

THE CLASSIFICATION OF CR MAPS FROM HYPERQUADRICS INTO TUBES OVER NULL CONES OF SYMMETRIC FORMS

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ABSTRACT. We classify CR maps from the hyperquadric of signature $l > 0$ in $\mathbb{C}^n$, $n \geq 3$, to the local model for the tube over the null cone of a symmetric form in $\mathbb{C}^{n+1}$, up to CR automorphisms of the source and target. In contrast to the setting of the Heisenberg hypersurface in $\mathbb{C}^3$ (i.e., the case $l = 0$), studied earlier in Reiter–Son [24], our analysis uncovers two new equivalence classes of CR maps of geometric rank one and one new class of geometric rank two in the case $n = 3$. In the case $n \geq 4$, we establish that all maps extend to local isometries of certain indefinite Kähler metrics. We further derive a classification of (local) proper holomorphic maps from the generalized unit ball $\mathbb{B}_l^n$ into a generalized version of the Lie ball $D_{m,l}^{IV}$ (the generalized classical domain of type IV).

1. INTRODUCTION

Our primary interest of this paper is the characterization of local CR maps from the hyperquadric of signature $l > 0$ in $\mathbb{C}^n$ to the tube over the null cone of a symmetric form in $\mathbb{R}^{n+1}$ which is homogeneous and Levi-degenerate of signature $(l, n-l-1, 1)$ in $\mathbb{C}^{n+1}$. This problem is closely related to the classification of local proper holomorphic mappings from the generalized ball into the domain the “generalized classical domain of type IV” (or the generalized Lie ball), which is ultimately motivated by the classical works of Henri Poincaré [19] and Alexander [1]. These two works focus on the classification of proper holomorphic maps on the complex unit balls of the same dimension. The different dimensional case was studied in various later works. In 1979, Webster [28] provided a rigidity result on the proper holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{B}^{n+1}$ which extend sufficiently smooth to a boundary point for $n \geq 3$. In 1982, Faran [7] completed the classification problem for the codimension one case by providing a classification of maps from 2-ball to 3-ball as four equivalence classes

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of maps. Later in 1986, Faran [8] proved that the proper holomorphic mappings which extend holomorphically from $\mathbb{B}^n$ to $\mathbb{B}^k$ with $n \geq 3, k \leq 2n - 2$ are linear fractional. The C^2 -smoothness case was treated in Huang [13]. For more results on proper holomorphic mapping between balls, the readers may refer to D'Angelo [6], Forstnerič [10], Della Sala et al [25], and well as Baouendi–Huang [4], Huang et al [14] for the case of hyperquadrics, and many references therein.

The problem to understand proper holomorphic mapping from the unit ball to the classical domains, or more generally, proper holomorphic mappings and holomorphic isometries between various type of classical domains has been studied extensively, see the beautiful survey of Mok [18] and the references therein. In the special case of balls and type IV domains, Xiao–Yuan [30] proved in 2020 the rigidity of proper holomorphic maps from the complex unit ball $\mathbb{B}^n$ to the type IV bounded symmetric domain D_m^{IV} (the m -dimensional Lie ball) with $n \geq 4, n + 1 \leq m \leq 2n - 3$, giving an explicit formula for the “nonstandard” isometry of Mok, see also Xiao [29]. The lower dimensional cases, namely the cases $n = 2$ and $n = 3$, were settled recently by Reiter–Son [22] and [24], respectively. The aim of this paper is to extend these works to obtain a classification of CR maps from the hyperquadric $\mathbb{H}_l^{2n-1}$ of signature l in $\mathbb{C}^n$ into the tube over the null cone $\mathcal{X}_l^{2n+1}$ of signature $(l, n - l - 1, 1)$ (the numbers of negative, positive, and zero eigenvalues of the Levi form). As an immediate consequence, we classify local proper holomorphic maps from the generalized ball into the generalized Lie ball $D_{m,l}^{\text{IV}}$ (or the generalized type IV domain).

We recall the definitions of indefinite product, the generalized classical domains with signature $l > 0$, namely the generalized ball $\mathbb{B}_l^n$ and the generalized Lie ball $D_{m,l}^{\text{IV}}$, as well as the hyperquadric $\mathbb{H}_l^{2n-1}$ and the local model $\mathcal{X}_l^{2n+1}$ for the tube over the null cone of a symmetric form. Precisely, for $a = (a_1, \dots, a_{n-1})$ and $b = (b_1, \dots, b_{n-1}) \in \mathbb{C}^{n-1}$, we define the indefinite product $\langle a, b \rangle_l$ of signature l to be:

$$\langle a, b \rangle_l = - \sum_{j=1}^l a_j b_j + \sum_{k=l+1}^{n-1} a_k b_k, \quad 0 \leq l < n - 1.$$

The real hyperquadric of signature l in $\mathbb{C}^n$ is the real hypersurface of real dimension $2n - 1$, denoted by $\mathbb{H}_l^{2n-1}$, and is defined as follows:

$$\mathbb{H}_l^{2n-1} = \{(z, w) = (z_1, \dots, z_{n-1}, w) \in \mathbb{C}^n \mid \text{Im } w - \langle z, \bar{z} \rangle_l = 0\}. \quad (1.1)$$

Of our special interest is the following real hypersurface in $\mathbb{C}^{n+1}$, which has real dimension $2n + 1$ and is denoted by $\mathcal{X}_l^{2n+1}$. Namely, we define

$$\mathcal{X}_l^{2n+1} = \{(z, \zeta, w) \in \mathbb{C}^{n+1} \mid (1 - |\zeta|^2) \text{Im } w - \langle z, \bar{z} \rangle_l - \text{Re}(\bar{\zeta} z z^t) = 0, |\zeta|^2 < 1\}, \quad (1.2)$$

where, as above, $z = (z_1, z_2, \dots, z_{n-1})$ and z^t is its transpose, so that $z z^t = z_1^2 + \dots + z_{n-1}^2$. In the case $n = 2$ and $l = 0$, this local model was found by Gaussier

and Merker, which was then shown to be locally equivalent to the tube over the future light cone by Fels–Kaup [9]. Based on this work of Fels–Kaup, we can easily construct a local equivalence of $\mathcal{X}_l^{2n+1}$ for $n \geq 3$ and the tube over a null cone of a symmetric form in $\mathbb{R}^{n+1}$. For example, when $n = 3$ and $l = 1$, the model $\mathcal{X}_1^7$ is locally equivalent to the tube $\mathcal{T}_1$ over the null cone of a symmetric form in $\mathbb{R}^4$ given by

$$\mathcal{C}_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid -x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0\}.$$

More precisely, as a real hypersurface in $\mathbb{C}^4$, $\mathcal{T}_1 = i\mathbb{R} \times \mathcal{C}_1$ is given by

$$\rho(z_1, z_2, z_3, w) := -(\operatorname{Re} w)^2 - (\operatorname{Re} z_1)^2 + (\operatorname{Re} z_2)^2 + (\operatorname{Re} z_3)^2, \quad (1.3)$$

while the rational map

$$(z, w) \mapsto \left(\frac{2iz_1}{1+w-z_3}, \frac{2z_2}{1+w-z_3}, \frac{1-w+z_3}{1+w-z_3}, \frac{2i(w+w^2+z_1^2-z_2^2+z_3-z_3^2)}{1+w-z_3} \right) \quad (1.4)$$

provides a local equivalence of the germ $(\mathcal{T}_1, p = (0, 0, -1/2, 1/2))$ to $(\mathcal{X}, 0)$. Since both models are locally homogeneous, equivalence of a pair of arbitrary points implies the equivalences of all pairs. Note also that, for the case $l > 0$, the hypersurface $\mathcal{X}_l^{2n+1}$ is no longer pseudoconvex. But it still plays an important role in the study of real hypersurfaces in complex spaces, especially in that of homogeneous 2-nondegenerate homogeneous CR manifolds in several recent papers by Gregorovič, Sykes, Porter–Zelenko, Santi, and others, see, e.g., Gregorovič [11], Gregorovič–Sykes [12], Porter–Zelenko [20], and Santi [26], and many references therein. It is known that $\mathcal{X}_1^7$ has the *third* largest dimension of symmetry algebras (dimension 15) among finitely nondegenerate real hypersurfaces in $\mathbb{C}^4$. This differs a bit from the case of tube over the future light cone in $\mathbb{C}^3$ whose symmetry algebras has *second* largest dimension, cf. Fels–Kaup [9].

The complex unit ball with signature l , denoted by $\mathbb{B}_l^n$, has also been much studied in the literature. In nonhomogeneous coordinates of $\mathbb{C}^n$, it is defined by

$$\mathbb{B}_l^n = \{z = (z_1, \dots, z_{n-1}, w) \in \mathbb{C}^n \mid 1 - |w|^2 - \langle z, \bar{z} \rangle_l > 0\}.$$

When $l = 0$, it is the unit ball in $\mathbb{C}^n$, but when $l > 0$, it is unbounded. The boundary of $\mathbb{B}_l^n$ is locally equivalent to the hyperquadric of signature l via the Cayley transform:

$$(z, w) \mapsto \left(\frac{2z}{w+i}, \frac{w-i}{w+i} \right).$$

Of our interest is also the “signature version” of the Lie ball, or generalized type IV domain, defined by

$$D_{m,l}^{\text{IV}} = \left\{ z = (z_1, \dots, z_m) \in \mathbb{C}^m \mid 1 - 2\langle z, \bar{z} \rangle_l + |zz^t|^2 > 0 \text{ and } \langle z, \bar{z} \rangle_l < 1 \right\}.$$

The generalized Lie ball is also unbounded when $l > 0$ (for example, the unbounded null cone $\{\langle z, \bar{z} \rangle_l = 0\}$ is contained in $D_{m,l}^{IV}$). But it also possesses many interesting properties as the bounded Lie ball (i.e., $l = 0$) does. Unfortunately, to the authors' best knowledge, the generalized Lie ball of signature $l > 0$ defined as above has not been studied in the literature. Let us point out that the smooth boundary part of $D_{n+1,l}^{IV}$ is locally CR equivalent to the local model $\mathcal{X}_l^{2n+1}$.

We denote the CR automorphism group of a real hypersurface M or the biholomorphism group of a domain D by $\text{Aut}(M)$ and $\text{Aut}(D)$, respectively. For example, for the hypersurfaces $\mathbb{H}_l^{2n-1}$ and $\mathcal{X}_l^{2n+1}$ as well as the complex domains $\mathbb{B}_l^n$ and $D_{m,l}^{IV}$, their CR automorphism groups and biholomorphism groups are $\text{Aut}(\mathbb{H}_l^n)$ and $\text{Aut}(\mathcal{X}_l^{n+1})$ as well as $\text{Aut}(\mathbb{B}_l^n)$ and $\text{Aut}(D_{m,l}^{IV})$, respectively. Similarly, the stability group at a point $p \in M$ of M consists of local CR automorphisms of M fixing p is denoted by $\text{Aut}_p(M)$. If A and B are two real hypersurfaces, then two germs of CR maps (H, p) and (H', q) from A to B are considered to be equivalent if there exists $\psi \in \text{Aut}(A)$, $\varphi \in \text{Aut}(B)$ such that $\psi(p) = q$, $\varphi(H(p)) = H'(q)$ and

$$H' = \varphi \circ H \circ \psi^{-1}.$$

Now, we state the main result of this paper. In the statement below, we write $z = (z_1, \dots, z_{n-1})$ and $f(z, w) = (f_1(z, w), \dots, f_{n-1}(z, w))$.

Theorem 1.1. *Let $n \geq 3$, $1 \leq l < n - 1$ and U be an open subset of $\mathbb{H}_l^{2n-1} \subset \mathbb{C}^n$. Let $H: U \rightarrow \mathcal{X}_l^{2n+1} \subset \mathbb{C}^{n+1}$ be a transversal CR map. Then*

I. If $n = 3$ and $l = 1$, then H is equivalent to exactly one of the germs at the origin of the following maps:

(i) $H_j(z, w) = \left(\frac{z + iwzP_j}{1 + \varepsilon_j w^2}, \frac{2zP_j z^t}{1 + \varepsilon_j w^2}, \frac{w}{1 + \varepsilon_j w^2} \right)$, where $z = (z_1, z_2)$, z^t is its transpose, and P_j is one of the following five matrices

$$P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and $\varepsilon_j = \det(P_j) \in \{-1, 0, 1\}$ is the determinant of P_j .

(ii) $I(z, w) = \frac{2(z, w, w)}{1 + \sqrt{1 - 4i(zz^t - iw^2)}}$.

II. If $n \geq 4$, then H is equivalent to exactly one of the germs at the origin of the following maps:

(i) $\ell^{(n)}(z, w) = (z, 0, w)$,

$$(ii) \ I^{(n)}(z, w) = \frac{2(z, w, w)}{1 + \sqrt{1 - 4i(zz^t - iw^2)}}, \text{ where } z = (z_1, z_2, \dots, z_{n-1}) \text{ and } z^t$$

is its transpose.

Remark 1. The maps $\ell^{(n)}$ and $I^{(n)}$, which already appeared in the signature zero case, are isometries of the “canonical” indefinite Kähler metric of the one-sided neighborhoods of the source and target. They are inequivalent since all automorphisms are rational. The inequivalences of pairs of maps related to polynomial maps H_2 and H_3 and rational maps H_4 and H_5 can be seen from the “geometric rank” and the CR Ahlfors tensor: Both $\ell^{(n)}$ and $I^{(n)}$ have vanishing geometric rank, H_2 and H_3 have geometric rank one while H_4 and H_5 have geometric rank two. The CR Ahlfors tensor of H_2 is nonnegative while that of H_3 is nonpositive. Likewise, the CR Ahlfors tensor of H_4 is positive definite while that of H_5 has two eigenvalues of opposite sign. This explains the pairwise inequivalence of all six maps in the case $n = 3$ and of two maps in the case $n \geq 4$. We will discuss this in more details later.

By a similar strategy as in Reiter–Son [22, 24], we obtain from Theorem 1.1 a list of local proper holomorphic maps from the generalized ball into the generalized Lie ball.

Corollary 1.2. *Let $n \geq 3$, $1 \leq l < n-1$ and $F: \mathbb{B}_l^n \rightarrow D_{n+1,l}^{IV}$ be a proper holomorphic map which extends smoothly to some boundary point $p \in \partial\mathbb{B}_l^n$.*

I. If $n = 3$ and $l = 1$, then F is equivalent to exactly one of the following maps:

$$(i) \ R_0(z, w) = \left(\frac{z_1}{\sqrt{2}}, \frac{z_2}{\sqrt{2}}, \frac{2w + 2w^2 - z_1^2 - z_2^2}{4(1+w)}, \frac{i(2w + 2w^2 + z_1^2 + z_2^2)}{4(1+w)} \right),$$

$$(ii) \ R_1(z, w) = \left(\frac{zA + wzB}{\sqrt{2}(1+w)}, \frac{w}{2} + \frac{zFz^t}{4(1+w)}, i \left(\frac{w}{2} - \frac{zFz^t}{4(1+w)} \right) \right), \text{ where}$$

$$A = \begin{pmatrix} 1 & 2i \\ -2i & 3 \end{pmatrix}, B = \overline{A^{-1}} = \begin{pmatrix} -3 & -2i \\ 2i & -1 \end{pmatrix}, \text{ and } F = \begin{pmatrix} -5 & 4i \\ 4i & 3 \end{pmatrix}.$$

$$(iii) \ R_2(z, w) = \left(\frac{-zB - wzA}{\sqrt{2}(1+w)}, \frac{w}{2} + \frac{zF^{-1}z^t}{4(1+w)}, i \left(\frac{w}{2} - \frac{zF^{-1}z^t}{4(1+w)} \right) \right), \text{ where}$$

A, B, and F are as above.

$$(iv) \ P_1(z, w) = \left(wz_1, z_2, \frac{w^2 - z_1^2}{2}, \frac{i(w^2 + z_1^2)}{2} \right),$$

$$(v) \ P_2(z, w) = \left(z_1, wz_2, \frac{w^2 - z_2^2}{2}, \frac{i(w^2 + z_2^2)}{2} \right).$$

$$(vi) \ I(z, w) = \left(\frac{z_1}{\sqrt{2}}, \frac{z_2}{\sqrt{2}}, \frac{w}{\sqrt{2}}, \frac{1 - \sqrt{1 - z_1^2 - z_2^2 - w^2}}{\sqrt{2}} \right).$$

II. If $n \geq 4$, then F is equivalent to exactly one of the following maps:

$$\begin{aligned}
(i) \quad R_0^{(n)}(z, w) &= \left(\frac{z}{\sqrt{2}}, \frac{2w + 2w^2 - zz^t}{4(1+w)}, \frac{i(2w + 2w^2 + zz^t)}{4(1+w)} \right), \\
(ii) \quad I^{(n)}(z, w) &= \left(\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}}, \frac{1 - \sqrt{1 - zz^t - w^2}}{\sqrt{2}} \right).
\end{aligned}$$

Remark 2. The maps $R_0^{(n)}$ and $I^{(n)}$ were previously studied for all $n \geq 2$ in the work of Xiao–Yuan [30]. These maps are isometries (up to a conformal constant factor) with respect to the Bergman metrics on the unit ball and the Lie ball. Both maps exhibit singularities at some points on the boundary of the unit ball.

When $l > 0$, the complex variety

$$W = \{(z, w) \in \mathbb{C}^n \mid w + 1 = 0\},$$

on which $R_0^{(n)}$ is singular, intersects both sides of the boundary $\partial\mathbb{B}_l^n$. Consequently, neither $R_0^{(n)}$ nor $I^{(n)}$ maps the entire domain $\mathbb{B}_l^n$ into $D_{n+1,l}^{\text{IV}}$. Yet, these two maps are still local isometries of “canonical” pseudo-Kähler-Einstein metrics of the generalized ball and Lie ball. Similar behavior is observed for the rational maps R_1 and R_2 . Thus, in the case $l > 0$ we have examples of local isometries of the pseudo-Kähler metrics that do not extend to a global one.

The polynomial maps P_1 and P_2 have similar formulas, but they are indeed inequivalent, as can be observed by comparing their CR Ahlfors tensors. Although P_1 is polynomial, it does not send the whole $\mathbb{B}_1^3$ into $D_{4,1}^{\text{IV}}$. The map P_2 sends $\mathbb{B}_1^3$ into the open set

$$\tilde{D}_{4,1}^{\text{IV}} := \left\{ z = (z_1, \dots, z_4) \in \mathbb{C}^4 \mid 1 - 2\langle z, \bar{z} \rangle_1 + |zz^t|^2 > 0 \right\},$$

and sends $\mathbb{B}_1^3 \cap \{|z_1|^2 < 1\}$ into $D_{4,1}^{\text{IV}}$. On the other hand, when being restricted to the “slice” $\mathbb{B}_1^3 \cap \{z_1 = 0\} \simeq \mathbb{B}^2$, two maps P_1 and P_2 induce two polynomial maps from $\mathbb{B}^2$ into D_3^{IV} discovered by Reiter–Son [22].

The rational maps R_1 and R_2 did not appear earlier in the literature. They have “geometric rank”, as defined in Section 2.3, equal one. We should point out that rank one maps do not exist in the case $l = 0$ considered in Reiter–Son [23].

The rest of the paper is organized as follows. In section 2, we will provide the formulas for some automorphisms in $\mathbb{H}_l^{2n-1}$ and $\mathcal{X}_l^{2n+1}$, and use them to normalize the CR maps from $\mathbb{H}_l^{2n-1}$ to $\mathcal{X}_l^{2n+1}$. In section 3, we will prove Theorem 1.1. In section 4, we will present a proof of Corollary 1.2 as well as some further extensions of Theorem 1.1. Finally, in section 5, we will discuss the higher (co-)dimensional case.

2. NORMALIZATION FOR THEOREM 1.1

2.1. The CR automorphism groups of $\mathbb{H}_l^{2n-1}$ and $\mathcal{X}_l^{2n+1}$. The automorphism group of a Levi-nondegenerate hyperquadric $\mathbb{H}_l^{2n-1}$ is well-known and parametrized as follows:

$$\phi_{c,r,s,U}(z, w) = \frac{(s(z + cw)U, s^2w)}{1 + rw - iw\langle c, \bar{c} \rangle_l - 2i\langle \bar{c}, z \rangle_l},$$

where $c \in \mathbb{C}^{n-1}$, $r \in \mathbb{R}$, $s > 0$ and $U \in SU(l, n-1)$, see Chern–Moser [5].

Next, we consider the automorphisms in $\mathcal{X}_l^{2n+1}$. They can be computed explicitly by integrating infinitesimal CR automorphisms, which are given by:

$$\begin{aligned} X_{-2} &= \frac{\partial}{\partial w}, \\ X_{-1} &= a \cdot \frac{\partial}{\partial z} - \zeta \left\langle \bar{a}, \frac{\partial}{\partial z} \right\rangle_l + 2i \langle \bar{a}, z \rangle_l \frac{\partial}{\partial w}, \quad a \in \mathbb{C}^{n-1}, \\ X_0^1 &= z \cdot \frac{\partial}{\partial z} + 2w \frac{\partial}{\partial w}, \\ X_0^2 &= iz \cdot \frac{\partial}{\partial z} + 2i\zeta \frac{\partial}{\partial \zeta}, \\ X_0^3 &= zA \cdot \frac{\partial}{\partial z}, \quad A = (a_{ij})_{1 \leq i, j \leq n-1}, \quad a_{ii} = 0, \\ &\quad i \neq j : a_{ij} + a_{ji} = 0, \epsilon_i a_{ij} + \epsilon_j \bar{a}_{ji} = 0, \\ X_0^4 &= -\bar{b}\zeta \left(z \cdot \frac{\partial}{\partial z} \right) + (b - \bar{b}\zeta^2) \frac{\partial}{\partial \zeta} + i\bar{b}zz^t \frac{\partial}{\partial w}, \quad b \in \mathbb{C}, \\ X_1 &= (iw\zeta - zz^t) \left(c \cdot \frac{\partial}{\partial z} \right) + 2(c \cdot z) \left(z \cdot \frac{\partial}{\partial z} \right) + iw \left\langle \bar{c}, \frac{\partial}{\partial z} \right\rangle_l \\ &\quad + 2[\langle \bar{c}, z \rangle_l + \zeta(c \cdot z)] \frac{\partial}{\partial \zeta} + 2(c \cdot z)w \frac{\partial}{\partial w}, \quad c \in \mathbb{C}^{n-1}, \\ X_2 &= w \left(z \cdot \frac{\partial}{\partial z} \right) - izzz^t \frac{\partial}{\partial \zeta} + w^2 \frac{\partial}{\partial w}. \end{aligned}$$

The Lie algebra generated by the vector fields above which vanish at the origin has real dimension $(n^2 + 2n + 4)/2$. The stability group $\text{Aut}_0(\mathcal{X}_l^{2n+1})$ can be computed by integrating the vector fields in its symmetry algebra which vanish at the origin. To make our formulas more concise, we denote by

$$D_l = \text{diag}(\underbrace{-1, \dots, -1}_l, \underbrace{1, \dots, 1}_{n-1-l}) \in \text{Mat}(n-1; \mathbb{C})$$

the diagonal matrix with signature l . Moreover, put

$$\delta = \delta(z, w) = 1 - (r' + iaD_l \bar{a}^t)w - 2izD_l \bar{a}^t + i\overline{aa^t}(w\zeta + izz^t),$$

for $a = (a_1, a_2) \in \mathbb{C}^2$, $u' \in \mathbb{C}$, $|u'| = 1$, $r' \in \mathbb{R}$, and $P \in O(l, n - l - 1)$. The stability group consists of holomorphic maps of the form $\gamma = (\eta, \gamma_n, \gamma_{n+1})$, where $\eta = (\gamma_1, \gamma_2, \dots, \gamma_{n-1})$ and

$$\begin{aligned} \eta(z, w) &= s'u'(z + wa - (w\zeta + izz^t)D_l \bar{a})P/\delta, \\ \gamma_n(z, w) &= u'^2 (\zeta - 2za^t - ia a^t w - (r' - i\bar{a}D_l a^t)(w\zeta + izz^t)) / \delta, \\ \gamma_{n+1}(z, w) &= s'^2 w / \delta, \end{aligned}$$

with $s' > 0$. This form of the CR automorphisms is similar to the case of the tube over the future light cone in $\mathbb{C}^{n+1}$. We parametrize elements of $\text{Aut}_0(\mathcal{X}_l^{2n+1})$ by

$$(z, w) \mapsto \psi'_{s', u', P, a, r'}(z, w) = \gamma(z, w).$$

This parametrization is used in the normalization discussed in the next section.

2.2. A partial normalization. Let $U \subset \mathbb{C}^n$ be a connected open neighborhood of the origin and $H = (f, \phi, g) = (f_1, f_2, \dots, f_{n-1}, \phi, g)$ a holomorphic map from U to $\mathbb{C}^{n+1}$, $H(0) = 0$, and $H(U \cap \mathbb{H}_l^{2n-1}) \subset \mathcal{X}_l^{2n+1}$. Then the following “mapping equation” holds:

$$\begin{aligned} & \left(g(z, \bar{w} + 2i\langle z, \bar{z} \rangle_l) - \bar{g}(\bar{z}, \bar{w}) \right) \left(1 - \phi(z, \bar{w} + 2i\langle z, \bar{z} \rangle_l) \bar{\phi}(\bar{z}, \bar{w}) \right) \\ & - 2i \left\langle f(z, \bar{w} + 2i\langle z, \bar{z} \rangle_l), \bar{f}(\bar{z}, \bar{w}) \right\rangle_l - i \bar{\phi}(\bar{z}, \bar{w}) F(z, \bar{w} + 2i\langle z, \bar{z} \rangle_l) \\ & - i \phi(z, \bar{w} + 2i\langle z, \bar{z} \rangle_l) \bar{F}(\bar{z}, \bar{w}) = 0, \end{aligned} \tag{2.5}$$

for all $(z, \bar{z}, \bar{w})$ in a suitable neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \times \mathbb{C}$. Here, we put $F = f_1^2 + \dots + f_{n-1}^2$ for short.

If $H: U \cap \mathbb{H}_l^{2n-1} \rightarrow \mathcal{X}_l^{2n+1}$, $H(0) = 0$ is a smooth CR map and if $l > 0$, then by the Lewy extension theorem (see [3]), H is the restriction of a holomorphic map in a neighborhood of the origin, which we also denote by H . Then (2.5) holds for this extended map.

Using the stability groups at the origin of the source and the target, we can bring the map of interest into the following partial normal form.

Proposition 2.1. *Let (H, p) be a germ at $p \in \mathbb{H}_l^{2n-1}$ of a smooth transversal CR map which sends the germ at p of $\mathbb{H}_l^{2n-1}$ into $\mathcal{X}_l^{2n+1}$. Then (H, p) is equivalent to*

the germ at the origin of a CR map $\tilde{H} = (f, \phi, g)$ which is of the following form:

$$\begin{cases} f(z, w) &= z + \frac{i}{2}w(zA) + vw^2 + O(3), \\ \phi(z, w) &= \lambda w + zBz^t + wz\mu^t + \sigma w^2 + O(3), \\ g(z, w) &= w + O(3), \end{cases} \quad (2.6)$$

where $A, B \in \text{Mat}(n-1; \mathbb{C})$, $v = (v_1, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$, $\mu = (\mu_1, \dots, \mu_{n-1}) \in \mathbb{C}^{n-1}$, and $\lambda, \sigma \in \mathbb{C}$. Moreover, $B = [b_{jk}]$ is symmetric with b_{11} being real.

Proof. Applying $\bar{z} = \bar{w} = 0$ in the mapping equation (2.5) yields $g(z, 0) = 0$. Taking the derivative of the mapping equation with respect to $\bar{w}$ and applying $\bar{z} = \bar{w} = 0$ implies that $\frac{\partial g}{\partial w}(0, 0) \in \mathbb{R}$. The transversality of H implies that $\frac{\partial g}{\partial w}(0, 0) \neq 0$.

Let

$$E_k = \left(\frac{\partial f_j}{\partial z_k} \right)_{1 \leq j \leq n-1}.$$

Taking the second-order derivatives of the mapping equation (2.5) with respect to the variables z_k and $\bar{z}_j$ with $1 \leq k, j \leq n-1$ and setting $z = \bar{z} = \bar{w} = 0$, we obtain that:

$$\langle E_k, \overline{E_j} \rangle_l = \frac{\partial g}{\partial w}(0, 0) \epsilon_k \delta_j^k, \quad (2.7)$$

where $\epsilon_k := \text{sgn}(k - l - 1/2)$.

Let $E \in \text{Mat}(n-1; \mathbb{C})$ be the matrix with E_k being its k^{th} column. Then we can deduce that

$$\overline{E}^T D_l E = \frac{\partial g}{\partial w}(0, 0) I_l. \quad (2.8)$$

Now we write $H_{[n]} = \varphi'_n \circ H_{[n-1]} \circ \varphi_n$, where $H_{[0]} = H$ and φ_n, φ'_n are two suitable automorphisms of $\mathbb{H}_l^{2n-1}$ and $\mathcal{X}_l^{2n+1}$, respectively. Taking the derivatives with respect to the variables z_i for $1 \leq i \leq n-1$ and setting $z = w = 0$, we obtain that:

$$\frac{\partial H_{[1]}}{\partial z}(0, 0) = \left(ss' u' EU, su'^2 \left(\frac{\partial \phi}{\partial z}(0, 0) U - 2iaEU \right), 0 \right).$$

After exchanging components, if needed, we can assume that $\frac{\partial g}{\partial w}(0, 0) > 0$. Now choose

$$s' = 1 / \left(s^{n-1} \sqrt{\frac{\partial g}{\partial w}(0, 0)} \right), \quad U = \overline{u'} E^{-1},$$

we can assume that $\frac{\partial f_i}{\partial z_j}(0, 0) = \delta_j^i$. Considering $H_{[2]} = \varphi'_2 \circ H_{[1]} \circ \varphi_2$ with the above parameters, we obtain that:

$$\frac{\partial H_{[2]}}{\partial z}(0, 0) = \left(I_{n-1}, su' \left(-2ia + \frac{\partial \phi}{\partial z}(0, 0) \right), 0 \right).$$

Choosing $a = -\frac{i}{2} \frac{\partial \phi}{\partial z}(0, 0)$ and considering $H_{[3]} = \varphi'_3 \circ H_{[2]} \circ \varphi_3$ with the above parameters, we obtain that:

$$\frac{\partial H_{[3]}}{\partial z}(0, 0) = (I_{n-1}, 0, 0).$$

Now taking the derivative with respect to the variable w of $H_{[3]}$ at $z = w = 0$, we obtain the following:

$$\frac{\partial H_{[3]}}{\partial w}(0, 0) = \left(c + su' \frac{\partial f}{\partial w}(0, 0), s^2 u'^2 \frac{\partial \phi}{\partial w}(0, 0), 1 \right).$$

Choosing $c = -su' \frac{\partial f}{\partial w}(0, 0)$, we obtain that:

$$\frac{\partial H_{[3]}}{\partial w}(0, 0) = \left(0, s^2 u'^2 \frac{\partial \phi}{\partial w}(0, 0), 1 \right).$$

Now we will consider the second-order derivatives of $H_{[3]}$ at $(0, 0)$. Taking the derivatives with respect to z_k twice and $\bar{z}_k$ once of the mapping equation at $z = \bar{z} = \bar{w} = 0$, we have:

$$\frac{\partial^2 f_k}{\partial z_k^2}(0, 0) = 2 \frac{\partial^2 g}{\partial z_k \partial w}(0, 0) \quad \forall k = 1, \dots, n-1. \quad (2.9)$$

Taking the derivatives with respect to z_k and $\bar{w}$ of the mapping equation at $z = \bar{z} = \bar{w} = 0$, we have:

$$\frac{\partial^2 g}{\partial z_k \partial w}(0, 0) = 0 \quad \forall k = 1, \dots, n-1. \quad (2.10)$$

Taking the derivatives with respect to z_k, z_j and $\bar{z}_k$ with $k \neq j$ of the mapping equation at $z = \bar{z} = \bar{w} = 0$, we have:

$$\frac{\partial^2 f_k}{\partial z_k \partial z_j}(0, 0) = \frac{\partial^2 g}{\partial z_k \partial w}(0, 0) \quad \forall i, j = 1, \dots, n-1, i \neq j. \quad (2.11)$$

Taking the derivatives with respect to z_k, z_j , and $\bar{z}_q$ with $k \neq q, j \neq k$ of the mapping equation at $z = \bar{z} = \bar{w} = 0$, we have:

$$\frac{\partial^2 f_q}{\partial z_k \partial z_j}(0, 0) = 0 \quad \forall k, j, q = 1, \dots, n-1, k \neq q, j \neq q. \quad (2.12)$$

Combining (2.9), (2.10), (2.11), (2.12), we obtain that all second-order derivatives of $f(z, w)$ with respect to z_k and z_j vanish at 0.

Now considering $H_{[4]} = \varphi'_4 \circ H_{[3]} \circ \varphi_4$ and taking the second-order derivatives with respect to w at $z = w = 0$, we obtain

$$\frac{\partial^2 g_{[4]}}{\partial w^2}(0, 0) = -2r + s^2 \left(2t + \frac{\partial^2 g}{\partial w^2}(0, 0) \right),$$

such that when choosing $r = s^2 \left(2t + \frac{\partial^2 g}{\partial w^2}(0, 0) \right) / 2$, we can assume that $\frac{\partial^2 g}{\partial w^2}(0, 0) = 0$. Furthermore, taking the second-order derivatives of $\phi_{[4]}$ with respect to z_1 , we obtain that:

$$\frac{\partial^2 \phi_{[4]}}{\partial z_1^2}(0, 0) = s^2 \left(-2it + \frac{\partial^2 \phi}{\partial z_1^2}(0, 0) \right).$$

After choosing $t = \text{Im} \frac{\partial^2 \phi}{\partial z_1^2}(0, 0) / 2$ we can assume that $\frac{\partial^2 \phi}{\partial z_1^2}(0, 0) \in \mathbb{R}$, showing that b_{11} is real. The proof is complete. $\square$

Proposition 2.2. *The partial normal form of a map germ (H, p) obtained in Proposition 2.1 satisfies $A = B$.*

Proof. We denote the entries of A and B by a_{ij} and b_{ij} , respectively. Taking the derivative of the mapping equation with respect to the variables $z_k, z_k, \bar{z}_k, \bar{z}_j$ with $1 \leq k \neq j \leq n-1$ at $z = \bar{z} = \bar{w} = 0$, and using the fact that B is a symmetric matrix, we have:

$$a_{kj} = \epsilon_k \epsilon_j \overline{b_{kj}} = \epsilon_k \epsilon_j \overline{b_{jk}} = a_{jk}. \quad (2.13)$$

This shows that A is symmetric.

Taking the derivatives of the mapping equation with respect to the variables $z_k, z_j, \bar{z}_k$, and $\bar{z}_j$ at $z = \bar{z} = \bar{w} = 0$, we have:

$$a_{kj} = b_{kj} = b_{jk} = a_{jk}. \quad (2.14)$$

From (2.13) and (2.14) we have $a_{kj} = b_{kj}$ is real if $\epsilon_k \epsilon_j = 1$, and $a_{kj} = b_{kj}$ is purely imaginary if $\epsilon_k \epsilon_j = -1$. Here, as above, $\epsilon_j = \text{sgn}(j - l - 1/2)$.

Taking the fourth-order derivative of the mapping equation with respect to the variables $z_k, z_k, \bar{z}_k$, and $\bar{z}_k$ at $z = \bar{z} = \bar{w} = 0$, we have:

$$2a_{kk} = b_{kk} + \overline{b_{kk}}. \quad (2.15)$$

Thus $a_{kk} \in \mathbb{R}$ for all $1 \leq k \leq n-1$. Taking the fourth-order derivatives of the mapping equation with respect to the variables $z_k, z_k, \bar{z}_j, \bar{z}_j$ with $1 \leq k \neq j \leq n-1$ at $z = \bar{z} = \bar{w} = 0$, we have:

$$\overline{b_{kk}} + b_{jj} = 0 \quad (2.16)$$

From (2.15) and (2.16) we deduce that:

$$a_{kk} + a_{jj} = 0, \quad \forall 1 \leq k \neq j \leq n-1. \quad (2.17)$$

From here we consider two cases depending on the value of n :

Case 1: If $n \geq 4$: Solving the system of equations (2.17) we obtain $a_{kk} = 0$ for all $1 \leq k \leq n-1$. Applying (2.15), (2.16) and the fact that $b_{11} = \frac{\partial^2 \phi}{\partial z_1^2}(0,0) \in \mathbb{R}$ we have $b_{kk} = 0$ for all $1 \leq k \leq n-1$. Thus we have $A = B$ and the diagonal entries equal to 0.

Case 2: If $n = 3$: Solving the system of equations (2.17) we obtain $a_{11} = -a_{22}$. Applying (2.15), (2.16) and the fact that $b_{11} \in \mathbb{R}$ we have $b_{kk} = a_{kk}$ for $j = 1, 2$. Thus we have $A = B$. The proof is complete. $\square$

For $1 \leq j, k \leq n-1$, we denote $a_{jk} = c_{jk}$ if $\epsilon_j \epsilon_k = 1$, and $a_{jk} = ic_{jk}$ if $\epsilon_j \epsilon_k = -1$. From Proposition 2.2 we have $c_{jk} \in \mathbb{R}$ for all $1 \leq j, k \leq n-1$. We will use the above normalization in Section 3.

2.3. Geometric rank and the CR Ahlfors tensor. In this section, we analyze a tensor defined for each transversal CR map between real hypersurfaces. In our situation, the tensor has an invariant property and provides an efficient way to prove the inequivalence of CR maps. Specifically, we shall use it in the proof of Corollary 1.2.

The CR Ahlfors tensor was first defined for CR maps between strictly pseudoconvex pseudohermitian manifolds by Lamel–Son [17]. The original construction is of differential geometric nature which is based on a construction of the conformal counterpart of Stowe [27]. It was subsequently used in several papers, see Reiter–Son [22, 23, 24].

2.3.1. The CR Ahlfors tensor. Let $G: M \rightarrow N$ be a CR transversal map extending holomorphically to a neighborhood of M . By the CR transversality, there exists a real-valued smooth function v such that

$$\rho_N \circ G = \pm e^v \rho_M. \quad (2.18)$$

In many situations, the complex Hessian of v is well-behaved and thus it is important for our purposes.

Let $N = \partial D_{m,l}^{\text{IV}}$ be the boundary of the generalized Lie ball and let $\rho_{D_{m,l}^{\text{IV}}} = 1 - 2\langle z, \bar{z} \rangle_l + |zz^t|$ be its defining function. Also, let $M = \partial \mathbb{B}_l^n$ be the boundary of the generalized ball with the defining function $\rho_{\mathbb{B}_l^n} = 1 - \langle z, \bar{z} \rangle_l$. Without loss of generality, we may assume that in (2.18) the positive sign occurs. At each point p , the complex Hessian of v gives rise to a Hermitian form on the holomorphic tangent space $T_p^{(1,0)}\mathbb{C}^n$. If $p \in M$, we identify $T_p^{(1,0)}M = T_p^{(1,0)}\mathbb{C}^n \cap \mathbb{C}T_p\mathbb{C}^n$. By this, we can

restrict the complex Hessian $v_{Z\bar{Z}}$ (where $Z = (z, w)$) to obtain a tensor $\mathcal{A}(G)$ defined on $T^{(1,0)}M$ and its complex conjugate. Thus,

$$\mathcal{A}(G) = v_{Z\bar{Z}} \Big|_{T^{(1,0)}\partial\mathbb{B}_l^n \times T^{(0,1)}\partial\mathbb{B}_l^n}.$$

Observe that $\mathcal{A}(G)$ is invariant under the pre-composition and composition with automorphisms of $\mathbb{B}_l^n$ and $D_{m,l}^{\text{IV}}$, respectively. In fact, this invariant property is a consequence of the fact that the CR automorphisms of the source and the target preserves certain (indefinite) Kähler metrics defined by Kähler potentials related to the defining functions.

Precisely, let ψ be an automorphism of $D_{m,l}^{\text{IV}}$ and let $G' = \psi \circ G$. From the fact that $\psi \in \text{Aut}(D_{m,l}^{\text{IV}})$ we can check directly that

$$\rho_{D_{m,l}^{\text{IV}}} \circ \psi = |q|^2 \rho_{D_{m,l}^{\text{IV}}}$$

for some rational function q (which can be explicitly computed) having no pole and zero on N . Indeed, we can verify this claim for ψ being one of the five maps ψ_j given in Section 2.1 (which generate the full group of automorphisms via compositions). Thus, we have

$$\rho_{D_{m,l}^{\text{IV}}} \circ G' = \rho_{D_{m,l}^{\text{IV}}} \circ \psi \circ G = (|q|^2 \rho_{D_{m,l}^{\text{IV}}}) \circ G = |q \circ G|^2 (\rho_{D_{m,l}^{\text{IV}}} \circ G) = |q \circ G|^2 e^v \rho_{B_l^n}.$$

Hence, we obtain

$$v' = v + \log(|q \circ G|^2).$$

Since q is holomorphic along N , $q \circ G$ is holomorphic along M , and thus the complex Hessian matrices of v and v' are the same, as desired.

Similarly, we can prove that the CR Ahlfors tensor is invariant under the pre-composition with an automorphisms of $\mathbb{B}_l^n$, we leave the details to the reader.

2.3.2. Geometric rank. As first noticed by Lamel and Son, the CR Ahlfors tensor defined in [17] is closely related to the notion of geometric rank of Huang [13]. This has been exploited in Reiter–Son [22, 24, 23]. In what follows, we shall use this to define the geometric rank for CR maps in our setting: For $N = \mathcal{X}_l^{2n+1}$, we take

$$\rho'(z, \zeta, w) = (1 - |\zeta|^2) \text{Im}(w) - \langle z, \bar{\zeta} \rangle_l - \text{Re}(\bar{\zeta} z z^t),$$

and for $M = \mathbb{H}_l^{2n-1}$, we take

$$\rho = \text{Im}(w) - \langle z, \bar{z} \rangle_l.$$

Assume that $H = (f, \phi, g)$ sends the germ at the origin of the hyperquadric into $\mathcal{X}_l^{2n+1}$ and has a partial normal form (2.6). Then there exists a real-valued function $Q(z, w, \bar{z}, \bar{w})$ defined in a neighborhood of the origin in $\mathbb{C}^n$ such that

$$(1 - |\phi(z, w)|^2)(g(z, w) - \bar{g}(\bar{z}, \bar{w})) - 2i\langle f(z, w), \bar{f}(\bar{z}, \bar{w}) \rangle_l$$

$$-i(\phi(z, w)\bar{F}(\bar{z}, \bar{w}) + \bar{\phi}(\bar{z}, \bar{w})F(z, w)) = Q(z, w, \bar{z}, \bar{w})(w - \bar{w} - 2i\langle z, \bar{z} \rangle_l). \quad (2.19)$$

Comparing the coefficients of w of the Taylor expansions at the origin of both sides of (2.19), we find that $Q(0) = 1$. Setting $w = \bar{w} = 0$, the equation reduces to

$$-2i\langle f(z, w), \bar{f}(\bar{z}, \bar{w}) \rangle_l - i(\phi(z, 0)\bar{F}(\bar{z}, 0) + \bar{\phi}(\bar{z}, 0)F(z, 0)) = -2iQ(z, 0, \bar{z}, 0)\langle z, \bar{z} \rangle_l.$$

We expand $Q(z, 0, \bar{z}, 0)$ in homogeneous terms of bidegree (k, l) :

$$Q(z, 0, \bar{z}, 0) = 1 + q_{(1,0)}(z) + q_{(0,1)}(\bar{z}) + q_{(2,0)}(z, \bar{z}) + q_{(0,2)}(z, \bar{z}) + q_{(1,1)}(z, \bar{z}) + O(3).$$

Comparing the terms of bidegrees $(1, 2)$ and $(2, 1)$, we find that

$$q_{(1,0)}(z) = q_{(0,1)}(\bar{z}) = 0.$$

Comparing the terms of bidegree $(2, 2)$, we find that

$$q_{(1,1)}(z, \bar{z})\langle z, \bar{z} \rangle_l = \frac{1}{2} \left((zBz^t)\bar{z}\bar{z}^t + \overline{(zBz^t)}(zz^t) \right). \quad (2.20)$$

Thus, if

$$q_{(1,1)}(z, \bar{z}) = \sum_{j,k=1}^{n-1} q_{j\bar{k}} z_j \bar{z}_k,$$

we easily find that

$$\epsilon_j q_{j\bar{k}} = b_{jk},$$

where, as before, $\epsilon_j = \text{sgn}(j - l - 1/2)$ (the signum function). In particular, since b_{ij} is symmetric, we find that $q_{j\bar{k}}$ is real if $\epsilon_j = \epsilon_k$ and purely imaginary otherwise.

Since $Q(0) = 1$ and all first order derivatives of Q at the origin vanish, we can easily find that

$$\mathcal{A}(H)|_0 (\partial_j|_0, \partial_{\bar{k}}|_0) = (\log(Q))_{j\bar{k}} = q_{j\bar{k}}. \quad (2.21)$$

We call the rank of $\mathcal{A}(H)|_0$, or equivalently, the rank of $[q_{j\bar{k}}]$, the *geometric rank* of H at the origin.

Remark 3. In the case $n \geq 4$, it follows from (2.20) and Huang's Lemma [13] that $q_{(1,1)}(z, \bar{z}) = 0$ and consequently $A = B = 0$. Hence, the CR Ahlfors tensor must vanish in this case.

2.4. Isometries. The Heisenberg hypersurface $\mathbb{H}^{2n-1}$ divides $\mathbb{C}^n$ into two half-spaces. The *Siegel upper half-space* is

$$\Omega_n^+ := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im } w > |z|^2\}.$$

It is an unbounded strictly pseudoconvex domain in $\mathbb{C}^n$. The Bergman kernel of Ω^+ is explicit, see, for example, Krantz [16]. The Bergman metric on Ω^+ is the Kähler metric given by the Kähler form

$$\omega_{\Omega^+} = -ni \partial \bar{\partial} \log (\operatorname{Im} w - |z|^2).$$

Then, Ω^+ with this “canonical” metric is a model for the complex hyperbolic space, which has constant negative holomorphic sectional curvature.

More generally, for each $0 < l < n - 1$, the real hyperquadric $\mathbb{H}_l^{2n-1}$ of signature $0 < l < n - 1$ divides $\mathbb{C}^n$ into two unbounded domains. Define the “upper” half-space to be

$$\Omega_{n,l}^+ := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Im} w > \langle z, \bar{z} \rangle_l\}.$$

The “canonical” pseudo-Kähler metric on Ω_l^+ is given by

$$\omega_{\Omega_{n,l}^+} = -ni \partial \bar{\partial} \log (\operatorname{Im} w - \langle z, \bar{z} \rangle_l). \quad (2.22)$$

As in the case of Siegel upper-half space, $\omega_{\Omega_{n,l}^+}$ is a model for the indefinite complex hyperbolic space, i.e., $\omega_{\Omega_{n,l}^+}$ is a pseudo-Kähler metric having a constant negative holomorphic sectional curvature.

A one-sided neighborhood of $\mathcal{X}_l^{2n+1}$ in $\mathbb{C}^{n+1}$ also possesses an interesting (pseudo) Kähler metric. We consider the “upper” domain

$$D_{n,l}^+ = \{(z, \zeta, w) \in \mathbb{C}^{n+1} \mid (1 - |\zeta|^2) \operatorname{Im} w - \langle z, \bar{z} \rangle_l - \operatorname{Re}(\bar{\zeta} z z^t) > 0, 1 - |\zeta|^2 > 0\}$$

and the “canonical” (pseudo) Kähler metric given by the fundamental form

$$\omega_{D_{n,l}^+} = -i \partial \bar{\partial} \log ((1 - |\zeta|^2) \operatorname{Im} w - \langle z, \bar{z} \rangle_l - \operatorname{Re}(\bar{\zeta} z z^t)). \quad (2.23)$$

When $l = 0$, this metric is locally isometric (up to a dimensional constant) to the Bergman metric on the classical domain of type IV, the Lie ball. When $l > 0$, Eq. (2.23) gives an interesting pseudo-Kähler metric with the Ricci form satisfying

$$\operatorname{Ric} = -n \omega_{D_{n,l}^+}.$$

Thus, it is (pseudo) Kähler–Einstein metric, and is locally isometric to a metric on the generalized Lie ball.

Similarly to Reiter–Son [24] for the case $l = 0$, we have a close relation between an isometry of a one-sided neighborhood of the canonical metrics and the vanishing of the geometric rank of the map on the real hypersurface.

Proposition 2.3. *Let $p \in \mathbb{H}_l^{2n-1}$ and let U be an open neighborhood of p in $\mathbb{C}^n$. Assume that $U \cap \mathbb{H}_l^{2n-1}$ is open connected and $H: U \cap \mathbb{H}_l^{2n-1} \rightarrow \mathcal{X}_l^{2n+1}$ is a smooth CR map. If H extends to an isometry from $U \cap \Omega_l^+$ into $D_{n+1,l}^+$ with respect to the pseudo-Kähler metrics described above, then H has vanishing geometric rank along $U \cap \mathbb{H}_l^{2n-1}$.*

Remark 4. By the Lewy extension theorem (see Baouendi–Ebenfelt–Rothschild [3]), if $0 < l < n - 1$, then H always extends holomorphically to both sides of $\mathbb{H}_l^{2n-1} \cap U$.

3. THE CLASSIFICATION OF CR MAPS FROM $\mathbb{H}_l^{2n-1}$ TO $\mathcal{X}_l^{2n+1}$

In this section we will prove Theorem 1.1. Let U be an open subset of $\mathbb{H}_l^{2n-1}$ and $H: U \rightarrow \mathcal{X}_l^{2n+1} \subset \mathbb{C}^{n+1}$ be a C^2 -smooth function. Note that the case $l = 0$ is known. Therefore, in the following $1 \leq l \leq n/2$.

By normalization, F is equivalent to a map of the form given in Proposition 2.1. We therefore assume that F is a holomorphic or formal map and already of this form. Similarly to Reiter–Son [22, 24], we first determine the map along the first Segre set.

Proposition 3.1. *With the assumptions and notations above, it holds that*

$$g(z, 0) = 0, \quad f(z, 0) = \frac{2z}{1 + \sqrt{1 - 4i\bar{\lambda}zz^t}}, \quad (3.24)$$

where as above $f = (f_1, \dots, f_{n-1})$, $z = (z_1, \dots, z_{n-1})$, and $zz^t = z_1^2 + z_2^2 + \dots + z_{n-1}^2$.

Proof. Setting $(\bar{z}, \bar{w}) = (0, 0)$ in the mapping equation, we obtain $g(z, 0) = 0$.

Define the following differential operators

$$L_j := \frac{\partial}{\partial \bar{z}_j} - 2i\epsilon_j z_j \frac{\partial}{\partial \bar{w}}, \quad j = 1, 2, \dots, n-1, \quad \epsilon_j = \text{sgn}(j - l - 1/2),$$

where $\text{sgn}(x)$ is the signum function. Observe that if $\varphi(z, w)$ is holomorphic, then

$$L_j(\varphi(z, \bar{w} + 2i\langle z, \bar{z} \rangle_l)) = 0.$$

Thus, applying L_j to the mapping equation (2.5) yields

$$(\phi\bar{\phi} - 1)(L_j\bar{g}) - 2i \sum_{k=1}^{n-1} \epsilon_k f_k (L_j\bar{f}_k) - iF(L_j\bar{\phi}) - i\phi(L_j\bar{F}) = 0, \quad (3.25)$$

where the “bared” functions are evaluated at $(\bar{z}, \bar{w})$ and the “unbared” functions are evaluated at $(z, \bar{w} + 2i\langle z, \bar{z} \rangle_l)$.

From the partial normal form (2.6), when evaluating at $(\bar{z}, \bar{w}) = (0, 0)$, it holds that

$$(L_j\bar{g})(z, 0, 0) = -2i\epsilon_j z_j, \quad (3.26)$$

$$(L_j\bar{f}_k)(z, 0, 0) = \delta_{jk}, \quad (3.27)$$

$$(L_j\bar{\phi})(z, 0, 0) = -2i\bar{\lambda}\epsilon_j z_j, \quad (3.28)$$

$$(L_j\bar{F})(z, 0, 0) = 0. \quad (3.29)$$

Thus, evaluating (3.25) at $\bar{z} = 0$ and $\bar{w} = 0$, we obtain:

$$z_j - f_j(z, 0) + iz_j\bar{\lambda}F(z, 0) = 0, \quad j = 1, 2, \dots, n-1. \quad (3.30)$$

where, as above, $F = f f^t$. Thus,

$$f_j(z, 0) = z_j(1 + i\bar{\lambda}F(z, 0)). \quad (3.31)$$

Taking the squares of both sides of (3.31) and summing over j , we obtain a quadratic equation for $F(z, 0)$. Namely,

$$F(z, 0) = z z^t (1 + i\lambda F(z, 0))^2. \quad (3.32)$$

From (3.32), we can solve for $F(z, 0)$ and then solve the system of equations (3.30) to obtain

$$f_j(z, 0) = \frac{2z_j}{1 + \sqrt{1 - 4i\bar{\lambda}z z^t}}, \quad j = 1, 2, \dots, n-1. \quad (3.33)$$

We complete the proof. $\square$

Proposition 3.2. *With the assumptions and notations as above, it holds that*

$$\frac{\partial g}{\partial w}(z, 0) = \frac{2}{1 + \sqrt{1 - 4i\bar{\lambda}z z^t}}.$$

Proof. Differentiating the mapping equation (2.5) with respect to $\bar{w}$ and setting $(\bar{z}, \bar{w}) = (0, 0)$ yield

$$-\bar{\lambda} \left(\sum_{j=1}^{n-1} f_j(z, 0)^2 \right) - i \left(-1 + \frac{\partial g}{\partial w}(z, 0) \right) = 0.$$

Using Proposition 3.1 and solving for $\frac{\partial g}{\partial w}(z, 0)$ yield

$$\frac{\partial g}{\partial w}(z, 0) = \frac{2}{1 + \sqrt{1 - 4i\bar{\lambda}z z^t}}. \quad (3.34)$$

The proof is complete. $\square$

From here, for simplicity, we divide our consideration into several cases depending on the value of λ and n .

Case 1: $\lambda = 0$.

In this case, from Propositions 3.1 and 3.2, it holds that

$$f(z, 0) = z, \quad g(z, 0) = 0, \quad g_w(z, 0) = 1.$$

Subcase 1.1: $n \geq 4$.

Proposition 3.3. *With the assumptions and notations as above, it holds that*

$$\sigma = 0, v = 0, \mu = 0, \text{ and } c_{jk} = 0 \text{ for all } 1 \leq j < k \leq n-1.$$

Proof. Applying L_k to (3.25), we obtain

$$\phi(L_k \bar{\phi})(L_j \bar{g}) + (\phi \bar{\phi} - 1)(L_k L_j \bar{g}) - 2i \langle f, L_k L_j \bar{f} \rangle_l - iF(L_k L_j \bar{\phi}) - i\phi(L_k L_j \bar{F}) = 0. \quad (3.35)$$

Setting $(\bar{z}, \bar{w}) = (0, 0)$, we have

$$\begin{aligned} L_k L_j \bar{F} &= 2\delta_{jk}, \\ L_k L_j \bar{g} &= 0, \\ L_k L_j \bar{\phi} &= \bar{a}_{jk} - 2i\epsilon_j z_j \bar{\mu}_k - 2i\epsilon_k z_k \bar{\mu}_j - 8\epsilon_j \epsilon_k z_j z_k \sigma, \\ L_k L_j \bar{f}_q &= -\epsilon_j z_j \bar{a}_{kq} - \epsilon_k z_k \bar{a}_{jq} - 8\epsilon_j \epsilon_k z_j z_k v_q. \end{aligned}$$

Substituting these into (3.35), with $k \neq j$ we obtain various polynomial equations of z . Equating the coefficients of both sides, we have $\sigma = 0, v = 0, \mu = 0$ and $c_{jk} = 0$ for all $1 \leq j < k \leq n-1$.

Next, substituting into (3.35) when $j = k$, we find that

$$\phi(z, 0) = 0.$$

The proof is complete. $\square$

Remark 5. We mentioned that it is also possible to deduce that $c_{jk} = 0$ by invoking the well-known Huang's Lemma [13]. On the other hand, the proof above breaks down for $n = 3$. In this case (which shall be treated below), we have only two differential operators L_1 and L_2 leading to only two equations that are not enough to conclude $c_{jk} = 0$.

Proposition 3.4. *With the assumptions and notations as above,*

$$\phi(z, w) = 0. \quad (3.36)$$

Remark 6. Once we prove $\phi(z, w) = 0$, our map reduces to $H = (f, 0, g)$. From the mapping equation, it follows that the map $\tilde{H} = (f, g)$ is a map between hyperquadrics of the same dimension. By the well-known 2-jet determination result in Chern–Moser [5], we can infer from the partial normal form (2.6) that $\tilde{H} = (z, w)$ and hence $H = (z, 0, w) = \ell^{(n)}(z, w)$, as desired.

Sketch of proof of Proposition 3.4. From the formula for the map H along the first Segre set: $H(z, 0) = (z, 0, 0)$, we can use a “reflection principle” argument as in Reiter [21] and Reiter–Son [22, 24] (cf. Baouendi–Ebenfelt–Rothschild [2]) to produce several holomorphic equations for components H , which, in turn, produce desired formula for H .

Namely, to compute $\phi(z, w)$, first we multiply the mapping equation with 2 and set $\bar{w} = 0$ to obtain

$$I = I(z, w, H(z, 2i\langle z, \bar{z} \rangle_l), \bar{H}(\bar{z}, 0)) = 0. \quad (3.37)$$

Substituting $\bar{H}(\bar{z}, 0) = (\bar{z}, 0, 0)$ yields an equation of the form

$$\tilde{I} = \tilde{I}(z, w, H(z, 2i\langle z, \bar{z} \rangle_l), \bar{z}) = 0. \quad (3.38)$$

For each $1 \leq j \leq n-1$, multiplying the equation above with $4z_j^2$ and setting

$$\bar{z}_j = \frac{w - 2i \sum_{1 \leq k \leq n-1, k \neq j} \epsilon_k z_k \bar{z}_k}{2i\epsilon_j z_j},$$

we obtain an equation of the form

$$\mathcal{I}_j(z, w, H(z, w), \bar{z}_1, \dots, \hat{\bar{z}}_j, \dots, \bar{z}_{n-1}) = 0,$$

where the variable $\bar{z}_j$ does not appear. Thus, we obtain $n-1$ such equations. Two of them lead to $\phi(z, w) = 0$. Indeed, taking the derivative of $\mathcal{I}_1$ with respect to $\bar{z}_2$, setting $\bar{z} = 0$, and multiplying by z_2 , we obtain the first equation. Taking the derivative of $\mathcal{I}_2$ with respect to $\bar{z}_1$, setting $\bar{z} = 0$ and multiplying by z_1 , we obtain the second equation. Taking the difference of the two equations give $\phi(z, w) = 0$. The proof is complete. $\square$

Thus, in the case $n \geq 4$, every map H in the partial normal form (2.6) with $\lambda = 0$ must be of the form $\ell^{(n)}(z, w) = (z, 0, w)$.

Subcase 1.2: $\lambda = 0$ and $n = 3$.

In this case, our consideration is somewhat similar to Reiter–Son [24]. For example, the following is almost the same as [24, Lemma 4.3]

Proposition 3.5. *With the assumptions and notations as above, $\sigma = 0, v = 0$, and $\mu = 0$.*

Proof. Applying L_1 and L_2 consecutively to the mapping equation (2.5), setting $\bar{z} = \bar{w} = 0$, and applying Propositions 3.1 and 3.2, we obtain a polynomial of z :

$$8\bar{v}_1 z_1^2 z_2 - 8\bar{v}_2 z_1 z_2^2 - (z_1^2 + z_2^2)(-iz_2 \bar{\mu}_1 + iz_1 \bar{\mu}_2 + 4z_1 z_2 \bar{\sigma}) = 0. \quad (3.39)$$

Now equating the coefficients of both sides, we have $\sigma = 0, v = 0$ and $\mu = 0$. This completes the proof. $\square$

Proposition 3.6. *Assume that H has a normal form (2.6) with $\lambda = 0$. Then*

$$\phi(z, 0) = zAz^t. \quad (3.40)$$

More precisely, if

$$A = \begin{pmatrix} \alpha & i\beta \\ i\beta & -\alpha \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{R}$, then $\phi(z, 0) = \alpha(z_1^2 - z_2^2) + 2i\beta z_1 z_2$.

Proof. Applying L_1 twice to the mapping equation and setting $\bar{z} = \bar{w} = 0$ we obtain that

$$\alpha(z_1^2 - z_2^2) + 2i\beta z_1 z_2 - \phi(z, 0) = 0,$$

from which we complete the proof. $\square$

From the formula for the map H along the first Segre set as above, we can use a “reflection principle” argument as above to produce several holomorphic equations for components f, ϕ and g of the map.

Proposition 3.7. *The following equations hold in a neighborhood of the origin of $\mathbb{C}^3$:*

$$\alpha w^2 \Phi - 4w z_1 f_1(z, w) + 4z_1^2 g(z, w) + i w^2 \phi(z, w) = 0 \quad (3.41)$$

$$\alpha w^2 \Phi + 4w z_2 f_2(z, w) - 4z_2^2 g(z, w) - i w^2 \phi(z, w) = 0 \quad (3.42)$$

$$w(z_2 \alpha - i z_1 \beta) \Phi - 2z_1(z_2 f_1(z, w) - z_1 f_2(z, w)) + i w z_2 \phi(z, w) = 0 \quad (3.43)$$

$$w(z_1 \alpha + i z_2 \beta) \Phi - 2z_2(z_2 f_1(z, w) - z_1 f_2(z, w)) - i w z_1 \phi(z, w) = 0 \quad (3.44)$$

$$(\alpha(z_1^2 - z_2^2) + 2i\beta z_1 z_2) \Phi - i(z_1^2 + z_2^2) \phi(z, w) = 0 \quad (3.45)$$

with $\Phi = g(z, w) \phi(z, w) + i(f_1(z, w)^2 + f_2(z, w)^2)$.

Sketch of the proof. Applying $\bar{w} = 0$ to the mapping equation and substituting $\bar{z}_1 = 0, \bar{z}_2 = \frac{w}{2iz_2}$ we obtain (3.42). Similarly, applying $\bar{w} = 0$ to the mapping equation and substituting $\bar{z}_2 = 0, \bar{z}_1 = \frac{-w}{2iz_1}$ we obtain (3.41). Now we define the function J by multiplying the mapping equation with 2 and applying $\bar{w} = 0$, we obtain an identity of the form

$$\mathcal{R}(z, w, H(z, w), \bar{H}(\bar{z}, 0)) = 0, \quad (3.46)$$

where $\mathcal{R}$ is explicit and polynomial in its arguments. We won't reproduce the explicit form of $\mathcal{R}$ here for simplicity. From this and the identity for $\bar{H}(\bar{z}, 0)$ obtained above, we get an identity of the form

$$\mathcal{R}_1(z, w, H(z, w), \bar{z}) = 0,$$

where, as above, $\mathcal{R}_1$ is also explicit and polynomial. Next, substituting $\bar{z}_1 = \frac{w - 2iz_2 \bar{z}_2}{-2iz_1}$ into this, we obtain an identity of the form

$$\mathcal{R}_2(z, w, H(z, w), \bar{z}_2) = 0,$$

with $\mathcal{R}_2$ is polynomial in its arguments (after clearing some denominator). Differentiating this with respect to $\bar{z}_2$ and setting $\bar{z}_2 = 0$, we obtain (3.43).

By the same procedure as above with the roles of z_1 and z_2 exchanged, we obtain (3.44).

Finally, multiplying (3.43) by z_2 , multiplying (3.44) by z_1 and taking the difference of the resulting equations, we obtain (3.45). The proof is complete. $\square$

Proposition 3.8. *Assume that $\lambda = 0$. It holds that*

$$f_1(z, w) = \frac{z_1}{w}g(z, w) + \frac{w(z_1\alpha + iz_2\beta)}{2(z_1^2 + z_2^2)}\Phi, \quad (3.47)$$

$$f_2(z, w) = \frac{z_2}{w}g(z, w) - \frac{w(z_2\alpha - iz_1\beta)}{2(z_1^2 + z_2^2)}\Phi, \quad (3.48)$$

$$\phi(z, w) = \frac{-i(\alpha(z_1^2 - z_2^2) + 2i\beta z_1 z_2)}{z_1^2 + z_2^2}\Phi. \quad (3.49)$$

Proof. We can rewrite the equations in Proposition 3.7 as a system of linear equations of 5 variables f_1, f_2, ϕ, g , and Φ . Solving the system of equations (3.41), (3.42), (3.45) we have the desired formulas above. $\square$

In order to fully determine $H(z, w)$, we need more holomorphic equations for its component. To this end, we shall determine H_w along the first Segre set in the next proposition.

Proposition 3.9. *Assume that $\lambda = 0$. It holds that*

$$\frac{\partial f_1}{\partial w}(z, 0) = \frac{i}{2}(z_1\alpha + iz_2\beta), \quad (3.50)$$

$$\frac{\partial f_2}{\partial w}(z, 0) = \frac{i}{2}(-z_2\alpha + iz_1\beta), \quad (3.51)$$

$$\frac{\partial \phi}{\partial w}(z, 0) = 0. \quad (3.52)$$

Proof. Applying L_1 and T to the mapping equation consecutively at $\bar{z} = \bar{w} = 0$, we obtain:

$$\frac{\partial f_1}{\partial w}(z, 0) = \frac{i}{2}(z_1\alpha + iz_2\beta).$$

Applying L_2 and T to the mapping equation consecutively at $\bar{z} = \bar{w} = 0$, we obtain:

$$\frac{\partial f_2}{\partial w}(z, 0) = \frac{i}{2}(-z_2\alpha + iz_1\beta).$$

Differentiating (3.45) with respect to w and setting $w = 0$, we obtain:

$$\frac{\partial \phi}{\partial w}(z, 0) = 0.$$

The proof for Proposition 3.9 is completed. $\square$

From the formulas for H and H_w along the first Segre set as above, we can produce another several holomorphic equations for components of the map. One of such equations is as follows.

Proposition 3.10. *Assume that $\lambda = 0$. It holds that*

$$w\alpha\Phi - z_1(2 + iw\alpha)f_1(z, w) + wz_1\beta f_2(z, w) + iw\phi(z, w) + 2z_1^2 = 0. \quad (3.53)$$

Proof. Evaluating (3.25) with $j = 1$ at $\bar{z}_1 = \frac{iw}{2z_1}$, $\bar{z}_2 = 0$ and $\bar{w} = 0$, we obtain (3.53). The proof is complete. $\square$

Theorem 3.11. *Let H be a holomorphic map in a neighborhood of the origin sending the germ at the origin of hyperquadric $\mathbb{H}_1^5$ into $\mathcal{X}_1^7$. Assume that H has the partial normal form (2.6) with $\lambda = 0$. Then H is given by*

$$H(z, w) = H_A(z, w) = \left(\frac{4z + 2iwzAz^t}{4 + |A|w^2}, \frac{4zAz^t}{4 + |A|w^2}, \frac{4w}{4 + |A|w^2} \right),$$

where

$$A = \begin{pmatrix} \alpha & i\beta \\ i\beta & -\alpha \end{pmatrix}, \quad (3.54)$$

with $\alpha, \beta \in \mathbb{R}$, so that $|A| = -\alpha^2 + \beta^2$. Conversely, each matrix A of the form (3.54) gives rise to a holomorphic map sending the hyperquadric into $\mathcal{X}_1^7$.

Thus, at this point we completely determine all rational holomorphic maps sending the 5-dimensional hyperquadric of signature 1 into $\mathcal{X}_1^7$ CR transversally.

Proof. Applying Proposition 3.8 to (3.53) and taking the numerator, we obtain an equation for $g(z, w)$ and $\Phi(z, w)$ as follows:

$$\begin{aligned} 0 = & 4(zz^t)^2\alpha g(z, w)^2 - 4z_1zz^t(z_1(w\alpha - 2i) + iwz_2\beta)g(z, w) \\ & + w(-8iz_1^4 + 4wz_1z_2\beta\Phi + w\alpha\Phi(4iz_2^2 + w^2(\alpha^2 - \beta^2)\Phi) - 2z_1^2(4iz_2^2 + w^2(\alpha^2 - \beta^2)\Phi)). \end{aligned} \quad (3.55)$$

Applying Proposition 3.8 to the equation $\Phi - (g(z, w)\phi(z, w) + i(f_1(z, w)^2 + f_2(z, w)^2)) = 0$ and taking the numerator, we obtain an equation of $g(z, w)$ and $\Phi(z, w)$:

$$4(z_1^2 + z_2^2)^2g(z, w)^2 + w^2\Phi(4i(z_1^2 + z_2^2) + w^2(\alpha^2 - \beta^2)\Phi) = 0. \quad (3.56)$$

Solving the equation from (3.55) and (3.56) we obtain

$$\begin{aligned} g(z, w) &= \frac{4w}{4 - w^2(\alpha^2 - \beta^2)}, \\ \Phi(z, w) &= \frac{4i(z_1^2 + z_2^2)}{4 - w^2(\alpha^2 - \beta^2)}. \end{aligned}$$

Substituting into Proposition 3.8 we have the formulas for $f_1(z, w)$, $f_2(z, w)$ and $\phi(z, w)$.

$$f_1(z, w) = \frac{z_1(4 + 2iw\alpha) - 2wz_2\beta}{4 - w^2(\alpha^2 - \beta^2)},$$

$$f_2(z, w) = \frac{z_2(4 - 2iw\alpha) - 2wz_1\beta}{4 - w^2(\alpha^2 - \beta^2)},$$

$$\phi(z, w) = \frac{4(\alpha(z_1^2 - z_2^2) + 2iz_1z_2\beta)}{4 - w^2(\alpha^2 - \beta^2)}.$$

Thus, $H = H_A$, as desired. Conversely, it can be checked directly that each map H_A is transversal to $\mathcal{X}_1^7$ and sends $\mathbb{H}_1^5 \setminus \text{Sing}(H_A)$ into $\mathcal{X}_1^7$. The proof is complete. $\square$

In the sequel, we shall show that this 2-parameter family of maps reduces to five equivalence classes, represented by H_{A_j} , $j = 1, 2, \dots, 5$, where

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2i \\ 2i & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -2 & -2i \\ -2i & 2 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}.$$

- (1) If $\alpha = \beta = 0$, we obtain the map $H_0(z, w) = (z, 0, w) = H_{A_1}(z, w)$.
- (2) If $\alpha = \beta > 0$, we write $\beta = 2s^2$ with $s > 0$ and consider the following automorphisms $\Psi_2(z, w) \in \text{Aut}(\mathbb{H}_1^5)$ and $\gamma_2 \in \text{Aut}(\mathcal{X}_1^7)$:

$$\Psi_2 = (sz_1, sz_2, s^2w)$$

and

$$\gamma_2 = (sz_1, sz_2, \zeta, s^2w).$$

Clearly, $\gamma_2 \circ H \circ \Psi_2^{-1} = H_{A_2}$.

- (3) If $-\alpha = \beta > 0$, then it can be shown similarly as above that H is equivalent to H_{A_2} . We omit the details.
- (4) If $\alpha = \beta < 0$, we write $\beta = -2s^2$ with $s > 0$ and consider the following automorphisms $\Psi_3(z, w) \in \text{Aut}(\mathbb{H}_1^5)$ and $\gamma_3 \in \text{Aut}(\mathcal{X}_1^7)$:

$$\Psi_3 = (-isz_2, isz_1, s^2w)$$

and

$$\gamma_3 = (-isz_2, isz_1, \zeta, s^2w).$$

Composing $\gamma_3 \circ H \circ \Psi_3^{-1}$ gives H_{A_3} .

- (5) If $-\alpha = \beta < 0$, then it can be shown similarly as above that H is equivalent to H_{A_3} . We omit the details.
- (6) If $\alpha^2 - \beta^2 > 0$, we put $\alpha = 2r^2 \cosh(s)$ and $\beta = 2r^2 \sinh(s)$, $r > 0$,

$$B = \begin{pmatrix} \cosh(s/2) & i \sinh(s/2) \\ -i \sinh(s/2) & \cosh(s/2) \end{pmatrix}, \quad B^{-1} = B^t$$

and consider the following automorphisms $\Psi_4(z, w) \in \text{Aut}(\mathbb{H}_1^5)$ and $\gamma_4 \in \text{Aut}(\mathcal{X}_1^7)$

$$\begin{aligned}\Psi_4(z, w) &= (rzB, r^2w), \\ \gamma_4(z, w) &= (rzB, \zeta, r^2w).\end{aligned}$$

Now composing $\gamma_4 \circ H \circ \Psi_4^{-1}$ gives us the mapping H_{A_4} .

- (7) If $\alpha^2 - \beta^2 < 0$, we put $\alpha = 2r^2 \sinh(s)$ and $\beta = 2r^2 \cosh(s)$. Now we consider the maps $\Psi_5(z, w) \in \text{Aut}(\mathbb{H}_1^3)$ and $\gamma_5 \in \text{Aut}(\mathcal{X}_1^7)$ which have the following formulas:

$$\begin{aligned}\Psi_5 &= \left(r \left(-z_1 \cosh\left(\frac{s}{2}\right) - iz_2 \sinh\left(\frac{s}{2}\right) \right), r \left(iz_1 \sinh\left(\frac{s}{2}\right) - z_2 \cosh\left(\frac{s}{2}\right) \right), r^2w \right) \\ &\text{and} \\ \gamma_5 &= \left(r \left(-z_1 \cosh\left(\frac{s}{2}\right) - iz_2 \sinh\left(\frac{s}{2}\right) \right), r \left(iz_1 \sinh\left(\frac{s}{2}\right) - z_2 \cosh\left(\frac{s}{2}\right) \right), \zeta, r^2w \right).\end{aligned}$$

Now composing $\gamma_5 \circ H \circ \Psi_5^{-1}$ gives us the mapping H_{A_5} .

Proposition 3.12. *For $j \neq k$, the germs at the origin of H_{A_j} and H_{A_k} are inequivalent.*

Proof. The geometric rank of H_A at the origin is equal to the rank of the matrix A . Thus, H_0 has vanishing geometric rank, H_{A_2} and H_{A_3} have geometric rank 1, while H_{A_4} and H_{A_5} have rank two at the origin.

To distinguish two maps with the same geometric rank, we can look at the eigenvalues of the CR Ahlfors tensors of the maps at the origin. By direct computations, the component of the CR Ahlfors tensor in the local holomorphic frame $\{\bar{L}_1, \bar{L}_2\}$ is given by

$$\mathcal{A}(H_A)|_0 = \begin{pmatrix} -\alpha & -i\beta \\ i\beta & -\alpha \end{pmatrix}.$$

From this, the inequivalences of H_{A_j} for different j are evident. $\square$

Case 2: $\lambda \neq 0$.

In this case, the map must be irrational.

Theorem 3.13. *Assume that the germ H is of the form (2.6) with $\lambda \neq 0$. Then*

$$H(z, w) = \frac{2(z, \lambda w, w)}{1 + \sqrt{1 - 4i\bar{\lambda}(zz^t - i\lambda w^2)}}. \quad (3.57)$$

Remark 7. The partial normal form (2.6) also determines the map uniquely in this case. It is interesting to point out that we also get a two-parameter analytic family of CR maps containing the linear map. Each map in the family is either equivalent to the linear map or the irrational map depending on whether $\lambda = 0$ or $\lambda \neq 0$.

In what follows, we will prove Theorem 3.13 via several propositions.

Proposition 3.14. *If $\lambda \neq 0$, then*

$$\sigma = 0, v = 0, \mu = 0 \text{ and } c_{ij} = 0$$

for all $1 \leq i < j \leq n - 1$.

Sketch of the proof. The idea of proof is similar to that of [24, Lemma 4.9]. We therefore only sketch the proof. For each j , applying L_j twice to the mapping equation we obtain $n - 1$ equations of the form (3.35) (with $k = j$). Taking the difference of the first equation (i.e, $j = 1$) times $(i + 4\bar{\lambda}z_2^2)$ and the second equation (i.e, $j = 2$) times $(i + 4\bar{\lambda}_1z_1^2)$ we obtain an equation of the form:

$$M(z) + N(z)\sqrt{1 - 4i\bar{\lambda}zz^t} = 0,$$

where $M(z)$ and $N(z)$ are polynomials in z . These calculations are quite lengthy and tedious, but can be done quickly with help of a computer algebra system.

Next, as $\lambda \neq 0$, we have

$$M(z) = N(z) = 0.$$

Now equating the coefficients both sides of the equation gives us the desired claim. $\square$

Proposition 3.15. *$\phi(z, 0) = 0$ and $\frac{\partial f}{\partial w}(z, 0) = 0$.*

Proof. Applying L_2 twice to the mapping equation, evaluating at $\bar{z} = \bar{w} = 0$, and combining with Propositions 3.2 and 3.14, we have

$$(-1 + 4i\bar{\lambda}z_2^2)\phi(z, 0) = 0.$$

Thus, $\phi(z, 0) = 0$. Next, applying T followed by L_j to the mapping equation (2.5), evaluating at $\bar{z} = \bar{w} = 0$, and combining with $\phi(z, 0) = 0$ we obtain $n - 1$ equations:

$$\left(1 - 4i\bar{\lambda}z_j^2 + \sqrt{1 - 4i\bar{\lambda}zz^t}\right) \frac{\partial f_j}{\partial w}(z, 0) - 4i\bar{\lambda}z_j \left(\sum_{1 \leq i \neq j \leq n-1} z_i \frac{\partial f_i}{\partial w}(z, 0)\right) = 0.$$

Solving this system of equations we obtain that $\frac{\partial f}{\partial w}(z, 0) = 0$. $\square$

Similarly to *Case 1*, by multiplying the mapping equation (2.5) with 2, setting $\bar{w} = 0$, and substituting the formulas for $\bar{H}(\bar{z}, 0)$, we obtain an equation of the form

$$J := J(z, w, H(z, 2i\langle z, \bar{z} \rangle_l), \bar{z}) = 0, \quad (3.58)$$

Now for each $1 \leq j \leq n - 1$, by multiplying J with z_j^2 , setting

$$\bar{z}_j = \frac{w - 2i \sum_{1 \leq k \leq n-1, k \neq j} \epsilon_k z_k \bar{z}_k}{2i\epsilon_j z_j},$$

and taking the numerators, we obtain an equation of the form

$$\mathcal{J}_j(z, w, H(z, w)) = 0,$$

for $j = 1, 2, \dots, n-1$. Again, the explicit formula for $\mathcal{J}$ is quite complicated and is not provided here. But it can be computed quickly by using a computer algebra system.

Proposition 3.16.

$$\frac{\partial \phi}{\partial w}(z, 0) = \frac{2\lambda}{1 + \sqrt{1 - 4i\bar{\lambda}zz^t}}, \quad (3.59)$$

$$\frac{\partial g}{\partial w}(z, 0) = \frac{2}{1 + \sqrt{1 - 4i\bar{\lambda}zz^t}}. \quad (3.60)$$

Proof. Taking the derivative of $\mathcal{J}_1$ with respect to w and applying $\bar{z} = 0$ and $w = \bar{w} = 0$, we have:

$$4iz_1^2 \left(\frac{2}{1 + \sqrt{1 - 4i\bar{\lambda}zz^t}} - \frac{\partial g}{\partial w}(z, 0) \right) = 0.$$

From this, we obtain (3.60). Taking the derivative of $\mathcal{J}_1$ with respect to w and applying $w = 0$, we obtain (3.59). The proof is complete. $\square$

Now we will divide into 2 smaller cases depending on the value of n .

Subcase 2.1: $n \geq 4$.

For pairwise distinct indices $1 \leq j, k, t \leq n-1$, we calculate the following expression

$$z_t \frac{\partial \mathcal{J}_j}{\partial \bar{z}_k} - \epsilon_k \epsilon_t z_k \frac{\partial \mathcal{J}_j}{\partial \bar{z}_t}$$

at $\bar{z}_m = 0$ for all $m \neq j$, to obtain that:

$$4z_j \left(z_j + \sqrt{z_j^2 - i\lambda w^2} \right) (z_k f_t(z, w) - z_t f_k(z, w)) = 0.$$

Thus we have $z_k f_t(z, w) = z_t f_k(z, w)$ for all $1 \leq k \neq t \leq n-1$. Therefore, to determine $f(z, w)$, we only need to determine $f_1(z, w)$.

Proposition 3.17. *With assumptions and notations as above,*

$$\phi(z, w) = \lambda g(z, w).$$

Proof. Taking the derivative of $\mathcal{J}_1$ with respect to $\bar{z}_2$, applying $\bar{z} = 0$, substitute $f_2(z, w)$ by $z_2 f_1(z, w)/z_1$ and taking the numerator we have:

$$w\lambda f_1(z, w) - z_1\lambda \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_1^2}}\right) g(z, w) + z_1\sqrt{1 - \frac{iw^2\lambda}{z_1^2}}\phi(z, w) = 0. \quad (3.61)$$

Taking the derivative of $\mathcal{J}_2$ with respect to $\bar{z}_1$, applying $\bar{z} = 0$, substitute $f_2(z, w)$ by $z_2 f_1(z, w)/z_1$ and taking the numerator we have:

$$w\lambda f_1(z, w) - z_1\lambda \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_2^2}}\right) g(z, w) + z_1\sqrt{1 - \frac{iw^2\lambda}{z_2^2}}\phi(z, w) = 0. \quad (3.62)$$

Finally, subtracting the equation (3.61) to (3.62), we obtain that $\lambda g(z, w) = \phi(z, w)$. $\square$

Proof of Theorem 3.13 for the case $n \geq 4$. Applying the operation L_1 to the mapping equation, setting $\bar{w} = 0, \bar{z}_1 = \frac{iw}{2z_1}, \bar{z}_j = 0$ for all $2 \leq j \leq n-1$, and substituting $f_j(z, w)$ by $z_j f_1(z, w)/z_1$, $1 \leq j \leq n-1$ and $g(z, w)$ by $w f_1(z, w)/z_1$, we obtain that

$$-z_1^2 + z_1 f_1(z, w) - i(z z^t - i\lambda w^2) \bar{\lambda} f_1(z, w)^2 = 0. \quad (3.63)$$

Solving this equation and combining with the condition $f_1(z, 0) = \frac{2z_1}{1 + \sqrt{1 - 4i\bar{\lambda} z z^t}}$ gives us

$$f_1(z, w) = \frac{2z_1}{1 + \sqrt{1 - 4i\bar{\lambda}(z z^t - i\lambda w^2)}}.$$

Now combining with Proposition 3.16 we complete the proof for Theorem 3.13. $\square$

Subcase 2.2: $n = 3$.

Then applying $\bar{z}_2 = 0$ in $\mathcal{J}_1$ we obtain that:

$$2iwz_1 \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_1^2}}\right) f_1(z, w) - iz_1^2 \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_1^2}}\right)^2 g(z, w) + w^2\phi(z, w) = 0.$$

Applying $\bar{z}_1 = 0$ in $\mathcal{J}_2$ we obtain that:

$$2iwz_2 \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_2^2}}\right) f_2(z, w) - iz_2^2 \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_2^2}}\right)^2 g(z, w) + w^2\phi(z, w) = 0.$$

Solving for $f_1(z, w)$ and $f_2(z, w)$ we have:

$$f_1(z, w) = \frac{z_1^2 \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_1^2}}\right)^2 g(z, w) + iw^2\phi(z, w)}{2wz_1 \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_1^2}}\right)}, \quad (3.64)$$

$$f_2(z, w) = \frac{z_2^2 \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_2^2}}\right)^2 g(z, w) + iw^2\phi(z, w)}{2wz_2 \left(1 + \sqrt{1 - \frac{iw^2\lambda}{z_2^2}}\right)}. \quad (3.65)$$

Proposition 3.18. $\phi(z, w) = \lambda g(z, w)$.

Proof. Taking the derivative of $\mathcal{J}_2$ with respect to $\bar{z}_1$, applying $\bar{z}_1 = 0$ and the formula (3.64), (3.65) we obtain the desired formula. $\square$

Proof of Theorem 3.13 for $n = 3$ and $l = 1$. Applying the operation L_1 to the mapping equation at $\bar{z}_1 = \frac{iw}{2z_1}, \bar{z}_2 = \bar{w} = 0$, using the formulas (3.64), (3.65) and Proposition 3.18 we obtain that

$$w^2 - wg(z, w) + (iz_1^2 + z_2^2 - i\lambda w^2)\bar{\lambda}g(z, w)^2 = 0.$$

Solving this equation we obtain that:

$$g(z, w) = \frac{2w}{1 + \sqrt{1 - 4i\bar{\lambda}(z_1^2 + z_2^2 - i\lambda w^2)}}. \quad (3.66)$$

Using (3.64), (3.65) and Proposition 3.18 we have:

$$\phi(z, w) = \frac{2\lambda w}{1 + \sqrt{1 - 4i\bar{\lambda}(z_1^2 + z_2^2 - i\lambda w^2)}}. \quad (3.67)$$

$$f_1(z, w) = \frac{2z_1}{1 + \sqrt{1 - 4i\bar{\lambda}(z_1^2 + z_2^2 - i\lambda w^2)}}. \quad (3.68)$$

$$f_2(z, w) = \frac{2z_2}{1 + \sqrt{1 - 4i\bar{\lambda}(z_1^2 + z_2^2 - i\lambda w^2)}}. \quad (3.69)$$

Theorem 3.13 is proved. $\square$

4. ON PROPER HOLOMORPHIC MAPS FROM $\mathbb{B}_l^n$ TO $D_{n+1,l}^{\text{IV}}$

In the case $m = n + 1$, by using Theorem 1.1, we have explicit formulas for local proper holomorphic maps from the generalized ball $\mathbb{B}_1^3$ into a generalized Lie ball $D_{3,1}^{\text{IV}}$. The following are (local) proper holomorphic maps from the generalized ball $\partial\mathbb{B}_l^n$ to the Siegel upper half-space $\Omega_{n,l}^+$:

$$\Upsilon_1(z, w) = \left(\frac{\sqrt{2}z_1}{1+w}, \frac{\sqrt{2}z_2}{1+w}, \frac{2i(1-w)}{1+w} \right), \quad (4.70)$$

$$\Upsilon_2(z, w) = \left(-\frac{\sqrt{2}z_1}{1+w}, \frac{\sqrt{2}z_2}{1+w}, \frac{2i(1-w)}{1+w} \right), \quad (4.71)$$

$$\Upsilon_3(z, w) = \left(\frac{\sqrt{2}z_1}{1+w}, -\frac{\sqrt{2}z_2}{1+w}, \frac{2i(1-w)}{1+w} \right), \quad (4.72)$$

$$\Upsilon_4(z, w) = \left(-\frac{z_1}{1-w}, \frac{z_2}{1-w}, \frac{i(1+w)}{1-w} \right), \quad (4.73)$$

$$\Upsilon_5(z, w) = \left(\frac{z_1}{1+w}, \frac{z_2}{1+w}, \frac{i(1-w)}{1+w} \right). \quad (4.74)$$

On the other hand, the following is a local biholomorphic map sending a piece of $\mathcal{X}_l^{n+1}$ into $D_{n+1,l}^{\text{IV}}$:

$$\Omega(z, w) = \left(\frac{2iz}{2i+w}, \frac{i - \frac{w}{2} - i\zeta - \frac{1}{2}(w\zeta + izz^t)}{2i+w}, \frac{-1 - \frac{iw}{2} - \zeta + \frac{i}{2}(w\zeta + izz^t)}{2i+w} \right), \quad (4.75)$$

where $zz^t = z_1^2 + z_2^2 + \cdots + z_{n-1}^2$ for short. Let $X(z, \zeta, w) = \{2z, \zeta, 4w\}$, which is an automorphism of $\mathcal{X}_1^7$.

For $n \geq 3$, similarly to the pseudoconvex case [30], we can verify that $R_0^{(n)}$ and $I^{(n)}$ are two local proper holomorphic maps sending a small one-sided neighborhood $\mathbb{B}_l^n$ into $D_{n+1,l}^{\text{IV}}$. When $n \geq 4$, these are two representatives of the two equivalence classes.

In the special case $n = 3$. The additional rational maps can be constructed by composing with the maps above. Precisely, we have the following maps as in Corollary 1.2:

$$\begin{aligned} R_0(z, w) &= \Omega \circ H_1 \circ \Upsilon_1, \\ R_1(z, w) &= \Omega \circ H_2 \circ \Upsilon_2, \end{aligned}$$

$$\begin{aligned}
R_2(z, w) &= \Omega \circ H_3 \circ \Upsilon_3, \\
P_1(z, w) &= \Omega \circ X \circ H_4 \circ \Upsilon_4, \\
P_2(z, w) &= \Omega \circ X \circ H_5 \circ \Upsilon_5,
\end{aligned}$$

Remark 8. Each of the rational maps $R_{1,2,3}$ has the same indeterminacy set

$$\{(z, w) \in \mathbb{C}^3 \mid w + 1 = 0, z_1^2 + z_2^2 = 0\},$$

which is contained in the boundary $\partial\mathbb{B}_1^3$ of the generalized ball. On the other hand, the pole set $\{(z, w) \in \mathbb{C}^3 \mid w + 1 = 0\}$ meets both sides of the boundary. This exhibits a difference from the signature zero case in which R_0 does not have a singularity in the unit ball $\mathbb{B}^3$.

On the open subset $U_j \subset \partial\mathbb{B}_1^3 \setminus \{w + 1 = 0\}$ whose points are mapped by R_j to a smooth point of $\partial D_{4,1}^{\text{IV}}$, $j = 1, 2, 3$, the geometric rank of R_j is constant: R_0 has vanishing geometric rank while $R_{1,2}$ both have geometric rank 1. This can be verified by direct but tedious calculations.

The polynomial maps $P_{1,2}$ both have geometric rank 2 at points which are mapped to a smooth point of the target.

We end this section by computing the CR Ahlfors tensor of a map. Let's take P_2 , for example. We will compute its components in the CR frame

$$Z_j = \frac{\partial \rho}{\partial w} \frac{\partial}{\partial z_j} - \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial w}, \quad j = 1, 2,$$

with $\rho = 1 - |w|^2 + |z_1|^2 - |z_2|^2$ being the “standard” defining function of the generalized ball. Thus, we have

$$Z_1 = -\bar{w} \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial w}, \quad Z_2 = -\bar{w} \frac{\partial}{\partial z_2} + \bar{z}_2 \frac{\partial}{\partial w}.$$

By direct calculations, we find that

$$\rho_{D_{3,1}^{\text{IV}}} \circ P_2 = (1 + |w|^2 + |z_1|^2 + |z_2|^2) \rho(z, w).$$

Hence, we put $Q(z, w) = 1 + |w|^2 + |z_1|^2 + |z_2|^2$ and $v(z, w) = \log(Q)$. The complex Hessian $v_{Z\bar{Z}}$ of v , where $Z = (z, w)$, can be computed exactly as in the computation of the Fubini-Study metric in the usual affine coordinate patch of the complex projective space $\mathbb{C}P^3$. Precisely,

$$v_{Z\bar{Z}} = \frac{1}{Q^2} \begin{bmatrix} 1 + |z_2|^2 + |w|^2 & -\bar{z}_1 z_2 & -\bar{z}_1 w \\ -\bar{z}_2 z_1 & 1 + |w|^2 + |z_1|^2 & -\bar{z}_2 w \\ -\bar{w} z_1 & -\bar{w} z_2 & 1 + |z_1|^2 + |z_2|^2 \end{bmatrix}.$$

We immediately see that P_2 has geometric rank 2. Moreover, $v_{Z\bar{Z}}$ is positive definite everywhere and so is the CR Ahlfors tensor of P_2 at all points which are mapped to a smooth points of the target.

Restricting this to the tangential CR vectors, we obtain the CR Ahlfors tensor. In terms of the frame above, its components are given by the following Hermitian matrix (actually, the matrix has real entries)

$$\mathcal{A}(P_2)_{j\bar{k}} = \frac{1}{Q^2} \begin{bmatrix} (|z_1|^2 + |w|^2)(1 + |z_2|^2) & |z_1|^2|z_2|^2 + |w|^2 + 2|w|^2|z_2|^2 \\ |z_1|^2|z_2|^2 + |w|^2 + 2|w|^2|z_2|^2 & (|w|^2 + |z_2|^2)(1 + |z_1|^2) + 4|w|^2|z_2|^2 \end{bmatrix} \quad (4.76)$$

We should note that the formula on the right hand side of (4.76) is only meaningful when being restricted to $\partial\mathbb{B}_1^3$ on which we have a relation between $|w|^2$, $|z_1|^2$, and $|z_2|^2$.

The computation of $\mathcal{A}(P_1)$ is almost the same. We start with

$$\rho_{D_{3,1}^{\text{IV}}} \circ P_2 = (1 + |w|^2 - |z_1|^2 - |z_2|^2)\rho(z, w).$$

Thus, the CR Ahlfors tensor $\mathcal{A}(P_1)$ is the restriction of the complex Hessian of $\log(1 + |w|^2 - |z_1|^2 - |z_2|^2)$, which is well-defined on a suitable open set. Although the rest of the computation is very similar, there is a difference: At every point outside a singular set of P_1 which is mapped to a smooth point of the target, the CR Ahlfors tensor of P_1 is nondegenerate, but not positive. This shows that P_1 and P_2 are not equivalent.

5. HIGHER CODIMENSIONAL CASE

In this section, we briefly discuss the case of higher but low codimension. In this case, one expects that under some conditions on the dimensions and signature, CR maps between hyperquadrics and the tube exhibit rigidity property. In fact, based on recent research on the rigidity of CR maps between spheres and hyperquadrics of Huang–Lu–Tang–Xiao [14] and Xiao [29], one can obtain a rigidity result for the case of CR maps from a sphere or a hyperquadric into the tube over the symmetric form of higher dimension and codimension. For the sake of completeness, we present two theorems below.

Theorem 5.1. *Let $m \geq n \geq 4$, $1 \leq l \leq l'$, and $l \leq (n-1)/2$. Assume that H is a smooth CR map from an connected open subset of $\mathbb{H}_l^{2n-1}$ into $\mathcal{X}_{l'}^{2m+1}$. Then $l \leq \min(l', m-l')$. Moreover, assume that one of the following conditions holds*

- (1) $l' < \min(2l-1, n-2)$,
- (2) $l' < 2l-1$ and $m-l' < n-1$,
- (3) $m-l' < 2(n-l-1)$ and $l' < n-2$,
- (4) $m-l' < 2(n-l-1)$ and $m-l' < n-1$.

Then H extends to a local holomorphic isometry of the indefinite “canonical” Kähler metrics of one-sided neighborhoods of the source and target.

The theorem above applies when, for example, $l = l' = 1$ and $m \leq 2n - 4$. It seems that these ranges of the dimensions and signatures are not optimal.

Proof. Consider the holomorphic map $\Psi: \mathbb{C}^{m+1} \rightarrow \mathbb{C}^{m+2}$ given by

$$\Psi(z, \zeta, w) = \left(z_1, \dots, z_{l'}, \frac{1}{2} (w\zeta + izz^t - i\zeta), z_{l'+1}, \dots, z_m, \frac{1}{2} (w\zeta + izz^t + i\zeta), w \right),$$

where $(z, \zeta, w) = (z_1, \dots, z_{m-1}, \zeta, w)$ and $zz^t = z_1^2 + \dots + z_{m-1}^2$. Then Ψ is transversal to $\mathbb{H}_{l'+1}^{2m+3}$ and sends $\mathcal{X}_{l'}^{2m+1}$ into $\mathbb{H}_{l'+1}^{2m+3}$, where

$$\mathbb{H}_{l'+1}^{2m+3} = \left\{ (z_1, \dots, z_{m+1}, w) \in \mathbb{C}^{m+2} \mid \tilde{\rho} := \operatorname{Im}(w) + \sum_{j=1}^{l'+1} |z_j|^2 - \sum_{j=l'+2}^{m+1} |z_j|^2 = 0 \right\},$$

is the real hyperquadric of signature $l' + 1$ in $\mathbb{C}^{m+2}$.

If $H: \mathbb{H}_l^{2n-1} \rightarrow \mathcal{X}_{l'}^{2m+1}$ is a CR transversal map, then $\tilde{H} := \Psi \circ H$ is a CR transversal map from $\mathbb{H}_l^{2n-1}$ to $\mathbb{H}_{l'+1}^{2m+3}$. Therefore, by the result of Huang-Lu-Tang-Xiao mentioned above, $\tilde{H}$ extends to an isometry of the indefinite complex hyperbolic metrics of one-sided neighborhoods of the hyperquadrics. On the other hand, H itself extends holomorphically to a neighborhood of p in $\mathbb{C}^n$ by the well-known Lewy extension theorem. Finally, the isometry of $\tilde{H}$ implies the isometry of H , as desired. $\square$

In the case $l = 0$ and $4 \leq n \leq m \leq 2n - 3$, we can use a result of Xiao [29] to obtain a rigidity of CR maps from the sphere. When $l' = 0$, the theorem below is just a result of Xiao-Yuan [30] (for $m \leq 2n - 4$) and Xiao [29] (for $m = 2n - 3$). The proof for the case $l' > 0$ follows the same strategy so it is omitted.

Theorem 5.2. *Let $m \geq n \geq 4$ and $0 \leq m < 2n - 3$. Assume that H is a smooth CR transversal map from a connected open subset of $\mathbb{H}_l^{2n-1}$ to $\mathcal{X}_{l'}^{2m+1}$. Then H extends to a local holomorphic isometry of the “canonical” pseudo-Kähler metric of a one-sided neighborhood of $\mathbb{H}_l^{2n-1}$ into the “canonical” pseudo-Kähler metric of a one-sided neighborhood of $\mathcal{X}_{l'}^{2m+1}$.*

A similar statement can be made for the boundaries of the generalized ball and generalized Lie ball. However it is not known at the moment if a local isometry in the generalized setting extends to a global holomorphic map defined on the whole generalized ball.

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