ON CONVERGENCE OF THE SECANT METHOD*

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Abstract. The secant method, as an important approach for solving nonlinear equations, is introduced in nearly all numerical analysis textbooks. However, most textbooks only briefly address the Q-order of convergence of this method, with few providing rigorous mathematical proofs. This paper establishes a rigorous proof for the Q-order of convergence of the secant method and theoretically compares its computational efficiency with that of Newton's method.

Key words. The secant method, Q-order of convergence

MSC codes. 41A25, 68W40

1. Introduction.

Numerous well-established numerical methods exist for finding roots of nonlinear equations $f(x) = 0$. The widely used Newton's method employs the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This method can commence iterative computation directly once an initial value x_0 is provided. The secant method, a derivative-free variant of Newton's method, replaces the derivative with a secant slope approximation at the nth iteration

$$f'(x_n) \leftarrow \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}},$$

yielding the iterative scheme

$$x_{n+1} = x_n - f(x_n)\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}. \tag{1.1}$$

Although this method requires two initial values, x_0 and x_1, to initiate iterations, it remains widely applicable. This stems from two key advantages: (i) enhanced robustness by eliminating derivative computations compared to Newton's method, and (ii) accelerated convergence relative to the bisection method.

Let α be a root of the equation $f(x) = 0$. For the iterative sequence $\{x_n\}$, we define the error sequence $\{e_n\}$ by $e_n = x_n - \alpha$. From a theoretical perspective, we seek initial values x_0, x_1 such that $\{x_n\}$ converges to α. Furthermore, we wish to find the Q-order of convergence p and the corresponding asymptotic error constant c for $\{x_n\}$, i.e., to prove that $\lim_{n\to\infty} \frac{E_{n+1}}{E_n^p} = c$, where $E_n = |e_n|$, see Definition 2.1.

When α is a simple root, although standard textbooks universally assert Q-superlinear convergence of the secant method for simple roots, rigorous proofs remain scarce. Two classical approaches exist: asymptotic error analysis and the Fibonacci approach. Díez [2] and Kincaid et al. [4] derived the quantitative relationship between the Q-order of convergence of the secant method and the golden ratio through asymptotic error analysis. However, this approach presupposes both the convergence of the

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* Yan Tan and Chenhao Ye are co-first authors with equal contributions while Qinghai Zhang is the corresponding author.

Funding: This reseach was supported in part by a grant from the Ministry of Science and Technology of China.

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secant method and sufficiently small errors at each step, assumptions that may not hold without appropriate initial conditions. Meanwhile, Atkinson [1] and Gautschi [3] developed the Fibonacci approach, which only estimates the R-order of convergence of the error, that is, the Q-order of convergence of this error upper bound, by tracking the error at each iteration. While this approach clearly illustrates the error reduction process and the convergence of $\{x_n\}$, it does not directly address the Q-order of convergence of the actual error itself. This distinction is crucial because the R-order of convergence of a sequence does not necessarily equal its Q-order of convergence, as demonstrated in the following example.

Example 1.1. For $0 < k < 1$, define $y_n = k^{n+(-1)^n}$ and $z_n = k^{n-1}$. Then for all $n \geq 1$, we have $0 < y_n \leq z_n$, so $\{z_n\}$ is an upper bound for $\{y_n\}$. The sequence $\{z_n\}$ has Q-linear convergence since $\frac{|z_{n+1}|}{|z_n|} = k$. However, the ratio $\frac{|y_{n+1}|}{|y_n|}$ oscillates between 1 and k^2 and thus does not converge; consequently, $\{y_n\}$ exhibits R-linear convergence but lacks Q-linear convergence.

When α is a root with multiplicity $m \geq 2$, related research is even more limited. The classic approach remains the asymptotic error analysis proposed by Díez [2], but it introduces further presuppositions. For $m = 2$, assuming the iterative sequence $\{x_n\}$ converges, it proves Q-linear convergence and finds the corresponding asymptotic error constant. For $m > 2$, it directly assumes that $\{x_n\}$ converges linearly to α and then finds the corresponding asymptotic error constant. As mentioned in the analysis for simple roots, these assumptions require appropriate initial conditions to hold. In fact, for some simple functions, even with straightforward initial conditions, the iteration may not converge, as shown in the following example.

Example 1.2. For the function $f(x) = x^2$, and $\alpha = 0$ is a 2-fold root of f. The corresponding iterative formula for the secant method is $x_{n+1} = x_n - \frac{x_n^2}{x_n + x_{n-1}}$. Let $k_n = \frac{e_{n+1}}{e_n} = \frac{x_{n+1}}{x_n}$, the sequence $\{k_n\}$ satisfies the iterative formula $k_n = 1 - \frac{k_{n-1}}{k_{n-1}+1}$. If the initial ratio $k_0 = \frac{x_1}{x_0}$ equals the fixed point $k^* := -\frac{1+\sqrt{5}}{2}$, then $k_n = k^*$ for all n, and thus $x_n = (k^*)^n x_0$. Since $|k^*| > 1$, the sequence $\{x_n\}$ diverges, despite the initial values x_0, x_1 being arbitrarily close to the root $\alpha = 0$.

Addressing the shortcomings in the current theory of the secant method, we pose the following questions:

(Q-1) For simple roots, how can we rigorously prove the convergence of the secant iterative sequence $\{x_n\}$ near α? How can we rigorously compute the Q-order of convergence and the corresponding asymptotic error constant for $\{x_n\}$?

(Q-2) For multiple roots, how can we provide appropriate sufficient conditions such that the secant iterative sequence $\{x_n\}$ near α converges? Under such sufficient conditions, how can we rigorously prove the Q-linear convergence of $\{x_n\}$ and find the corresponding asymptotic error constant?

In this paper, we provide positive answers to all above questions.

For (Q-1), Theorem 3.4 provides a rigorous proof of Q-superlinear convergence. It retains the advantage of explicit error tracking while eliminating vague asymptotic analysis and insufficient upper-bound estimates. By constructing the proof directly on the error terms and leveraging properties of the Fibonacci sequence, it rigorously establishes the convergence of $\{x_n\}$ and explicitly derives the Q-order of convergence for the error itself.

For (Q-2), Section 4.1 gives simple sufficient conditions ($k_0 \in (0, 1) \cup (1, +\infty)$ and $e_0 > 0$ is sufficiently small) to fully prove that the iterative sequence $\{x_n\}$ converges linearly to α and to determine the corresponding asymptotic error constant. Furthermore, in Section 4.2, we attempt to relax the sufficient conditions on the initial values under which $\{x_n\}$ still converges linearly. Theorem 4.8 and Theorem 4.14 provide sufficient (but not necessary) conditions on the initial values k_0 and e_0 for the Q-linear convergence of the iterative sequence $\{x_n\}$ when m is odd and even, respectively.

Thus, the main contributions of this work are:

- Providing rigorous proofs for the Q-order of convergence and the existence of the corresponding asymptotic error constants for $\{x_n\}$ in both simple and multiple root cases, addressing the gaps in existing textbooks and literature.
- For the multiple root case, supplementing appropriate sufficient conditions on the initial values, serving as theoretical guidance for selecting initial values in the secant method. Practically, these conditions help avoid choosing inappropriate initial values that would lead to a divergent iterative sequence $\{x_n\}$.

The rest of this paper is structured as follows. Section 2 gives some fundamental lemmas and relevant notation; Section 3 presents the convergence analysis for simple roots, including a comparative analysis of computation time with Newton's method, fully addressing (Q-1); Section 4 provides the convergence analysis for multiple roots, covering the Q-linear convergence of the iterative sequence near multiple roots and sufficient conditions on the initial values to ensure convergence of the iterative sequence, thereby addressing all aspects of (Q-2); Section 5 concludes the paper.

2. Preliminaries and notation.

In this section, we introduce some fundamental concepts to be used later.

2.1. Q-order of convergence.

A sequence $\{x_n\}$ is said to *converge* to L if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\quad \text{s.t.}\quad \forall n > N,\ |x_n - L| \le \epsilon,$$

denoted by $\lim_{n\to\infty} x_n = L$. The speed of convergence of $\{x_n\}$ to L is measured by

Definition 2.1 (Q-order of convergence). *A convergent sequence $\{x_n\}$ is said to* converge *to L with* Q-order p $(p \ge 1)$ *if*

$$\lim_{n\to\infty} \tfrac{|x_{n+1}-L|}{|x_n-L|^p} = c > 0,$$

where the constant c is called the asymptotic error constant (AEC). *In particular, $\{x_n\}$ has* Q-linear convergence *if $p = 1$ and $0 < c < 1$,* Q-superlinear convergence *if $1 < p < 2$, and* Q-quadratic convergence *if $p = 2$.*

Throughout this paper, the term converges linearly is used to mean Q-linear convergence, as no ambiguity arises in the present context.

2.2. Fibonacci numbers.

The following definition introduces Fibonacci numbers, which are closely related to the Q-superlinear convergence of the secant method.

Definition 2.2. *The sequence $\{F_n\}$ of* Fibonacci numbers *is defined as*

$$F_0 = 0, \quad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1}.$$

By analyzing the recurrence relation, we derive its closed-form expression.

Theorem 2.3 (Binet's formula). *Denote the golden ratio by $r_0 := \frac{1+\sqrt{5}}{2}$ and let*

$r_1 := 1 - r_0 = \frac{1-\sqrt{5}}{2}$. *Then*

$$F_n = \tfrac{r_0^n - r_1^n}{\sqrt{5}}. \tag{2.1}$$

COROLLARY 2.4. *The Fibonacci sequence in Definition 2.2 satisfies*

$$F_{n+1} = r_0 F_n + r_1^n.$$

Proof. This follows from $r_1 = r_0 - \sqrt{5}$ and (2.1). ∎

2.3. Approximating rational functions. We first introduce some necessary notation and preliminary results.

Notation 2.5 (Range notation). For a finite sequence $\{a_i\}_{i=1}^n$, define

$$\begin{aligned}\mathrm{range}(a_1, a_2, \ldots, a_n) &:= (\min(a_1, \ldots, a_n), \max(a_1, \ldots, a_n)),\\ \mathrm{range}[a_1, a_2, \ldots, a_n] &:= [\min(a_1, \ldots, a_n), \max(a_1, \ldots, a_n)].\end{aligned}$$

Notation 2.6 (Big-O notation). Let $f(x)$ and $g(x)$ be functions defined in a neighborhood of a point x_0. We write $f(x) = O(g(x))$ as $x \to x_0$ if there exist constants $C > 0$ and $\delta > 0$ such that for all x with $|x - x_0| < \delta$, $|f(x)| \le C\,|g(x)|$. When no confusion arises, we omit "as $x \to x_0$" and simply write $f(x) = O(g(x))$.

THEOREM 2.7 (Taylor's theorem). *Suppose that f has continuous derivatives up to order $n+1$ on an interval containing a. Then, for any x in this interval,*

$$f(x) = f(a) + f'(a)(x-a) + \tfrac{f''(a)}{2!}(x-a)^2 + \cdots + \tfrac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where $f^{(k)}(a)$ denotes the kth derivative of f at a, $(x-a)^k$ is the kth power centered at a, and for some $\xi \in \mathrm{range}(a, x)$,

$$R_n(x) := \tfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Equivalently, $R_n(x) = O\left((x-a)^{n+1}\right)$ as $x \to a$.

THEOREM 2.8 (Lagrange's mean value theorem). *If a function $f \in \mathcal{C}[a, b]$ and f' exists on (a, b), then*

$$\exists \xi \in (a, b) \quad s.t. \qquad f'(\xi) = \tfrac{f(b)-f(a)}{b-a}.$$

COROLLARY 2.9. *Consider a function $f(x) \in \mathcal{C}^1[a, b]$, $a \neq b$, and for any $x \in (a, b)$, $f'(x) \neq 0$. Then f is an injective function, that is, for any $y, z \in [a, b]$, if $y \neq z$, then $f(y) \neq f(z)$.*

Proof. Assume for the sake of contradiction that $f(y) = f(z)$. By Theorem 2.8, there exists $w \in \mathrm{range}(y, z) \subseteq (a, b)$ such that

$$0 = f(y) - f(z) = f'(w)(y - z).$$

Since $y \neq z$, we obtain $f'(w) = 0$, which contradicts $f'(x) \neq 0$ on (a, b). ∎

THEOREM 2.10 (Cauchy's mean value theorem). *If functions $f, g \in \mathcal{C}[a, b]$ and f', g' exist on (a, b), and satisfy that for any $x \in (a, b)$, $g'(x) \neq 0$. Then there exists a point $\xi \in (a, b)$ such that*

$$\tfrac{f(b)-f(a)}{g(b)-g(a)} = \tfrac{f'(\xi)}{g'(\xi)}.$$

LEMMA 2.11. *Let functions $A(s) : \mathcal{S} \to \mathbb{R}$ and $B(t) : \mathcal{T} \to \mathbb{R}$ satisfy the following conditions:*

1. $\exists L > 0$ *s.t.* $\forall t \in \mathcal{T}$, $|B(t)| \geq L$;
2. $\exists M > 0$ *s.t.* $\forall s \in \mathcal{S}$, $\forall t \in \mathcal{T}$, $\left|\frac{A(s)}{B(t)}\right| \leq M$.

Then

$$\forall \epsilon > 0, \exists \delta > 0 \ \text{ s.t. } \ \forall |x|, |y| < \delta, \forall s \in \mathcal{S}, \forall t \in \mathcal{T}, \quad \left|\frac{A(s)+O(x)}{B(t)+O(y)} - \frac{A(s)}{B(t)}\right| < \epsilon.$$

Proof. The definitions of $O(x)$ and $O(y)$ yield

$$\begin{aligned}
&\exists \delta_x > 0, M_x > 0 \ \text{ s.t. } \ \forall |x| < \delta_x, \ |O(x)| \leq M_x|x|, \\
&\exists \delta_y > 0, M_y > 0 \ \text{ s.t. } \ \forall |y| < \delta_y, \ |O(y)| \leq M_y|y|.
\end{aligned}$$

Combined with $|B(t)| \geq L > 0$ for all $t \in \mathcal{T}$, we have

$$\forall t \in \mathcal{T}, \ \forall |y| < \delta_0, \quad |O(y)| \leq M_y|y| < \tfrac{L}{2} \leq \tfrac{|B(t)|}{2},$$

where $\delta_0 := \min\left(\delta_y, \frac{L}{2M_y}\right)$. Hence the triangle inequality shows

$$\forall t \in \mathcal{T}, \ \forall |y| < \delta_0, \quad |B(t) + O(y)| \geq |B(t)| - |O(y)| > |B(t)| - \tfrac{|B(t)|}{2} = \tfrac{|B(t)|}{2}.$$

For any $\epsilon > 0$, let $\delta_1 := \frac{L\epsilon}{2(M+1)(M_x+M_y)}$ and $\delta := \min(\delta_x, \delta_y, \delta_0, \delta_1) > 0$. Then for any $|x|, |y| < \delta$, $s \in \mathcal{S}$, and $t \in \mathcal{T}$, by the triangle inequality,

$$\begin{aligned}
|B(t)O(x) - A(s)O(y)| &\leq |B(t)||O(x)| + |A(s)||O(y)| \\
&\leq |B(t)| \cdot M_x|x| + |A(s)| \cdot M_y|y| \\
&< (|B(t)|M_x + |A(s)|M_y)\delta \\
&< (|A(s)| + |B(t)|)(M_x + M_y)\delta.
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
\left|\frac{A(s)+O(x)}{B(t)+O(y)} - \frac{A(s)}{B(t)}\right| = \left|\frac{B(t)O(x)-A(s)O(y)}{B(t)(B(t)+O(y))}\right| &< \frac{(|A(s)|+|B(t)|)(M_x+M_y)\delta}{\frac{1}{2}|B(t)||B(t)|} \\
&= \frac{2}{|B(t)|}\left(\left|\frac{A(s)}{B(t)}\right| + 1\right)(M_x + M_y)\delta,
\end{aligned}$$

which, together with $\delta \leq \delta_1 = \frac{L\epsilon}{2(M+1)(M_x+M_y)}$, implies $\left|\frac{A(s)+O(x)}{B(t)+O(y)} - \frac{A(s)}{B(t)}\right| < \epsilon$. ∎

LEMMA 2.12. *For $m \geq 2$, the characteristic equation*

$$p_m(k) := k^m + k^{m-1} - 1 = 0 \tag{2.2}$$

has a unique solution $c_{m,0}$ in $(0, 1)$.

Proof. Since $-1 = p_m(0) < 0 < p_m(1) = 1$, the intermediate value theorem shows that (2.2) has a solution $c_{m,0}$ in $(0, 1)$. For $k \in (0, 1)$ and $m \geq 2$, we have $p_m'(k) = k^{m-2}(mk + m - 1) > 0$, thus $p_m(k)$ is strictly monotonically increasing on $(0, 1)$, which ensures the uniqueness of $c_{m,0}$. ∎

3. Analysis of a simple root. This section establishes a rigorous proof for the Q-order of convergence of the secant method in the case of a simple root.

Let α denote the root of function $f(x)$, $\{x_n\}$ be the sequence of iterates near α, and

$$e_n := x_n - \alpha, \tag{3.1}$$
$$E_n := |x_n - \alpha| \tag{3.2}$$

be the iteration errors.

Hypothesis 3.1. We exclude the trivial case in which the iteration reaches the root in finitely many steps; that is, we assume $x_n \neq \alpha$ for all $n \in \mathbb{N}$. Consequently,

$$\forall n \in \mathbb{N}, \quad x_n \neq \alpha, \quad e_n \neq 0, \quad \text{and} \quad E_n > 0. \tag{3.3}$$

Given this hypothesis, the following lemma establishes that the iteration of the secant method (1.1) is well-defined under certain conditions.

LEMMA 3.2. *Consider a function $f(x) \in \mathcal{C}^2(\mathcal{B})$ with domain $\mathcal{B} := [\alpha - \delta, \alpha + \delta]$, and α is a simple root of f, i.e., f satisfies $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then there exists $\delta_0 \in (0, \delta)$ such that for any $x_{n-1} \neq x_n$ satisfying $x_{n-1}, x_n \in \mathcal{B}_0 := [\alpha - \delta_0, \alpha + \delta_0]$, we have $f(x_{n-1}) \neq f(x_n)$, and the secant method iteration (1.1) yields $x_{n+1} \neq x_n$.*

Proof. The continuity of f' and the assumption $f'(\alpha) \neq 0$ imply

$$\exists \delta_0 \in (0, \delta) \ \text{ s.t. } \ \forall x \in \mathcal{B}_0, \ f'(x) \neq 0.$$

Together with Corollary 2.9, $x_{n-1} \neq x_n$, and $x_{n-1}, x_n \in \mathcal{B}_0$, we have $f(x_{n-1}) \neq f(x_n)$. Similarly, the assumption (3.3) and $x_n, \alpha \in \mathcal{B}_0$ yield that $f(x_n) \neq f(\alpha) = 0$. According to the secant method iteration (1.1) and $x_{n-1} \neq x_n$, we obtain

$$|x_{n+1} - x_n| = \left| \tfrac{f(x_n)}{f(x_{n-1}) - f(x_n)} (x_{n-1} - x_n) \right| \geq \tfrac{|f(x_n)|}{|f(x_{n-1})| + |f(x_n)|} |x_{n-1} - x_n| > 0. \qquad \square$$

Next, we analyze the error relation of the secant method for simple roots.

LEMMA 3.3 (Error relation of the secant method for simple roots). *Consider a function $f(x) \in \mathcal{C}^2(\mathcal{B})$, and α is a simple root of f. If $x_{n-1} \neq x_n$, then for the secant method (1.1), there exist $\xi_n \in \mathrm{range}(x_{n-1}, x_n)$ and $\zeta_n \in \mathrm{range}(x_{n-1}, x_n, \alpha)$ such that*

$$x_{n+1} - \alpha = (x_n - \alpha)(x_{n-1} - \alpha) \tfrac{f''(\zeta_n)}{2f'(\xi_n)}.$$

Proof. Applying algebraic manipulations to the secant method (1.1) yields

$$x_{n+1} - \alpha = (x_n - \alpha)(x_{n-1} - \alpha) \frac{\frac{f[x_{n-1}, x_n] - f[x_n, \alpha]}{x_{n-1} - \alpha}}{f[x_{n-1}, x_n]}, \tag{3.4}$$

where $f[a, b] := \frac{f(a) - f(b)}{a - b}$. By Theorem 2.8 and the condition $x_{n-1} \neq x_n$, there exists $\xi_n \in \mathrm{range}(x_{n-1}, x_n)$ such that

$$f[x_{n-1}, x_n] = \tfrac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n} = f'(\xi_n). \tag{3.5}$$

Define a function $g(x) := f[x, x_n]$. Using assumption (3.3) and applying Theorem 2.8 to $g(x)$, it follows that

$$\tfrac{f[x_{n-1}, x_n] - f[\alpha, x_n]}{x_{n-1} - \alpha} = \tfrac{g(x_{n-1}) - g(\alpha)}{x_{n-1} - \alpha} = g'(\beta)$$

for some $\beta \in \text{range}(x_{n-1}, \alpha)$. Computing the derivative of $g(\beta)$ and using Theorem 2.7, we have

$$g'(\beta) = (f[\beta, x_n])' = \tfrac{f(x_n)-f(\beta)-(x_n-\beta)f'(\beta)}{(x_n-\beta)^2} = \tfrac{f''(\zeta_n)}{2} \tag{3.6}$$

for some $\zeta_n \in \text{range}(\beta, x_n) \subseteq \text{range}(x_{n-1}, x_n, \alpha)$. The proof is completed by substituting (3.5) and (3.6) into (3.4). ∎

The error relation in Lemma 3.3 establishes the Q-superlinear convergence of the secant method in the case of a simple root.

THEOREM 3.4 (Convergence of the secant method for simple roots). *Consider a function $f(x) \in \mathcal{C}^2(\mathcal{B})$, and α is a simple root of f. If both x_0 and x_1 are chosen sufficiently close to the root α, $x_0 \neq x_1$ and $f''(\alpha) \neq 0$, then the sequence of iterates $\{x_n\}$ in the secant method converges to α with Q-order $p = r_0 \approx 1.618$ and AEC $c = m_\alpha^{r_0-1}$, i.e.,*

$$\lim_{n\to\infty} \tfrac{E_{n+1}}{E_n^{r_0}} = m_\alpha^{r_0-1},$$

where E_n is as defined in (3.2) and $m_\alpha := \left|\frac{f''(\alpha)}{2f'(\alpha)}\right| > 0$.

Proof. We first prove that the sequence $\{x_n\}$ generated by the secant method converges to the root α.

Define $M := \frac{\max_{x\in\mathcal{B}_0}|f''(x)|}{2\min_{x\in\mathcal{B}_0}|f'(x)|}$, where $\mathcal{B}_0$ is as described in Lemma 3.2. M is well-defined because $\min_{x\in\mathcal{B}_0}|f'(x)| > 0$ and $\max_{x\in\mathcal{B}_0}|f''(x)| < +\infty$, and we have $M > 0$ from $f''(\alpha) \neq 0$. Since both x_0 and x_1 are chosen sufficiently close to α, we can let $\delta_1 := \min(\delta_0, \frac{1}{M}) \in (0, \delta)$ and

$$E_0 = |x_0 - \alpha| < \delta_1, \quad E_1 = |x_1 - \alpha| < \delta_1.$$

The assumption (3.3) gives that for all $\ell \in \mathbb{N}$, $E_\ell > 0$. Moreover, we claim that for all $\ell \in \mathbb{N}$, $E_\ell < \delta_1$, and for all $\ell \in \mathbb{N}^+$, $x_{\ell-1} \neq x_\ell$. We already have that $E_0, E_1 < \delta_1$ and $x_0 \neq x_1$. Assume by the induction hypothesis that $E_{n-1}, E_n < \delta_1$ and $x_{n-1} \neq x_n$. Then it suffices to show $E_{n+1} < \delta_1$ and $x_n \neq x_{n+1}$. On the one hand, it follows that

$$E_{n+1} = E_n E_{n-1} \frac{|f''(\zeta_n)|}{2|f'(\xi_n)|} \leq E_n E_{n-1} M < E_n < \delta_1,$$

where the first step follows from Lemma 3.3, the second step from the definition of M and $\zeta_n, \xi_n \in \text{range}(x_{n-1}, x_n, \alpha) \subseteq (\alpha - \delta_1, \alpha + \delta_1) \subseteq \mathcal{B}_0$, and the third step from $ME_{n-1} < M\delta_1 \leq 1$. On the other hand, $E_{n-1}, E_n < \delta_1$ implies $x_{n-1}, x_n \in \mathcal{B}_0$, which, together with Lemma 3.2, yields $x_n \neq x_{n+1}$. Therefore, the induction step is completed, and we get

$$E_{n+1} = E_n E_{n-1} m_n, \tag{3.7}$$

where $m_n := \left|\frac{f''(\zeta_n)}{2f'(\xi_n)}\right|$. Multiplying both sides of (3.7) by M gives

$$ME_{n+1} = ME_n E_{n-1} m_n \leq ME_n ME_{n-1}. \tag{3.8}$$

Define $\eta := \max(ME_0, ME_1)$; we have $\eta < M\delta_1 \leq 1$. Then induction on (3.8) shows that

$$ME_n \leq ME_{n-1} ME_{n-2} < \eta^{F_n + F_{n-1}} = \eta^{F_{n+1}},$$

where $\{F_n\}$ is the Fibonacci sequence as in Definition 2.2. Thus, $E_n < \frac{1}{M}\eta^{F_{n+1}}$. From Theorem 2.3 and $|r_1| < 1$, it holds that $F_n \sim \frac{r_0^n}{\sqrt{5}}$ as $n \to \infty$. Since $\eta < 1$, we have $\lim_{n\to\infty} E_n = 0$, which shows that $\{x_n\}$ converges to α.

Next, we prove that the convergence order is $p = r_0$ and the AEC is $c = m_\alpha^{r_0-1}$.

By performing induction on the recurrence relation (3.7), we obtain the closed-form expression for E_n as follows:

$$\begin{aligned} E_n &= E_1^{F_n} E_0^{F_{n-1}} m_1^{F_{n-1}} \cdots m_{n-2}^{F_2} m_{n-1}^{F_1}, \\ E_{n+1} &= E_1^{F_{n+1}} E_0^{F_n} m_1^{F_n} \cdots m_{n-1}^{F_2} m_n^{F_1}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{E_{n+1}}{E_n^{r_0}} &= E_1^{F_{n+1}-r_0F_n} E_0^{F_n-r_0F_{n-1}} m_1^{F_n-r_0F_{n-1}} \cdots m_{n-1}^{F_2-r_0F_1} m_n^{F_1} \\ &= E_1^{r_1^n} E_0^{r_1^{n-1}} m_1^{r_1^{n-1}} \cdots m_{n-1}^{r_1^1} m_n^1, \end{aligned}$$

where the second step follows from Corollary 2.4. Since $\{x_n\}$ converges to α, this yields $\lim_{n\to\infty} m_n = m_\alpha > 0$, which means

$$\exists N \in \mathbb{N} \ \text{ s.t. } \ \forall n > N, \ m_n \in \left(\tfrac{1}{l} m_\alpha, l m_\alpha\right), \tag{3.9}$$

where $l > 1$ is an arbitrary constant. We define

$$\begin{aligned} A_n &:= E_1^{r_1^n} \cdot E_0^{r_1^{n-1}} \cdot m_1^{r_1^{n-1}} \cdot m_2^{r_1^{n-2}} \cdots m_N^{r_1^{n-N}}, \\ B_n &:= m_{N+1}^{r_1^{n-N-1}} \cdot m_{N+2}^{r_1^{n-N-2}} \cdots m_{n-1}^{r_1^1} \cdot m_n^1 \end{aligned}$$

so that $\frac{E_{n+1}}{E_n^{r_0}} = A_n B_n$. Then $|r_1| < 1$ implies $\lim_{n\to\infty} A_n = 1$. As for B_n, (3.9) gives

$$\begin{aligned} B_n &\le \begin{cases} (lm_\alpha)^1 \left(\frac{1}{l} m_\alpha\right)^{r_1} \cdots (lm_\alpha)^{r_1^{n-N-1}} & \text{if } n-N-1 \text{ is even;} \\ (lm_\alpha)^1 \left(\frac{1}{l} m_\alpha\right)^{r_1} \cdots \left(\frac{1}{l} m_\alpha\right)^{r_1^{n-N-1}} & \text{if } n-N-1 \text{ is odd} \end{cases} \\ &= l^{1-r_1+r_1^2-\cdots+(-r_1)^{n-N-1}} \cdot m_\alpha^{1+r_1+r_1^2+\cdots+r_1^{n-N-1}} \\ &= l^{\sum_{i=0}^{n-N-1}(-r_1)^i} \cdot m_\alpha^{\sum_{i=0}^{n-N-1} r_1^i}, \end{aligned}$$

where the discrimination of the even-numbered factors in the first line comes from the fact that $r_1 < 0$. Taking $n \to \infty$ in the above inequality yields

$$\lim_{n\to\infty} B_n \le l^{\frac{1}{1+r_1}} m_\alpha^{\frac{1}{1-r_1}} = l^{\frac{1}{1+r_1}} m_\alpha^{r_0-1}. \tag{3.10}$$

By similar arguments, we derive

$$\lim_{n\to\infty} B_n \ge l^{-\frac{1}{1+r_1}} m_\alpha^{r_0-1}. \tag{3.11}$$

Since inequalities (3.10) and (3.11) hold for any $l > 1$, taking $l \to 1^+$ in both implies $\lim_{n\to\infty} B_n = m_\alpha^{r_0-1}$. Therefore,

$$\lim_{n\to\infty} \frac{E_{n+1}}{E_n^{r_0}} = \lim_{n\to\infty} A_n \lim_{n\to\infty} B_n = m_\alpha^{r_0-1},$$

which, together with Definition 2.1, shows that the secant method converges with Q-order $p = r_0$ and the AEC $c = m_\alpha^{r_0-1}$. ∎

Furthermore, the application of asymptotic analysis to the secant method has also demonstrated analogous conclusions [4].

Here, to compare the secant method with Newton's method, we present the Q-order of convergence of Newton's method without proof. In fact, most numerical analysis textbooks already provide detailed and rigorous mathematical proofs for the Q-order of convergence of Newton's method.

THEOREM 3.5 (Convergence of Newton's method for simple roots). *Consider a function $f(x) \in \mathcal{C}^2(\mathcal{B})$, and α is a simple root of f. If x_0 is chosen sufficiently close to the root α, then the sequence of iterates $\{x_n\}$ in Newton's method converges to α with Q-order $p = 2$ and AEC $c = m_\alpha$, i.e.,*

$$\lim_{n\to\infty} \tfrac{E_{n+1}}{E_n^2} = m_\alpha.$$

Clearly, in terms of the Q-order of convergence, Newton's method outperforms the secant method. However, this does not imply that for the same problem and the same accuracy requirement, Newton's method requires less computational time than the secant method. Hence, we compare the time required by Newton's method and the secant method to achieve the same accuracy requirement.

COROLLARY 3.6. *Consider solving $f(x) = 0$ near a root α. Let m and $s \cdot m$ be the time to evaluate $f(x)$ and $f'(x)$, respectively. If their initial values are chosen sufficiently close to the root α, then the minimum time required to obtain the desired absolute accuracy ϵ with Newton's method and the secant method are respectively given by*

$$\begin{aligned} T_N &= (1+s)m\lceil \log_2 K\rceil, \\ T_S &= m\lceil \log_{r_0} K\rceil, \end{aligned}$$

where $K := \frac{\log(m_\alpha \epsilon)}{\log(m_\alpha E_0)}$ and $\lceil\cdot\rceil$ denotes the rounding-up operator, which maps a real number x to the smallest integer greater than or equal to x.

Proof. See [1, Sec. 2.3].

Therefore, for the same problem and precision requirements, if the secant method outperforms Newton's method in computational time, i.e., $\frac{T_S}{T_N} < 1$, we can derive the condition $s > \frac{\log 2}{\log r_0} - 1 \approx 0.44$.

4. Analysis of a multiple root. We adopt the notation and Hypothesis 3.1 introduced at the beginning of Section 3. Without loss of generality, let $\alpha = 0$; otherwise, consider the function $\hat{f}(x) := f(x+\alpha)$ instead of $f(x)$. Therefore, by (3.1) and (1.1), the error $e_n = x_n - \alpha = x_n$ satisfies the secant method iteration

$$e_{n+1} = e_n - f(e_n)\tfrac{e_{n-1}-e_n}{f(e_{n-1})-f(e_n)}. \tag{4.1}$$

Let $\mathcal{B} = [-\delta, \delta]$ be a neighborhood containing the zero of the function $f(x)$.

4.1. The Q-linear convergence near a multiple root. To properly characterize the Q-linear convergence of $\{e_n\}$, i.e., the limit $\lim\limits_{n\to\infty} \frac{E_{n+1}}{E_n} = \lim\limits_{n\to\infty} \frac{|e_{n+1}|}{|e_n|}$ exists and belongs to $(0, 1)$, we define

$$k_n := \tfrac{e_{n+1}}{e_n}. \tag{4.2}$$

The assumption (3.3) ensures that k_n is well-defined and $k_n \neq 0$. Since $e_0 = x_0$ and $k_0 = \frac{e_1}{e_0} = \frac{x_1}{x_0}$, there is a bijection between the initial values x_0, x_1 and k_0, e_0. Thus, we may equivalently take k_0, e_0 as the initial values and study the conditions for the convergence of the secant method error sequence $\{e_n\}$ in the neighborhood $\mathcal{B}$.

We first assume that e_0 and e_1 have the same sign, so that $k_0 > 0$. For the case $k_0 < 0$, Theorems 4.8 and 4.14 show that if k_0 and e_0 satisfy appropriate conditions, analogous convergence results still hold. Without loss of generality, we further assume $e_0 > 0$; otherwise, one may consider the function $\tilde{f}(x) := f(-x)$, which reduces the problem to the case of $e_0 > 0$.

We will prove that if the initial values k_0, e_0 satisfy certain conditions, then the sequence $\{k_n\}$ converges to $c_{m,0} \in (0,1)$, where $c_{m,0}$ is as described in Lemma 2.12.

First, the following lemma shows that under certain conditions, the iteration formula (4.1) is well-defined.

LEMMA 4.1. *Consider a function $f(x) \in \mathcal{C}^{m+1}(\mathcal{B})$ with $m \geq 2$, and 0 is an m-fold root of f, i.e., f satisfies $j = 0, 1, \ldots, m-1,\ f^{(j)}(0) = 0$ and $f^{(m)}(0) \neq 0$. Then there exists $\delta_0 \in (0, \delta]$ such that for any $x \in (0, \delta_0)$, $f'(x) \neq 0$. Furthermore, for $e_{n-1} \neq e_n$ satisfying $e_{n-1}, e_n \in (0, \delta_0)$, we have $f(e_{n-1}) \neq f(e_n)$, and the secant method iteration (4.1) yields $e_{n+1} \neq e_n$.*

Proof. Theorem 2.7 and $j = 0, 1, \ldots, m-1,\ f^{(j)}(0) = 0$ imply that for any $x \in \mathcal{B}$, there exists $\xi_x \in \text{range}(0, x) \subseteq \mathcal{B}$ such that

$$f'(x) = x^{m-1} \left(\frac{f^{(m)}(0)}{(m-1)!} + \frac{f^{(m+1)}(\xi_x)}{m!} x \right).$$

Since $f(x) \in \mathcal{C}^{m+1}(\mathcal{B})$, it follows that for $i = 0, 1, \ldots, m+1$, $f^{(i)}(x)$ is continuous on $\mathcal{B}$. Combined with the fact that $\mathcal{B}$ is a closed interval, the absolute value of $f^{(i)}(x)$ has a finite nonzero upper bound

$$\forall i = 0, 1, \ldots, m+1, \quad M_{f,\mathcal{B}}^{(i)} := \max_{x \in \mathcal{B}} \left| f^{(i)}(x) \right| + 1. \tag{4.3}$$

Thus

$$\forall x \in \mathcal{B},\ \left| f^{(m+1)}(\xi_x) \right| < M_{f,\mathcal{B}}^{(m+1)} \Longrightarrow \forall x \in \mathcal{B},\ 1 - \frac{|f^{(m+1)}(\xi_x)|}{M_{f,\mathcal{B}}^{(m+1)}} > 0.$$

Since $f^{(m)}(0) \neq 0$, set $\delta_0 = \min \left(\frac{m \left| f^{(m)}(0) \right|}{M_{f,\mathcal{B}}^{(m+1)}}, \delta \right) \in (0, \delta]$. Then for any $x \in (0, \delta_0)$, by the triangle inequality,

$$\begin{aligned}
\left| \frac{f^{(m)}(0)}{(m-1)!} + \frac{f^{(m+1)}(\xi_x)}{m!} x \right| &\geq \left| \frac{f^{(m)}(0)}{(m-1)!} \right| - \left| \frac{f^{(m+1)}(\xi_x)}{m!} x \right| \geq \left| \frac{f^{(m)}(0)}{(m-1)!} \right| - \left| \frac{f^{(m+1)}(\xi_x)}{m!} \right| \delta_0 \\
&\geq \frac{|f^{(m)}(0)|}{(m-1)!} - \frac{|f^{(m+1)}(\xi_x)| \cdot |f^{(m)}(0)|}{M_{f,\mathcal{B}}^{(m+1)} (m-1)!} \\
&= \frac{|f^{(m)}(0)|}{(m-1)!} \cdot \left(1 - \frac{|f^{(m+1)}(\xi_x)|}{M_{f,\mathcal{B}}^{(m+1)}} \right) > 0,
\end{aligned}$$

which gives $|f'(x)| = |x|^{m-1} \left| \frac{f^{(m)}(0)}{(m-1)!} + \frac{f^{(m+1)}(\xi_x)}{m!} x \right| > 0$.

For $e_{n-1} \neq e_n$ satisfying $e_{n-1}, e_n \in (0, \delta_0) \subseteq [0, \delta_0]$, by Corollary 2.9, combined with the condition that for any $x \in (0, \delta_0)$, $f'(x) \neq 0$, we have $f(e_{n-1}) \neq f(e_n)$. Hence, $0 < |f(e_{n-1}) - f(e_n)| < 2M_{f,\mathcal{B}}^{(0)}$, where $M_{f,\mathcal{B}}^{(0)}$ is defined as in (4.3). Similarly, for $e_n \neq 0$ satisfying $e_n, 0 \in [0, \delta_0]$, we have $f(e_n) \neq f(0) = 0$. According to the secant method recurrence formula (4.1) and $e_{n-1} \neq e_n$, we obtain

$$|e_{n+1} - e_n| = \left| \tfrac{f(e_n)}{f(e_{n-1}) - f(e_n)} (e_{n-1} - e_n) \right| > \tfrac{|f(e_n)|}{2M^{(0)}_{f,\mathcal{B}}} |e_{n-1} - e_n| > 0. \qquad \square$$

Next, we prove the convergence of $\{e_n\}$.

THEOREM 4.2. *Consider a function $f(x) \in \mathcal{C}^{m+1}(\mathcal{B})$ with $m \geq 2$, and 0 is an m-fold root of f. If the initial values k_0 and e_0 are chosen such that $k_0 \in (0,1) \cup (1, +\infty)$ and $e_0 > 0$ is sufficiently small, then the secant method error $e_n > 0$, and $\{e_n\}$ converges to 0.*

Proof. Algebraic manipulation of the secant method recurrence formula (4.1) yields

$$\begin{aligned} e_{n+1} &= e_n - f(e_n) \tfrac{e_{n-1} - e_n}{f(e_{n-1}) - f(e_n)} = e_n \tfrac{f(e_{n-1}) - f(e_n) - \frac{f(e_n)(e_{n-1} - e_n)}{e_n}}{f(e_{n-1}) - f(e_n)} \\ &= e_n \tfrac{f(e_{n-1}) - \frac{e_{n-1}}{e_n} f(e_n)}{f(e_{n-1}) - f(e_n)}. \end{aligned}$$

Hence (4.2) gives

$$k_n = \tfrac{f(e_{n-1}) - \frac{f(e_{n-1}k_{n-1})}{k_{n-1}}}{f(e_{n-1}) - f(e_{n-1}k_{n-1})} = e_{n-1} \tfrac{\frac{f(e_{n-1})}{e_{n-1}} - \frac{f(e_n)}{e_n}}{f(e_{n-1}) - f(e_n)}. \tag{4.4}$$

By Theorem 2.7 and the fact that for $j = 0, 1, \ldots, m-1$, $f^{(j)}(0) = 0$, we have

$$f(x) = x^m \left(\tfrac{f^{(m)}(0)}{m!} + O(x) \right), \tag{4.5}$$

$$f'(x) = x^{m-1} \left(\tfrac{f^{(m)}(0)}{(m-1)!} + O(x) \right), \tag{4.6}$$

$$f''(x) = x^{m-2} \left(\tfrac{f^{(m)}(0)}{(m-2)!} + O(x) \right), \tag{4.7}$$

where each $O(x)$ term satisfies the existence of a constant $M^{(m+1)}_{f,\mathcal{B}} > 0$ as defined in (4.3) such that $|O(x)| \leq M^{(m+1)}_{f,\mathcal{B}} |x|$.

Since $k_0 > 0$, we have $e_1 = k_0 e_0 > 0$. Note that for $\ell \in \{0, 1\}$, we have $0 < e_\ell \leq \eta := \max(e_0, e_1) = \max(e_0, k_0 e_0) = O(e_0)$. Since $|e_0|$ is assumed to be sufficiently small, it follows that η is also sufficiently small. Hence, we let $\eta \in (0, \delta_0)$, where δ_0 is defined as in Lemma 4.1. Moreover, $k_0 \neq 1$ implies $e_0 \neq e_1$. We claim that

$$\forall \ell \in \mathbb{N},\ e_\ell \in (0, \eta],\ \text{and } \forall \ell \in \mathbb{N}^+,\ e_{\ell-1} \neq e_\ell. \tag{4.8}$$

The claim holds for $\ell = 1$ and $e_0 \in (0, \eta]$. Assume it holds for indices up to $n \geq 1$; that is, $e_{n-1}, e_n \in (0, \eta] \subseteq (0, \delta_0)$ and $e_{n-1} \neq e_n$. Then Lemma 4.1 ensures that for all $x \in (0, \delta_0)$, $f'(x) \neq 0$ and $e_n \neq e_{n+1}$. Thus, it suffices to show $e_{n+1} \in (0, \eta]$. By Theorem 2.10 and (4.4), there exists $t_n \in \text{range}(e_n, e_{n-1})$ such that

$$k_n = \tfrac{e_{n+1}}{e_n} = e_{n-1} \tfrac{\left(\frac{f}{x}\right)'(t_n)}{f'(t_n)} = e_{n-1} \tfrac{t_n f'(t_n) - f(t_n)}{t_n^2 f'(t_n)} = e_{n-1} p(t_n), \tag{4.9}$$

where $p(x) := \frac{x f'(x) - f(x)}{x^2 f'(x)}$. Substituting (4.5) and (4.6) into $p(x)$ yields

$$p(x) = \tfrac{m x^m f^{(m)}(0) - x^m f^{(m)}(0) + O(x^{m+1})}{m x^{m+1} f^{(m)}(0) + O(x^{m+2})} = \tfrac{m-1}{mx} \cdot \tfrac{1 + O(x)}{1 + O(x)}.$$

Lemma 2.11 shows that for

$$\epsilon = \tfrac{1}{2m} > 0,\ \exists \delta_1 > 0 \ \text{ s.t. } \ \forall |x| < \delta_1, \quad \left|\tfrac{1+O(x)}{1+O(x)} - 1\right| < \epsilon.$$

Combined with $t_n \in \operatorname{range}(e_{n-1}, e_n) \subseteq (0, \eta)$ and sufficiently small $\eta > 0$ such that $|t_n| < \eta < \delta_1$, we obtain $\left|\frac{1+O(t_n)}{1+O(t_n)} - 1\right| < \epsilon$. Therefore

$$p(t_n) \in \left(\tfrac{m-1}{mt_n}(1-\epsilon), \tfrac{m-1}{mt_n}(1+\epsilon)\right). \tag{4.10}$$

Taking the derivative of $p(x)$ gives

$$\begin{aligned} p'(x) &= \tfrac{x^3 f'(x)f''(x) - 2x^2 f'(x)f'(x) + 2xf(x)f'(x) - x^3 f'(x)f''(x) + x^2 f(x)f''(x)}{(x^2 f'(x))^2} \\ &= \tfrac{-2x(f'(x))^2 + 2f(x)f'(x) + xf(x)f''(x)}{x^3(f'(x))^2}. \end{aligned}$$

Substituting (4.5), (4.6), and (4.7) into $p'(x)$ yields

$$\begin{aligned} p'(x) &= \tfrac{(-2m^2x^{2m-1} + 2mx^{2m-1} + m(m-1)x^{2m-1})(f^{(m)}(0))^2 + O(x^{2m})}{m^2x^{2m+1}(f^{(m)}(0))^2 + O(x^{2m+2})} \\ &= \tfrac{1}{x^2} \cdot \tfrac{1-m+O(x)}{m+O(x)}. \end{aligned}$$

Lemma 2.11 shows that for

$$\epsilon = \tfrac{1}{2m} > 0,\ \exists \delta_2 > 0 \ \text{ s.t. } \ \forall |x| < \delta_2, \quad \left|\tfrac{1-m+O(x)}{m+O(x)} - \tfrac{1-m}{m}\right| < \epsilon.$$

For $x \in \operatorname{range}(e_{n-1}, e_n) \subseteq (0, \eta)$, since $\eta > 0$ is sufficiently small, it follows that $|x| < \eta < \delta_2$ and

$$p'(x) < \tfrac{1}{x^2}\left(\tfrac{1-m}{m} + \epsilon\right) = \tfrac{1}{x^2} \cdot \tfrac{3-2m}{2m} < 0.$$

Hence $p(x)$ is strictly monotonically decreasing on $\operatorname{range}(e_{n-1}, e_n)$, which implies $p(t_n) \in \operatorname{range}(p(e_{n-1}), p(e_n))$. Furthermore, together with (4.10), we have

$$p(t_n) \in \left(\tfrac{m-1}{m}(1-\epsilon)\min\left(\tfrac{1}{e_{n-1}}, \tfrac{1}{e_n}\right), \tfrac{m-1}{m}(1+\epsilon)\max\left(\tfrac{1}{e_{n-1}}, \tfrac{1}{e_n}\right)\right).$$

Therefore, (4.9) gives

$$\tfrac{e_{n+1}}{e_n} \in \left(\tfrac{m-1}{m}(1-\epsilon)\min\left(1, \tfrac{e_{n-1}}{e_n}\right), \tfrac{m-1}{m}(1+\epsilon)\max\left(1, \tfrac{e_{n-1}}{e_n}\right)\right), \tag{4.11a}$$

$$\tfrac{e_{n+1}}{e_{n-1}} \in \left(\tfrac{m-1}{m}(1-\epsilon)\min\left(\tfrac{e_n}{e_{n-1}}, 1\right), \tfrac{m-1}{m}(1+\epsilon)\max\left(\tfrac{e_n}{e_{n-1}}, 1\right)\right). \tag{4.11b}$$

Let $\mu := \frac{m-1}{m}(1+\epsilon)$. When $e_n > e_{n-1}$, (4.11a) shows that

$$\tfrac{e_{n+1}}{e_n} \in \left(\tfrac{m-1}{m}\tfrac{e_{n-1}}{e_n}(1-\epsilon), \tfrac{m-1}{m}(1+\epsilon)\right),$$

which implies $0 < e_{n+1} < \mu e_n \leq \mu \max(e_{n-1}, e_n)$. Similarly, when $e_n < e_{n-1}$, by (4.11b) we get $0 < e_{n+1} < \mu e_{n-1} \leq \mu \max(e_{n-1}, e_n)$. Hence, it always holds that $0 < e_{n+1} < \mu \max(e_{n-1}, e_n) \leq \mu\eta$. Note that for $m \geq 2$, we have

$$\mu = \tfrac{m-1}{m}(1+\epsilon) = \tfrac{2m^2-m-1}{2m^2} \in (0, 1),$$

which yields $e_{n+1} \in (0, \eta]$. Thus, the induction step is completed, and we obtain $0 < e_{n+1} < \mu \max(e_{n-1}, e_n)$. It follows that

$$\begin{aligned} e_{2n} &< \mu \max(e_{2n-2}, e_{2n-1}), \\ e_{2n+1} &< \mu \max(e_{2n-1}, e_{2n}) \leq \mu \max(e_{2n-2}, e_{2n-1}), \end{aligned}$$

which means $\max(e_{2n}, e_{2n+1}) \leq \mu \max(e_{2n-2}, e_{2n-1})$. By mathematical induction, we get $\max(e_{2n}, e_{2n+1}) \leq \mu^n \max(e_0, e_1) = \mu^n \eta$. Therefore

$$\lim_{n\to+\infty} e_{2n} \leq \lim_{n\to+\infty} \mu^n \eta = 0, \quad \lim_{n\to+\infty} e_{2n+1} \leq \lim_{n\to+\infty} \mu^n \eta = 0.$$

Combined with $e_n > 0$, we conclude that $\{e_n\}$ converges to 0. ◻

When the subscript ℓ is sufficiently large, the following lemma shows that k_ℓ belongs to some closed subinterval of $(0, 1)$.

Lemma 4.3. *Consider a function $f(x) \in \mathcal{C}^{m+1}(\mathcal{B})$ with $m \geq 2$, and 0 is an m-fold root of f. If the initial values k_0 and e_0 are chosen such that $k_0 \in (0,1) \cup (1, +\infty)$ and $e_0 > 0$ is sufficiently small, then there exist $N > 0$ and $r \in \left(\frac{2m-3}{2m}, 1\right)$ such that for any $\ell > N$, it holds that $k_\ell \in \left[\frac{2m-3}{2m}, r\right]$, where k_ℓ is as defined in (4.2).*

Proof. By Theorem 4.2, we know that $e_n > 0$ and $e_n \to 0$, hence $k_n > 0$ for all $n \in \mathbb{N}$. Furthermore, there must exist some $N \in \mathbb{N}$ such that $k_N < 1$. Indeed, if $k_n \geq 1$ for all $n \in \mathbb{N}$, then $e_{n+1} \geq e_n$, so $\{e_n\}$ would be non-decreasing and bounded below by $e_0 > 0$, contradicting $e_n \to 0$. Substituting $n = N + 1$ into (4.4), we obtain

$$k_{N+1} = \frac{f(e_N) - \frac{f(e_N k_N)}{k_N}}{f(e_N) - f(e_N k_N)} = 1 + \frac{f(e_N k_N) - \frac{f(e_N k_N)}{k_N}}{f(e_N) - f(e_N k_N)}.$$

Therefore, there exist $\hat{k}_N \in (k_N, 1)$, $\xi_N \in (0, e_N k_N)$, and $\zeta_N \in (0, e_N \hat{k}_N)$ such that

$$\begin{aligned} 1 - k_{N+1} &= \frac{\frac{f(e_N k_N)}{k_N} - f(e_N k_N)}{f(e_N) - f(e_N k_N)} = \frac{\frac{1}{k_N}(1-k_N) f(e_N k_N)}{(1-k_N) e_N f'(e_N \hat{k}_N)} \\ &= \frac{1}{k_N e_N} \cdot \frac{\frac{f^{(m)}(0)}{m!}(k_N e_N)^m + \frac{f^{(m+1)}(\xi_N)}{(m+1)!}(k_N e_N)^{m+1}}{\frac{f^{(m)}(0)}{(m-1)!}(\hat{k}_N e_N)^{m-1} + \frac{f^{(m+1)}(\zeta_N)}{m!}(\hat{k}_N e_N)^m} \\ &= \left(\frac{k_N}{\hat{k}_N}\right)^{m-1} \frac{1 + k_N \frac{f^{(m+1)}(\xi_N)}{(m+1) f^{(m)}(0)} e_N}{m + \hat{k}_N \frac{f^{(m+1)}(\zeta_N)}{f^{(m)}(0)} e_N} = \left(\frac{k_N}{\hat{k}_N}\right)^{m-1} \frac{1 + O(e_N)}{m + O(e_N)}, \end{aligned}$$

where the second step follows from $f(e_N) - f(e_N k_N) = (e_N - e_N k_N) f'(e_N \hat{k}_N)$; the third step holds because $k_N < 1$ allows canceling the nonzero common factor $1 - k_N$ from the numerator and denominator, followed by applying the Taylor expansion of $f(e_N k_N)$ and $f'(e_N \hat{k}_N)$ at 0 using Theorem 2.7; and the last step follows from $\left| k_N \frac{f^{(m+1)}(\xi_N)}{(m+1) f^{(m)}(0)} \right| \leq \frac{M_{f,\mathcal{B}}^{(m+1)}}{(m+1)|f^{(m)}(0)|}$ and $\left| \hat{k}_N \frac{f^{(m+1)}(\zeta_N)}{f^{(m)}(0)} \right| \leq \frac{M_{f,\mathcal{B}}^{(m+1)}}{|f^{(m)}(0)|}$. Lemma 2.11 shows that for

$$\epsilon = \frac{1}{2m} > 0, \ \exists \delta_1 > 0 \ \text{ s.t. } \ \forall |e| < \delta_1, \quad \left| \frac{1 + O(e)}{m + O(e)} - \frac{1}{m} \right| < \epsilon.$$

From (4.8) and the fact that η is sufficiently small so that $|e_N| \leq \eta < \delta_1$, we obtain $\left| \frac{1+O(e_N)}{m+O(e_N)} - \frac{1}{m} \right| < \epsilon$. This implies

$$1 - k_{N+1} \begin{cases} > \left(\frac{k_N}{\hat{k}_N}\right)^{m-1} \left(\frac{1}{m} - \epsilon\right) = \left(\frac{k_N}{\hat{k}_N}\right)^{m-1} \frac{1}{2m} > 0; \\ < \left(\frac{k_N}{\hat{k}_N}\right)^{m-1} \left(\frac{1}{m} + \epsilon\right) = \left(\frac{k_N}{\hat{k}_N}\right)^{m-1} \frac{3}{2m} < \frac{3}{2m}, \end{cases} \tag{4.12}$$

which is equivalent to $\frac{2m-3}{2m} < k_{N+1} < 1$.

Let $r := \max\left(k_{N+1}, 1 - \frac{(2m-3)^{m-1}}{(2m)^m}\right) \in \left(\frac{2m-3}{2m}, 1\right)$. We claim that for all $\ell > N$, $k_\ell \in \left[\frac{2m-3}{2m}, r\right]$. For $\ell = N+1$, $k_{N+1} \leq r$ gives $\frac{2m-3}{2m} \leq k_{N+1} \leq r$. Assume by the induction hypothesis that for $\ell = n > N$, $k_n \in \left[\frac{2m-3}{2m}, r\right]$. Then for $\ell = n+1$, similar to the derivation of (4.12), we have

$$\left(\frac{k_n}{\hat{k}_n}\right)^{m-1} \frac{1}{2m} < 1 - k_{n+1} < \left(\frac{k_n}{\hat{k}_n}\right)^{m-1} \frac{3}{2m},$$

where $\hat{k}_n \in (k_n, 1)$. Thus, on the one hand,

$$k_{n+1} > 1 - \left(\frac{k_n}{\hat{k}_n}\right)^{m-1} \frac{3}{2m} > 1 - \frac{3}{2m} = \frac{2m-3}{2m}.$$

On the other hand, $\frac{2m-3}{2m} \leq k_n < \hat{k}_n < 1$ yields $\frac{k_n}{\hat{k}_n} > \frac{2m-3}{2m}$ and

$$k_{n+1} < 1 - \left(\frac{k_n}{\hat{k}_n}\right)^{m-1} \frac{1}{2m} < 1 - \left(\frac{2m-3}{2m}\right)^{m-1} \frac{1}{2m} = 1 - \frac{(2m-3)^{m-1}}{(2m)^m} \leq r.$$

Therefore, $k_{n+1} \in \left[\frac{2m-3}{2m}, r\right]$, and the induction holds. □

Next, we prove that $\{k_n\}$ converges. Motivated by fixed-point theory, we analyze the recurrence (4.4). For any $\ell, n \in \mathbb{N}$, we write

$$\begin{aligned}
k_{\ell+1} - k_{n+1} &= \frac{f(e_\ell) - \frac{f(e_\ell k_\ell)}{k_\ell}}{f(e_\ell) - f(e_\ell k_\ell)} - \frac{f(e_n) - \frac{f(e_n k_n)}{k_n}}{f(e_n) - f(e_n k_n)} \\
&= \left(\frac{f(e_\ell) - \frac{f(e_\ell k_\ell)}{k_\ell}}{f(e_\ell) - f(e_\ell k_\ell)} - \frac{f(e_\ell) - \frac{f(e_\ell k_n)}{k_n}}{f(e_\ell) - f(e_\ell k_n)}\right) + \left(\frac{f(e_\ell) - \frac{f(e_\ell k_n)}{k_n}}{f(e_\ell) - f(e_\ell k_n)} - \frac{f(e_n) - \frac{f(e_n k_n)}{k_n}}{f(e_n) - f(e_n k_n)}\right) \\
&= (d_\ell(k_\ell) - d_\ell(k_n)) + (g_n(e_\ell) - g_n(e_n)),
\end{aligned}$$

where

$$d_\ell(k) := \frac{f(e_\ell) - \frac{f(e_\ell k)}{k}}{f(e_\ell) - f(e_\ell k)}, \quad g_n(e) := \frac{f(e) - \frac{f(e k_n)}{k_n}}{f(e) - f(e k_n)}. \tag{4.13}$$

By Theorem 2.8, there exist $\hat{k}_{\ell,n} \in \text{range}(k_\ell, k_n), \hat{e}_{\ell,n} \in \text{range}(e_\ell, e_n)$ such that

$$k_{\ell+1} - k_{n+1} = d'_\ell(\hat{k}_{\ell,n})(k_\ell - k_n) + g'_n(\hat{e}_{\ell,n})(e_\ell - e_n). \tag{4.14}$$

Furthermore, the definition of d_ℓ implies

$$\forall \ell \in \mathbb{N}, \quad k_{\ell+1} = d_\ell(k_\ell). \tag{4.15}$$

The following lemma shows that, for sufficiently small e_ℓ, the derivative $d'_\ell(k)$ is uniformly bounded away from 1 in absolute value.

Lemma 4.4. *Consider a function $f(x) \in \mathcal{C}^{m+1}(\mathcal{B})$ with $m \geq 2$, and 0 is an m-fold root of f. For any fixed $r \in \left(\frac{2m-3}{2m}, 1\right)$, there exists $\gamma_0 \in (0, \delta)$ such that for all $k \in \left[\frac{2m-3}{2m}, r\right]$, $e_\ell \in (0, \gamma_0)$, the derivative $d'_\ell(k)$ satisfies $|d'_\ell(k)| < \frac{m}{m+1}$, where $d_\ell(k)$ is defined in (4.13).*

Proof. Taking the derivative of $d_\ell(k)$ yields

$$\begin{aligned}
d'_\ell(k) &= \frac{\frac{1}{k^2}(e_\ell k^2 f(e_\ell) f'(e_\ell k) - e_\ell k f(e_\ell) f'(e_\ell k) + f(e_\ell) f(e_\ell k) - f(e_\ell k) f(e_\ell k))}{(f(e_\ell) - f(e_\ell k))^2} \\
&= \frac{(mk^{m-1} - mk^{m-2} + k^{m-2} - k^{2m-2}) e_\ell^{2m} + O(e_\ell^{2m+1})}{(1-k^m)^2 e_\ell^{2m} + O(e_\ell^{2m+1})} \\
&= \frac{mk^{m-1} - (m-1)k^{m-2} - k^{2m-2} + O(e_\ell)}{(1-k^m)^2 + O(e_\ell)}
\end{aligned}$$

where the second step applies Taylor expansions to $f(e_\ell)$, $f(e_\ell k)$, and $f'(e_\ell k)$ at 0 using equations (4.5) and (4.6), and then absorbs the $O((e_\ell k)^j)$ terms into $O(e_\ell^j)$ since $k < 1$.

When $m = 2$, we have $k \geq \frac{1}{4}$. By Lemma 2.11 for $\epsilon = \frac{2}{75}$, there exists $\gamma_0 \in (0, \delta)$ such that whenever $e_\ell \in (0, \gamma_0)$, we have $|d'_\ell(k)| = \frac{1+O(e_\ell)}{(1+k)^2+O(e_\ell)} < \frac{4^2}{5^2} + \epsilon = \frac{2}{3} = \frac{m}{m+1}$, which directly proves the result.

For $m \geq 3$, define

$$b(k) := \frac{mk^{m-1}-(m-1)k^{m-2}-k^{2m-2}}{(1-k^m)^2}, \tag{4.16}$$

We next prove that for $k \in \left[\frac{2m-3}{2m}, r\right]$, $|b(k)| < \frac{m-1}{m}$. Further simplifying $b(k)$ yields

$$\begin{aligned} b(k) &= \frac{mk^{m-1}-(m-1)k^{m-2}-k^{2m-2}}{(1-k^m)^2} = -\frac{k^{m-2}(1-k)(m-1-k\sum_{j=0}^{m-2}k^j)}{(1-k^m)^2} \\ &= -\frac{k^{m-2}(1-k)\sum_{j=1}^{m-1}(1-k^j)}{(1-k^m)^2} = -\frac{k^{m-2}(1-k)^2(\sum_{i=0}^{m-2}\sum_{j=0}^{i}k^j)}{(1-k)^2(\sum_{j=0}^{m-1}k^j)^2} \\ &= -\frac{k^{m-2}(\sum_{i=0}^{m-2}\sum_{j=0}^{i}k^j)}{(\sum_{j=0}^{m-1}k^j)^2}, \end{aligned}$$

where the last step follows from the condition $k \leq r < 1$, by canceling the common factor $(1-k)^2$ in both the numerator and denominator. Observe that for $k > 0$, we have $k^{m-2}(\sum_{i=0}^{m-2}\sum_{j=0}^{i}k^j) > 0$ and $(\sum_{j=0}^{m-1}k^j)^2 > 0$, so $b(k) < 0$. Thus, it suffices to show that $b(k) > -\frac{m-1}{m}$. By bounding $b(k)$, we obtain

$$b(k) = -\frac{k^{m-2}\sum_{i=0}^{m-2}(\sum_{j=0}^{i}k^j)}{(\sum_{j=0}^{m-1}k^j)(\sum_{j=0}^{m-1}k^j)} > -\frac{k^{m-2}\sum_{i=0}^{m-2}(\sum_{j=0}^{m-1}k^j)}{(\sum_{j=0}^{m-1}k^j)(\sum_{j=0}^{m-1}k^j)} = -\frac{(m-1)k^{m-2}}{\sum_{j=0}^{m-1}k^j}.$$

Given that $0 < k < 1$, the inequality $k^{m-3} + k^{m-1} > 2k^{m-2}$ holds, and we derive

$$b(k) > -\frac{(m-1)k^{m-2}}{\sum_{j=0}^{m-1}k^j} > -\frac{(m-1)k^{m-2}}{\sum_{j=0}^{m-4}k^j+3k^{m-2}} > -\frac{(m-1)k^{m-2}}{mk^{m-2}} = -\frac{m-1}{m}, \tag{4.17}$$

where the third step follows from that for $j = 0, 1, \dots, m-4$, we have $k^j > k^{m-2}$.

Since for $k \in \left[\frac{2m-3}{2m}, r\right]$, $|b(k)| < \frac{m-1}{m}$ and $\left|(1-k^m)^2\right| \geq \left|(1-r^m)^2\right| > 0$, Lemma 2.11 shows that for $\epsilon = \frac{1}{m(m+1)} > 0$, there exists $\gamma_0 \in (0, \delta)$ such that whenever $e_\ell \in (0, \gamma_0)$, $|d'_\ell(k) - b(k)| < \epsilon$. Together with the triangle inequality, we have

$$|d'_\ell(k)| \leq |b(k)| + |d'_\ell(k) - b(k)| < \frac{m-1}{m} + \epsilon = \frac{m}{m+1}. \qquad \square$$

Similarly, the following lemma shows that for $k_n \in \left[\frac{2m-3}{2m}, r\right]$, and a sufficiently small e, the derivative $|g'_n(e)|$ is uniformly bounded.

Lemma 4.5. *Consider a function $f(x) \in \mathcal{C}^{m+1}(\mathcal{B})$ with $m \geq 2$, and 0 is an m-fold root of f. For any fixed $r \in \left(\frac{2m-3}{2m}, 1\right)$, define the constant*

$$G := \frac{4M_{f,\mathcal{B}}^{(m+1)}}{(1-r^m)\left|f^{(m)}(0)\right|} + 1 > 0, \tag{4.18}$$

where $M_{f,\mathcal{B}}^{(m+1)}$ is as in (4.3). There exists $\gamma_1 \in (0, \delta)$ such that for all $k_n \in \left[\frac{2m-3}{2m}, r\right]$, $e \in (0, \gamma_1)$, the derivative $g'_n(e)$ satisfies $|g'_n(e)| < G$, where $g_n(e)$ is defined in (4.13).

Proof. Taking the derivative of $g_n(e)$ yields

$$g'_n(e) = \frac{\left(\frac{1}{k_n}-1\right)\left(f'(e)f(ek_n)-k_nf(e)f'(ek_n)\right)}{(f(e)-f(ek_n))^2}.$$

Consider the Taylor expansions of $f(e), f'(e), f(ek_n)$, and $f'(ek_n)$ at 0. By Theorem 2.7, there exist $\xi_0, \xi_1 \in (0, e)$ and $\zeta_0, \zeta_1 \in (0, ek_n) \subseteq (0, e)$ such that:

$$\begin{aligned}
f(e) &= \tfrac{f^{(m)}(0)}{m!}e^m + \tfrac{f^{(m+1)}(\xi_0)}{(m+1)!}e^{m+1},\\
f'(e) &= \tfrac{f^{(m)}(0)}{(m-1)!}e^{m-1} + \tfrac{f^{(m+1)}(\xi_1)}{m!}e^m,\\
f(ek_n) &= \tfrac{f^{(m)}(0)}{m!}e^m k_n^m + \tfrac{f^{(m+1)}(\zeta_0)}{(m+1)!}e^{m+1}k_n^{m+1},\\
f'(ek_n) &= \tfrac{f^{(m)}(0)}{(m-1)!}e^{m-1}k_n^{m-1} + \tfrac{f^{(m+1)}(\zeta_1)}{m!}e^m k_n^m.
\end{aligned}$$

From the above four equations, we compute

$$\begin{aligned}
&(m!)^2 f'(e) f(ek_n)\\
&= \left(mf^{(m)}(0)e^{m-1} + f^{(m+1)}(\xi_1)e^m\right)\left(f^{(m)}(0)k_n^m e^m + \tfrac{1}{m+1}f^{(m+1)}(\zeta_0)k_n^{m+1}e^{m+1}\right)\\
&= \Big[m\left(f^{(m)}(0)\right)^2 k_n^m e^{2m-1}\\
&\quad + f^{(m)}(0)\left(f^{(m+1)}(\xi_1)k_n^m + f^{(m+1)}(\zeta_0)\tfrac{m}{m+1}k_n^{m+1}\right)e^{2m} + O(e^{2m+1})\Big],\\
&(m!)^2 k_n f(e) f'(ek_n)\\
&= \left(f^{(m)}(0)e^m + \tfrac{1}{m+1}f^{(m+1)}(\xi_0)e^{m+1}\right)\left(mf^{(m)}(0)k_n^m e^{m-1} + f^{(m+1)}(\zeta_1)k_n^{m+1}e^m\right)\\
&= \Big[m\left(f^{(m)}(0)\right)^2 k_n^m e^{2m-1}\\
&\quad + f^{(m)}(0)\left(f^{(m+1)}(\zeta_1)k_n^{m+1} + f^{(m+1)}(\xi_0)\tfrac{m}{m+1}k_n^m\right)e^{2m} + O(e^{2m+1})\Big],
\end{aligned}$$

and

$$\begin{aligned}
&\left(f(e) - f(ek_n)\right)^2\\
&= \tfrac{1}{(m!)^2}\Big[\left(f^{(m)}(0)e^m + \tfrac{1}{m+1}f^{(m+1)}(\xi_0)e^{m+1}\right)\\
&\qquad - \left(f^{(m)}(0)k_n^m e^m + \tfrac{1}{m+1}f^{(m+1)}(\zeta_0)k_n^{m+1}e^{m+1}\right)\Big]^2\\
&= \tfrac{1}{(m!)^2}\left[f^{(m)}(0)(1-k_n^m)e^m + \tfrac{1}{m+1}\left(f^{(m+1)}(\xi_0) - f^{(m+1)}(\zeta_0)k_n^{m+1}\right)e^{m+1}\right]^2\\
&= \tfrac{1}{(m!)^2}\left[\left(f^{(m)}(0)\right)^2(1-k_n^m)^2 e^{2m} + O(e^{2m+1})\right],
\end{aligned}$$

where the $O(e^{2m+1})$ in the last step of the above three equations is due to the fact that the absolute values of the coefficients of higher-order terms all have constant upper bounds:

$$\begin{aligned}
\left|f^{(m+1)}(\xi_1)\cdot\tfrac{1}{m+1}f^{(m+1)}(\zeta_0)k_n^{m+1}\right| &\le \tfrac{1}{m+1}\left(M_{f,\mathcal{B}}^{(m+1)}\right)^2|k_n|^{m+1}\\
&\le \tfrac{1}{m+1}\left(M_{f,\mathcal{B}}^{(m+1)}\right)^2,\\
\left|\tfrac{1}{m+1}f^{(m+1)}(\xi_0)\cdot f^{(m+1)}(\zeta_1)k_n^{m+1}\right| &\le \tfrac{1}{m+1}\left(M_{f,\mathcal{B}}^{(m+1)}\right)^2|k_n|^{m+1}\\
&\le \tfrac{1}{m+1}\left(M_{f,\mathcal{B}}^{(m+1)}\right)^2,\\
\left|2\cdot f^{(m)}(0)(1-k_n^m)\cdot\tfrac{1}{m+1}\left(f^{(m+1)}(\xi_0) - f^{(m+1)}(\zeta_0)k_n^{m+1}\right)\right|&\\
&\le 2\cdot|f^{(m)}(0)|(1+|k_n|^m)\cdot\tfrac{M_{f,\mathcal{B}}^{(m+1)}}{m+1}\\
&\le \tfrac{8M_{f,\mathcal{B}}^{(m+1)}|f^{(m)}(0)|}{m+1},\\
\left|\tfrac{1}{m+1}\left(f^{(m+1)}(\xi_0) - f^{(m+1)}(\zeta_0)k_n^{m+1}\right)\right|^2 &\le \left|\tfrac{1}{m+1}M_{f,\mathcal{B}}^{(m+1)}(1+|k_n|^{m+1})\right|^2\\
&\le \left(\tfrac{2M_{f,\mathcal{B}}^{(m+1)}}{m+1}\right)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}&f'(e)f(ek_n) - k_n f(e)f'(ek_n)\\ &= \tfrac{1}{(m!)^2} f^{(m)}(0)\left(f^{(m+1)}(\xi_1) - \tfrac{m}{m+1}f^{(m+1)}(\xi_0) + \tfrac{m}{m+1}f^{(m+1)}(\zeta_0)k_n - f^{(m+1)}(\zeta_1)k_n\right)\\ &\quad\cdot (1-k_n)k_n^{m-1}e^{2m} + O(e^{2m+1}).\end{aligned}$$

Substituting the expressions for $f'(e)f(ek_n) - k_n f(e)f'(ek_n)$ and $\big(f(e) - f(ek_n)\big)^2$ into $g_n'(e)$, and using the conditions $f^{(m)}(0) \neq 0$, $e \neq 0$, and $k_n \leq r < 1$, we cancel the common factor $\frac{1}{(m!)^2}f^{(m)}(0)(1-k_n)e^{2m}$ from the numerator and denominator of $g_n'(e)$ to obtain

$$g_n'(e) = \frac{\left(f^{(m+1)}(\xi_1) - \frac{m}{m+1}f^{(m+1)}(\xi_0) + \frac{m}{m+1}f^{(m+1)}(\zeta_0)k_n - f^{(m+1)}(\zeta_1)k_n\right)k_n^{m-1} + O(e)}{f^{(m)}(0)(1-k_n^m)\left(\sum_{j=0}^{m-1}k_n^j\right) + O(e)}.$$

Since $\left|f^{(m)}(0)(1-k_n^m)\left(\sum_{j=0}^{m-1}k_n^j\right)\right| > \left|f^{(m)}(0)\right|\cdot|1-r^m|\cdot 1 > 0$ and

$$\left|\frac{\left(f^{(m+1)}(\xi_1) - \frac{m}{m+1}f^{(m+1)}(\xi_0) + \frac{m}{m+1}f^{(m+1)}(\zeta_0)k_n - f^{(m+1)}(\zeta_1)k_n\right)k_n^{m-1}}{f^{(m)}(0)(1-k_n^m)\left(\sum_{j=0}^{m-1}k_n^j\right)}\right| < \frac{4M_{f,\mathcal{B}}^{(m+1)}}{\left|f^{(m)}(0)\right|\cdot(1-r^m)},$$

Lemma 2.11 shows that for $\epsilon = 1 > 0$, there exists $\gamma_1 \in (0,\delta)$ such that for all $e \in (0,\gamma_1)$,

$$\begin{aligned}|g_n'(e)| &< \left|\frac{\left(f^{(m+1)}(\xi_1) - \frac{m}{m+1}f^{(m+1)}(\xi_0) + \frac{m}{m+1}f^{(m+1)}(\zeta_0)k_n - f^{(m+1)}(\zeta_1)k_n\right)k_n^{m-1}}{f^{(m)}(0)(1-k_n^m)\left(\sum_{j=0}^{m-1}k_n^j\right)}\right| + \epsilon\\ &< \frac{4M_{f,\mathcal{B}}^{(m+1)}}{(1-r^m)\left|f^{(m)}(0)\right|} + \epsilon = G,\end{aligned}$$

where the constant G is as shown in (4.18). ∎

Finally, we combine the above conclusions to prove that the secant method converges linearly near multiple roots of the function f.

THEOREM 4.6. *Consider a function $f(x) \in \mathcal{C}^{m+1}(\mathcal{B})$ with $m \geq 2$, and 0 is an m-fold root of f. If the initial values k_0 and e_0 are chosen such that $k_0 \in (0,1)\cup(1,+\infty)$ and $e_0 > 0$ is sufficiently small, then the secant method error sequence $\{e_n\}$ converges linearly, and its AEC $c = c_{m,0}$, i.e.,*

$$\lim_{n\to\infty}\frac{E_{n+1}}{E_n} = \lim_{n\to\infty}|k_n| = c_{m,0},$$

where $c_{m,0}$ is as defined in Lemma 2.12.

Proof. By Theorem 4.2 and Lemma 4.3, it follows that

$$\begin{aligned}\forall\epsilon > 0,\quad &\exists N_0 \text{ s.t. } \forall \ell, n \geq N_0,\ |e_\ell - e_n| < \tfrac{\epsilon}{2G(m+1)};\\ &\exists N_1 \text{ s.t. } \forall n \geq N_1,\ 0 < e_n < \min(\gamma_0,\gamma_1);\\ &\exists N_2 \text{ s.t. } \forall n \geq N_2,\ |k_n| = k_n \in \left[\tfrac{2m-3}{2m}, r\right],\end{aligned}$$

where G is as described in Lemma 4.5; γ_0, γ_1 are as described in Lemma 4.4 and Lemma 4.5, respectively; $r \in \left(\frac{2m-3}{2m}, 1\right)$ is as described in Lemma 4.3. We define $N_3 := \max(N_0, N_1, N_2)$, then for all $n \geq N_3$ and $\ell > n$, $\mathrm{range}(k_\ell, k_n) \subseteq \left[\frac{2m-3}{2m}, r\right]$ and $\mathrm{range}(e_\ell, e_n) \subseteq (0, \min(\gamma_0,\gamma_1))$. Applying Lemmas 4.4 and 4.5, respectively, we get

$$\forall \ell \geq N_3,\ \forall k \in \left[\tfrac{2m-3}{2m}, r\right],\ |d_\ell'(k)| < \tfrac{m}{m+1}, \tag{4.19}$$

$$\text{and}\quad \forall n \geq N_3,\ \forall e \in (0,\gamma_1),\ |g_n'(e)| < G. \tag{4.20}$$

Since $\frac{m}{m+1} < 1$, it holds that

$$\exists N > N_3 \ \text{ s.t. } \ \forall n \geq N, \ 2\left(\tfrac{m}{m+1}\right)^{n+1-N_3} < \tfrac{\epsilon}{2}.$$

For any $n \geq N > N_3$ and $\ell > n$, from (4.14), we know $\hat{k}_{\ell,n} \in \text{range}(k_\ell, k_n) \subseteq \left[\frac{2m-3}{2m}, r\right]$ and $\hat{e}_{\ell,n} \in \text{range}(e_\ell, e_n) \subseteq (0, \gamma_1)$. Given the above inclusions, we derive

$$\begin{aligned} |k_{\ell+1} - k_{n+1}| &= |d'_\ell(\hat{k}_{\ell,n})(k_\ell - k_n) + g'_n(\hat{e}_{\ell,n})(e_\ell - e_n)| \\ &\leq |d'_\ell(\hat{k}_{\ell,n})(k_\ell - k_n)| + |g'_n(\hat{e}_{\ell,n})(e_\ell - e_n)| \\ &< \tfrac{m}{m+1}|k_\ell - k_n| + G|e_\ell - e_n|, \end{aligned}$$

where the second step follows from the triangle inequality, and the last from the fact that $\hat{k}_{\ell,n} \in \left[\frac{2m-3}{2m}, r\right]$ and (4.19) imply $|d'_\ell(\hat{k}_{\ell,n})| < \frac{m}{m+1}$, while $\hat{e}_{\ell,n} \in (0, \gamma_1)$ and (4.20) yield $|g'_n(\hat{e}_{\ell,n})| < G$. Applying induction to $|k_{\ell+1} - k_{n+1}| < \frac{m}{m+1}|k_\ell - k_n| + G|e_\ell - e_n|$, we obtain

$$\begin{aligned} &|k_{\ell+1} - k_{n+1}| \\ <\ &\left(\tfrac{m}{m+1}\right)^{n+1-N_3} |k_{\ell-n+N_3} - k_{N_3}| + G\textstyle\sum_{i=0}^{n-N_3} \left(\tfrac{m}{m+1}\right)^{i} |e_{\ell-i} - e_{n-i}| \\ <\ &\left(\tfrac{m}{m+1}\right)^{n+1-N_3} \cdot 2 + G \cdot \tfrac{1}{1-\frac{m}{m+1}} \cdot \tfrac{\epsilon}{2G(m+1)} < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, $\{k_n\}$ is a Cauchy sequence, and it converges to some $c \in \left[\frac{2m-3}{2m}, r\right]$.

Next, we compute $d_\ell(k)$ on the domain $k \in \left[\frac{2m-3}{2m}, r\right]$:

$$\begin{aligned} d_\ell(k) = \tfrac{f(e_\ell) - \frac{f(e_\ell k)}{k}}{f(e_\ell) - f(e_\ell k)} &= \frac{\left(\frac{f^{(m)}(0)}{m!}e_\ell^m + O(e_\ell^{m+1})\right) - \left(\frac{f^{(m)}(0)}{m!}e_\ell^m k^{m-1} + O((e_\ell k)^{m+1})\right)}{\left(\frac{f^{(m)}(0)}{m!}e_\ell^m + O(e_\ell^{m+1})\right) - \left(\frac{f^{(m)}(0)}{m!}e_\ell^m k^{m} + O((e_\ell k)^{m+1})\right)} \\ &= \tfrac{(1-k^{m-1})e_\ell^m + O(e_\ell^{m+1}) + O((e_\ell k)^{m+1})}{(1-k^m)e_\ell^m + O(e_\ell^{m+1}) + O((e_\ell k)^{m+1})}, \end{aligned}$$

where the second step applies Taylor expansion of $f(e_\ell)$ and $f(e_\ell k)$ at 0 using (4.5), and the last follows from $\frac{f^{(m)}(0)}{m!} \neq 0$. Since $k \leq r < 1$ has a constant upper bound, the term $O((e_\ell k)^{m+1})$ can be absorbed into $O(e_\ell^{m+1})$. Then, dividing both the numerator and denominator by e_ℓ^m yields

$$d_\ell(k) = \tfrac{(1-k^{m-1}) + O(e_\ell)}{(1-k^m) + O(e_\ell)}. \tag{4.21}$$

Define

$$h_m(k) := \tfrac{1-k^{m-1}}{1-k^m} \tag{4.22}$$

so that for $0 < k \leq r < 1$, $|1 - k^m| \geq 1 - r^m > 0$ and $|h_m(k)| = \frac{1-k^{m-1}}{1-k^m} < 1$. By Lemma 2.11, for any $\epsilon_d > 0$, there exists $\delta_d > 0$ such that for all $k \in \left[\frac{2m-3}{2m}, r\right]$, provided that $|e_\ell| < \delta_d$, we have $|d_\ell(k) - h_m(k)| < \epsilon_d$. Since the sequence $\{e_\ell\}$ converges to 0, there exists $N_d > N$ such that for all $\ell > N_d$, $|e_\ell| < \delta_d$. In summary,

$$\forall \epsilon_d > 0, \exists N_d > 0 \ \text{ s.t. } \ \forall \ell > N_d, \forall k \in \left[\tfrac{2m-3}{2m}, r\right], \ |d_\ell(k) - h_m(k)| < \epsilon_d,$$

which means that $d_\ell(k)$ converges uniformly to $h_m(k)$ on $\left[\frac{2m-3}{2m}, r\right]$. It holds that

$$h_m(c) = \lim_{k \to c} h_m(k) = \lim_{\ell \to \infty} h_m(k_\ell) = \lim_{\ell \to \infty} d_\ell(k_\ell) = \lim_{\ell \to \infty} k_{\ell+1} = c,$$

where the first step follows from the continuity of $h_m(k)$, the second step from the assumption $\lim_{\ell\to\infty} k_\ell = c \in \left[\frac{2m-3}{2m}, r\right]$, the third step from the uniform convergence of $d_\ell(k)$, and the fourth step from (4.15). Therefore, $c \in \left[\frac{2m-3}{2m}, r\right] \subseteq (0,1)$ satisfies

$$h_m(c) - c = \frac{1-c^{m-1}-c^m}{\sum_{j=0}^{m-1} c^j} = -\frac{p_m(c)}{\sum_{j=0}^{m-1} c^j} = 0 \Longrightarrow p_m(c) = 0, \tag{4.23}$$

which, together with Lemma 2.12, shows that c is the unique solution $c_{m,0}$ of the equation $p_m(k) = 0$ in $(0,1)$. Thus, $\lim_{n\to\infty} k_n = c_{m,0}$, completing the proof. ∎

4.2. Sufficient conditions on initial values for Q-linear convergence. Next, we consider the case where $e_0 = x_0$ and $e_1 = x_1$ have opposite signs, i.e., $k_0 = \frac{e_1}{e_0} < 0$. By Theorem 4.6, we know that for initial values $k_0 \in (0,1) \cup (1,+\infty)$ and $e_0 > 0$ sufficiently small, the secant method error sequence $\{e_n\}$ converges linearly. The following lemma shows that $k_0 \in (-1,0)$ together with sufficiently small $|e_0|$ is a sufficient condition for Q-linear convergence of the secant method error sequence $\{e_n\}$. We do not consider the case $k_0 = 0$, which corresponds to premature termination of the secant method iteration as described in Hypothesis 3.1.

LEMMA 4.7. *Consider a function $f(x) \in \mathcal{C}^{m+1}(\mathcal{B})$ with $m \geq 2$, and 0 is an m-fold root of f. If the initial values k_0 and e_0 are chosen such that $k_0 \in (-1,0) \cup (0,1) \cup (1,+\infty)$ and $|e_0|$ is sufficiently small, then the error sequence $\{e_n\}$ converges linearly.*

Proof. When $k_0 \in (0,1) \cup (1,+\infty)$, without loss of generality, assume $e_0 > 0$; otherwise, consider the function $\tilde{f}(x) = f(-x)$. By Theorem 4.6, if e_0 is sufficiently small, then the error sequence $\{e_n\}$ converges linearly.

When $k_0 \in (-1,0)$, we first show that $k_1 > 0$. Applying (4.21) and (4.15) with $\ell = 0$, $k = k_0$, we obtain

$$k_1 = d_0(k_0) = \frac{1-k_0^{m-1}+O(e_0)}{1-k_0^m+O(e_0)}. \tag{4.24}$$

Lemma 2.11 shows that for

$$\epsilon_0 = \frac{1-k_0^{m-1}}{1-k_0^m} > 0, \exists \delta_0 > 0 \ \text{ s.t. } \ \forall |x| < \delta_0,\ \frac{1-k_0^{m-1}+O(x)}{1-k_0^m+O(x)} > \frac{1-k_0^{m-1}}{1-k_0^m} - \epsilon_0 = 0,$$

which, together with sufficiently small $|e_0|$ such that $|e_0| < \delta_0$, implies $k_1 > 0$.

Next, we show that $k_1 \neq 1$.

When m is even, we have $-1 < k_0^{m-1} < 0$ and $0 < k_0^m < 1$, which yields

$$1 < \frac{1}{1-k_0^m} < \frac{1-k_0^{m-1}}{1-k_0^m}.$$

By Lemma 2.11, for

$$\epsilon_1 = \frac{-k_0^{m-1}}{1-k_0^m} > 0, \exists \delta_1 > 0 \ \text{ s.t. } \ \forall |x| < \delta_1,\ 1 < \frac{1-k_0^{m-1}}{1-k_0^m} - \epsilon_1 < \frac{1-k_0^{m-1}+O(x)}{1-k_0^m+O(x)}.$$

Provided that $|e_0|$ is sufficiently small such that $|e_0| < \delta_1$, we conclude $1 < k_1$.

When m is odd, we have $-1 < k_0^m < 0$ and $0 < k_0^{m-1} < 1$, which gives

$$\frac{1-k_0^{m-1}}{1-k_0^m} < \frac{1}{1-k_0^m} < 1.$$

From Lemma 2.11, for

$$\epsilon_2 = 1 - \frac{1-k_0^{m-1}}{1-k_0^m} > 0, \exists \delta_2 > 0 \ \text{ s.t. } \ \forall |x| < \delta_2,\ \frac{1-k_0^{m-1}+O(x)}{1-k_0^m+O(x)} < \frac{1-k_0^{m-1}}{1-k_0^m} + \epsilon_2 = 1.$$

Combined with sufficiently small $|e_0|$ such that $|e_0| < \delta_2$, we obtain $k_1 < 1$.

In summary, when $k_0 \in (-1, 0)$ and $|e_0|$ is chosen sufficiently small such that $|e_0| < \min(\delta_0, \delta_1, \delta_2)$, we obtain $k_1 \in (0, 1) \cup (1, +\infty)$, and $|e_1| = |k_0||e_0| = O(e_0)$ is also sufficiently small. Considering $k_0 \leftarrow k_1, e_0 \leftarrow e_1$ as new initial values for iteration of the secant method, we now have $k_0 \in (0, 1) \cup (1, +\infty)$ and $|e_0|$ is sufficiently small. Under these conditions, it was previously proven that $\{e_n\}$ converges linearly. ∎

In fact, the condition that $k_0 \in (-1, 0) \cup (0, 1) \cup (1, +\infty)$ and e_0 satisfying $|e_0|$ is sufficiently small is only a sufficient condition for the Q-linear convergence of $\{e_n\}$. Furthermore, depending on the parity of m, the admissible range of k_0 can be extended: for sufficiently small $|e_0|$, the sequence $\{e_n\}$ still converges linearly.

We first consider the case where m is odd. For notational convenience, when discussing odd multiple roots, we replace m with $2m + 1$, where $m \in \mathbb{N}^+$, that is, 0 is a $(2m + 1)$-fold root of $f \in \mathcal{C}^{2m+2}(\mathcal{B})$. The following theorem provides sufficient conditions for the Q-linear convergence of $\{e_n\}$ with respect to the values of k_0 and e_0 in the neighborhood of odd multiple roots.

THEOREM 4.8. *Consider a function $f(x) \in \mathcal{C}^{2m+2}(\mathcal{B})$ with $m \in \mathbb{N}^+$, and 0 is a $(2m + 1)$-fold root of f. If the initial values k_0 and e_0 are chosen such that $k_0 \notin \{-1, 0, 1\}$ and $|e_0|$ is sufficiently small, then the error sequence $\{e_n\}$ converges linearly.*

Proof. By Lemma 4.7, it is sufficient to consider the case $k_0 < -1$. Applying (4.24) with m replaced by $2m + 1$ yields

$$k_1 = d_0(k_0) = \tfrac{1-k_0^{2m}+O(e_0)}{1-k_0^{2m+1}+O(e_0)}.$$

When $k_0 < -1$, we have $1 - k_0^{2m} < 0 < 1 - k_0^{2m+1}$, which implies $\frac{1-k_0^{2m}}{1-k_0^{2m+1}} < 0$ and

$$\tfrac{1-k_0^{2m}}{1-k_0^{2m+1}} = \tfrac{1-|k_0|^{2m}}{1+|k_0|^{2m+1}} > \tfrac{1-|k_0|^{2m+1}}{1+|k_0|^{2m+1}} > -1.$$

This shows that $\frac{1-k_0^{2m}}{1-k_0^{2m+1}} \in (-1, 0)$. Then Lemma 2.11 yields that for

$$\begin{aligned} &\epsilon = \min\left(-\tfrac{1-k_0^{2m}}{1-k_0^{2m+1}}, 1 + \tfrac{1-k_0^{2m}}{1-k_0^{2m+1}}\right) > 0,\ \exists \delta_0 > 0 \\ \text{s.t.}\quad &\forall |x| < \delta_0,\ -1 \le \tfrac{1-k_0^{2m}}{1-k_0^{2m+1}} - \epsilon < \tfrac{1-k_0^{2m}+O(x)}{1-k_0^{2m+1}+O(x)} < \tfrac{1-k_0^{2m}}{1-k_0^{2m+1}} + \epsilon \le 0. \end{aligned}$$

Combined with the condition that $|e_0|$ is sufficiently small such that $|e_0| < \delta_0$, it follows that $-1 < k_1 < 0$. Since $|e_1| = |k_0||e_0| = O(e_0)$ is also sufficiently small, by taking $k_0 \leftarrow k_1$ and $e_0 \leftarrow e_1$ as new initial values for the iteration of the secant method, the initial values k_0, e_0 satisfy the conditions of Lemma 4.7. Consequently, the error sequence $\{e_n\}$ converges linearly. ∎

Next, we consider the case where m is even. For notational convenience, when discussing even multiple roots, we replace m with $2m$, where $m \in \mathbb{N}^+$, i.e., 0 is a $2m$-fold root of $f \in \mathcal{C}^{2m+1}(\mathcal{B})$. Note that, in this setting, it is not true that for arbitrary $k_0 \notin \{-1, 0, 1\}$, the error sequence $\{e_n\}$ converges linearly whenever $|e_0|$ is sufficiently small. To establish more general sufficient conditions for the convergence of the secant method in this case, we first introduce the following lemma and constants.

LEMMA 4.9. *For $m \ge 1$, replacing m by $2m$ in $p_m(k)$ as defined in (2.2) yields the characteristic equation*

$$p_{2m}(k) = k^{2m} + k^{2m-1} - 1 = 0. \tag{4.25}$$

This equation has exactly two real solutions, and one of them is $c_{2m,0} \in (0,1)$*. If we denote the other solution as* $c_{2m,1}$*, then* $c_{2m,1} \in (-2,-1)$*. Furthermore, the characteristic equation*

$$q_{2m}(k) := p_{2m}(k) - 1 = k^{2m} + k^{2m-1} - 2 = 0 \tag{4.26}$$

has a unique solution $c_{2m,2}$ *on* $[-2, c_{2m,1})$*. In this case, it holds that*

$$\forall x \in (-\infty, c_{2m,2}),\ \forall y \in (c_{2m,2}, -1), \quad q_{2m}(x) > q_{2m}(c_{2m,2}) = 0 > q_{2m}(y). \tag{4.27}$$

Proof. Since $p'_{2m}(k) = k^{2m-2}(2mk + 2m - 1)$ is negative on $\big(-\infty, -\frac{2m-1}{2m}\big)$ and positive on $\big(-\frac{2m-1}{2m}, 0\big) \cup (0,+\infty)$, $p_{2m}(k)$ is strictly monotonically decreasing on $\big(-\infty, -\frac{2m-1}{2m}\big)$ and strictly monotonically increasing on $\big(-\frac{2m-1}{2m}, 0\big)$ and $(0,+\infty)$. Note that $p_{2m}(0) = -1 < 0 < 1 = p_{2m}(1)$. The intermediate value theorem and Lemma 2.12 yield that (4.25) has a unique solution $c_{2m,0}$ on $(0,1)$. Moreover, for $k \in \big[-\frac{2m-1}{2m}, 0\big)$, $p_{2m}(k) < p_{2m}(0) < 0$, i.e., (4.25) has no real solution on $\big[-\frac{2m-1}{2m}, 0\big)$; for $k \in [1,+\infty)$, $p_{2m}(k) \ge p_{2m}(1) > 0$, i.e., (4.25) has no real solution on $[1,+\infty)$. Similarly, from

$$p_{2m}(-2) = 2^{2m} - 2^{2m-1} - 1 = 2^{2m-1} - 1 \ge 1 > 0 > -1 = p_{2m}(-1), \tag{4.28}$$

we know that (4.25) has a unique solution $c_{2m,1} \in (-2,-1)$ on $\big(-\infty, -\frac{2m-1}{2m}\big)$.

For the characteristic equation (4.26), since $q'_{2m}(k) = p'_{2m}(k) < 0$ on $[-2, c_{2m,1}) \subseteq \big(-\infty, -\frac{2m-1}{2m}\big)$, $q_{2m}(k)$ is strictly monotonically decreasing on $[-2, c_{2m,1})$. When $m = 1$, (4.26) becomes $q_2(k) = k^2 + k - 2 = 0$, which has a unique solution $c_{2,2} = -2$ on $[-2, c_{2,1})$. When $m > 1$, from (4.28), we have

$$q_{2m}(-2) = p_{2m}(-2) - 1 \ge 2^{2m-1} - 2 > 0 > -1 = p_{2m}(c_{2m,1}) - 1 = q_{2m}(c_{2m,1}).$$

Then, by the intermediate value theorem, equation (4.26) has a solution $c_{2m,2}$ on $(-2, c_{2m,1})$. Combined with the fact that $q_{2m}(k)$ is strictly monotonically decreasing on $(-\infty,-1)$, this solution is unique, and (4.27) holds. ◻

Using the constants introduced in Lemma 4.9, the following lemma extends the sufficient conditions for the Q-linear convergence of $\{e_n\}$ with respect to the values of k_0 and e_0 in the neighborhood of even multiple roots.

Lemma 4.10. *Consider a function* $f(x) \in \mathcal{C}^{2m+1}(\mathcal{B})$ *with* $m \in \mathbb{N}^+$*, and* 0 *is a* $2m$*-fold root of* f*. If the initial values* k_0 *and* e_0 *are chosen such that* $k_0 \in (-\infty, c_{2m,2}) \cup (-1,0) \cup (0,1) \cup (1,+\infty)$ *and* $|e_0|$ *is sufficiently small, then the error sequence* $\{e_n\}$ *converges linearly.*

Proof. By Lemma 4.7, it suffices to consider the case $k_0 \in (-\infty, c_{2m,2})$. Applying (4.24) with m replaced by $2m$ yields

$$k_1 = d_0(k_0) = \tfrac{1-k_0^{2m-1}+O(e_0)}{1-k_0^{2m}+O(e_0)}.$$

From $k_0 < c_{2m,2} < -1$, it follows that $1 - k_0^{2m-1} > 0 > 1 - k_0^{2m}$, which implies $\frac{1-k_0^{2m-1}}{1-k_0^{2m}} < 0$. Moreover, (4.27) gives

$$\tfrac{1-k_0^{2m-1}}{1-k_0^{2m}} + 1 = \tfrac{2-k_0^{2m-1}-k_0^{2m}}{1-k_0^{2m}} = \tfrac{q_{2m}(k_0)}{k_0^{2m}-1} > \tfrac{q_{2m}(c_{2m,2})}{k_0^{2m}-1} = 0.$$

Hence $\frac{1-k_0^{2m-1}}{1-k_0^{2m}} \in (-1,0)$. Then Lemma 2.11 shows that for

$$\begin{aligned} &\epsilon = \min\left(-\tfrac{1-k_0^{2m-1}}{1-k_0^{2m}}, 1+\tfrac{1-k_0^{2m-1}}{1-k_0^{2m}}\right) > 0,\ \exists \delta_0 > 0 \\ \text{s.t.}\quad &\forall |x| < \delta_0,\ -1 \le \tfrac{1-k_0^{2m-1}}{1-k_0^{2m}} - \epsilon < \tfrac{1-k_0^{2m-1}+O(x)}{1-k_0^{2m}+O(x)} < \tfrac{1-k_0^{2m-1}}{1-k_0^{2m}} + \epsilon \le 0. \end{aligned}$$

Combined with the fact that $|e_0|$ is sufficiently small such that $|e_0| < \delta_0$, we have $-1 < k_1 < 0$. Since $|e_1| = |k_0||e_0| = O(e_0)$ is also sufficiently small, we can consider taking $k_0 \leftarrow k_1$ and $e_0 \leftarrow e_1$ as new initial values for the iteration of the secant method. Then k_0, e_0 satisfy the conditions of Lemma 4.7, and thus the error sequence $\{e_n\}$ converges linearly. ∎

However, we can appropriately construct functions and initial values such that the error sequence $\{e_n\}$ of the secant method does not converge.

If, before the first element k_ℓ in the sequence $\{k_n\}$ satisfies the convergence condition $k_\ell \in (-\infty, c_{2m,2}) \cup (-1,0) \cup (0,1) \cup (1,+\infty)$ of Lemma 4.10, there exists $i < \ell$ such that $k_i \in \{-1, c_{2m,1}, c_{2m,2}\}$. Then, for the function $f(x) = x^{2m}$ with any $e_0 \neq 0$, the error sequence $\{e_n\}$ does not converge, as demonstrated in the following examples.

Example 4.11. Consider the function $f(x) = x^{2m}$ with $m \in \mathbb{N}^+$, and assume $k_{n-1} \neq \pm 1$. From the recurrence relation (4.4), we obtain

$$k_n = \frac{f(e_{n-1}) - \frac{f(e_{n-1}k_{n-1})}{k_{n-1}}}{f(e_{n-1}) - f(e_{n-1}k_{n-1})} = \frac{1-k_{n-1}^{2m-1}}{1-k_{n-1}^{2m}} = h_{2m}(k_{n-1}), \tag{4.29}$$

where $h_{2m}(k)$ is defined as (4.22). Note that from (4.23), the fixed points of $h_{2m}(k)$ satisfy the characteristic equation $p_{2m}(k) = 0$. Then by Lemma 4.9, the two fixed points of $h_{2m}(k)$ are $c_{2m,0}$ and $c_{2m,1}$.

Taking $k_0 = c_{2m,1} \in (-2,-1)$ and any $e_0 \neq 0$, which, together with iteration (4.29), yields that for all $n \in \mathbb{N}$, $k_n = k_0 = c_{2m,1}$. Because $|c_{2m,1}| > 1$, it follows that $|e_n| = |c_{2m,1}|^n |e_0| \to +\infty$ as $n \to \infty$. Hence, the error sequence $\{e_n\}$ diverges.

Example 4.12. Consider the function $f(x) = x^{2m}$ with $m \in \mathbb{N}^+$. If we take $k_0 = -1$ and $e_0 \neq 0$, then $e_1 = -e_0$, and by formula (4.1), we have $e_2 = \infty$.

If we take $k_0 = c_{2m,2}$ and $e_0 \neq 0$, then by substituting $q_{2m}(c_{2m,2}) = c_{2m,2}^{2m} + c_{2m,2}^{2m-1} - 2 = 0$ into formula (4.29), we can compute

$$k_1 = h_{2m}(k_0) = \frac{1-c_{2m,2}^{2m-1}}{1-c_{2m,2}^{2m}} = \frac{c_{2m,2}^{2m}-1}{1-c_{2m,2}^{2m}} = -1,$$

so $e_2 = k_1 e_1 = -e_1$, and by formula (4.1), we have $e_3 = \infty$.

In these cases, for any $e_0 \neq 0$, the error sequence $\{e_n\}$ diverges.

To pave the way for establishing more general sufficient conditions for the convergence of the secant method for general functions $f(x) \in \mathcal{C}^{2m+1}(\mathcal{B})$, we first analyze the prototypical case $f(x) = x^{2m}$ with $m \in \mathbb{N}^+$.

LEMMA 4.13. *Consider the function $f(x) = x^{2m}$ with $m \in \mathbb{N}^+$. For given initial values $k_0 \neq 1$ and $e_0 \neq 0$, let $\{k_n\}$ be the sequence obtained by applying the secant method. Let $\mathcal{B}_{k,2m} := (c_{2m,2}, c_{2m,1}) \cup (c_{2m,1}, -1)$, where the constants $c_{2m,0}, c_{2m,1}$, and $c_{2m,2}$ are defined as in Lemma 4.9. Then*

- *(i) there exists an index $\ell \ge 0$ such that $k_\ell \notin \mathcal{B}_{k,2m}$;*
- *(ii) Let ℓ be the smallest such index. Then the error sequence $\{e_n\}$ converges linearly if and only if $k_\ell \notin \{-1, 0, 1, c_{2m,1}, c_{2m,2}\}$.*

Proof. The recurrence for $\{k_n\}$ is given by $k_n = h_{2m}(k_{n-1})$ as derived in (4.29). Taking the derivative of $h_{2m}(k)$ yields

$$h'_{2m}(k) = \tfrac{2mk^{2m-1}-(2m-1)k^{2m-2}-k^{4m-2}}{(1-k^{2m})^2}. \tag{4.30}$$

We first prove that there exists $k_\ell \notin \mathcal{B}_{k,2m}$ in the sequence $\{k_n\}$. Suppose, for contradiction, that for all $\ell \in \mathbb{N}$, $k_\ell \in \mathcal{B}_{k,2m}$. Then, by the definition of $\mathcal{B}_{k,2m}$ and Example 4.11, k_ℓ cannot be either of the two fixed points $c_{2m,0}$ and $c_{2m,1}$ of $h_{2m}(k)$. Hence $h_{2m}(k_\ell) \neq k_\ell$, which, combined with (4.29), implies $k_{\ell+1} \neq k_\ell$.

For $m = 1$, we have $h_2(k) = \frac{1}{1+k}$, $c_{2,1} = -\frac{1+\sqrt{5}}{2}$, and $c_{2,2} = -2$. If $k_\ell \in \left(-\frac{3}{2}, -1\right)$, then $k_{\ell+1} = h_2(k_\ell) < -2$, which contradicts $k_{\ell+1} \in \mathcal{B}_{k,2m}$. If $k_\ell \in \left(-2, -\frac{5}{3}\right)$, then $k_{\ell+1} = h_2(k_\ell) \in \left(-\frac{3}{2}, -1\right)$ gives $k_{\ell+2} = h_2(k_{\ell+1}) < -2$, which contradicts $k_{\ell+2} \in \mathcal{B}_{k,2m}$. Hence for all $\ell \in \mathbb{N}$, $k_\ell \in \left[-\frac{5}{3}, c_{2,1}\right) \cup \left(c_{2,1}, -\frac{3}{2}\right]$. Note that for $k \in \left[-\frac{5}{3}, -\frac{3}{2}\right]$, $|h'_2(k)| = \frac{1}{(1+k)^2} \geq \frac{9}{4} > 2$. Theorem 2.8 and $k_\ell \neq k_{\ell-1}$ show that there exists $\hat{k}_{\ell,\ell-1} \in \operatorname{range}(k_\ell, k_{\ell-1}) \subseteq \left[-\frac{5}{3}, -\frac{3}{2}\right]$ such that

$$|k_{\ell+1} - k_\ell| = |h_2(k_\ell) - h_2(k_{\ell-1})| = |h'_2(\hat{k}_{\ell,\ell-1})(k_\ell - k_{\ell-1})| > 2|k_\ell - k_{\ell-1}|,$$

where the first step follows from (4.29). By induction, we obtain $|k_{\ell+1} - k_\ell| > 2^\ell |k_1 - k_0|$. Therefore, when ℓ is sufficiently large such that $2^\ell|k_1 - k_0| > -1 - c_{2,2}$, it holds that $|k_{\ell+1} - k_\ell| > -1 - c_{2,2} = \sup_{k_x,k_y \in \mathcal{B}_{k,2m}} |k_x - k_y|$, which contradicts $k_\ell, k_{\ell+1} \in \mathcal{B}_{k,2m}$.

For $m > 1$, from (4.27), we have

$$\forall k \in (c_{2m,2}, -1), \quad q_{2m}(k) = k^{2m} + k^{2m-1} - 2 = |k|^{2m} - |k|^{2m-1} - 2 < 0.$$

Hence, (4.30) yields that for any $k \in (c_{2m,2}, -1)$,

$$\begin{aligned}
&(1-k^{2m})^2(|h'_{2m}(k)| - 1) \\
&= 2m|k|^{2m-1} + (2m-1)|k|^{2m-2} + |k|^{4m-2} - 1 + 2|k|^{2m} - |k|^{4m} \\
&= 2m|k|^{2m-1} + (2m-1)|k|^{2m-2} + |k|^{4m-2} - 1 + |k|^{2m}(2 - |k|^{2m}) \\
&> 2m|k|^{2m-1} + (2m-1)|k|^{2m-2} + |k|^{4m-2} - 1 + |k|^{2m}(-|k|^{2m-1}) \\
&= 2m|k|^{2m-1} + (2m-1)|k|^{2m-2} - 1 + |k|^{2m-1}(|k|^{2m-1} - |k|^{2m}) \\
&> 2m|k|^{2m-1} + (2m-1)|k|^{2m-2} - 1 - 2|k|^{2m-1} \\
&= 2(m-1)|k|^{2m-1} + (2m-1)|k|^{2m-2} - 1 \\
&> 0 + 3|k|^{2m-2} - 1 > 1,
\end{aligned}$$

where the first step follows from the fact that $2mk^{2m-1}$, $-(2m-1)k^{2m-2}$ and $-k^{4m-2}$ are negative, and the third and fifth steps from $|k|^{2m} - |k|^{2m-1} - 2 < 0$. This shows that

$$\forall k \in (c_{2m,2}, -1), \quad |h'_{2m}(k)| > 1 + \tfrac{1}{(1-k^{2m})^2} > C := 1 + \tfrac{1}{(2^{2m}-1)^2} > 1,$$

where the second step follows from $k \in (c_{2m,2}, -1) \subseteq (-2, -1)$. Then Theorem 2.8 and $k_\ell \neq k_{\ell-1}$ imply that there exists $\hat{k}_{\ell,\ell-1} \in \operatorname{range}(k_\ell, k_{\ell-1}) \subseteq (c_{2m,2}, -1)$ such that

$$|k_{\ell+1} - k_\ell| = |h_{2m}(k_\ell) - h_{2m}(k_{\ell-1})| = |h'_{2m}(\hat{k}_{\ell,\ell-1})(k_\ell - k_{\ell-1})| > C|k_\ell - k_{\ell-1}|,$$

where the first step follows from (4.29). By induction, we obtain $|k_{\ell+1} - k_\ell| > C^\ell |k_1 - k_0|$. Therefore, when ℓ is sufficiently large such that $C^\ell |k_1 - k_0| > -1 - c_{2m,2}$, it holds that $|k_{\ell+1} - k_\ell| > -1 - c_{2m,2} = \sup_{k_x,k_y \in \mathcal{B}_{k,2m}} |k_x - k_y|$, which contradicts $k_\ell, k_{\ell+1} \in \mathcal{B}_{k,2m}$.

In summary, there exists $k_\ell \notin \mathcal{B}_{k,2m}$ in the sequence $\{k_n\}$.

Next, let k_ℓ denote the first value in $\{k_n\}$ that belongs to $\mathbb{R} \setminus \mathcal{B}_{k,2m}$. We prove that $k_\ell \notin \{-1, 0, 1, c_{2m,1}, c_{2m,2}\}$ is a necessary and sufficient condition for Q-linear convergence of the error sequence $\{e_n\}$.

Necessity: Examples 4.11 and 4.12 have verified that when $k_\ell \in \{-1, c_{2m,1}, c_{2m,2}\}$, the error sequence $\{e_n\}$ diverges. If $k_\ell = 1$, then $e_\ell = e_{\ell+1}$, the secant method formula (4.1) is not well-defined. The case $k_\ell = 0$ is excluded by Hypothesis 3.1.

Sufficiency: For any $k \in (-\infty, c_{2m,2})$, since $c_{2m,2} < -1$, we have $k^{2m} - 1 > 0$ and $h_{2m}(k) < 0$. Then (4.27) gives

$$h_{2m}(k) = \tfrac{1-k^{2m-1}}{1-k^{2m}} = \tfrac{2-k^{2m-1}-k^{2m}}{1-k^{2m}} - 1 = \tfrac{q_{2m}(k)}{k^{2m}-1} - 1 > \tfrac{q_{2m}(c_{2m,2})}{k^{2m}-1} - 1 = -1.$$

Moreover, when $k \notin \{-1, 0, 1\}$, algebraic computation yields

$$h_{2m}(k) = \tfrac{1-k^{2m-1}}{1-k^{2m}} = \tfrac{\sum_{j=0}^{2m-2} k^j}{\sum_{j=0}^{2m-1} k^j} = 1 - \tfrac{k^{2m-1}}{\sum_{j=0}^{2m-1} k^j} = 1 - \tfrac{1}{\sum_{j=0}^{2m-1} \frac{1}{k^j}} = 1 - \tfrac{1-\frac{1}{k}}{1-\frac{1}{k^{2m}}}.$$

From the above, we can verify that

$$\begin{aligned}
&\forall k \in (-\infty, c_{2m,2}),\ h_{2m}(k) \in (-1, 0), && \forall k \in (-1, 0),\ h_{2m}(k) \in (1, +\infty),\\
&\forall k \in (1, +\infty),\ h_{2m}(k) \in (0, 1), && \forall k \in (0, 1),\ h_{2m}(k) \in \left(\tfrac{2m-1}{2m}, 1\right),\\
&\forall k \in \left(\tfrac{2m-1}{2m}, 1\right),\ h_{2m}(k) \in \left(\tfrac{2m-1}{2m}, a\right) \subseteq \left[\tfrac{2m-1}{2m}, a\right],
\end{aligned}$$

where $a := 1 - \frac{(2m-1)^{2m-1}}{2m^{2m}-(2m-1)^{2m}} \in \left(\frac{2m-1}{2m}, 1\right)$. Hence from $k_\ell \notin \{-1, 0, 1, c_{2m,1}, c_{2m,2}\}$ and $k_\ell \notin \mathcal{B}_{k,2m}$, i.e., $k_\ell \in (-\infty, c_{2m,2}) \cup (-1, 0) \cup (0, 1) \cup (1, +\infty)$, it follows that

$$\begin{aligned}
k_{\ell+1} &= h_{2m}(k_\ell) \in (-1, 0) \cup (0, 1) \cup (1, +\infty),\\
k_{\ell+2} &= h_{2m}(k_{\ell+1}) \in (0, 1) \cup (1, +\infty),\\
k_{\ell+3} &= h_{2m}(k_{\ell+2}) \in (0, 1),\\
k_{\ell+4} &= h_{2m}(k_{\ell+3}) \in \left(\tfrac{2m-1}{2m}, 1\right),\\
k_{\ell+5} &= h_{2m}(k_{\ell+4}) \in \left(\tfrac{2m-1}{2m}, a\right) \subseteq \left[\tfrac{2m-1}{2m}, a\right].
\end{aligned}$$

By induction, for any $N > 0$, $k_{N+\ell+4} \in \left[\frac{2m-1}{2m}, a\right]$. Since $a \in \left(\frac{2m-1}{2m}, 1\right)$ implies $a \in \left(\frac{4m-3}{4m}, 1\right)$, replacing m and r with $2m$ and a, respectively, in $b(k)$ as defined in (4.16) yields $b(k) = h'_{2m}(k)$. Then $\mathrm{dom}(h'_{2m}) = \left[\frac{2m-1}{2m}, a\right] \subseteq \left[\frac{4m-3}{4m}, a\right] = \mathrm{dom}(b)$. Therefore, from (4.17) and $b(k) < 0$, we obtain

$$\forall k \in \left[\tfrac{2m-1}{2m}, a\right], \quad |h'_{2m}(k)| < \tfrac{2m-1}{2m}.$$

For any $N > 0$ and $L > \ell + 5$, if there exists an integer $\hat{i}$ with $0 < \hat{i} \le L - \ell - 5$ such that $|k_{N+L-\hat{i}} - k_{L-\hat{i}}| = 0$, then (4.29) means that $|k_{N+L-\hat{i}+1} - k_{L-\hat{i}+1}| = |h_{2m}(k_{N+L-\hat{i}}) - h_{2m}(k_{L-\hat{i}})| = 0$. By induction, it holds that for all $\hat{L} \ge L - \hat{i}$, $|k_{N+\hat{L}} - k_{\hat{L}}| = 0$, and hence for any $N > 0$, $\lim_{L\to\infty} |k_{N+L} - k_L| = 0$. Now assume that for all $0 < i \le L - \ell - 5$, $|k_{N+L-i} - k_{L-i}| \ne 0$. Then Theorem 2.8 yields

$$\begin{aligned}
&\exists \hat{k}_{N+L-i,L-i} \in \mathrm{range}(k_{N+L-i}, k_{L-i}) \subseteq \left[\tfrac{2m-1}{2m}, a\right]\\
\text{s.t.}\quad &|k_{N+L-i+1} - k_{L-i+1}| = |h_{2m}(k_{N+L-i}) - h_{2m}(k_{L-i})|\\
&= |h'_{2m}(\hat{k}_{N+L-i,L-i})(k_{N+L-i} - k_{L-i})| \le \tfrac{2m-1}{2m} |k_{N+L-i} - k_{L-i}|,
\end{aligned}$$

where the first step follows from (4.29). Therefore, an induction implies that $|k_{N+L} - k_L| \leq \left(\frac{2m-1}{2m}\right)^{L-\ell-5} |k_{N+\ell+5} - k_{\ell+5}|$, which shows

$$\begin{aligned} \forall N > 0, \quad \lim_{L\to\infty} |k_{N+L} - k_L| &\leq \lim_{L\to\infty} \left(\tfrac{2m-1}{2m}\right)^{L-\ell-5} |k_{N+\ell+5} - k_{\ell+5}| \\ &\leq \lim_{L\to\infty} \left(\tfrac{2m-1}{2m}\right)^{L-\ell-5} \cdot 2 = 0. \end{aligned}$$

This means that $\{k_n\}$ is a Cauchy sequence, and it converges to some $\hat{c} \in \left[\frac{2m-1}{2m}, a\right]$.

As in (4.23), $\hat{c}$ must be a fixed point of $h_{2m}(k)$. Example 4.11 shows that $h_{2m}(k)$ has a unique fixed point $c_{2m,0} \in \left[\frac{2m-1}{2m}, a\right]$, which forces that $\{k_n\}$ converges to $\hat{c} = c_{2m,0} \in (0, 1)$. Hence for any $e_0 \neq 0$, the error sequence $\{e_n\}$ converges linearly. ∎

The result of Lemma 4.13 can be generalized to establish a sufficient condition for Q-linear convergence of the error sequence $\{e_n\}$ when 0 is a $2m$-fold root of a more general function $f(x) \in \mathcal{C}^{2m+1}(\mathcal{B})$.

THEOREM 4.14. *Consider a function $f(x) \in \mathcal{C}^{2m+1}(\mathcal{B})$ with $m \in \mathbb{N}^+$, and 0 is a $2m$-fold root of f. For given initial values k_0, e_0, define $\tilde{k}_0 := k_0$, and let the sequence $\{\tilde{k}_n\}$ be defined by the iteration $\tilde{k}_{n+1} := h_{2m}(\tilde{k}_n)$, where h_{2m} is defined in (4.22). If the sequence $\{\tilde{k}_n\}$ is such that the first term $\tilde{k}_\ell$ that satisfies $\tilde{k}_\ell \notin \mathcal{B}_{k,2m}$ also satisfies $\tilde{k}_\ell \notin \{-1, 0, 1, c_{2m,1}, c_{2m,2}\}$, then for sufficiently small $|e_0|$, the error sequence $\{e_n\}$ converges linearly.*

Proof. By Lemma 4.13, there exists $N \geq 0$ such that for all $n < N, \tilde{k}_n \in \mathcal{B}_{k,2m}$ and $\tilde{k}_N \notin \mathcal{B}_{k,2m}$.

For $N = 0$, we have $k_0 = \tilde{k}_0 \notin \mathcal{B}_{k,2m}$ and $k_0 \notin \{-1, 0, 1, c_{2m,1}, c_{2m,2}\}$, that is, $k_0 \in (-\infty, c_{2m,2}) \cup (-1, 0) \cup (0, 1) \cup (1, +\infty)$. Lemma 4.10 shows that when $|e_0|$ is sufficiently small, the error sequence $\{e_n\}$ converges linearly.

For $N > 0$, we prove by induction that

$$\forall n \leq N, \forall \epsilon_n > 0,\ \exists \delta_n > 0 \ \text{ s.t. } \ \forall |e_0| < \delta_n,\ |k_n - \tilde{k}_n| < \epsilon_n.$$

When $n = 0$, $k_0 = \tilde{k}_0$, hence, $|k_0 - \tilde{k}_0| < \epsilon_0$ always holds. Now assume that the statement holds for $0, 1, \ldots, n < N$. Then we prove that the statement holds for $n + 1$, i.e.,

$$\forall \epsilon_{n+1} > 0, \exists \delta_{n+1} > 0 \ \text{ s.t. } \ \forall |e_0| < \delta_{n+1},\ |k_{n+1} - \tilde{k}_{n+1}| < \epsilon_{n+1}.$$

Since for $\ell \leq n < N$, $\tilde{k}_\ell \in \mathcal{B}_{k,2m} \subseteq (c_{2m,2}, -1)$, by the inductive hypothesis, for

$$\epsilon_\ell = \frac{1}{2}\min(|\tilde{k}_\ell - c_{2m,2}|, |\tilde{k}_\ell + 1|) > 0, \exists \delta_\ell > 0, \ \text{ s.t. } \ \forall |e_0| < \delta_\ell,\ |k_\ell - \tilde{k}_\ell| < \epsilon_\ell. \tag{4.31}$$

Denote $\delta_{n+1,0} := \min_{\ell=0}^{n}(\delta_\ell) > 0$. Then when $|e_0| < \delta_{n+1,0}$, it follows that for all $\ell \leq n$, $|k_\ell - \tilde{k}_\ell| < \epsilon_\ell$. Combined with $\epsilon_\ell \leq \frac{1}{2}|\tilde{k}_\ell + 1|$ and $\tilde{k}_\ell < -1$, we obtain

$$\begin{aligned} k_\ell < \tilde{k}_\ell + \epsilon_\ell \leq \tilde{k}_\ell + \tfrac{1}{2}|\tilde{k}_\ell + 1| &= \tilde{k}_\ell + \tfrac{1}{2}(-1 - \tilde{k}_\ell) \\ &= -1 + \tfrac{1}{2}(\tilde{k}_\ell + 1) = -1 - \tfrac{1}{2}|\tilde{k}_\ell + 1| \leq -1 - \epsilon_\ell < -1. \end{aligned} \tag{4.32}$$

Similarly, using $\epsilon_\ell \leq \frac{1}{2}|\tilde{k}_\ell - c_{2m,2}|$ and $\tilde{k}_\ell > c_{2m,2}$, we can prove that $k_\ell > c_{2m,2}$. Therefore, $k_\ell \in (c_{2m,2}, -1)$. Moreover, since $c_{2m,2} \in [-2, -1)$, it follows that for all $\ell \leq n$, we have $|k_\ell| < 2$, and

$$|e_n| = |e_0| \textstyle\prod_{\ell=0}^{n-1} |k_\ell| < 2^n |e_0| < 2^N |e_0| = O(e_0). \tag{4.33}$$

Since $n < N$, we have $\tilde{k}_n \in \mathcal{B}_{k,2m} \subseteq (-2,-1)$. The function $h_{2m}(k)$ is continuous at $\tilde{k}_n$, so the definition of continuity gives

$$\forall \epsilon_{n+1} > 0, \exists \varepsilon_{n+1} > 0 \ \text{ s.t. } \ \forall |k - \tilde{k}_n| < \varepsilon_{n+1},\ |h_{2m}(k) - h_{2m}(\tilde{k}_n)| < \frac{\epsilon_{n+1}}{2}.$$

By the inductive hypothesis, for

$$\varepsilon_{n+1} > 0, \exists \delta_{n+1,1} > 0 \ \text{ s.t. } \ \forall |e_0| < \delta_{n+1,1},\ |k_n - \tilde{k}_n| < \varepsilon_{n+1},$$

which implies

$$\forall \epsilon_{n+1} > 0, \exists \delta_{n+1,1} > 0 \ \text{ s.t. } \ \forall |e_0| < \delta_{n+1,1},\ |h_{2m}(k_n) - h_{2m}(\tilde{k}_n)| < \frac{\epsilon_{n+1}}{2}.$$

From (4.31) and (4.32), we obtain that $|e_0| < \delta_{n+1,0} \le \delta_n$ ensures $k_n < a_n := -1 - \epsilon_n < -1$. Since $k_n > c_{2m,2}$, we have $a_n \in (k_n, -1) \subseteq (c_{2m,2}, -1)$. For any $k \in (c_{2m,2}, -1)$, $1 - k^{2m-1} > 0$, $1 - k^{2m} < 0$, and $2mk^{2m-1} - (2m-1)k^{2m-2} - k^{4m-2} < 0$, which, together with (4.22) and (4.30), yields

$$h_{2m}(k) = \tfrac{1-k^{2m-1}}{1-k^{2m}} < 0, \quad h'_{2m}(k) = \tfrac{2mk^{2m-1}-(2m-1)k^{2m-2}-k^{4m-2}}{(1-k^{2m})^2} < 0.$$

Hence, $h_{2m}(k)$ is strictly monotonically decreasing on $(c_{2m,2}, -1)$. From $c_{2m,2} < k_n < a_n < -1$, we have $h_{2m}(a_n) < h_{2m}(k_n) < 0$, which implies $\left|\frac{1-k_n^{2m-1}}{1-k_n^{2m}}\right| = |h_{2m}(k_n)| < |h_{2m}(a_n)| = \left|\frac{1-a_n^{2m-1}}{1-a_n^{2m}}\right|$. Combined with $\left|1 - k_n^{2m}\right| > a_n^{2m} - 1 > 0$, Lemma 2.11 and (4.33) give

$$\begin{aligned} &\forall \epsilon_{n+1} > 0,\ \exists \delta_{n+1,2} > 0 \ \text{ s.t. } \ \forall |e_0| < \delta_{n+1,2}, \\ &\left|\tfrac{1-k_n^{2m-1}+O(e_n)}{1-k_n^{2m}+O(e_n)} - \tfrac{1-k_n^{2m-1}}{1-k_n^{2m}}\right| = \left|\tfrac{1-k_n^{2m-1}+O(e_0)}{1-k_n^{2m}+O(e_0)} - \tfrac{1-k_n^{2m-1}}{1-k_n^{2m}}\right| < \tfrac{\epsilon_{n+1}}{2}. \end{aligned}$$

Therefore, for any $\epsilon_{n+1} > 0$, there exists $\delta_{n+1} := \min(\delta_{n+1,0}, \delta_{n+1,1}, \delta_{n+1,2}) > 0$ such that when $|e_0| < \delta_{n+1}$, by the triangle inequality,

$$\begin{aligned} |k_{n+1} - \tilde{k}_{n+1}| &= \left|k_{n+1} - h_{2m}(k_n) + h_{2m}(k_n) - \tilde{k}_{n+1}\right| \\ &= \left|\tfrac{1-k_n^{2m-1}+O(e_n)}{1-k_n^{2m}+O(e_n)} - \tfrac{1-k_n^{2m-1}}{1-k_n^{2m}} + h_{2m}(k_n) - h_{2m}(\tilde{k}_n)\right| \\ &\le \left|\tfrac{1-k_n^{2m-1}+O(e_n)}{1-k_n^{2m}+O(e_n)} - \tfrac{1-k_n^{2m-1}}{1-k_n^{2m}}\right| + \left|h_{2m}(k_n) - h_{2m}(\tilde{k}_n)\right| \\ &< \tfrac{\epsilon_{n+1}}{2} + \tfrac{\epsilon_{n+1}}{2} = \epsilon_{n+1}, \end{aligned}$$

which completes the induction step.

Therefore, for $n = N$, since $\tilde{k}_N \notin \mathcal{B}_{k,2m}$ and $\tilde{k}_N \notin \{-1, 0, 1, c_{2m,1}, c_{2m,2}\}$, we have $\tilde{k}_N \in (-\infty, c_{2m,2}) \cup (-1,0) \cup (0,1) \cup (1,+\infty)$. On the one hand, taking $\epsilon_N = \min(|\tilde{k}_N - c_{2m,2}|, |\tilde{k}_N + 1|, |\tilde{k}_N|, |\tilde{k}_N - 1|) > 0$, there exists $\delta_N > 0$ such that when $|e_0| < \delta_N$, we have $|k_N - \tilde{k}_N| < \epsilon_N$. Then, following a derivation similar to (4.32), we obtain

$$k_N \in (-\infty, c_{2m,2}) \cup (-1,0) \cup (0,1) \cup (1,+\infty).$$

On the other hand, as in (4.33), it holds that $e_N < 2^N|e_0| = O(e_0)$ is sufficiently small. Consider taking $k_0 \leftarrow k_N, e_0 \leftarrow e_N$ as new initial values for the iteration of the secant method. Then k_0, e_0 satisfy the conditions of Lemma 4.10, and thus the error sequence $\{e_n\}$ converges linearly. ∎

5. Conclusion. This paper has presented a novel approach for rigorously establishing the Q-order of convergence of the secant method for both simple and multiple roots. In contrast to the proofs found in existing textbooks and literature, the method proposed herein offers greater rigor. Notably, for the case of multiple roots, it provides sufficient conditions for the convergence of the secant method, thereby furnishing theoretical guidance for the selection of initial values in the secant method.

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