

Target Controllability Score

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Abstract—We introduce the target controllability score (TCS), a concept for evaluating node importance under actuator constraints and designated target objectives, formulated within a virtual system setting. The TCS consists of the target volumetric controllability score (VCS) and the target average energy controllability score (AECS), each defined as an optimal solution to a convex optimization problem associated with the output controllability Gramian. We establish the existence and uniqueness (for almost all time horizons) and develop a projected gradient method for their computation. To enable scalability, we construct a target-only reduced virtual system and derive non-asymptotic bounds showing that weak cross-coupling and a low or negative logarithmic norm of the system matrix yield accurate approximations of target VCS/AECS, particularly over short or moderate time horizons. Experiments on human brain networks reveal a clear trade-off: at short horizons, both target VCS and target AECS are well approximated by their reduced formulations, while at long horizons, target AECS remains robust but target VCS deteriorates.

Index Terms—Brain networks, controllability, convex optimization, network centrality, reduced-order modeling

I. INTRODUCTION

Identifying the roles of individual nodes in complex systems—such as brain networks, social systems, or infrastructure systems—remains a central challenge in network science [1]–[6]. To address this, a wide range of centrality measures have been proposed. While these measures differ in scope and methodology, they are commonly classified into three categories: structural, spectral, and dynamics-aware. Structural and spectral measures—including degree, betweenness, closeness, eigenvector centrality, and PageRank—rely on network topology and, in some cases, edge weights, but ultimately provide static evaluations determined by the underlying structure [7], [8]. By contrast, dynamics-aware measures go beyond topology by relying on the system matrix to capture time evolution. This dynamical perspective is crucial in systems where the propagation of signals, information, or energy evolves over time, meaning that node importance cannot be fully understood from static connectivity alone [9]–[11].

Among such dynamics-aware approaches, controllability plays a central role, as it captures the fundamental ability of a system to be steered through external interventions [12], [13]. Qualitative controllability—often referred to as structural controllability—focuses on whether a system is controllable for almost all numerical choices of edge weights, based solely on the sparsity pattern of the system matrix together with the actuator placement [9], [14]. Graph theoretic tools such as maximum matchings and Dugale–Mendelsohn decomposition are employed to determine the minimum number of inputs

required for a network system to be controllable and to identify which state nodes should be actuated [15]–[20]. In practice, however, structural controllability provides only a coarse guarantee. Even if a network is controllable in the qualitative sense, achieving control may require prohibitively large input energy, rendering the system effectively uncontrollable from a practical standpoint [21]–[23]. This limitation highlights the need to move beyond a binary feasibility perspective and to quantify controllability in terms of the control effort required to steer the system.

In contrast, quantitative controllability enables the evaluation of the size of the reachable set and the amount of control energy required to reach a desired state, once the actual edge weights and time horizons are specified [22]–[27]. However, existing dynamics-aware indices such as the Volumetric Control Energy (VCE) and Average Control Energy (ACE) centralities [23] have inherent limitations. They rely on single-input controllability Gramians, which often become nearly singular in large-scale networks [21], [22], [28]–[31]. As a result, these measures may fail to consistently reflect node importance—a shortcoming noted in [32, Remarks 2 and 3] and [33, Section IV-B].

To address this limitation, Sato and Terasaki [32] introduced the controllability score (CS), a novel centrality metric defined for linear dynamical networks of the form

$$\dot{x}(t) = Ax(t), \quad (1)$$

which quantifies each state node’s ability to steer the system toward desired targets. Here, $x(t) \in \mathbb{R}^n$ collects the state variables assigned to the n nodes of the network, and the system matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ describes the weighted network topology, where each entry a_{ij} specifies how the state of node j influences that of node i . This initial formulation was further developed in later work [33], which broadened the theoretical foundation of the CS and, through applications to brain networks, demonstrated that it uncovers node importance behaviors markedly different from those identified by conventional centrality measures.

The CS was formulated under an idealized setting in which virtual inputs could, in principle, be applied to all state nodes of system (1). To formalize this, the virtual system

$$\dot{x}(t) = Ax(t) + \text{diag}(\sqrt{p_1}, \dots, \sqrt{p_n})u(t) \quad (2)$$

was introduced, establishing a one-to-one correspondence between virtual input nodes u_i and state nodes x_i , thereby assigning to each node x_i a weight p_i that reflects the degree to which it can be directly actuated. However, external control signals are rarely available beyond a limited subset of nodes, a constraint encountered across a wide range of domains, including neuroscience and systems biology [34]–[37]. Restricting the CS to this accessible subset thus provides a

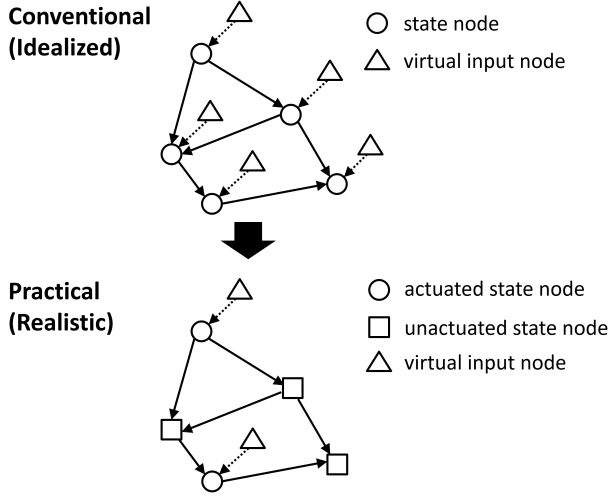


Fig. 1. Comparison between the conventional and practical settings.

principled way to identify intervention points that can steer the system's dynamics. This distinction between the conventional and practical settings is summarized in Fig. 1.

To address the limitation that only a subset of nodes is typically accessible for actuation, we introduce the target controllability score (TCS). This metric quantifies controllability-based importance for designated target nodes rather than all nodes. Unlike the original CS, TCS is specifically designed for scenarios where actuators are restricted to selected locations and the control objective concerns particular nodes. This formulation not only aligns the metric with realistic actuator placement and task objectives but also improves interpretability by focusing on a lower-dimensional target subspace. Nevertheless, even when restricted to a designated subset of target nodes, computing the TCS for large-scale networks is computationally prohibitive. This is because it is defined using the output controllability Gramian, whose computation involves evaluating matrix exponentials or equivalent procedures with cubic complexity in the network dimension n , although specialized methods can mitigate this cost.

A natural approach to resolve this difficulty is to construct a reduced model retaining only the target variables. In this reduced representation, the system dynamics are projected onto the subspace spanned by the chosen targets, yielding a model of dimension m , the number of target nodes. We then analyze the error in the CS computed from this reduced model, and identify structural properties of the original system matrix that ensure the error remains small. When such conditions are satisfied, the reduced model provides an efficient and interpretable surrogate for evaluating TCS in large-scale networks, substantially broadening the applicability of TCS to real-world systems.

The main contribution of this paper are summarized as follows:

- 1) We introduce two novel concepts—target volumetric controllability score (VCS) and target average energy controllability score (AECS)—defined on the designated target nodes via the output controllability Gramian introduced in [38], formulated as solutions to optimization problems.

These measures inherit the classical geometric and energy interpretations, while focusing on the task-relevant coordinates. We prove the existence and uniqueness of target VCS/AECS for any system matrix A in system (1) and for almost all time horizons, by modifying the proof of [33, Theorem 1]. The uniqueness ensures interpretability, comparability, and reproducibility. For computation, we propose a projected gradient scheme that solves the corresponding optimization problems to obtain the target VCS/AECS.

- 2) We develop a reduced virtual system that approximates the original target controllability scoring problem. We provide theoretical results showing that the accuracy of this reduction depends primarily on the strength of cross-coupling between target and non-target nodes and on the short-term growth or contraction rate of the dynamics, as quantified by the logarithmic norm. When the cross-coupling is weak and the dynamics exhibit slow or contractive behavior, the reduced system closely matches the original target VCS/AECS over short or moderate time horizons. We further show how additive approximation bounds can be converted into direct estimates of the error norms between the target and reduced formulations of VCS and AECS.
- 3) We validate our framework using human brain network data from 88 individuals. The experiments reveal a consistent pattern: at short horizons, both target VCS and target AECS are well approximated by their reduced formulations, but at long horizons, target AECS remains robust while target VCS exhibits substantial discrepancies. Moreover, target AECS identifies the same key brain regions across time horizons, demonstrating temporal stability, whereas VCS selects markedly different regions depending on the horizon, indicating sensitivity to time scale. These findings highlight a trade-off between the two measures: target VCS better reflects local coupling properties in short-term dynamics, while target AECS offers more robust characterization across varying horizons.

The remainder of this paper is organized as follows. In Section II, we define target VCS and target AECS, and present several results along with a computation method for evaluating them, which can be readily derived from the existing results presented in [32]. In Section III, we discuss the uniqueness of target VCS/AECS for any matrix $A \in \mathbb{R}^{n \times n}$. In Section IV, we propose approximation methods for the target VCS/AECS and conduct an error analysis. To this end, we derive a reduced-order model and analyze the error between the controllability Gramian associated with the reduced model and the output controllability Gramian of the original system. In Section V, we validate the proposed framework using human brain network data from 88 individuals and demonstrate a trade-off between short-term sensitivity of the target VCS and long-term robustness of the target AECS. Finally, Section VI concludes this paper.

Notation: The sets of real numbers and positive real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_{>0}$, respectively. For a matrix $X \in \mathbb{R}^{m \times n}$, $X^\top$ denotes the transpose of X . For a square

matrix $A \in \mathbb{R}^{n \times n}$, $\det A$ and $\text{tr}(A)$ denote the determinant and diagonal sum of A , respectively. For a symmetric matrix A , $A \succeq O$ (resp. $A \succ O$) denotes that A is positive semidefinite (resp. positive definite), and $A \preceq O$ (resp. $A \prec O$) denotes that A is negative semidefinite (resp. negative definite). More generally, for two symmetric matrices A and B , we write $A \preceq B$ (resp. $A \succeq B$) if $B - A$ (resp. $A - B$) is positive semidefinite. This is the standard Loewner partial order. For symmetric matrices X and Y ,

$$|X| \preceq Y \quad \Leftrightarrow \quad -Y \preceq X \preceq Y.$$

The symbols of $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ are the largest and smallest eigenvalues of a symmetric matrix X , respectively. The symbol I denotes the identity matrix of appropriate size. Given a vector $v = (v_i) \in \mathbb{R}^n$, $\|v\|$ and $\text{diag}(v_1, \dots, v_n)$ denote the usual Euclidean norm $\|v\| = \sqrt{v^\top v}$ and the diagonal matrix with the diagonal elements $v_1, \dots, v_n$, respectively. Instead of $\text{diag}(v_1, \dots, v_n)$, we also use $\text{diag}(v)$. For a matrix $A \in \mathbb{R}^{m \times n}$, we define its operator norm (or spectral norm) induced by the Euclidean norm as $\|A\| := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$. Equivalently, $\|A\|$ is the largest singular value of A . The symbol $\mathbf{1}$ denotes the all-ones column vector.

II. TARGET CONTROLLABILITY SCORE

The limitation, as noted in Section I, is that the original formulation assumes inputs can be applied to all state nodes of system (1), whereas in practice inputs are typically restricted to a limited subset of nodes. In this setting, system (2) is not an appropriate virtual representation.

Therefore, we introduce the following alternative formulation:

$$\begin{cases} \dot{x}(t) = Ax(t) + B(p)u(t) \\ y(t) = Cx(t) \end{cases} \quad (3)$$

where

$$B(p) := \begin{pmatrix} \sqrt{p_1} & 0 & \dots & 0 \\ 0 & \sqrt{p_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{p_m} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad (4)$$

$$C := (I_m \ 0) = \sum_{i=1}^m e_i^{(m)} (e_i^{(n)})^\top,$$

where p_i and $e_i^{(k)}$ denote a nonnegative real value and the i th standard basis vector in $\mathbb{R}^k$, respectively. The output matrix C specifies the target nodes: it serves as a selector that projects the full state vector $x(t)$ onto the subspace spanned by the designated targets. Consequently, the output $y(t) = Cx(t)$ contains only the components of the state corresponding to the chosen targets, so that controllability is evaluated only with respect to this restricted subset, rather than the entire network. This formulation implies that a virtual input is applied to the nodes indicated by $\bigcirc$ in Fig. 1, and only the states of these $\bigcirc$

nodes are observed. In other words, no virtual input is applied to the nodes marked by $\square$, nor are the states of these $\square$ nodes observed.

When the number of accessible input nodes m is small, the overall controllability of the full state equation $\dot{x}(t) = Ax(t) + B(p)u(t)$ becomes extremely weak, and the system is practically uncontrollable [21], [22], [28]–[31]. Therefore, instead of evaluating full-state controllability, we consider controllability only through the output $y(t) = Cx(t)$, which captures how the designated target nodes can be influenced by the available inputs. The formal definition of this output controllability will be introduced in Section II-A.

Remark 1: The specification of $B(p)$ and C implies no loss of generality. In (3), the first m state nodes of $x(t)$ are designated as target nodes. However, let P be a permutation matrix that reorders the coordinates of $x(t)$ so that the state coordinates corresponding to the target nodes specified by $B(p)$ are moved to the first m positions. By performing the coordinate transformation $\tilde{x}(t) := Px(t)$, the system can be rewritten into the same form as (3) with the transformed matrices $\tilde{A} = PAP^\top$ and $\tilde{B}(p) = PB(p)$. The output matrix in the transformed coordinates is $\tilde{C} = CP^\top = (I_m \ 0)$. Therefore, any selection of target nodes can be transformed into the above canonical form without loss of generality. For notational simplicity, we will henceforth adopt this canonical representation.

A. Output Controllability

Before introducing the output controllability Gramian used to define target VCS/AECS, we recall the notion of output controllability introduced in [38]. System (3) is said to be output controllable on $[0, T]$ if, for any initial state $x(0)$ and any desired terminal output y_T , there exists an input $u(\cdot)$ that steers the system so that $y(T) = y_T$ within the finite horizon T . In contrast, the usual notion of controllability requires that the entire state vector $x(t)$ can be driven to an arbitrary terminal state. Output controllability is therefore a weaker property, concerning only those components of the state observed through the output matrix C .

As shown in [38, Theorem I], system (3) is output controllable on $[0, T]$ if and only if the output controllability Gramian

$$W(p, T) = C\widetilde{W}(p, T)C^\top \quad (5)$$

is nonsingular; equivalently, $\text{rank } W(p, T) = m$. Here, $\widetilde{W}(p, T)$ is the usual controllability Gramian of system (3) and is given by

$$\begin{aligned} \widetilde{W}(p, T) &= \int_0^T \exp(At)B(p)B(p)^\top \exp(A^\top t) dt \\ &= \sum_{i=1}^m p_i \widetilde{W}_i(T), \end{aligned} \quad (6)$$

with

$$\widetilde{W}_i(T) := \int_0^T \exp(At)e_i^{(n)}(e_i^{(n)})^\top \exp(A^\top t) dt. \quad (7)$$

From (5) and (6), we obtain

$$W(p, T) = \sum_{i=1}^m p_i W_i(T), \quad (8)$$

where

$$W_i(T) := C \widetilde{W}_i(T) C^\top. \quad (9)$$

Equivalently, [38, Theorem III] shows that system (3) is output controllable on $[0, T]$ if and only if the output controllability matrix

$$(CB(p) \quad CAB(p) \quad \cdots \quad CA^{n-1}B(p)) \quad (10)$$

has full row rank m . Thus, the output controllability property of system (3) is independent of T . For this reason, we simply say that system (3) is output controllable, without explicitly referring to the horizon $[0, T]$.

This weaker notion of controllability is crucial in our setting, as we are concerned with steering only the designated target nodes rather than the full state vector.

Before proceeding, we establish a basic property of the matrices $W_i(T)$, which will be used in subsequent analysis.

Lemma 1: For any $T > 0$ and any $i \in \{1, \dots, m\}$, the matrix $W_i(T)$ in (9) satisfies $W_i(T) \succeq O$ and $W_i(T) \neq O$.

Proof: From (9),

$$W_i(T) = \int_0^T v_i(t) v_i(t)^\top dt,$$

where $v_i(t) := C \exp(At) e_i^{(n)} \in \mathbb{R}^m$. Thus, $W_i(T) \succeq O$.

Moreover, since C is defined in (4),

$$v_i(0) = C \exp(A \cdot 0) e_i^{(n)} = C e_i^{(n)} = e_i^{(m)} \neq 0.$$

Because $v_i(t)$ is continuous in t , there exists $\delta > 0$ such that $v_i(t) \neq 0$ for $t \in [0, \delta]$. If $T \geq \delta$, then

$$\int_0^T \|v_i(t)\|^2 dt \geq \int_0^\delta \|v_i(t)\|^2 dt > 0.$$

If $T < \delta$, then $v_i(t) \neq 0$ for all $t \in [0, T]$, and hence $\int_0^T \|v_i(t)\|^2 dt > 0$. In either case, we have $\int_0^T \|v_i(t)\|^2 dt > 0$. Thus,

$$\text{tr}(W_i(T)) = \int_0^T \text{tr}(v_i(t) v_i(t)^\top) dt = \int_0^T \|v_i(t)\|^2 dt > 0,$$

which implies $W_i(T) \neq O$. $\square$

B. Geometric and Energy-Based Interpretation of Output Controllability

In the classical setting, the controllability Gramian $\widetilde{W}(p, T)$ admits natural geometric and energy-based interpretations. Specifically, it characterizes the volume of the ellipsoid

$$\widetilde{\mathcal{E}}(p, T) := \{x \in \mathbb{R}^n \mid x^\top \widetilde{W}(p, T)^{-1} x \leq 1\},$$

which represents the set of states that can be reached within unit control energy, and it quantifies the average input energy required to steer the state to points on the unit sphere in $\mathbb{R}^n$.

In the target setting, however, our concern is not the entire state $x(t)$ but only the designated target nodes specified by the

output matrix C . Accordingly, the relevant object is the output controllability Gramian $W(p, T)$ defined in (5). This Gramian determines the shape of the ellipsoid

$$\mathcal{E}(p, T) := \{y \in \mathbb{R}^m \mid y^\top W(p, T)^{-1} y \leq 1\},$$

which represents the set of target outputs reachable within unit energy. The square root of $\det W(p, T)$ is proportional to the volume of $\mathcal{E}(p, T)$, quantifying how widely the output can be driven with unit energy. Moreover, $\text{tr}(W(p, T)^{-1})$ is proportional to the average of the minimum input energy $y_T^\top W(p, T)^{-1} y_T$ required to steer the output from the origin to y_T uniformly distributed on the unit sphere in $\mathbb{R}^m$. Hence, $\det W(p, T)$ and $\text{tr}(W(p, T)^{-1})$ respectively capture the volume of the reachable output set and the average input energy required to reach target outputs.

These geometric and energy-based viewpoints form the foundation for the definitions of the target VCS and target AECS, respectively.

C. Target VCS and Target AECS

For any positive number T , we define two convex sets on $\mathbb{R}^m$:

$$\begin{aligned} X_T &:= \{p \in \mathbb{R}^m \mid W(p, T) \succ O\}, \\ \Delta_m &:= \left\{ (p_i) \in \mathbb{R}^m \mid \begin{array}{l} \sum_{i=1}^m p_i = 1, \\ 0 \leq p_i \quad (i = 1, \dots, m) \end{array} \right\}. \end{aligned} \quad (11)$$

The set X_T is an open subset of $\mathbb{R}^m$, whereas Δ_m is a closed subset of $\mathbb{R}^m$.

To define the target VCS and target AECS, we consider the following optimization problem for a given $T > 0$:

$$\begin{aligned} &\text{minimize} && h_T(p) \\ &\text{subject to} && p \in X_T \cap \Delta_m. \end{aligned} \quad (12)$$

Here, h_T denotes either f_T or g_T on X_T , defined as

$$f_T(p) := -\log \det W(p, T), \quad g_T(p) := \text{tr}(W(p, T)^{-1}).$$

The logarithm in $f_T(p)$ improves numerical stability, since $\det W(p, T)$ —the product of the eigenvalues of $W(p, T)$ —can approach zero when several eigenvalues are extremely small. The negative sign allows both $f_T(p)$ and $g_T(p)$ to be formulated as minimization problems in a unified way. Note that $p \in X_T$ implies that system (3) is output controllable on $[0, T]$, as mentioned in Section II-A. Moreover, $p \in \Delta_m$ represents the relative importance assigned to each target node.

We define the target VCS as an optimal solution $p^* \in X_T \cap \Delta_m$ to problem (12) with $h_T = f_T$, and the target AECS as an optimal solution with $h_T = g_T$. From the perspective in Section II-B, the target VCS and target AECS respectively capture the roles of target nodes in enlarging $\mathcal{E}(p, T)$ and in reducing the control energy to reach the unit sphere in $\mathbb{R}^m$. This perspective clarifies how target VCS and target AECS generalize the classical counterparts by restricting the analysis to a designated subset of nodes rather than all nodes.

If an optimal solution to problem (12) is unique, the target VCS/AECS can serve as a centrality measure for network

system (1), since larger values of p_i indicate the greater importance of target node x_i in output controllability. Indeed, the gradients of $f_T(p)$ and $g_T(p)$ are given by

$$\begin{aligned}(\nabla f_T(p))_i &= -\text{tr}(W(p, T)^{-1}W_i(T)), \\ (\nabla g_T(p))_i &= -\text{tr}(W(p, T)^{-1}W_i(T)W(p, T)^{-1}),\end{aligned}$$

which imply that

$$\begin{aligned}(\nabla f_T(p))_i &= -\text{tr}(W(p, T)^{-1/2}W_i(T)W(p, T)^{-1/2}) < 0, \\ (\nabla g_T(p))_i &< 0\end{aligned}$$

for any $p \in X_T$, because Lemma 1 guarantees $W_i(T) \neq O$. This shows that increasing p_i enlarges the ellipsoid $\mathcal{E}(p, T)$ and reduces the average energy required for output steering.

Target VCS/AECS can be calculated using Algorithm 1, which is a modification from the method proposed in [32]. There are two key differences:

- While [32] considered the case $m = n$, Algorithm 1 allows a general number of target nodes $m \leq n$.
- Algorithm 1 is based on the output controllability Gramians (9), whereas [32] used the standard controllability Gramians (7).

In this algorithm, Π_{Δ_m} at step 3 denotes the efficient projection onto the standard simplex Δ_m in (11), as detailed in [39]. Note that the initial point $p^{(0)} := (1/m, \dots, 1/m)$ in Algorithm 1 guarantees that matrix (10) is of full row rank, namely, the corresponding virtual system (3) is output controllable.

Since the intersection $X_T \cap \Delta_m$ is neither closed nor open as demonstrated in [32, Section III-A], the existence of an optimal solution to problem (12) is not guaranteed a priori. Nevertheless, it can be established in the following manner: Suppose that $p^{(0)} \in X_T \cap \Delta_m$ is given, and define the sublevel set

$$\mathcal{F}_T^{(0)} := \{p \in \mathbb{R}^m \mid h_T(p) \leq h_T(p^{(0)})\} \cap (X_T \cap \Delta_m).$$

Arguing as in the proof of [32, Lemma 1], we obtain

$$\mathcal{F}_T^{(0)} = \{p \in \Delta_m \mid h_T(p) \leq h_T(p^{(0)})\}.$$

Therefore, problem (12) is equivalent to

$$\begin{aligned}\text{minimize} \quad & h_T(p) \\ \text{subject to} \quad & p \in \mathcal{F}_T^{(0)}.\end{aligned}\tag{13}$$

Since Δ_m is compact and h_T is continuous on X_T , the set $\mathcal{F}_T^{(0)}$ is a nonempty compact subset of $\mathbb{R}^m$. Thus, by Weierstrass' theorem, problem (13) has an optimal solution (see [40, Proposition A.8]), and problem (12) also admits an optimal solution.

We can prove the following proposition in the same manner as [32, Theorem 6].

Proposition 1: Suppose that an optimal solution to problem (12) is unique. Let $\{p^{(k)}\}$ be a sequence generated by Algorithm 1 with $\varepsilon = 0$. Then,

$$\lim_{k \rightarrow \infty} p^{(k)} = p^*,$$

where p^* is the optimal solution to problem (12), namely, the target VCS/AECS.

Algorithm 1 A projected gradient method

Input: Output Controllability Gramians $W_1(T), \dots, W_m(T)$ in (9), $p^{(0)} := (1/m, \dots, 1/m) \in X_T \cap \Delta_m$, and $\varepsilon \geq 0$.

Output: target VCS/AECS.

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1: for  $k = 0, 1, \dots$  do
2:    $q^{(k)} := p^{(k)} - \alpha^{(k)} \nabla h_T(p^{(k)})$ , where  $\alpha^{(k)}$  is defined by
     using Algorithm 2.
3:    $p^{(k+1)} := \Pi_{\Delta_m}(q^{(k)})$ .
4:   if  $\|p^{(k)} - p^{(k+1)}\| \leq \varepsilon$  then
5:     return  $p^{(k+1)}$ .
6:   end if
7: end for
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Algorithm 2 Armijo rule along the projection arc

Input: $\sigma, \rho \in (0, 1)$ and $\alpha > 0$.

Output: Step size.

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1:  $\tilde{p}^{(k)} := \Pi_{\Delta_m}(p^{(k)} - \alpha \nabla h_T(p^{(k)}))$ .
2: if  $h_T(\tilde{p}^{(k)}) \leq h_T(p^{(k)}) + \sigma \nabla h_T(p^{(k)})^\top (\tilde{p}^{(k)} - p^{(k)})$  then
3:   return  $\alpha^{(k)} := \alpha$ .
4: else
5:    $\alpha \leftarrow \rho \alpha$ , and go back to step 1.
6: end if
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Remark 2: According to [33, Theorem 6], Algorithm 1 is guaranteed to converge linearly to the optimal solution of problem (12) under mild assumptions. Using an argument analogous to that in [33, Section III-C], it can be shown that for sufficiently small $T > 0$, the optimal solution to problem (12) is approximately $(1/m, \dots, 1/m)$.

III. UNIQUENESS OF TARGET CONTROLLABILITY SCORE

We now turn to the question of the uniqueness of the target VCS/AECS, that is, an optimal solution to problem (12). Such uniqueness is fundamental for ensuring interpretability, comparability, and reproducibility when target VCS/AECS is employed as a centrality measure for target nodes.

To this end, we use the Hessians of $f_T(p)$ and $g_T(p)$, which are expressed as

$$\begin{aligned}(\nabla^2 f_T(p))_{ij} &= \text{tr}(W(p, T)^{-1}W_i(T)W(p, T)^{-1}W_j(T)),\end{aligned}\tag{14}$$

$$\begin{aligned}(\nabla^2 g_T(p))_{ij} &= \text{tr}(W(p, T)^{-1}W_i(T)W(p, T)^{-1}W_j(T)W(p, T)^{-1}) \\ &\quad + \text{tr}(W(p, T)^{-1}W_j(T)W(p, T)^{-1}W_i(T)W(p, T)^{-1}),\end{aligned}\tag{15}$$

respectively. For the uniqueness analysis, it is useful to rewrite these Hessians in quadratic form. The following lemma provides such a representation.

Lemma 2: Suppose that $T > 0$ and $p \in X_T$ are given. Then, for any $x \in \mathbb{R}^m$,

$$x^\top \nabla^2 f_T(p)x = \text{tr}(G(p, x, T)^2), \quad (16)$$

$$x^\top \nabla^2 g_T(p)x = 2\text{tr}(W(p, T)^{-1}G(p, x, T)^2), \quad (17)$$

where

$$G(p, x, T) := W(p, T)^{-1/2}W(x, T)W(p, T)^{-1/2}.$$

Proof: It follows from (14) that for any $x \in \mathbb{R}^m$,

$$\begin{aligned} x^\top \nabla^2 f_T(p)x &= \sum_{i,j=1}^m x_i (\nabla^2 f_T(p))_{ij} x_j \\ &= \text{tr}(W(p, T)^{-1}W(x, T)W(p, T)^{-1}W(x, T)). \end{aligned}$$

Thus, (16) holds. Similarly, it follows from (15) that for any $x \in \mathbb{R}^m$,

$$\begin{aligned} x^\top \nabla^2 g_T(p)x &= \sum_{i,j=1}^m x_i (\nabla^2 g_T(p))_{ij} x_j \\ &= 2\text{tr}(W(p, T)^{-1}W(x, T)W(p, T)^{-1}W(x, T)W(p, T)^{-1}). \end{aligned}$$

Thus, (17) holds. $\square$

Lemma 2 yields the following result, which is pivotal for proving the uniqueness of the target VCS and AECS. Although its proof closely follows the arguments of [32, Lemma 2 and Theorem 1], we include it here for completeness.

Lemma 3: Let $T > 0$ be arbitrary. If

$$W(x, T) = O \quad \Rightarrow \quad x = 0, \quad (18)$$

then an optimal solution to problem (12) is unique.

Proof: Since $X_T \cap \Delta_m$ is convex, the uniqueness of an optimal solution to problem (12) follows if $h_T(p)$ is strictly convex on X_T , as shown in [40, Proposition 1.1.2].

First, let $h_T(p) := f_T(p)$. From Lemma 2, $x^\top \nabla^2 f_T(p)x = 0$ implies that $G(p, x, T) = O$, meaning that $W(x, T) = O$. By assumption (18), we then have $x = 0$. This means that $f_T(p)$ is strictly convex on X_T .

Next, let $h_T(p) := g_T(p)$. From Lemma 2, $x^\top \nabla^2 g_T(p)x = 0$ implies that $G(p, x, T)^2 = O$, meaning that

$$\begin{aligned} W(x, T)W(p, T)^{-1}W(x, T) &= O \\ \Leftrightarrow V(p, x, T)^\top V(p, x, T) &= O, \end{aligned}$$

where $V(p, x, T) := W(p, T)^{-1/2}W(x, T)$. Thus, $x^\top \nabla^2 g_T(p)x = 0$ implies that $V(p, x, T) = O$, which yields $W(x, T) = O$, and hence $x = 0$ by assumption (18). Therefore, $g_T(p)$ is strictly convex on X_T . $\square$

Since $W(p, T)$ is given in (8), assumption (18) is equivalent to requiring that the output controllability Gramians $W_1(T), \dots, W_m(T)$ in (9) are linearly independent over $\mathbb{R}$.

Using Lemma 3, we can derive the following theorem. Although the proof parallels that of [33, Theorem 1], it replaces the controllability Gramian (6) used there with the output controllability Gramian (5), and the argument has been adjusted accordingly.

Theorem 1: For all $A \in \mathbb{R}^{n \times n}$ and almost all $T > 0$, there exists a unique solution to problem (12).

Proof: From Lemma 3, it is sufficient to show that for almost all $T > 0$ and all $x = (x_i) \in \mathbb{R}^m$, $W(x, T) = O$ yields $x = 0$. Thus, we assume $W(x, T) = O$. Note that $W_i(T)$ can be expressed as

$$W_i(T) = \int_0^T P(t) e_i^{(n)} (e_i^{(n)})^\top P(t)^\top dt$$

with $P(t) := C \exp(At)$. For $i = 1, 2, \dots, m$, the (i, i) -th component of $W(x, T)$ is obtained as

$$\begin{aligned} (W(x, T))_{ii} &= (e_i^{(m)})^\top W(x, T) e_i^{(m)} \\ &= \sum_{j=1}^m x_j \cdot \int_0^T ((e_i^{(m)})^\top P(t) e_j^{(n)})^2 dt \\ &= \sum_{j=1}^m x_j \cdot \int_0^T P_{ij}(t)^2 dt. \end{aligned} \quad (19)$$

Eq. (19) implies that $W(x, T) = O$ yields

$$R(T)x = 0, \quad (20)$$

where

$$R(T) := \int_0^T \begin{pmatrix} P_{11}(t)^2 & P_{12}(t)^2 & \cdots & P_{1m}(t)^2 \\ P_{21}(t)^2 & P_{22}(t)^2 & \cdots & P_{2m}(t)^2 \\ \vdots & \vdots & \ddots & \vdots \\ P_{m1}(t)^2 & P_{m2}(t)^2 & \cdots & P_{mm}(t)^2 \end{pmatrix} dt.$$

If $\det R(T) \neq 0$, (20) implies $x = 0$. Thus, it suffices to show that $\det R(T) \neq 0$ for almost all $T > 0$. Note that $R(0) = 0$ and

$$\begin{aligned} \frac{dR}{dT}(0) &= \begin{pmatrix} (e_1^{(m)})^\top C e_1^{(n)} & (e_1^{(m)})^\top C e_2^{(n)} & \cdots & (e_1^{(m)})^\top C e_m^{(n)} \\ (e_2^{(m)})^\top C e_1^{(n)} & (e_2^{(m)})^\top C e_2^{(n)} & \cdots & (e_2^{(m)})^\top C e_m^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ (e_m^{(m)})^\top C e_1^{(n)} & (e_m^{(m)})^\top C e_2^{(n)} & \cdots & (e_m^{(m)})^\top C e_m^{(n)} \end{pmatrix} \\ &= I, \end{aligned}$$

where the second equality follows from (4). Therefore, the fact that $\det R(T) \neq 0$ for almost all $T > 0$ follows by exactly the same argument as in the proof of [33, Theorem 1]. $\square$

By Theorem 1, target VCS/AECS can be used as centrality measures for target nodes.

Note that we cannot replace ‘‘almost all T ’’ in Theorem 1 with ‘‘all T ’’, because there is an example where a solution to problem (12) is not unique, as shown in [32, Section IV].

IV. APPROXIMATE COMPUTATION OF TARGET VCS/AECS

Although $W_i(T)$ in (9) is of size $m \times m$, computing it requires first obtaining $\tilde{W}_i(T)$ in (7), which is an $n \times n$ matrix. Thus, in practice, one still needs to compute and store large matrices when $n \gg m$. As a result, for large-scale systems, solving optimization problem (12) to obtain the target VCS/AECS becomes computationally intractable. This motivates the development of approximate methods that can efficiently estimate these scores without explicitly forming large-scale Gramians.

To this end, starting from virtual system (3), we set $x(t) = C^\top x_{\text{red}}(t)$ and obtain

$$\dot{x}_{\text{red}}(t) = A_{11}x_{\text{red}}(t) + \text{diag}(\sqrt{p_1}, \dots, \sqrt{p_m})u, \quad (21)$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ and } A_{11} = CAC^\top \in \mathbb{R}^{m \times m}. \quad (22)$$

This reduced virtual system describes the dynamics projected onto the first m coordinates, where the input matrix is diagonal and each virtual input u_i acts directly on the corresponding reduced state $x_{\text{red},i}$. Thus, the reduced virtual system in (21) contains exactly m state nodes, and the controllability Gramian $W_{\text{red}}(p, T)$ of virtual system (21) is given by

$$W_{\text{red}}(p, T) = \sum_{i=1}^m p_i W_{i,\text{red}}(T), \quad (23)$$

$$W_{i,\text{red}}(T) := \int_0^T \exp(A_{11}t) e_i^{(m)} (e_i^{(m)})^\top \exp(A_{11}^\top t) dt.$$

The standard CS [32], [33] is an optimal solution to the following optimization problem for a given $T > 0$:

$$\begin{array}{ll} \text{minimize} & h_T^{\text{red}}(p) \\ \text{subject to} & p \in X_T^{\text{red}} \cap \Delta_m. \end{array} \quad (24)$$

Here,

$$X_T^{\text{red}} := \{p \in \mathbb{R}^m \mid W_{\text{red}}(p, T) \succ O\},$$

and $h_T^{\text{red}}(p)$ denotes either $f_T^{\text{red}}(p)$ or $g_T^{\text{red}}(p)$ on X_T^{red} , defined as

$$\begin{aligned} f_T^{\text{red}}(p) &:= -\log \det W_{\text{red}}(p, T), \\ g_T^{\text{red}}(p) &:= \text{tr}(W_{\text{red}}(p, T)^{-1}). \end{aligned}$$

The VCS/AECS, i.e., an optimal solution to problem (24), can be obtained using Algorithm (1) by replacing the output controllability Gramians $W_1(T), \dots, W_m(T)$ with $W_{1,\text{red}}(T), \dots, W_{m,\text{red}}(T)$. It has been proven in [33, Theorem 1] that for all $A_{11} \in \mathbb{R}^{m \times m}$ and for almost all $T > 0$, problem (24) admits a unique solution.

In the following, we investigate under which conditions on the system matrix A in (1) the standard CS of reduced system (21) serves as a good approximation to the TCS of original system (3).

A. Error and Structural Analysis between the Reduced and Output Controllability Gramians

In this subsection, we provide both quantitative error bounds and a structural interpretation of the gap between reduced controllability Gramian (23) and output controllability Gramian (9) by analyzing

$$\|W_{\text{red}}(p, T) - W(p, T)\| \leq \varepsilon_T(p), \quad (25)$$

where

$$\varepsilon_T(p) := \sum_{i=1}^m p_i \|\Delta W_i(T)\|, \quad (26)$$

$$\Delta W_i(T) := W_{i,\text{red}}(T) - W_i(T). \quad (27)$$

To obtain an upper bound for $\|\Delta W_i(T)\|$, we first establish an integral representation of the difference

$$X(t) := \exp(A_{11}t) - C \exp(At) C^\top \quad (28)$$

using a variation-of-constants argument. For this purpose, define

$$E := \begin{pmatrix} 0 & -A_{12} \end{pmatrix}. \quad (29)$$

Lemma 4: Assume (4), (22), (28), and (29). Then,

$$X(t) = \int_0^t \exp(A_{11}(t-s)) E \exp(As) C^\top ds. \quad (30)$$

Proof: Let $Y(t) := C \exp(At)$. Then,

$$\begin{aligned} \dot{Y}(t) &= CA \exp(At) \\ &= A_{11}Y(t) - E \exp(At) \end{aligned} \quad (31)$$

with $Y(0) = C$. Here, the second equality follows from

$$CA = \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} = A_{11}C - E.$$

The solution to (31) is given by

$$Y(t) = \exp(A_{11}t)Y(0) - \int_0^t \exp(A_{11}(t-s)) E \exp(As) ds. \quad (32)$$

Substituting (32) into the right hand side of

$$X(t) = (\exp(A_{11}t)C - Y(t))C^\top,$$

we obtain (30). $\square$

To obtain an upper bound for $\|\Delta W_i(T)\|$, we next derive exponential bounds on the matrix exponentials $\exp(At)$ and $\exp(A_{11}t)$, based on the logarithmic norm defined by

$$\mu(A) := \lambda_{\max} \left(\frac{A + A^\top}{2} \right).$$

Lemma 5: Assume (22). Then, for all $t \geq 0$,

$$\|\exp(At)\| \leq e^{\mu(A)t}, \quad \|\exp(A_{11}t)\| \leq e^{\mu(A_{11})t} \leq e^{\mu(A)t}.$$

Proof: For any fixed vector $c \in \mathbb{R}^n$ (independent of t), define $z(t) := \exp(At)c$. Then,

$$\begin{aligned} \frac{d}{dt} \|z(t)\|^2 &= 2z(t)^\top A z(t) \\ &= z(t)^\top (A^\top + A) z(t) \leq 2\mu(A) \|z(t)\|^2, \end{aligned}$$

and integrating this differential inequality gives

$$\|z(t)\| \leq e^{\mu(A)t} \|z(0)\| = e^{\mu(A)t} \|c\|.$$

Thus,

$$\|\exp(At)\| = \sup_{c \neq 0} \frac{\|\exp(At)c\|}{\|c\|} \leq e^{\mu(A)t} \quad (t \geq 0).$$

The same argument applied to A_{11} yields

$$\|\exp(A_{11}t)\| \leq e^{\mu(A_{11})t} \quad (t \geq 0).$$

Since $\mu(A_{11}) \leq \mu(A)$, the desired inequality follows. In fact, since $(A_{11} + A_{11}^\top)/2$ is a principal submatrix of $(A + A^\top)/2$, Cauchy's interlacing theorem (see [41, Theorem 4.3.28]) yields

$$\mu(A_{11}) = \lambda_{\max}\left(\frac{A_{11} + A_{11}^\top}{2}\right) \leq \lambda_{\max}\left(\frac{A + A^\top}{2}\right) = \mu(A).$$

This completes the proof. $\square$

If the logarithmic norm satisfies $\mu(A) < 0$, then both A and A_{11} are stable, since Lemma 5 implies that

$$\|\exp(At)\| \leq e^{\mu(A)t} \rightarrow 0, \quad \|\exp(A_{11}t)\| \leq e^{\mu(A)t} \rightarrow 0$$

as $t \rightarrow \infty$. However, stability does not, in general, transfer between a matrix and its principal submatrix: for example, $A = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$ is stable whereas $A_{11} = 1$ is unstable, while $A = \begin{pmatrix} -1 & 10 \\ 0 & 1 \end{pmatrix}$ is unstable even though $A_{11} = -1$ is stable. These examples demonstrate that relying solely on the stability of either block can be misleading. In contrast, verifying $\mu(A) < 0$ guarantees stability of both the full and reduced systems, highlighting the importance of analyzing $\mu(A)$ as a unified stability criterion.

Using Lemmas 4 and 5, we obtain the following expression and bound on $\Delta W_i(T)$.

Theorem 2: Assume (4), (22), and (29). For any $T > 0$, define $\Delta W_i(T)$ as (27). Then,

$$\Delta W_i(T) = \int_0^T G_i(t) dt, \quad (33)$$

where

$$\begin{aligned} G_i(t) &= \int_0^t \exp(A_{11}(t-s)) E \exp(As) e_i^{(n)}(e_i^{(m)})^\top \exp(A_{11}^\top t) ds \\ &+ \int_0^t \exp(A_{11}t) e_i^{(m)}(e_i^{(n)})^\top \exp(A^\top s) E^\top \exp(A_{11}^\top(t-s)) ds. \end{aligned} \quad (34)$$

Then,

$$\|\Delta W_i(T)\| \leq \Phi_{\mu(A)}(T) \|A_{12}\|, \quad (35)$$

where

$$\Phi_{\mu(A)}(T) := \begin{cases} \frac{e^{2\mu(A)T} (2\mu(A)T - 1) + 1}{2\mu(A)^2} & (\mu(A) \neq 0) \\ T^2 & (\mu(A) = 0). \end{cases} \quad (36)$$

In particular, if $\mu(A) < 0$, then $\lim_{T \rightarrow \infty} \Delta W_i(T)$ exists and

$$\lim_{T \rightarrow \infty} \|\Delta W_i(T)\| \leq \frac{1}{2\mu(A)^2} \|A_{12}\|.$$

Proof : First, we establish (33). Definitions (27) and (28) imply that

$$\begin{aligned} \Delta W_i(T) &= \int_0^T \left(\exp(A_{11}t) e_i^{(m)}(e_i^{(m)})^\top \exp(A_{11}^\top t) \right. \\ &\quad \left. - C \exp(At) e_i^{(n)}(e_i^{(n)})^\top \exp(A^\top t) C^\top \right) dt \\ &= \int_0^T \left(X(t) e_i^{(m)}(e_i^{(m)})^\top \exp(A_{11}^\top t) \right. \\ &\quad \left. + C \exp(At) C^\top e_i^{(m)}(e_i^{(m)})^\top X(t)^\top \right) dt, \end{aligned} \quad (37)$$

where the second equality follows from the identity $C^\top e_i^{(m)} = e_i^{(n)}$. Applying Lemma 4 to (37) yields (33).

Next, we derive (35). Using (34), sub-multiplicativity of the operator norm $\|\cdot\|$, $\|e_i^{(n)}(e_i^{(m)})^\top\| = 1$, $\|E\| = \|A_{12}\|$, and Lemma 5, the first integrand in (34) can be bounded as

$$\begin{aligned} &\|\exp(A_{11}(t-s)) E \exp(As) e_i^{(n)}(e_i^{(m)})^\top \exp(A_{11}^\top t)\| \\ &\leq \|\exp(A_{11}(t-s))\| \cdot \|E\| \cdot \|\exp(As)\| \cdot \|e_i^{(n)}(e_i^{(m)})^\top\| \\ &\quad \cdot \|\exp(A_{11}^\top t)\| \\ &\leq e^{\mu(A)(t-s)} \|A_{12}\| e^{\mu(A)s} e^{\mu(A)t} = \|A_{12}\| e^{2\mu(A)t}. \end{aligned}$$

The second integrand in (34) is bounded in the same way, and hence

$$\|G_i(t)\| \leq 2 \int_0^t \|A_{12}\| e^{2\mu(A)s} ds = 2 \|A_{12}\| t e^{2\mu(A)t}.$$

Therefore,

$$\|\Delta W_i(T)\| \leq \int_0^T \|G_i(t)\| dt = 2 \|A_{12}\| \int_0^T t e^{2\mu(A)t} dt,$$

which yields (35). $\square$

Choice (4) is not only the natural setting for defining the target controllability score—since the target nodes are then identified with the first m state coordinates—but also serves as a structural assumption that enables the simple and interpretable error estimates established in Theorem 2. In this case, leading to the block decomposition (22) in which the approximation error depends only on the off-diagonal block A_{12} . This structure permits the simple estimate (35). For general C , by contrast, the error representation depends on multiple matrix blocks (A_{12} , A_{21} , A_{22}) and on $\|C\|$.

The bound in (35) shows that the approximation error scales linearly with the cross-coupling magnitude $\|A_{12}\|$, with a time- and dynamics-dependent prefactor $\Phi_{\mu(A)}(T)$. In block decomposition (22), the block A_{12} encodes the direct influence from the lower-level subsystem to the higher-level subsystem. Its norm therefore quantifies the inter-layer coupling: a small $\|A_{12}\|$ implies weak upward influence, so the reduced Gramian $W_{\text{red}}(p, T)$ closely approximates the full output-controllability Gramian $W(p, T)$. Our bound in Theorem 2 shows that for networks with sparse or weak upward connections the controllability analysis may safely ignore the peripheral dynamics (See Fig. 2). This expectation is numerically supported in Section V, where the results indicate that the bound provides a useful estimate for small horizons T , and that networks with

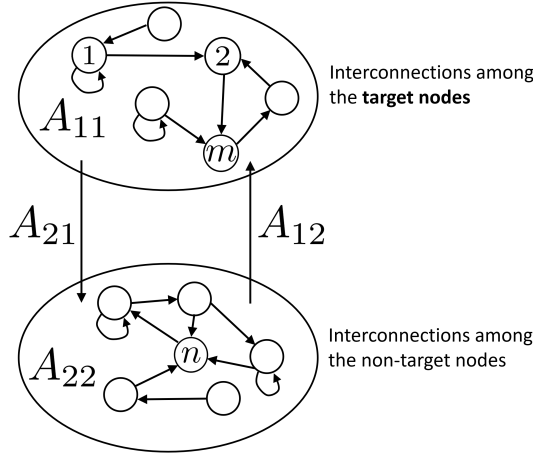


Fig. 2. Illustration of the block decomposition of A and the role of A_{12} . The block A_{12} represents upward coupling from the lower-level subsystem to the higher-level subsystem. When $\|A_{12}\|$ is small and the exponential factor $\Phi_{\mu(A)}(T)$ is sufficiently small, the reduced Gramian $W_{\text{red}}(p, T)$ provides a close approximation to the full output controllability Gramian $W(p, T)$.

smaller $\|A_{12}\|$ tend to exhibit a smaller approximation error in the TCS.

This block-structured interpretation is not merely theoretical: many real-world networks naturally exhibit such hierarchical or layered organization, where interactions between subsystems are sparse or asymmetric. Examples include brain connectomes, power grids, and multi-scale biochemical networks [42]–[44].

Remark 3: The key difference between the error analysis of [45] and our approach lies in the definition of the output operator. In [45], the output is not taken as individual node states but rather as aggregate variables formed from the non-actuated part of the network. This design is motivated by applications concerned with the controllability of collective behaviors, and it imposes a structural constraint on the output matrix C : it must represent such aggregated quantities. Consequently, the resulting bounds involve general norms of both A and C unlike our setting.

B. Relative (Multiplicative) Gramian Error Analysis

Our next goal is to convert additive Gramian error (25) into a relative multiplicative error analysis. An additive error bound measures the absolute difference between W_{red} and W , whereas a multiplicative error bound quantifies their relative deviation. Since the objective function $h_T(p)$ of optimization problem (12) involves $\det(W)$ and W^{-1} , its sensitivity is naturally governed by relative deviations of W , rather than by absolute differences. For this reason, multiplicative error bounds are directly relevant to the analysis of the target VCS/AECS. The additive error estimates established earlier provide the foundation for deriving the multiplicative comparisons presented below.

To derive a multiplicative error bound, we normalize additive error upper bound (26) by the smallest eigenvalue

$\lambda_{\min}(W(p, T))$, thereby defining

$$\delta_T(p) := \frac{\varepsilon_T(p)}{\lambda_{\min}(W(p, T))}, \quad (38)$$

for each $T > 0$ and $p \in X_T \cap \Delta_m$. Note that $\lambda_{\min}(W(p, T)) > 0$ for any $p \in X_T$, because $p \in X_T$ guarantees that corresponding virtual system (3) is output controllable, as explained in Section II-A.

Lemma 6 (Pointwise error bound): For each $T > 0$ and $p \in X_T \cap \Delta_m$, define $\delta_T(p)$ in (38). Then,

$$|W_{\text{red}}(p, T) - W(p, T)| \preceq \delta_T(p)W(p, T). \quad (39)$$

Proof: Since for any symmetric matrix S we have $|S| \preceq \|S\|I$, bound (25) implies

$$|W_{\text{red}}(p, T) - W(p, T)| \preceq \varepsilon_T(p)I, \quad (40)$$

by setting $S = W_{\text{red}}(p, T) - W(p, T)$. By definition, $W(p, T) \succeq \lambda_{\min}(W(p, T))I$ and thus

$$\varepsilon_T(p)I \preceq \delta_T(p)W(p, T).$$

Using this in (40), we obtain (39). $\square$

Even if $\lambda_{\min}(W(p, T)) > 0$ holds for any $p \in X_T \cap \Delta_m$, the global output controllability margin

$$\inf_{p \in X_T \cap \Delta_m} \lambda_{\min}(W(p, T)) \quad (41)$$

may still be zero, because the feasible set $X_T \cap \Delta_m$ is in general not closed, as mentioned in Section II-C. Consequently, the infimum need not be attained, and sequences approaching the boundary may drive $\lambda_{\min}(W(p, T))$ arbitrarily close to zero.

Example 1: Consider system (3) with $n = 3$, $m = 2$, and $A = \text{diag}(-1, 1/2, -3)$. Then $\exp(At) = \text{diag}(e^{-t}, e^{t/2}, e^{-3t})$, and

$$\begin{aligned} \widetilde{W}_i(T) &= \int_0^T \exp(At) e_i^{(3)} (e_i^{(3)})^\top \exp(A^\top t) dt \\ &= \begin{cases} \text{diag}(\phi_{-1}(T), 0, 0) & (i = 1) \\ \text{diag}(0, \phi_{1/2}(T), 0) & (i = 2) \end{cases} \end{aligned}$$

where $\phi_\gamma(T) := \frac{e^{2\gamma T} - 1}{2\gamma}$ ($\gamma \neq 0$). Hence

$$W_i(T) = C \widetilde{W}_i(T) C^\top = \begin{cases} \text{diag}(\phi_{-1}(T), 0) & (i = 1), \\ \text{diag}(0, \phi_{1/2}(T)) & (i = 2), \end{cases}$$

and thus

$$\begin{aligned} W(p, T) &= p_1 W_1(T) + p_2 W_2(T) \\ &= \text{diag}(p_1 \phi_{-1}(T), p_2 \phi_{1/2}(T)). \end{aligned}$$

Therefore

$$\begin{aligned} \lambda_{\min}(W(p, T)) &= \min\{p_1 \phi_{-1}(T), p_2 \phi_{1/2}(T)\}, \\ X_T \cap \Delta_2 &= \{(p_1, p_2) \mid p_1 > 0, p_2 > 0, p_1 + p_2 = 1\}. \end{aligned}$$

For any $p \in X_T \cap \Delta_2$, we have $\lambda_{\min}(W(p, T)) > 0$. Nevertheless, the sequence $\{p^{(k)}\}$ defined by $p^{(k)} = \left(\frac{1}{k+2}, \frac{k+1}{k+2}\right)$ satisfies $p^{(k)} \in X_T \cap \Delta_2$ for all $k \in \{0, 1, 2, \dots\}$ and

$$\begin{aligned} \lambda_{\min}(W(p^{(k)}, T)) &= \min \left\{ \frac{1}{k+2} \phi_{-1}(T), \frac{k+1}{k+2} \phi_{1/2}(T) \right\} \\ &\leq \frac{1}{k+2} \phi_{-1}(T) \xrightarrow{k \rightarrow \infty} 0, \end{aligned}$$

and hence $\inf_{p \in X_T \cap \Delta_2} \lambda_{\min}(W(p, T)) = 0$, even though $\lambda_{\min}(W(p, T)) > 0$ at any $p \in X_T \cap \Delta_2$. $\square$

To obtain a uniform error bound from the pointwise error bound in Lemma 6, we restrict the feasible set to a nonempty compact subset $Z \subset X_T \cap \Delta_m$. In contrast to (41), the local output controllability margin

$$\inf_{p \in Z} \lambda_{\min}(W(p, T)) \quad (42)$$

is strictly positive. In fact, since $p \mapsto W(p, T)$ is affine and $\lambda_{\min}(\cdot)$ is continuous on the space of symmetric matrices, the map $p \mapsto \lambda_{\min}(W(p, T))$ is continuous on Z . Moreover, $W(p, T) \succ 0$ for all $p \in Z$, so strict positivity follows by continuity and compactness.

From Lemma 6 and (42), the following corollary holds.

Corollary 1 (Uniform error bound): Assume (22) and Z is a nonempty compact subset of $X_T \cap \Delta_m$. Define

$$\delta_T^* := \frac{\Phi_{\mu(A)}(T) \|A_{12}\|}{\inf_{p \in Z} \lambda_{\min}(W(p, T))}, \quad (43)$$

where $\Phi_{\mu(A)}(T)$ is defined in (36). Then, for all $p \in Z$,

$$|W_{\text{red}}(p, T) - W(p, T)| \preceq \delta_T^* W(p, T). \quad (44)$$

Proof : Since $\sum_i p_i = 1$ for $p \in \Delta_m$, Theorem 2 implies that

$$\varepsilon_T(p) \leq \max_{i \in \{1, \dots, m\}} \|\Delta W_i(T)\| \leq \Phi_{\mu(A)}(T) \|A_{12}\|,$$

Hence, for all $p \in Z$,

$$\begin{aligned} \delta_T(p) &= \frac{\varepsilon_T(p)}{\lambda_{\min}(W(p, T))} \\ &\leq \frac{\Phi_{\mu(A)}(T) \|A_{12}\|}{\inf_{p \in Z} \lambda_{\min}(W(p, T))} = \delta_T^*. \end{aligned}$$

Thus, Lemma 6 yields (44) for all $p \in Z$. $\square$

This corollary serves as a theoretical basis for analyzing the approximation quality of target VCS/AECS.

C. Approximation Guarantees for Target VCS/AECS

In this subsection, we establish theoretical guarantees on the accuracy of the reduced-model approximation for the target VCS/AECS. Specifically, we provide bounds that quantify the discrepancy between the target controllability scores and their approximations, thereby ensuring that the proposed methods are both reliable and interpretable.

To conduct the following discussion rigorously, we define the sets

$$\begin{aligned} D^{\text{VCS}} &:= \left\{ T \in \mathbb{R}_{>0} \left| \begin{array}{l} \text{problem (12) with } h_T = f_T \\ \text{admits a unique solution at } T \end{array} \right. \right\}, \\ D^{\text{AECS}} &:= \left\{ T \in \mathbb{R}_{>0} \left| \begin{array}{l} \text{problem (12) with } h_T = g_T \\ \text{admits a unique solution at } T \end{array} \right. \right\}, \\ D_{\text{red}}^{\text{VCS}} &:= \left\{ T \in \mathbb{R}_{>0} \left| \begin{array}{l} \text{problem (24) with } h_T^{\text{red}} = f_T^{\text{red}} \\ \text{admits a unique solution at } T \end{array} \right. \right\}, \\ D_{\text{red}}^{\text{AECS}} &:= \left\{ T \in \mathbb{R}_{>0} \left| \begin{array}{l} \text{problem (24) with } h_T^{\text{red}} = g_T^{\text{red}} \\ \text{admits a unique solution at } T \end{array} \right. \right\}. \end{aligned}$$

From Theorem 1 and [33, Theorem 1], their complements have Lebesgue measure zero. Thus, the complements of

$$D^{\text{VCS}} \cap D_{\text{red}}^{\text{VCS}} \quad \text{and} \quad D^{\text{AECS}} \cap D_{\text{red}}^{\text{AECS}}$$

also have Lebesgue measure zero. Note that D^{VCS} (resp. D^{AECS}) and $D_{\text{red}}^{\text{VCS}}$ (resp. $D_{\text{red}}^{\text{AECS}}$) are not identical, as shown in Appendix A.

The following theorem shows how closely the target VCS is approximated by its reduced counterpart. The proof relies on Corollary 1.

Theorem 3: Assume (22) and $T \in D^{\text{VCS}} \cap D_{\text{red}}^{\text{VCS}}$. Let p^{VCS} and $p_{\text{red}}^{\text{VCS}}$ be the unique optimal solutions to problems (12) with $h_T = f_T$ and (24) with $h_T^{\text{red}} = f_T^{\text{red}}$, respectively, where $p_{\text{red}}^{\text{VCS}} \in X_T$. Define $Z := \{p^{\text{VCS}}, p_{\text{red}}^{\text{VCS}}\}$. Then, for $p \in Z$,

$$f_T(p) - m \log(1 + \delta_T^*) \leq f_T^{\text{red}}(p), \quad (45)$$

where δ_T^* is defined in (43). If $\delta_T^* < 1$, then for $p \in Z$,

$$f_T^{\text{red}}(p) \leq f_T(p) - m \log(1 - \delta_T^*) \quad (46)$$

and there exists $\mu_T^{\text{VCS}} > 0$ such that

$$\|p_{\text{red}}^{\text{VCS}} - p^{\text{VCS}}\| \leq 2 \sqrt{\frac{\varepsilon_T^{\text{VCS}}}{\mu_T^{\text{VCS}}}} \quad (47)$$

where

$$\varepsilon_T^{\text{VCS}} := m \cdot \max\{\log(1 + \delta_T^*), -\log(1 - \delta_T^*)\}.$$

Proof : First, we bound the objective gap on Z . Corollary 1 implies

$$W_{\text{red}}(p, T) \preceq (1 + \delta_T^*) W(p, T),$$

and therefore, by [41, Corollary 7.7.4],

$$\det W_{\text{red}}(p, T) \leq (1 + \delta_T^*)^m \det W(p, T). \quad (48)$$

Taking logarithms of both sides of (48) then yields (45). Similarly, since Corollary 1 implies

$$(1 - \delta_T^*) W(p, T) \preceq W_{\text{red}}(p, T),$$

we obtain (46) when $\delta_T^* < 1$. From (45) and (46), for $p \in Z$,

$$|f_T^{\text{red}}(p) - f_T(p)| \leq \varepsilon_T^{\text{VCS}}.$$

Then, under the assumption $p_{\text{red}}^{\text{VCS}} \in X_T$,

$$\begin{aligned} f_T(p_{\text{red}}^{\text{VCS}}) &\leq f_T^{\text{red}}(p_{\text{red}}^{\text{VCS}}) + \varepsilon_T^{\text{VCS}} \\ &\leq f_T^{\text{red}}(p^{\text{VCS}}) + \varepsilon_T^{\text{VCS}} \\ &\leq f_T(p^{\text{VCS}}) + 2\varepsilon_T^{\text{VCS}}. \end{aligned} \quad (49)$$

Next, we translate the objective gap into an error bound for the optimization variables, that is, the distance between the optimal solutions $p_{\text{red}}^{\text{VCS}}$ and p^{VCS} , by exploiting the strong convexity of f_T . As discussed in [33, Lemma 5], $f_T(p)$ is strongly convex¹ on the convex hull of Z for any $T \in D^{\text{VCS}}$. Thus, there exists $\mu_T^{\text{VCS}} > 0$ such that

$$f_T(p_{\text{red}}^{\text{VCS}}) - f_T(p^{\text{VCS}}) \geq \frac{\mu_T^{\text{VCS}}}{2} \|p_{\text{red}}^{\text{VCS}} - p^{\text{VCS}}\|^2,$$

since p^{VCS} is the optimal solution to problem (12) with $h_T = f_T$ (refer [46, Section 5.2]). Combining this with (49) (under $\delta_T^* < 1$) implies (47). $\square$

Note that the assumption $p_{\text{red}}^{\text{VCS}} \in X_T$ in Theorem 3 guarantees that $f_T(p_{\text{red}}^{\text{VCS}})$ is well-defined. The assumption is necessary, because $p_{\text{red}}^{\text{VCS}} \in X_T^{\text{red}}$ does not imply $p_{\text{red}}^{\text{VCS}} \in X_T$ in general. However, if $p_{\text{red}}^{\text{VCS}}$ is a positive vector, $p_{\text{red}}^{\text{VCS}} \in X_T$ automatically holds.

The following theorem shows how closely the target AECS is approximated by its reduced counterpart. The proof relies on Corollary 1.

Theorem 4: Assume (22) and $T \in D^{\text{AECS}} \cap D_{\text{red}}^{\text{AECS}}$. Let $p_{\text{red}}^{\text{AECS}}$ and $p_{\text{red}}^{\text{AECS}}$ be the unique optimal solutions to problems (12) with $h_T = g_T$ and (24) with $h_T^{\text{red}} = g_T^{\text{red}}$, respectively, where $p_{\text{red}}^{\text{AECS}} \in X_T$. Let $Z := \{p_{\text{red}}^{\text{AECS}}, p_{\text{red}}^{\text{AECS}}\}$. Then, for $p \in Z$,

$$\frac{1}{1 + \delta_T^*} g_T(p) \leq g_T^{\text{red}}(p), \quad (50)$$

where δ_T^* is defined in (43). If $\delta_T^* < 1$, then for $p \in Z$,

$$g_T^{\text{red}}(p) \leq \frac{1}{1 - \delta_T^*} g_T(p) \quad (51)$$

and there exists $\mu_T^{\text{AECS}} > 0$ such that

$$\|p_{\text{red}}^{\text{AECS}} - p^{\text{AECS}}\| \leq 2\sqrt{\gamma} \sqrt{\frac{\varepsilon_T^{\text{AECS}}}{\mu_T^{\text{AECS}}}} \quad (52)$$

where

$$\varepsilon_T^{\text{AECS}} := \max \left\{ \frac{\delta_T^*}{1 + \delta_T^*}, \frac{\delta_T^*}{1 - \delta_T^*} \right\}$$

and $\gamma > 0$ is a constant, which is an upper bound of g_T on Z .

Proof: First, we bound the objective gap on Z . Corollary 1 implies

$$W_{\text{red}}^{-1}(p, T) \succeq \frac{1}{1 + \delta_T^*} W^{-1}(p, T)$$

for $p \in Z$. Thus, by [41, Corollary 7.7.4], we obtain (50). Moreover, if $\delta_T^* < 1$, then

$$W_{\text{red}}^{-1}(p, T) \preceq \frac{1}{1 - \delta_T^*} W^{-1}(p, T),$$

which implies (51). From (50) and (51), for $p \in Z$,

$$|g_T^{\text{red}}(p) - g_T(p)| \leq \varepsilon_T^{\text{AECS}} g_T(p). \quad (53)$$

Since there exists $\gamma > 0$ such that $g_T(p) \leq \gamma$ for any $p \in Z$, (53) implies that

$$|g_T^{\text{red}}(p) - g_T(p)| \leq \varepsilon_T^{\text{AECS}} \gamma.$$

for any $p \in Z$. Thus, under the assumption $p_{\text{red}}^{\text{AECS}} \in X_T$,

$$\begin{aligned} g_T(p_{\text{red}}^{\text{AECS}}) &\leq g_T^{\text{red}}(p_{\text{red}}^{\text{AECS}}) + \varepsilon_T^{\text{AECS}} \gamma \\ &\leq g_T^{\text{red}}(p^{\text{AECS}}) + \varepsilon_T^{\text{AECS}} \gamma \\ &\leq g_T(p^{\text{AECS}}) + 2\varepsilon_T^{\text{AECS}} \gamma. \end{aligned} \quad (54)$$

Next, we translate the objective gap into an error bound for the optimization variables, that is, the distance between the optimal solutions $p_{\text{red}}^{\text{AECS}}$ and p^{AECS} , by exploiting the strong convexity of g_T . As discussed in [33, Lemma 5], g_T is strongly convex on the convex hull of Z for any $T \in D^{\text{AECS}}$. Thus, there exists $\mu_T^{\text{AECS}} > 0$ such that

$$g_T(p_{\text{red}}^{\text{AECS}}) - g_T(p^{\text{AECS}}) \geq \frac{\mu_T^{\text{AECS}}}{2} \|p_{\text{red}}^{\text{AECS}} - p^{\text{AECS}}\|^2,$$

since p^{AECS} is the optimal solution to problem (12) with $h_T = g_T$ (refer [46, Section 5.2]). Combining this with (54) (under $\delta_T^* < 1$) implies (52). $\square$

Note that the assumption $p_{\text{red}}^{\text{AECS}} \in X_T$ in Theorem 4 guarantees that $g_T(p_{\text{red}}^{\text{AECS}})$ is well-defined, analogous to the assumption $p_{\text{red}}^{\text{VCS}} \in X_T$ in Theorem 3.

Theorems 3 and 4 provide useful insights when considering problem (24) instead of problem (12). In fact, for example, consider a graph-Laplacian dynamics $\dot{x}(t) = -Lx(t)$, where L denotes the graph Laplacian corresponding to a directed graph. Then, the logarithm norm of $-L$ satisfies $\mu(-L) = 0$. In fact, since $L\mathbf{1} = 0$, we have $\frac{L+L^\top}{2}\mathbf{1} = 0$, which shows that $\mathbf{1}$ is an eigenvector corresponding to the smallest eigenvalue 0 of the positive semidefinite matrix $\frac{L+L^\top}{2}$. Thus,

$$\mu(-L) = -\lambda_{\min} \left(\frac{L+L^\top}{2} \right) = 0,$$

and $\Phi_{\mu(-L)}(T)$ defined as (36) is $\Phi_{\mu(-L)}(T) = T^2$. Therefore, if $p_{\text{red}}^{\text{VCS}} \in X_T$ and $d := \min\{\lambda_{\min}(W(p_{\text{red}}^{\text{VCS}}, T)), \lambda_{\min}(W(p_{\text{red}}^{\text{VCS}}, T))\}$, we have

$$\delta_T^* = \frac{T^2}{d} \|A_{12}\| \quad (55)$$

in Theorem 3. This implies that, under the assumptions of small T and large d , if $\|A_{12}\|$ is sufficiently small, then $\delta_T^* < 1$ holds and (47) yields

$$p_{\text{red}}^{\text{VCS}} \approx p^{\text{VCS}}.$$

Similarly, under the corresponding assumptions for AECS, (52) yields

$$p_{\text{red}}^{\text{AECS}} \approx p^{\text{AECS}}.$$

¹In [33, Lemma 5], no explicit restriction on T was stated. Strictly speaking, however, T should be taken from the range in which problem (12) admits a unique solution.

This observation for the graph-Laplacian dynamics is validated in Section V.

Remark 4: Note that the quantity d in (55) is desirable to be as large as possible. This means that the virtual systems (3) corresponding to $p = p^{\text{VCS}}$ and $p = p_{\text{red}}^{\text{VCS}}$ exhibit high output controllability. Since both p^{VCS} and $p_{\text{red}}^{\text{VCS}}$ are designed to maximize the volume of the reachable ellipsoid as mentioned in Section II-C, the associated Gramians are expected to have large minimal eigenvalues (i.e., high output controllability), and hence d is expected to be large. In the case of AECS, it is related to the average minimum input energy required to reach points on the unit sphere, and the virtual systems (3) corresponding to $p = p^{\text{AECS}}$ and $p = p_{\text{red}}^{\text{AECS}}$ also exhibit high output controllability. In this case as well, the quantity corresponding to d is expected to be large.

V. NUMERICAL EXPERIMENTS USING REAL-WORLD HUMAN BRAIN NETWORK DATA

We evaluated target VCS/AECS, along with their reduced approximations, using real-world data from human brain networks, available at <https://doi.org/10.17605/OSF.IO/YW5VF>, as provided in [47]. This dataset contains structural connectivity matrices for 88 individuals. Each individual's brain network is represented by a 90×90 matrix, where the entry at row i and column j specifies the probability of a connection from the i th region of interest (ROI) to the j th ROI, as defined by the Automatic Anatomical Labeling atlas. Thus, the dataset comprises brain networks for 88 individuals, each consisting of 90 nodes corresponding to distinct brain regions. We note that this is the same dataset used in [33, Section IV].

We model the individual blood oxygen level-dependent (BOLD) signal dynamics as the continuous-time linear system

$$\dot{x}^{(i)}(t) = A^{(i)}x^{(i)}(t) \quad (i = 1, \dots, 88), \quad (56)$$

where each component of $x^{(i)}(t)$ denotes the BOLD signal of each ROI at time t for the i th individual. The system matrix $A^{(i)}$ is defined by $A^{(i)} := -\mathcal{L}^{(i)}$, where $\mathcal{L}^{(i)}$ is the graph Laplacian

$$\mathcal{L}^{(i)} := \text{diag} \left(\sum_{j=1}^{90} \mathcal{C}_{1j}^{(i)}, \dots, \sum_{j=1}^{90} \mathcal{C}_{90j}^{(i)} \right) - \mathcal{C}^{(i)}.$$

Here $\mathcal{C}^{(i)} \in \mathbb{R}^{90 \times 90}$ denotes the transpose of the structural connectivity matrix for the i th individual.

Based on system model (56), we construct the target selection matrices for both VCS and AECS as follows. For each individual $i \in \{1, \dots, 88\}$ and for a given time horizon $T > 0$, let $\{s_1^{\text{VCS}}(T), \dots, s_m^{\text{VCS}}(T)\}$ and $\{s_1^{\text{AECS}}(T), \dots, s_m^{\text{AECS}}(T)\}$ denote the indices of the top m ROIs ranked by the average VCS and AECS values at horizon T , respectively. Without loss of generality, we relabel the coordinates so that the selected ROIs appear in the first m positions of the state vector, as explained in Section II. For each i , this yields a block partition of the system matrix

$$A_{\bullet}^{(i)} = \begin{pmatrix} A_{11,\bullet}^{(i)} & A_{12,\bullet}^{(i)} \\ A_{21,\bullet}^{(i)} & A_{22,\bullet}^{(i)} \end{pmatrix}, \quad A_{11,\bullet}^{(i)} \in \mathbb{R}^{m \times m},$$

where $\bullet \in \{\text{VCS}, \text{AECS}\}$ indicates whether the target set is determined by VCS or AECS. Here, $A_{11,\bullet}^{(i)}$ represents the dynamics restricted to the selected target set, whereas $A_{12,\bullet}^{(i)}$ captures the coupling from non-target nodes into the targets.

For each individual $i \in \{1, \dots, 88\}$, let $(p^{\text{VCS}})^{(i)}$ and $(p^{\text{AECS}})^{(i)}$ denote the target VCS and target AECS, respectively, and let $(p_{\text{red}}^{\text{VCS}})^{(i)}$ and $(p_{\text{red}}^{\text{AECS}})^{(i)}$ denote their reduced-system counterparts.

Remark 5: Standard models of brain activity include nonlinearities in the mapping from neural activity to the hemodynamic/BOLD response [48], [49]. In this work, however, we adopt a linear Laplacian-based dynamics described in (56) as a first-order approximation near the resting state. A previous study has shown that, for resting-state low-frequency correlations, linear diffusion on the structural network can capture the principal second-order statistics and, in some cases, match or even outperform more complex nonlinear neural-mass models in predicting functional connectivity from structural connectivity [50]. This approach has also been supported by prior work in the literature [29]–[31], [51].

A. Comparison of Target and Non-Target Coupling for VCS and AECS

To assess the influence of non-target regions on the target dynamics, we evaluated the operator norms $\|A_{12,\bullet}^{(i)}\|$ across all individuals. For each selection method $\bullet \in \{\text{VCS}, \text{AECS}\}$, define $\{\|A_{12,\bullet}^{(i)}\|\}_{i=1}^{88}$ as the collection of values obtained across all subjects. We summarize these results using the sample mean $\overline{\|A_{12,\bullet}\|} := \frac{1}{88} \sum_{i=1}^{88} \|A_{12,\bullet}^{(i)}\|$ and the corresponding population standard deviation $\sqrt{\frac{1}{88} \sum_{i=1}^{88} \left(\|A_{12,\bullet}^{(i)}\| - \overline{\|A_{12,\bullet}\|} \right)^2}$.

Table I reveals a consistent pattern across all examined configurations (T, m) : the cross-coupling magnitudes $\|A_{12,\bullet}\|$ associated with VCS are uniformly smaller than those associated with AECS, irrespective of the choice of time horizon T or the number of selected targets m . This suggests that VCS-based target selection tends to identify subsets of regions whose dynamics are less influenced by non-target nodes, resulting in weaker interference from the complementary block compared to AECS-based selection.

It should be noted, however, that this advantage in terms of cross-coupling does not necessarily imply better approximation accuracy of the target VCS/AECS, as shown in the next subsection.

B. Approximation Error Between Target and Reduced Formulations

Let $\{\|\text{diff}_{\bullet}^{(i)}\|\}_{i=1}^{88}$ denote the norm difference between the TCS and the reduced-system CS for $\bullet \in \{\text{VCS}, \text{AECS}\}$. That is, for each subject i ,

$$\|\text{diff}_{\bullet}^{(i)}\| = \|(p^{\bullet})^{(i)} - (p_{\text{red}}^{\bullet})^{(i)}\|.$$

We report the sample mean $\overline{\|\text{diff}_{\bullet}\|} := \frac{1}{88} \sum_{i=1}^{88} \|\text{diff}_{\bullet}^{(i)}\|$ and the corresponding population standard deviation $\sqrt{\frac{1}{88} \sum_{i=1}^{88} \left(\|\text{diff}_{\bullet}^{(i)}\| - \overline{\|\text{diff}_{\bullet}\|} \right)^2}$.

TABLE I
CROSS-COUPLING MAGNITUDES (MEAN $\pm$ STANDARD-DEVIATION) ACROSS 88 SUBJECTS FOR DIFFERENT (T, m) CONFIGURATIONS.

T	m	$\ A_{12, \text{VCS}}\ $ (mean $\pm$ std)	$\ A_{12, \text{AECS}}\ $ (mean $\pm$ std)
1	3	$3.124 \times 10^{-1} \pm 8.392 \times 10^{-2}$	$7.364 \times 10^{-1} \pm 8.055 \times 10^{-2}$
1	10	$2.568 \times 10^{-1} \pm 3.512 \times 10^{-2}$	$8.004 \times 10^{-1} \pm 8.385 \times 10^{-2}$
1	30	$3.939 \times 10^{-1} \pm 4.134 \times 10^{-2}$	$8.123 \times 10^{-1} \pm 8.745 \times 10^{-2}$
100	3	$1.218 \times 10^{-1} \pm 3.743 \times 10^{-2}$	$7.420 \times 10^{-1} \pm 8.076 \times 10^{-2}$
100	10	$2.409 \times 10^{-1} \pm 4.249 \times 10^{-2}$	$8.004 \times 10^{-1} \pm 8.385 \times 10^{-2}$
100	30	$3.828 \times 10^{-1} \pm 4.343 \times 10^{-2}$	$8.120 \times 10^{-1} \pm 8.564 \times 10^{-2}$

Table II summarizes the norm differences between the target and reduced formulations of VCS and AECS across 88 subjects under various (T, m) configurations. At $T = 1$, VCS exhibits substantially smaller differences than AECS for $m = 3, 10, 30$, indicating that the reduced formulation closely approximates the target VCS in the short-horizon setting. This observation is consistent with the theoretical implications of Theorem 2 (for $\mu(A) = 0$), Section IV-C, and Section V-A, which suggest that VCS should yield smaller errors when the cross-coupling term $\|A_{12}\|$ is small.

However, as T increases, the situation changes markedly. When $T = 100$, VCS produces significantly larger discrepancies than AECS regardless of the value of m . This can be explained by the factor $\Phi_{\mu(A)}(T) = T^2$ in Theorem 2, which grows rapidly with T and renders the error bounds of Theorems 3 and 4 ineffective. As a result, the expected advantage of VCS based on $\|A_{12}\|$ no longer holds at long horizons, leading to the nontrivial outcome that VCS, despite being favorable at $T = 1$, performs worse than AECS at $T = 100$. By contrast, AECS achieves relatively consistent accuracy across both short and long horizons, irrespective of m , thereby demonstrating its robustness in approximating target AECS across varying time scales.

Figures 3–6 collectively show that the reduced-system approximation is faithful for target VCS/AECS at the short horizon ($T = 1$), but only AECS remains well approximated at the long horizon ($T = 100$), whereas VCS exhibits sizable discrepancies between the target and reduced formulations. This qualitative pattern aligns with the node-level evidence in Table III: Target AECS (TAECs) selects exactly the same top 5 regions at $T = 1$ and $T = 100$, indicating temporal stability of the target set, while target VCS (TVCS) selects markedly different regions across horizons (e.g., middle orbital gyrus, cingulum, cuneus, pallidum at $T = 1$ versus amygdala and Heschl’s gyrus at $T = 100$), revealing strong sensitivity to T . The mismatch is particularly evident at $(T, m) = (100, 30)$, where the reduced VCS elevates Node 36 (Right Cingulum Post) and Node 69 (Left Paracentral Lobule) into its top 5, even though neither appears in the corresponding TVCS results—clear evidence that, at longer horizons, the reduced VCS fails to track its target formulation, whereas AECS retains robust agreement across time scales.

VI. CONCLUSIONS

A. Summary

This paper introduced the target controllability score (TCS) as a dynamics-aware centrality metric tailored to scenarios

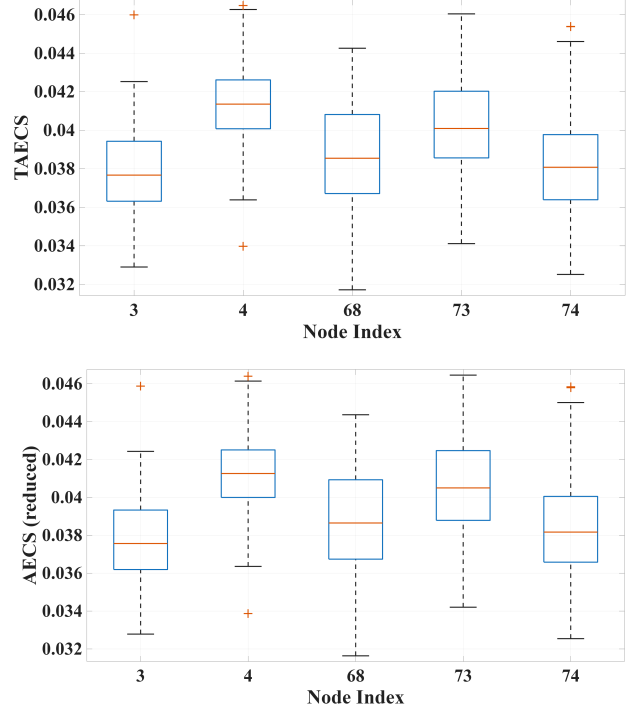


Fig. 3. Boxplots of the top 5 nodes: target AECS (top) and its reduced-system approximation (bottom) for $(T, m) = (1, 30)$.

in which only a designated subset of nodes is actuated and only task-relevant outputs are evaluated. We formalized target VCS/AECS as solutions to finite-horizon optimization problems built on the output controllability Gramian, and established existence and (for almost all horizons) uniqueness. To enable scalable computation, we proposed a projected-gradient scheme and a reduced virtual system that retains the target variables while capturing their coupling to the rest of the network. Our theory shows that, for a fixed time horizon, the approximation error is mainly determined by the strength of the cross-coupling between target and non-target nodes and by the short-term contraction or growth rate of the dynamics, as quantified by the logarithmic norm. Empirical studies on human brain networks (88 individuals) corroborate the theoretical analysis. At short horizons, both target VCS and target AECS are accurately approximated by their reduced counterparts. At longer horizons, target AECS remains robust—consistently identifying the same key regions—whereas target VCS becomes sensitive to time scale and exhibits larger approximation discrepancies.

TABLE II
DIFFERENCES BETWEEN TARGET AND REDUCED FORMULATIONS (MEAN $\pm$ STD OF $\|\text{diff}_\bullet\|$) OVER 88 SUBJECTS FOR VARIOUS (T, m) CONFIGURATIONS.

T	m	$\ \text{diff}_{\text{VCS}}\ $ (mean $\pm$ std)	$\ \text{diff}_{\text{AECS}}\ $ (mean $\pm$ std)
1	3	$0.000 \times 10^0 \pm 0.000 \times 10^0$	$1.827 \times 10^{-3} \pm 7.529 \times 10^{-4}$
1	10	$4.465 \times 10^{-5} \pm 4.940 \times 10^{-5}$	$1.132 \times 10^{-3} \pm 2.391 \times 10^{-4}$
1	30	$6.471 \times 10^{-5} \pm 3.177 \times 10^{-5}$	$6.887 \times 10^{-4} \pm 8.818 \times 10^{-5}$
100	3	$2.281 \times 10^{-2} \pm 2.142 \times 10^{-2}$	$4.177 \times 10^{-3} \pm 2.046 \times 10^{-3}$
100	10	$4.082 \times 10^{-2} \pm 1.841 \times 10^{-2}$	$3.252 \times 10^{-3} \pm 9.036 \times 10^{-4}$
100	30	$1.992 \times 10^{-2} \pm 5.357 \times 10^{-3}$	$4.399 \times 10^{-3} \pm 5.215 \times 10^{-4}$

TABLE III
NODE INDICES AND BRAIN REGIONS IN THE TOP 5 RANKINGS OF TAECS OR TVCS ($m = 30$).

Node Index	Brain Region	TAECS ($T = 1, 100$)	TVCS ($T = 1$)	TVCS ($T = 100$)
3	Left Superior Frontal Gyrus	✓		
4	Right Superior Frontal Gyrus	✓		
22	Right Olfactory Cortex			✓
25	Left Middle Orbital Gyrus		✓	
36	Right Cingulum Post		✓	
41	Left Amygdala			✓
42	Right Amygdala			✓
45	Left Cuneus		✓	
68	Right Precuneus	✓		
73	Left Putamen	✓		
74	Right Putamen	✓		
75	Left Pallidum		✓	
76	Right Pallidum		✓	
79	Left Heschl's Gyrus			✓
80	Right Heschl's Gyrus			✓

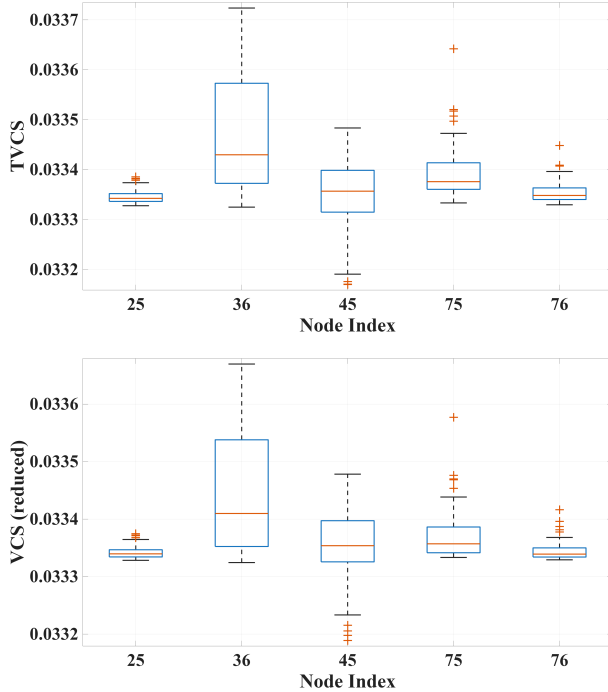


Fig. 4. Boxplots of the top 5 nodes: target VCS (top) and its reduced-system approximation (bottom) for $(T, m) = (1, 30)$.

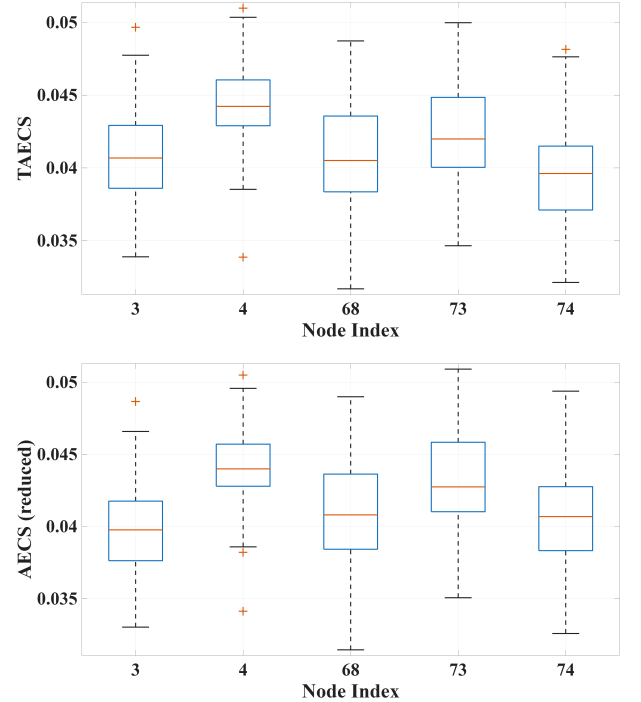


Fig. 5. Boxplots of the top 5 nodes: target AECS (top) and its reduced-system approximation (bottom) for $(T, m) = (100, 30)$.

B. Future Work

Building on the empirical and theoretical findings summarized in Section VI-A, we outline two future directions:

- **Explaining AECS robustness in T :** Explain why the ap-

proximation error for the target AECS (i.e., $\|\text{diff}_{\text{AECS}}\|$) exhibits markedly weaker dependence on T than that of the target VCS (i.e., $\|\text{diff}_{\text{VCS}}\|$) as shown in Table II, and identify structural and dynamical conditions under which

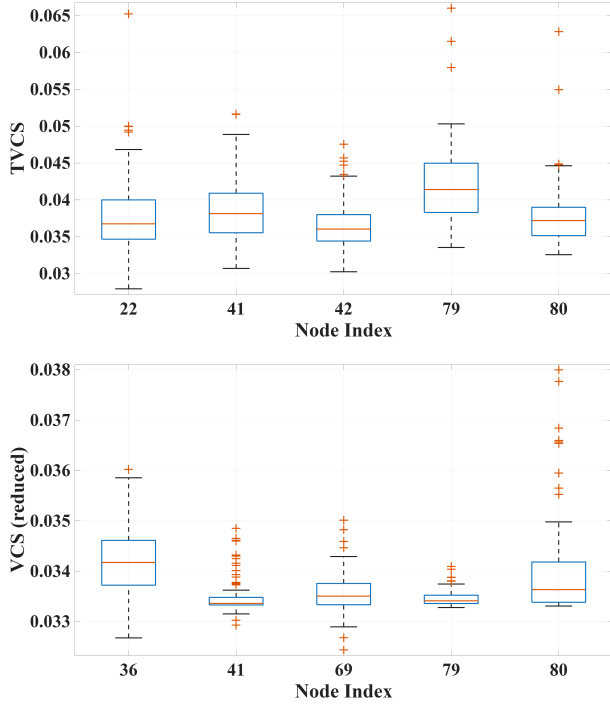


Fig. 6. Boxplots of the top 5 nodes: target VCS (top) and its reduced-system approximation (bottom) for $(T, m) = (100, 30)$.

the reduced AECS yields T -uniformly accurate estimates of the target AECS. To this end, extend Theorems 3 and 4 by deriving tighter, asymptotically sharp, and horizon-explicit bounds on the error norms between the target VCS/AECS and their reduced approximations, with particular emphasis on the large- T regime.

- **Toward infinite-horizon formulations:** To perform a rigorous large- T analysis, it may be first necessary to clarify how the underlying optimization problems behave in the limit $T \rightarrow \infty$. Although the finite-horizon objective functions are not formally defined in this limit, our experiments (Table II) on Laplacian dynamics with a zero eigenvalue (marginally stable systems) indicate that the corresponding optimal solutions—target VCS/AECS—remain well-defined and bounded even for large T . This observation suggests that the infinite-horizon case can be made mathematically meaningful. Therefore, before deriving asymptotically sharp error bounds, an important next step is to formulate infinite-horizon ($T \rightarrow \infty$) versions of target VCS/AECS as optimization problems with finite, well-posed objective functions, and to establish the existence and uniqueness of their solutions, particularly for marginally stable or unstable systems.

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APPENDIX

A. Example: Non-Identity of the Uniqueness Sets

Consider the setting with $n = 3$, $m = 2$, and

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, as shown in [32, Section IV], the VCS/AECS of $\dot{x}_{\text{red}}(t) = A_{11}x_{\text{red}}(t)$ are not unique for $T = \pi$.

In contrast, the target VCS/AECS of $\dot{x}(t) = Ax(t)$ are unique for any $T > 0$. In fact, by a direct calculation, we have

$$\exp(At) = \begin{pmatrix} 1 & t & -t \\ -t & 1 - \frac{t^2}{2} & \frac{t^2}{2} \\ -t & -\frac{t^2}{2} & 1 + \frac{t^2}{2} \end{pmatrix}.$$

Thus, $W_1(T)$ and $W_2(T)$, defined in (9), are given by

$$W_1(T) = \begin{pmatrix} T & -\frac{T^2}{2} \\ -\frac{T^2}{2} & \frac{T^3}{3} \end{pmatrix},$$

$$W_2(T) = \begin{pmatrix} \frac{T^3}{3} & \frac{T^2}{2} - \frac{T^4}{6} \\ \frac{T^2}{2} - \frac{T^4}{6} & T - \frac{T^3}{3} + \frac{T^5}{20} \end{pmatrix},$$

respectively. Therefore, $W_1(T)$ and $W_2(T)$ are linearly independent over $\mathbb{R}$ for any $T > 0$. Namely, the only solution (a, b) to $aW_1(T) + bW_2(T) = 0$ is $a = b = 0$ for any $T > 0$. Hence, Lemma 3 guarantees that the target VCS/AECS of $\dot{x}(t) = Ax(t)$ are unique for any $T > 0$.

Therefore, D^{VCS} (resp. D^{AECS}) and $D_{\text{red}}^{\text{VCS}}$ (resp. $D_{\text{red}}^{\text{AECS}}$) are not identical, in general.

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