# MATRIX POINTS ON VARIETIES

ASVIN G, YIFENG HUANG, RUOFAN JIANG, YIFAN WEI

Abstract. We study the cohomology of $C_n(X)$, the moduli space of commuting $n$-by-$n$ matrices satisfying the equations defining a variety $X$. This space can be viewed as a non-commutative Weil restriction from the algebra of $n$-by-$n$ matrices to the ground field. We introduce a "Fermionic" counterpart $S_n(X)$, constructed as a convolution $X^n \times^{S_n} \mathrm{GL}_n/\mathrm{T}_n$. Our main result establishes that a natural map $\sigma\colon S_n(X) \to C_n(X)$ induces an isomorphism on $\ell$-adic cohomology under mild conditions on $X$ or the characteristic of the field. This confirms a heuristic derived from the classical theory of Weil restrictions and highlights a version of Boson-Fermion correspondence. Furthermore, we derive explicit combinatorial formulae for the Betti numbers of $C_n(X)$ and a Macdonald-type generating series. Finally we prove a Hermitian variant of our main result.

## 1. Introduction

1.1. **Motivation: Non-commutative Weil restriction.** Let $X$ be a $k$-scheme, if $L$ is a separable extension of degree $n$ with Galois group $G$, and $\widehat{L}$ is its Galois closure, the Weil restriction of $X_L$ to $k$ represents the following functor (for $k$-scheme $S$)

$$\mathrm{Res}_k^L(X_L)(S) = \mathrm{Hom}_k(S_L, X)$$

It can be verified that $\mathrm{Res}_k^L(X_L)$ is represented by the twisted product $X^n \times^G \mathrm{Spec}(\widehat{L})$ (the diagonal $G$-quotient of the product $X^n \times \mathrm{Spec}(\widehat{L})$, where $G$ acts on the $X^n$ factor by permuting $n = \mathrm{Hom}_k(L, \widehat{L})$.)

In this paper, we explore an analogue of this construction in a non-commutative setting. We view the inclusion of a field $k$ into the algebra of $n$-by-$n$ matrices, $\mathbb{M}_n(k)$, as a non-commutative analogue of a degree $n$ "separable field extension", and we study the Weil restriction functor from the matrix algebra to the ground field.

Let $X = \mathrm{Spec}\, R$ be a finite type affine $k$-scheme. We define the functor $C_n(X)$ on commutative $k$-algebras $S$ by

$$C_n(X)(S) = \mathrm{Hom}_{k\text{-alg}}(R, \mathbb{M}_n(S)).$$

If $R$ is generated by commuting variables $x_i$ satisfying relations $P_j(x_i) = 0$, then $C_n(X)$ represents the space of commuting $n$-by-$n$ matrices $X_i$ satisfying the same relations $P_j(X_i) = 0$. This space is representable by a finite type affine scheme.

The space $C_n(X)$ admits a moduli interpretation: it parametrizes length $n$ coherent sheaves $M$ on $X$ equipped with a global framing, i.e., an isomorphism $\iota\colon \Gamma(X, M) \xrightarrow{\sim} k^n$. This interpretation generalizes to arbitrary quasi-projective schemes $X$. The scheme $C_n(X)$ is a smooth cover of the Artin stack $\mathrm{Coh}_n(X)$, the moduli space of length $n$ coherent sheaves on $X$.

1.2. **The Bosonic and Fermionic Spaces.** There is a natural support map $p_2\colon C_n(X) \to \operatorname{Sym}^n X$ (see [FR24]), obtained by taking the simultaneous eigenvalues of the commuting matrices. The fibers of this map allow for non-trivial Jordan structures, meaning that "quantum states" (eigenspaces) can fuse together to form generalized eigenspaces. In this sense, we consider $C_n(X)$ as a *Bosonic* quantization of $\operatorname{Sym}^n X$.

To complement this, we introduce a *Fermionic* partner. Let $\mathrm{T}_n$ be the subgroup of diagonal matrices in $\mathrm{GL}_n$. The symmetric group $S_n$ acts on $\mathrm{GL}_n/\mathrm{T}_n$ via the Weyl group identification $N(\mathrm{T}_n)/\mathrm{T}_n \approx S_n$. We define $S_n(X)$ as the quotient of $X^n \times \mathrm{GL}_n/\mathrm{T}_n$ by the diagonal action of $S_n$:

$$S_n(X) = X^n \times^{S_n} \mathrm{GL}_n/\mathrm{T}_n.$$

The space $S_n(X)$ also maps to $\operatorname{Sym}^n X$. The fibers of $p_1\colon S_n(X) \to \operatorname{Sym}^n X$ parametrize direct sum decompositions of $k^n$ into one-dimensional subspaces corresponding to the points in the support. Here, the quantum states are forced to remain distinct (the eigenspaces are not allowed to fuse together); hence $S_n(X)$ is Fermionic.

There is a natural map $\sigma\colon S_n(X) \to C_n(X)$. It takes as input an ordered tuple of points $(x_i)$ on $X$ and an ordered direct-sum decomposition $\bigoplus_{i=1}^n L_i = k^n$ (parametrized by $\mathrm{GL}_n/\mathrm{T}_n$), and produces a framed finite sheaf by endowing $L_i$ with the structure of a skyscraper sheaf at $x_i$.

1.3. **Main Results.** The central result of this paper is that the Bosonic and Fermionic spaces are cohomologically equivalent.

**Theorem 1.1.** *Suppose we are in either of the following situations:*

- *$k$ is of characteristic zero, or*
- *$X$ is locally homogeneous, i.e., any local ring $R$ at a closed point of $X$ admits a non-negative grading.*

*Then the map $\sigma\colon S_n(X) \to C_n(X)$ induces isomorphisms*

$$H^i_{\text{ét}}(C_n(X), \mathbb{Q}_\ell) \xrightarrow{\sim} H^i_{\text{ét}}(S_n(X), \mathbb{Q}_\ell)$$

*for all $i$.*

We conjecture that the assumptions on $X$ or $k$ can be removed (see Conjecture 2.10). For the rest of this paper, we always assume we are in the situation stated in Theorem 1.1 unless specified otherwise. Note that the assumption on $X$ automatically holds when $X$ is smooth.

This result validates the Weil restriction heuristics: we may think of $k \hookrightarrow \mathbb{M}_n(k)$ as a non-commutative version of a separable degree $n$ "field extension". Let's pretend that $S_n$ is the Galois group of this field extension, and that there exists a Galois closure $\widehat{\mathbb{M}_n}(k)$ of the extension $k \hookrightarrow \mathbb{M}_n(k)$. Then from the classical theory of Weil restrictions, we expect

$$C_n(X) = X^n \times^{S_n} \text{“}\operatorname{Spec} \widehat{\mathbb{M}_n}(k)\text{”}.$$

On the other hand, the cohomology groups $H^*(\mathrm{GL}_n/\mathrm{T}_n)$ form the regular $S_n$ representation, except that it's graded in some non-trivial way (see Theorem 3.3). Thus, Theorem 1.1 implies that cohomologically, $C_n(X)$ behaves as $X^n \times^{S_n} \mathrm{GL}_n/\mathrm{T}_n$, aligning with the expectation that $H^*(\mathrm{GL}_n/\mathrm{T}_n)$ serves as the cohomological model for the "Galois closure" of "$\operatorname{Spec} \mathbb{M}_n(k)$" $\to \operatorname{Spec} k$.

The construction of $S_n(X)$ allows for explicit computation of its cohomology using combinatorial methods involving symmetric functions. We define the Poincare polynomial $P_u(M) = \sum_i \dim H^i(M)(-u)^i$. Let $\phi_n(q) = \prod_{i=1}^n (1-q^i)$.

**Theorem 1.2.** *The Poincare polynomial of $C_n(X)$ is given by*

$$P_u(C_n(X)) = \phi_n(u^2)\mathrm{sp}_{u^2}(\mathbf{ch}_u(X^n)),$$

*where $\mathbf{ch}_u(X^n)$ is the graded $S_n$-character on the cohomology of $X^n$ (Definition 3.1), and $\mathrm{sp}_{u^2}$ is the principal specialization of symmetric polynomials $f(x_1, x_2, \dots) \mapsto f(1, u^2, u^4, \dots)$.*

Furthermore, we can package these polynomials into a generating series which admits a beautiful infinite product formula in terms of the Betti zeta function $\zeta_X^{\mathrm{B}}(t) = \sum_{n=0}^\infty P_u(\mathrm{Sym}^n X)t^n$, which itself admits a product factorization by a celebrated formula of Macdonald [Mac62b], see Theorem 3.13.

**Theorem 1.3.** *We have the following product formula (note that $P_u(\mathrm{Coh}_n(X)) = \frac{P_u(C_n(X))}{\phi_n(u^2)}$ by Equation (1)):*

$$1 + \sum_{n=1}^\infty P_u(\mathrm{Coh}_n(X))t^n = \prod_{i=0}^\infty \zeta_X^{\mathrm{B}}(u^{2i}t).$$

Finally, we establish a variant of our main result in the Hermitian setting when $X$ is defined over $\mathbb{R}$. Let $HC_n(X)$ be the space of commuting Hermitian matrices satisfying the equations of $X$, and $HS_n(X) = X(\mathbb{R})^n \times^{S_n} \mathrm{U}_n/\mathrm{UT}_n$.

**Theorem 1.4.** *The natural map $\sigma_H \colon HS_n(X) \to HC_n(X)$ induces an isomorphism on rational cohomology.*

1.4. **Context and Related Works.** The map $\sigma$ is $\mathrm{GL}_n$-equivariant. Taking the stacky $\mathrm{GL}_n$ quotient relates our construction to the moduli stack of coherent sheaves $\mathrm{Coh}_n(X)$ (it can be seen that $S_n(X)/\mathrm{GL}_n$ is isomorphic to $\mathrm{Sym}^n(X \times B\mathbb{G}_m)$):

$$\begin{tikzcd} \mathrm{Sym}^n(X \times B\mathbb{G}_m) \arrow[rr, "\sigma"] \arrow[dr] & & \mathrm{Coh}_n(X) \arrow[dl] \\ & B\mathrm{GL}_n & \end{tikzcd}$$

Thus, we obtain two related spectral sequences computing the equivariant cohomology of $S_n(X)$ and $C_n(X)$:

$$\begin{tikzcd} H^q(B\mathrm{GL}_n, H^p(C_n(X))) \arrow[r, Rightarrow] \arrow[d, "\approx"', "\sigma^*"] & H^{p+q}(\mathrm{Coh}_n(X)) \arrow[d, "\approx"', "\sigma^*"] \\ H^q(B\mathrm{GL}_n, H^p(S_n(X))) \arrow[r, Rightarrow] & H^{p+q}(\mathrm{Sym}^n(X \times B\mathbb{G}_m)). \end{tikzcd}$$

The fact that the rational cohomology of $\mathrm{Coh}_n(X)$ agrees with $\mathrm{Sym}^n(X \times B\mathbb{G}_m)$ is classically known in the case of a smooth curve $X$ (see, e.g., Heinloth [Hei12]).

Our result also implies a strong equivariant formality for the $\mathrm{GL}_n$ action on $C_n(X)$. The derived direct image of the constant sheaf along $\pi \colon \mathrm{Coh}_n(X) \to B\mathrm{GL}_n$ splits as a direct sum of shifted constant sheaves (Corollary 2.12). Consequently the Leray spectral sequences for the equivariant cohomology collapse at the second page, and we obtain the formula for the Poincare polynomial of the stack:

$$P_u(\mathrm{Coh}_n(X)) = P_u(B\mathrm{GL}_n)P_u(C_n(X)) = \mathrm{sp}_{u^2}(\mathbf{ch}_u(X^n)). \tag{1}$$

Matrix points recover, as special cases, several classical examples of commuting varieties: $C_n(\mathbb{A}^N)$ is the variety of commuting $N$ tuples in $\mathfrak{gl}_n$, $C_n(\mathbb{G}_m^N)$ is the variety of commuting $N$ tuples in $\mathrm{GL}_n$, $\mathrm{HC}_n(\mathbb{A}^N)$ is the real variety of commuting $N$ tuples in $\mathfrak{u}_n$, and, less obviously but not hard to check, that $\mathrm{HC}_n((S^1)^N)$ is the real variety of commuting $N$ tuples in $U_n$, where $S^1 = \operatorname{Spec}\mathbb{R}[x,y]/(x^2+y^2-1)$. Our results thus give the Poincaré polynomials of these real varieties, in line with the results of [RS19] in type $A$.

Matrix points on more general varieties have been explored by earlier works of Chien-Hao Liu and Shing-Tung Yau in the context of D-branes in string theory [LY10]. The moduli space $C_n(X)$ (as representation space $\mathsf{rep}_n(R)$) is also seen in the works of Lieven Le Bruyn and Jens Hemelaer [Le 12; HB16]. However, to the authors' knowledge, the rational cohomology of the space $C_n(X)$ has not been computed before. In fact, even the simple question of determining the Betti numbers of the space of $n$-by-$n$ matrices avoiding several eigenvalues was likely open, but can be computed using our combinatorial formula (Theorem 1.2) applied to the space $\mathbb{A}^1 - \{a_1, a_2, \dots, a_r\}$.

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1.6. **Outline.** In Section 2, we analyze the geometry of the map $\sigma$ relatively over $\operatorname{Sym}^n X$ and prove Theorem 1.1. Section 3 develops the necessary combinatorial tools involving graded characters and symmetric functions, leading to the proofs of Theorem 1.2 and Theorem 1.3. Section 4 discusses the Hermitian variant and proves Theorem 1.4.

## 2. The proof

In this section, following a suggestion by Dima Arinkin, we prove our main theorem by analyzing the map $\sigma\colon S_n(X) \to C_n(X)$ relatively over the symmetric product $\operatorname{Sym}^n X$. We assume that we are in the situation where either $k$ is of characteristic zero, or that any local ring $R$ at a closed point of $X$ is non-negatively graded.

2.1. **The Relative Picture.** Let $p_1\colon S_n(X) \to \operatorname{Sym}^n X$ and $p_2\colon C_n(X) \to \operatorname{Sym}^n X$ denote the respective natural morphisms. The map $\sigma$ commutes with these projections. We aim to show that $\sigma$ induces an isomorphism on the derived direct images of the constant sheaf.

**Theorem 2.1.** *Under the assumptions of Theorem 1.1, the map $\sigma$ induces an isomorphism $\sigma^*\colon Rp_{2*}\underline{\mathbb{Q}_\ell} \xrightarrow{\sim} Rp_{1*}\underline{\mathbb{Q}_\ell}$ in the derived category $D^b_c(\operatorname{Sym}^n X)$.*

The heart of our proof is the computation of the stalk of $Rp_{2*}\underline{\mathbb{Q}_\ell}$ using base change for the morphism $\mathrm{Coh}_n(X) \to \operatorname{Sym}^n X$. Theorem 1.1 follows immediately from the sheaf theoretic statement by taking cohomology.

2.2. **Analysis of the Fibers.** Let $D_\lambda$ be a geometric point on $\operatorname{Sym}^n X$ corresponding to a zero-cycle $\lambda_1[x_1] + \cdots + \lambda_m[x_m]$, where $\lambda = (\lambda_1, \dots, \lambda_m)$ is a partition of $n$. Let $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_m} \subset S_n$ be the corresponding Young subgroup.

**Lemma 2.2.** *The fiber* $p_1^{-1}(D_\lambda)$ *of* $S_n(X) \to \operatorname{Sym}^n X$ *is isomorphic to the quotient* $(\mathrm{GL}_n/\mathrm{T}_n)/S_\lambda$.

The description of the fiber of $C_n(X) \to \operatorname{Sym}^n X$ is more involved. Let $R_i$ be the local ring of $X$ at $x_i$ with maximal ideal $\mathfrak{m}_i$. Define $\mathrm{Nil}_{\lambda_i,x_i}$ to be the variety underlying $C_{\lambda_i}(R_i/\mathfrak{m}_i{}^{\lambda_i})$. This space parametrizes length $\lambda_i$ modules supported at $x_i$ with a framing.

**Lemma 2.3.** *The fiber* $p_2^{-1}(D_\lambda)$ *is isomorphic to the associated bundle*

$$\mathrm{GL}_n \times^{\prod_i \mathrm{GL}_{\lambda_i}} \left(\prod_{i=1}^m \mathrm{Nil}_{\lambda_i,x_i}\right).$$

*Proof.* A point in $p_2^{-1}(D_\lambda)$ corresponds to a framed sheaf $(M,\iota)$ whose support is given by $D_\lambda$. Such a sheaf decomposes as $M = \bigoplus M_i$, where $M_i$ is supported at $x_i$ and has length $\lambda_i$. The framing $\iota$ induces a decomposition $k^n = \bigoplus V_i$ where $V_i = \Gamma(M_i)$. The space of such modules $M_i$ with a fixed basis of $V_i$ is parametrized by $\mathrm{Nil}_{\lambda_i,x_i}$. The group $G_\lambda = \prod_i \mathrm{GL}(V_i) \approx \prod_i \mathrm{GL}_{\lambda_i}$ acts on the choice of basis of $V_i$ inside $k^n$. The fiber is thus the quotient of $\mathrm{GL}_n \times \prod_i \mathrm{Nil}_{\lambda_i,x_i}$ by the action of $G_\lambda$, where $G_\lambda$ acts on $\mathrm{GL}_n$ by right multiplication (via the standard inclusion $G_\lambda \subset \mathrm{GL}_n$) and on $\prod \mathrm{Nil}_{\lambda_i,x_i}$ diagonally. $\square$

2.3. **Computation of stalks.** Let $z$ be a closed point on $\operatorname{Sym}^n X$, we show here that the stalks of $Rp_{1*}\underline{\mathbb{Q}_\ell}$ and $Rp_{2*}\underline{\mathbb{Q}_\ell}$ at $z$ can be identified with the cohomology of the fibers. Note that this is not automatic since neither $p_1$ nor $p_2$ is proper.

**Proposition 2.4** (Stalk of $Rp_{1*}\underline{\mathbb{Q}_\ell}$). *Let $z$ be a closed point on* $\operatorname{Sym}^n X$*. Let* $S_n(X)_z$ *denote the fiber* $p_1^{-1}(z)$*. The base change morphism* $(Rp_{1*}\underline{\mathbb{Q}_\ell})_z \to H^*(S_n(X)_z)$ *is an isomorphism.*

*Proof.* Let $\widehat{S}_n(X)$ denote the product $X^n \times \mathrm{GL}_n/\mathrm{T}_n$, and $\widehat{p}_1$ denote the composition of the projection $\widehat{S}_n(X) \to X^n$ and the finite quotient $X^n \to \operatorname{Sym}^n X$. We have the following commutative diagram

$$\begin{array}{ccc} \widehat{S}_n(X) & \longrightarrow & X^n \\ \big\downarrow{\scriptstyle \mu} & & \big\downarrow \\ S_n(X) & \xrightarrow{p_1} & \operatorname{Sym}^n X \end{array}$$

The complex $R\widehat{p}_{1*}\underline{\mathbb{Q}_\ell}$ carries an $S_n$-action via the $S_n$ quotient map $\mu$, such that $(Rp_{1*}\underline{\mathbb{Q}_\ell})_z = (R\widehat{p}_{1*}\underline{\mathbb{Q}_\ell})_z^{S_n}$ for every closed point $z$ on $\operatorname{Sym}^n X$. It now suffices to show that $(R\widehat{p}_{1*}\underline{\mathbb{Q}_\ell})_z = H^*(\widehat{p}_2^{-1}(z))$, which follows from proper base change for the morphism $X^n \to \operatorname{Sym}^n X$ and projection formula for $\widehat{S}_n(X) \to X^n$. $\square$

Computation of the stalk of $Rp_{2*}\underline{\mathbb{Q}_\ell}$ is a more difficult question, we decompose the morphism $p_2$ as a composite of the quotient morphism $q\colon C_n(X) \to \mathrm{Coh}_n(X)$ and the Jordan-Hölder morphism $\mathsf{JH}\colon \mathrm{Coh}_n(X) \to \operatorname{Sym}^n X$. Let $z$ be a closed point on $\operatorname{Sym}^n X$, let $C_n(X)_z$ be the preimage $p_2^{-1}(z)$ and let $\mathrm{Coh}_n(X)_z$ be the preimage $\mathsf{JH}^{-1}(z)$. We have the following commutative diagrams:

$$\begin{tikzcd} C_n(X)_z \arrow[r, "i'"] \arrow[d, "q'"] & C_n(X) \arrow[d, "q"] \\ \mathrm{Coh}_n(X)_z \arrow[r, "i"] & \mathrm{Coh}_n(X) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Coh}_n(X)_z \arrow[r, "i"] \arrow[d] & \mathrm{Coh}_n(X) \arrow[d, "\mathsf{JH}"] \\ \{z\} \arrow[r, hook] & \mathrm{Sym}^n X \end{tikzcd}$$

First, we note that base change holds in the first diagram for the constant sheaf on $C_n(X)$, which is the content of the following lemma.

**Lemma 2.5** (Base change for stacky quotient)**.** *Let $X$ and $Y$ be $G$-varieties and $f\colon Y \to X$ a morphism of $G$-varieties with $g\colon Y/G \to X/G$ the associated map on stacks. Let $q\colon X \to X/G$ and $q'\colon Y \to Y/G$ be the quotient maps. Then base change morphism $g^*Rq_*\underline{\mathbb{Q}}_\ell \to Rq'_*\underline{\mathbb{Q}}_\ell$ is an isomorphism in $D^b_c(Y/G)$.*

*Proof sketch.* This follows from the universal case where $X = pt$. □

Thus $i^*Rq_*\underline{\mathbb{Q}}_\ell \to Rq'_*\underline{\mathbb{Q}}_\ell$ is an isomorphism. To compute the stalk $Rp_{2*}\underline{\mathbb{Q}}_\ell$ at $z$, it suffices to show base change works in the second diagram for $Rq_*\underline{\mathbb{Q}}_\ell$ on $\mathrm{Coh}_n(X)$. This is achieved by approximating the morphism $\mathsf{JH}$ by proper morphisms. To be precise, recall that in [Kin24] it was shown that base change for constructible sheaves holds for a class of morphisms called *approximately proper* (AP) morphisms.

**Definition 2.6.** Define a class AP of morphisms between finite type Artin stacks to be the minimal one satisfying the following conditions:

(AP1) All proper morphisms represented by Deligne–Mumford stacks are in AP.
(AP2) AP is closed under composition.
(AP3) Being in AP can be checked étale locally on the target.
(AP4) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism and $\mathfrak{Z}$ be a non-empty finite type Artin stack. Assume that the composition

$$\mathfrak{X} \times \mathfrak{Z} \xrightarrow{\mathrm{pr}} \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$$

is in AP. Then $f$ is in AP.
(AP5) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism. Assume that for each integer $n$, there exists a vector bundle $\mathcal{E}_n$ over $\mathfrak{X}$ and an open subset $U_n \subset \mathcal{E}_n$ with $\mathrm{codim}_{\mathcal{E}_n}((\mathcal{E}_n \setminus U_n)|_x) > n$ for each $x \in \mathfrak{X}$ such that the composition

$$U_n \to \mathcal{E}_n \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$$

is in AP. Then $f$ is in AP.

A morphism contained in AP is called *approximately proper.*

**Theorem 2.7** (Proposition 2.2 of [Kin24])**.** *In the following Cartesian square of finite type Artin stacks, let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in AP. Let $\mathcal{F}$ be an object in $D^+_c(\mathfrak{X})$, then the base change morphism $g^*Rf_*\mathcal{F} \to Rf'_*g'^*\mathcal{F}$ is an isomorphism.*

$$\begin{tikzcd} \mathfrak{T} \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r, "g'"] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{T} \arrow[r, "g"] & \mathfrak{Y} \end{tikzcd}$$

The following proposition has been kindly communicated to us by Tasuki Kinjo.

**Proposition 2.8.** *The morphism $\mathsf{JH}\colon \mathrm{Coh}_n(X) \to \mathrm{Sym}^n X$ is in AP.*

*Proof.* For each positive integer $m$, let $\mathcal{V}_m \to \mathrm{Coh}_n(X)$ be the vector bundle parameterizing a length $n$ sheaf $\mathcal{Z}$ on $X$ along with a morphism $\mathcal{O}_X^m \to \mathcal{Z}$. The quot scheme $\mathrm{Quot}_X(\mathcal{O}_X^m, n)$ is an open substack of $\mathcal{V}_m$ parameterizing surjective morphisms $\mathcal{O}_X^m \to \mathcal{Z}$. We make two observations:

(1) The morphism $\mathrm{Quot}_X(\mathcal{O}_X^m, n) \to \mathrm{Sym}^n(X)$ is proper.
(2) The codimension of the complement of $\mathrm{Quot}_X(\mathcal{O}_X^m, n)$ in $\mathcal{V}_m$ grows to infinity as $m \to \infty$.

Observation (1) can be seen by compactifying $X$, and making use of properness of $\mathrm{Quot}_X(\mathcal{O}_X^m, n)$ for projective $X$. For observation (2), notice that the $\mathrm{GL}_n$-cover of $\mathcal{V}_m$ is $C_n(X) \times (\mathbb{A}^n)^m$ where the factor $(\mathbb{A}^n)^m$ record $m$ sections to the length $n$ finite sheaf, and $\mathrm{GL}_n$ acts diagonally. Let $W_m \subset (\mathbb{A}^n)^m$ be the determinantal variety parameterizing a non-spanning $m$-tuple of vectors in $\mathbb{A}^n$, the dimension of $W_m$ equals $(m+1)(n-1)$ (for $m$ large enough). The $\mathrm{GL}_n$-cover of the complement $\mathcal{V}_m \setminus \mathrm{Quot}_X(\mathcal{O}_X^m, n)$ is $C_n(X) \times W_m$, hence its codimension in $C_n(X) \times (\mathbb{A}^n)^m$ grows to infinity as $m \to \infty$.

Thus by AP1 and AP5 we see that $\mathsf{JH}$ is in AP. □

To conclude the discussion, we can now compute the stalk of $Rp_{2*}\underline{\mathbb{Q}}_\ell$.

**Proposition 2.9** (Stalk of $Rp_{2*}\underline{\mathbb{Q}}_\ell$)**.** *Let $z$ be a closed point on $\mathrm{Sym}^n X$. Let $C_n(X)_z$ denote the fiber $p_2^{-1}(z)$. The base change morphism $(Rp_{2*}\underline{\mathbb{Q}}_\ell)_z \to H^*(C_n(X)_z)$ is an isomorphism.*

2.4. **Local Contractibility and Proof of Theorem 2.1.** Another key input for the proof is the contractibility of the local factors $\mathrm{Nil}_{\lambda_i,x_i}$.

**Conjecture 2.10.** *If $R$ is an Artinian local algebra over $k$, the underlying variety of $C_n(R)$ has trivial $\ell$-adic cohomology.*

This conjecture holds under the assumptions of Theorem 1.1.

**Theorem 2.11.** *Let $R$ be an Artinian local algebra over $k$. The underlying variety of $C_n(R)$ has trivial $\ell$-adic cohomology if either $k$ has characteristic zero or $R$ is non-negatively graded.*

*Proof.* If $R$ is non-negatively graded, the grading induces a $\mathbb{G}_m$-action on $R$. This action extends to $C_n(R)$, contracting it to the "center point" (the trivial representation where all generators act by zero). Hence, $C_n(R)$ has trivial $\ell$-adic cohomology.

If $k$ has characteristic zero, we use Geometric Invariant Theory. The $\mathrm{GL}_n$ action on $C_n(R)$ has a unique closed orbit (the trivial representation). By a result of Neeman [Nee85], there exists an equivariant deformation retraction of the complex points of $C_n(R)$ onto this closed point. Hence $C_n(R)$ has trivial cohomology. □

*Proof of Theorem 2.1.* By Theorem 2.11, the local factors $\mathrm{Nil}_{\lambda_i,x_i}$ all have trivial cohomology. Thus the cohomology of $p_2^{-1}(D_\lambda)$ is captured by the semi-simple locus $Z \subset p_2^{-1}(D_\lambda)$, which is isomorphic to $\mathrm{GL}_n/\prod_i \mathrm{GL}_{\lambda_i}$.

The map $\sigma$ restricted to the fiber $p_1^{-1}(D_\lambda)$ maps surjectively onto this semi-simple locus $Z$. Specifically, $\sigma$ induces a map between $\mathrm{GL}_n$-homogeneous spaces:

$$\sigma\colon (\mathrm{GL}_n/\mathrm{T}_n)/S_\lambda \to \mathrm{GL}_n/\prod_i \mathrm{GL}_{\lambda_i}.$$

This map is a fibration. The typical fiber is isomorphic to

$$\left(\prod_i \mathrm{GL}_{\lambda_i}\right) / \left(\prod_i N(\mathrm{T}_{\lambda_i})\right) \approx \prod_i (\mathrm{GL}_{\lambda_i}/N(\mathrm{T}_{\lambda_i})).$$

The spaces $\mathrm{GL}_m/N(\mathrm{T}_m)$ are known to have trivial rational cohomology (see Theorem 3.3). Therefore, the Leray spectral sequence for the fibration induced by $\sigma$ degenerates, and $\sigma^*$ induces an isomorphism $H^*(p_2^{-1}(D_\lambda)) \xrightarrow{\sim} H^*(p_1^{-1}(D_\lambda))$.

By the base change theorems Propositions 2.4 and 2.9, the stalks of $Rp_{1*}\underline{\mathbb{Q}}_\ell$ and $Rp_{2*}\underline{\mathbb{Q}}_\ell$ are canonically identified with cohomology of fibers of $p_1, p_2$. Thus we conclude $\sigma^*\colon Rp_{2*}\underline{\mathbb{Q}}_\ell \to Rp_{1*}\underline{\mathbb{Q}}_\ell$ is an isomorphism. □

2.5. **Equivariant Formality.** As a consequence of the main theorem, we obtain the strong equivariant formality mentioned in the introduction.

**Corollary 2.12** (Strong equivariant formality)**.** *Let* $\pi\colon \mathrm{Coh}_n(X) \to B\mathrm{GL}_n$ *denote the canonical morphism. We have* $R\pi_*\underline{\mathbb{Q}}_\ell \approx \bigoplus_i H^i(C_n(X)) \otimes \underline{\mathbb{Q}}_\ell[-i]$ *in* $D_c^b(B\mathrm{GL}_n)$.

*Proof.* Let $\pi'$ denote the canonical morphism $S_n(X)/\mathrm{GL}_n \to B\mathrm{GL}_n$. By Theorem 1.1, $\sigma^*$ induces an isomorphism $R\pi_*\underline{\mathbb{Q}}_\ell \xrightarrow{\sim} R\pi'_*\underline{\mathbb{Q}}_\ell$. It suffices to show the result for $\pi'$.

Let $\widehat{S}_n(X) = X^n \times \mathrm{GL}_n/\mathrm{T}_n$, and $\widehat{\pi}'\colon \widehat{S}_n(X)/\mathrm{GL}_n \to B\mathrm{GL}_n$. Since $R\pi'_*\underline{\mathbb{Q}}_\ell$ is the $S_n$-invariant part of $R\widehat{\pi}'_*\underline{\mathbb{Q}}_\ell$, it is a direct summand. It suffices to prove that $R\widehat{\pi}'_*\underline{\mathbb{Q}}_\ell$ splits.

Note that $\widehat{S}_n(X)/\mathrm{GL}_n = X^n \times B\mathrm{T}_n$. The map $\widehat{\pi}'$ factors as $X^n \times B\mathrm{T}_n \to B\mathrm{T}_n \to B\mathrm{GL}_n$. The first map is a trivial fibration. The second map $B\mathrm{T}_n \to B\mathrm{GL}_n$ factors as $B\mathrm{T}_n \to \mathrm{GL}_n \backslash \mathcal{B}_n \to B\mathrm{GL}_n$, where $\mathcal{B}_n$ is the flag variety. The map $B\mathrm{T}_n \to \mathrm{GL}_n \backslash \mathcal{B}_n$ is an affine space bundle (the stacky quotient of $\mathrm{GL}_n/\mathrm{T}_n \to \mathcal{B}_n$), hence acyclic for the constant sheaf. The map $\mathrm{GL}_n \backslash \mathcal{B}_n \to B\mathrm{GL}_n$ is a smooth projective morphism between Artin stacks with affine diagonal. By the decomposition theorem for Artin stacks (see [Sun12]), the derived direct image splits as a direct sum of shifted cohomology sheaves. Thus $R\widehat{\pi}'_*\underline{\mathbb{Q}}_\ell$ splits. □

## 3. Combinatorics

This section is dedicated to the character theoretic computation of the cohomologies of the convoluted product $S_n(X) = X^n \times^{S_n} \mathrm{GL}_n/\mathrm{T}_n$. To do so, we develop necessary language to talk about the $S_n$ characters showing up in the cohomology of $H^*(X^n)$ via a power structure on the generating series of graded $S_n$ representations (as $n$ varies). Meanwhile, we observed that the graded $S_n$ characters on the cohomology of $\mathrm{GL}_n/\mathrm{T}_n$ can be packaged nicely using *principal specializations*, this description behaves especially well with power structure, allowing us to derive a compact formula for the cohomology of $S_n(X)$, and an infinite product formula for its generating series as $n$ varies.

Since the technicalities of étale cohomology is irrelevant in this section, we use $\mathbb{Q}$ to denote the char 0 coefficient field of cohomologies, and use $H^*$ to mean either singular or étale cohomology depending on the context.

**Notation.**

| | |
|---|---|
| $\lambda \vdash n$ | an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_m)$ of $n$. |
| $\ell(\lambda)$ or $\lvert\lambda\rvert$ | the length of a partition |

| | |
|---|---|
| $S_n$ | the permutation group on $n$ letters |
| $S_\lambda \subset S_n$ | Young subgroup $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_m}$ of $S_n$ |
| $\mathsf{SymPoly}$ | the ring of symmetric polynomials with rational coefficients in infinitely many variables $x_1, x_2, x_3, \ldots$ |
| $s_\lambda$ | Schur basis of $\mathsf{SymPoly}$ |
| $h_n$, $h_\lambda$ | complete homogeneous polynomials $h_n$, and complete homogeneous basis $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_m}$ in $\mathsf{SymPoly}$ |
| $p_n$, $p_\lambda$ | power-sum symmetric polynomials $p_n = \sum_k x_k^n$, and $p_\lambda = \prod_i p_{\lambda_i}$. |
| $\phi_n(q)$, $\phi(q)$ | $\phi_n(q) = (q;q)_n = \prod_{i=1}^n (1-q^i)$, $\phi(q) = \phi_\infty(q)$ |
| $\langle V, W \rangle$ | the Hall inner product of two $S_n$ representations $\langle V, W \rangle = \dim(V \otimes W)^{S_n}$ |
| $\mathcal{B}_n$, $\mathcal{P}_\lambda$ | the complete flag variety, and the partial flag variety associated with a partition $\lambda$ |
| $P_u(M)$ | the Poincare polynomial $\sum_i \dim H^i(M)(-u)^i$ of a space $M$. |

### 3.1. Graded characters and cohomology of flag varieties.

**Definition 3.1** (Frobenius character)**.** The *Frobenius character* map is a ring isomorphism

$$\mathbf{ch}\colon \bigoplus_{n=0}^{\infty} \mathrm{Rep}_{\mathbb{Q}}(S_n) \to \mathsf{SymPoly}$$

from the ring of representations of $S_n$ under the induction product, to the ring of symmetric polynomials over $\mathbb{Q}$.

Under $\mathbf{ch}$, the Specht module $V_\lambda$ corresponding to a partition $\lambda \vdash n$ is sent to the Schur function $s_\lambda$. Using the $p$-basis of $\mathsf{SymPoly}$, the Frobenius character of an $S_n$ representation $V$ has the following expression

$$\mathbf{ch}(V) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{tr}(\sigma) p_\sigma$$

where $p_\sigma$ is the power sum symmetric function associated with the cycle type of $\sigma$.

More generally, if $V = \bigoplus_{i=0}^{\infty} V_i$ is a graded $S_n$-representation over $\mathbb{Q}$, we define its graded character $\mathbf{ch}_u$ as an alternating series

$$\mathbf{ch}_u(V) = \sum_{i=0}^{\infty} \mathbf{ch}(V_i)(-u)^i$$

where $u$ is a formal variable.

**Definition 3.2** (Hall inner product)**.** There exists an inner product $\langle -,- \rangle$ on the ring of symmetric polynomials $\mathsf{SymPoly}$ where Schur functions $s_\lambda$ serves as an orthonormal basis (while $p_\lambda$ are orthogonal basis). This inner product is compatible with the inner product $\langle U, V \rangle_{S_n}$ of $S_n$-representations under the Frobenius character map. We can extend $\langle -,- \rangle$ bilinearly to graded characters

$$\langle -,- \rangle\colon \mathsf{SymPoly}[[u]] \times \mathsf{SymPoly}[[u]] \to \mathbb{Q}[[u]]$$

where the formal variable $u$ is treated as a scalar for the inner product.

**Convention:** For simplicity, when $S_n$ acts on the cohomology groups of a space $Y$, we write $\mathbf{ch}_u(Y)$ for $\mathbf{ch}_u(H^*(Y))$.

The cohomology ring of $\mathrm{GL}_n/\mathrm{T}_n$ is isomorphic to the cohomology ring of the flag variety $\mathcal{B}_n$ since $\mathrm{GL}_n/\mathrm{T}_n \to \mathcal{B}_n$ is an affine space bundle. The free $S_n$ action on $\mathrm{GL}_n/\mathrm{T}_n$ induces an $S_n$ action on its cohomology ring. The following presentation is due to Borel [Bor53]:

$$H^*(\mathrm{GL}_n/\mathrm{T}_n) = H^*(\mathcal{B}_n) = \mathbb{Q}[x_1, x_2, \dots, x_n]/(e_1, e_2, \dots, e_n), \quad S_n \text{ permutes } x_i\text{'s}$$

In Garsia-Procesi [GP92], the graded $S_n$ characters for the cohomology of Springer fibers were computed. In particular we obtain a convenient formula for the graded $S_n$ characters on a flag variety.

**Theorem 3.3** (Graded character of the flag variety)**.** *Let $\phi_n(q) = \prod_{i=1}^n (1-q^i)$, let $\mathrm{sp}_q(f) = f(1, q, q^2, \dots)$ denote the principal specialization of a symmetric polynomial $f$. We have*

$$\mathbf{ch}_u(\mathcal{B}_n) = \sum_{\lambda \vdash n} \phi_n(u^2)\mathrm{sp}_{u^2}(s_\lambda)s_\lambda$$

*Remark* 3.1. In the above theorem, setting $u = 1$ recovers the decomposition of the regular representation $\mathbb{Q}[S_n]$ into Specht modules.

**Corollary 3.4.** *Let $V$ be a graded $S_n$-rep, we have*

$$\langle \mathbf{ch}_u(\mathcal{B}_n), V \rangle = \phi_n(u^2)\mathrm{sp}_{u^2}(\mathbf{ch}_u(V)).$$

*Consequently, for a topological space $X$, we have*

$$P_u(S_n(X)) = \phi_n(u^2)\mathrm{sp}_{u^2}(\mathbf{ch}_u(X^n)) \tag{2}$$

To compute the number of $\mathbb{F}_q$-points on $S_n(X)$, we need to consider the $S_n$ character on $H^*(X^n)$ along with an endomorphism $F$ (the arithmetic Frobenius over $\mathbb{F}_q$). This motivates the following enhancement of Definition 3.1.

**Definition 3.5** ($F$-enhanced Frobenius character)**.** Let $V$ be a graded $S_n$ representation $V = \bigoplus_{i \geq 0} V_i$ over $\mathbb{Q}$, along with a graded endomorphism $F$ which commutes with the action of $S_n$. We call such $V$ a *graded $S_n$-rep enhanced by $F$*. The *$F$-enhanced Frobenius character* of $V$ to is defined as

$$\mathbf{ch}_{F,u}(V) = \sum_{i \geq 0} (-u)^i \sum_{\sigma \in S_n} \frac{1}{n!} \mathrm{tr}(F\sigma | V_i) p_\sigma$$

where $p_\sigma$ is the power sum symmetric function associated with the cycle type of $\sigma$. Note that $\mathrm{tr}(F\sigma)$ only depends on the cycle type of $\sigma$ due to the commutativity of $F$ and $S_n$, thus the inner sum is essentially over the conjugacy classes of $\sigma$.

*Remark* 3.2. When $F = \mathrm{id}$, $\mathbf{ch}_{F,u}(V) = \mathbf{ch}_u(V)$ reduces to the graded character defined in Definition 3.1.

*Remark* 3.3. Let $F = F_s + F_n$ be the Jordan decomposition of $F$ on $V$, where $F_s$ is the semi-simple part. We clearly have $\mathrm{tr}(F\sigma|V_i) = \mathrm{tr}(F_s\sigma|V_i)$. Hence for practical purposes we may always assume $F$ is semi-simple when computing $\mathbf{ch}_{F,u}(V)$.

**Lemma 3.6.** *Let $V$ a graded $S_n$-rep enhanced by $F_1$, and $W$ a graded $S_n$-rep enhanced by $F_2$. The tensor product $V \otimes W$ is naturally bi-graded, and enhanced* *by an endomorphism $F = F_1 \otimes F_2$. Thus $F$ naturally acts on the bi-graded vector space $(V \otimes W)^{S_n}$ as well. We have*

$$\sum_{i,j\geq 0} (-u_1)^i (-u_2)^j \operatorname{tr}(F|(V_i \otimes W_j)^{S_n}) = \langle \mathbf{ch}_{F_1,u_1}(V), \mathbf{ch}_{F_2,u_2}(W)\rangle$$

*where the inner product on the right hand side is the Hall inner product extended bi-linearly to graded characters.*

*Proof.* As both sides of the equality are additive for short exact sequences in $V$ and $W$ (respectively), it suffices to prove it in the case when $V$ and $W$ are irreducible $S_n$-reps, and both $F_1$ and $F_2$ act as scalars (upon base change to $\overline{\mathbb{Q}}$). In which case the equality can be checked directly using the definition of $\mathbf{ch}_{F,u}$. □

Let $X$ be a variety over $\mathbb{F}_q$, the cohomology groups $H^*(X^n)$ form a graded $S_n$ representation with a commuting arithmetic Frobenius endomorphism $F$ commuting with $S_n$. Same for the space $\mathrm{GL}_n/\mathrm{T}_n$. The number of $\mathbb{F}_q$ points on the quotient $S_n(X)$ is determined by the alternating trace of $F$ on its cohomologies, which can be computed using the previous lemma.

**Corollary 3.7.** *The number of $\mathbb{F}_q$ points on $S_n(X)$ equals*

$$|\mathrm{GL}_n(\mathbb{F}_q)| \cdot \mathrm{sp}_{q^{-1}}(\mathbf{ch}_{F,1}(X^n))$$

*Proof.* By Lemma 3.6 and the Behrend trace formula (trace formula for cohomology groups on smooth varieties), we have

$$\begin{aligned} S_n(X)(\mathbb{F}_q) &= q^{n^2}\langle \mathbf{ch}_{F,1}(X^n), \mathbf{ch}_{F,1}(\mathrm{GL}_n/\mathrm{T}_n)\rangle \\ &= q^{n^2}\phi_n(q^{-1})\mathrm{sp}_{q^{-1}}(\mathbf{ch}_{F,1}(X^n)) \\ &= |\mathrm{GL}_n(\mathbb{F}_q)| \cdot \mathrm{sp}_{q^{-1}}(\mathbf{ch}_{F,1}(X^n)) \end{aligned}$$

where the second equality follows from the same calculation as in Corollary 3.4. □

3.2. **Generating Series and Zeta function.** So far we have not used the (induction product) ring structure on SymPoly, for what follows the ring structure becomes essential. Let $(V, F)$ be a fixed graded vector space with a graded endomorphism $F$. By abuse of notation, we denote the natural endomorphism $F^{\otimes n}$ on $V^{\otimes n}$ still by $F$. We equip the **signed $S_n$ representation** on $V^{\otimes n}$ as the left $S_n$ action satisfying:

$$(ij)(\cdots v_i \otimes v_j \cdots) = (-1)^{\deg(v_i)\deg(v_j)}(\cdots v_j \otimes v_i \cdots)$$

for $(ij) \in S_n$ a transposition, and $v_i, v_j$ homogeneous. By [Mac62a], such an action exists uniquely. (There is a direct albeit more technical definition in [Mac62a], which shows that it is clearly well-defined, but the current form is enough to give an algorithm to compute the action on any pure tensor.)

We now compute the generating series of $F$-enhanced characters of $V^{\otimes n}$.

**Lemma 3.8.** *Let $(V, F)$ be a graded vector space with a graded endomorphism, let $\mathcal{A} \subset V$ be a homogeneous eigen-basis of the semi-simple part of $F$ on $V$, and $\chi\colon \mathcal{A} \to \overline{\mathbb{Q}}$ be the corresponding eigen-values.*

*Let $\sigma \in S_n$ be a permutation whose cycle type corresponds to a partition (in exponential form) $\lambda = (1^{a_1}2^{a_2}\cdots n^{a_n})$. Let $\operatorname{tr}_u(F\sigma|V^{\otimes n}) = \sum_i \operatorname{tr}(F\sigma|(V^{\otimes n})_i)(-u)^i$*

*be the graded trace of $F\sigma$ on $V^{\otimes n}$. We have*

$$\mathrm{tr}_u(F\sigma|V^{\otimes n}) = \prod_{i\geq 1}\left(\sum_{\alpha\in\mathcal{A}}(-1)^{\deg(\alpha)}u^{i\deg(\alpha)}\chi(\alpha)^i\right)^{a_i}.$$

*Proof.* As mentioned earlier, we may assume $F = F_s$ is already semi-simple. Thus $V^{\otimes n}$ has a homogeneous basis $\{\bigotimes_{i=1}^n \alpha_i \mid \text{for } \alpha_i \in \mathcal{A}\}$. Note that under this basis, the matrix for $F\sigma$ is *monomial*, meaning that there's only one non-zero entry for each row and each column. Thus the only contribution to trace comes from basis vectors $\bigotimes_{i=1}^n \alpha_i$ satisfying

$$\alpha_i = \alpha_{\sigma(i)} \text{ for all } i.$$

In other words, if $\sigma$ is decomposed into $r$ cycles of lengths $\lambda_1, \lambda_2, \ldots, \lambda_r$, the set $\{\alpha_i\}_{i=1}^n$ will only have $r$ distinct elements $\{\beta_j\}_{j=1}^r$, and we may compute

$$\begin{aligned} F\sigma\left(\bigotimes\nolimits_i \alpha_i\right) &= (-1)^{\sum_{j=1}^r(\lambda_j-1)\deg(\beta_j)}F\left(\bigotimes\nolimits_i \alpha_i\right) \\ &= \prod_{j=1}^r(-1)^{(\lambda_j-1)\deg(\beta_j)}\chi(\beta_j)^{\lambda_j}\left(\bigotimes\nolimits_i \alpha_i\right) \end{aligned}$$

and we get

$$\begin{aligned} \mathrm{tr}_u(F\sigma|V^{\otimes n}) &= \sum_{\beta_1,\ldots,\beta_r\in\mathcal{A}}\prod_{j=1}^r(-1)^{(\lambda_j-1)\deg(\beta_j)}\chi(\beta_j)^{\lambda_j}(-u)^{\lambda_j\deg(\beta_j)} \\ &= \prod_{j=1}^r\sum_{\beta\in\mathcal{A}}(-1)^{\deg(\beta)}\chi(\beta)^{\lambda_j}u^{\lambda_j\deg(\beta)} \\ &= \prod_i\left(\sum_{\beta\in\mathcal{A}}(-1)^{\deg(\beta)}\chi(\beta)^i u^{i\deg(\beta)}\right)^{a_i}. \end{aligned}$$

□

**Theorem 3.9.** *We have (by passing to $\overline{\mathbb{Q}}$)*

$$\sum_{n\geq 0}\mathbf{ch}_{F,u}(V^{\otimes n})t^n = \prod_\chi\left(1+\sum_{n=1}^\infty h_n\chi^n t^n\right)^{[V_\chi]} \tag{3}$$

*where $V_\chi = \bigoplus_{i\geq 0}V_{i,\chi}$ is the graded generalized eigenspace associated with the eigenvalue $\chi$ of $F$, and $[V_\chi] = \sum_{i\geq 0}(-u)^i\dim V_{i,\chi}$. The exponentiation on the right hand side simply means:*

$$\left(1+\sum_{n=1}^\infty h_n\chi^n t^n\right)^{[V_\chi]} = \prod_i\left(1+\sum_{n=1}^\infty h_n\chi^n(u^it)^n\right)^{(-1)^i\dim V_{i,\chi}}$$

*Proof.* Keeping notations the same as in the previous lemma, let $\sigma_\lambda$ denote any permutation of cycle type $\lambda$, and recalling that $z_\lambda = \prod_{i\geq 1}i^{a_i}a_i!$ for $\lambda = 1^{a_1}2^{a_2}\ldots$ and

$$\exp\left(\sum_{n\geq 1}\frac{p_n}{n}t^n\right) = 1+\sum_{n\geq 1}h_nt^n,$$

we have

$$\begin{aligned}
\sum_{n\geq 0}\mathbf{ch}_{F,u}(V^{\otimes n})t^n &= \sum_{\lambda}\operatorname{tr}_u(F\sigma_\lambda|V^{\otimes|\lambda|})\frac{p_\lambda}{z_\lambda}t^{|\lambda|}\\
&= \sum_{a_1,a_2,\dots\geq 0}\prod_{i\geq 1}\left(\sum_{\alpha\in\mathcal{A}}(-1)^{\deg(\alpha)}u^{i\deg(\alpha)}\chi(\alpha)^i t^i\right)^{a_i}\frac{p_i^{a_i}}{i^{a_i}a_i!}\\
&= \prod_{i\geq 1}\sum_{a\geq 0}\left(\frac{p_i}{i}\sum_{\alpha\in\mathcal{A}}(-1)^{\deg(\alpha)}u^{i\deg(\alpha)}\chi(\alpha)^i t^i\right)^{a}/a!\\
&= \exp\left(\sum_{i\geq 1}\frac{p_i}{i}\sum_{\alpha\in\mathcal{A}}(-1)^{\deg(\alpha)}u^{i\deg(\alpha)}\chi(\alpha)^i t^i\right)\\
&= \exp\left(\sum_{\alpha\in\mathcal{A}}(-1)^{\deg(\alpha)}\sum_{i\geq 1}\frac{p_i(u^{\deg(\alpha)}\chi(\alpha))^i t^i}{i}\right)\\
&= \prod_{\alpha\in\mathcal{A}}\left(1+\sum_{n\geq 1}h_n(u^{\deg(\alpha)}\chi(\alpha))^n t^n\right)^{(-1)^{\deg(\alpha)}}
\end{aligned}$$

which is equivalent to what we want. □

**Definition 3.10** (The groupoid generating series $Z_X(t)$)**.** Let $X$ be a quasi-projective variety over a finite field of size $q$. Recall that the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ acts on the set of rational points $C_n(X)(\mathbb{F}_q)$. We define $Z_X(t)$ to be the *groupoid generating series*

$$Z_X(t) = 1+\sum_{n=1}^{\infty}\frac{|C_n(X)(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|}t^n$$

We arrive at a new combinatorial proof of the following result.

**Theorem 3.11** ([Hua23])**.** *When $X$ is a smooth curve over a finite field of size $q$, the groupoid generating series $Z_X(t)$ has an infinite product formula*

$$Z_X(t) = \prod_{i=1}^{\infty}\zeta_X(q^{-i}t)$$

*Proof.* Note that by our main result Theorem 1.1 $C_n(X)$ and $S_n(X)$ have the same cohomology. In the case when $X$ is a smooth curve, $C_n(X)$ is a smooth variety of dimension $n^2$, thus by trace formula we have $C_n(X)(\mathbb{F}_q) = S_n(X)(\mathbb{F}_q)$. Now we can evaluate $Z_X(t)$ using Corollary 3.7

$$\begin{aligned}
1+\sum_{n=1}^{\infty}\frac{S_n(X)(\mathbb{F}_q)}{\mathrm{GL}_n(\mathbb{F}_q)}t^n &= 1+\sum_{n=1}^{\infty}\mathrm{sp}_{q^{-1}}(\mathbf{ch}_{F,1}(X^n))t^n\\
&= \mathrm{sp}_{q^{-1}}\left(\sum_{n\geq 0}\mathbf{ch}_{F,1}(X^n)t^n\right)
\end{aligned}$$

By Theorem 3.9, and the fact that $\mathrm{sp}_{q^{-1}}$ is a ring homomorphism, we can further evaluate (where $V = \bigoplus_r H^r(X)$)

$$\begin{aligned}\mathrm{sp}_{q^{-1}}\left(\sum_{n\geq 0}\mathbf{ch}_{F,1}(X^n)t^n\right) &= \prod_\chi \mathrm{sp}_{q^{-1}}\left(1+\sum_{n=1}^\infty h_n\chi^n t^n\right)^{[V_\chi]}\\ &= \prod_\chi\prod_{i\geq 0}\left(\frac{1}{1-q^{-i}\chi t}\right)^{[V_\chi]}\\ &= \prod_{i\geq 1}\zeta_X(q^{-i}t)\end{aligned}$$

where the second equality follows from the sum-product formula for generating series of $h_n$ ($x_n$ are the polynomial variables in $\mathsf{SymPoly}$)

$$\sum_{n\geq 0} h_n t^n = \prod_{n=1}^\infty \frac{1}{1-x_n t}$$

and the third equality follows from the product formula of the Weil zeta function

$$\zeta_X(t) = \prod_\chi\left(\frac{1}{1-\chi q t}\right)^{[V_\chi]}. \qquad \square$$

3.3. **Betti Zeta function.** Finally we prove a generating series for Poincare series of $C_n(X)$ using similar computations as above. To fully demonstrate the parallelism, let us also define the Betti analog of the zeta function.

**Definition 3.12** (Betti analog of zeta function)**.** Let $X$ be a variety over an algebraically closed field, we define its Betti zeta function $\zeta_X^{\mathrm{B}}(t)$ to be the generating series of Poincare series of $\mathrm{Sym}^n X$

$$\zeta_X^{\mathrm{B}}(t) = \sum_{n=0}^\infty P_u(\mathrm{Sym}^n X)t^n$$

The following Betti analog of Weil conjecture was proved by I. G. Macdonald in 1962.

**Theorem 3.13** ([Mac62b])**.** *Let $X$ be a variety over an algebraically closed field, let $P_u(X) = \sum_{i\geq 0}\dim H^i(X)(-u)^i$ be the Poincare polynomial of $X$. We have*

$$\zeta_X^{\mathrm{B}}(t) = \left(\frac{1}{1-t}\right)^{P_u(X)} = \prod_{i\geq 0}\left(\frac{1}{1-u^i t}\right)^{(-1)^i\dim H^i(X)}$$

Parallel to Theorem 3.11, we have the following Betti version.

**Theorem 3.14.** *We have the following product formula*

$$1+\sum_{n=1}^\infty P_u(\mathrm{Coh}_n(X))t^n = \prod_{i\geq 0}\left(\frac{1}{1-u^{2i}t}\right)^{(-1)^i\dim H^i(X)} = \prod_{i=0}^\infty \zeta_X^{\mathrm{B}}(u^{2i}t) \tag{4}$$

*Proof.* The Poincare polynomial of $C_n(X)$ equals that of $S_n(X)$, thus $P_u(C_n(X)) = \phi_n(u^2)\mathrm{sp}_{u^2}(\mathbf{ch}_u(X^n))$ by Equation (2). Furthermore, the degeneration result Corollary 2.12 implies that $P_u(\mathrm{Coh}_n(X)) = P_u(C_n(X))P_u(B\mathrm{GL}_n)$. Thus $P_u(\mathrm{Coh}_n(X))$ simply equals to $\mathrm{sp}_{u^2}(\mathbf{ch}_u(X^n))$.

Now an application of Theorem 3.9 (for $F = id$), and the same computation as in Theorem 3.11 shows

$$1 + \sum_{n=1}^{\infty} \mathrm{sp}_{u^2}(\mathbf{ch}_u(X^n))t^n = \prod_{i \geq 0} \left( \frac{1}{1 - u^{2i}t} \right)^{(-1)^i \dim H^i(X)} = \prod_{i=0}^{\infty} \zeta_X^{\mathrm{B}}(u^{2i}t)$$

where the last equality follows from Macdonald's formula (Theorem 3.13). □

**Corollary 3.15.** *When $X$ is connected, the Betti numbers of $C_n(X)$ stabilizes as $n \to \infty$*

$$\lim_{n \to \infty} P_u\left(C_n(X)\right) = \phi(u^2) \lim_{t \to 1} (t-1) \prod_{i=0}^{\infty} \zeta_X^{\mathrm{B}}(u^{2i}t)$$

*Proof.* The stability of $P_u(C_n(X))/\phi_n(u^2)$ follows from representation stability of $H^*(X^n)$ and $H^*(\mathrm{GL}_n/\mathrm{T}_n)$. The residue at $t = 1$ of Equation (4) creates a telescoping series, hence the corollary. □

## 4. A Hermitian variant

We conclude by presenting a Hermitian version of our main result, demonstrating the robustness of the Boson-Fermion duality phenomenon we observe. Let $X = \operatorname{Spec} R$ be an affine variety over the field of real numbers $\mathbb{R}$.

The variety $C_n(X)(\mathbb{C})$ is equipped with an involution $(-)^\dagger$, sending commuting complex matrices to their conjugate transposes. Canonically, $(-)^\dagger$ takes a homomorphism $\phi \colon R \to \mathbb{M}_n(\mathbb{C})$ to the composite $R \xrightarrow{\phi} \mathbb{M}_n(\mathbb{C}) \xrightarrow{(-)^\dagger} \mathbb{M}_n(\mathbb{C})$. (Thus we may take $X$ to be a general variety defined over $\mathbb{R}$.)

**Definition 4.1.** The space of Hermitian points $HC_n(X) \subset C_n(X)(\mathbb{C})$ is the fixed locus of the involution $(-)^\dagger$.

In other words, $HC_n(X)$ is the set of commuting Hermitian matrices satisfying the defining equations of $X$. Since the spectrum of Hermitian matrices consists of real numbers, the support map restricts to $HC_n(X) \to \mathrm{Sym}^n(X(\mathbb{R}))$.

We define the corresponding Fermionic space using the unitary group $\mathrm{U}_n$ and its subgroup of diagonal unitary matrices $\mathrm{UT}_n$.

**Definition 4.2.** The Hermitian Fermionic space is $HS_n(X) = X(\mathbb{R})^n \times_{S_n} \mathrm{U}_n/\mathrm{UT}_n$.

Analogous to the map $\sigma$, we have a morphism $\sigma_H \colon HS_n(X) \to HC_n(X)$ over $\mathrm{Sym}^n(X(\mathbb{R}))$. This map takes a tuple of real points and a decomposition into orthogonal lines (parametrized by $\mathrm{U}_n/\mathrm{UT}_n$) and constructs the corresponding semisimple Hermitian operator.

*Remark* 4.1. The spaces $HS_n(X)$ and $HC_n(X)$ are in fact real Galois twists of the original $S_n(X)$ and $C_n(X)$. On $S_n(X)$ we twist the complex conjugation by the inverse transpose operation on the factor $\mathrm{GL}_n/\mathrm{T}_n$, and on $C_n(X)$ we twist by the transpose operation. Note that the orginal $\sigma$ is invariant under these involutions, hence $\sigma_H : HS_n(X) \to HC_n(X)$ is the Galois twist of $\sigma$.

**Theorem 4.3** (Theorem 1.4)**.** *Let $p_1 : HS_n(X) \to \mathrm{Sym}^n X(\mathbb{R})$ and $p_2 : HC_n(X) \to \mathrm{Sym}^n X(\mathbb{R})$ denote the canonical projections, we have a sheaf theoretic isomorphism $\sigma_H^* \colon Rp_{2*}\underline{\mathbb{Q}} \approx Rp_{1*}\underline{\mathbb{Q}}$. In particular, $\sigma_H$ induces isomorphism between the rational cohomology of $HC_n(X)$ and $HS_n(X)$.*

*Proof.* We analyze the map $\sigma_H$ fiberwise over $\mathrm{Sym}^n(X(\mathbb{R}))$. A well-known result in linear algebra states that commuting Hermitian matrices are simultaneously diagonalizable by a unitary transformation. This implies:

(1) The map $\sigma_H$ is surjective.
(2) All finite sheaves appearing in $HC_n(X)$ are semisimple (they have no nilpotent parts).
(3) The following maps in the diagram are all proper maps

$$\begin{tikzcd} HS_n(X) \arrow[rr, "\sigma_H"] \arrow[dr, "p_1"'] & & HC_n(X) \arrow[dl, "p_2"] \\ & \mathrm{Sym}^n X(\mathbb{R}) & \end{tikzcd}$$

Let $D \in \mathrm{Sym}^n(X(\mathbb{R}))$ correspond to a partition $\lambda \vdash n$. The fiber of $HC_n(X) \to \mathrm{Sym}^n(X(\mathbb{R}))$ above $D$ consists of matrices unitarily equivalent to a fixed diagonal matrix with eigenvalues determined by $D$. This fiber is a single orbit under the unitary group $\mathrm{U}_n$. The stabilizer is the product of unitary groups corresponding to the eigenspaces, $\prod_i \mathrm{U}_{\lambda_i}$. Thus, the fiber is isomorphic to the homogeneous space $\mathrm{U}_n/\prod_i \mathrm{U}_{\lambda_i}$.

The fiber of $HS_n(X) \to \mathrm{Sym}^n(X(\mathbb{R}))$ above $D$ is isomorphic to $(\mathrm{U}_n/\mathrm{UT}_n)/S_\lambda$.

The map $\sigma_H$ restricts to a surjective map on the fibers:

$$\sigma_H \colon (\mathrm{U}_n/\mathrm{UT}_n)/S_\lambda \to \mathrm{U}_n/\prod_i \mathrm{U}_{\lambda_i}.$$

This is a fibration. The fibers are isomorphic to $\left(\prod_i \mathrm{U}_{\lambda_i}\right)/N(\prod_i \mathrm{UT}_{\lambda_i})$. Similar to the complex case, the spaces $\mathrm{U}_m/N(\mathrm{UT}_m)$ have trivial rational cohomology. Therefore, $\sigma_H$ induces an isomorphism on the cohomology of the fibers. By proper base change theorem, fiberwise cohomology isomorphism implies isomorphism of stalks $(Rp_{2*}\underline{\mathbb{Q}})_D \to (Rp_{1*}\underline{\mathbb{Q}})_D$, thus we obtain the sheaf theoretic isomorphism. □

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