

A 2-SYSTOLIC INEQUALITY ON NON-RATIONAL COMPACT KÄHLER SURFACES WITH POSITIVE SCALAR CURVATURE

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ABSTRACT. In this note, we prove a 2-systolic inequality on non-rational compact positive scalar curvature Kähler surfaces admitting a nonconstant holomorphic map to a positive-genus compact Riemann surface. According to the classification of positive scalar curvature Kähler surfaces, any such surface must be a ruled surface fibred over a complex curve with positive genus.

1. INTRODUCTION

Systolic geometry studies the relationship between the minimal size of non-trivial homological cycles in a Riemannian manifold and its global geometric properties. Given a closed Riemannian manifold (M, h) , the k -systole is defined as

$$\mathrm{sys}_k(M, h) := \inf \{ \mathrm{Vol}_g(Z) \mid Z \subset M \text{ embedded}, [Z] \neq 0 \in H_k(M; \mathbb{Z}) \}.$$

Consider a Riemannian manifold (M, h) with positive scalar curvature (PSC for abbreviation), Schoen–Yau [SY79] confirmed that any area-minimizing surface in M is homeomorphic to either S^2 or $\mathbb{RP}^2$. The $\mathbb{RP}^2$ case was studied in [BBN10], which proved the sharp upper bound of the area for an embedded area-minimizing projective plane,

$$\mathrm{Area}_h(\Sigma) \cdot \min_M S_M \leq 12\pi$$

with equality if $M \simeq \mathbb{RP}^3$. In the S^2 case, Bray–Brendle–Neves [BBN10] established the following important rigidity result concerning the π_2 -systole and the minimum of the scalar curvature $\min S_M$ of a PSC 3-manifold (M, h) :

Theorem 1.1 (Bray-Brendle-Neves, [BBN10]). Let (M^3, h) be a closed, orientable Riemannian 3-manifold with positive scalar curvature. Then the following inequality holds:

$$\mathrm{sys}_{\pi_2}(M, h) \cdot \min_M S_M \leq 8\pi. \tag{1.1}$$

Moreover, equality holds if and only if M^3 is isometrically covered by $S^2 \times S^1$ with the round metric on S^2 , product with the flat metric on S^1 .

The sharp bound in (1.1) was recently refined by Xu [Xu25], who proved that it is not only rigid but also exhibits a strict quantitative gap away from the model case. Further developments on the interplay between the 2-systole and positive scalar curvature include [Zhu20, Ric20, Ori25].

In [Ste22], Stern introduced the following inequality for a non-constant S^1 -valued harmonic map u on a 3-manifold (M, h) using the level set method,

$$2\pi \int_{\theta \in S^1} \chi(\Sigma_\theta) \geq \frac{1}{2} \int_{\theta \in S^1} \int_{\Sigma_\theta} (|du|^{-2} |\text{Hess}(u)|^2 + S_M) \quad (1.2)$$

and generalized the Bray–Brendle–Neves’ systolic inequality for the homological 2-systole.

Motivated by Stern’s approach, we adapt the level set method to compact Kähler surfaces fibred over a Riemann surface with positive genus, and obtain a Bray–Brendle–Neves type inequality for the homological 2-systole. In particular, we have the following:

Theorem 1.2. Let (X, ω) be a compact PSC Kähler surface admitting a non-constant holomorphic map $f : X \rightarrow C$ to a compact Riemann surface C with genus $g(C) \geq 1$. Then, we have

$$\min_X S_X \cdot \text{sys}_2(X, \omega) \leq 8\pi. \quad (1.3)$$

Moreover, the equality holds if and only if X is isometrically covered by $\mathbb{CP}^1 \times \mathbb{C}$ equipped with the product of the standard Fubini–Study metric on $\mathbb{CP}^1$ and a flat metric on $\mathbb{C}$, so that $\text{sys}_2(X, \omega)$ is achieved by $\mathbb{CP}^1$ -fibre.

A direct consequence of the above result is:

Corollary 1.3. Let (X, ω) be a compact PSC Kähler surface admitting a non-constant holomorphic map $f : X \rightarrow C$ to a compact Riemann surface C with genus $g(S) \geq 2$. Then, we have

$$\min_X S_X \cdot \text{sys}_2(X, \omega) < 8\pi. \quad (1.4)$$

It is worth-noting that a compact Kähler surface X admits a PSC metric if and only if X is obtained from $\mathbb{P}^2$ or $\mathbb{P}(E)$ by a finite sequence of blow ups, where E is a rank 2 holomorphic vector bundle over a compact Riemann surface: For minimal compact Kähler surfaces (which are not the blow-up of other Kähler surfaces), Yau [Yau74] showed that the existence of a Kähler PSC metric is equivalent to X is ruled or $\mathbb{P}^2$ (see also [LeB95]). LeBrun conjectured that this statement is still valid after allowing the blowup. The remaining gap—whether blowing up preserves the *sign* of the scalar curvature was settled recently by Brown, who proved that Kähler blow-ups preserve the sign of scalar curvature and completed the classification [Bro24, Thm B].

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2. THE KÄHLER SURFACE FIBRED OVER A RIEMANN SURFACE WITH POSITIVE GENUS

Let $f : (X^n, \omega) \rightarrow C$ be a non-constant holomorphic map from a compact Kähler manifold to a compact Riemann surface C with genus $g \geq 1$. Thanks to the uniformization theorem, we can always choose a metric ω_0 on C with non-positive Gauss curvature.

Taking $z \in C$ be any point, then we have $f^{-1}(z) := D_z$ is a Cartier divisor in X with complex codimension 1. Indeed, D_z defines a line bundle $\mathcal{O}(D_z)$ with the first Chern class $c_1(\mathcal{O}(D_z))$ represented by $f^*\omega_0$ after normalization. In the subsequent part, we only consider the smooth part of D_z , and denote it by D_z directly.

Recall the adjunction formula for a smooth divisor D , the canonical bundle K_D satisfies

$$K_D = (K_X \otimes \mathcal{O}(D))|_D. \quad (2.1)$$

Since D is the fiber of a holomorphic map over C , the normal bundle $\mathcal{N}_D \cong \mathcal{O}(D)|_D$ is trivial. By taking the first Chern class of the adjunction formula, we obtain

$$c_1(D) = c_1(X)|_D,$$

which implies

$$\text{Ric}_D(\omega) = \text{Ric}_X(\omega)|_D. \quad (2.2)$$

Denote ν by the unit normal vector field of D of type $(1, 0)$, we obtain the traced Gauss equation

$$S_D(\omega) = \text{tr}_\omega \text{Ric}_D(\omega) = \text{tr}_\omega \text{Ric}_X(\omega)|_D = S_X(\omega) - \text{Ric}_X(\omega)(\nu, \bar{\nu}).$$

Moreover, since $\nu = \nabla^{1,0}f/|\nabla^{1,0}f|$, we obtain

$$\text{Ric}_X(\omega) (\nabla^{1,0}f, \nabla^{0,1}f) = |\nabla^{1,0}f|^2 (S_X(\omega) - S_D(\omega)). \quad (2.3)$$

Recall the Bochner formula for holomorphic maps, and the co-area formula:

Lemma 2.1 (Bochner formula). Let $f : (X, \omega) \rightarrow (N, \tilde{\omega})$ be holomorphic, then

$$\Delta|\partial f|^2 = |\nabla\partial f|^2 + \langle \text{Ric}(\omega), f^*\tilde{\omega} \rangle - \text{tr}_\omega^2(f^* \text{Rm}(\tilde{\omega})). \quad (2.4)$$

where $\text{Ric}(\omega)$ is the Ricci form of X , and $\text{Rm}(\tilde{\omega})$ is the curvature form of N .

Lemma 2.2 (Co-area formula). Let (X^n, ω) be a compact Kähler manifold and let (C, ω_0) be a compact Riemann surface such that

$$\int_C \omega_0 = \frac{1}{n},$$

after normalization. Then for any $g \in C^\infty(X)$ and $z \in E$ regular value of f , we have

$$\int_X g \omega^n = \int_C \left(\int_{f^{-1}(z)} \frac{g}{|\partial f|^2} \omega^{n-1} \right) \omega_0. \quad (2.5)$$

When $(N, \tilde{\omega}) = (C, \omega_0)$, we have

$$\Delta|\partial f|^2 \geq |\nabla\partial f|^2 + \text{Ric}_X(\omega) (\nabla^{1,0}f, \nabla^{0,1}f),$$

with equality holds if $\text{Rm}(\omega_0) \equiv 0$. Combining with the traced Gauss equation (2.3), we obtain

$$\Delta|\partial f|^2 \geq |\nabla\partial f|^2 + |\nabla^{1,0}f|^2 (S_X(\omega) - S_D(\omega)). \quad (2.6)$$

Combining with the co-area formula and (2.6), we can then see the following identity:

Lemma 2.3. Let (X^n, ω) be a compact Kähler manifold and let (C, ω_0) be a compact Riemann surface with genus $g \geq 1$ endowed with a constant curvature metric ω_0 . Suppose that $f : X \rightarrow C$ is a non-trivial holomorphic map. Then for any $\phi \in C^\infty(X)$, we have

$$\int_C \left[\int_{D_z} \phi^2 \left(\frac{|\nabla \partial f|^2}{|\partial f|^2} + S_X - S_{D_z} \right) \omega^{n-1} \right] \omega_0 \leq -n \int_X \sqrt{-1} \partial(\phi^2) \wedge \bar{\partial} |\partial f|^2 \wedge \omega^{n-1}. \quad (2.7)$$

Moreover, the equality holds if and only if (S, ω_0) is an elliptic curve with a flat metric.

Proof. Let $C = A \cup B$, where A contains the set for all critical values of f . Then, multiplying any smooth function ϕ^2 on the both sides and integrating over $f^{-1}(B)$, we have

$$\int_{f^{-1}(B)} \phi^2 \left[|\nabla \partial f|^2 + |\nabla^{1,0} f|^2 (S_X(\omega) - S_{D_z}(\omega)) \right] \omega^n \leq \int_{f^{-1}(B)} \phi^2 (\Delta |\partial f|^2) \omega^n. \quad (2.8)$$

Indeed, the right-hand side of (2.8) gives

$$\int_{f^{-1}(B)} \phi^2 (\Delta |\partial f|^2) \omega^n = n \int_{f^{-1}(B)} \phi^2 \sqrt{-1} \partial \bar{\partial} |\partial f|^2 \wedge \omega^{n-1}$$

We then apply the co-area formula to the left-hand side of (2.8), which yields

$$\begin{aligned} & \int_{f^{-1}(B)} \phi^2 \left[|\nabla \partial f|^2 + |\nabla^{1,0} f|^2 (S_X - S_{D_z}) \right] \omega^n \\ &= n \int_B \left[\int_{D_z} \left(\frac{|\nabla \partial f|^2}{|\partial f|^2} + S_X - S_D \right) \omega^{n-1} \right] \omega_0 \end{aligned}$$

By Sard's theorem, we can take the measure of A arbitrarily small, and this gives

$$n \int_X \phi^2 \sqrt{-1} \partial \bar{\partial} |\partial f|^2 \wedge \omega^{n-1} = -n \int_X \sqrt{-1} \partial(\phi^2) \wedge \bar{\partial} |\partial f|^2 \wedge \omega^{n-1}.$$

Consequently, we have

$$\int_C \left[\int_{D_z} \phi^2 \left(\frac{|\nabla \partial f|^2}{|\partial f|^2} + S_X - S_{D_z} \right) \omega^{n-1} \right] \omega_0 \leq -n \int_X \sqrt{-1} \partial(\phi^2) \wedge \bar{\partial} |\partial f|^2 \wedge \omega^{n-1},$$

which implies the desired equality. $\square$

3. THE 2-SYSTOLE IN KÄHLER SURFACE

This section is devoted to the study of the (homological) 2-systole in Kähler surfaces. We begin by recalling the fundamental definition of the k -systole in Riemannian geometry. Let (M, h) be a closed Riemannian manifold of dimension $n \geq k$. The k -systole $\text{sys}_k(M, h)$ is defined as the infimum of the volumes of all integral k -cycles representing nontrivial homology classes:

$$\text{sys}_k(M, h) := \inf \{ \text{Vol}_h(Z) \mid Z \subset M \text{ embedded}, [Z] \neq 0 \in H_k(M; \mathbb{Z}) \}.$$

In the context of Kähler geometry, additional structure enriches this concept. Let (X, ω) be a compact Kähler surface. The 2-systole is the least area among nonseparating real surfaces in X .

Definition 3.1 (2-systole in Kähler surfaces). For a compact Kähler surface (X, ω) , the 2-systole can be defined by

$$\text{sys}_2(X, \omega) = \inf \{ \text{Vol}_\omega(Z) \mid Z \subset X \text{ embedded}, [Z] \neq 0 \in H_2(X; \mathbb{Z}) \}.$$

The following result gives a Bray-Brendle-Neves type inequality [BBN10] for 2-systole for compact PSC Kähler surfaces over a Riemann surface with genus $g \geq 1$.

Theorem 3.2. Let (X, ω) be a compact Kähler surface admitting a non-constant holomorphic map $f : X \rightarrow C$ to a complex curve C with genus $g(C) \geq 1$. Then, we have

$$\min_X S_X \cdot \text{sys}_2(X, \omega) \leq 8\pi. \quad (3.1)$$

Moreover, the equality holds if and only if X is isometrically covered by $\mathbb{CP}^1 \times \mathbb{C}$ equipped with the product of the standard Fubini–Study metric on $\mathbb{CP}^1$ and a flat metric on $\mathbb{C}$, so that $\text{sys}_2(X, \omega)$ is achieved by $\mathbb{CP}^1$ -fibre.

Proof. By taking $\phi = 1$ in (2.7), we obtain

$$\int_C \left[\int_{D_z} \left(\frac{|\nabla \partial f|^2}{|\partial f|^2} + S_X \right) \omega \right] \omega_0 \leq \int_C \left(\int_{D_z} S_{D_z} \cdot \omega \right) \omega_0.$$

It follows from the Gauss-Bonnet formula,

$$\begin{aligned} 4\pi \int_C \chi(D_z) \omega_0 &= \int_C \left(\int_{D_z} S_{D_z} \cdot \omega \right) \omega_0 \\ &\geq \int_C \left(\int_{D_z} S_X \cdot \omega \right) \omega_0 \\ &\geq \min_X S_X \cdot \int_C \text{Vol}_\omega(D_z) \omega_0. \end{aligned}$$

Denote $N(z)$ by the number of the homological non-zero irreducible components of D_z , we then have

$$\text{Vol}_\omega(D_z) \geq N(z) \cdot \text{sys}_2(X, \omega).$$

Meanwhile, since $D_z \simeq \mathbb{CP}^1$, we have

$$\chi(D_z) = 2N(z).$$

Thus, we have

$$\begin{aligned} 8\pi \int_C N(z) \omega_0 &= 4\pi \int_C \chi(D_z) \omega_0 \\ &\geq \min_X S_X \cdot \int_C \text{Vol}_\omega(D_z) \omega_0 \\ &\geq \min_X S_X \cdot \text{sys}_2(X, \omega) \int_C N(z) \omega_0, \end{aligned}$$

which gives the desired result. The equality holds in case C admits a flat metric and ∇f is parallel along D_z , with each irreducible component of D_z is $\mathbb{CP}^1$. $\square$

We finally see a simple but interesting example:

Example 3.3. Let $X = \mathbb{P}^1 \times C$ be a compact complex surface, where C is a compact Riemann surface of genus $g \geq 2$. Equip X with the product Kähler metric

$$\omega = \omega_{\text{FS}} \oplus \omega_C,$$

where on $\mathbb{P}^1$ we take the Fubini–Study metric normalized by

$$\text{Vol}_{\omega_{\text{FS}}}(\mathbb{P}^1) = \pi, \quad S_{\mathbb{P}^1} = 8,$$

and on C we choose a constant scalar curvature metric with

$$S_C = -8 + \varepsilon \quad \text{for some } \varepsilon \in (0, 8).$$

Then the product scalar curvature $S_X = S_{\mathbb{P}^1} + S_C = 8 + (-8 + \varepsilon) = \varepsilon$ is constant, hence $\min_X S_X = \varepsilon$ and X has positive scalar curvature.

Next, compare the areas of the two basic complex curves:

- For the $\mathbb{P}^1$ -fiber $F = \mathbb{P}^1 \times \{p\}$, calibration by ω gives $\text{Vol}_\omega(F) = \pi$.
- For the C -fiber $C_p = \{q\} \times C$, Gauss–Bonnet formula yields

$$\int_C K_C \omega = 2\pi\chi(C) = 2\pi(2 - 2g) = -4\pi(g - 1).$$

Then we obtain

$$\text{Vol}_\omega(C_p) = \frac{8\pi(g - 1)}{8 - \varepsilon}.$$

For $g \geq 2$ and $\varepsilon \in (0, 8)$ one has $\text{Area}_\omega(C_p) > \pi$, so the 2-systole is realized by the $\mathbb{P}^1$ -fiber:

$$\text{sys}_2(X, \omega) = \min \{ \text{Area}_\omega(F), \text{Area}_\omega(C_p) \} = \pi.$$

Consequently,

$$\min_X S_X \cdot \text{sys}_2(X, \omega) = \varepsilon \cdot \pi < 8\pi.$$

In particular, this product is independent of the genus g , and it can be made arbitrarily close to 8π by letting $\varepsilon \uparrow 8$.

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