SELFLESS INCLUSIONS OF C^*-ALGEBRAS

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Abstract. We introduce and study a natural notion of selflessness for inclusions of C*-probability spaces, which in particular implies that all intermediate C*-algebras are selfless in the sense of [Rob25]. We identify natural sources of selfless inclusions in the realms of $\mathcal{Z}$-stable and free product C*-algebras. As an application of this, we prove selflessness for a new family of C*-probability spaces outside the regime of free products and group C*-algebras. These include the reduced free unitary compact quantum groups.

1. Introduction

We continue the recent program on *selflessness* in C*-algebra theory by extending this notion to inclusions and by identifying new examples of selfless C*-probability spaces arising from quantum groups. Selfless C*-probability spaces were introduced by the fourth author in [Rob25]. This framework unravels certain old problems in C*-algebra theory, as demonstrated in the recent work of Amrutam, Gao, Kunnawalkam Elayavalli, and Patchell [AGKEP25]. Since then, rapid progress has been made in the field, resulting in the development of several new techniques and applications [KES25, Vig25, HKER25, RTV25, Oza25]. Notably, Ozawa very recently discovered a new approach to proving selflessness, and used it to resolve several open questions concerning this notion [Oza25].

We introduce and study a natural notion of *selfless inclusions* of C*-probability spaces, extending [Rob25]. We call an inclusion of C*-probability spaces $B \subset (A, \rho)$ selfless if there exist a free ultrafilter ω and a C*-probability space (C, κ), with $C \neq \mathbb{C}$, such that the first factor embedding $(B \subset A) \to (B * C \subset A * C)$ is existential (Definition 2.3). Roughly put, while the notion of selfless C*-probability space asserts the existence of nontrivial elements in an ultrapower A^ω that are free from A in a nondegenerate way, the notion of selfless inclusion additionally keeps track of a C*-subalgebra B of A from which these freely independent elements are sourced.

If an inclusion $B \subset (A, \rho)$ is selfless, then all the intermediate C*-subalgebras $B \subset C \subset A$ are selfless. This gives a way to identify selfless C*-algebras by realizing them as intermediate C*-subalgebras of selfless inclusions. Note also that, since selfless C*-algebras are simple, every intermediate C*-subalgebra of a selfless inclusion is simple. Thus, selfless inclusions are C*-irreducible in the sense investigated by Rørdam in [Rør23b].

We prove several results on selfless inclusions that closely parallel results from [Rob25] on selfless C*-probability spaces. We also show that some of Ozawa's methods from [Oza25] can be straightforwardly adapted to prove selflessness for purely infinite and for $\mathcal{Z}$-stable inclusions in the sense of Sarkowicz [Sar25]; see Theorems 2.7 and 2.9.

In [Oza25], Ozawa defines the PHP (Powers–Haagerup–Pisier) property for groups and shows that groups with this property are C*-selfless. Here we formulate this property in the context of an inclusion and show that it again implies selflessness. Very recently, Flores, Klisse, Ó Cobhthaigh, and Pagliero proved that reduced free products satisfying an Avitzour-type condition are selfless [FKÓCP25]. We revisit their argument here, more explicitly framing it as a verification of the PHP property of an inclusion, and adapting it to suit our application to free complexifications (Theorem 3.4).

The free complexification construction was introduced by Banica in the context of compact quantum groups [Ban08]. Some notable compact quantum groups, such as the free unitary compact quantum group, can be described as free complexifications [Ban97]. We show that, under Avitzour-type conditions, (reduced) free complexifications arise as intermediate C*-subalgebras of a selfless inclusion, and are therefore selfless. This applies to the reduced free unitary quantum group:

Theorem 1.1 (Theorem 4.2)**.** *The reduced free unitary compact quantum group $A_u(n)$ is selfless for $n \geq 2$.*

By [Rob25, Theorem 3.1] (see also [BS16, FHL+21]), this yields the following corollary:

Corollary 1.2. *For $n \geq 2$, $A_u(n)$ has stable rank one and strict comparison of positive elements by its unique trace.*

We also show, with the same strategy, that the reduced half-liberated unitary quantum group $A_u^*(n)$ introduced in [BDD11] is selfless.

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2. Selfless Inclusions

We work in the setting of C*-probability spaces (A, ρ); i.e., unital C*-algebras endowed with a distinguished state. We always assume that the state induces a faithful GNS representation of the C*-algebra, with the exception of ultrapower states. Morphisms between C*-probability spaces are unital *-homomorphisms preserving the respective states. We often omit reference to the state of a C*-probability space, e.g., in the formation of reduced free products, if it can be inferred from the context. We note however that all free products below are reduced free products with respect to the given states.

Given a C*-probability space (A, ρ) and an ultrafilter ω (over some set), we obtain the ultrapower C*-probability space (A^ω, ρ^ω), where A^ω is the C*-algebra ultrapower of A and ρ^ω the limit state (along ω) induced by ρ.

By an inclusion of C*-probability spaces we understand a pair of C*-probability spaces (A, ρ) and $(B, \rho|_B)$, where B is a unital C*-subalgebra of A and the state on B is the restriction of the state on A. To denote it, we write $B \subset A$ or $B \subset (A, \rho)$. By an embedding of $B_1 \subset (A_1, \rho_1)$ in $B_2 \subset (A_2, \rho_2)$ we understand a unital embedding of C*-algebras $\theta\colon A_1 \to A_2$ that is state preserving, i.e. $\rho_2\theta = \rho_1$, and such that $\theta(B_1) \subseteq B_2$.

Let $\theta\colon (B_1 \subset (A_1, \rho_1)) \to (B_2 \subset (A_2, \rho_2))$ be an embedding of inclusions. We call θ existential if there exists an ultrafilter ω and an embedding

$$\sigma\colon (B_2 \subset (A_2, \rho_2)) \to (B_1^\omega \subset (A_1^\omega, \rho_1^\omega))$$

such that $\sigma \circ \theta$ agrees with the diagonal inclusion of A_1 in A_1^ω. Equivalently, there exists a C*-algebra embedding $\sigma\colon A_2 \to A_1^\omega$ such that σ is state preserving, $\sigma \circ \theta$ agrees with the diagonal inclusion, and $\sigma(B_2) \subseteq B_1^\omega$. Note that in this case the induced embeddings $\theta\colon A_1 \to A_2$ and $\theta|_{B_1}\colon B_1 \to B_2$ are both existential embeddings of C*-probability spaces. See [Rob25, Section 1] for a discussion of existential embeddings of C*-probability spaces.

The following lemma is elementary.

Lemma 2.1. *Let θ_1 and θ_2 be embeddings between inclusions of C*-probability spaces such that $\theta_2 \circ \theta_1$ is well-defined.*

(i) If θ_1 and θ_2 are existential, then $\theta_2 \circ \theta_1$ is also existential.
(ii) If $\theta_2 \circ \theta_1$ is existential, then so is θ_1.

We also have the following corollary of "free exactness" (cf. [Rob25, Corollary 1.9]).

Theorem 2.2. *A reduced free product of existential embeddings of inclusions is again existential. That is, given inclusions $B_i \subset (A_i, \rho_i)$, $D_i \subset (C_i, \psi_i)$ for $i = 1, 2$, and existential embeddings*

$$\theta_i\colon (B_i \subset (A_i, \rho_i)) \to (D_i \subset (C_i, \psi_i))$$

for $i = 1, 2$, the induced map

$$\theta_1 * \theta_2\colon (B_1 * B_2 \subset A_1 * A_2) \to (D_1 * D_2 \subset C_1 * C_2)$$

is existential.

Definition 2.3. *We call an inclusion of C*-probability spaces $B \subset (A, \rho)$ selfless if there exist a free ultrafilter ω and a C*-probability space (C, κ), with $C \neq \mathbb{C}$, such that the first factor embedding of inclusions*

$$\theta\colon (B \subset A) \to (B * C \subset A * C)$$

is existential. More concretely, $B \subset A$ is selfless if there exist an ultrafilter ω, a C-probability space (C, κ) with $C \neq \mathbb{C}$, and a state-preserving embedding of C*-algebras $\sigma\colon A * C \to A^\omega$, such that $\sigma \circ \theta$ agrees with the diagonal inclusion and $\sigma(B * C) \subseteq B^\omega$ (equivalently, $\sigma(C) \subseteq B^\omega$).*

Notice that a C*-probability space (A, ρ) is selfless in the sense of [Rob25] if and only if $A \subset (A, \rho)$ is a selfless inclusion, and that if $B \subset (A, \rho)$ is a selfless inclusion, then both (A, ρ) and $(B, \rho|_B)$ are selfless C*-probability spaces.

Lemma 2.4. *Let $\phi\colon (B_1 \subset A_1) \to (B_2 \subset A_2)$ be an existential embedding of inclusions and suppose that $B_2 \subset A_2$ is selfless. Then so is $B_1 \subset A_1$.*

Proof. By assumption, there exist $C \neq \mathbb{C}$ such that the first factor embedding $\theta\colon (B_2 \subset A_2) \to (B_2 * C \subset A_2 * C)$ is existential. Thus, the embedding

$$\theta \circ \phi\colon (B_1 \subset A_1) \to (B_2 * C \subset A_2 * C)$$

is existential (Lemma 2.1 (i)). Since $\theta\circ\phi$ ranges in $B_1 * C \subset A_1 * C$, its corestriction to $B_1 * C \subset A_1 * C$ is also existential (Lemma 2.1 (ii)). This shows that $B_1 \subset A_1$ is selfless. $\square$

In the following theorem we sometimes omit reference to the state, but the reader should bear in mind that we work in the category of C*-probability spaces and state preserving unital *-homomorphisms.

Theorem 2.5. *Let $B \subset (A, \rho)$ be a selfless inclusion of C*-probability spaces. The following are true:*

- *(i) If $B \subset C \subset A$, then $B \subset C$ and $C \subset A$ are selfless. In particular, C is selfless.*
- *(ii) Given an inclusion of C*-probability spaces $D_1 \subset D_2$, the reduced free product inclusion $B * D_1 \subset A * D_2$ is selfless.*
- *(iii) The C*-probability space (C, κ) witnessing selflessness of $B \subset (A, \rho)$ in Definition 2.3 may be chosen to be $(C^*_r(\mathbb{F}_\infty), \tau)$.*
- *(iv) Choose any $2/\sqrt{5} < \alpha < 1$. Then for every $a \in A$ there exist unitaries $u_1, \ldots, u_5 \in B$ such that*

$$\Big\| \frac{1}{5} \sum_{i=1}^{5} u_i a u_i^* - \rho(a)1 \Big\| \leq \alpha \|a\|. \tag{1}$$

  Consequently, the inclusion $B \subset A$ has the relative Dixmier property with respect to ρ; i.e., $\rho(a)1 \in \overline{\mathrm{co}}\{uau^* : u \in U(B)\}$ for all $a \in A$.
- *(v) Assume that ρ is faithful. Then ρ can be B-excised in the following sense: there exists a net $(b_\lambda)_\lambda$ of positive elements of norm 1 in B such that $\|b_\lambda^{\frac{1}{2}} a b_\lambda^{\frac{1}{2}} - \rho(a) b_\lambda\| \to 0$ and $\rho(a b_\lambda a^*) \to 0$ for all $a \in A$.*

Proof. (i) This is straightforward.

(ii) Since $B \subset A$ is selfless, the first factor embedding $(B \subset A) \to (B*C \subset A*C)$ is existential for some nontrivial C. Since the reduced free product of existential embeddings (of inclusions) is again existential (Theorem 2.2), and the identity embedding $(D_1 \subset D_2) \to (D_1 \subset D_2)$ is trivially existential, the embedding of inclusions

$$(B * D_1 \subset A * D_2) \to (B * C * D_1 \subset A * C * D_2)$$

is existential. It follows that $B * D_1 \subset A * D_2$ is a selfless inclusion.

(iii) Let $C \neq \mathbb{C}$ be such that the first factor embedding $(B \subset A) \to (B*C \subset A*C)$ is existential. Taking reduced free product with the identity $(C \subset C) \to (C \subset C)$, we get that

$$(B * C \subset A * C) \to (B * C * C \subset A * C * C)$$

is existential, and so

$$(B \subset A) \to (B * C * C \subset A * C * C)$$

is existential. Applying this argument repeatedly, we get that

$$(B \subset A) \to (B * C^{*n} \subset A * C^{*n})$$

is existential. For large enough n, C^{*n} contains an embedding of $C_r^*(\mathbb{F}_\infty)$ (by [Rob25, Lemma 2.5]). Thus,

$$(B \subset A) \to (B * C_r^*(\mathbb{F}_\infty) \subset A * C_r^*(\mathbb{F}_\infty))$$

is existential, as desired.

(iv) Avitzour's proof of simplicity of the reduced free product $A * C_r^*(\mathbb{F}_\infty)$ finds for every $a \in A \subseteq A * C_r^*(\mathbb{F}_\infty)$ unitaries $u_1, u_2, \ldots, u_5 \in C_r^*(\mathbb{F}_\infty)$ such that (1) holds with $\alpha = \frac{2}{\sqrt{5}}$ (see [Avi82, Proposition 3.1]). Applying $\sigma\colon A * C_r^*(\mathbb{F}_\infty) \to A^\omega$, and using that the image of $C_r^*(\mathbb{F}_\infty)$ is contained in B^ω, we get (1) with $\alpha = \frac{2}{\sqrt{5}}$ and unitaries in B^ω. Lifting these unitaries to sequences of unitaries in B, we obtain (1) with any $\frac{2}{\sqrt{5}} < \alpha < 1$ and unitaries in B.

(v) It will suffice to show that for a set $\mathcal{S}$ whose linear span is dense in A, and for every finite set $F \subseteq \mathcal{S}$ and $\epsilon > 0$, there exists $b \in B_+$ of norm 1 such that $\|b^{\frac{1}{2}} a b^{\frac{1}{2}} - \rho(a) b\| < \epsilon$ for all $a \in F$. We will show this with

$$\mathcal{S} = \{1\} \cup \{a \in A_+ : \|a\| = 1 \text{ and } ad = 0 \text{ for some nonzero } d \in A_+\}.$$

Given $a \in A_+$, define $d_\rho(a) = \lim_n \rho(a^{\frac{1}{n}})$. Note that if $ad = 0$ for some nonzero $d \in A_+$, then $d_\rho(a) + d_\rho(d) = d_\rho(a + d) \leq 1$. On the other hand, since ρ is faithful, $d_\rho(d) > 0$. It follows that if $a \in \mathcal{S}$, then $d_\rho(a) < 1$.

Let $\epsilon > 0$ and $F \subseteq \mathcal{S}$. Choose $\delta > 0$ such that $d_\rho(a) < 1 - \delta$ for all $a \in F$. Choose a $c \in C_r^*(\mathbb{F}_\infty)_+$ of norm 1 such that $d_\tau(c) < \min(\delta, \epsilon^2)$, which exists since $C_r^*(\mathbb{F}_\infty)$ is simple and nonelementary. Then, in $A * C_r^*(\mathbb{F}_\infty)$ we have

$$\|c^{\frac{1}{2}}(a - \rho(a))c^{\frac{1}{2}}\| < \epsilon$$

for all $a \in \{1\} \cup F$, by [Rob25, Lemma 8.1]. Also, by the free independence of A and c, we have

$$(\rho * \tau)(aca) = \rho(a^2)\tau(c) < \epsilon^2,$$

for all $a \in \{1\} \cup F$. Let $\sigma\colon A * C_r^*(\mathbb{F}_\infty) \to A^\omega$ be such that $\sigma \circ \theta$ is the diagonal inclusion of A in A^ω and $\sigma(C_r^*(\mathbb{F}_\infty)) \subseteq B^\omega$. Set $b = \sigma(c) \in B^\omega$, and lift b to $(b_i)_i$, where each b_i is a positive element of norm 1. Then $\|b_i^{\frac{1}{2}}(a - \rho(a))b_i^{\frac{1}{2}}\| < \epsilon$ and $\rho(ab_ia) < \epsilon^2$ for all $a \in \{1\} \cup F$ and ω-almost all i. □

An inclusion $B \subset (A, \rho)$ is said to have relative excision, in the sense of Rørdam, if ρ can be excised (in A) with a net of positive contractions coming from B [Rør23b, Definition 3.14]. (We also say in this case that ρ is B-excised.) Rørdam showed that if B is simple, unital, and infinite dimensional, and Γ a discrete group with an outer action on B, then $B \subset B \rtimes_r \Gamma$ has relative excision with respect to every state that factors through the conditional expectation onto B [Rør23b, Lemma 5.7]. Theorem 2.5 (v) implies that if the inclusion $B \subset (A, \rho)$ is selfless and ρ is faithful, then $B \subset (A, \rho)$ has relative excision.

We require the following lemma due to Ozawa.

Lemma 2.6. *([Oza25, Lemma 12]). Let (A, ρ) be a C^*-probability space and $(\mathcal{T}, \omega)$ the Toeplitz probability space (the universal C^*-algebra generated by an isometry). Then $A * \mathcal{T}$ is the universal C^*-algebra generated by A and an isometry T (the generator of the Toeplitz algebra) that satisfies $T^*aT = \rho(a)1$ for $a \in A$. The free* *product state ψ on $A * \mathcal{T}$ is the unique state that satisfies $\psi|_A = \rho$ and $\psi(aTT^*a^*) = 0$ for all $a \in A$.*

Proof. See Example 4.6.11 and Exercise 4.8.1 in [BO08]. □

Theorem 2.7. *Let $B \subset (A, \rho)$ be an inclusion such that B is simple and purely infinite and ρ can be B-excised by a net of positive contractions (b_λ) also satisfying that $\rho(ab_\lambda a^*) \to 0$ for all $a \in A$. Then $B \subset (A, \rho)$ is selfless.*

Proof. We follow Ozawa's proof that simple purely infinite C*-algebras are selfless [Oza25, Theorem 3].

Using that B has real rank zero, we can assume that b_λ has finite spectrum (and norm 1) for all λ. Then, with p_λ the projection $1_{\{1\}}(b_\lambda)$ obtained by functional calculus, we have $b_\lambda^{\frac{1}{2}} p_\lambda = p_\lambda$. We deduce that (p_λ) still excises ρ and $\rho(ap_\lambda a^*) \to 0$ for all $a \in A$.

Using now that B is simple and purely infinite, we choose isometries (T_λ) in B such that $T_\lambda T_\lambda^* \leq p_\lambda$ for all λ. The net (T_λ) still excises ρ and satisfies that $\rho(aT_\lambda T_\lambda^* a^*) \to 0$ for all a. Choose a free ultrafilter ω over the index set for λ containing the Fréchet filter. Letting $T \in B^\omega$ be the image of (T_λ), we obtain an isometry such that $T^*aT = \rho(a)1$ and $\rho(aTT^*a^*) = 0$ for all $a \in A$. It follows by Lemma 2.6 that there exists a state preserving isomorphism $\sigma \colon A * \mathcal{T} \to C^*(A, T)$ that restricts to the identity on A (with $\mathcal{T}$ denoting the Toeplitz algebra endowed with the vacuum state). Since $\sigma(\mathcal{T}) \subseteq B^\omega$, this shows that $B \subset (A, \rho)$ is a selfless inclusion. □

Note that the condition $\rho(ab_\lambda a^*) \to 0$ for all $a \in A$ in the theorem above is necessary to reach the conclusion when ρ is faithful, by Theorem 2.5 (v), and may always be necessary. On the other hand, if ρ is a diffuse state, then any sequence of positive contractions excising it must converge weakly to 0 [AAP86, Corollary 2.15]. Thus, in this case, the condition $\rho(ab_\lambda a^*) \to 0$ for all $a \in A$ holds automatically.

Let $\mathcal{Z}$ denote the Jiang–Su C*-algebra. In [Sar25], Sarkowicz calls an inclusion $B \subset A$ $\mathcal{Z}$-stable if it is isomorphic to the tensor inclusion $B \otimes \mathcal{Z} \subset A \otimes \mathcal{Z}$.

Lemma 2.8. *If $B \subset A$ is $\mathcal{Z}$-stable, then the first factor embedding*

$$\theta \colon (B \subset A) \to (B \otimes \mathcal{Z} \subset A \otimes \mathcal{Z})$$

is existential. That is, there exists an embedding $\sigma \colon A \otimes \mathcal{Z} \to A^\omega$, for some ultrafilter ω, such that $\sigma \circ \theta$ agrees with the diagonal inclusion and $\sigma(B \otimes \mathcal{Z}) \subseteq B^\omega$.

Proof. This is well known for a single $\mathcal{Z}$-stable C*-algebra. We now keep track of a C*-subalgebra. Set $D = \mathcal{Z}^{\infty\otimes}$ and $D_n = \mathcal{Z}^{n\otimes} \otimes 1 \otimes \cdots \subseteq D$. Choose "permutation" isomorphisms $\sigma_n \colon D \otimes \mathcal{Z} \to D$ for $n = 1, 2, \ldots$ that are the identity on $D_n \otimes 1$. Now tensor with A to get isomorphisms $\mathrm{id}_A \otimes \sigma_n$ from $(A \otimes D) \otimes \mathcal{Z}$ to $A \otimes D$. Choose a free ultrafilter ω on $\mathbb{N}$ and use $(\sigma_n)_n$ to define $\sigma \colon (A \otimes D) \otimes \mathcal{Z} \to (A \otimes D)^\omega$ such that $\sigma \circ \theta$ is the diagonal inclusion. We have shown that $\theta \colon A \otimes D \to (A \otimes D) \otimes \mathcal{Z}$ is existential. Observe now that the image of $(1 \otimes 1_D) \otimes \mathcal{Z}$ under σ is contained in $(1 \otimes D)^\omega$. It follows that for any $1 \in B \subseteq A$, the first factor embedding

$$(B \otimes D \subset A \otimes D) \to ((B \otimes D) \otimes \mathcal{Z} \subset (A \otimes D) \otimes \mathcal{Z})$$

is existential. But $D \cong \mathcal{Z}$, and $B \subset A$ is isomorphic to $B \otimes \mathcal{Z} \subset A \otimes \mathcal{Z}$ by assumption. The lemma thus follows. □

Theorem 2.9. *Let $B \subset (A, \rho)$ be a $\mathcal{Z}$-stable inclusion of C*-probability spaces that has the relative Dixmier property, where A is exact and ρ is $U(B)$-invariant. Then $B \subset (A, \rho)$ is a selfless inclusion.*

Proof. We follow Ozawa's proof [Oza25, Theorem 3] for the case $B = A$.

Let $(\mathcal{C}, \tau)$ denote the free semicircular system with countably many generators. By the previous lemma, there exists $\sigma_1 \colon A \otimes \mathcal{Z} \to A^\omega$ that is the diagonal inclusion on $A \otimes 1$ and maps $B \otimes \mathcal{Z}$ to B^ω. (Note: from the proof of the lemma, ω can be any free ultrafilter over $\mathbb{N}$.) By [Oza23, Theorem 4.1], $\mathcal{C}$ embeds in $\mathcal{Z}^\omega$. Tensoring with A and using its exactness (see e.g. [Hay15, the proof of Proposition 3.11(ii)], [Mal12, Lemma in the appendix], [vH25, Lemma 2.18]), we obtain an embedding of $A \otimes \mathcal{C}$ in $(A \otimes \mathcal{Z})^\omega$. Composing it with σ_1^ω, we get $\sigma \colon A \otimes \mathcal{C} \to (A^\omega)^\omega \cong A^{\omega \otimes \omega}$ such that $A \ni a \mapsto \sigma(a \otimes 1)$ agrees with the diagonal inclusion and $1 \otimes \mathcal{C}$ is mapped into $B^{\omega\otimes\omega}$. Relabel $\omega \otimes \omega$ as ω (as it will be irrelevant which specific ultrafilter we are working with), so as to now have $\sigma \colon A \otimes \mathcal{C} \to A^\omega$. Let us argue that σ is state preserving; i.e., $\rho^\omega \sigma = \rho \otimes \tau$. For $b \in A' \cap B^\omega$, the functional $A \ni a \mapsto \rho^\omega(ab)$ is $U(B)$ invariant. By the relative Dixmier property of $B \subset A$, any such functional is a scalar multiple of ρ. Hence, $\rho^\omega(ab) = \rho(a)\rho^\omega(b)$ for all $a \in A$, $b \in A' \cap B^\omega$, which implies that

$$\rho^\omega \sigma(a \otimes c) = \rho(a)\rho^\omega(\sigma(1 \otimes c)) = \rho(a)\tau(c),$$

for all $a \in A$ and $c \in \mathcal{C}$ (where we have used the uniqueness of trace in $\mathcal{C}$). Thus, σ is state preserving. This means that the first factor embedding

$$(B \subset (A, \rho)) \to (B \otimes \mathcal{C} \subset (A \otimes \mathcal{C}, \rho \otimes \tau))$$

is an existential embedding of inclusions of C*-probability spaces. Trivially, the diagonal embedding

$$(B \otimes \mathcal{C} \subset A \otimes \mathcal{C}) \to ((B \otimes \mathcal{C})^\omega \subset (A \otimes \mathcal{C})^\omega)$$

is also existential. Composing them, we obtain that the embedding of $B \subset (A, \rho)$ in

$$(B \otimes \mathcal{C})^\omega \subset ((A \otimes \mathcal{C})^\omega, (\rho \otimes \tau)^\omega)$$

is existential. We will be done once we have shown that this embedding can be factored through the first factor embedding of $B \subset A$ in $B * \mathcal{C}_1 \subset A * \mathcal{C}_1$, where $\mathcal{C}_1$ is the C*-probability space generated by a single semicircular element. Put differently, we will be done once we have found a semicircular element $s \in (B \otimes \mathcal{C})^\omega$ such that $C^*(A, s) \cong A * \mathcal{C}_1$.

The proof continues verbatim along the lines of Ozawa's proof, so we will only indicate the main steps. Using the relative Dixmier property of $B \subset A$, we choose collections of unitaries $(u_{i,k})_{k=1}^{N_i}$ in B such that the unitary mixing operators

$$A \ni a \mapsto \frac{1}{N_i} \sum_{k=1}^{N_i} u_{i,k} a u_{i,k}^*$$

converge pointwise to $A \ni a \mapsto \rho(a)1$. We arrange for $(u_{i,k})_{k=1}^{N_i}$ to be closed under conjugation; i.e., $(u_{i,k}^*)_{k=1}^{N_i}$ and $(u_{i,k})_{k=1}^{N_i}$ agree up to a permutation. A similar argument to the previous paragraph shows that the embedding of inclusions

$$(B \subset A) \to (B \otimes C_r^*(\mathbb{F}_\infty) \subset A \otimes C_r^*(\mathbb{F}_\infty))$$

is existential. Thus, by replacing the $u_{i,k}$ with $u_{i,k} \otimes t_k$ (where the t_k are the generators of $\mathbb{F}_\infty$), using the resulting free independence, and applying Voiculescu's inequality [Jun05, Proposition 7.4], we may also arrange for

$$\frac{1}{N_i}\sum_{k=1}^{N_i} u_{i,k} a u_{i,k} \to 0$$

for all $a \in A$ with $\rho(a) = 0$.

Let $l_k \in \mathcal{O}_\infty$ for $k = 1, \dots$ be the isometries generating $\mathcal{O}_\infty$. Identify $\mathcal{C}$ with the C*-subalgebra generated by $\{(l_k + l_k^*)/2 : k = 1, \dots\}$ and τ with the restriction of the vacuum state on $\mathcal{O}_\infty$ to $\mathcal{C}$. Define $T_i \in B \otimes \mathcal{O}_\infty$ as

$$T_i = \frac{1}{\sqrt{N_i}}\sum_{k=1}^{N_i}(u_{i,k} + u_{i,k}^*) \otimes l_k,$$

and

$$s_i = \frac{1}{2}(T_i + T_i^*) \in B \otimes \mathcal{C}.$$

Let $s = (s_i)_i \in (B \otimes \mathcal{C})^\omega$. By Ozawa's proof $s \in (B \otimes \mathcal{C})^\omega$ is a semicircular element such that $C^*(A, s) \cong A * \mathcal{C}_1$. □

Example 2.10. Let $\mathcal{O}_2$ denote the Cuntz algebra generated by two isometries, and let $M_{2^\infty} \subseteq \mathcal{O}_2$ denote the canonically embedded UHF algebra [RS02, 4.2]. Let ρ be the state on $\mathcal{O}_2$ obtained via the expectation $E\colon \mathcal{O}_2 \to M_{2^\infty}$ followed by the trace τ on M_{2^∞}. Then, by Theorem 2.9, the inclusion $M_{2^\infty} \subset (\mathcal{O}_2, \rho)$ is selfless. (That is, the selflessness of $(\mathcal{O}_2, \rho)$ can be certified by Haar unitaries coming from M_{2^∞}.) Indeed, by [Sar25, Example 6.8], the inclusion $M_{2^\infty} \subset \mathcal{O}_2$ is $\mathcal{Z}$-stable. In the lemma below we show that it has the relative Dixmier property. The other conditions of Theorem 2.9 can be routinely checked.

Lemma 2.11. *The inclusion $M_{2^\infty} \subset (\mathcal{O}_2, \rho)$ has the relative Dixmier property.*

Proof. Let s_1, s_2 denote the isometries that generate $\mathcal{O}_2$. It suffices to show that a dense set of elements $a \in \mathcal{O}_2$ can be $U(M_{2^\infty})$-averaged to $(\tau \circ E)(a)$. Thus, we may assume that

$$a = \sum_{k=1}^{m}(s_1^*)^k b_{-k} + b_0 + \sum_{k=1}^{m} b_k s_1^k$$

where $b_k \in M_{2^\infty}$ for all k. Consider a term bs_1^k. For $u \in U(M_{2^\infty})$, we have

$$u^* b s_1^k u = (u^* b \lambda^k(u)) s_1^k$$

where $\lambda\colon M_{2^\infty} \to M_{2^\infty}$ is the Bernoulli right-shift endomorphism. We thus see that the set $M_{2^\infty} s_1^k$ is convex and invariant under conjugation by unitaries in M_{2^∞}. This holds similarly for the terms $(s_1^*)^k b$. Therefore, by a process of successive averaging, we can consider each of the terms of the above sum separately. Since M_{2^∞} has the Dixmier property, b_0 can be $U(M_{2^\infty})$-averaged to $\tau(b_0)$. Let us show that the remaining terms can be $U(M_{2^\infty})$-averaged to 0. Equivalently, we must show that each $b \in M_{2^\infty}$ can be averaged to 0 in the following sense:

$$T_i(b) = \frac{1}{N_i}\sum_{j=1}^{N_i} u_{i,j}^* b \lambda^k(u_{i,j}) \to 0,$$

for $u_{i,j} \in U(M_{2^\infty})$. This is precisely the type of unitary averaging relative to an automorphism studied by Rørdam in [Rør23a], except λ^k is here an endomorphism rather than an automorphism.

Fix $k \in \mathbb{N}$. Set $M_{2^n} := M_2(\mathbb{C})^{\otimes n} \otimes 1 \otimes 1 \cdots \subset M_{2^\infty}$ for all n. Suppose without loss of generality that $b \in M_{2^n}$ for some n. Notice that if we choose the unitaries $u_{i,j}$ of the form $1^{\otimes n} \otimes u'_{i,j}$, so that they belong to $M'_{2^n} \cap M_{2^\infty}$, then $T_i(b) = bT_i(1)$. We thus see that it suffices to consider the case $b = 1$.

Let $\epsilon > 0$. By [RS02, Proposition 5.1.3], there exist $N \in \mathbb{N}$ and projections $p_1, \ldots, p_{2^k} \in M_{2^N}$ such that $1 = \sum_{i=1}^{2^k} p_i$ and $\|\lambda^k(p_i) - p_{i+k}\| < \epsilon$ for all i (with indices taken modulo 2^k). For $\omega = (\omega_i)_{i=1}^{2^k} \in \mathbb{T}^{2^k}$ define $u(\omega) = \sum_{i=1}^{2^k} \omega_i p_i$. Then

$$y := \int_{\mathbb{T}^{2^k}} u(\omega)^* \lambda^k(u(\omega)) d\omega = \sum_{i=1}^{2^k} p_i \lambda^k(p_i)$$

(cf. [Rør23a, Lemma 2.13]). By the orthogonality of the projections $(p_i)_i$, we have $\|y\| = \max_i \|p_i \lambda^k(p_i)\| < \epsilon$, where we have used that $\lambda^k(p_i) \approx_\epsilon p_j$ for $j \neq i$. Since y is a limit of averages of the form $T_i(1)$, and ϵ is arbitrary, we are done. □

3. The PHP property and reduced free products

Following Ozawa in the group case, we define property PHP (Powers–Haagerup–Pisier) for a C^*-probability space (A, ρ) represented faithfully on a Hilbert space H, and more generally, for an inclusion $B \subset (A, \rho)$.

Definition 3.1. *(Cf.* [Oza25, Section 8].*) Let (A, ρ) be a C^*-probability space and assume that $A \subseteq \mathbb{B}(H)$. Let $B \subset A$ be a C^*-subalgebra. Let us say that the inclusion $B \subset (A, \rho)$ has the PHP property if for all finite sets $F \subset \ker(\rho)$, all $\epsilon > 0$, and all $n \in \mathbb{N}$, there exist u_i, P_i, P_i^+, for $i = 1, \ldots, n$ such that $u_i \in U(B)$, $P_i, P_i^+ \in \mathbb{B}(H)$ are projections with $P_i^+ \leq P_i$, and the following hold:*

(1) $(P_i)_{i=1}^n$ is a pairwise orthogonal family of projections;
(2) $u_i P_i^+ u_i^* \leq P_i^+$ and $u_i(1 - P_i)u_i^* \leq P_i^+$ for all i;
(3) $\|P_i x P_j\| < \epsilon$ for all $x \in F$ and all i, j.

We note that to verify the PHP property it suffices to choose the finite sets F from a set $\mathcal{S}$ whose linear span is dense in $\ker \rho$. A further reduction is the following:

Lemma 3.2. *If we can verify the PHP property for $n = 3$, then we can verify it for all $n \in \mathbb{N}$.*

Proof. Note that if property PHP holds for some n, it also holds for all $m \leq n$, by simply discarding some of the P_i, P_i^+, u_i. Thus, we must show that it holds for arbitrarily large n.

Let $F \subseteq \ker \rho$ be a finite set and $\epsilon > 0$. Suppose that we have u_i, P_i, P_i^+ for $i = 1, 2, 3$ that satisfy (1), (2), (3) in the definition of the PHP property. Since $P_j \leq 1 - P_i$ for $i \neq j$, we have by (2) that $u_i P_j u_i^* \leq P_i^+$ for $i \neq j$. Consider the family of six projections $P_{i,j} = u_i P_j u_i^*$, for $i \neq j$. Observe that they are pairwise orthogonal, since if $i \neq i'$ then $P_{i,j}$ and $P_{i',j'}$ are subprojections of P_i and $P_{i'}$, respectively, which are orthogonal, while if $j \neq j'$, then $P_{i,j}$ and $P_{i,j'}$ are conjugates of P_j and $P_{j'}$ by the same unitary. The norm bounds on the cut-downs $P_{i,j} x P_{i',j'}$ continue to hold, since the new projections are subprojections of the original projections. This shows (3) for the new family of projections. Finally, to check (2), choose $P^+_{i,j} = u_i P^+_j u_i^*$ and $u_{i,j} = u_i u_j u_i^*$. More generally, the same procedure allows us to pass from PHP for n to PHP for $n^2 - n$. Thus starting with $n = 3$, we obtain the PHP for arbitrarily large values of n. □

The following holds as in [Oza25, Theorem 14]. We briefly sketch the proof.

Theorem 3.3. *If $B \subset (A, \rho) \subset \mathbb{B}(H)$ has property PHP, then $B \subset (A, \rho)$ is a selfless inclusion.*

Proof. Let $F \subset \ker(\rho)$ be a finite set, $\epsilon > 0$, and $n \in \mathbb{N}$. Choose P_i, P_i^+, u_i for $i = 1, \ldots, n$ that satisfy (1), (2), (3) of the PHP relative to F and ϵ. Define $T \in \mathbb{B}(H)$ by

$$T = \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} (P_i^+ u_i + u_i^*(1 - P_i^+)).$$

Then

$$T + T^* = \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} (u_i + u_i^*) \in B.$$

Moreover, as argued in [Oza25, Theorem 14], we have

$$(1 - \frac{1}{2n}) \le T^* T \le 1$$

and

$$\|T^* x T\| < \epsilon, \quad \rho(x T T^* x^*) < \epsilon \text{ for all } x \in F.$$

By taking a suitable ultralimit, we obtain an isometry $T \in \mathbb{B}(H)^{\mathcal{U}}$ as required in the hypothesis of Lemma 2.6. Thus there is an embedding $A * \mathcal{T}$ into $\mathbb{B}(H)^{\mathcal{U}}$ which restricts to the diagonal embedding on A. This embedding maps $T + T^*$ to a self-adjoint operator in $B^{\mathcal{U}}$, witnessing the selflessness of $B \subset (A, \rho)$. □

It is proved in [FKÓCP25] that reduced free products satisfying an Avitzour-type condition are selfless. In the theorem below we revisit the same argument, which originates in [Oza25], explicitly framing it as a verification of the PHP property. We have tailored the statement of this theorem to suit our application to free complexifications in the next section.

Theorem 3.4. *Let $(A, \rho) = (A_1, \rho_1) * (A_2, \rho_2)$ be a reduced free product. Let $z \in A_1$ and $a, b, c \in A_2$ be unitaries satisfying that*

- $\rho_1(z) = \rho_2(a) = \rho_2(b) = \rho_2(c) = \rho_2(c^{-1}b) = \rho_2(c^{-1}a) = 0,$
- *z belongs to the centralizer of ρ_1, and a, b, c belong to the centralizer of ρ_2.*

Let $B = C^*(z^{-1}az, b, c)$. Then $B \subset (A, \rho)$ has the PHP property, and consequently it is a selfless inclusion.

Before proving this theorem we make some preliminary remarks on reduced words in reduced free products. Let $(A, \rho) = (A_1, \rho_1) * (A_2, \rho_2)$ be as in the theorem, and identify A_1 and A_2 as C*-subalgebras of A. We call $w \in A$ an alternating centered word, or a reduced word, if

$$w = c_1 c_2 \cdots c_l$$

where $c_i \in A_{j(i)} \ominus \mathbb{C}1$ for all i and $j\colon \{1, \ldots, l\} \to \{1, 2\}$ is an alternating function; i.e., $j(i) \neq j(i+1)$ for all $1 \le i \le l-1$. If w is a reduced word, then its representation as an alternating centered product as above is unique. The number of factors (i.e., letters) in this representation is called the length of the word. From the construction of the reduced free product we have that A is spanned by 1 and the reduced words. In the proof below we shall use the following facts, which are easily established by induction.

Fact 1: Let $w, w' \in A$ be reduced words. Then $w \perp w'$ (where $\langle w, w'\rangle := \rho((w')^* w)$) if w and w' have different length, while if they have the same length, and say $w = c_1 c_2 \cdots c_l$, $w' = c_1' c_2' \cdots c_l'$, then

$$\langle w, w'\rangle = \langle c_1, c_1'\rangle \cdots \langle c_l, c_l'\rangle.$$

We conclude that if two reduced words have orthogonal letters at the same position, then they are orthogonal.

Fact 2: If w_1, x, w_2 are reduced words such that w_1 and w_2 are at least as long as x, then every non-scalar summand in the reduced expansion of $w_1 x w_2$ is of one of the following forms:

- a reduced word $w_1' y w_2'$, where w_1' is an initial segment of w_1, w_2' is a terminal segment of w_2, and y is a reduced word (possibly empty),
- an initial segment of w_1,
- a terminal segment of w_2.

Proof of Theorem 3.4. We set $H = L^2(A, \rho)$ and identify A with a C*-subalgebra of $\mathbb{B}(H)$ via the GNS representation.

For a reduced word $w \in A$, denote by $B_w \subseteq H$ the closed linear span of the reduced words that start with w, i.e., of the form $w\eta$ for some reduced word η. Denote by P_w the orthogonal projection onto B_w.

Let $F \subseteq A \ominus \mathbb{C}1$ be a finite set and $\epsilon > 0$. As remarked above, to verify the PHP property it suffices to let F range through finite subsets of a set with dense linear span in $A \ominus \mathbb{C}1$. Thus, we may assume that F consists of reduced words. We can also simultaneously conjugate the elements of F by a unitary leaving $A \ominus \mathbb{C}1$ invariant (i.e., in the centralizer of ρ) and work with this new set. Observe that, given a reduced word x, the element

$$x' = (cz^{-1}az)^{-N_0} x (cz^{-1}az)^{N_0}$$

is a linear combination of nontrivial reduced words, by the invariance of the reduced free product state ρ under conjugation by $z^{\pm 1}$ and a, c. Moreover, using Fact 2 stated above, we can choose a large enough N_0 such that x' is a linear combination of reduced words that either

- start in z^{-1} and end in z, or
- have either the forms $(cz^{-1}az)^l$ or $az(cz^{-1}az)^l$ for $l > 0$,
- are inverses of the words in the previous bullet point.

Let N_0 be large enough so that the above applies to every $x \in F$. Replacing F by $(cz^{-1}az)^{-N_0} F (cz^{-1}az)^{N_0}$, and then replacing this new set by the reduced words supporting its elements, we may (and will) assume that F consists of finitely many words as in the three bullet points above.

To verify the PHP property, we set

$$w_i = (bz^{-1}az)^i (cz^{-1}az)(bz^{-1}az)^{3-i}, \quad w_i^+ = w_i(bz^{-1}az)$$

for $i = 1, 2, 3$. We then choose

$$P_i = P_{w_i}, \quad P_i^+ = P_{w_i^+}, \quad u_i = w_i^+ c^{-1} w_i^*,$$

for $i = 1, 2, 3$ (by a previous lemma, it suffices to verify the PHP with $n = 3$). Note that $u_i \in C^*(z^{-1}az, b, c) = B$ for all i.

To check the orthogonality between the projections P_1, P_2, P_3, we note that the first appearance of the letter c in each word w_1, w_2, w_3 is matched at the same position with a b in the other words. Since $b \perp c$, we deduce from Fact 1 stated above that the words starting in w_i and w_j for $i \neq j$ are orthogonal, i.e., $B_{w_i} \perp B_{w_j}$. This proves (1) of the PHP property.

To prove (2) of the PHP property, fix $1 \leq i \leq 3$ and set $w = w_i$, $w^+ = w_i^+$, and $u_w = u_i$. The second property in the definition of the PHP can be reformulated as $u_w B_{w^+} \subseteq B_{w^+}$ and $u_w(H \ominus B_w) \subseteq B_{w^+}$.

To prove $u_w B_{w^+} \subseteq B_{w^+}$ it suffices to do so on the reduced words spanning B_{w^+}. Let $\xi = w^+\eta$ be a reduced word in B_{w^+}. Then

$$\begin{aligned} u_w\xi &= (w^+c^{-1}w^*)w^+\eta \\ &= (w^+c^{-1}w^*)w(bz^{-1}az)\eta \\ &= w^+(c^{-1}b)z^{-1}az\eta. \end{aligned}$$

Notice that $w^+(c^{-1}b)$ is a reduced word ending in $c^{-1}b \in A_2 \ominus \mathbb{C}$ (since w^+ ends in A_1 and $b \perp c$). So $w^+(c^{-1}b)z^{-1}az\eta$ is already reduced and starts with w^+. This verifies the first inclusion.

Consider now the inclusion $u_w(H \ominus B_w) \subseteq B_{w^+}$. Again, to prove this inclusion it suffices to do so on the reduced words spanning $H \ominus B_w$, i.e., reduced words not starting in w. Take a reduced word $\xi \in H \ominus B_w$. Then

$$u_w\xi = w^+c^{-1}w^*\xi.$$

We claim that $w^*\xi$ belongs to the span of reduced words that start in $A_1 \ominus \mathbb{C}$. To show this, it suffices to check that it is orthogonal to every reduced word that starts in $A_2 \ominus \mathbb{C}$ and to 1. Let ξ' be either 1 or a reduced centered word that starts in $A_2 \ominus \mathbb{C}$. Since w ends in z, $w\xi' \in B_w$, and so $\langle w^*\xi, \xi'\rangle = \langle \xi, w\xi'\rangle = 0$. The claim follows. It now follows that $w^+c^{-1}(w^*\xi)$ belongs to the span of reduced words that start in w^+, as desired.

Finally, let us prove (3) in the definition of the PHP property, i.e., that $P_{w_i}xP_{w_j} = 0$ for all i, j and all $x \in F$. Note that $P_{w_i}xP_{w_j} = 0$ is equivalent to

$$xw_i\eta \perp w_j\eta' \tag{2}$$

for all i, j and reduced words $w_i\eta$ and $w_j\eta'$. Suppose first that x is a reduced word starting in z^{-1} and ending in z. Then xw_i is already a reduced word starting in $z^{-1} \in A_1 \ominus \mathbb{C}$, while $w_j\eta'$ starts in $b \in A_2 \ominus \mathbb{C}$. We thus have the desired orthogonality (2). Suppose that $x = (cz^{-1}az)^l$ for some nonzero $l > 0$. Then $xw_i\eta$ is a reduced word that starts in c, while $w_j\eta'$ starts in b. Since $b \perp c$, we again have (2). Suppose that $x = az(cz^{-1}az)^l$ for $l > 0$. Then $xw_i\eta$ is a reduced word whose third letter is a c, matched in the position by an a in $w_j\eta'$. We thus have (2). Finally, the inverses of these two types of words can be dealt with by rewriting (2) as $w_i\eta \perp x^{-1}w_j\eta'$. □

4. Free complexifications and selfless compact quantum groups

Let us discuss the operation of free complexification, introduced by Banica [Ban08], and further studied in [Rau12], [TW17]. Here we work with reduced free products rather than universal ones, in contrast to [Ban08].

Let (A,ρ) be a C*-probability space. Let $X\subseteq A$ be a set generating A as a unital C*-algebra; i.e., such that $A=C^*(X,1)$. Form the reduced free product $(A,\rho)*(C(\mathbb{T}),\lambda)$ (where λ denotes the state induced by the Lebesgue measure). Let

$$Xz := \{xz : x\in X\}\subseteq A*C(\mathbb{T}),$$

where z is the canonical generator of $C(\mathbb{T})$. We call

$$\widetilde{A}=C^*(1,Xz)\subseteq A*C(\mathbb{T})$$

the free complexification of (A,ρ,X). We endow $\widetilde{A}$ with the state $(\rho*\lambda)\big|_{\widetilde{A}}$ and the generating set Xz. Note that, although not explicitly reflected in our notation, the free complexification $\widetilde{A}$ depends both on ρ and X.

We denote by $\mathcal{P}_XA\subseteq A$ the C*-subalgebra generated by $\{1\}$ and

$$XX^*=\{xy^*: x,y\in X\}.$$

We call $(\mathcal{P}_XA, XX^*)$ the projective version of (A,X) (see [Ban22]). Note that, although it is not generally the case that $A\subseteq\widetilde{A}$, we do have $\mathcal{P}_XA\subseteq\widetilde{A}$. In the theorem below we also make reference to the C*-subalgebra $\mathcal{P}_{X^*}A$ generated by $\{1\}$ and X^*X; i.e., the projective version of (A,X^*). We note that $z^{-1}(\mathcal{P}_{X^*}A)z\subseteq\widetilde{A}$.

Theorem 4.1. *Let (A,X,ρ) be a C*-probability space endowed with a generating set X. Suppose that there exist unitaries $a\in\mathcal{P}_{X^*}A$ and $b,c\in\mathcal{P}_XA$ in the centralizer of ρ and such that $\rho(a)=\rho(b)=\rho(c)=\rho(c^{-1}a)=\rho(c^{-1}b)=0$. Then the inclusion $\widetilde{A}\subset A*C(\mathbb{T})$ is selfless. In particular, the free complexification $\widetilde{A}$ is selfless.*

Proof. Consider the reduced free product $(A,\rho)*(C(\mathbb{T}),\lambda)$. We readily check that the hypotheses of Theorem 3.4 are verified for this free product with $A_1=C(\mathbb{T})$, $A_2=A$, and the unitaries $z\in C(\mathbb{T})$ and $a,b,c\in A$. Thus, defining $B=C^*(z^{-1}az,b,c)$, we deduce that the inclusion $B\subset A*C(\mathbb{T})$ is selfless. On the other hand, it is easily checked that $z^{-1}az,b,c\in\widetilde{A}$. Thus, the inclusion $\widetilde{A}\subset A*C(\mathbb{T})$ is selfless. □

Let us recall the definition of the reduced free unitary and free orthogonal compact quantum groups.

Let $n\in\mathbb{N}$, with $n\geq 2$. The reduced free unitary compact quantum group is obtained as follows: start from the universal C*-algebra $\mathfrak{A}$ generated by elements $\{u_{i,j}: i,j=1,\dots,n\}$ such that, with $u=(u_{ij})_{ij}$, the relations

$$u^*u=uu^*=1_n,\quad (u^t)^*(u^t)=u^t(u^t)^*=1_n$$

are satisfied (where u^t is the transpose of u). This C*-algebra comes equipped with a comultiplication $\Delta:\mathfrak{A}\to\mathfrak{A}\otimes_{\min}\mathfrak{A}$ defined by $\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$. It also comes with a unique Haar state τ which is tracial and invariant under Δ; i.e., for all $a\in\mathfrak{A}$, $(\tau\otimes\iota)\Delta(a)=(\iota\otimes\tau)\Delta(a)=\tau(a)$ (see [Wor87, BC07]). Take the GNS representation of this universal C*-algebra with respect to the Haar trace. The resulting C*-algebra is the reduced free unitary compact quantum group. We denote it by $A_u(n)$.

Similarly, the reduced free orthogonal compact quantum group is obtained starting from the universal C*-algebra generated by elements $\{v_{ij}\}_{i,j=1}^n$ satisfying

$$v^*v=vv^*=1_n,\qquad v_{ij}^*=v_{ij}\ \text{ for all } i,j,$$

with $v = (v_{ij})$ (which again comes equipped with the comultiplication $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$) and passing to the GNS representation with respect to its (tracial) Haar state. We denote the reduced free orthogonal compact quantum group by $A_o(n)$.

Banica showed [Ban97] (see also [Ban08, Theorem 3.4]), that $A_u(n) \cong \widetilde{A_o(n)}$, for $n \geq 2$. More concretely, $A_u(n)$ is isomorphic to the free complexification of $A_o(n)$, relative to the Haar state and generating set X given by the entries of $v = (v_{i,j})_{ij}$. For $n = 2$, Banica showed in [Ban97] that $A_u(2)$ can be alternatively obtained as the free complexification of $(C(SU(2)), \lambda, \{v_{ij}\}_{ij})$, where in this case λ is given by the Haar measure on $SU(2)$ and $v \in M_2(C(SU(2)))$ is the fundamental representation of $SU(2)$; i.e., the inclusion map $SU(2) \to M_2(\mathbb{C})$.

Theorem 4.2. *$A_u(n)$ is selfless for all $n \geq 2$.*

Proof. First assume $n > 2$. As remarked above, $A_u(n)$ is the free complexification of $(A_o(n), \tau)$ relative to the generating set $X = \{v_{ij} : i, j\}$. Since X is a selfadjoint set, $\mathcal{P}_X A_o(n) = \mathcal{P}_{X^*} A_o(n)$. We will be done once we have shown that $\mathcal{P}_X A_o(n)$ contains a Haar unitary u, as then we can apply Theorem 4.1 with $a = b = u$ and $c = u^2$. It is known from [Ban97, Proposition 1] that $s = \sum_{i=1}^n v_{ii}$ has a semicircular distribution. It follows that $|s| := \sqrt{s^2} \in \mathcal{P}_X A_o(n)$ has a quarter-circular distribution (which is diffuse). Thus, through functional calculus, we can find a Haar unitary in $\mathcal{P}_X A_o(n)$.

Assume that $n = 2$. In this case $A_u(2)$ is the free complexification of $(C(SU(2)), \lambda)$, where the generating set X is the entries of the fundamental representation $v \in M_2(C(SU(2)))$ and λ is induced by the Haar measure on $SU(2)$. We have

$$\mathcal{P}_{X^*} C(SU(2)) = \mathcal{P}_X C(SU(2)) \cong C(SO(3)),$$

where the first equality holds by the commutativity of $C(SU(2))$ and the second one by [Ban97, Théorème 5]. Since the Haar measure on $SO(3)$ is diffuse, $C(SO(3))$ contains a Haar unitary u. As in the previous case, choosing $a = b = u$ and $c = u^2$, we can apply Theorem 4.1 to reach the desired conclusion. □

Another class of compact quantum groups that arise as free complexifications are the half-liberated unitary quantum groups introduced in [BDD11]. We briefly recall the definition. For $n \in \mathbb{N}$, the half-liberated unitary compact quantum group is the universal C*-algebra generated by the entries of an $n \times n$ matrix u, subject to the same relations as the free unitary compact quantum group—namely, that u and u^t are unitaries—and, in addition, the relations

$$ab^*c = cb^*a \qquad \text{for all } a, b, c \in \{u_{ij} : i, j = 1, \ldots, n\}.$$

We denote by $A_u^*(n)$ the reduced half-liberated quantum unitary group, i.e., the image of the universal one under the GNS representation induced by the Haar state.

Corollary 4.3. *The reduced half-liberated unitary compact quantum groups $A_u^*(n)$ are selfless for $n \geq 2$.*

Proof. By [BB17, Proposition 3.7], $A_u^*(n)$ is isomorphic to the free complexification of $(C(U(n)), \tau)$, where $U(n)$ is the group of $n \times n$ unitary matrices, τ is the Haar state, and the generating set X is the entries of the fundamental representation of $U(n)$. (Note: In [BB17], the universal half-liberated unitary quantum group is denoted by $C(U_n^\times)$, so $A_u^*(n)$ is the reduced version of $C(U_n^\times)$.) The projective versions $\mathcal{P}_X C(U(n))$ and $\mathcal{P}_{X^*} C(U(n))$ agree and are isomorphic to $C(PU(n))$, where $PU(n) := U(n)/\mathbb{T}$ (see again [BB17, Proposition 3.7]). Since the Haar measure on $PU(n)$ is diffuse for $n \geq 2$, we can find a Haar unitary $u \in C(PU(n))$. Hence, the hypotheses of Theorem 4.1 are satisfied with $a = b = u$ and $c = u^2$. $\square$

For further examples of compact quantum groups obtained as free complexifications see [Ban08, Theorem 3.4], [Rau12]. There is also a construction of free d-complexification where the reduced free product with $C(\mathbb{T})$ is replaced with a reduced free product with $\mathbb{C}^d$. Examples of compact quantum groups obtained as free d-complexifications are obtained in [TW17, Theorem 6.16].

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