AMORPHOUS SETS AND DUAL DEDEKIND FINITENESS

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Abstract. A set A is dually Dedekind finite if every surjection from A onto A is injective; otherwise, A is dually Dedekind infinite. An amorphous set is an infinite set that cannot be partitioned into two infinite subsets. A strictly amorphous set is an amorphous set in which every partition has only finitely many non-singleton blocks. It is proved consistent with ZF (i.e., the Zermelo–Fraenkel set theory without the axiom of choice) that there exists an amorphous set A whose power set $\mathscr{P}(A)$ is dually Dedekind infinite, which gives a negative solution to a question proposed by Truss [J. Truss, *Fund. Math.* 84, 187–208 (1974)]. Nevertheless, we prove in ZF that, for all strictly amorphous sets A and all natural numbers n, $\mathscr{P}(A)^n$ is dually Dedekind finite, which generalizes a result of Goldstern.

1. Introduction

In 1888, Dedekind [1] defined an infinite set as one that is equinumerous with a proper subset of itself. Such sets are now referred to as *Dedekind infinite*. Sets that are not Dedekind infinite are called *Dedekind finite*. In the presence of the axiom of choice, a Dedekind finite set is the same as a finite set, where a finite set is defined as one that is equinumerous with some natural number. However, without the axiom of choice, there may exist a Dedekind-finite infinite set.

In 1958, Levy [8] investigated the relationships between various notions of finiteness in the absence of the axiom of choice. An *amorphous set* is an infinite set that cannot be partitioned into two infinite subsets. Levy referred to amorphous sets and finite sets collectively as Ia-finite sets.

A set A is *dually Dedekind finite* if every surjection from A onto A is injective; otherwise, A is *dually Dedekind infinite*. The notion of dual Dedekind finiteness was first introduced by Truss [13], who used Δ_5 to denote the class of cardinalities of all dually Dedekind finite sets. It turns out that dual Dedekind finiteness is a particularly interesting notion of finiteness, which has been extensively investigated in [2, 3, 5, 9, 10, 12].

We use $\mathscr{P}(A)$ and $\operatorname{fin}(A)$ to denote, respectively, the power set and the set of all finite subsets of A. Since the class of dually Dedekind finite sets is closed under finite unions (see [13, Theorem 1(iv)]), it follows that for any amorphous set A, $\mathscr{P}(A)$ is dually Dedekind finite if and only if $\operatorname{fin}(A)$ is (see also [5, Theorem 2.15(3)]).

In the paragraph preceding Theorem 4 of [13], Truss proposed the conjecture that for all amorphous sets A, $\operatorname{fin}(A)$ is dually Dedekind finite. This question was also proposed in [12, Problem 4.1], [5, Question 1] and [11, Question 4.8]. In this article, we provide a negative solution to the above question.

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Theorem 1.1. *It is consistent with* ZF *that there exists an amorphous set* A *such that both* $\mathscr{P}(A)$ *and* $\operatorname{fin}(A)$ *are dually Dedekind infinite.*

Truss [14] defined a *strictly amorphous set* as an amorphous set in which every partition has only finitely many non-singleton blocks. Goldstern [3] defined a *strongly amorphous set* as an infinite set A such that, for every positive integer k, every k-ary relation on A is first-order definable from finitely many parameters in the language of equality. He also proved that for every strongly amorphous set A, $\mathscr{P}(A)$ is dually Dedekind finite. It is easy to see that every strongly amorphous set is strictly amorphous. In this article, using ideas in [14], we show that the converse also holds. Furthermore, we extend Goldstern's result by proving that, for every strictly amorphous set A and every natural number n, $\mathscr{P}(A)^n$ is dually Dedekind finite. This result also provides a partial answer to [11, Question 4.8].

2. An amorphous set with a dually Dedekind infinite power set

We shall employ the method of permutation models. We refer the reader to [4, Chap. 8] or [7, Chap. 4] for an introduction to the theory of permutation models. Permutation models are not models of ZF; they are models of ZFA (the Zermelo–Fraenkel set theory with atoms). We shall construct a permutation model in which the atoms form an amorphous set whose power set is dually Dedekind infinite. Then, by the Jech–Sochor embedding theorem (see [4, Theorem 17.1] or [7, Theorem 6.1]), we obtain Theorem 1.1.

We take the set of atoms to be

$$A = \{a_v \mid v \in V\},$$

where V is an infinite vector space over the field $\mathbb{F}_2$. For simplicity, we identify each a_v with v, and hence identify A with V. Let $\mathcal{G}$ be the general linear group of A; that is, the group of all invertible linear transformations of A.

Every permutation π of A extents recursively to a permutation of the universe by

$$\pi x = \{\pi z \mid z \in x\}.$$

Then x belongs to the permutation model $\mathcal{V}$ determined by $\mathcal{G}$ if and only if $x \subseteq \mathcal{V}$ and x has a *finite support*, that is, a finite subset $E \subseteq A$ such that every permutation $\pi \in \mathcal{G}$ fixing E pointwise also fixes x.

For $S \subseteq A$, we write $\operatorname{span}(S)$ for the linear span of S. We also write 0 for the zero vector and $\operatorname{Sub}(A)$ for the set of all linear subspaces of A.

Lemma 2.1. *In* $\mathcal{V}$*,* A *is amorphous.*

Proof. Let $B \in \mathcal{V}$ be a subset of A and let $E \subseteq A$ be a finite support of B. Clearly, for any distinct $u, v \in A \setminus \operatorname{span}(E)$, there exists a $\pi \in \mathcal{G}$ that fixes $\operatorname{span}(E)$ pointwise and moves u to v, and thus $u \in B$ if and only if $v \in B$, since E supports B. Hence, $B \subseteq \operatorname{span}(E)$ or $A \setminus B \subseteq \operatorname{span}(E)$, which implies that B is finite or cofinite. □

Lemma 2.2. *In* $\mathcal{V}$*,* $\operatorname{fin}(A)$*, and consequently* $\mathscr{P}(A)$*, is dually Dedekind infinite.*

Proof. In $\mathcal{V}$, we construct a non-injective surjection $f : \operatorname{fin}(A) \to \operatorname{fin}(A)$ as follows. For each $S \in \operatorname{fin}(A)$, define

$$f(S) = \begin{cases} S \setminus \bigcup\{W \in \operatorname{Sub}(A) \mid W \subseteq S \text{ with } |W| \text{ maximal}\} & \text{if } 0 \in S, \\ S \cup \{0\} & \text{otherwise.} \end{cases}$$

Since every $\pi \in \mathcal{G}$ is a linear transformation, it follows that every $\pi \in \mathcal{G}$ fixes f, which implies $f \in \mathcal{V}$. f is not injective since $f(W) = \varnothing$ for all finite $W \in \mathrm{Sub}(A)$.

We show that f is surjective as follows. Take an arbitrary $T \in \mathrm{fin}(A)$. If $0 \in T$, then $T = f(T \setminus \{0\}) \in \mathrm{ran}(f)$. Suppose $0 \notin T$. We need to construct an $S \in \mathrm{fin}(A)$ such that $f(S) = T$. Let $n = |T|$. We define by recursion a sequence $\langle u_i \rangle_{i \leqslant n}$ of vectors as follows. Let $j \leqslant n$ and assume that $\langle u_i \rangle_{i<j}$ has already been defined. Since A is infinite, we can choose

$$u_j \in A \setminus \mathrm{span}(T \cup \{u_i \mid i < j\}).$$

Let $U = \mathrm{span}(\{u_i \mid i \leqslant n\})$ and let $S = T \cup U$. By construction, it is easy to see that $\mathrm{span}(T) \cap U = \{0\}$, which implies that for every $W \in \mathrm{Sub}(A)$ with $W \subseteq S$, either $W \subseteq T \cup \{0\}$ or $W \subseteq U$. Since $|T \cup \{0\}| = n + 1 < 2^{n+1} = |U|$, it follows from the definition of f that $f(S) = T$. $\square$

Now, Theorem 1.1 follows immediately from Lemmas 2.1 and 2.2 together with the Jech–Sochor embedding theorem.

3. Amorphous sets of projective type and dual Dedekind finiteness

In this section, we extend the result of the previous section by showing that every amorphous set of projective type has a dually Dedekind infinite power set. We first recall the definitions (see [14, p. 198] and [6, Section 4.6]).

A *pregeometry* on a set A is a function $\mathrm{cl} : \mathrm{fin}(A) \to \mathrm{fin}(A)$ satisfying the following conditions:

(1) cl is a closure operator; that is, for all $S, T \in \mathrm{fin}(A)$, $S \subseteq \mathrm{cl}(S) = \mathrm{cl}(\mathrm{cl}(S))$, and if $S \subseteq T$ then $\mathrm{cl}(S) \subseteq \mathrm{cl}(T)$. $S \in \mathrm{fin}(A)$ is called *closed* if $\mathrm{cl}(S) = S$.
(2) cl has the exchange property; that is, for all $S \in \mathrm{fin}(A)$ and $a, b \in A \setminus \mathrm{cl}(S)$, $a \in \mathrm{cl}(S \cup \{b\})$ if and only if $b \in \mathrm{cl}(S \cup \{a\})$.
(3) cl is locally homogeneous; that is, for all closed $S, T \in \mathrm{fin}(A)$ with $S \subseteq T$ and all $a, b \in T \setminus S$, there is a permutation π of T that preserves cl on $\mathrm{fin}(T)$, fixes S pointwise, takes a to b, and is such that, for all closed $U \in \mathrm{fin}(A)$ with $T \subseteq U$, π can be extended to U so as to preserve cl on $\mathrm{fin}(U)$.

If in addition $\mathrm{cl}(\varnothing) = \varnothing$ and $\mathrm{cl}(\{a\}) = \{a\}$ for all $a \in A$, then the pregeometry is called a *geometry*. We say that cl is *degenerate* if $\mathrm{cl}(S) = \bigcup_{a \in S} \mathrm{cl}(\{a\})$ for all non-empty $S \in \mathrm{fin}(A)$. An amorphous set is said to be of *projective type* if it admits a non-degenerate pregeometry. Note that if A is an infinite vector space over a finite field, then $\mathrm{span} : \mathrm{fin}(A) \to \mathrm{fin}(A)$ is a non-degenerate pregeometry on A.

Let $S, T \in \mathrm{fin}(A)$. S is *independent over* T if $a \notin \mathrm{cl}(T \cup (S \setminus \{a\}))$ for all $a \in S$. S is *independent* if it is independent over $\varnothing$. By the exchange property, $\{a_i \mid i < n\}$ is independent over T if and only if $a_j \notin \mathrm{cl}(T \cup \{a_i \mid i < j\})$ for all $j < n$.

Lemma 3.1. *Let* cl *be a pregeometry on an infinite set A and let $S, T \in \mathrm{fin}(A)$ be independent. If $|S| = |T|$, then $|\mathrm{cl}(S)| = |\mathrm{cl}(T)|$.*

Proof. Without loss of generality, assume that $S \cap T = \varnothing$ and $S \cup T$ is independent. Suppose $S = \{a_i \mid i < n\}$ and $T = \{b_i \mid i < n\}$. We prove by induction on $j \leqslant n$ that $|\mathrm{cl}(S)| = |\mathrm{cl}(\{a_i \mid i < n - j\} \cup \{b_i \mid n - j \leqslant i < n\})|$ as follows. The base case of the induction is trivial. Assume the conclusion holds for $j < n$. To establish the step for $j + 1$, it suffices to show that

$$|\mathrm{cl}(X \cup \{a_{n-j-1}\})| = |\mathrm{cl}(X \cup \{b_{n-j-1}\})|, \tag{3.1}$$

where

$$X = \{a_i \mid i < n - j - 1\} \cup \{b_i \mid n - j \leqslant i < n\}.$$

Since $a_{n-j-1}, b_{n-j-1} \in \operatorname{cl}(S \cup T) \setminus \operatorname{cl}(X)$, by the local homogeneity, there is a permutation π of $\operatorname{cl}(S \cup T)$ that preserves cl on $\operatorname{fin}(\operatorname{cl}(S \cup T))$, fixes $\operatorname{cl}(X)$ pointwise, and takes a_{n-j-1} to b_{n-j-1}. Since $\pi[X \cup \{a_{n-j-1}\}] = X \cup \{b_{n-j-1}\}$, it follows that $\pi[\operatorname{cl}(X \cup \{a_{n-j-1}\})] = \operatorname{cl}(X \cup \{b_{n-j-1}\})$, from which (3.1) follows. Finally, by setting $j = n$, we obtain $|\operatorname{cl}(S)| = |\operatorname{cl}(T)|$. □

Theorem 3.2. *For every amorphous set A of projective type, $\operatorname{fin}(A)$, and consequently $\mathscr{P}(A)$, is dually Dedekind infinite.*

Proof. Let cl be a non-degenerate pregeometry on A. Then there exists a non-empty $E \in \operatorname{fin}(A)$ with $|E|$ minimal such that $\operatorname{cl}(E) \neq \bigcup_{a \in E} \operatorname{cl}(\{a\})$. Then E must be independent. We claim that for every independent $X \in \operatorname{fin}(A)$ with $|X| = |E|$,

$$\operatorname{cl}(X) \neq \bigcup_{a \in X} \operatorname{cl}(\{a\}). \tag{3.2}$$

In fact, since E is independent, it follows that $E \subseteq A \setminus \operatorname{cl}(\varnothing)$. By the exchange property, the sets $\operatorname{cl}(\{a\}) \setminus \operatorname{cl}(\varnothing)$ $(a \in E)$ are pairwise disjoint. Hence,

$$|\operatorname{cl}(\varnothing)| + \sum_{a \in E} |\operatorname{cl}(\{a\}) \setminus \operatorname{cl}(\varnothing)| = \left| \bigcup_{a \in E} \operatorname{cl}(\{a\}) \right| < |\operatorname{cl}(E)|.$$

For similar reasons, it follows from Lemma 3.1 that

$$\begin{aligned} \left| \bigcup_{a \in X} \operatorname{cl}(\{a\}) \right| &= |\operatorname{cl}(\varnothing)| + \sum_{a \in X} |\operatorname{cl}(\{a\}) \setminus \operatorname{cl}(\varnothing)| \\ &= |\operatorname{cl}(\varnothing)| + \sum_{a \in E} |\operatorname{cl}(\{a\}) \setminus \operatorname{cl}(\varnothing)| < |\operatorname{cl}(E)| = |\operatorname{cl}(X)|, \end{aligned}$$

and hence (3.2) follows.

Clearly, $|E| \geqslant 2$. Let D be a subset of E obtained by removing two points. Let $\mathcal{C} = \{W \in \operatorname{fin}(A) \mid \operatorname{cl}(D \cup W) = W\}$. We construct a non-injective surjection $f : \operatorname{fin}(A) \to \operatorname{fin}(A)$ as follows. For each $S \in \operatorname{fin}(A)$, define

$$f(S) = \begin{cases} S \setminus \bigcup\{W \in \mathcal{C} \mid W \setminus \operatorname{cl}(D) \subseteq S \text{ with } |W| \text{ maximal}\} & \text{if } S \cap \operatorname{cl}(D) = \varnothing, \\ S & \text{otherwise.} \end{cases}$$

Note that f is not injective since $f(W \setminus \operatorname{cl}(D)) = \varnothing$ for all $W \in \mathcal{C}$.

We show that f is surjective as follows. Let $T \in \operatorname{fin}(A)$. If $T \cap \operatorname{cl}(D) \neq \varnothing$, then $T = f(T) \in \operatorname{ran}(f)$. Suppose $T \cap \operatorname{cl}(D) = \varnothing$. We need to construct an $S \in \operatorname{fin}(A)$ such that $f(S) = T$. Let $n = |T|$. We define by recursion a sequence $\langle a_i \rangle_{i \leqslant n}$ of elements of A as follows. Let $j \leqslant n$ and assume that $\langle a_i \rangle_{i < j}$ has already been defined. Since A is infinite, we can choose

$$a_j \in A \setminus \operatorname{cl}(D \cup T \cup \{a_i \mid i < j\}).$$

Then $\{a_i \mid i \leqslant n\}$ is independent over $D \cup T$. Let $U = \operatorname{cl}(D \cup \{a_i \mid i \leqslant n\})$ and let $S = T \cup (U \setminus \operatorname{cl}(D))$. It is easy to see that $\operatorname{cl}(D \cup T) \cap U = \operatorname{cl}(D)$.

We conclude the proof by showing $f(S) = T$. By definition of f, it suffices to show that whenever $W \in \mathcal{C}$ satisfies $W \setminus \operatorname{cl}(D) \subseteq S$ and $|W| \geqslant |U|$, we have $W = U$. Assume towards a contradiction that there exists $W \in \mathcal{C}$ such that $W \setminus \operatorname{cl}(D) \subseteq S$, $|W| \geqslant |U|$, and $W \neq U$. Then $W \subseteq T \cup U$, and therefore $W \cap T \neq \varnothing$. Since

$|W| \geqslant |U| \geqslant |\operatorname{cl}(D)| + n + 1 > |\operatorname{cl}(D) \cup T|$, it follows that $W \cap (U \setminus \operatorname{cl}(D)) \neq \varnothing$. Let $b \in W \cap T$ and let $c \in W \cap (U \setminus \operatorname{cl}(D))$. Let $X = D \cup \{b, c\}$. X is independent and $|X| = |E|$. By (3.2), there exists $d \in \operatorname{cl}(X) \setminus \bigcup_{a \in X} \operatorname{cl}(\{a\})$. By the minimality of $|E|$, $d \notin \operatorname{cl}(D \cup \{b\})$ and $d \notin \operatorname{cl}(D \cup \{c\})$. Now, if $d \in T$, then by the exchange property we have $c \in \operatorname{cl}(D \cup T)$; and if $d \in U$, then by the exchange property we have $b \in U$. In either case, we obtain a contradiction to $\operatorname{cl}(D \cup T) \cap U = \operatorname{cl}(D)$. $\square$

4. Strictly amorphous sets and strongly amorphous sets

In this section, we show that strictly amorphous sets and strongly amorphous sets coincide. Let $\mathscr{L}_{\doteq}$ denote the first-order language of equality. For each set E, let $\mathscr{L}_{\doteq}^{E}$ be the extension of $\mathscr{L}_{\doteq}$ obtained by adding constant symbols $\dot{e}$ for each $e \in E$.

Lemma 4.1. *Let A be a strictly amorphous set. For every positive integer k and every $R \subseteq A^k$, there is a finite subset E_R of A such that for some quantifier-free $\mathscr{L}_{\doteq}^{E_R}$-formula $\phi(x_1, \dots, x_k)$ we have*

$$R = \bigl\{ \langle a_1, \dots, a_k \rangle \in A^k \bigm| A \models \phi[a_1, \dots, a_k] \bigr\}.$$

Proof. We define E_R for each $R \subseteq A^k$ by recursion on k as follows.

Let $R \subseteq A$. Then R is finite or cofinite since A is amorphous. Define

$$E_R = \begin{cases} R & \text{if } R \text{ is finite,} \\ A \setminus R & \text{otherwise.} \end{cases}$$

Clearly, if $E_R = \{e_1, \dots, e_n\}$, then the quantifier-free $\mathscr{L}_{\doteq}^{E_R}$-formula

$$\phi(x) = \begin{cases} x \doteq \dot{e}_1 \vee \cdots \vee x \doteq \dot{e}_n & \text{if } R \text{ is finite,} \\ \neg(x \doteq \dot{e}_1 \vee \cdots \vee x \doteq \dot{e}_n) & \text{otherwise,} \end{cases}$$

satisfies $R = \{a \in A \mid A \models \phi[a]\}$.

Let k be a positive integer and let $R \subseteq A^{k+1}$. For each $a \in A$, let

$$R_a = \{\langle a_1, \dots, a_k \rangle \in A^k \mid \langle a, a_1, \dots, a_k \rangle \in R\}.$$

By the induction hypothesis, for each $a \in A$, the set E_{R_a} is defined so that the required condition holds. For each $b \in A$, define by recursion

$$\begin{aligned} F_{b,0} &= \{b\}, \\ F_{b,n+1} &= F_{b,n} \cup \bigcup_{a \in F_{b,n}} E_{R_a}, \\ F_b &= \bigcup_{n \in \omega} F_{b,n}. \end{aligned}$$

Clearly, each $F_{b,n}$ is finite. Since A is amorphous, there are no surjections from A onto ω, and hence there exists an $l \in \omega$ such that $F_b = F_{b,l}$ is finite. Again, since A is amorphous, there is a unique $m \in \omega$ such that the set

$$B = \{b \in A \mid |F_b| = m\}$$

is cofinite.

Consider

$$\pi = \{F_b \cap B \mid b \in B\} \cup \{\{b\} \mid b \in A \setminus B\}.$$

We claim that π is a partition of A. Let $b, c \in B$ be such that $(F_b \cap B) \cap (F_c \cap B) \neq \varnothing$. Let $d \in F_b \cap F_c \cap B$. An easy induction shows that $F_{d,n} \subseteq F_b \cap F_c$ for every $n \in \omega$, so $F_d \subseteq F_b \cap F_c$. Since $b, c, d \in B$, $|F_b| = |F_c| = |F_d| = m$, and hence $F_b = F_c = F_d$, which implies that $F_b \cap B = F_c \cap B$.

Since A is strictly amorphous, all but finitely many blocks of π are singletons. Let

$$C = \{b \in B \mid F_b \cap B \text{ is a singleton}\}.$$

Then C is cofinite, and for every $a \in C$, $F_a \cap B = \{a\}$, so $E_{R_a} \subseteq (A \setminus B) \cup \{a\}$, which implies that there exists a quantifier-free $\mathscr{L}_{\doteq}^{A\setminus B}$-formula $\psi(x, x_1, \dots, x_k)$ such that

$$R_a = \big\{ \langle a_1, \dots, a_k \rangle \in A^k \bigm| A \models \psi[a, a_1, \dots, a_k] \big\};$$

let Γ_a be the set of all such ψ. Let $\sim$ be the equivalence relation on C defined by

$$a \sim b \qquad \text{if and only if} \qquad \Gamma_a = \Gamma_b.$$

Since there are only finitely many inequivalent quantifier-free $\mathscr{L}_{\doteq}^{A\setminus B}$-formulas with free variables among $x, x_1, \dots, x_k$, it follows that there are only finitely many $\sim$-equivalence classes. Since A is amorphous, there is a unique $\sim$-equivalence class D which is cofinite. Finally, we define

$$E_R = \bigcup_{b \in A \setminus D} F_b.$$

We conclude the proof by showing that E_R has the required properties. Suppose that $E_R = \{e_1, \dots, e_n\}$. For every $i = 1, \dots, n$, since $E_{R_{e_i}} \subseteq E_R$, by the induction hypothesis, there exists a quantifier-free $\mathscr{L}_{\doteq}^{E_R}$-formula $\phi_i(x_1, \dots, x_k)$ such that

$$R_{e_i} = \big\{ \langle a_1, \dots, a_k \rangle \in A^k \bigm| A \models \phi_i[a_1, \dots, a_k] \big\}.$$

Since $A \setminus B \subseteq A \setminus D \subseteq E_R$ and D is a $\sim$-equivalence class, it follows that there exists a quantifier-free $\mathscr{L}_{\doteq}^{E_R}$-formula $\psi(x, x_1, \dots, x_k)$ such that for every $a \in A \setminus E_R$,

$$R_a = \big\{ \langle a_1, \dots, a_k \rangle \in A^k \bigm| A \models \psi[a, a_1, \dots, a_k] \big\}.$$

Hence, the quantifier-free $\mathscr{L}_{\doteq}^{E_R}$-formula

$$\phi(x, x_1, \dots, x_k) = \bigvee_{i=1}^{n} (x \doteq \dot{e}_i \wedge \phi_i) \vee \left(\bigwedge_{i=1}^{n} \neg(x \doteq \dot{e}_i) \wedge \psi \right)$$

satisfies

$$R = \big\{ \langle a, a_1, \dots, a_k \rangle \in A^{k+1} \bigm| A \models \phi[a, a_1, \dots, a_k] \big\}. \qquad \square$$

Theorem 4.2. *A set A is strictly amorphous if and only if it is strongly amorphous.*

Proof. Suppose that A is strictly amorphous. By Lemma 4.1, for every positive integer k and every k-ary relation R on A, there is a quantifier-free $\mathscr{L}_{\doteq}$-formula $\phi(x_1, \dots, x_k, y_1, \dots, y_n)$ such that

$$R = \big\{ \langle a_1, \dots, a_k \rangle \in A^k \bigm| A \models \phi[a_1, \dots, a_k, e_1, \dots, e_n] \big\}$$

for some $e_1, \dots, e_n \in A$. Hence, A is strongly amorphous.

For the other direction, suppose that A is strongly amorphous. Every subset of A is first-order definable from finitely many parameters in $\mathscr{L}_{\doteq}$, and hence is either finite or cofinite. Therefore, A is amorphous. Let π be an arbitrary partition of A. Since the equivalence relation induced by π is first-order definable from a finite set

E of parameters in $\mathscr{L}_{\doteq}$, it follows that either all elements of $A \setminus E$ lie in a single block of π, or each element of $A \setminus E$ forms a singleton block of π. Hence, all but finitely many blocks of π are singletons. Thus, A is strictly amorphous. □

5. Strictly amorphous sets and dual Dedekind finiteness

In this section, we show that every finite power of the power set of a strictly amorphous set is dually Dedekind finite. A set A is *power Dedekind finite* if the power set of A is Dedekind finite; otherwise, A is *power Dedekind infinite*. According to Kuratowski's celebrated theorem (see [4, Proposition 5.3]), a set A is power Dedekind infinite if and only if there exists a surjection from A onto ω. The class of power Dedekind finite sets is closed under finite products (see [13, Theorem 1(vi)]). Clearly, every amorphous set is power Dedekind finite, and so is every finite power of an amorphous set.

Lemma 5.1. *Let $n \in \omega$. For each $i < n$, let $e_i \in \omega$ and let $X_i = \bigcup_{l\in\omega} X_{i,l}$, where $X_{i,l}$ is power Dedekind finite for all $l \in \omega$. If $X = \prod_{i<n} X_i^{e_i}$ is dually Dedekind infinite and $o \notin X$, then there exist a surjection $g : X \to \{o\} \cup X$ and functions $h, p_{i,j} : \omega \to \omega$ ($i < n, j < e_i$) such that*

1. *h is increasing,*
2. *every $p_{i,j}$ is monotonic, and at least one of $p_{i,j}$ is unbounded,*
3. *for every $k \in \omega$, $g^{-h(k)-1}[\{o\}] \subseteq \prod_{i<n}\prod_{j<e_i} X_{i,p_{i,j}(k)}$.*

Proof. See [9, Lemma 2.4]. □

We write $[A]^k$ for the set of k-element subsets of A. For a tuple $\bar{a} = \langle a_1, \ldots, a_k\rangle$, we write $\{\bar{a}\}$ for the set $\{a_1, \ldots, a_k\}$.

Corollary 5.2. *Let A be a power Dedekind finite set and let $n \in \omega$. If $\mathrm{fin}(A)^n$ is dually Dedekind infinite, then there exist a surjection $g : \mathrm{fin}(A)^n \to \{\varnothing\} \cup \mathrm{fin}(A)^n$ and functions $h, p_i : \omega \to \omega$ ($i < n$) such that*

1. *h is increasing,*
2. *every p_i is monotonic, and at least one of p_i is unbounded,*
3. *for every $m \in \omega$, $g^{-h(m)-1}[\{\varnothing\}] \subseteq \prod_{i<n}[A]^{p_i(m)}$.*

Proof. In Lemma 5.1, take $o = \varnothing$, $e_i = 1$, $X_i = \mathrm{fin}(A)$, and $X_{i,l} = [A]^l$ for all $i < n$ and all $l \in \omega$. □

Theorem 5.3. *For all strictly amorphous sets A and all natural numbers n, $\mathrm{fin}(A)^n$, and consequently $\mathscr{P}(A)^n$, is dually Dedekind finite.*

Proof. Since A is amorphous, $\mathscr{P}(A)$ is equinumerous to $2 \times \mathrm{fin}(A)$, which implies that $\mathscr{P}(A)^n$ is equinumerous to $2^n \times \mathrm{fin}(A)^n$. Since the class of dually Dedekind finite sets is closed under finite unions (see [13, Theorem 1(iv)]), it follows that $\mathrm{fin}(A)^n$ is dually Dedekind finite if and only if $\mathscr{P}(A)^n$ is. We prove that $\mathrm{fin}(A)^n$ is dually Dedekind finite as follows.

We assume the contrary and aim for a contradiction. By Corollary 5.2, there exist a surjection $g : \mathrm{fin}(A)^n \to \{\varnothing\} \cup \mathrm{fin}(A)^n$ and functions $h, p_i : \omega \to \omega$ ($i < n$) such that

1. h is increasing,
2. every p_i is monotonic, and at least one of p_i is unbounded,
3. for every $m \in \omega$, $g^{-h(m)-1}[\{\varnothing\}] \subseteq \prod_{i<n}[A]^{p_i(m)}$.

Without loss of generality, assume that p_0 is unbounded.

For each $\bar{k} = \langle k_0, \dots, k_{n-1} \rangle, \bar{l} = \langle l_0, \dots, l_{n-1} \rangle \in \omega^n \setminus \{\bar{0}\}$ and each $m \in \omega \setminus \{0\}$, let

$$R_{\bar{k},\bar{l},m} = \left\{ \langle \bar{a}_0, \dots, \bar{a}_{n-1}, \bar{b}_0, \dots, \bar{b}_{n-1} \rangle \;\middle|\; \begin{array}{l} \bar{a}_i \in A^{k_i} \text{ and } \bar{b}_i \in A^{l_i} \text{ for every } i < n, \text{ and} \\ g^m(\{\bar{a}_0\}, \dots, \{\bar{a}_{n-1}\}) = \langle \{\bar{b}_0\}, \dots, \{\bar{b}_{n-1}\} \rangle \end{array} \right\};$$

by Lemma 4.1, we can define $E_{\bar{k},\bar{l},m} \in \operatorname{fin}(A)$ such that for some $\mathscr{L}_{\doteq}^{E_{\bar{k},\bar{l},m}}$-formula $\phi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{y}_0, \dots, \bar{y}_{n-1})$ we have

$$R_{\bar{k},\bar{l},m} = \big\{ \langle \bar{a}_0, \dots, \bar{a}_{n-1}, \bar{b}_0, \dots, \bar{b}_{n-1} \rangle \;\big|\; A \models \phi[\bar{a}_0, \dots, \bar{a}_{n-1}, \bar{b}_0, \dots, \bar{b}_{n-1}] \big\}. \tag{5.1}$$

Note that $\{E_{\bar{k},\bar{l},m} \mid \bar{k}, \bar{l} \in \omega^n \setminus \{\bar{0}\}, m \in \omega \setminus \{0\}\}$ is a countable subset of $\operatorname{fin}(A)$, and hence it is finite, since $\operatorname{fin}(A)$ is Dedekind finite. Let

$$E = \bigcup \{E_{\bar{k},\bar{l},m} \mid \bar{k}, \bar{l} \in \omega^n \setminus \{\bar{0}\}, m \in \omega \setminus \{0\}\}.$$

Then E is a finite subset of A such that, for each $\bar{k}, \bar{l} \in \omega^n \setminus \{\bar{0}\}$ and each $m \in \omega \setminus \{0\}$, there exists an $\mathscr{L}_{\doteq}^{E}$-formula $\phi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{y}_0, \dots, \bar{y}_{n-1})$ that satisfies (5.1).

Since p_0 is unbounded, there exists an increasing sequence $\langle m_j \rangle_{j \in \omega}$ such that $0 < p_0(m_j) < p_0(m_{j+1})$ for all $j \in \omega$. Since g is surjective, there exists a finite sequence $\langle \bar{T}_j \rangle_{j \leqslant 2^{2^{n+|E|}}}$ of elements of $\operatorname{fin}(A)^n$ such that, for all $j < 2^{2^{n+|E|}}$,

$$g^{h(m_0)+1}(\bar{T}_0) = \varnothing,$$
$$g^{h(m_{j+1})-h(m_j)}(\bar{T}_{j+1}) = \bar{T}_j.$$

Since $g^{-h(m)-1}[\{\varnothing\}] \subseteq \prod_{i<n} [A]^{p_i(m)}$ for all $m \in \omega$, we have $\bar{T}_j \in \prod_{i<n} [A]^{p_i(m_j)}$ for all $j \leqslant 2^{2^{n+|E|}}$.

Let $\bar{S} = \bar{T}_{2^{2^{n+|E|}}} = \langle S_0, \dots, S_{n-1} \rangle$, and let $\sim$ be the equivalence relation on A defined by

$$a \sim b \iff \text{either } a = b \in E \text{ or } a, b \notin E \text{ and } \forall i < n (a \in S_i \leftrightarrow b \in S_i).$$

Clearly, the number of finite equivalence classes of $\sim$ is less than $2^{n+|E|}$. For each $j < 2^{2^{n+|E|}}$, let $\bar{T}_j = \langle T_{j,0}, \dots, T_{j,n-1} \rangle$. Since $|T_{j,0}| = p_0(m_j)$ and $\langle p_0(m_j) \rangle_{j \in \omega}$ is increasing, it follows that there exists an $r < 2^{2^{n+|E|}}$ such that $T_{r,0}$ is not a union of equivalence classes of $\sim$; that is, there are $c, d \in A$ such that $c \sim d$, $c \in T_{r,0}$ and $d \notin T_{r,0}$. Clearly, $c, d \notin E$. Let π be the transposition that interchanges c and d. Then π fixes E pointwise, $\pi[S_i] = S_i$ for all $i < n$, and $\pi[T_{r,0}] \neq T_{r,0}$.

For each $i < n$, let $k_i = |S_i|$ and $l_i = |T_{r,i}|$, and let $m = h(m_{2^{2^{n+|E|}}}) - h(m_r)$. Then $g^m(\bar{S}) = \bar{T}_r$. For each $i < n$, let $\bar{a}_i \in A^{k_i}$ and $\bar{b}_i \in A^{l_i}$ be such that $S_i = \{\bar{a}_i\}$ and $T_{r,i} = \{\bar{b}_i\}$, respectively. Then

$$\langle \bar{a}_0, \dots, \bar{a}_{n-1}, \bar{b}_0, \dots, \bar{b}_{n-1} \rangle \in R_{\bar{k},\bar{l},m}.$$

Since isomorphisms preserve all formulas, it follows from (5.1) that

$$\langle \pi\bar{a}_0, \dots, \pi\bar{a}_{n-1}, \pi\bar{b}_0, \dots, \pi\bar{b}_{n-1} \rangle \in R_{\bar{k},\bar{l},m}.$$

Hence, $g^m(\bar{S}) = \langle \pi[T_{r,0}], \dots, \pi[T_{r,n-1}] \rangle$, contradicting $\pi[T_{r,0}] \neq T_{r,0}$. $\square$

6. Open questions

We conclude the article with the following two questions. First, we do not know whether the converse of Theorem 3.2 holds.

Question 6.1. Does ZF prove that every amorphous set having a dually Dedekind infinite power set is of projective type?

Second, we wonder whether the consistency result in [9] can be generalized to the power sets of amorphous sets.

Question 6.2. Is it consistent with ZF that there exists a family $\langle A_n \rangle_{n\in\omega}$ of amorphous sets such that, for all $n \in \omega$, $\mathscr{P}(A_n)^n$ is dually Dedekind finite, whereas $\mathscr{P}(A_n)^{n+1}$ is dually Dedekind infinite?

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