On certain semigroups of finite oriented and order-decreasing partial transformations

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Abstract

Let $\mathcal{PORD}_n$ be the semigroup consisting of all oriented and order-decreasing partial transformations on the finite chain $X_n = \{1 < \cdots < n\}$. Let $\mathcal{IORD}_n$ be the subsemigroup of $\mathcal{PORD}_n$ consisting of all injective transformations of $\mathcal{PORD}_n$. For $2 \leq r \leq n$, let $\mathcal{PORD}(n,r) = \{\alpha \in \mathcal{PORD}_n : |\mathrm{im}\,(\alpha)| \leq r\}$ and $\mathcal{IORD}(n,r) = \{\alpha \in \mathcal{IORD}_n : |\mathrm{im}\,(\alpha)| \leq r\}$. In this paper, we determine some minimal generating sets and ranks of $\mathcal{PORD}(n,r)$ and $\mathcal{IORD}(n,r)$, and moreover, we characterize the maximal subsemigroups of $\mathcal{PORD}(n,r)$ and $\mathcal{IORD}(n,r)$.

Key words: Orientation-preserving; orientation-reserving; oriented; partial transformation; injective transformation; generating set; rank; maximal subsemigroup.

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1 Introduction

For $n \in \mathbb{N}$, let $\mathcal{PT}_n$, $\mathcal{I}_n$, and $\mathcal{T}_n$ be the partial, injective, and full transformation semigroups on the finite chain $X_n = \{1, \ldots, n\}$ under its natural order, respectively. An element $\alpha \in \mathcal{PT}_n$ with the domain $\mathrm{dom}\,(\alpha) = \{x_1 < \cdots < x_t\}$ $(0 \leq t \leq n)$ (note that α is the empty transformation 0_n if $t = 0$) is called *order-preserving* (*order-reversing*) if $x_1\alpha \leq \cdots \leq x_t\alpha$ $(x_t\alpha \leq \cdots \leq x_1\alpha)$, and *orientation-preserving* (*orientation-reversing*) if the sequence $(x_1\alpha, \ldots, x_t\alpha)$ is cyclic (anti-cyclic), i.e. there exists no more than one subscript i such that $x_{i+1}\alpha < x_i\alpha$ $(x_i\alpha < x_{i+1}\alpha)$, where $x_{t+1} = x_1$. Furthermore, $\alpha \in \mathcal{PT}_n$ is called *order-decreasing* (*extensive* or *order-increasing*) if $x\alpha \leq x$ $(x \leq x\alpha)$ for all $x \in \mathrm{dom}\,(\alpha)$, *monotone* if α is

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order-preserving or order-reversing, and *oriented* if α is orientation-preserving or orientation-reversing. We denote the subsemigroups of $\mathcal{PT}_n$ consisting of all order-preserving transformations by $\mathcal{PO}_n$, consisting of all monotone transformations by $\mathcal{PM}_n$, consisting of all order-decreasing (extensive) transformations by $\mathcal{PD}_n$ ($\mathcal{PE}_n$), consisting of all orientation-preserving transformations by $\mathcal{POP}_n$, and consisting of all oriented transformations by $\mathcal{POR}_n$. We also let

$$\begin{aligned} &\mathcal{PC}_n = \mathcal{PO}_n \cap \mathcal{PD}_n, && \mathcal{PMD}_n = \mathcal{PM}_n \cap \mathcal{PD}_n, \\ &\mathcal{POPD}_n = \mathcal{POP}_n \cap \mathcal{PD}_n, && \mathcal{PORD}_n = \mathcal{POR}_n \cap \mathcal{PD}_n, \\ &\mathcal{IC}_n = \mathcal{PC}_n \cap \mathcal{I}_n, && \mathcal{IOPD}_n = \mathcal{POPD}_n \cap \mathcal{I}_n, \\ &\mathcal{IORD}_n = \mathcal{PORD}_n \cap \mathcal{I}_n, \end{aligned}$$

and let $\mathcal{U}(n,r) = \{\alpha \in \mathcal{U}_n : |\mathrm{im}\,(\alpha)| \leq r\}$ for every subsemigroup $\mathcal{U}_n$ of $\mathcal{PT}_n$ and $1 \leq r \leq n$. It is shown in [27, Lemma 1.1] that $\mathcal{D}_n = \mathcal{T}_n \cap \mathcal{PD}_n$ and $\mathcal{E}_n = \mathcal{T}_n \cap \mathcal{PE}_n$ are isomorphic semigroups. By a similar mapping to the one used in the proof of [27, Lemma 1.1], one can easily show that $\mathcal{PD}_n$ and $\mathcal{PE}_n$, and so $\mathcal{PORD}_n$ and $\mathcal{PORE}_n = \mathcal{POR}_n \cap \mathcal{PE}_n$ are isomorphic semigroups. Thus, it is enough to consider just $\mathcal{PORD}_n$.

We denote the smallest subsemigroup of a semigroup S containing a non-empty subset A of S by $\langle A \rangle$. If $\langle A \rangle = S$, then A is called a *generating set* of S. A generating set of S with a minimum number of elements is called a *minimal generating set*, the size of a minimal generating set of S is called the *rank* of S, and denoted by $\mathrm{rank}\,(S)$. An element z of a semigroup S is called *undecomposable* in S if, for all $x, y \in S$, $z = xy$ implies $z = x$ or $z = y$. Then it is clear from the definition that every generating set of S contains all undecomposable elements of S. The set of all idempotents in a non-empty subset T of S is denoted by $E(T)$, i.e. $E(T) = \{e \in T : e^2 = e\}$. It is clear that $\alpha \in \mathcal{PD}_n$ is an idempotent if and only if $\min(x\alpha^{-1}) = x$ for each $x \in \mathrm{im}\,(\alpha)$. For other terms in semigroup theory, which are not explained here, we refer to [18, 23].

It is well-known that $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$ have ranks 4, 3, and 3, respectively. Gomes and Howie [21] showed that the ranks of the semigroup of all order-preserving full transformations $\mathcal{O}_n$ and order-preserving partial transformations $\mathcal{PO}_n$ are equal to $n+1$ and $2n$, respectively. In [17], Fernandes, Gomes and Jesus showed that the rank of the monoid $\mathcal{POP}_n$ is equal to 3. Dimitrova and Koppitz [11] showed that the rank of $\mathcal{POE}_n = \mathcal{PO}_n \cap \mathcal{PE}_n$ is equal to $2n$. More recently, the rank of the semigroup $\mathcal{POPD}_n$ was determined by Ayık, Ayık, Bugay, and Dağdeviren in [1]. Ayık, Ayık, Dimitrova, and Koppitz determined the ranks of the semigroups $\mathcal{PMD}_n$, $\mathcal{IMD}_n$, and their ideals in [3], as well as of the semigroup $\mathcal{ORD}_n$ and its ideals in [4]. Finding ranks of certain transformation semigroups is an important problem considered by many authors (see, for example, [5, 8, 9, 16, 20, 24, 25, 26, 28, 30]). In the present paper, we will determine some minimal generating sets and ranks of $\mathcal{PORD}(n,r)$ and $\mathcal{IORD}(n,r)$ for $3 \leq r \leq n$.

Moreover, we will characterize the maximal subsemigroups of $\mathcal{PORD}(n,r)$ as well as of $\mathcal{IORD}(n,r)$ for any $3 \leq r \leq n$. By a maximal subsemigroup of a semigroup S, we mean a maximal element, under set inclusion, within the family of all proper subsemigroups of S. In 1966, Bayramov [6] proved that S is a maximal subsemigroup of $\mathcal{T}_n$ if and only if S is either $\mathcal{T}(n, n-2) \cup \mathcal{S}_n$ or $\mathcal{T}(n, n-1) \cup G$, where G is a maximal subgroup of the symmetric group $\mathcal{S}_n$. In 1968, Graham, Graham, and Rhodes [22] described forms to which the maximal subsemigroups of a finite semigroup belong. Determining which of these forms appear in a given finite semigroup is difficult. Donoven, Mitchell, and Wilson [14] presented an algorithm for computing the maximal subsemigroups of a finite semigroup. Since it is important to know the maximal subsemigroups for concrete semigroups, there are numerous papers about finding maximal subsemigroups of particular classes of transformation semigroups (see, for example, [3, 4, 10, 11, 12, 13, 15, 19, 29, 30]).

2 Preliminaries

The *fix* and *kernel* sets of $\alpha \in \mathcal{PT}_n$ are defined by $\text{fix}\,(\alpha) = \{x \in \text{dom}\,(\alpha) : x\alpha = x\}$ and $\ker(\alpha) = \{(x, y) \in \text{dom}\,(\alpha) \times \text{dom}\,(\alpha) : x\alpha = y\alpha\}$, respectively. We use the notations 1_n and 0_n to denote the identity and empty (zero) mappings on X_n, respectively. Let Y be a non-empty subset of X_n. Then the restriction map of an element α in $\mathcal{PT}_n$ to $Y \cap \text{dom}\,(\alpha)$ is denoted by $\alpha_{|_Y}$. In particular, the restriction map of 1_n to Y is denoted by 1_Y, and called the *partial identity* on Y.

Let U, V and Y be any non-empty subsets of X_n. If $u < v$ for all $u \in U$ and $v \in V$, then we write $U < V$; and if $P = \{U_1, \ldots, U_r\}$ $(1 \leq r \leq n)$ is a family of non-empty disjoint subsets of Y such that $Y = \bigcup\limits_{i=1}^{r} U_i$, then P is called a *partition* of Y with r parts. For a non-zero $\alpha \in \mathcal{PT}_n$, notice that

$$\text{Ker}\,(\alpha) = \{y\alpha^{-1} : y \in \text{im}\,(\alpha)\}$$

is a partition of $\text{dom}\,(\alpha)$. A partition $P = \{U_1, \ldots, U_r\}$ of Y is called an *ordered* partition of Y if $U_i < U_{i+1}$ for all $1 \leq i \leq r - 1$, and a *convex* partition of Y if for each $1 \leq i \leq r$, U_i is convex, i.e. for every $u, v \in U_i$ and $y \in Y$, $u \leq y \leq v$ implies $y \in U_i$. For a non-zero $\alpha \in \mathcal{PM}_n$, if $\text{im}\,(\alpha) = \{x_1 < \cdots < x_r\}$ $(1 \leq r \leq n)$, then it is clear that $\text{Ker}\,(\alpha)$ is a convex and ordered partition of $\text{dom}\,(\alpha)$. In addition, $\text{Ker}\,(\alpha) = (x_1\alpha^{-1} < \cdots < x_r\alpha^{-1})$ if $\alpha \in \mathcal{PO}_n$, and otherwise, $\text{Ker}\,(\alpha) = (x_r\alpha^{-1} < \cdots < x_1\alpha^{-1})$.

Let $\mathcal{PR}_n$ denote the subset of $\mathcal{PT}_n$ consisting of all orientation-reversing partial transformations. It is shown in [7, Lemma 1.1] that for $\alpha \in \mathcal{PT}_n$ with $\text{dom}\,(\alpha) = \{a_1 < \cdots < a_t\}$, the sequence $(a_1\alpha, \ldots, a_t\alpha)$ is both cyclic and anti-cyclic if and only if $|\text{im}\,(\alpha)| \leq 2$. Thus, we have

$$\mathcal{POP}_n \cap \mathcal{PR}_n = \{\alpha \in \mathcal{POR}_n : |\text{im}\,(\alpha)| \leq 2\}. \tag{1}$$

It is also shown in [7, Proposition 2.3] that a non-constant oriented full transformation α is order-preserving (order-reversing) if and only if $1\alpha < n\alpha$ $(n\alpha < 1\alpha)$. Similarly, it is shown that a non-constant $\alpha \in \mathcal{POR}_n$ is order-preserving (order-reversing) if and only if $(\min(\text{dom}\,(\alpha)))\alpha < (\max(\text{dom}\,(\alpha)))\alpha$ $((\max(\text{dom}\,(\alpha)))\alpha < (\min(\text{dom}\,(\alpha)))\alpha)$.

We suppose that $n \geq 4$ since $\mathcal{PORD}_n = \mathcal{POPD}_n = \mathcal{PC}_n$ for $n = 1, 2$, and $\mathcal{PORD}_3 = \mathcal{POPD}_3$, and that $3 \leq r \leq n - 1$ since $\mathcal{PORD}(n, 1) = \mathcal{POPD}(n, 1) = \mathcal{PC}(n, 1)$, $\mathcal{PORD}(n, 2) = \mathcal{POPD}(n, 2)$, $\mathcal{PORD}(n, n-1) = \mathcal{PORD}_n \setminus \{1_n\}$ and $\mathcal{PORD}(n, n) = \mathcal{PORD}_n$ throughout this paper unless otherwise stated.

Next we let

$$\mathcal{PMD}_n^* = \mathcal{PMD}_n \setminus \mathcal{PC}_n, \qquad \mathcal{POPD}_n^* = \mathcal{POPD}_n \setminus \mathcal{PC}_n,$$
$$\mathcal{PORD}_n^* = \mathcal{PORD}_n \setminus \mathcal{POPD}_n, \qquad \mathcal{PRD}_n^* = \mathcal{PORD}_n^* \setminus \mathcal{PMD}_n,$$

and for $1 \leq r \leq s \leq n$, let

$$[r, s] = \{r, r + 1, \ldots, s\}.$$

In [1, Section 2], the *order-preserving degree* of $\alpha \in \mathcal{POPD}_n$ is defined by

$$\text{opd}\,(\alpha) = \max\{m : \alpha_{|_{X_m}} \in \mathcal{PO}_m\} = \max\{m : \alpha_{|_{X_m}} \in \mathcal{PC}_m\},$$

and noticed that for $\alpha \in \mathcal{POPD}_n^*$, if $\mathrm{opd}\,(\alpha) = m$ and $\mathrm{dom}\,(\alpha) = \{x_1 < \cdots < x_t\}$, then $2 \leq t \leq n$, $2 \leq m \leq n-1$, and there exists a unique $1 \leq s \leq t-1$ such that $1 \leq x_{s+1}\alpha \leq \cdots \leq x_t\alpha \leq x_1\alpha \leq \cdots \leq x_s\alpha \leq m$ and $x_{s+1} = m+1$. Thus, $\alpha_{|X_m} \in \mathcal{PC}_m$, and if $k = x_1\alpha$, then $\alpha_{|[m+1,n]}$ is an order-preserving partial transformation from $[m+1, n]$ to X_k such that $m+1 \in \mathrm{dom}\,(\alpha_{|[m+1,n]})$. Now we define the *order-reserving degree* of $\alpha \in \mathcal{PORD}_n^*$ by

$$\mathrm{ord}\,(\alpha) = \max\{m : \alpha_{|X_m} \in \mathcal{PMD}_n \text{ and } \max(\mathrm{im}\,(\alpha)) = (m+1)\alpha\}.$$

(This definition is a simple generalization of the definition given in [4, Section 2].) For $\alpha \in \mathcal{PRD}_n^*$, if $\mathrm{ord}\,(\alpha) = m$ and $\mathrm{dom}\,(\alpha) = \{x_1 < \cdots < x_t\}$, then similarly, we notice that $3 \leq t \leq n$, $2 \leq m \leq n-1$, and there exists unique $1 \leq s \leq t-1$ such that $x_{s+1} = m+1$ and $\mathrm{im}\,(\alpha) \subseteq [x_s\alpha, x_{s+1}\alpha]$. More precisely, $\alpha \in \mathcal{PORD}_n^*$ has the tabular form:

$$\alpha = \left(\begin{array}{ccc|ccc} x_1 & \cdots & x_s & x_{s+1} & \cdots & x_t \\ x_1\alpha & \cdots & x_s\alpha & x_{s+1}\alpha & \cdots & x_t\alpha \end{array}\right) \tag{2}$$

with the property that

$$1 \leq x_s\alpha \leq \cdots \leq x_1\alpha \leq x_t\alpha \leq \cdots \leq x_{s+1}\alpha \leq x_{s+1} = m+1.$$

Thus, $\alpha_{|X_m} \in \mathcal{PMD}_m$, and if $k = x_1\alpha$, then $\alpha_{|[m+1,n]}$ is an order-reversing partial transformation from $[m+1, n]$ to $[k, m+1]$ such that $m+1 \in \mathrm{dom}\,(\alpha_{|[m+1,n]})$.

Let $\alpha \in \mathcal{PRD}_n^*$ with $\mathrm{ord}\,(\alpha) = m$. As noticed in [3, Equation (1)], we have

$$|\mathrm{im}\,(\alpha_{|X_m})| \leq \max(\mathrm{im}\,(\alpha_{|X_m})) = (\min(\mathrm{dom}\,(\alpha_{|X_m})))\alpha \leq \min(\mathrm{dom}\,(\alpha_{|X_m})),$$

and so we conclude that

$$1 \leq |\mathrm{im}\,(\alpha_{|X_m})| \leq \left\lfloor \frac{m+1}{2} \right\rfloor \tag{3}$$

where $\left\lfloor \frac{m+1}{2} \right\rfloor$ denotes the largest integer less than or equal to $\frac{m+1}{2}$. Notice that if $\alpha = \begin{pmatrix} 3 & 5 & 7 \\ 3 & 5 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 3 & 5 & 7 \\ 3 & 5 & 4 \end{pmatrix}$ in $\mathcal{PORD}_7$, then $\mathrm{opd}\,(\alpha) = 6$ and $\mathrm{ord}\,(\beta) = 4$, but $\mathrm{ord}\,(\alpha)$ and $\mathrm{opd}\,(\beta)$ are not defined since $\alpha \notin \mathcal{PORD}_7^*$ and $\beta \notin \mathcal{POPD}_7$.

Let r_n denote the maximum size of image of a orientation-reversing and order-decreasing partial mapping on X_n, i.e.

$$r_n = \max\{|\mathrm{im}\,(\alpha)| : \alpha \in \mathcal{PRD}_n^*\}.$$

Then we have the following result.

Proposition 1 $r_n = n - \left\lfloor \frac{n}{3} \right\rfloor$.

Proof Let $k = \left\lfloor \frac{n}{3} \right\rfloor$, and so by the Division Algorithm, we have $n = 3k + i$ for $0 \leq i \leq 2$. If we consider the mapping

$$\gamma_n = \left(\begin{array}{cccc|cccc} k+i-1 & k+i & \cdots & 2k+2i-3 & 2k+i & 2k+i+1 & \cdots & n \\ k+i-1 & k+i-2 & \cdots & 1 & 2k+i & 2k+i-1 & \cdots & k+i \end{array}\right),$$

then it is clear that $\gamma_n \in \mathcal{PRD}_n^*$ and $|\mathrm{im}\,(\gamma_n)| = 2k + i = n - k$, and hence, $r_n \geq n - \left\lfloor \frac{n}{3} \right\rfloor$.

Let $\alpha \in \mathcal{PRD}_n^*$, and suppose that $|\mathrm{im}\,(\alpha)| = r \geq n - k$. If $\max(\mathrm{im}\,(\alpha)) = t$, then there exists $0 \leq j \leq k$ such that $\min(t\alpha^{-1}) = n - k + j = 2k + i + j$ since

$\min(t\alpha^{-1}) \geq t \geq r \geq n-k$, and so by (2), we obtain $m = \mathrm{ord}\,(\alpha) = 2k+i+j-1$. Since $r = |\mathrm{im}\,(\alpha)| \leq |\mathrm{im}\,(\alpha_{|_{X_m}})| + |\mathrm{im}\,(\alpha_{|_{[m+1,n]}})|$, it follows from (3) that

$$\begin{aligned} r &\leq \left\lfloor \frac{2k+i+j}{2} \right\rfloor + (k-j+1) = 2k+i-\left((j-1)-\left\lfloor \frac{j-i}{2} \right\rfloor\right) \\ &= 2k+i-\left\lfloor \frac{j-1+i}{2} \right\rfloor \leq 2k+i = n-k, \end{aligned}$$

and so $r = n-k$. Therefore, we conclude that $r_n = n - \left\lfloor \frac{n}{3} \right\rfloor$. $\square$

For all $\delta, \lambda \in \mathcal{D}_n$, it is shown in [24] that $\mathrm{fix}\,(\delta\lambda) = \mathrm{fix}\,(\delta) \cap \mathrm{fix}\,(\lambda)$. Also we give the following lemma without a proof since its proof is similar to the proof of [24, Lemma 1.1].

Lemma 2 $\mathrm{fix}\,(\delta\lambda) = \mathrm{fix}\,(\delta) \cap \mathrm{fix}\,(\lambda)$ *for all* $\delta, \lambda \in \mathcal{PD}_n$. $\square$

For every subsemigroup $\mathcal{U}_n$ of $\mathcal{PT}_n$ and $0 \leq r \leq n$, let

$$E_r(\mathcal{U}_n) = \{\alpha \in E(\mathcal{U}_n) : |\mathrm{im}\,(\alpha)| = r\}.$$

For each $1 \leq r \leq n$, it is determined in [1, Proposition 1] that $|E_r(\mathcal{POPD}_n)| = \sum_{s=r}^{n} \binom{n}{s}\binom{s}{r} = \binom{n}{r}2^{n-r}$. Now we have the following result.

Proposition 3 *For each* $\alpha \in \mathcal{PORD}_n^*$, *we have* $|\mathrm{fix}\,(\alpha)| \leq 2$. *Thus,* $E(\mathcal{PORD}_n) = E(\mathcal{POPD}_n)$, *and so for each* $1 \leq r \leq n$, *we have* $|E_r(\mathcal{PORD}_n)| = \binom{n}{r}2^{n-r}$.

Proof For any $\alpha \in \mathcal{PORD}_n^*$, let $\mathrm{ord}\,(\alpha) = m$ and $\mathrm{dom}\,(\alpha) = \{x_1 < \cdots < x_t\}$. Then $1 \leq x_s\alpha \leq \cdots \leq x_1\alpha \leq x_t\alpha \leq \cdots \leq x_{s+1}\alpha \leq x_{s+1} = m+1$ for some $1 \leq s \leq t-1$. Since $x_1\alpha \leq x_1$ and α is order-reversing on $\mathrm{dom}\,(\alpha) \cap X_m$, there exists at most one fixed point of α in X_m. Moreover, since $(m+1)\alpha \leq m+1$ and α is order-reversing on $\mathrm{dom}\,(\alpha) \cap [m+1,n]$, there exists at most one fixed point of α in $[m+1,n]$. Therefore, α has at most two fixed points in X_n.

For any $\alpha \in \mathcal{PORD}_n^*$, if α is an idempotent, then $|\mathrm{im}\,(\alpha)| = |\mathrm{fix}\,(\alpha)| \leq 2$, and so by (1), we obtain $\alpha \in E(\mathcal{POPD}_n)$. Thus, $E(\mathcal{PORD}_n) = E(\mathcal{POPD}_n)$ and so $|E_r(\mathcal{PORD}_n)| = \binom{n}{r}2^{n-r}$ for each $1 \leq r \leq n$. $\square$

3 Minimal generating set and rank of $\mathcal{PORD}(n,r)$ and $\mathcal{IORD}(n,r)$

For each $2 \leq r \leq n-1$, it is proven in [1, Proposition 5] that each element of $E_r = E_r(\mathcal{POPD}_n)$ is undecomposable in $\mathcal{POPD}(n,r)$. We also have the following proposition. Although the proof of [1, Proposition 5] is also valid for $\mathcal{PORD}(n,r)$, we want to give a short and easy proof to make it easier for the reader.

Proposition 4 *Each element of* E_r *is undecomposable in* $\mathcal{PORD}(n,r)$.

Proof For any $\xi \in E_r$, suppose that $\xi = \alpha\beta$ for some $\alpha, \beta \in \mathcal{PORD}(n,r)$. By Lemma 2, since $\mathrm{fix}\,(\xi) = \mathrm{im}\,(\xi)$ is a subset of both $\mathrm{fix}\,(\alpha)$ and $\mathrm{fix}\,(\beta)$, we get $3 \leq r \leq |\mathrm{fix}\,(\alpha)|, |\mathrm{fix}\,(\beta)|$, and so from Proposition 3, both α and β must be in $\mathcal{POPD}(n,r)$ which contradicts to the result given in [1, Proposition 5]. $\square$

Next we consider some idempotents in $\mathcal{POPD}(n,r)$ whose image size are strictly less than r. For each $1 \le p \le n-2$ and each $p+1 \le q \le p+r-2 \le n-1$, we define

$$\xi^r_{p,q} \quad = \quad \begin{pmatrix} p & p+1 & \cdots & q & q+1 \\ p & p+1 & \cdots & q & p \end{pmatrix} \in E(\mathcal{POPD}(n,r)), \qquad (4)$$

and we let

$$F_r = \{\, \xi^r_{p,q} : 1 \le p \le n-2 \text{ and } p+1 \le q \le p+r-2 \le n-1 \,\}.$$

First notice that $|\mathrm{im}\,(\xi^r_{p,q})| = q-p+1 \le r-1$, and so $E_r \cap F_r = \emptyset$. For each $3 \le r \le n-1$, it is determined in [1, Equation (3)] that $|F_r| = \frac{(2n-r-1)(r-2)}{2}$. Moreover, it is proven in [1, Proposition 6] that each generating set of $\mathcal{POPD}(n,r)$ must contain a mapping such that the restriction to $[p, q+1]$ is $\xi^r_{p,q}$ for each $1 \le p \le n-2$ and $p+1 \le q \le p+r-2 \le n-1$. By using this fact, we have the following proposition.

Proposition 5 *Let A be a generating set of $\mathcal{PORD}(n,r)$. Then, for each $1 \le p \le n-2$ and each $p+1 \le q \le p+r-2 \le n-1$, there exists an $\alpha \in A$ such that $\alpha_{|_{[p,q+1]}} = \xi^r_{p,q}$.*

Proof Suppose that A is a generating set of $\mathcal{PORD}(n,r)$. Let $1 \le p \le n-2$ and $p+1 \le q \le p+r-2 \le n-1$. Then, there exist $\alpha_1, \ldots, \alpha_t \in A$ such that $\xi^r_{p,q} = \alpha_t \cdots \alpha_1$. Since $|\mathrm{fix}\,(\xi^r_{p,q})| = q-p+1$, if $q \ge p+2$, then $|\mathrm{fix}\,(\xi^r_{p,q})| = 3$, and so it follows from Lemma 2 and Proposition 3 that $\alpha_1, \ldots, \alpha_t \in \mathcal{POPD}(n,r)$, and so from [1, Proposition 6], we have $\alpha_{i|_{[p,q+1]}} = \xi^r_{p,q}$ for at least one $1 \le i \le t$.

Suppose $q = p+1$. For each $1 \le i \le t$, it follows from Lemma 2 that $p\alpha_i = p$ and $(p+1)\alpha_i = p+1$. Since $\mathrm{Ker}\,(\xi^r_{p,q}) = \{\{p, p+2\}, \{p+1\}\}$ and each α_i is order-decreasing, it follows that either $(p+2)\alpha_i = p+2$ or $(p+2)\alpha_i = p$ for every $1 \le i \le t$. Thus, we also have $\alpha_{i|_{[p,q+1]}} = \xi^r_{p,q}$ for at least one $1 \le i \le t$. $\square$

It is proven in [1, Theorem 8] that $E_r \cup F_r$ is a minimal generating set of $\mathcal{POPD}(n,r)$, and so $\mathrm{rank}\,(\mathcal{POPD}(n,r)) = \binom{n}{r}2^{n-r} + \frac{(2n-r-1)(r-2)}{2}$. In particular, $\mathrm{rank}\,(\mathcal{POPD}_n) = \frac{n^2+n+2}{2}$.

Now we focus on $\mathcal{PORD}^*_n$. Let γ in $\mathcal{PORD}^*_n$ with $\mathrm{fix}\,(\gamma) = \{p,q\}$ where $1 \le p < q \le n$. Then notice that $\mathrm{dom}\,(\gamma) \setminus \{p,q\} \ne \emptyset$ since $|\mathrm{dom}\,(\gamma)| \ge |\mathrm{im}\,(\gamma)| \ge 3$. Moreover, for each $t \in \mathrm{dom}\,(\gamma) \setminus \{p,q\}$, since γ is order-decreasing, it is easy to see that $p < t$, and so

$$\mathrm{dom}\,(\gamma) \subseteq [p,n].$$

Also it is easy to see that $1 \le t\gamma \le p\gamma = p$ for all $t \in \mathrm{dom}\,(\gamma) \cap [p, q-1]$, and that $1 \le p = p\gamma \le t\gamma \le q\gamma = q$ for all $t \in \mathrm{dom}\,(\gamma) \cap [q,n]$, and so

$$1 \le p \le q-2 \le n-2, \quad \mathrm{im}\,(\gamma) \subseteq [1,q] \ \text{ and } \ \mathrm{ord}\,(\gamma) = q-1.$$

For every $1 \le p \le q-2 \le n-2$, we let

$$k = \min\{p-1, q-p-1\} \ \text{ and } \ l = \min\{q-p-1, n-q\},$$

and define the mapping

$$\gamma_{p,q} \quad = \quad \left(\begin{array}{cccc|cccc} p & p+1 & \cdots & p+k & q & q+1 & \cdots & q+l \\ p & p-1 & \cdots & p-k & q & q-1 & \cdots & q-l \end{array} \right) \in \mathcal{PORD}_n. \qquad (5)$$

First notice that $p-k = 1$ or $p+k = q-1$, and that $q-l = p+1$ or $q+l = n$. Thus, for every $\gamma \in \mathcal{PORD}^*_n$ with $\mathrm{fix}\,(\gamma) = \{p,q\}$, we have $|\mathrm{im}\,(\gamma)| \le |\mathrm{im}\,(\gamma_{p,q})|$. Also notice that $\gamma_{1,n} = \begin{pmatrix} 1 & n \\ 1 & n \end{pmatrix} \in \mathcal{POPD}_n$. However, if we let

$$G_n = \{\gamma_{p,q} : 1 \le p \le q-2 \le n-2\} \setminus \{\gamma_{1,n}\},$$

then is easy to see that G_n is a subset of $\mathcal{PORD}_n^*$, and that

$$|G_n| = -1 + \sum_{p=1}^{n-2} \sum_{q=p+2}^{n} 1 = -1 + \sum_{p=1}^{n-2} (n-p-1) = \frac{n(n-3)}{2}.$$

For example, $G_4 = \{\gamma_{1,3}, \gamma_{2,4}\}$ where $\gamma_{1,3} = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 3 & 2 \end{pmatrix}$ and $\gamma_{2,4} = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 1 & 4 \end{pmatrix}$. As Proposition 5, we also have the following proposition.

Proposition 6 *Let A be a generating set of $\mathcal{PORD}(n,r)$. Then, for every $1 \leq p \leq n-2$ and $p+2 \leq q \leq n$, there exists an $\alpha \in A \cap \mathcal{PORD}^*(n,r)$ such that* $\mathrm{fix}\,(\alpha) = \{p,q\}$.

Proof Let A be a generating set of $\mathcal{PORD}(n,r)$. For every $1 \leq p \leq q-2 \leq n-2$, there exist $\alpha_1, \ldots, \alpha_t \in A$ such that $\gamma_{p,q} = \alpha_1 \cdots \alpha_t$. By Lemma 2, we have $\{p,q\} \subseteq \mathrm{fix}\,(\alpha_i)$ for each $1 \leq i \leq t$. Since the products of orientation-preserving mappings are also orientation-preserving, for at least one of $1 \leq j \leq t$, α_j must be orientation-reversing. Since orientation-reversing mappings have at most two fix points, we conclude that $\mathrm{fix}\,(\alpha_j) = \{p,q\}$, as required. □

Next we determine which $\gamma_{p,q}$'s are undecomposable in $\mathcal{PORD}_n$ in the following lemma.

Lemma 7 *Let $1 \leq p \leq q-2 \leq n-2$. Then $\gamma_{p,q}$ is undecomposable in $\mathcal{PORD}_n$ if and only if $q-1, n \in \mathrm{dom}\,(\gamma_{p,q})$, i.e. $\mathrm{dom}\,(\gamma_{p,q}) = [p,n]$.*

Proof ($\Rightarrow$) Let $Y = \mathrm{dom}\,(\gamma_{p,q})$, and suppose that $q-1 \notin Y$, i.e. $k = \min\{p-1, q-p-1\} = p-1 < q-p-1$ as defined above. Then we observe that $p+k = 2p-1 \in Y$ and $(2p-1)\gamma_{p,q} = 1$. If we let $Z_1 = Y \cup \{q-1\}$ and consider the partial identity 1_Y and the mapping $\beta_1 : Z_1 \to \mathrm{im}\,(\gamma_{p,q})$ defined by

$$x\beta_1 = \begin{cases} x\gamma_{p,q} & \text{if } x \in Y, \\ 1 & \text{if } x = q-1, \end{cases}$$

then $\gamma_{p,q}$ is different from both 1_Y and β_1, and $\gamma_{p,q} = 1_Y\beta_1$. Thus, we conclude that $\gamma_{p,q}$ is not undecomposable in $\mathcal{PORD}_n$.

Suppose that $n \notin Y$, i.e. $l = \min\{q-p-1, n-q\} = q-p-1 < n$ as defined above. Then we observe that $q+l = 2q-p-1 \in Y$ and $(2q-p-1)\gamma_{p,q} = p+1$. If we let $Z_2 = Y \cup \{n\}$ and consider the mapping $\beta_2 : Z_2 \to \mathrm{im}\,(\gamma_{p,q})$ defined by

$$x\beta_2 = \begin{cases} x\gamma_{p,q} & \text{if } x \in Y, \\ p & \text{if } x = n, \end{cases}$$

then similarly, $\gamma_{p,q} = 1_Y\beta_2$ is not undecomposable in $\mathcal{PORD}_n$.

($\Leftarrow$) Suppose that $q-1, n \in \mathrm{dom}\,(\gamma_{p,q})$, i.e. $\mathrm{dom}\,(\gamma_{p,q}) = [p,n]$. Assume that there are $\alpha, \beta \in \mathcal{PORD}_n$ such that $\gamma_{p,q} = \alpha\beta$. By Lemma 2, $\{p,q\}$ is a subset of both $\mathrm{fix}\,(\alpha)$ and $\mathrm{fix}\,(\beta)$. First notice that $[p,n] \subseteq \mathrm{dom}\,(\alpha)$, and that α is an injective mapping on $[p,n]$ since $\gamma_{p,q}$ is injective.

Suppose that α is orientation-preserving, and so β is orientation-reversing. Then α is order-preserving on $[p,q]$ otherwise $t\alpha < p < q$ for some $t \in (p,q)$ which contradicts with orientation-preservation of α. Since $p = p\alpha \leq (p+1)\alpha \leq (p+1)$, it follows from the injectivity of α on $[p,n]$ that $(p+1)\alpha = (p+1)$. By continuing in this fashion, we obtain $(p+i)\alpha = (p+i)$ for each $1 \leq i \leq q-p-1$. One can similarly show that $(q+j)\alpha = (q+j)$ for each $1 \leq j \leq n-q$, and so we conclude that $\alpha_{|_{[p,n]}} = 1_{p,n}$, and so $[p,n] \subseteq \mathrm{dom}\,(\beta)$ and $\beta_{|_{[p,n]}} = \gamma_{p,q}$. Since β is orientation-reversing, we get $\mathrm{dom}\,(\beta) \subseteq [p,n]$. It follows that $\mathrm{dom}\,(\beta) = [p,n] = \mathrm{dom}\,(\gamma_{p,q})$, and so $\beta = \beta_{|_{[p,n]}} = \gamma_{p,q}$.

Suppose that α is orientation-reversing, and so β is orientation-preserving. Then α is order-reversing on both $[p, q-1]$ and $[q, n]$. Since

$$p-1 = ((p+1)\alpha)\beta \leq (p+1)\alpha \leq p\alpha = p,$$

it follows from the injectivity of α on $[p, n]$ that $(p+1)\alpha = (p-1)$. By continuing in this fashion, we obtain $(p+i)\alpha = (p-i)$ for each $1 \leq i \leq q-p-1$. One can similarly continue and prove that $\alpha = \gamma_{p,q}$, and so $\gamma_{p,q}$ is undecomposable in $\mathcal{PORD}_n$. $\square$

If $n - \lfloor \frac{n}{3} \rfloor \leq r \leq n-1$, then by Proposition 1, notice that G_n of $\mathcal{PORD}(n, r)$. Now we state and prove one of the main results of this paper.

Theorem 8 *$E_r \cup F_r \cup G_n$ is a minimal generating set of $\mathcal{PORD}(n, r)$, and so*

$$\operatorname{rank}(\mathcal{PORD}(n, r)) = \binom{n}{r} 2^{n-r} + \frac{(2n-r-1)(r-2)}{2} + \frac{n(n-3)}{2}$$

for each $n - \lfloor \frac{n}{3} \rfloor \leq r \leq n-1$.

Proof Let $n - \lfloor \frac{n}{3} \rfloor \leq r \leq n-1$. Since $\mathcal{POPD}(n, r) = \langle E_r \cup F_r \rangle$ and $\mathcal{PORD}^*_n = \mathcal{PORD}(n, r) \setminus \mathcal{POPD}(n, r)$, it is enough to show that $\mathcal{PORD}^*_n \subseteq \langle \mathcal{POPD}(n, r) \cup G_n \rangle$. Let $\alpha \in \mathcal{PORD}^*_n$ with $\operatorname{ord}(\alpha) = m$ and $\operatorname{im}(\alpha) = \{a_1, a_2, \ldots, a_t\}$ where $3 \leq t \leq r$. Then we have the following sequence:

$$1 \leq a_s < a_{s-1} < \cdots < a_1 < a_t < a_{t-1} < \cdots < a_{s+1} \leq m+1.$$

for an unique $1 \leq s \leq t-1$. If we take $A_1 = \{x \in X_m : x\alpha = a_1\}$, $A_i = a_i\alpha^{-1}$ for each $2 \leq i \leq t$ and $A_{t+1} = \{x \in [m+1, n] : x\alpha = a_1\}$ (A_{t+1} may be the empty set), then α can be written in the following form:

$$\alpha = \left(\begin{array}{cccc|cccc} A_1 & A_2 & \cdots & A_s & A_{s+1} & \cdots & A_t & A_{t+1} \\ a_1 & a_2 & \cdots & a_s & a_{s+1} & \cdots & a_t & a_1 \end{array} \right)$$

with the property that either

- $(A_1, \ldots, A_s, A_{s+1}, \ldots, A_t)$, whenever $A_{t+1} = \emptyset$, or
- $(A_1, \ldots, A_s, A_{s+1}, \ldots, A_t, A_{t+1})$, whenever $A_{t+1} \neq \emptyset$,

is a convex and ordered partition of $\operatorname{dom}(\alpha)$. In addition, since α is order-decreasing and $\min(A_{s+1}) = m+1$, if we take $a_1 = p$ to simplify notation, then we observe that

(*i*) $a_i \leq p - i + 1$ for every $1 \leq i \leq s$,

(*ii*) $p + i - 1 \leq \min(A_i)$ for every $1 \leq i \leq s$,

(*iii*) $a_{s+j} \leq m + 2 - j$ for every $1 \leq j \leq t - s$, and

(*iv*) $m + j \leq \min(A_{s+j})$ for every $1 \leq j \leq t - s$.

After all these observations, we define the mappings

$$\beta = \left(\begin{array}{cccccccc|c} A_1 & A_2 & \cdots & A_s & A_{s+1} & A_{s+2} & \cdots & A_t & A_{t+1} \\ p & p+1 & \cdots & p+s-1 & m+1 & m+2 & \cdots & m+t-s & p \end{array} \right)$$

and

$$\delta = \left(\begin{array}{cccccccc} p-s+1 & \cdots & p-1 & p & m-t+s+2 & \cdots & m & m+1 \\ a_s & \cdots & a_2 & a_1 & a_t & \cdots & a_{s+2} & a_{s+1} \end{array} \right).$$

By (ii), since $p+s-1 \leq \min(A_s) \leq m$, it follows from (ii) and (iv) that $\beta \in \mathcal{POPD}(n,r)$. By (iii), since $p = a_1 < a_t \leq m-t+s+2$, i.e. $p < m-t+s+2$, it follows from (i) and (iii) that $\delta \in \mathcal{PC}(n,r) \subseteq \mathcal{POPD}(n,r)$. Also we define

$$\gamma = \left(\begin{array}{cccc|cccc} p & p+1 & \cdots & p+s-1 & m+1 & m+2 & \cdots & m+t-s \\ p & p-1 & \cdots & p-s+1 & m+1 & m & \cdots & m-t+s+2 \end{array}\right). \quad (6)$$

Since $|\mathrm{im}\,(\gamma)| = t \geq 3$, it is clear that $\gamma \in \mathcal{PORD}_n^*$ and γ is an injective mapping with $\mathrm{fix}\,(\gamma) = \{p, m+1\}$, and that $\alpha = \beta\gamma\delta$. If $\gamma \in G_n$, then the proof is completed. Let $Y = \mathrm{dom}\,(\gamma)$. Since $p+s-1 \leq \min(A_s) \leq m$ and $1 \leq a_s \leq p-s+1$, we have $s-1 \leq \min\{p-1, m-p\}$, and since $m+t-s \leq \min(A_t) \leq n$ and $p = a_1 < a_t \leq m-t+s+2$, we have $t-s-1 \leq \min\{n-(m+1), (m+1)-p-1\}$. Thus, it follows from the definition of $\gamma_{p,m+1}$ which is given in (5) that $Y \subseteq \mathrm{dom}\,(\gamma_{p,m+1})$, and that $\gamma = 1_Y\gamma_{p,m+1}$ which proves that $\gamma \in \langle \mathcal{POPD}(n,r) \cup G_n\rangle = \langle E_r \cup F_r \cup G_n\rangle$, and so $\mathcal{PORD}(n,r) = \langle E_r \cup F_r \cup G_n\rangle$.

Finally, it follows from Propositions 4, 5 and 6 that $E_r \cup F_r \cup G_n$ is a minimal generating set. Moreover, since E_r, F_r and G_n are disjoint subsets, we have $\mathrm{rank}\,(\mathcal{PORD}(n,r)) = \binom{n}{r}2^{n-r} + \frac{(2n-r-1)(r-2)}{2} + \frac{n(n-3)}{2}$, as claimed. $\square$

Since $\mathrm{rank}\,(\mathcal{POPD}_n) = \frac{n^2+n+2}{2}$, we also have the following immediate corollary.

Corollary 9 $\mathrm{rank}\,(\mathcal{PORD}_n) = n^2 - n + 1$. $\square$

If $p = n - 2\lfloor \frac{n}{3} \rfloor - 1$ and $q = n - \lfloor \frac{n}{3} \rfloor$, then, since $|\mathrm{im}\,(\gamma_{p,q})| = n - \lfloor \frac{n}{3} \rfloor$, we observe that G_n is not a subset of $\mathcal{PORD}(n,r)$, whenever $1 \leq r < n - \lfloor \frac{n}{3} \rfloor$.

Let $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$ and let $1 \leq p \leq q-2 \leq n-2$. If $\alpha \in \mathcal{PORD}(n,r)$ is injective with $\mathrm{fix}\,(\alpha) = \{p,q\}$ and if we take $s = |\mathrm{dom}\,(\alpha) \cap [p, q-1]|$ and $u = |\mathrm{dom}\,(\alpha) \cap [q,n]|$, then we notice that $s \leq \min\{p, q-p, r-1\}$ and $u \leq \min\{r-s, q-p, n-q+1\}$. Now, for each $1 \leq s \leq \min\{p, q-p, r-1\}$, we let $u = \min\{r-s, q-p, n-q+1\}$ and define the mapping

$$\gamma_{p,q}^{r,s} = \left(\begin{array}{cccc|cccc} p & p+1 & \cdots & p+s-1 & q & q+1 & \cdots & q+u-1 \\ p & p-1 & \cdots & p-s+1 & q & q-1 & \cdots & q-u+1 \end{array}\right). \quad (7)$$

Since $|\mathrm{im}\,(\gamma_{p,q}^{r,s})| = s+u \leq s+(r-s) = r$, we notice that $\gamma_{p,q}^{r,s} \in \mathcal{PORD}(n,r)$. For each $1 \leq p \leq n-2$, we also notice that $\gamma_{p,n}^{r,1} = \left(\begin{array}{c|c} p & n \\ p & n \end{array}\right) \in \mathcal{POPD}(n,r)$. However, if we let

$$H_n^r = \{\gamma_{p,q}^{r,s} : 1 \leq p \leq q-2 \leq n-2,\ \ 1 \leq s \leq \min\{p, q-p, r-1\} \text{ and } (q,s) \neq (n,1)\},$$

then it is easy to see that H_n^r is a subset of $\mathcal{PORD}_n^*$ since $u \geq 2$, whenever $s = 1$. Furthermore, since $p+1 = q-(q-p)+1 \leq q-u+1$, by (7), it is also easy to see that every element of H_n^r is injective, that is H_n^r is a subset of $\mathcal{IORD}(n,r)$. With these notations, we have the following proposition:

Proposition 10 *Let* $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$ *and* $1 \leq p \leq q-2 \leq n-2$, *and let* A *be a generating set of* $\mathcal{PORD}(n,r)$. *Then, for each* $1 \leq s \leq \min\{p, q-p, r-1\}$, *if* $(q,s) \neq (n,1)$, *then there exists an* $\alpha \in A$ *such that* $\alpha_{|_Y} = \gamma_{p,q}^{r,s}$, *where* $Y = \mathrm{dom}\,(\gamma_{p,q}^{r,s})$.

Proof Let $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$. Then, there exist $\alpha_1, \ldots, \alpha_k \in A$ such that $\gamma_{p,q}^{r,s} = \alpha_1 \cdots \alpha_k$. By Lemma 2, we have $\{p,q\} \subseteq \mathrm{fix}\,(\alpha_i)$ for each $1 \leq i \leq k$. Moreover, note that $(q,s) \neq (n,1)$ justifies that $\gamma_{p,q}^{r,s}$ is orientation-reserving but not orientation-preserving. So, since the products of orientation-preserving mappings are also orientation-preserving, there exists $1 \leq j \leq k$ such that α_j is orientation-reversing, and so by Proposition 3, $\mathrm{fix}\,(\alpha_j) = \{p, q\}$. Without loss of generality, we suppose that $\beta = \alpha_1 \cdots \alpha_{j-1}$ is orientation-preserving. (We take $\beta = 1_n$ if $j = 1$.) Since $\gamma_{p,q}^{r,s}$ is order-decreasing and injective, it follows that the restriction of β to $Y = \mathrm{dom}\,(\gamma_{p,q}^{r,s})$ is order-decreasing, orientation-preserving and injective, and so we immediately have $\beta_{|_Y} = 1_Y$. Thus, we also have the equality $\gamma_{p,q}^{r,s} = \alpha_j \cdots \alpha_k$. Since $\mathrm{dom}\,(\alpha_j) \subseteq [p, n]$ and $\mathrm{fix}\,(\alpha_j) = \{p, q\}$, it is clear that α_j is order-reversing on both $[p, p+s-1] \subseteq \mathrm{dom}\,(\alpha) \cap [p, q-1]$ and $[q, q+u-1] \subseteq \mathrm{dom}\,(\alpha) \cap [q, n]$. Since α_j is also injective on Y, it follows that $p - 1 \leq (p + 1)\alpha_j < p\alpha_j = p$, i.e. $(p + 1)\alpha_j = p - 1$, whenever $s \geq 2$. If we continue in this fashion, we obtain the equality $\alpha_{j|_Y} = \gamma_{p,q}^{r,s}$, as claimed. $\square$

Now we have the following similar theorem to Theorem 8.

Theorem 11 *Let $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$. Then $E_r \cup F_r \cup H_n^r$ is a minimal generating set of $\mathcal{PORD}(n, r)$ for each $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$.*

Proof With the notations and assumptions given in the proof of Theorem 8, it is enough to show that $\gamma \in \langle \mathcal{POPD}(n, r) \cup H_n^r \rangle$ since $\alpha = \beta\gamma\delta$. As defined in (6) and (7), it is clear that $1 \leq s \leq \min\{p, m+1-p, r-1\}$ and that $t-s \leq u = \min\{r-s, m+1-p, n-m\}$ since $3 \leq t \leq r$, and so $Y = \mathrm{dom}\,(\gamma) \subseteq \mathrm{dom}\,(\gamma_{p,m+1}^{r,s})$. Now it is clear from their definitions that $\gamma = 1_Y \gamma_{p,m+1}^{r,s}$, and so $\mathcal{PORD}(n, r) = \langle E_r \cup F_r \cup H_n^r \rangle$. Therefore, by Propositions 4, 5 and 10, $E_r \cup F_r \cup H_n^r$ is a minimal generating set. $\square$

Recall that $E(\mathcal{I}_n) = \{1_Y : \emptyset \neq Y \subseteq X_n\} \cup \{0_n\}$ is the set of all idempotents of $\mathcal{I}_n$. For $1 \leq r \leq n - 1$ and $2 \leq a \leq n$, let Y be any subset of $X_n \setminus \{a - 1, a\}$ with size $r - 1$ and let $\delta_Y^a : Y \cup \{a\} \to X_n$ be defined by

$$x\delta_Y^a = \begin{cases} x & \text{if} \quad x \in Y \\ a - 1 & \text{if} \quad x = a \end{cases}$$

for all $x \in Y \cup \{a\}$. Moreover, for $1 \leq r \leq n - 1$, let

$$\begin{aligned} EI_r &= \{1_Y : Y \text{ is a subset of } X_n \text{ with size } r\} \text{ and let} \\ FI_r &= \{\delta_Y^a : 2 \leq a \leq n \text{ and } Y \subseteq X_n \setminus \{a - 1, a\} \text{ with } |Y| = r - 1\}. \end{aligned}$$

For $1 \leq r \leq n - 1$, it is proven in [2, Theorem 4] that $EI_r \cup FI_r$ is the unique minimal generating set of $\mathcal{IC}(n, r)$, and that $\mathrm{rank}\,(\mathcal{IC}(n, r)) = \binom{n}{r} + r\binom{n-1}{r}$. In particular, it is concluded in [2, Corollary 5] that $\mathrm{rank}\,(\mathcal{IC}_n) = 2n$.

For $2 \leq k \leq n - 1$, let Z be any subset of $X_n \setminus \{1, n\}$ with size $k - 1$. If $\max(Z) = a$ and $\min(Z) = b$, then let $\zeta_Z : Z \cup \{a + 1\} \to X_n$ be defined by

$$x\zeta_Z = \begin{cases} x & \text{if} \quad x \in Z \\ b - 1 & \text{if} \quad x = a + 1 \end{cases}$$

for all $x \in Z \cup \{a + 1\}$. Moreover, for $2 \leq k \leq n - 1$, let

$$\begin{aligned} GI_k &= \{\zeta_Z : Z \text{ is a subset of } X_n \setminus \{1, n\} \text{ with size } k - 1\} \text{ and let} \\ GI_k^c &= \{\zeta_Z : Z \text{ is a convex subset of } X_n \setminus \{1, n\} \text{ with size } k - 1\}, \end{aligned}$$

which is a subset of GI_k. For $1 \leq r \leq n - 1$, it is proven in [2, Theorem 8] that $EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI_k^c \right)$ is the unique minimal generating set of $\mathcal{IOPD}(n, r)$, and that $\mathrm{rank}\,(\mathcal{IOPD}(n, r)) = \binom{n}{r} + n\binom{n-2}{r-1} + (r - 2)n - \frac{r^2-r-2}{2}$. In particular, it is concluded in [2, Corollary 9] that $\mathrm{rank}\,(\mathcal{IOPD}_n) = \frac{n^2+n+2}{2}$.

Now we state the following lemma without a proof since the proofs of [2, Lemmas 3 and 7] are also valid in $\mathcal{IORD}(n, r)$.

Lemma 12 *(i) For $1 \leq r \leq n-1$, each element of $EI_r \cup FI_r$ is undecomposable in $\mathcal{IORD}(n,r)$.*

(ii) For $2 \leq r \leq n-1$, each element of $GI_r \cup \left(\bigcup_{k=2}^{r-1} GI_k^c\right)$ is undecomposable in $\mathcal{IORD}(n,r)$. $\square$

Next we let

$$\mathcal{IORD}_n^* = \mathcal{IORD}_n \setminus \mathcal{IOPD}_n.$$

Since the proof of Proposition 6 is also valid in $\mathcal{IORD}(n,r)$, we just state the following proposition.

Proposition 13 *Let A be a generating set of $\mathcal{IORD}(n,r)$. Then, for every $1 \leq p \leq n-2$ and $p+2 \leq q \leq n$, there exists an $\alpha \in A \cap \mathcal{IORD}^*(n,r)$ such that* $\mathrm{fix}\,(\alpha) = \{p,q\}$. $\square$

Moreover, we have the following lemma.

Lemma 14 *For $1 \leq p \leq n-2$ and $p+2 \leq q \leq n$, $\gamma_{p,q}$ is undecomposable in $\mathcal{IORD}_n$.*

Proof Let $1 \leq p \leq q-2 \leq n-2$ and let $Y = \mathrm{dom}\,(\gamma_{p,q})$. If $q-1, n \in Y$, then it follows from Lemma 7 that $\gamma_{p,q}$ is undecomposable in $\mathcal{PORD}_n$, and so in $\mathcal{IORD}_n$. Thus, it is enough to consider the cases: $q-1 \notin Y$ or $n \notin Y$.

Suppose that $q-1 \notin Y$, i.e. $k = \min\{p-1, q-p-1\} = p-1 < q-p-1$ as defined in (5). Thus, $p+k = 2p-1 \in Y$, i.e. $Y \cap [p, q-1] = [p, 2p-1]$ and $(2p-1)\gamma_{p,q} = 1$. Then assume that there are $\alpha, \beta \in \mathcal{IORD}_n$ such that $\gamma_{p,q} = \alpha\beta$. By Lemma 2, we have $\mathrm{fix}\,(\gamma_{p,q}) = \{p,q\} \subseteq \mathrm{fix}\,(\alpha) \cap \mathrm{fix}\,(\beta)$. Moreover, notice that $Y \subseteq \mathrm{dom}\,(\alpha)$, and that α is injective on Y since $\gamma_{p,q}$ is injective. Suppose α is orientation-preserving, then β is orientation-reversing. In fact, as shown in the proof of Lemma 7, it is similarly shown that $\alpha_{|_Y} = 1_Y$, and so $Y \subseteq \mathrm{dom}\,(\beta)$ and $1_Y\beta = \gamma_{p,q}$. Since $\beta \in \mathcal{IORD}_n^*$ with $\mathrm{fix}\,(\beta) = \{p,q\}$, it follows from the definition of $\gamma_{p,q}$, given in (5), that $|\mathrm{dom}\,(\beta)| = |\mathrm{im}\,(\beta)| \leq |\mathrm{im}\,(\gamma_{p,q})| = |Y|$, and so $\mathrm{dom}\,(\beta) = Y$, Thus, we have $\gamma_{p,q} = \beta$.

Suppose that α is orientation-reversing. Since $\mathrm{fix}\,(\alpha) = \{p,q\}$, it similarly follows that $\mathrm{dom}\,(\alpha) = Y$. Since the mappings are injective, as shown in the proof of Lemma 7, we have $\gamma_{p,q} = \alpha$. Thus, $\gamma_{p,q}$ is undecomposable in $\mathcal{IORD}_n$.

Since the proof for $n \notin Y$ is dual to the proof for $q-1 \notin Y$, the proof is completed. $\square$

With above notations, we are now able to give the following result:

Theorem 15 *$EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI_k^c\right) \cup G_n$ is the unique minimal generating set of $\mathcal{IORD}(n,r)$, and so*

$$\mathrm{rank}\,(\mathcal{IORD}(n,r)) = \binom{n}{r} + n\binom{n-2}{r-1} + (r-2)n - \frac{r^2-r-2}{2} + \frac{n(n-3)}{2}$$

for each $n - \lfloor \frac{n}{3} \rfloor \leq r \leq n-1$. In particular, $\mathrm{rank}\,(\mathcal{IORD}_n) = n^2 - n + 1$.

Proof Let $n - \lfloor \frac{n}{3} \rfloor \leq r \leq n-1$. Since $\mathcal{IOPD}(n,r) = \langle EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI_k^c\right)\rangle$, it is enough to show that $\mathcal{IORD}_n^* \subseteq \langle \mathcal{IOPD}(n,r) \cup G_n \rangle$. Let $\alpha \in \mathcal{IORD}_n^*$ with

$\operatorname{ord}(\alpha) = m$ and $\operatorname{im}(\alpha) = \{a_1, a_2, \ldots, a_t\}$ where $3 \le t \le r$. Then we have the following sequence:

$$1 \le a_s < a_{s-1} < \cdots < a_1 < a_t < a_{t-1} < \cdots < a_{s+1} \le m+1.$$

for an unique $1 \le s \le t-1$. If we take $b_i = a_i\alpha^{-1}$ for each $1 \le i \le t$, then α can be written in the following tabular form:

$$\alpha = \left(\begin{array}{cccc|ccc} b_1 & b_2 & \cdots & b_s & b_{s+1} & \cdots & b_t \\ a_1 & a_2 & \cdots & a_s & a_{s+1} & \cdots & a_t \end{array} \right)$$

with the property that $1 \le b_1 < \cdots < b_s < b_{s+1} = m+1 < b_{s+2} < \cdots < b_t$. In addition, if we take $a_1 = p$, then we observe that

(*i*) $a_i \le p - i + 1$ for every $1 \le i \le s$,

(*ii*) $p + i - 1 \le b_i$ for every $1 \le i \le s$,

(*iii*) $a_{s+j} \le m + 2 - j$ for every $1 \le j \le t - s$, and

(*iv*) $m + j \le b_{s+j}$ for every $1 \le j \le t - s$.

After all these observations, if we define the mapping

$$\beta = \left(\begin{array}{ccccccc} b_1 & b_2 & \cdots & b_s & b_{s+1} & b_{s+2} & \cdots & b_t \\ p & p+1 & \cdots & p+s-1 & m+1 & m+2 & \cdots & m+t-s \end{array} \right)$$

and consider δ as defined in the proof of Theorem 8, then it similarly follows from (*ii*) and (*iv*) that $\beta \in \mathcal{IC}(n,r)$, and it similarly follows from (*i*) and (*iii*) that $\delta \in \mathcal{IC}(n,r)$. Moreover, if we consider γ as defined in (6), then it is clear that $\gamma \in \mathcal{IORD}_n^*$, $\operatorname{fix}(\gamma) = \{p, m+1\}$, and $\alpha = \beta\gamma\delta$. Thus, it similarly follows from the definition of $\gamma_{p,m+1}$, which is given in (5), Lemmas 12 and 14 that $EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI_k^c \right) \cup G_n$ is the unique minimal generating set of $\mathcal{IORD}(n,r)$, and so by [2, Theorem 8], we have

$$\operatorname{rank}(\mathcal{IORD}(n,r)) = \binom{n}{r} + n\binom{n-2}{r-1} + (r-2)n - \frac{r^2 - r - 2}{2} + \frac{n(n-3)}{2}.$$

In particular, $\operatorname{rank}(\mathcal{IORD}_n) = \frac{n^2+n+2}{2} + \frac{n(n-3)}{2} = n^2 - n + 1$ since $\mathcal{IORD}_n = \mathcal{IORD}(n, n-1) \cup \{1_n\}$ and 1_n is undecomposable in $\mathcal{IORD}_n$. $\square$

Let $3 \le r < n - \lfloor \frac{n}{3} \rfloor$ and let $1 \le p \le q - 2 \le n - 2$. For each $1 \le s \le \min\{p, q-p, r-1\}$, consider $\gamma_{p,q}^{r,s}$, as defined in (7), and H_n^r. Since the proofs of Proposition 10 and Theorem 11 are also valid for the following theorem, we omit its proof.

Theorem 16 *Let* $3 \le r < n - \lfloor \frac{n}{3} \rfloor$. *Then* $EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI_k^c \right) \cup H_n^r$ *is a minimal generating set of* $\mathcal{IORD}(n,r)$. $\square$

Notice that $\gamma_{5,7}^{5,1} = \left(\begin{array}{c|cc} 5 & 7 & 8 \\ 5 & 7 & 6 \end{array} \right) \in \mathcal{IORD}(9,5)$ is not undecomposable in $\mathcal{IORD}(9,5)$ since $1_Y \left(\begin{array}{cc|cc} 5 & 6 & 7 & 8 \\ 5 & 4 & 7 & 6 \end{array} \right) = \gamma_{5,7}^{5,1}$, where $Y = \{5,7,8\}$. However, we have the following lemma.

Lemma 17 *Let* $3 \le r < n - \lfloor \frac{n}{3} \rfloor$*,* $1 \le p \le n-2$ *and* $p+2 \le q \le n$*. For each* $1 \le s \le \min\{p, q-p, r-1\}$*, if* $\min\{r-s, q-p, n-q+1\} = r-s$*, then* $\gamma_{p,q}^{r,s}$ *is undecomposable in* $\mathcal{IORD}(n,r)$*.*

Proof We have $1 \le p \le q-2 \le n-2$. Let $Y = \mathrm{dom}\,(\gamma_{p,q}^{r,s})$ and $Z = \mathrm{im}\,(\gamma_{p,q}^{r,s})$. For each $1 \le s \le \min\{p, q-p, r-1\}$, if $\min\{r-s, q-p, n-q+1\} = r-s$, then by (7), we first notice that $(q,s) \neq (n,1)$, and that

$$\gamma_{p,q}^{r,s} = \left(\begin{array}{cccc|cccc} p & p+1 & \cdots & p+s-1 & q & q+1 & \cdots & q+r-s-1 \\ p & p-1 & \cdots & p-s+1 & q & q-1 & \cdots & q-r+s+1 \end{array} \right),$$

and so $|\mathrm{dom}\,(\gamma_{p,q}^{r,s})| = r$. Assume that there are $\alpha, \beta \in \mathcal{IORD}(n,r)$ such that $\gamma_{p,q}^{r,s} = \alpha\beta$. Then it is clear that $\mathrm{dom}\,(\alpha) = Y$ and $\mathrm{im}\,(\beta) = Z$. Then, after all these experiences, one can easily show that either $\alpha = 1_Y$ and $\beta = \gamma_{p,q}^{r,s}$, or $\alpha = \gamma_{p,q}^{r,s}$ and $\beta = 1_Z$. $\square$

4 Maximal subsemigroups of $\mathcal{PORD}(n,r)$ and $\mathcal{IORD}(n,r)$

In this section, we will characterize the maximal subsemigroups of $\mathcal{PORD}(n,r)$ as well as of $\mathcal{IORD}(n,r)$ for $3 \le r \le n$.

For $1 \le p \le n-2$ and $p+1 \le q \le p+r-2 \le n-1$, let

$$F_{p,q}^r = \{\, \alpha \in \mathcal{PORD}(n,r) : \alpha_{|_{[p,q+1]}} = \xi_{p,q}^r \}$$

and for $1 \le p \le q-2 \le n-2$ with $(p,q) \neq (1,n)$, let

$$G_{p,q} = \{\alpha \in \mathcal{PORD}^*(n,r) : \alpha_{|_{\mathrm{dom}\,(\gamma_{p,q})}} = \gamma_{p,q}\}.$$

If $3 \le r < n - \lfloor \frac{n}{3} \rfloor$, then let

$$H_{p,q}^{r,s} = \{\alpha \in \mathcal{PORD}^*(n,r) : \alpha_{|_{\mathrm{dom}\,(\gamma_{p,q}^{r,s})}} = \gamma_{p,q}^{r,s}\}$$

for $1 \le p \le q-2 \le n-2$ and $1 \le s \le \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$.

Lemma 18 *Let* $n - \lfloor \frac{n}{3} \rfloor \le r \le n-1$*,* $1 \le p \le n-2$*, and* $p+1 \le q \le p+r-2 \le n-1$*. Then* $\mathcal{PORD}(n,r) \setminus F_{p,q}^r$ *is a maximal subsemigroup of* $\mathcal{PORD}(n,r)$*.*

Proof Assume that there are $\alpha, \beta \in \mathcal{PORD}(n,r) \setminus F_{p,q}^r$ such that $\alpha\beta \in F_{p,q}^r$. Then $\alpha\beta_{|_{[p,q+1]}} = \xi_{p,q}^r$, i.e. $1_{[p,q+1]}\alpha\beta = \xi_{p,q}^r$, and so by Lemma 2, we have $x\alpha = x$ and $x\beta = x$ for all $x \in [p,q]$. Therefore, we obtain $\alpha_{|_{[p,q]}} = \beta_{|_{[p,q]}} = \xi_{p,q}{}_{|_{[p,q]}}$. From $p = (q+1)\xi_{p,q}^r = (q+1)1_{[p,q+1]}\alpha\beta = (q+1)\alpha\beta$ and the injectivity of $\xi_{p,q}^r$ on $[p+1, q+1]$, we obtain $(q+1)\alpha \in \{p, q+1\}$. If $(q+1)\alpha = p$, then $\alpha_{|_{[p,q+1]}} = \xi_{p,q}^r$, which contradicts $\alpha \notin F_{p,q}^r$. If $(q+1)\alpha = q+1$, then $(q+1)\beta = p$, and so $\beta_{|_{[p,q+1]}} = \xi_{p,q}^r$, which contradicts $\beta \notin F_{p,q}^r$.

Let $\alpha \in F_{p,q}^r$. Since $\alpha_{|_{[p,q+1]}} = \xi_{p,q}^r$, we have $\xi_{p,q}^r = 1_{[p,q+1]}\alpha \in \langle (\mathcal{PORD}(n,r) \setminus F_{p,q}^r) \cup \{\alpha\} \rangle$, and so $E_r \cup F_r \cup G_n \subseteq \langle (\mathcal{PORD}(n,r) \setminus F_{p,q}^r) \cup \{\alpha\} \rangle$. Therefore, $\mathcal{PORD}(n,r) \setminus F_{p,q}^r$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$ since $\langle E_r \cup F_r \cup G_n \rangle = \mathcal{PORD}(n,r)$ by Theorem 8. $\square$

Lemma 19 *Let* $n - \lfloor \frac{n}{3} \rfloor \le r \le n-1$ *and* $1 \le p \le q-2 \le n-2$ *with* $(p,q) \neq (1,n)$*. Then* $\mathcal{PORD}(n,r) \setminus G_{p,q}$ *is a maximal subsemigroup of* $\mathcal{PORD}(n,r)$*.*

Proof Assume that there are $\alpha, \beta \in \mathcal{PORD}(n,r) \setminus G_{p,q}$ such that $\alpha\beta \in G_{p,q}$. Then $\alpha\beta_{|_{\mathrm{dom}\,(\gamma_{p,q})}} = \gamma_{p,q}$, i.e. $1_{\mathrm{dom}\,(\gamma_{p,q})}\alpha\beta = \gamma_{p,q}$, and so by Lemma 2, we have $x\alpha = x$ and $x\beta = x$ for all $x \in \{p, q\}$ since $\mathrm{fix}\,(\gamma_{p,q}) = \{p, q\}$. Then, since $\mathrm{dom}\,(\gamma_{p,q}) \subseteq \mathrm{dom}\,(\alpha)$, we have

$$\alpha_{|_{[p,p+k]}} = \begin{pmatrix} p & p+1 & \cdots & p+k \\ p & p-1 & \cdots & p-k \end{pmatrix} \quad \text{or} \quad \alpha_{|_{[p,p+k]}} = 1_{[p,p+k]}.$$

On the other hand

$$\alpha_{|_{[q,q+l]}} = \begin{pmatrix} q & q+1 & \cdots & q+l \\ q & q-1 & \cdots & q-l \end{pmatrix} \quad \text{or} \quad \alpha_{|_{[q,q+l]}} = 1_{[q,q+l]}.$$

Since $\alpha \in \mathcal{PORD}(n,r)$, we can conclude that $\alpha_{|_{\mathrm{dom}\,(\gamma_{p,q})}}$ is the partial identity on $\mathrm{dom}\,(\gamma_{p,q})$ or $\alpha_{|_{\mathrm{dom}\,(\gamma_{p,q})}} = \gamma_{p,q}$. Since $\alpha \notin G_{p,q}$, we obtain $\alpha_{|_{\mathrm{dom}\,(\gamma_{p,q})}} = 1_{\mathrm{dom}\,(\gamma_{p,q})}$. Hence, $\gamma_{p,q} = \alpha\beta_{|_{\mathrm{dom}\,(\gamma_{p,q})}} = \beta_{|_{\mathrm{dom}\,(\gamma_{p,q})}}$, which contradicts $\beta \notin G_{p,q}$. Consequently, we obtain that $\mathcal{PORD}(n,r) \setminus G_{p,q}$ is a subsemigroup of $\mathcal{PORD}(n,r)$.

Now, let $\alpha \in G_{p,q}$. Since $\alpha_{|_{\mathrm{dom}\,(\gamma_{p,q})}} = \gamma_{p,q}$, we have $\gamma_{p,q} = 1_{\mathrm{dom}\,(\gamma_{p,q})}\alpha \in \langle(\mathcal{PORD}(n,r) \setminus G_{p,q}) \cup \{\alpha\}\rangle$, and so $E_r \cup F_r \cup G_n \subseteq \langle(\mathcal{PORD}(n,r) \setminus G_{p,q}) \cup \{\alpha\}\rangle$. Therefore, $\mathcal{PORD}(n,r) \setminus G_{p,q}$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$ since $\langle E_r \cup F_r \cup G_n\rangle = \mathcal{PORD}(n,r)$ by Theorem 8. $\square$

Similarly to Lemma 19, one can prove the following.

Lemma 20 *Let $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$, $1 \leq p \leq q-2 \leq n-2$, and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$. Then $\mathcal{PORD}(n,r) \setminus H_{p,q}^{r,s}$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$.* $\square$

Now, we can characterize the maximal subsemigroups of $\mathcal{PORD}(n,r)$ for $3 \leq r \leq n$.

Theorem 21 *Let $n - \lfloor \frac{n}{3} \rfloor \leq r \leq n-1$ and let S be a subsemigroup of $\mathcal{PORD}(n,r)$. Then S is a maximal subsemigroup of $\mathcal{PORD}(n,r)$ if and only if S belongs to one of the following types:*

(1) *$S = \mathcal{PORD}(n,r) \setminus \{\varepsilon\}$, for each $\varepsilon \in E_r$.*

(2) *$S = \mathcal{PORD}(n,r) \setminus F_{p,q}^r$, for all $1 \leq p \leq n-2$ and $p+1 \leq q \leq p+r-2 \leq n-1$.*

(3) *$S = \mathcal{PORD}(n,r) \setminus G_{p,q}$, for all $1 \leq p \leq q-2 \leq n-2$ with $(p,q) \neq (1,n)$.*

Proof Let $\varepsilon \in E_r$. By Proposition 4, since ε is undecomposable in $\mathcal{PORD}(n,r)$, we can conclude that $\mathcal{PORD}(n,r) \setminus \{\varepsilon\}$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$. By Lemma 18, we know that $\mathcal{PORD}(n,r) \setminus F_{p,q}^r$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$ for all $1 \leq p \leq n-2$ and $p+1 \leq q \leq p+r-2 \leq n-1$. By Lemma 19, we know that $\mathcal{PORD}(n,r) \setminus G_{p,q}$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$ for all $1 \leq p \leq q-2 \leq n-2$.

Conversely, let S be a maximal subsemigroup of $\mathcal{PORD}(n,r)$. Suppose that $E_r \nsubseteq S$, i.e. there is $\varepsilon \in E_r \setminus S$. Since $\mathcal{PORD}(n,r) \setminus \{\varepsilon\}$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$, we obtain that $S = \mathcal{PORD}(n,r) \setminus \{\varepsilon\}$.

Suppose that $E_r \subseteq S$ and $F_{p,q}^r \cap S = \emptyset$ for some $1 \leq p \leq n-2$ and $p+1 \leq q \leq p+r-2 \leq n-1$. Since $\mathcal{PORD}(n,r) \setminus F_{p,q}^r$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$, we obtain that $S = \mathcal{PORD}(n,r) \setminus F_{p,q}^r$.

Suppose that $E_r \subseteq S$ and $F_{p,q}^r \cap S \neq \emptyset$, i.e. S contains at least one $\alpha \in \mathcal{PORD}(n,r)$ such that $\alpha_{|_{[p,q+1]}} = \xi_{p,q}^r$, for all $1 \leq p \leq n-2$ and $p+1 \leq q \leq p+r-2 \leq n-1$. Thus, since $1_{[p,q+1]} \in \mathcal{PC}(n,r) = \langle E_r\rangle \subseteq S$, we have $1_{[p,q+1]}\alpha = \xi_{p,q}^r \in S$ for all $1 \leq p \leq n-2$ and $p+1 \leq q \leq p+r-2 \leq n-1$. Therefore, we obtain $F_r \subseteq S$. Since $\mathcal{PORD}(n,r) = \langle E_r \cup F_r \cup G_n \rangle$ and $E_r \cup F_r \subseteq S$, we have that $G_n \nsubseteq S$.

Assume that $G_{p,q} \cap S \neq \emptyset$, i.e. S contains at least one $\alpha \in \mathcal{PORD}^*(n,r)$ such that $\alpha|_{\mathrm{dom}\,(\gamma_{p,q})} = \gamma_{p,q}$, for all $1 \leq p \leq q-2 \leq n-2$ with $(p,q) \neq (1,n)$. Thus, since $1_{\mathrm{dom}\,(\gamma_{p,q})} \in S$, we have $\gamma_{p,q} = 1_{\mathrm{dom}\,(\gamma_{p,q})}\alpha \in S$ for all $1 \leq p \leq q-2 \leq n-2$ with $(p,q) \neq (1,n)$. Therefore, we obtain $G_n \subseteq S$, which is a contradiction. Hence, there are $1 \leq p \leq q-2 \leq n-2$ with $(p,q) \neq (1,n)$ such that $G_{p,q} \cap S = \emptyset$. Finally, we obtain $S = \mathcal{PORD}(n,r) \setminus G_{p,q}$, since $\mathcal{PORD}(n,r) \setminus G_{p,q}$ is a maximal subsemigroup of $\mathcal{PORD}(n,r)$. □

Clearly, $\mathcal{PORD}_n = \mathcal{PORD}(n,n-1) \cup \{1_n\}$. Hence, we obtain the following result:

Theorem 22 *Let S be a subsemigroup of $\mathcal{PORD}_n$. Then S is a maximal subsemigroup of $\mathcal{PORD}_n$ if and only if $S = \mathcal{PORD}(n,n-1)$ or there exists a maximal subsemigroup T of $\mathcal{PORD}(n,n-1)$ such that $S = T \cup \{1_n\}$.* □

Similarly to Theorem 21, one can prove the following theorem.

Theorem 23 *Let $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$ and let S be a subsemigroup of $\mathcal{PORD}(n,r)$. Then S is a maximal subsemigroup of $\mathcal{PORD}(n,r)$ if and only if S belongs to one of the following types:*

(1) *$S = \mathcal{PORD}(n,r) \setminus \{\varepsilon\}$, for each $\varepsilon \in E_r$.*

(2) *$S = \mathcal{PORD}(n,r) \setminus F^r_{p,q}$, for all $1 \leq p \leq n-2$ and $p+1 \leq q \leq p+r-2 \leq n-1$.*

(3) *$S = \mathcal{PORD}(n,r) \setminus H^{r,s}_{p,q}$, for all $1 \leq p \leq q-2 \leq n-2$ and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$.* □

Now, we will characterize the maximal subsemigroups of $\mathcal{IORD}(n,r)$.

First, let $n - \lfloor \frac{n}{3} \rfloor \leq r \leq n-1$. Since $EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI^c_k \right) \cup G_n$ is a generating set of $\mathcal{IORD}(n,r)$ consisting entirely of undecomposable elements, by Lemmas 12 and 14, we get immediately the following:

Theorem 24 *Let $n - \lfloor \frac{n}{3} \rfloor \leq r \leq n-1$ and let S be a subsemigroup of $\mathcal{IORD}(n,r)$. Then S is a maximal subsemigroup of $\mathcal{IORD}(n,r)$ if and only if $S = \mathcal{IORD}(n,r) \setminus \{\alpha\}$, for each $\alpha \in EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI^c_k \right) \cup G_n$.* □

Clearly, $\mathcal{IORD}_n = \mathcal{IORD}(n,n-1) \cup \{1_n\}$. Hence, we obtain the following result:

Theorem 25 *Let S be a subsemigroup of $\mathcal{IORD}_n$. Then S is a maximal subsemigroup of $\mathcal{IORD}_n$ if and only if $S = \mathcal{IORD}_n \setminus \{\alpha\}$ for each $\alpha \in EI_{n-1} \cup FI_{n-1} \cup GI_{n-1} \cup \left(\bigcup_{k=2}^{n-2} GI^c_k \right) \cup G_n \cup \{1_n\}$.* □

Now, let $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$. For $1 \leq p \leq q-2 \leq n-2$ and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$, let

$$HI^{r,s}_{p,q} = H^{r,s}_{p,q} \cap \mathcal{IORD}(n,r).$$

Lemma 26 *Let $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$, $1 \leq p \leq q-2 \leq n-2$, and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$. Then $\mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}$ is a maximal subsemigroup of $\mathcal{IORD}(n,r)$.*

Proof Let $Y = \mathrm{dom}\,(\gamma^{r,s}_{p,q}) = [p, p+s-1] \cup [q, q+u-1]$, where $u = \min\{r-s, q-p, n-q+1\}$. Obviously, we have $HI^{r,s}_{p,q} = \{\gamma^{r,s}_{p,q}\}$ if $\min\{r-s, q-p, n-q+1\} = r-s$, i.e. $|Y| = r$. So in this case, $\mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q} = \mathcal{IORD}(n,r) \setminus \{\gamma^{r,s}_{p,q}\}$ is a maximal subsemigroup of $\mathcal{IORD}(n,r)$, since $\gamma^{r,s}_{p,q}$ is undecomposable in $\mathcal{IORD}(n,r)$ by Lemma 17.

Let now $|Y| < r$. Assume that there are $\alpha, \beta \in \mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}$ such that $\alpha\beta \in HI^{r,s}_{p,q}$. Then $\alpha\beta_{|_Y} = \gamma^{r,s}_{p,q}$, i.e. $1_Y\alpha\beta = \gamma^{r,s}_{p,q}$, and so by Lemma 2, we have $x\alpha = x$ and $x\beta = x$ for all $x \in \{p,q\}$ since $\mathrm{fix}\,(\gamma^{r,s}_{p,q}) = \{p,q\}$. Then, since $\mathrm{dom}\,(\gamma^{r,s}_{p,q}) = [p, p+s-1] \cup [q, q+u-1] \subseteq \mathrm{dom}\,(\alpha)$, we have

$$\alpha_{|_{[p,p+s-1]}} = \begin{pmatrix} p & p+1 & \cdots & p+s-1 \\ p & p-1 & \cdots & p-s+1 \end{pmatrix} \quad \text{or} \quad \alpha_{|_{[p,p+s-1]}} = 1_{[p,p+s-1]}.$$

On the other hand

$$\alpha_{|_{[q,q+u-1]}} = \begin{pmatrix} q & q+1 & \cdots & q+u-1 \\ q & q-1 & \cdots & q-u+1 \end{pmatrix} \quad \text{or} \quad \alpha_{|_{[q,q+u-1]}} = 1_{[q,q+u-1]}.$$

Since $\alpha \in \mathcal{IORD}(n,r)$, we can conclude that $\alpha_{|_Y}$ is the partial identity on $\mathrm{dom}\,(\gamma^{r,s}_{p,q})$ or $\alpha_{|_Y} = \gamma^{r,s}_{p,q}$. Since $\alpha \notin HI^{r,s}_{p,q}$, we obtain $\alpha_{|_Y} = 1_{\mathrm{dom}\,(\gamma^{r,s}_{p,q})}$. Hence, $\gamma^{r,s}_{p,q} = \alpha\beta_{|_Y} = \beta_{|_{\mathrm{dom}\,(\gamma^{r,s}_{p,q})}} = \gamma^{r,s}_{p,q}$, which contradicts $\beta \notin HI^{r,s}_{p,q}$. Consequently, we obtain that $\mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}$ is a subsemigroup of $\mathcal{IORD}(n,r)$.

Let $\alpha \in HI^{r,s}_{p,q}$. Since $\alpha_{|_Y} = \gamma^{r,s}_{p,q}$, we have $\gamma^{r,s}_{p,q} = 1_Y\alpha \in \langle(\mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}) \cup \{\alpha\}\rangle$, and so $EI_r \cup FI_r \cup GI_r \cup \left(\bigcup\limits_{k=2}^{r-1} GI^c_k\right) \cup H^r_n \subseteq \langle(\mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}) \cup \{\alpha\}\rangle$. Therefore, $\mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}$ is a maximal subsemigroup of $\mathcal{IORD}(n,r)$ since $\langle EI_r \cup FI_r \cup GI_r \cup \left(\bigcup\limits_{k=2}^{r-1} GI^c_k\right) \cup H^r_n\rangle = \mathcal{IORD}(n,r)$ by Theorem 16. □

Theorem 27 *Let $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$ and let S be a subsemigroup of $\mathcal{IORD}(n,r)$. Then S is a maximal subsemigroup of $\mathcal{IORD}(n,r)$ if and only if S belongs to one of the following types:*

(1) $S = \mathcal{IORD}(n,r) \setminus \{\alpha\}$, for each $\alpha \in EI_r \cup FI_r \cup GI_r \cup \left(\bigcup\limits_{k=2}^{r-1} GI^c_k\right)$.

(2) $S = \mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}$, for all $1 \leq p \leq q-2 \leq n-2$ and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$.

Proof Let $\alpha \in EI_r \cup FI_r \cup GI_r \cup \left(\bigcup\limits_{k=2}^{r-1} GI^c_k\right)$. Since α is undecomposable in $\mathcal{IORD}(n,r)$ by Lemma 12, we can conclude that $\mathcal{IORD}(n,r) \setminus \{\alpha\}$ is a maximal subsemigroup of $\mathcal{IORD}(n,r)$. By Lemma 26, we know that $\mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}$ is a maximal subsemigroup of $\mathcal{IORD}(n,r)$, for all $1 \leq p \leq q-2 \leq n-2$ and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$.

Conversely, let S be a maximal subsemigroup of $\mathcal{IORD}(n,r)$. Suppose that $(EI_r \cup FI_r \cup GI_r \cup \left(\bigcup\limits_{k=2}^{r-1} GI^c_k\right)) \nsubseteq S$, i.e. there is $\alpha \in (EI_r \cup FI_r \cup GI_r \cup \left(\bigcup\limits_{k=2}^{r-1} GI^c_k\right)) \setminus S$. Since $\mathcal{IORD}(n,r) \setminus \{\alpha\}$ is a maximal subsemigroup of $\mathcal{IORD}(n,r)$, we obtain that $S = \mathcal{IORD}(n,r) \setminus \{\alpha\}$.

Suppose that $(EI_r \cup FI_r \cup GI_r \cup \left(\bigcup\limits_{k=2}^{r-1} GI^c_k\right)) \subseteq S$ and $HI^{r,s}_{p,q} \cap S \neq \emptyset$, i.e. S contains at least one $\alpha \in \mathcal{IORD}^*(n,r)$ such that $\alpha_{|_{\mathrm{dom}\,(\gamma^{r,s}_{p,q})}} = \gamma^{r,s}_{p,q}$, for all $1 \leq p \leq$

$q-2 \leq n-2$ and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$. Thus, since $1_{\mathrm{dom}\,(\gamma^{r,s}_{p,q})} \in IC(n,r) = \langle EI_r \cup FI_r \rangle \subseteq S$, we have $\gamma^{r,s}_{p,q} = 1_{\mathrm{dom}\,(\gamma^{r,s}_{p,q})}\alpha \in S$, for all $1 \leq p \leq q-2 \leq n-2$ and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$. Therefore, we obtain $H^r_n \subseteq S$. Since $\mathcal{IORD}(n,r) = \langle EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI^c_k\right) \cup H^r_n \rangle$ and $(EI_r \cup FI_r \cup GI_r \cup \left(\bigcup_{k=2}^{r-1} GI^c_k\right)) \subseteq S$, we obtain a contradiction with maximality of S. Hence, there are $1 \leq p \leq q-2 \leq n-2$ and $1 \leq s \leq \min\{p, q-p, r-1\}$ with $(q,s) \neq (n,1)$ such that $HI^{r,s}_{p,q} \cap S = \emptyset$. Finally, we obtain $S = \mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}$, since $\mathcal{IORD}(n,r) \setminus HI^{r,s}_{p,q}$ is a maximal subsemigroup of $\mathcal{IORD}(n,r)$. $\square$

Since we have not found any explicit formula for the cardinality of $|H^r_n|$, we have not written the ranks of $\mathcal{PORD}(n,r)$ and $\mathcal{IORD}(n,r)$ for the case $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$.

Open Problem. Does there exist an explicit formula for the cardinality of $|H^r_n|$ for each $3 \leq r < n - \lfloor \frac{n}{3} \rfloor$?

Declaration of Interest

No potential conflict of interest was reported by the authors.

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