Forbidding the subdivided claw as a subgraph or a minor

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Abstract

Let Y be the *subdivided claw*, the 7-vertex tree obtained from a claw $K_{1,3}$ by subdividing each edge exactly once. We characterize the graphs that do not have Y as a subgraph, or, equivalently, do not have Y as a minor. This work was motivated by a problem involving VCD minors. A graph H is a *vertex contraction-deletion minor*, or *VCD minor*, of a graph G if H can be obtained from G by a sequence of vertex deletions or contractions of all edges incident with a single vertex. Our result is a key step in describing $K_{1,3}$-VCD-minor-free line graphs. Moreover, our characterization raises some general questions about the structure of graphs forbidding a specific tree as a minor.

1 Introduction

There are several results on the structure of graphs that exclude a tree or forest as a minor. However, there is little data on what graphs excluding specific small trees look like. Information of this kind can help to generate new general questions. In this paper we consider the *subdivided claw*, the 7-vertex graph obtained by subdividing each edge of $K_{1,3}$, which we denote by Y. We characterize the Y-minor-free graphs, which are equivalent to the graphs without Y as a subgraph. Our characterization raises some general questions that we discuss at the end of the paper.

For this paper, all graphs will be simple. Any notation not defined here follows [12]. We will consider several well-known graph relations whose definitions follow. A graph H is a *subgraph* of a graph G if H can be obtained from G by a sequence of vertex and edge deletions. A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of vertex deletions, edges deletions, and edge contractions. A graph H is a *topological minor* of a graph G if H is isomorphic to a subdivision of a subgraph of G.

It is often desirable to relate a particular class of graphs to a set of forbidden substructures. There are a variety of such results using different graph relations. Bipartite graphs are a characterized by forbidding odd cycles as subgraphs. Kuratowski [8] characterized planar graphs using the forbidden topological minors K_5 and $K_{3,3}$. Wagner [11] proved that the same two graphs were forbidden as minors. In [1], Beineke proved that line graphs can be characterized by nine forbidden induced subgraphs. Chordal graphs are characterized by having no induced cycles of length 4 or more. In [9], Robertson and Seymour proved that every minor-closed class has a finite list of forbidden minors.

Since Y is a tree that has only one vertex of degree at least 3, having Y as a subgraph is equivalent to having Y as a topological minor. Further, since each vertex of Y has degree at most 3, having Y as a

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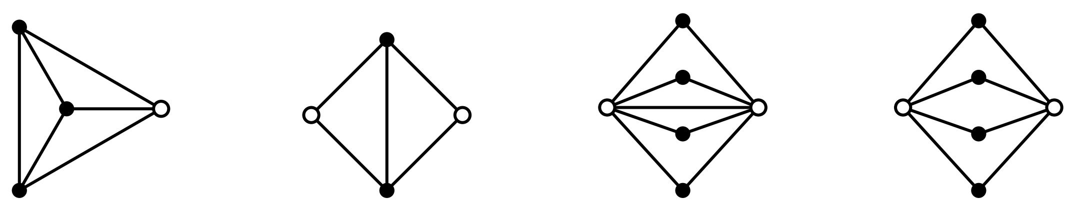
topological minor is equivalent to having Y as a minor.

Our initial motivation for studying graphs without Y as a subgraph or minor arose from a question involving a graph relation called a *vertex-contraction-deletion minor*. A *vertex contraction* of a vertex v in a graph G is the operation of identifying v and all neighbors of v into a single vertex. This is equivalent to contracting all edges incident with v. A graph H is an *vertex-contraction-deletion minor* (VCD minor) of a graph G if H can be obtained from G by a sequence of vertex deletions and vertex contractions. The VCD minor relation is a slightly weaker relation than the *t-minor* relation [4]. We are interested in characterizing $K_{1,3}$-VCD-minor free graphs, and an important subclass of this family is the $K_{1,3}$-VCD-minor free line graphs. Forbidding a $K_{1,3}$ as a VCD minor in a line graph $G = L(H)$ where $\alpha'(H) \geq 4$ is equivalent to forbidding Y as a subgraph of H. The results in this paper will be used in a forthcoming paper on this topic.

We note that forbidding Y as a subgraph has already been used to characterize the class of trees known as *caterpillars*, in which there is a path (the *spine*) such that every edge is incident to a vertex of that path. We will use this result as a key part of our proof.

Theorem 1.1 (West [12, Theorem 2.2.19]; see also [7, p. 172])**.** *A tree does not have Y as a subgraph if and only if it is a caterpillar.*

We will now define some structures necessary to describe Y-subgraph free graphs. Let $t_1 \geq 0$ and $t_2 \geq 2$ be integers. A *bead* is one of the following graphs: K_4, $K_{2,1,1}$, $K_{1,1,t_1}$, or K_{2,t_2}. Note that a $K_{1,1,t_1}$ bead can be a single edge, $K_{1,1,0}$, or a triangle, $K_{1,1,1}$. Each bead has one (K_4) or two (the other beads) special vertices called *primary vertices*; the other vertices are *secondary* vertices. Beads can be "strung" together at primary vertices. This means we can identify a primary vertex in one bead with a primary vertex in a different bead. Each primary vertex can only be identified with one other primary vertex. The primary vertices in $K_{1,1,t_1}$ are the parts of size one. The primary vertices in K_{2,t_2} and $K_{2,1,1}$ are the vertices in the part of size two. One vertex of K_4 is designated as a primary vertex. Note that for us $K_{2,1,1}$ is not the same as $K_{1,1,2}$ because each has a different set of primary vertices. The beads are illustrated in Figure 1.1, where the white vertices are the primary vertices.

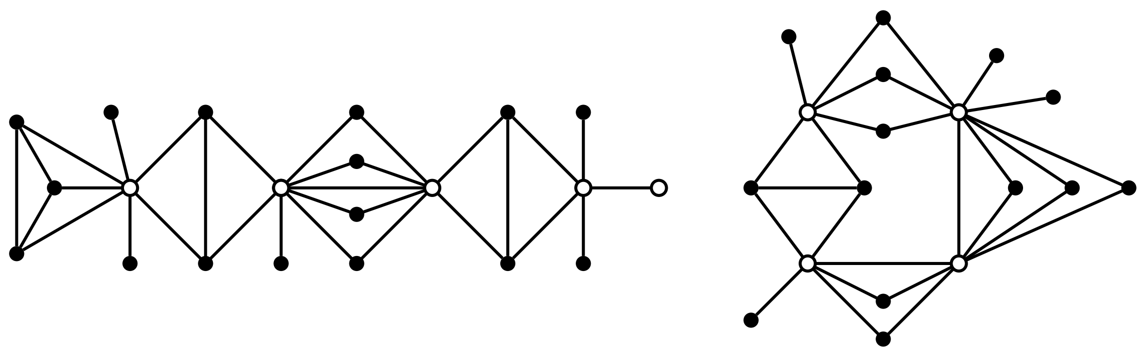


(a) K_4 (b) $K_{2,1,1}$ (c) $K_{1,1,t_1}$ with $t_1 = 4$ (d) K_{2,t_2} with $t_2 = 4$

Figure 1.1: Beads

Consider a connected graph G created by stringing beads together. We form a bipartite auxiliary graph A by taking one black vertex for each bead and one white vertex for each primary vertex, where a bead vertex is joined to its primary vertices. Then G is a *strand* if A is a path, and a *necklace* if A is a cycle. For a strand, we could have K_4 with one primary vertex as the first and/or last bead; necklaces do not have K_4 as a bead. A necklace with exactly two beads cannot have both beads of the form $K_{1,1,t_1}$ because stringing two beads of this kind would create parallel edges, and we only allow simple graphs. A vertex of degree 1 is a *leaf* or *pendant vertex*, and an edge incident with a leaf is a *pendant edge*. A *spiked necklace (strand)* is a necklace (strand) that is allowed to have additional pendant edges called *spikes* incident with primary vertices that are in exactly two beads. Figure 1.2 illustrates a spiked strand and a spiked necklace; again, the white vertices are the primary vertices.

We now explain some of the restrictions imposed by our definitions. We could allow $K_{2,1}$ as a bead, but this can be considered as two $K_{1,1,0}$ beads strung together. We could allow spikes incident with primary vertices that are in only one bead, which must be a bead at the end of a strand, but then one of those spikes can be considered as a new $K_{1,1,0}$ bead. Thus, these restrictions eliminate ambiguity about the number of beads in a spiked strand. There are still some ambiguities about whether certain graphs should be regarded as spiked strands or spiked necklaces, which we do not try to eliminate. These occur for spiked necklaces with at most four beads.



(a) Spiked Strand

(b) Spiked Necklace

Figure 1.2: Structures in $\mathcal{B}$

A graph is a member of the family $\mathcal{B}$ if and only if it is either K_1 or built by stringing together beads into a spiked necklace or strand. All graphs in $\mathcal{B}$ are connected.

We say that two nonadjacent vertices u and v in a graph G are *clones* in G if $N(u) = N(v)$. This is an equivalence relation and the equivalences classes are called *clone classes*. A *leaf class* is a clone class whose elements are leaves in G. By *cloning* or *duplicating* a vertex v in G, we mean adding new vertices $v_1, \dots, v_k$ to form G' such that $v, v_1, \dots, v_k$ are pairwise nonadjacent and $N_{G'}(v_i) = N_G(v)$ for $1 \le i \le k$.

We can now state the main theorem of this paper.

Theorem 1.2. *A connected graph G has no subgraph isomorphic to Y (is Y-minor-free) if and only if either*

(a) *G is obtained from a graph with at most six vertices by cloning leaves, or*

(b) *G belongs to the family $\mathcal{B}$.*

We divide the forwards direction of the proof of Theorem 1.2 into two cases, depending on whether G has a long cycle or not. Section 2 proves some basic structural results that apply in both cases, Section 3 addresses the simpler case where there is no long cycle (giving spiked strands), and Section 4 addresses the case where there is a long cycle (giving spiked necklaces). We complete the proof in Section 5. In Section 6, we make some observations on the pathwidth and growth constant of graphs without a Y, which raise some larger issues.

2 Basic Structural Results

In this section, we prove some basic facts about the structure of a connected graph G that does not have Y as a subgraph.

Let $P = v_0v_1 \dots v_\ell$ be a fixed longest path in G. By a *chord* we mean a chord of P, i.e., $e = v_jv_k \in E(G)$ with $k \geq j+2$; we call e an *i-chord* if $k - j = i$. The vertices $v_{j+1}, \dots, v_{k-1}$ are the *enclosed vertices* of e and we say that e *covers* the edges of the path $v_jv_{j+1} \dots v_k$.

If we extend P to a spanning tree T of G, then Theorem 1.1 implies that T is a caterpillar. Therefore, every vertex of T is either in $V(P)$ or has a neighbor in $V(P)$, i.e., $V(P)$ is a dominating set for T, and hence for G. Let $L_i = N_G(v_i) - V(P)$, so that $V(G) - V(P) = \bigcup_{i=0}^{\ell} L_i$. Our analysis focuses on the relationships between the sets L_i and on the existence of particular chords of the path P.

We first prove two lemmas based on P being a longest path.

Lemma 2.1. *L_0 and L_ℓ are empty.*

Proof. If either L_0 or L_ℓ is non-empty, then P would not be the longest path in G. The conclusion follows. □

Lemma 2.2. *For $i \in \{0, 1, \dots, \ell-1\}$, $L_i \cap L_{i+1}$ is empty.*

Proof. Suppose $L_i \cap L_{i+1} \neq \emptyset$. Then G has a longer path than P using two of the edges incident with a vertex in $L_i \cap L_{i+1}$. Thus $L_i \cap L_{i+1} = \emptyset$, as required. □

Now we prove that $\mathcal{L} = \bigcup\limits_{i=1}^{\ell-1} L_i = V(G) - V(P)$ is an independent set.

Lemma 2.3. *There is no edge of G with both endpoints in $\mathcal{L}$.*

Proof. Suppose G has an edge e with both endpoints in $\mathcal{L}$. If one endpoint of e is in L_1 or $L_{\ell-1}$, then G has a longer path than P. We may therefore assume that e has endpoints $w \in L_i$ and $x \in L_j$ for $2 \leq i \leq j \leq \ell-2$. Then the paths $v_iv_{i-1}v_{i-2}$, $v_iv_{i+1}v_{i+2}$ and v_iwx form a Y subgraph, which is a contradiction. Thus, G has no edge with both endpoints in $\mathcal{L}$. □

Thus, every edge of G is incident with a vertex of P, i.e., $V(P)$ is a vertex cover of G. We now proceed based on the value of ℓ. If $\ell = 0$ or 1 then G is either K_1 or $K_2 \cong K_{1,1,0}$, and hence satisfies both conclusions of Theorem 1.2. We consider $\ell = 2$ or 3 in Lemma 2.4 and $\ell = 4$ in Lemma 2.5, so that we can then focus on the general case where $\ell \geq 5$.

Lemma 2.4. *If $\ell = 2$ or 3, then G satisfies Theorem 1.2(a).*

Proof. Suppose $\ell = 2$. Then Lemma 2.1 implies that L_0 and L_2 are empty. Since $V(P)$ is a vertex cover by Lemma 2.3, the only vertices of G not on P are in L_1. If $L_1 = \emptyset$, then G is either P_3 or a triangle and satisfies Theorem 1.2(a). If $L_1 \neq \emptyset$, then G is obtained by duplicating leaves in a graph of order at most 4, and also satisfies Theorem 1.2(a).

We may therefore assume that $\ell = 3$. Then Lemma 2.1 implies that L_0 and L_3 are empty. Since G has no edges with both endpoints in $\mathcal{L}$ and $L_1 \cap L_2$ is empty, it follows that G is obtained by duplicating leaves in a graph with at most six vertices and satisfies Theorem 1.2(a). □

If $L_i \cap L_{i+2} \neq \emptyset$, then for each $w \in L_i \cap L_{i+2}$, we call the path v_iwv_{i+2} a *vee*. We call v_i and v_{i+2} the *endpoints* of the vee, v_{i+1} the *enclosed vertex* of the vee, and we say that the vee *covers* the edges of the path $v_iv_{i+1}v_{i+2}$.

Lemma 2.5. *If $\ell = 4$, then G satisfies Theorem 1.2(a), or G is a spiked strand and satisfies Theorem 1.2(b).*

Proof. Lemma 2.1 implies that $L_0 = L_4 = \emptyset$, and Lemma 2.2 implies that $L_1 \cap L_2 = L_2 \cap L_3 = \emptyset$.

Claim 1. We may assume that $v_0v_4 \notin E(G)$. If $v_0v_4 \in E(G)$, there is a cycle C with $V(C) = V(P)$, so if $L_1 \cup L_2 \cup L_3 \neq \emptyset$ there is a path longer than P, which is impossible, and otherwise G has five vertices and Theorem 1.2(a) holds.

Claim 2. None of the following happen: (a) $L_1 \neq \emptyset$ and $v_0v_2 \in E(G)$, (b) $L_2 \neq \emptyset$ and either $v_0v_3 \in E(G)$ or $v_1v_4 \in E(G)$, (c) $L_3 \neq \emptyset$ and $v_2v_4 \in E(G)$. In each case there is a path longer than P, which is impossible.

Suppose first that $L_1 \cap L_3 \neq \emptyset$. If $L_2 \neq \emptyset$ then there is a path longer than P, which is impossible, so $L_2 = \emptyset$. By Claims 1 and 2 $v_0v_4, v_0v_2, v_2v_4 \notin E(G)$, so the only possible chords are v_0v_3, v_1v_3 and v_1v_4. If $v_1v_3 \in E(G)$ there is a $K_{1,1,t}$ bead, $t \geq 2$, with primary vertices v_1, v_3 and secondary vertices including v_2, $L_1 \cap L_3$ and possibly v_0 (if $v_0v_3 \in E(G)$) or v_4 (if $v_1v_4 \in E(G)$); the remaining edges form at most two $K_{1,1,0}$ beads and spikes. If $v_1v_3 \notin E(G)$ we have a similar situation but with a $K_{2,t}$ bead, $t \geq 2$. In either case G is a spiked strand.

Now suppose that $L_1 \cap L_3 = \emptyset$. So all vertices in $L_1 \cup L_2 \cup L_3$ are leaves. If at most one of L_1, L_2, L_3 is nonempty then Theorem 1.2(a) holds. If L_1, L_2, L_3 are all nonempty then by Claims 1 and 2 the only possible chord is v_1v_3; whether this chord is present or not, Theorem 1.2(a) holds. So we may assume that exactly two of L_1, L_2, L_3 are nonempty.

If $L_1, L_3 \neq \emptyset$ and $L_2 = \emptyset$, then by Claims 1 and 2 the only possible chords are v_0v_3, v_1v_3, v_1v_4. If $v_1v_3 \in E(G)$ there is a $K_{1,1,t}$ bead, $t \geq 1$, with primary vertices v_1, v_3 and secondary vertices including v_2 and possibly v_0 (if $v_0v_3 \in E(G)$) or v_4 (if $v_1v_4 \in E(G)$); the remaining edges form at most two $K_{1,1,0}$ beads and spikes. If $v_1v_3 \notin E(G)$ we have a similar situation but with a $K_{2,t}$ bead, $t \geq 1$. If $t = 1$ then $K_{2,1}$ is not a legitimate bead, but the path $v_1v_2v_3$ can be considered as two $K_{1,1,0}$ beads strung together. In all cases G is a spiked strand.

If $L_1, L_2 \neq \emptyset$ and $L_3 = \emptyset$, then by Claims 1 and 2 the only possible chords are v_1v_3 and v_2v_4. No matter which of these two chords are present, Theorem 1.2(a) holds because the elements of L_1 are clones of v_0.

The case where $L_2, L_3 \neq \emptyset$ and $L_1 = \emptyset$ is symmetric to the previous case, so the proof is complete. □

The results for small ℓ can be summarized as follows.

Proposition 2.6. *Suppose G is a connected graph with no Y subgraph and $P = v_0v_1 \dots v_\ell$ is a longest path in G, where $\ell \leq 4$. Then G either satisfies Theorem 1.2(a), or G is a spiked strand and satisfies Theorem 1.2(b).*

We henceforth (in the remainder of this section and in Sections 3 and 4) assume that $\ell \geq 5$. We first consider intersections between the sets L_i.

Lemma 2.7. *If $L_i \cap L_j \neq \emptyset$ where $i < j$, then $j = i + 2$, or $i = 1$ and $j = \ell - 1$.*

Proof. By Lemmas 2.1 and 2.2 we have $i \geq 1$ and $i + 2 \leq j \leq \ell - 1$. Let $w \in L_i \cap L_j$ and suppose that $j > i+2$. If $i \geq 2$, then the paths $v_iv_{i-1}v_{i-2}$, $v_iv_{i+1}v_{i+2}$ and v_iwv_j form a Y subgraph. If $j \leq \ell-2$ there is a symmetric Y subgraph. The only remaining case is $i = 1$ and $j = \ell - 1$, so either this holds or $j = i+2$. □

Lemma 2.8. *If i, j, and k are distinct, then $L_i \cap L_j \cap L_k = \emptyset$.*

Proof. Suppose that $L_i \cap L_j \cap L_k \neq \emptyset$, where $i < j < k$. Lemmas 2.1 and 2.7 imply that $i \geq 1$, $j \geq i+2$, and $j+2 \leq k \leq \ell - 1$. If $w \in L_i \cap L_j \cap L_k$, then the paths wv_iv_{i+1}, wv_jv_{j+1}, wv_kv_{k+1} form a Y subgraph. Thus, $L_i \cap L_j \cap L_k = \emptyset$, as required. □

In this lemma, we prove that if P has a vee, then all neighbors of the enclosed vertex are on P.

Lemma 2.9. *Suppose $1 \leq i \leq \ell - 1$. Then we do not have $L_i \neq \emptyset$ and $L_{i-1} \cap L_{i+1} \neq \emptyset$.*

Proof. Suppose that $L_i \neq \emptyset$ and $L_{i-1} \cap L_{i+1} \neq \emptyset$. Then $2 \leq i \leq l - 2$ because $L_0 = L_\ell = \emptyset$. Let $w \in L_i$ and $u \in L_{i-1} \cap L_{i+1}$. Then $w \neq u$ by Lemma 2.2. If $i \geq 3$ then the paths $v_{i-1}v_{i-2}v_{i-3}$, $v_{i-1}v_iw$, $v_{i-1}uv_{i+1}$ form a Y subgraph, which is a contradiction. If $i \leq \ell - 3$ there is a symmetric Y subgraph. Since $\ell \geq 5$, these cases cover all values of i. □

Two vees *cross* if one vee has endpoints v_i and v_{i+2} and the other has one endpoint v_{i+1}. The following is an immediate consequence of Lemma 2.9.

Corollary 2.10. *Two distinct vees do not cross.*

While distinct vees do not cross, it is possible to have multiple vees that cover the same pair of edges.

To conclude this section, we characterize the possible chords of P.

Lemma 2.11. *The possible chords of P are v_iv_{i+2} for $0 \leq i \leq \ell - 2$, v_0v_3, $v_{\ell-3}v_\ell$, $v_0v_{\ell-1}$, v_0v_ℓ, $v_1v_{\ell-1}$, and v_1v_ℓ.*

Proof. Suppose we have a chord v_iv_j with $i \geq 0$, $i+2 \leq j \leq \ell$. If $i \geq 2$ and $i+3 \leq j \leq \ell - 1$ the paths $v_iv_{i-1}v_{i-2}$, $v_iv_{i+1}v_{i+2}$, $v_iv_jv_{j+1}$ form a Y subgraph. There is a symmetric Y subgraph if $i \geq 1$ and $i+3 \leq j \leq \ell - 2$. If $i \geq 2$ and $i+4 \leq j \leq \ell$ the paths $v_iv_{i-1}v_{i-2}$, $v_iv_{i+1}v_{i+2}$, $v_iv_jv_{j-1}$ form a Y subgraph. There is a symmetric Y subgraph if $i \geq 0$ and $i+4 \leq j \leq \ell - 2$. The only remaining cases are $j = i+2$, $(i,j) = (0,3)$ or $(\ell-3,\ell)$, and $i \in \{0,1\}$ and $j \in \{\ell-1,\ell\}$. □

3 No Long Cycle

In this section and the next we impose an additional restriction on P: out of all longest paths $v_0v_1 \ldots v_\ell$ in G, P should minimize $\deg(v_0) + \deg(v_\ell)$. This condition is used in Lemmas 3.7 and 3.8 to simplify our analysis a little. The results of Section 2 are still valid for all such P.

In this section, we consider the case where P does not have a chord from v_0 or v_1 to $v_{\ell-1}$ or v_ℓ, and $L_1 \cap L_{\ell-1} = \emptyset$. These assumptions hold throughout this section, as well as the assumption that $\ell \geq 5$. Thus, the only chords allowed are v_0v_3, $v_{\ell-3}v_\ell$, or v_iv_{i+2} for some i. Roughly, this means that G does not have a a long cycle, and this yields spiked strands. We will address the case where G has a long cycle in Section 4.

We now show that P does not have three consecutive 2-chords.

Lemma 3.1. *The path P does not have three 2-chords v_iv_{i+2}, $v_{i+1}v_{i+3}$ and $v_{i+2}v_{i+4}$, where $0 \leq i \leq \ell - 4$.*

Proof. Assume that the three given 2-chords exist.

Suppose first that $\ell > 5$. If $i = 0$, then the paths $v_4v_5v_6$, $v_4v_3v_1$, $v_4v_2v_0$ form a Y subgraph. If $i = \ell - 4$ there is a symmetric Y subgraph. If $i \in \{1, \ldots, \ell-5\}$, then the paths $v_{i+2}v_iv_{i-1}$, $v_{i+2}v_{i-1}v_{i+1}$, $v_{i+2}v_{i+4}v_{i+5}$ form a Y subgraph.

Suppose now that $\ell = 5$. Note that either $i = 0$ or $i = \ell - 4 = 1$. These cases are symmetric. Without loss of generality, suppose $i = 0$. If $L_j = \emptyset$ for $1 \leq j \leq 4$, then G is a graph of order 6 and satisfies Theorem 1.2(a). We may therefore assume that at least one of $L_j \neq \emptyset$ for $1 \leq j \leq 4$. If $L_1 \neq \emptyset$ or $L_2 \neq \emptyset$, then G has a Y subgraph centered at v_3. If $L_3 \neq \emptyset$, then G has a Y centered at v_2. We may therefore assume that $L_4 \neq \emptyset$ and that $L_1, L_2, L_3 = \emptyset$. Let $w \in L_4$. We consider the additional possible chords of P in G. If G has no chords incident with v_5, then all vertices in L_4 are clones of v_5, so G satisfies Theorem 1.2(a). Under the assumptions of this section, the only possible chords incident with v_5 are v_2v_5 and v_3v_5. If $v_2v_5 \in E(G)$, then the paths v_3v_4w, $v_3v_2v_5$, $v_3v_1v_0$ form a Y subgraph. If $v_3v_5 \in E(G)$, then we have a case symmetric to the case where $L_1 \neq \emptyset$, with a Y centered at v_2. ☐

For most 2-chords, all neighbors of the enclosed vertex are on P.

Lemma 3.2. *If $v_{i-1}v_{i+1} \in E(G)$ where $2 \leq i \leq \ell - 2$, then $L_i = \emptyset$.*

Proof. Recall that $\ell \geq 5$. If $3 \leq i \leq \ell - 2$ then G has a Y subgraph centered at v_{i-1}. If $i = 2$, then G has a Y subgraph centered at v_{i+1}. ☐

Two chords v_iv_j with $i < j$ and v_pv_q with $p < q$ *cross* if $i < p < j < q$ or $p < i < q < j$. We show that we may assume that a 2-chord does not cross a 3-chord.

Lemma 3.3. *If a 2-chord of P crosses a 3-chord v_0v_3 or $v_{\ell-3}v_\ell$ of P, then $\ell = 5$ and G satisfies Theorem 1.2(a).*

Proof. Suppose the 3-chord is v_0v_3; the argument for $v_{\ell-3}v_\ell$ is symmetric. Then the 2-chord is v_2v_4. If $\ell \geq 6$, then G has a Y centered at v_4. So we may assume that $\ell = 5$. Lemmas 2.1 and 3.2 imply that $L_0 = L_3 = L_5 = \emptyset$. If $L_1 \neq \emptyset$, then G has a Y centered at v_2. If $L_2 \neq \emptyset$, then G has a Y centered at v_3. So we may assume that $L_1 = L_2 = \emptyset$. If $L_4 = \emptyset$ or there is no chord incident with v_5, then all vertices in L_4 are clones of v_5, and Theorem 1.2(a) holds. Therefore, we may suppose that $L_4 \neq \emptyset$, and either $v_3v_5 \in E(G)$ and there is a Y centered at v_2, or $v_2v_5 \in E(G)$ and there is a Y centered at v_3. ☐

Lemma 3.4. *If the two 3-chords v_0v_3 and $v_{\ell-3}v_\ell$ cross, then $\ell = 5$ and G satisfies Theorem 1.2(a).*

Proof. Suppose that v_0v_3 and $v_{\ell-3}v_\ell$ cross. This does not happen if $\ell \geq 6$, so $\ell = 5$ and the second chord is v_2v_5. If there is $w \in L_1$ then the paths v_2v_1w, $v_2v_3v_0$, $v_2v_5v_4$ form a Y. If there is $w \in L_2$ then the paths $v_3v_0v_1$, v_3v_2w, $v_3v_4v_5$ form a Y. The cases of $w \in L_3$ and $w \in L_4$ also give Y subgraphs by symmetric arguments. Hence $\mathcal{L} = \emptyset$ and Theorem 1.2(a) holds. ☐

A chord and a vee *cross* if the enclosed vertex of the vee is an endpoint of the chord. A 2-chord and a vee do not cross.

Lemma 3.5. *A 2-chord of P does not cross a vee of P.*

Proof. Suppose $v_{i-1}v_{i+1}$ is a 2-chord of P that crosses a vee. Then either $1 \leq i \leq \ell - 2$ and $L_i \cap L_{i+2} \neq \emptyset$, or $2 \leq i \leq \ell - 1$ and $L_{i-2} \cap L_i \neq \emptyset$. The two cases are symmetric, so it suffices to prove the first. Lemma 3.2 implies that i must be 1. Then there is a Y centered at v_3, which is a contradiction. ☐

Also, a 3-chord and a vee do not cross.

Lemma 3.6. *A 3-chord v_0v_3 or $v_{\ell-3}v_\ell$ of P does not cross a vee of P.*

Proof. Suppose the 3-chord v_0v_3 crosses a vee; the argument for $v_{\ell-3}v_\ell$ is symmetric. Then $L_2 \cap L_4 \neq \emptyset$. If $w \in L_2 \cap L_4$ then the paths $v_3v_0v_1$, v_3v_2w, $v_3v_4v_5$ form a Y, which is a contradiction. ☐

Lemma 3.7. *If P has a 2-chord v_0v_2, then $L_0 = L_1 = \emptyset$. Similarly, if P has a 2-chord $v_{\ell-2}v_\ell$, then $L_{\ell-1} = L_\ell = 0$.*

Proof. The two situations are symmetric, so we consider only the first. We know $L_0 = \emptyset$. We know that $L_1 \cap L_3 = \emptyset$ since vees do not cross 2-chords, so all vertices in L_1 are leaves. If there is $w \in L_1$, then the path $wv_1v_2 \dots v_\ell$ is another longest path, but with degree sum of the terminal vertices smaller than for P, a contradiction. Thus, $L_1 = \emptyset$. ☐

Lemma 3.8. *If P has a 3-chord v_0v_3, then $L_0 = L_2 = \emptyset$ and $L_1 \subseteq L_3$. Similarly, if P has a 3-chord $v_{\ell-3}v_\ell$, then $L_{\ell-2} = L_\ell = 0$ and $L_{\ell-1} \subseteq L_{\ell-3}$.*

Proof. The two situations are symmetric, so we consider only the first. We know $L_0 = \emptyset$. If there is $w \in L_2$, then there is a path $wv_2v_1v_0v_3v_4 \dots v_\ell$ longer than P, so $L_2 = \emptyset$. If there is $z \in L_1 - L_3$ then the path $zv_1v_2 \dots v_\ell$ is another longest path, but with degree sum of the terminal vertices smaller than for P, a contradiction. Thus, $L_1 \subseteq L_3$. ☐

We can now complete the case where G does not have a long cycle.

Proposition 3.9. *Suppose G is a connected graph with no Y subgraph and $P = v_0v_1 \dots v_\ell$ is a longest path in G, and subject to that $\deg(v_0) + \deg(v_\ell)$ is as small as possible. Suppose also that $\ell \geq 5$, G has no chord from $\{v_0, v_1\}$ to $\{v_{\ell-1}, v_\ell\}$, and $L_1 \cap L_{\ell-1} = \emptyset$. Then G either satisfies Theorem 1.2*(a)*, or G is a spiked strand and satisfies Theorem 1.2*(b)*.*

Proof. If G has order 6, then G satisfies Theorem 1.2(a). If G has a 3-chord that crosses a 2-chord or another 3-chord, then by Lemmas 3.3 and 3.4 G also satisfies Theorem 1.2(a). We may therefore assume that G has order at least 7, and that no 3-chord crosses a 2-chord or another 3-chord. We also know by Corollary 2.10 and Lemmas 3.1, 3.5 and 3.6 that there are no three consecutive 2-chords, and that vees do not cross other vees or 2-chords or 3-chords. We also know that edges not on P are 2-chords or 3-chords (specifically v_0v_3 or $v_{\ell-3}v_\ell$), or belong to a vee, or are pendant edges incident with a vertex of P that is not v_0 or v_ℓ.

Therefore, we can partition the edges of P into subpaths of length at most three as follows: (a) triples of edges covered by a 3-chord (which may also be covered by 2-chords or vees), or the following sets of edges not covered by a 3-chord: (b) triples of edges covered by either of a pair of crossing 2-chords, (c) pairs of edges covered by a 2-chord and possibly also by vees, (d) pairs of edges covered only by one or more vees, and (e) single other edges (not covered by any chords or vees). We show that all other non-pendant edges of G combine with one of these subpaths to form a bead, and that the pendant edges form spikes, so we have a spiked strand.

For subpaths $v_iv_{i+1}v_{i+2}v_{i+3}$ of type (b), Lemma 3.2 implies that $L_{i+1} = L_{i+2} = \emptyset$, except perhaps for L_1 when $i = 0$ and $L_{\ell-1}$ when $i = \ell - 3$. These two exceptions are symmetric, so we discuss only the first. If $i = 0$ and there is $w \in L_1$, then the paths v_3v_1w, $v_3v_2v_0$, $v_3v_4v_5$ form a Y subgraph, which is a contradiction. Therefore $L_{i+1} = L_{i+2} = \emptyset$ in all cases. Moreover, no additional chords cross v_iv_{i+2} or $v_{i+1}v_{i+3}$, so $N(v_{i+1}) = \{v_i, v_{i+2}, v_{i+3}\}$ and $N(v_{i+2}) = \{v_i, v_{i+1}, v_{i+3}\}$. The subpath $v_iv_{i+1}v_{i+2}v_{i+3}$ and the two 2-chords form a $K_{2,1,1}$ bead with primary vertices v_i and v_{i+3}.

For subpaths $v_iv_{i+1}v_{i+2}$ of type (c), $L_{i+1} = \emptyset$ by Lemma 3.2 if $1 \leq i \leq \ell - 3$, and by Lemma 3.7 if $i = 0$ or $\ell - 2$. There are also no chords crossing v_iv_{i+2}, so $N(v_{i+1}) = \{v_i, v_{i+2}\}$. The subpath, the 2-chord, and any vees that cover the subpath form a $K_{1,1,t}$ bead where $t = |L_i \cap L_{i+2}| + 1 \geq 1$, and v_i and v_{i+2} are primary vertices.

For subpaths $v_i v_{i+1} v_{i+2}$ of type (d), Lemmas 2.9 and 3.5 imply that $N(v_{i+1}) = \{v_i, v_{i+2}\}$. This subpath and the vees that cover it form a $K_{2,t}$ bead where $t = |L_i \cap L_{i+2}| + 1 \geq 2$, with primary vertices v_i and v_{i+2}.

Subpaths $v_i v_{i+1}$ of type (e) form a $K_{1,1,0}$ bead by themselves, with primary vertices v_i and v_{i+1}.

The most complicated case is subpaths of type (a), which must be $v_0 v_1 v_2 v_3$ or $v_{\ell-3} v_{\ell-2} v_{\ell-1} v_\ell$. The situations are symmetric, so we consider only the first. By Lemma 3.8, $L_0 = L_2 = \emptyset$ and $L_1 \subseteq L_3$. Consider the subgraph H induced by $\{v_0, v_1, v_2, v_3\}$, which contains the 4-cycle $C = (v_0 v_1 v_2 v_3)$. If $H = C$ then C and the vees that cover $v_1 v_2 v_3$ form a $K_{2,t}$ bead with $t = |L_1 \cap L_3| + 2 \geq 2$ and primary vertices v_1 and v_3. If $H = C \cup \{v_1 v_3\}$ then we similarly have a $K_{1,1,t}$ bead with primary vertices v_1 and v_3. Suppose now that $v_0 v_2 \in E(H)$. If there is $w \in L_1 = L_1 \cap L_3$ then the paths $v_3 w v_1$, $v_3 v_2 v_0$, $v_3 v_4 v_5$ form a Y subgraph, so we must have $L_1 = \emptyset$. If $H = C \cup \{v_0 v_2\}$ then H is a $K_{2,1,1}$ bead with primary vertices v_1 and v_3, and if $H = C \cup \{v_0 v_2, v_1 v_3\}$ then H is a K_4 bead with primary vertex v_3. In all situations, since we have accounted for all edges from v_i to L_i for $0 \leq i \leq 2$, and there are no chords crossing $v_0 v_3$, the bead is joined to the rest of G only at the primary vertex v_3.

The above accounts for all edges of P and non-pendant edges not in P, and shows moreover that any pendant edges not in P must be adjacent to primary vertices. Therefore, we have a spiked strand. $\square$

4 Long Cycle

We now consider the case that P has a chord, or an intersection between L_1 and $L_{\ell-1}$, that creates a long cycle. As in Section 3, we assume that P is a longest path $v_0 v_1 \ldots v_\ell$ with $\ell \geq 5$, and subject to that $\deg(v_0) + \deg(v_\ell)$ is smallest. But now we assume that either P has a chord from v_0 or v_1 to $v_{\ell-1}$ or v_ℓ, or $L_1 \cap L_{\ell-1} \neq \emptyset$. We first show that there is a long cycle that is *edge-dominating*, i.e., every edge is incident with a vertex of this cycle.

Lemma 4.1. *The graph G has an edge-dominating cycle of length at least $\ell - 1$, and hence at least 4.*

Proof. First notice that $v_1 v_2 \ldots v_{\ell-1}$ is an edge-dominating path in G with $\ell - 1$ vertices. Therefore, any cycle that contains this path is an edge-dominating cycle of length at least $\ell - 1$. If we have a chord $v_i v_j$ with $i \in \{0, 1\}$ and $j \in \{\ell - 1, \ell\}$ then $(v_i v_{i+1} \ldots v_j)$ is such a cycle. If we have $w \in L_1 \cap L_{\ell-1}$ then $(w v_1 v_2 \ldots v_{\ell-1})$ is such a cycle. $\square$

Our arguments in the rest of this section rely only on the hypothesis that G is a graph with no subgraph isomorphic to Y and with an edge-dominating cycle of length at least 4. We choose $C = (u_0 u_1 u_2 \ldots u_{k-1})$ to be an edge-dominating cycle in G of maximum length $k \geq 4$. The subscript i of u_i is considered to belong to $\mathbb{Z}_k = \{0, 1, 2, \ldots, k-1\}$ with operations modulo k. For $i, j \in \mathbb{Z}_k$ we let $d(i, j)$ denote the cyclic distance between i and j, i.e., if $i \leq j$ then $d(i, j) = \min(j - i, k + i - j)$ and if $i > j$ then $d(i, j) = d(j, i)$. Let $N_i = N_G(v_i) - V(C)$ and $\mathcal{N} = \bigcup_{i=0}^{k-1} N_i$.

We prove a series of lemmas similar to those in Sections 2 and 3. The arguments here are sometimes simpler since we do not have to worry about "end effects". Since the arguments are very similar we omit some details. We focus on the relationships between the sets N_i and on the existence of particular chords of the cycle C. Henceforth (until the end of this section) an *i-chord* is an edge $u_j u_m$ with $d(j, m) = i$, and a *vee* is a path $u_j w u_m$ where $d(j, m) = 2$ and $w \in N_j \cap N_m$.

Because C is edge-dominating we have the following.

Observation 4.2. *There are no edges of G with both endpoints in $\mathcal{N}$.*

We now show that neighbors on C do not form a triangle with a vertex outside of C.

Lemma 4.3. *For $i \in \mathbb{Z}_k$, $N_i \cap N_{i+1} = \emptyset$.*

Proof. If $w \in N_i \cap N_{i+1} \neq \emptyset$, then replacing v_iv_{i+1} by v_iwv_{i+1} gives an edge-dominating cycle longer than C. Thus, $N_i \cap N_{i+1} = \emptyset$, as required. □

We deal with the smallest value of k separately.

Lemma 4.4. *If $k = 4$, then G satisfies Theorem 1.2(a), or G is a spiked necklace and satisfies Theorem 1.2(b).*

Proof. By Lemma 4.3 we have $N_i \cap N_{i+1} = \emptyset$ for $i \in \mathbb{Z}_4$. Suppose G has one of the following: $N_0 \cap N_2 \neq \emptyset$ and $N_1 \cap N_3 \neq \emptyset$; $N_0 \cap N_2 \neq \emptyset$ and $u_1u_3 \in E(G)$; or $N_1 \cap N_3 \neq \emptyset$ and $u_0u_2 \in E(G)$. Then G has a longer edge-dominating cycle. So we may assume none of these occur.

Suppose that G has a vee. By symmetry, we may assume that $N_0 \cap N_2 \neq \emptyset$; then $N_1 \cap N_3 = \emptyset$ and $u_1u_3 \notin E(G)$. If $N_1 \neq \emptyset$ and $N_3 \neq \emptyset$, then G has a Y subgraph. So at least one of N_1, N_3 is empty: without loss of generality assume that $N_3 = \emptyset$. If $u_0u_2 \in E(G)$ then G may be regarded as a spiked necklace with three beads. Two are $K_{1,1,0}$ beads u_0u_1, u_1u_2 and the other is a $K_{1,1,t}$ bead with primary vertices u_0 and u_2, formed by u_0u_2, $u_0u_3u_2$ and all vees u_0wu_2, with $t = |N_0 \cap N_2| + 1 \geq 2$. If $u_0u_2 \notin E(G)$ then in a similar way G may be regarded as a spiked necklace with two $K_{1,1,0}$ beads and a $K_{2,t}$ bead, $t \geq 2$. We may therefore assume that G does not have a vee.

If G has no chords, then G is a cycle with potential pendant edges at each vertex, which is a spiked necklace. If G has exactly one chord, we may suppose without loss of generality that it is u_0u_2. If G has pendant edges attached to at most two vertices of C, then G satisfies Theorem 1.2(a). If G has pendant edges at u_1, u_3, and at least one of u_0 or u_2, then G has a Y subgraph, a contradiction. If G has pendant edges at u_0, u_2, and one of u_1 or u_3, then it is a spiked necklace with two $K_{1,1,0}$ beads and a $K_{1,1,1}$ bead. We may therefore assume that G has two chords.

Then G has pendant edges attached to at most two vertices in C, otherwise G has a Y. Thus, G satisfies Theorem 1.2(a). □

We henceforth (for the rest of this section) assume that $k \geq 5$. Because $k \geq 5$, if we have a vee then there is a unique subpath of C of length 2 joining the endpoints of the vee. We say the middle vertex of this path is *enclosed* by the vee, and the edges of this path are *covered* by the vee. We can similarly define the enclosed vertex and covered edges for a 2-chord.

We first consider intersections between the N_i.

Lemma 4.5. *If $N_i \cap N_j \neq \emptyset$ where $i \neq j$ then $d(i, j) = 2$.*

Proof. By Lemma 4.3 we cannot have $d(i, j) = 1$. When $k = 5$ this means that $d(i, j) = 2$. We may therefore assume that $k \geq 6$. If $d(i, j) \geq 3$ and $w \in N_i \cap N_j$, then the paths u_iwu_j, $u_iu_{i-1}u_{i-2}$, $u_iu_{i+1}u_{i+2}$ form a Y subgraph, a contradiction. The conclusion follows. □

Lemma 4.6. *If i, j, and m are distinct, then $N_i \cap N_j \cap N_m = \emptyset$.*

Proof. Suppose there is $w \in N_i \cap N_j \cap N_m$. By Lemma 4.5, $d(i, j) = d(j, m) = d(i, m) = 2$. This can only happen when $k = 6$, and then there is a Y subgraph centered at w, which is a contradiction. □

We now show that if C has a vee, then all neighbors of the enclosed vertex are on C.

Lemma 4.7. *Suppose $i \in \mathbb{Z}_k$. Then we do not have $N_i \neq \emptyset$ and $N_{i-1} \cap N_{i+1} \neq \emptyset$.*

Proof. Suppose that $w \in N_i$ and $v \in N_{i-1} \cap N_{i+1}$. Then $w \neq v$ by Lemma 4.5. The paths $u_{i-1}u_{i-2}u_{i-3}$, $u_{i-1}u_iw$, and $u_{i-1}vu_{i+1}$ form a Y subgraph, which is a contradiction. □

Corollary 4.8. *Two distinct vees do not cross.*

Again, while distinct vees do not cross, there may be multiple vees that cover the same pair of edges.

Now we show that all chords are 2-chords.

Lemma 4.9. *There are no i-chords for $i \geq 3$.*

Proof. Suppose there is an i-chord e for $i \geq 3$. If $k = 5$, this is not possible. If $k = 6$, then e is a 3-chord. If $V(G) = V(C)$ then Theorem 1.2(a) holds, so we may assume there is at least one vertex w not on C. Then in each of the two possible cases (w is or is not adjacent to an end of e) there is a Y subgraph in G, which is a contradiction. We may therefore assume that $k \geq 7$. Then G has a Y centered at either endpoint of e, a contradiction. Therefore, there is no such i-chord. □

We now show that C does not have three consecutive 2-chords (this is similar to Lemma 3.1).

Lemma 4.10. *The cycle C does not have three 2-chords u_iu_{i+2}, $u_{i+1}u_{i+3}$ and $u_{i+2}u_{i+4}$.*

Proof. Suppose three such 2-chords exist; without loss of generality we may assume they are u_0u_2, u_1u_3 and u_2u_4. If $k \geq 7$, then G has a Y centered at u_2. If $k = 6$ then there must be $v \in N_i$ for some i, otherwise G satisfies Theorem 1.2(a). There are four possible cases $i = 2, 3, 4, 5$ (up to symmetry) and in each case G has a Y subgraph.

Suppose now that $k = 5$. We may assume that there is some $v \in N_i$ and $w \in N_j$ with $v \neq w$ and $i \neq j$, otherwise all vertices not in C are a clone class of leaves, and Theorem 1.2(a) holds. The cycle C and the three 2-chords form a wheel centered at u_2 with rim $R = (u_0u_1u_3u_4)$. Thus, there are three cases up to symmetry: $i = 2$ and $j \neq 2$; u_i and u_j are adjacent in R; and u_i and u_j are opposite in R. In each case there is a Y subgraph.

Hence C does not have the three specified 2-chords. □

For all 2-chords, all neighbors of the enclosed vertex are on C (this is similar to Lemma 3.2).

Lemma 4.11. *If $u_{i-1}u_{i+1} \in E(G)$, then $N_i = \emptyset$.*

Proof. Suppose that $u_{i-1}u_{i+1} \in E(G)$, and assume there is $v \in N_i$. Without loss of generality we may assume that $i = 1$. If $k \geq 6$, then G contains a Y subgraph centered at u_0, a contradiction.

Suppose now that $k = 5$. We may assume that there is $w \in N_j$ for some $j \neq 1$, otherwise all vertices not in C are a clone class of leaves, and Theorem 1.2(a) holds. There are two possible cases $j = 2, 3$ (up to symmetry) and in both cases there is a Y subgraph.

Hence $N_{i+1} = \emptyset$. □

Corollary 4.12. *A 2-chord does not cross a vee.*

We can now complete the case where G has a long cycle. Note that we have counterparts for all of the results in Sections 2 and 3 that deal with the existence and behavior of 2-chords and vees and neighbors of their enclosed vertices, so we can use the same arguments as in Proposition 3.9.

Proposition 4.13. *Suppose G is a connected graph with no Y subgraph and with an edge-dominating cycle C of length at least 4. Then G either satisfies Theorem 1.2(a), or G is a spiked necklace and satisfies Theorem 1.2(b).*

Proof. If $k = 4$, then the result holds by Lemma 4.4. We may therefore assume that $k \geq 5$.

We know that the only chords are 2-chords, there are no three consecutive 2-chords, and vees do not cross other vees or 2-chords. We also know that edges not on C are 2-chords, or belong to a vee, or are pendant edges incident with a vertex of C. Moreover, vertices enclosed by vees or 2-chords do not have any neighbors not on C.

Therefore, we can partition the edges of C into subpaths of length at most three as follows: (b) triples of edges covered by either of a pair of crossing 2-chords, (c) pairs of edges covered by a 2-chord and possibly also by vees, (d) pairs of edges covered only by one or more vees, and (e) single other edges (not covered by any chords or vees). (We no longer have the subpaths of type (a) in the proof of Proposition 3.9, since we no longer have 3-chords.) The same arguments as in Proposition 3.9 (modified to use the appropriate results from this section, rather than from Sections 2 and 3) show that all other non-pendant edges of G combine with one of these subpaths to form a bead, and that the pendant edges form spikes, so we have a spiked necklace. □

5 Main Result

We now restate and prove Theorem 1.2.

Theorem 1.2. *A connected graph G has no subgraph isomorphic to Y (is Y-minor-free) if and only if either*

(a) *G is obtained from a graph with at most six vertices by cloning leaves, or*

(b) *G belongs to the family $\mathcal{B}$.*

Proof. First suppose that G is a graph that does not have Y as a subgraph. Let $P = v_0v_1 \dots v_\ell$ be a longest path in G that minimizes $\deg(v_0) + \deg(v_\ell)$. If $\ell \leq 4$ then the result holds by Proposition 2.6. If $\ell \geq 5$ and P does not have any chords from $\{v_0, v_1\}$ to $\{v_{\ell-1}, v_\ell\}$, and $L_1 \cap L_{\ell-1} = \emptyset$, then the result holds by Proposition 3.9. Otherwise, the result holds by Lemma 4.1 and Proposition 4.13.

Conversely, suppose that G satisfies Theorem 1.2(a) or Theorem 1.2(b). Assume that G contains Y as a subgraph. If G satisfies Theorem 1.2(a), then at most one vertex in each leaf class can be used in the Y. Therefore if we delete all but one vertex in each leaf class to obtain a graph G', then G' still contains Y as a subgraph. This is a contradiction because G' has at most six vertices.

Now suppose that G satisfies Theorem 1.2(b). Then the Y subgraph may be centered at a primary vertex or at a secondary vertex of degree 3. There are only two types of secondary vertices of degree 3. A Y cannot be centered at a secondary vertex v of K_4 because K_4 does not have two paths of length 2 disjoint except at v. A Y cannot be centered at a secondary vertex v of $K_{2,1,1}$ because once the three edges incident with v are used, one of the paths cannot be further extended.

We may therefore assume that the Y is centered at a primary vertex v. Note that pendant edges at v cannot be used in the Y subgraph. In order to obtain a Y subgraph, some bead would need to have two paths of length 2 beginning at the primary vertex v and disjoint except at v. However, no bead has two such paths. Hence G does not have Y as a subgraph, as required. □

6 Pathwidth and Growth Constant of Classes of Graphs

In this section we consider the pathwidth and growth constant of Y-minor-free graphs, putting these in context relative to known results.

A sequence $\mathcal{V} = (V_1, V_2, \dots, V_t)$ is a *path-decomposition* of a graph G if $V_i \subseteq V(G)$, for every edge $e \in E(G)$, V_i contains both endpoints for some $1 \le i \le t$, and for $1 \le i < j < k \le t$, $V_i \cap V_k \subseteq V_j$. Each set V_i is called a *bag*. The *width* of $\mathcal{V}$ is $\max\{|V_t| - 1 \ : \ 1 \le i \le t\}$. The *pathwidth* of G, denoted $\mathrm{pw}(G)$, is defined as the smallest width over all path-decompositions of G.

The notions of path-decompositions and pathwidth were defined by Robertson and Seymour in [10], and they proved that the class of graphs obtained by forbidding a forest F as a minor has bounded pathwidth. This was improved to a sharp bound as follows.

Lemma 6.1 (Bienstock et al. [3], short proof by Diestel [5])**.** *If F is an n-vertex forest, then the pathwidth of F-minor-free graphs is at most $n-2$. Furthermore, K_{n-1} shows that this bound is sharp.*

In [6], Dujmovic et al. give a related result, showing that for a given tree T of radius r there are some c and graph H of pathwidth at most $2r-1$ such that every T-minor-free graph is a subgraph of the strong product $H \boxtimes K_c$.

In [7], Kinnersley and Langston observed that the graphs with pathwidth at most 1 have K_3 or Y as a forbidden minor. Their main result was a characterization of graphs with pathwidth at most 2 in terms of 110 forbidden minors, one of which is K_4, whose pathwidth is obviously 3. Since K_4 is part of the $\mathcal{B}$ family, the best possible upper bound on pathwidth of graphs in $\mathcal{B}$ would be 3, and in fact this is the correct bound.

Lemma 6.2. *The pathwidth of a graph in the class of graphs that do not have Y as subgraph is at most 5. The pathwidth of a graph in the family $\mathcal{B}$ is at most 3.*

Proof. Lemma 6.1 implies that the pathwidth of Y-minor-free graphs is at most 5. We will now show that the pathwidth of a graph in the family $\mathcal{B}$ is at most 3. Note that K_4 is a possible bead. So, $\mathrm{pw} \ge 3$. For a strand, let $B_1, \dots, B_r$ be the beads with primary vertices $v_1, \dots, v_{r+1}$. If $B_1 \cong K_4$, then $v_1 = v_2$ or if $B_r \cong K_4$, then $v_r = v_{r+1}$. We will create the bags of the decomposition based on bead type. If $B_1 \cong K_4$, then the first bag is the vertex set of K_4 and the next bag will have v_1 and at most 3 pendant vertices. We continue creating bags with v_1 and pendant vertices until all pendant vertices incident with v_1 are included in a bag. A similar argument holds for $B_t \cong K_4$. At a primary vertex v_i, we consider all pendant vertices attached to that primary vertex before proceeding to the bead vertices. If $B_i \cong K_{2,1,1}$, then we have two bags one with v_i and the two vertices in B_i adjacent to it and another with those two vertices and v_{i+1}. If $B_i \cong K_{1,1,t_1}$, then each bag can consists of the two primary vertices and one of the t vertices until we have a bag for each of the t vertices. K_{2,t_2} is a subgraph of $K_{1,1,t_1}$ so a similar process works.

Let $B_1, \dots, B_r$ be the beads of a necklace with primary vertices $v_1, \dots, v_r$ where B_r has primary vertices v_1 and v_r. Note that v_1 is in each bag of the decomposition. We process the beads and pendant vertices as in the strands case. Note the bags for each individual $K_{2,1,1}$, $K_{1,1,t_1}$, and K_{2,t_2} bead only needs 3 vertices. Since v_1 is in each bag, it follows that we have a path decomposition where each bag has at most four vertices. Hence $\mathrm{pw} \le 3$. ☐

In the previous lemma, pathwidth 3 is required for spiked strands with a K_4 bead, or for general spiked necklaces, but a spiked strand that does not have a K_4 bead will have pathwidth at most 2. Notice that the graphs in $\mathcal{B}$ have a smaller upper bound on pathwidth than the connected graphs obtained from graphs on at most 6 vertices. We wonder if this is a general phenomenon.

Question 6.3. *If you forbid a tree T on n vertices as a minor, the sharp upper bound on pathwidth is $n-2$. Does a stronger upper bound on pathwidth apply to connected T-minor-free graphs not derived in some sense from graphs on at most $n-1$ vertices? More specifically, do connected T-minor-free graphs with sufficiently high diameter (or perhaps radius) have a stronger upper bound on pathwidth?*

Next, we discuss growth constants. The *growth constant* of a family of graphs with g_n labeled elements of order n is defined by Bernardi, Noy, and Welsh [2] as $\limsup_{n\to\infty}(g_n/n!)^{1/n}$. This is known to be finite for minor-closed classes of graphs. Bernardi, Noy, and Welsh observe that the growth constant of a minor-closed class of graphs $\mathcal{G}$ is the same as for the subset of $\mathcal{G}$ consisting of its connected elements.

For Y-minor-free graphs the results of Bernardi, Noy, and Welsh give a lower bound on the growth constant. One of the classes that they investigate is the class of *thick caterpillars*, which are caterpillars where for each edge of the spine we can add one vertex adjacent to both of its ends (forming a triangle). The growth constant of thick caterpillars is the inverse of the positive solution of $(z+z^2)\exp(z)=1$, which is $\delta\approx 2.25159$. Thick caterpillars are a type of spiked strand, and hence Y-minor-free, so the growth constant of Y-minor-free graphs must be at least δ.

Question 6.4. *What is the growth constant of the class of Y-minor-free graphs? Is it δ, or is it larger than δ? More generally, what can be said about the growth constant for graphs without a fixed tree T (or fixed forest F) as a minor?*

Determining the growth constant for Y-minor-free graphs may be relatively easy given our structure theorem, but we have not pursued this.

References

[1] L. W. Beineke. Characterizations of derived graphs. *J. Combinatorial Theory*, 9:129–135, 1970.

[2] O. Bernardi, M. Noy, and D. Welsh. Growth constants of minor-closed classes of graphs. *J. Combin. Theory Ser. B*, 100(5):468–484, 2010.

[3] D. Bienstock, N. Robertson, P. Seymour, and R. Thomas. Quickly excluding a forest. *J. Combin. Theory Ser. B*, 52(2):274–283, 1991.

[4] Henning Bruhn and Maya Stein. t-perfection is always strong for claw-free graphs. *SIAM J. Discrete Math.*, 24(3):770–781, 2010.

[5] R. Diestel. Graph minors. I. A short proof of the path-width theorem. Comment on: "Graph minors. I. Excluding a forest" [J. Combin. Theory Ser. B **35** (1983), no. 1, 39–61; MR0723569 (85d:05148)] by N. Robertson and P. D. Seymour. *Combin. Probab. Comput.*, 4(1):27–30, 1995.

[6] V. Dujmović, R. Hickingbotham, G. Joret, P. Micek, P. Morin, and D. R. Wood. The excluded tree minor theorem revisited. *Combin. Probab. Comput.*, 33(1):85–90, 2024.

[7] N. G. Kinnersley and M. A. Langston. Obstruction set isolation for the gate matrix layout problem. *Discrete Applied Mathematics*, 54(2):169–213, 1994.

[8] C. Kuratowski. Sur le problème des courbes gauches en topologie. *Fund. Math.*, 15:271–283, 1930.

[9] N. Robertson and P. D. Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.

[10] N. Robertson and P.D. Seymour. Graph minors. I. excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983.

[11] K. Wagner. Über eine Erweiterung eines Satzes von Kuratowski. *Deutsche Mathematik*, 2:280–285, 1937.

[12] D. B. West. *Introduction to Graph Theory.* Prentice Hall, Upper Saddle River, N.J., second edition, 2001.