_s[\rho]$ inherit the required scalar regularity.

As a simple example of such transformations, if for a $\mathbf{\Gamma} \in C_t^3 C_x^4$ we take $\mathcal{L} = \nabla \times$, then $\mathbf{\Gamma}' = \mathcal{L}[\mathbf{\Gamma}] = \nabla \times \mathbf{\Gamma}$, $\mathcal{L}_s \equiv 0$, and the transformed solution $[\text{EM}(\mathcal{L}[\mathbf{\Gamma}])]$ satisfies:

$$\varphi' = 0, \mathbf{A}' = \nabla \times \mathbf{A} = \mathbf{B}, \mathbf{E}' = \nabla \times \mathbf{E}, \mathbf{B}' = \nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}), \rho' = 0, \mathbf{J}' = \nabla \times \mathbf{J}, \mathbf{G}' = \nabla \times \mathbf{G}.$$

We can see here that taking the curl of the Γ -potential induces, in effect, the curl of the entire electromagnetic configuration. In other words, the curl operation applied to Γ is carried through to all EM quantities—potentials, fields, and sources. Consequently, the components not tied to curl (the longitudinal/gradient part) are removed in the transformed configuration—e.g., φ and ρ vanish—leaving only the solenoidal (divergence-free) content relevant to curl.

For brevity, we write $[\mathbf{EM}'] = \nabla \times [\mathbf{EM}]$, or, more generally, $[\mathbf{EM}'] = \mathcal{L}[\mathbf{EM}]$. Equivalently, Theorem 5 can be expressed as $[\mathbf{EM}(\mathcal{L}[\mathbf{I}])] = \mathcal{L}[\mathbf{EM}(\mathbf{I})]$ (assuming the regularity condition holds).

Under the hypotheses of Theorem 5, finite compositions of admissible operators are also admissible (subject to the regularity condition). Examples include the vector Laplacian ∇^2 , the d'Alembert operator $\square$, the time derivative ∂_t , isotropic scalar filters $f(\nabla^2)$ or $f(\partial_t, \nabla^2)I$ (applied componentwise), and mixed forms such as $f(\partial_t, \nabla^2)I + g(\partial_t, \nabla^2)(\nabla \times)$.

These higher-order, operator-induced, structure-preserving transformations can be useful when vorticity-like features or other topological content are central.

5. Algebraic Perspective on Structure of the Electromagnetic Solution Space

Building on the linear structure established in Section 4, we now explore the algebraic structures of the electromagnetic solution space as revealed through the Γ -representation.

From Theorems 1 and 2, every regular vacuum EM solution corresponds uniquely to a Γ -potential (modulo the symmetries described in Theorem 3), and vice versa. Thus, the space of all EM solutions—denoted $\mathcal{E}$ —can be viewed as isomorphic to the vector space

$$\mathcal{V} \triangleq C_t^3 C_x^3(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$$

modulo an equivalence relation $\sim$ induced by symmetries:

$$\mathcal{E} \cong \mathcal{V} / \sim.$$

The equivalence relation $\sim$ depends on which physical attributes of the EM solution are deemed invariant under transformation. Theorem 3 identifies a hierarchy of such symmetries, each inducing a distinct quotient structure on the vector field space $\mathcal{V}$:

- If two Γ -potentials differ by a static, divergence-free field, then they yield exactly the same potentials, fields, and sources. This defines the finest equivalence relation (Theorem 3(a)), and the resulting quotient $(\mathcal{V}/\sim)_a$ classifies Γ -potentials that represent identical EM solutions in full.
- If we regard two EM solutions as equivalent whenever they produce the same fields and sources—even if their scalar and vector potentials differ via a gauge transformation—then the relevant equivalence relation (Theorem 3(b)) is coarser. The quotient space $(\mathcal{V}/\sim)_b$ corresponds to field-source equivalence under gauge symmetry (preserving the Lorenz gauge; with standard decay at infinity for $\mathbf{B}_2 - \mathbf{B}_1$).
- Finally, if we consider two solutions equivalent whenever they originate from the same source $(\rho, \mathbf{J})$, we obtain the coarsest equivalence class (Theorem 3(c)). The quotient $(\mathcal{V}/\sim)_c$ captures source-preserving transformations, under which the fields may differ due to the allowed addition of source-free solutions (i.e., free-space wavefields).

These nested equivalence relations reflect the layered symmetry of electrodynamics, and their respective quotient structures reveal different aspects of the solution space geometry—ranging from strict identity to field-level and source-level equivalence.

This isomorphic relation between the EM solution space and the vector field space $\mathcal{V}$ (modulo symmetry) invites the application of well-established results from vector field analysis to the study of electromagnetic solutions.

As an illustration, the Helmholtz decomposition asserts that any $\mathbf{\Gamma} \in \mathcal{V}$ can be expressed as the sum of a curl-free part and a divergence-free part (under suitable boundary conditions or sufficient decay at infinity). Applied to a Γ -potential, we write:

$$\mathbf{\Gamma} = -\nabla\psi + \nabla \times \mathbf{C} \triangleq \mathbf{\Gamma}_{irr} + \mathbf{\Gamma}_{sol}.$$

This leads to a decomposition of an EM configuration:

$$[\text{EM}(\mathbf{\Gamma})] = [\text{EM}(\mathbf{\Gamma}_{irr})] + [\text{EM}(\mathbf{\Gamma}_{sol})]$$

By Eqs. (12)–(17), $[\text{EM}(\mathbf{\Gamma}_{irr})]$ (the curl-free component) is purely electric, with $\mathbf{B}_{\Gamma_{irr}} = 0$; whereas $[\text{EM}(\mathbf{\Gamma}_{sol})]$ (the divergence-free component) satisfies $\varphi_{\Gamma_{sol}} = 0$ and $\rho_{\Gamma_{sol}} = 0$ (Eqs. (11, 16)). In this way, an EM solution is decomposed into two physically distinct parts. Such a decomposition may facilitate analysis of the physical origin of fields and offers a flexible means to isolate or synthesize particular features in complex electromagnetic environments.

Beyond decomposition, the same framework may support the construction of topologically structured solutions—for example, those with knotted or linked field lines—by engineering Γ -potentials to inherit prescribed topological invariants from known vector field configurations.

In this sense, the Γ -representation endows the set of electromagnetic solutions with a natural vector-space structure: with addition and scalar multiplication defined via Γ , the set forms a real vector space. The symmetry classes in Theorem 3—full identity, gauge equivalence, and source equivalence—select corresponding equivalence relations and quotient spaces; these layered quotients precisely encode which physical distinctions are retained in a given context. From this vantage, the Γ -representation links the analysis of Maxwell’s equations to the algebra of vector fields, supporting the identification of invariant subspaces, decomposition into simpler components, and the study of operator actions on the solution space.

6. Application of Γ -Representation in Media — with Connections to the Hertz Potentials

As a concrete application of the Γ -representation, we revisit the structure of electromagnetic fields in isotropic, homogeneous, linear media—the typical regime that classical Hertz potentials are applied to. This setting also offers a brief comparison between the Γ -potential and the classical Hertz potentials, illustrating where they coincide and where they diverge.

In such media, the constitutive relations are

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}, \quad (26)$$

where $\mathbf{P}$ and $\mathbf{M}$ are the electric polarization and magnetization vectors, which, in a linear medium, are parallel and proportional to $\mathbf{E}$ and $\mathbf{B}$ respectively. From Eq. (26), we have

$$\mathbf{P} = (\varepsilon - \varepsilon_0) \mathbf{E}, \quad \mathbf{M} = \left(\frac{1}{\mu_0} - \frac{1}{\mu} \right) \mathbf{B}. \quad (27)$$

As Stratton shows (*Electromagnetic Theory* [3], Sec. 1.6), Maxwell's equations in such media can be written as

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{1}{\varepsilon_0}(\rho_0 - \nabla \cdot \mathbf{P}), \quad \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \left(\mathbf{J}_0 + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \right).\end{aligned}\tag{28}$$

In alignment with this formulation, Stratton states [3]: “*the presence of rigid material bodies in an electromagnetic field may be completely accounted for by an equivalent distribution of charge of density $-\nabla \cdot \mathbf{P}$, and an equivalent distribution of current of density $\frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$.*”

Accordingly, electromagnetic fields in such media are equivalent to fields (in vacuum) with the following effective charge and current densities (which satisfy the continuity equation (Eq. (1)) and thus constitute a valid source pair):

$$\rho = \rho_0 - \nabla \cdot \mathbf{P}, \quad \mathbf{J} = \mathbf{J}_0 + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}\tag{29}$$

There are two approaches for constructing the solution. First, we can split the electric-polarization and magnetization contributions, solve each subsystem, and then superpose the results to obtain the total solution. This is the route taken by the classical Hertz potential framework. Within this route, the Γ -representation also provides a parallel, more uniform scheme for handling each contribution. Second, there is a unified approach unique to the Γ -representation: starting directly from the total effective sources $(\rho, \mathbf{J})$ in Eq. (29), compute the total Γ -source $\mathbf{G}$ via Eqs. (16)–(17) and solve the single vector wave equation (Eq. (11)) to obtain the corresponding Γ -potential, after which the full EM solution follows from the Γ -representation formulas (Eqs. (12)–(15)).

Below we analyze both approaches.

Leveraging the linearity and additive structure of EM configurations discussed earlier, we may decompose the effective sources $(\rho, \mathbf{J})$ into three groups:

- Free sources: $\rho_0, \mathbf{J}_0$;
- Electric polarization sources: $\rho_e = -\nabla \cdot \mathbf{P}, \mathbf{J}_e = \frac{\partial \mathbf{P}}{\partial t}$;
- Magnetization sources: $\rho_m = 0, \mathbf{J}_m = \nabla \times \mathbf{M}$.

Each group satisfies the continuity equation individually and hence defines a valid source pair. By finding the fields and potentials for each group and adding them, we obtain the total fields and potentials.

For Group 1 (free sources $(\rho_0, \mathbf{J}_0)$), we can construct a Γ -potential from

$$\varepsilon_0 \mu_0 \frac{\partial^2 \Gamma_0}{\partial t^2} - \nabla^2 \Gamma_0 = \mathbf{G}_0.\tag{30}$$

Here $\mathbf{G}_0$ is the Γ -source associated with $(\rho_0, \mathbf{J}_0)$ via Eqs. (16)–(17). Since the influence of the material medium is already encoded through the equivalent polarization–magnetization sources (Eq. (29)), this

subproblem is treated within the vacuum framework; accordingly, the vacuum permittivity ε_0 and permeability μ_0 (rather than the medium parameters ε, μ) appear in Eq. (30).

In the case $\rho_0 = 0, \mathbf{J}_0 = 0$, the Γ -source $\mathbf{G}_0$ vanishes and $\mathbf{\Gamma}_0$ reduces to a homogeneous vector wavefield. Together with the homogeneous contributions from the other two groups, this part is then to be determined by the boundary and initial conditions.

For Group 2, the electric polarization sources, $(\rho_e, \mathbf{J}_e) = \left(-\nabla \cdot \mathbf{P}, \frac{\partial \mathbf{P}}{\partial t}\right)$ match Eqs. (16)–(17) when the Γ -source is defined as $\mathbf{G}_e = \mathbf{P}/\varepsilon_0$. The corresponding wave equation is:

$$\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{\Gamma}_e}{\partial t^2} - \nabla^2 \mathbf{\Gamma}_e = \mathbf{G}_e = \frac{\mathbf{P}}{\varepsilon_0} \quad (31)$$

and the fields are given by Eqs. (14), (14a), and (15):

$$\mathbf{E}_e = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{\Gamma}_e}{\partial t^2} + \nabla(\nabla \cdot \mathbf{\Gamma}_e) = \nabla \times (\nabla \times \mathbf{\Gamma}_e) - \frac{\mathbf{P}}{\varepsilon_0} \quad (32)$$

$$\mathbf{B}_e = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{\Gamma}_e) \quad (33)$$

As a comparison, the Hertz-potential approach treats this subsystem slightly differently. Instead of viewing it as a vacuum case, the electric Hertz potential $\mathbf{\Pi}_e$ is defined in the medium [5], using the medium parameters ε, μ :

$$\varphi = -\nabla \cdot \mathbf{\Pi}_e, \quad \mathbf{A} = \varepsilon \mu \frac{\partial \mathbf{\Pi}_e}{\partial t}, \quad \varepsilon \mu \frac{\partial^2 \mathbf{\Pi}_e}{\partial t^2} - \nabla^2 \mathbf{\Pi}_e = \frac{\mathbf{P}}{\varepsilon_0}.$$

Comparing with Eqs. (12)–(13) and (31) for $\mathbf{\Gamma}_e$,

$$\varphi_e = -\nabla \cdot \mathbf{\Gamma}_e, \quad \mathbf{A}_e = \frac{1}{c^2} \frac{\partial \mathbf{\Gamma}_e}{\partial t}, \quad \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{\Gamma}_e}{\partial t^2} - \nabla^2 \mathbf{\Gamma}_e = \frac{\mathbf{P}}{\varepsilon_0}$$

we see that $\mathbf{\Pi}_e$ and $\mathbf{\Gamma}_e$ are related by a time-scale transformation. Let

$$\kappa \triangleq \frac{\varepsilon \mu}{\varepsilon_0 \mu_0} = \frac{c^2}{v^2},$$

and set the rescaled time $\tau = \kappa t$. Then $\partial_t = \kappa \partial_\tau$, and the two definitions match via

$$\mathbf{\Gamma}_e(\mathbf{x}, t) = \mathbf{\Pi}_e(\mathbf{x}, \tau) \quad (\text{up to a time-independent term fixed by initial data}).$$

Consequently, the field-reconstruction formulas (Eqs. (14)–(15)) for $\mathbf{\Pi}_e$ coincide with those for $\mathbf{\Gamma}_e$ after replacing c by $v = 1/\sqrt{\varepsilon \mu} = c/\sqrt{\kappa}$ (or equivalently, replacing t by $\tau = \kappa t$). The corresponding wave equations transform consistently under the same time rescaling.

As a side note, other references (e.g., [3], [4]) adopt slightly different definitions for the electric Hertz potential; in those conventions the time-scaling factor differs from κ , but the comparison with $\mathbf{\Gamma}_e$ proceeds analogously.

In short, in this medium setting, the electric Hertz potential $\boldsymbol{\Pi}_e$ and the Γ -potential $\boldsymbol{\Gamma}_e$ have slightly different forms but carry the same physical content. $\boldsymbol{\Pi}_e$ can essentially be viewed as $\boldsymbol{\Gamma}_e$ expressed in a medium-normalized time.

For Group 3, the physical source pair is $(\rho_m, \mathbf{J}_m) = (0, \nabla \times \mathbf{M})$. Recall from Eq. (27) that $\mathbf{M}$ is proportional to $\mathbf{B}$ with a constant coefficient. Since $\nabla \cdot \mathbf{B} = 0$, we have $\nabla \cdot \mathbf{M} = 0$, and $\mathbf{M}$ inherits the regularity of $\mathbf{B}$.

If we define a vector field

$$\mathbf{G}_m = \frac{1}{\varepsilon_0} \int_{t_0}^t (\nabla \times \mathbf{M}) d\tau, \quad (34)$$

and take it as a Γ -source, Eqs. (16)–(17) are satisfied with $(\rho_m, \mathbf{J}_m)$:

$$\varepsilon_0 \frac{\partial \mathbf{G}_m}{\partial t} = \nabla \times \mathbf{M} = \mathbf{J}_m, \quad \nabla \cdot \mathbf{G}_m = 0 = \rho_m.$$

The wave equation for the corresponding Γ -potential $\boldsymbol{\Gamma}_m$ is

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{\Gamma}_m}{\partial t^2} - \nabla^2 \boldsymbol{\Gamma}_m = \mathbf{G}_m = \frac{1}{\varepsilon_0} \int_{t_0}^t (\nabla \times \mathbf{M}) d\tau. \quad (35)$$

Solving (35) for $\boldsymbol{\Gamma}_m$, the full solution then follows from Eqs. (12)–(17).

To connect with the magnetic Hertz potential, we can take an alternative route. Consider an auxiliary Γ -source $\mathbf{G}_a = \mu_0 \mathbf{M}$, and let the corresponding (auxiliary) Γ -potential (denoted as $\boldsymbol{\Gamma}_a$) solve

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{\Gamma}_a}{\partial t^2} - \nabla^2 \boldsymbol{\Gamma}_a = \mathbf{G}_a = \mu_0 \mathbf{M}. \quad (36)$$

The associated (auxiliary) EM solution $[\text{EM}_a] = [\varphi_a, \mathbf{A}_a, \mathbf{E}_a, \mathbf{B}_a, \rho_a, \mathbf{J}_a]$ is then given by Eqs. (12)–(17); in particular,

$$\varphi_a = -\nabla \cdot \boldsymbol{\Gamma}_a, \mathbf{A}_a = \frac{1}{c^2} \frac{\partial \boldsymbol{\Gamma}_a}{\partial t}, \mathbf{E}_a = -\frac{1}{c^2} \frac{\partial^2 \boldsymbol{\Gamma}_a}{\partial t^2} + \nabla(\nabla \cdot \boldsymbol{\Gamma}_a), \mathbf{B}_a = \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \boldsymbol{\Gamma}_a), \rho_a = 0, \mathbf{J}_a = \frac{1}{c^2} \frac{\partial \mathbf{M}}{\partial t}. \quad (37)$$

Now, define a linear operator

$$\mathcal{L} \triangleq c^2 \left(\int_{t_0}^t \cdot d\tau \right) \circ (\nabla \times). \quad (38)$$

Assume $t_0 = -\infty$ with causal decay (or impose zero initial data at a finite t_0) so that time integration produces no boundary term. Under this assumption, $\mathcal{L}$ commutes with ∂_t , $\nabla \times$, and $\square$, and it admits the scalar companion (described in Theorem 5) $\mathcal{L}_s \equiv 0$. With sufficient regularity inherited from $\mathbf{M}$, Theorem 5 applies. Furthermore, the operator $\mathcal{L}$ (and its companion $\mathcal{L}_s$) maps $\boldsymbol{\Gamma}_a$ and $[\text{EM}_a]$ precisely to the Γ -potential $\boldsymbol{\Gamma}_m$ and the full solution $EM(\boldsymbol{\Gamma}_m)$ obtained from Eq. (35):

$$\boldsymbol{\Gamma}_m \triangleq \mathcal{L}[\boldsymbol{\Gamma}_a], EM(\boldsymbol{\Gamma}_m) = \mathcal{L}[EM(\boldsymbol{\Gamma}_a)].$$

This is confirmed by applying $\mathcal{L}$ and $\mathcal{L}_s$ to the potentials, fields and sources in Eq. (37), which yields

$$\begin{aligned}
\varphi_m &= \mathcal{L}_s[\varphi_a] = 0, \\
\mathbf{A}_m &= \mathcal{L}[\mathbf{A}_a] = \mathcal{L}\left[\frac{1}{c^2} \frac{\partial \mathbf{\Gamma}_a}{\partial t}\right] = \nabla \times \int_{t_0}^t \frac{\partial \mathbf{\Gamma}_a}{\partial t} d\tau = \nabla \times \mathbf{\Gamma}_a, \\
\mathbf{E}_m &= \mathcal{L}[\mathbf{E}_a] = \mathcal{L}\left[-\frac{1}{c^2} \frac{\partial^2 \mathbf{\Gamma}_a}{\partial t^2} + \nabla(\nabla \cdot \mathbf{\Gamma}_a)\right] = -\frac{\partial}{\partial t}(\nabla \times \mathbf{\Gamma}_a), \\
\mathbf{B}_m &= \mathcal{L}[\mathbf{B}_a] = \mathcal{L}\left[\frac{1}{c^2} \frac{\partial}{\partial t}(\nabla \times \mathbf{\Gamma}_a)\right] = \nabla \times \nabla \times \mathbf{\Gamma}_a, \\
\rho_m &= \mathcal{L}_s[\rho_a] = 0, \\
\mathbf{J}_m &= \mathcal{L}[\mathbf{J}_a] = \mathcal{L}\left[\frac{1}{c^2} \frac{\partial \mathbf{M}}{\partial t}\right] = \nabla \times \mathbf{M},
\end{aligned}$$

and the transformed Γ -source

$$\mathbf{G}_m = \mathcal{L}[\mathbf{G}_a] = c^2 \int_{t_0}^t (\nabla \times \mathbf{G}_a) d\tau = \frac{1}{\varepsilon_0} \int_{t_0}^t (\nabla \times \mathbf{M}) d\tau,$$

which exactly matches Eq. (35).

In the Hertz formalism (e.g., [5]), the magnetic Hertz potential $\mathbf{\Pi}_m$ is introduced via:

$$\varphi = 0, \mathbf{A} = \nabla \times \mathbf{\Pi}_m, \mu\varepsilon \frac{\partial^2 \mathbf{\Pi}_m}{\partial t^2} - \nabla^2 \mathbf{\Pi}_m = \mu \mathbf{M}.$$

Comparing with the auxiliary Γ -potential $\mathbf{\Gamma}_a$ in Eq. (36):

$$\varphi_a = 0, \mathbf{A}_a = \nabla \times \mathbf{\Gamma}_a, \frac{1}{c^2} \frac{\partial^2 \mathbf{\Gamma}_a}{\partial t^2} - \nabla^2 \mathbf{\Gamma}_a = \mu_0 \mathbf{M},$$

we see that $\mathbf{\Pi}_m$ and $\mathbf{\Gamma}_a$ are equivalent through a time-scale normalization similar to the one in the electric case. Meanwhile, the effective Γ -potential $\mathbf{\Gamma}_m$ arises from $\mathbf{\Gamma}_a$ by the operator-induced transformation $\mathcal{L}$. The resulting EM solution from $\mathbf{\Gamma}_m$ coincides with that obtained from $\mathbf{\Pi}_m$.

As shown in the analysis of Group 2 and Group 3, the Γ -representation provides an equivalent (but more uniform and flexible) description that reproduces the same physical fields as the Hertz framework. In this sense, the Γ -potential can be viewed as a natural generalization of the classical Hertz potentials.

Moreover, by linearity and uniformity, the Γ -representation permits a single-pass approach: with the Γ -source for each group, i.e., $\mathbf{G}_0$ (Eq. (30)), $\mathbf{G}_e$ (Eq. (31)), and $\mathbf{G}_m$ (Eq. (34)), form the total Γ -source $\mathbf{G}_{tot} = \mathbf{G}_0 + \mathbf{G}_e + \mathbf{G}_m$, solve the single vector wave equation (Eq. (11)) for the total Γ -potential, and then recover the entire EM solution from Eqs. (12)–(17). This route is not available in the classical Hertz framework.

7. Summary and Outlook

In this work, we introduced the Γ -representation as a unified, constructive framework for describing electromagnetic configurations, in which fields, potentials, and sources are systematically reconstructed from a single vector wavefield. This extends the classical concept of Hertz potentials by generalizing the representation to encompass all electromagnetic configurations, enabling all quantities and their interrelations—including Maxwell's equations—to be encoded in a single vector wavefield. The mapping between electromagnetic configurations and the vector wavefields reveals layered symmetries and a linear

algebraic structure, enabling systematic tools for transformation and classification. The framework remains compatible with classical results while offering new viewpoints on solution construction and organization.

While electromagnetic fields and sources are traditionally viewed as the only physically observable quantities, phenomena such as the Aharonov–Bohm effect suggest that the electromagnetic potentials—scalar potential φ and vector potential $\mathbf{A}$ —may themselves possess physical significance beyond their role as mathematical intermediaries. In this context, the Γ -potential extends the classical three-tiered hierarchy (sources, fields, and EM potentials) by adding a fourth layer that generates the EM potentials themselves. Should the EM potentials indeed have ontological status—i.e., possess physical reality—the Γ -potential may then serve as an auxiliary structure for them, much as the EM potentials underlie the observable fields, offering a fresh perspective on the foundations of electromagnetic theory.

Although the present work adopts a non-covariant (3+1) formulation to highlight the structural derivations and physical interpretation, the Γ -representation can be embedded naturally in a covariant four-dimensional framework. In such a setting, the Γ -potential may be reinterpreted as a spacetime vector field satisfying a covariant wave equation, potentially allowing a reformulation within relativistic field theory. Its transformation properties, relation to four-potentials, and compatibility with Lorentz invariance may merit further investigation.

Acknowledgments. The author is grateful to Markus Penz for insightful comments on regularity—particularly for noting that the regularity assumption in an earlier draft was insufficient—and for encouraging the “total-source $\rightarrow \Gamma$ -potential” construction in the media case. His suggestions substantially clarified the presentation.

DATA AVAILABILITY — No datasets were generated or analyzed in this study. All mathematical derivations needed to verify the results are contained in the manuscript.

CONFLICT OF INTEREST — The author declares no conflict of interest.

AUTHOR CONTRIBUTIONS — Conceptualization, Methodology, Formal analysis, Writing—original draft, Writing—review & editing: Ting Yi.

ETHICS APPROVAL — Not applicable. This theoretical work involves no human or animal subjects.

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