Can outcome communication explain Bell nonlocality?

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A central aspect of quantum information is that correlations between spacelike separated observers sharing entangled states cannot be reproduced by local hidden variable (LHV) models, a phenomenon known as Bell nonlocality. If one wishes to explain such correlations by classical means, a natural possibility is to allow communication between the parties. In particular, LHV models augmented with two bits of classical communication can explain the correlations of any two-qubit state. Would this still hold if communication is restricted to measurement outcomes? While in certain scenarios with a finite number of inputs the answer is yes, we prove that if a model must reproduce all projective measurements, then for any qubit–qudit state the answer is no. In fact, a qubit-qudit under projective measurements admits an LHV model with outcome communication if and only if it already admits an LHV model without communication. On the other hand, we also show that when restricted sets of measurements are considered (for instance, when the qubit measurements are in the upper hemisphere of the Bloch ball), outcome communication does offer an advantage. This exemplifies that trivial properties in standard LHV scenarios, such as deterministic measurements and outcome-relabelling, play a crucial role in the outcome communication scenario.

Introduction.—In a seminal work, John Bell proved that the correlations between spatially separated experiments predicted by quantum theory cannot be reproduced by classical models satisfying a natural notion of locality [1]. This phenomenon, known as Bell nonlocality, constitutes one of the basis of our current understanding of nature, and it is a central aspect of quantum information theory [2–6].

The class of classical models originally considered by Bell, termed local hidden variable (LHV) models, represents the behaviour of classical systems that have interacted in the past, but are then physically separated and unable to communicate [Fig. 1(a)]. As the model is not powerful enough to explain all quantum correlations, a natural question is to investigate what additional resources one would need in order to reproduce them. If the observers are allowed to freely communicate, for instance, any nonsignalling behaviour — hence, any quantum behaviour — can be reproduced by classical models.

In the past few decades, many efforts have been dedicated to understanding what is the minimal amount of communication needed to achieve quantum correlations from classical strategies [Fig. 1(b)]. Early efforts [7–9] and a series of improvements [10–12] led to a proof that one classical bit is sufficient to reproduce the statistics of a maximally entangled two-qubit state under projective measurements, and that two bits of communication are sufficient to reproduce projective measurements on arbitrary two-qubit states [13]. Later, it was also shown that two bits of communication are sufficient to classically simulate arbitrary positive operator-valued measure (POVM) measurements on any two-qubit state [14], and it is possible that even a single bit of communication is enough to do so [15].

In this work, we take a different approach, and instead of investigating the minimal amount of communication needed to reproduce all quantum correlations, we fix *what* can be communicated by the parties and explore which quantum correlations can then be reached [16, 17]. In the first place, if one party is allowed to communicate their measurement setting, then all nonsignalling correlations can be reproduced by classical models [8]. On the other hand, if only the *measurement outcome* is communicated by one party to another [Fig. 1(c)], the answer becomes less obvious.

Such local hidden variable models augmented with outcome communication were investigated in scenarios with

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a finite number of measurement choices [16, 17]. In this case, they can be significantly more powerful than the local model and, in particular, can reproduce any nonsignalling behaviour in the minimal scenario, including a Popescu-Rohrlich box [18]. While previous research has analysed the power of outcome communication at the level of quantum behaviours, much less is known about models that aim to reproduce the statistics of all possible measurements on a bipartite quantum state.

This motivates a fundamental question: how well can local hidden variable models augmented with outcome communication reproduce all correlations of a quantum state? Surprisingly, we show that when the communicating party performs arbitrary two-outcome measurements, a state is local if and only if it is local with communication of outcomes. As a special case, outcome communication cannot explain the correlations of any nonlocal qubit-qudit state under projective measurements. Notably, our proof of this result strongly depends on the requirement that the local model with communication must also reproduce the correlations generated when the communicating party performs a deterministic measurement, i.e., a measurement that always returns the same outcome regardless of the underlying state. This naturally raises the question of whether the result still holds if such a measurement is not considered. To explore this, we analyse the case of two-qubit Werner states, which reveals particularities of outcome communication models that are not seen in standard Bell scenarios.

The outcome communication model.— We start by introducing the scenario and notation. A bipartite Bell scenario consists of two experimenters, Alice and Bob, who can locally measure their systems [19, 20]. In each run of the experiment, they independently choose their measurements, labelled respectively by x and y, and collect the corresponding outcomes, denoted by a and b. By repeating the experiment multiple times and later gathering their data, they can infer the conditional probabilities $p(ab|xy)$. We refer to the set of probabilities $p(ab|xy)$ for all a, b, x and y as the *behaviour* of the system.

Different models have been proposed to explain the potential correlations observed in the experiment's statistics under various assumptions [8, 13, 14, 16, 17, 21, 22]. When Alice and Bob are free to choose their inputs and no communication occurs during the experiment, a natural model emerges, called the LHV model [3].

Within this model, since communication is forbidden, all correlations between the parties must be due to a common past, which is represented by a hidden variable λ [see Fig. 1(a)]. Conditioned on this common past, the observed statistics must factorise. More precisely, the behaviour $p(ab|xy)$ has a local hidden variable model (i.e., it is an LHV behaviour) if there is an additional variable λ, along with probability distributions $p(\lambda)$, $p_A(a|x\lambda)$ and $p_B(b|y\lambda)$, such that

$$p(ab|xy) = \sum_{\lambda} p(\lambda)p_A(a|x\lambda)p_B(b|y\lambda),\ \forall a, b, x, y. \quad (1)$$

When a behaviour admits an LHV model, we say that this behaviour is Bell local, or simply, local. These behaviours satisfy the nonsignalling conditions, that is, they are such that $\sum_a p(ab|xy) = \sum_a p(ab|x'y)$ and $\sum_b p(ab|xy) = \sum_b p(ab|xy')$.

As is well known, there are behaviours obtained through local measurements on entangled quantum states that do not admit decomposition by a local hidden variable model. These are called nonlocal behaviours [19, 20]. A natural attempt to better understand quantum correlations is thus to consider less restrictive models [16, 17, 21, 22].

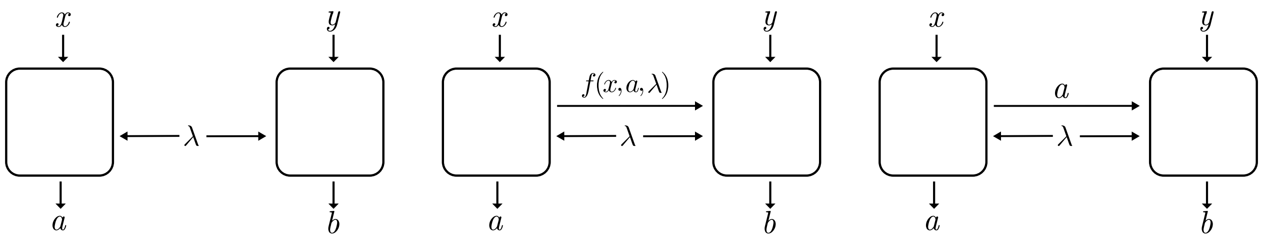
Our focus is on the model where, in addition to the common past λ, correlations can arise from Alice communicating her outcomes to Bob [Fig. 1(c)].

Definition 1 (LHV+Out model)**.** A behaviour $p(ab|xy)$ admits an LHV+Out model (or a local hidden variable model with communication of outcomes) when there is an additional variable λ, along with probability distributions $p(\lambda)$, $p_A(a|x\lambda)$ and $p_B(b|ay\lambda)$, such that

$$p(ab|xy) = \sum_{\lambda} p(\lambda)p_A(a|x\lambda)p_B(b|ay\lambda),\ \forall a, b, x, y. \quad (2)$$

Trivially, all local behaviours are also LHV+Out, but the converse is not true. In particular, any local behaviour is nonsignalling, but the LHV+Out model includes some signalling behaviours. Less obviously, one can also find nonsignalling LHV+Out behaviours that are not LHV. Consider, for instance, the Popescu-Rohrlich (PR) boxes [18] in the paradigmatic Clauser-Horne-Shimony-Holt (CHSH) scenario [20], which are known to be (extremely) nonlocal. The PR-box is a nonsignalling behaviour where the outcomes and inputs respect the relation $a \oplus b = xy$, where $\oplus$ is the modulo two bit addition. To see that they are LHV+Out, let $\lambda \in \{0, 1\}$, $p(\lambda) = 1/2$, $p_A(a|x, \lambda) = \delta(a, x + \lambda)$, $p_B(b|a, y, \lambda) = \delta(b, (a \oplus \lambda)(y \oplus 1) \oplus \lambda)$. Then, it follows that $\sum_{\lambda=1}^{2} p(\lambda)p_A(a|x\lambda)p_B(b|ay\lambda) = \frac{1}{2}\delta(a \oplus b, xy)$. In fact, in the CHSH scenario, all nonsignalling behaviours are LHV+Out [17], showing that outcome communication is a very powerful resource in some Bell scenarios. However, in general scenarios, not all nonsignalling behaviours are LHV+Out. It is even possible to obtain quantum behaviours that are not LHV+Out [16], a fact that was also verified experimentally [23]. Figure 2 illustrates the relationships between these sets of behaviours.

Similarly to the case of Bell nonlocality, the set of LHV+Out behaviours is a polytope. Therefore, it is fully characterised by a finite number of extremal points,



(a) Standard Bell scenario (b) Bell scenario with general communication (c) Bell scenario with outcome communication

Figure 1. Different causal structures for bipartite Bell scenarios. (a) Standard Bell scenario, where the parties are correlated solely through a shared hidden variable λ and no communication occurs. (b) Bell scenario with general communication from Alice to Bob, where Alice's message $f(x, a, \lambda)$ may depend on both her input and outcome and the hidden variable λ. (c) Bell scenario with outcome communication, where Alice's message to Bob is restricted to her measurement outcome.

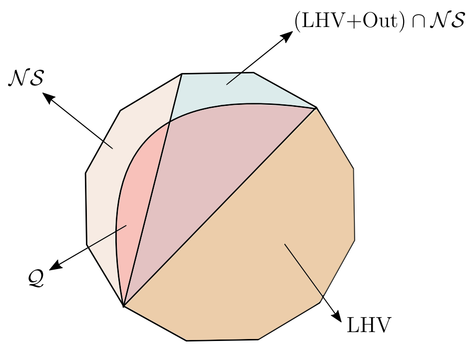


Figure 2. Pictorial representation of the relations between the different sets of correlations: LHV denotes the of local correlations; $\mathcal{Q}$ the set of quantum correlations; $\mathcal{NS}$ the set of nonsignalling correlations; and (LHV+Out) $\cap\,\mathcal{NS}$ the set of local correlations augmented with outcome communication, restricted to the nonsignalling space. Note that LHV $\subset \mathcal{Q}$ and LHV $\subset$ (LHV+Out) $\cap\,\mathcal{NS}$. However, in general, $\mathcal{Q}$ and (LHV+Out) $\cap\,\mathcal{NS}$ are not subsets of one another.

which are the deterministic strategies. A behaviour $p(ab|xy)$ is LHV+Out (see Definition 1) if and only if it is a convex mixture of local deterministic processes with communication of outcomes:

$$p(ab|xy) = \sum_\lambda p(\lambda)\, D_A(a|x\lambda) D_B(b|ay\lambda), \tag{3}$$

where $D_A(\,\cdot\,|x\lambda)$ and $D_B(\,\cdot\,|ay\lambda)$ are deterministic probability distributions. The proof is analogous to that of Fine's theorem [24].

The notion of LHV behaviour [Eq. (1)] can be passed on to quantum states: A bipartite state ρ admits an LHV model if, for any sets of local measurements $\{A_{a|x}\}$ and $\{B_{b|y}\}$, the behaviour with probabilities $p(ab|xy) = \mathrm{tr}\left[\left(A_{a|x} \otimes B_{b|y}\right)\rho\right]$ has an LHV model [Eq. (1)], where $\{A_{a|x}\}$ is a measurement if $A_{a|x} \succcurlyeq 0$ for any a, x, with $\sum_a A_{a|x} = \mathbb{1}$ for any x, and analogously for $\{B_{b|y}\}$. Characterising the set of states that admits an LHV model is a central goal in nonlocality and the topic of an extensive literature, e.g., [25–31]. We extend this definition to the case of the LHV+Out model.

Definition 2 (LHV+Out states)**.** A bipartite quantum state ρ admits an LHV+Out model for the set of measurements $\{A_{a|x}\}$ and $\{B_{b|y}\}$ when the behaviour $p(ab|xy) = \mathrm{tr}\big[(A_{a|x} \otimes B_{b|y})\rho\big]$ admits an LHV+Out model [Eq. (2)]. We say that ρ admits an LHV+Out model when it admits an LHV+Out model for all possible sets of measurements $\{A_{a|x}\}$ and $\{B_{b|y}\}$.

As is usual in nonlocality, it is often relevant to analyse the case where the sets of $\{A_{a|x}\}$ and $\{B_{b|y}\}$ are the sets of all projective measurements, i.e., when $A_{a|x}A_{a'|x} = \delta_{a,a'}A_{a|x}$ and $B_{b|y}B_{b'|y} = \delta_{b,b'}B_{b|y}$ for any x, y. We remark that, as of today, there is no example where non-projective measurements, i.e., positive operator-valued measurements (POVM), are necessary for nonlocality [32].

Remark 1. It follows from the definitions that any state that admits an LHV model also admits an LHV+Out model for any set of measurements. On the other hand, since the LHV+Out model for behaviours is generally more powerful than the LHV model, there could exist nonlocal states that admit an LHV+Out model. This should perhaps even be expected, as the LHV+Out model allows for a considerable amount of signalling, and will be further discussed at a later point.

To show that a given behaviour $p(ab|xy)$ admits an LHV+Out model, it is enough to check whether $p(ab|xy)$ is inside the polytope given by the convex hull of Eq. (3). On the other hand, showing that a state ρ admits an LHV+Out model is a much more challenging problem, since one has to consider all possible measurements that may be performed by Alice and Bob. This difference is analogous to the standard LHV scenario where no communication is allowed [27–30].

Outcome communication cannot explain nonlocality.— One of our main results in this section shows the rather counter-intuitive fact that outcome communication provides no help in explaining the statistics of projective measurements on qubit-qudit states. Put another way, it shows that a qubit-qudit state has an LHV+Out model for projective measurements if and only if it has an LHV model.

Before presenting this result, we first address a more elementary question: under which conditions can we construct an LHV model from an LHV+Out model? We formalise this in the following theorem, which will serve as a key tool for proving our main result.

Theorem 1. *Let $p(ab|xy)$ be a nonsignalling behaviour where Alice's outcomes are dichotomic ($a \in \{\pm 1\}$). Suppose that $p(ab|xy)$ admits an LHV+Out model and that there exists an input x' such that $p_A(\,\cdot\,|x')$ is deterministic. Then, $p(ab|xy)$ admits an LHV model.*

The proof of Theorem 1 is provided in the Appendix. We now turn to our main result.

Result 1. *Let $\{A_{a|x}\}$ be a set of dichotomic measurements containing the deterministic measurement $\{\mathbb{1}, 0\}$, and $\{B_{b|y}\}$ be an arbitrary set of measurements. A bipartite quantum state $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ admits an LHV+Out model for the measurements $\{A_{a|x}\}$ and $\{B_{b|y}\}$ if and only if ρ admits an LHV model for the measurements $\{A_{a|x}\}$ and $\{B_{b|y}\}$.*

In particular, if $\rho \in \mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^d)$ is a qubit-qudit state, then under projective measurements, ρ admits an LHV model if and only if it admits an LHV+Out model.

The proof of Result 1 follows directly from Theorem 1. When Alice performs the deterministic measurements, her marginal is necessarily deterministic, regardless of the quantum state ρ. Hence, if the associated behaviour admits an LHV+Out model, Theorem 1 ensures that it must also admit an LHV model. As for the second part of Result 1, notice that the set of all projective measurements includes the deterministic measurement $\{\mathbb{1}, 0\}$, and for qubits, projective measurements are necessarily dichotomic.

Although the LHV+Out model is signalling and can even simulate PR boxes in the CHSH scenario, our main result shows that it offers no advantage over purely local models for qubit-qudit states. In contrast, if we allow the transmitted bit to be a function $f(x, a, \lambda) \in \{0, 1\}$ of Alice's input and the shared variable [see Fig. 1(b)], instead of being directly the outcome, then any behaviour of a two-qubit maximally entangled state can be reproduced [13, 14]. This shows an extreme difference in the advantage that these two types of communication allow in the task of simulating quantum correlations.

A curious feature of Result 1 is the prominent role played by a deterministic quantum measurement $\{\mathbb{1}, 0\}$. Its proof hinges on the fact that Alice's deterministic measurement must also be compatible with an LHV+Out model, which strongly constrains Bob's response functions $p_B(b|ay\lambda)$ [see Eq. (8)] and makes it possible to recover a standard LHV model from the LHV+Out model.

However, to our knowledge, no other example in the literature of Bell nonlocality assigns such a crucial role to deterministic measurements. In fact, in standard LHV models for quantum states, it is straightforward to account for a deterministic measurement by coarse-graining the outcomes of a nondeterministic one. More fundamentally, deterministic measurements usually are not even considered genuinely quantum, since they can be implemented by simply discarding the quantum system and outputting the outcome that occurs with unit probability. This raises the question of whether deterministic measurements are really the underlying factors leading to the equivalence between LHV and LHV+Out states, or whether a more subtle condition can be identified.

It is well known that the extremal qubit dichotomic measurements are the deterministic measurements $\{\mathbb{1}, 0\}$, $\{0, \mathbb{1}\}$, and rank-1 projective measurements which can be represented in the Bloch sphere $M_\pm = \frac{\mathbb{1} \pm \vec{n}\cdot\vec{\sigma}}{2}$ [33]. It is thus natural to investigate whether a qubit-qudit state which is LHV+Out with respect to rank-1 projective measurements but not with respect to the deterministic measurements can violate a Bell inequality. We show that this is not the case for the arguably most natural candidate two-qubit states, namely, the two-qubit Werner states $W(v) = v\,|\psi^-\rangle\!\langle\psi^-| + (1-v)\frac{\mathbb{1}}{4}$, where $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and $v \in [0, 1]$.

Result 2. *For the set of all rank-1 projective measurements, a Werner state $W(v)$ admits an LHV+Out model if and only if it admits an LHV model.*

This result follows from an adaptation of Ref. [34, Lemma II.1] (see Appendix for details). In contrast, if Alice's measurements are restricted to lie on a single hemisphere of the Bloch sphere, the situation changes: LHV and LHV+Out models for the Werner state exhibit a distinct separation.

Result 3. *If Alice's measurements are all projective measurements lying on the upper hemisphere of the Bloch sphere, the Werner state $W(v)$ admits an LHV+Out model for $v \leqslant 0.69828$. Meanwhile, it is known to violate a Bell inequality for $v > 0.69604$ [35].*

Curiously, in the LHV case, having a model for projective measurements within a single hemisphere already suffices to extend it to all dichotomic measurements, since antipodal measurements are equivalent via outcome relabelling. Result 3 shows this is not the case for LHV+Out models.

To construct this LHV+Out model, we adapted computational techniques for studying local models (namely, polytope approximations [27–30, 36] and Frank-Wolfe algorithms [30, 37, 38]) to the LHV+Out scenario. These techniques are discussed in the Appendix and in the Supplemental Material, and an explicit model is provided in a repository [39].

Our results suggest that a key requirement behind the equivalence between LHV and LHV+Out is the presence of antipodal measurements. That is, whenever an LHV+Out model is demanded for a measurement $x = \{A_0, A_1\}$, if one simultaneously requires a model for the swapped measurement $x' = \{A_1, A_0\}$, then LHV+Out seems to imply LHV. This condition becomes more transparent when phrased in terms of behaviours:

Open Question 1. *Let $p(ab|xy)$ be a nonsignalling behaviour where $x \in \{0, ..., 2m_x - 1\}$, $a \in \{\pm 1\}$ and $p(ab|xy) = p(-a, b|x+m_x, y)$, with addition modulo $2m_x$. If $p(ab|xy)$ admits an LHV+Out model, does it also admit an LHV model?*

Our findings provide evidence pointing towards an affirmative answer to the above question. First, in the simplest case where Bob's measurements are also dichotomic and both Alice and Bob's marginals are uniform, the discussion presented in the Appendix establishes that the answer is indeed affirmative. Second, using linear programming, we computationally verified the same conclusion in scenarios where $m_x \leqslant 4$. Finally, we have sampled a large set of random behaviours respecting the hypothesis of the open question, and no counterexample was found.

Discussion.— In this work, we defined the notion of LHV+Out models for quantum states. We then established general conditions under which a behaviour that admits an LHV+Out model also admits an LHV model (Theorem 1), which allowed us to identify scenarios in which the LHV and LHV+Out models are equivalent at the level of quantum states. In particular, we showed that when Alice's measurements are dichotomic, a quantum state admits an LHV model if and only if it admits an LHV+Out model. As a consequence, any qubit–qudit state under projective measurements satisfies this equivalence (Result 1). Notably, the equivalence established in Result 1 relies on the presence of a deterministic measurement on Alice. This naturally led us to investigate whether such deterministic measurements are indeed necessary for the equivalence between LHV+Out and LHV models. For the case of two-qubit Werner states, we showed that for rank-1 projective measurements the equivalence holds regardless (Result 2). However, restricting Alice's measurements to a single hemisphere of the Bloch sphere leads to a strict separation between the two models (Result 3). These findings suggest that the key requirement behind the equivalence is the presence of antipodal measurements, precisely formulated in Open Question 1.

From the proof techniques used, we suspect that Result 1 strongly relies on the fact that one party is restricted to dichotomic measurements. Also, note that a quantum measurement with two outcomes is completely described by a single measurement operator, since the other one is implicitly defined by the measurement normalisation condition. This is not the case for measurements with more than two outcomes, and this particularity of dichotomic measurements is one of the reasons for results like Gleason's theorem [40] not to hold on the dichotomic case. We thus expect LHV+Out models beyond dichotomic measurements to be a rich and interesting area to be explored, which should help us to understand quantum correlations in communication scenarios.

Finally, an interesting observation is that marginalising over Alice's outcome in an LHV+Out model yields a classical model in the context of entanglement-assisted classical communication (EACC) scenarios [41–44]. Therefore, the techniques and results developed in this work may find further applications in the study of EACC protocols, particularly in understanding the classical simulation of quantum correlations assisted by entanglement [45, 46].

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APPENDIX

PROOF OF THEOREM 1

Theorem 1. *Let $p(ab|xy)$ be a nonsignalling behaviour where Alice's outcomes are dichotomic ($a \in \{\pm 1\}$). Suppose that $p(ab|xy)$ admits an LHV+Out model and that there exists an input x' such that $p_A(\cdot|x')$ is deterministic. Then, $p(ab|xy)$ admits an LHV model.*

Proof. Without loss of generality, assume that $p_A(1|x') = 1$. The case $p_A(-1|x') = 1$ is analogous.

Since $p(ab|xy)$ admits an LHV+Out model, we have

$$p(ab|xy) = \sum_\lambda p(\lambda) p_A(a|x\lambda) p_B(b|ay\lambda), \ \forall a, b, x, y. \tag{4}$$

We claim that $p(ab|xy)$ admits the following LHV model:

$$p(ab|xy) = \sum_\lambda p(\lambda) p_A(a|x\lambda) \widetilde{p}_B(b|y\lambda), \ \forall a, b, x, y. \tag{5}$$

where $p(\lambda)$ and $p_A(a|x\lambda)$ are the same as in Eq. (4), and $\widetilde{p}_B(b|y\lambda) := p_B(b|1y\lambda)$ $\forall b, y, \lambda$. For $a = 1$, Eq. (4) directly implies Eq. (5). Thus, it remains to verify the case $a = -1$. Marginalising Eq. (4) over Bob's outcomes, and setting $a = 1$, $x = x'$, we obtain

$$1 = p_A(1|x') = \sum_\lambda p(\lambda) p_A(1|x'\lambda). \tag{6}$$

Hence, for all λ, we must have $p_A(1|x'\lambda) = 1$ and consequently $p_A(-1|x'\lambda) = 0$.

By the nonsignalling property,

$$p(-1b|xy) = p_B(b|y) - p(1b|xy). \tag{7}$$

On the other hand, marginalising Eq. (4) over Alice's outcomes at input x', we find

$$\begin{aligned} p_B(b|y) &:= \sum_a p(ab|x'y) = \sum_{a,\lambda} p(\lambda) p_A(a|x'\lambda) p_B(b|ay\lambda) \\ &= \sum_\lambda p(\lambda) p_B(b|1y\lambda). \end{aligned} \tag{8}$$

Combining Eqs. (4), (7) and (8), we have

$$\begin{aligned} p(-1b|xy) &= p_B(b|y) - p(1b|xy) \\ &= \sum_\lambda p(\lambda) \left[1 - p_A(1|x\lambda)\right] p_B(b|1y\lambda) \\ &= \sum_\lambda p(\lambda) p_A(-1|x\lambda) \widetilde{p}_B(b|y\lambda). \end{aligned} \tag{9}$$

which matches the model in Eq. (5) for $a = -1$. $\square$

PROOF OF RESULT 2

Result 2. *For the set of all rank-1 projective measurements, a Werner state $W(v)$ admits an LHV+Out model if and only if it admits an LHV model.*

While Result 2 was stated for the specific case of Werner states, the next theorem establishes that the same equivalence remains valid in a broader context. It applies to all two-qubit states with maximally mixed marginals, namely the Bell-diagonal family. Result 2 follows from the following theorem.

Theorem 2. *Let $\rho \in \mathcal{L}(\mathbb{C}_2 \otimes \mathbb{C}_2)$ be a two-qubit state with uniform partial states, that is, $\mathrm{tr}_A(\rho) = \mathrm{tr}_B(\rho) = \frac{\mathbb{1}}{2}$. Under rank-1 projective measurements, ρ admits an LHV model if and only if it admits an LHV+Out model.*

Proof. The statement is proven by showing that if ρ violates a Bell inequality under rank-1 projective measurements, then it also violates an LHV+Out inequality (the converse follows from Definition 1). First, recall that any two-qubit state violating a Bell inequality under projective measurements also violates a dichotomic Bell inequality (all extremal dichotomic measurements are projective [33]). Moreover, since ρ has uniform marginals, we have $\langle A_x \rangle_\rho = \langle B_y \rangle_\rho = 0$ for any observables $A_x = A_{0|x} - A_{1|x}$ and $B_y = B_{0|y} - B_{1|y}$, where $\{A_{a|x}\}, \{B_{b|y}\}$ are rank-1 projective measurements. Hence, any violation can be expressed through a full-correlator Bell inequality of the form

$$\mathcal{O}_{\mathrm{LHV}} = \sum_{x=1}^{m} \sum_{y=1}^{n} M_{xy} \langle A_x B_y \rangle \leqslant L(M), \tag{10}$$

where $M = \{M_{xy}\}$ is a real coefficient matrix. The local (LHV) bound is

$$L(M) := \max_{\substack{\vec{a} \in \{-1,1\}^m \\ \vec{b} \in \{-1,1\}^n}} \sum_{x=1}^{m} \sum_{y=1}^{n} M_{xy}\, a_x b_y. \tag{11}$$

If outcome communication from Alice to Bob is allowed, Bob's deterministic response may depend on both his setting y and Alice's outcome a_x. The corresponding LHV+Out bound is

$$L_{\mathrm{Out}}(M) := \max_{\substack{\vec{a} \in \{-1,1\}^m \\ \vec{b} \in \{-1,1\}^{n \times 2}}} \sum_{x=1}^{m} \sum_{y=1}^{n} M_{xy}\, a_x\, b_{y,a_x}. \tag{12}$$

We can manipulate the expression of $L_{\text{Out}}(M)$ to get:

$$\begin{aligned}
L_{\text{Out}}(M) &:= \max_{\substack{\vec{a}\in\{-1,1\}^n\\ \vec{b}\in\{-1,1\}^{n\times 2}}} \sum_{x=1}^{m}\sum_{y=1}^{n} M_{xy} a_x b_{y,a_x} \\
&= \max_{\substack{\vec{a}\in\{-1,1\}^n\\ \vec{b}\in\{-1,1\}^{n\times 2}}} \left[\sum_{y=1}^{n}\sum_{x:a_x=+1} M_{xy} b_{y,+1} - \sum_{y=1}^{m}\sum_{x:a_x=-1} M_{xy} b_{y,-1}\right] \\
&= \max_{\substack{\vec{a}\in\{-1,1\}^n\\ \vec{b}\in\{-1,1\}^{n\times 2}}} \left[\sum_{y=1}^{n} b_{y,+1}\sum_{x:a_x=+1} M_{xy} - \sum_{y=1}^{m} b_{y,-1}\sum_{x:a_x=-1} M_{xy}\right] \\
&= \max_{\vec{a}\in\{-1,1\}^n} \left[\sum_{y=1}^{n}\left|\sum_{x:a_x=+1} M_{xy}\right| + \sum_{y=1}^{m}\left|\sum_{x:a_x=-1} M_{xy}\right|\right] \\
&= L_2(M),
\end{aligned} \tag{13}$$

where $L_2(M)$ is the classical bound of the associated prepare-and-measure scenario inequality introduced in Ref. [34]. In particular, Lemma II.1 therein shows that for the symmetrised matrix

$$M' = \begin{pmatrix} M \\ -M \end{pmatrix},$$

one has the identity $L_2(M') = L(M') = 2L(M)$, thus for this case $L_{\text{Out}}(M') = 2L(M)$. Therefore, the following Bell expression:

$$\mathcal{O}_{\text{LHV+Out}} = \sum_{x=1}^{m}\sum_{y=1}^{n} M_{xy}\langle A_x B_y\rangle - \sum_{x=1}^{m}\sum_{y=1}^{n} M_{xy}\langle A_{m+x} B_y\rangle, \tag{14}$$

has LHV and LHV+Out bounds both equal $2L(M)$.

On the other hand, since ρ violates (10), there exist observables such that

$$Q_\rho(M) := \sum_{x=1}^{m}\sum_{y=1}^{n} M_{xy}\langle A_x B_y\rangle_\rho > L(M). \tag{15}$$

Defining the extended observables

$$\widetilde{A}_x = \begin{cases} A_x & 1 \leqslant x \leqslant m, \\ -A_x & m+1 \leqslant x \leqslant 2m, \end{cases} \tag{16}$$

we obtain

$$\begin{aligned}
\mathcal{O}_{\text{LHV+Out}} &= \sum_{x=1}^{m}\sum_{y=1}^{n} M_{xy}\left\langle \widetilde{A}_x B_y\right\rangle_\rho \\
&\quad - \sum_{x=m+1}^{2m}\sum_{y=1}^{n} M_{xy}\left\langle \widetilde{A}_x B_y\right\rangle_\rho \\
&= 2\sum_{x=1}^{m}\sum_{y=1}^{n} M_{xy}\langle A_x B_y\rangle_\rho \\
&= 2Q_\rho(M) > 2L(M),
\end{aligned} \tag{17}$$

therefore violating the LHV+Out bound. In conclusion, if ρ violates any Bell inequality [Eq. (10)], one can construct an LHV+Out inequality [Eq. (14)] that is also violated by ρ. □

CONSTRUCTING LHV+OUT MODELS

In order to obtain the LHV+Out model, we adapt the computational methods originally developed for LHV models [27–29, 36]. In a nutshell, the idea is to work with an inner approximation of the set of quantum measurements and then certify whether the corresponding behaviour lies inside the LHV+Out polytope. For this purpose, we extended the Frank–Wolfe method, previously used to certify LHV behaviours [30], to the LHV+Out scenario [47]. The only modification required is to enlarge Bob's response functions so that they can explicitly depend on Alice's outcome. Moreover, the implementation can be made more efficient by employing the symmetrisation techniques described in Ref. [38, 48]. Note that models obtained with this method are self-contained, and can be verified independently of the method we use here.

With this approach, for a carefully chosen set of 401 projective measurements per party, restricted to the upper hemisphere of the Bloch sphere, we obtained a model that very closely reproduces the statistics of the Werner state with visibility $v = 0.7071$, the Euclidean distance between the model and the quantum distribution being only $\epsilon \approx 2 \times 10^{-4}$. Consequently, one can guarantee that by slightly reducing the visibility to $v = \frac{1}{1+\epsilon} \cdot 0.7071 \approx 0.70695$, the resulting statistics admit an exact LHV+Out model for the chosen measurements [30].

To extend the result beyond this finite set, we employ the convex-hull technique and further enlarge Bob's measurement set by including, for each measurement direction $\vec{u}$, its antipode $-\vec{u}$. In the standard LHV case, this doubling of the measurement sets can be applied symmetrically to both parties and is known to increase the inscribed radius of the convex hull. In the LHV+Out scenario, however, the procedure can only be applied to Bob's measurements, since Alice communicates her input to Bob and cannot freely replace her measurement vectors by their antipodes without modifying the communication strategy. With this construction, we conclude that the Werner state $W(v)$ admits an LHV+Out model for all projective measurements on Bob's side and for projective measurements restricted to the upper hemisphere on Alice's side, up to $v \leqslant 0.69828$. Since $W(v_{\text{NL}})$ is known to be nonlocal for $v_{\text{NL}} \geqslant 0.69604$ [35], this establishes Result 3.

A detailed description of these arguments and an explicit construction of the LHV+Out model are discussed in the Supplemental Material and provided in a repository [39].

Supplemental Material

PROOF OF RESULT 3

We begin by presenting an equivalent way of defining an LHV+Out model within the correlator representation. This representation is more economical and ergonomic than the full probability representation, and will later simplify the construction of the model. The following proposition shows how to define an LHV+Out behaviour in terms of the correlators representation.

Proposition 1. *A nonsignalling behaviour $p(ab|xy)$ with binary outcomes has an LHV+Out model if, and only if, there exists a probability distribution $p(\lambda)$ and correlators $\{\langle a_x^\lambda\rangle\}_{x,\lambda}, \{\langle b_{y,1}^\lambda\rangle\}_{y,\lambda}, \{\langle b_{y,-1}^\lambda\rangle\}_{y,\lambda} \subseteq \{\pm 1\}$ such that*

$$\langle a_x\rangle = \sum_\lambda p(\lambda)\langle a_x^\lambda\rangle \tag{18a}$$

$$\langle b_y\rangle = \sum_\lambda p(\lambda)\langle b_{y,\langle a_x^\lambda\rangle}^\lambda\rangle, \quad \forall x \tag{18b}$$

$$\langle a_x b_y\rangle = \sum_\lambda p(\lambda)\langle a_x^\lambda\rangle\langle b_{y,\langle a_x^\lambda\rangle}^\lambda\rangle. \tag{18c}$$

Here, $\langle a_x\rangle$ and $\langle b_y\rangle$ are called the marginal correlators for Alice and Bob, respectively, while $\langle a_x b_y\rangle$ is the two-body correlator.

Proof. Let start with the "$\Rightarrow$" direction. As $p(ab|xy)$ has an LHV+Out model, then there exists a probability distribution $p(\lambda)$ and deterministic probability distributions $D_A(a|x\lambda)$ and $D_B(b|ay\lambda)$, such that

$$p(ab|xy) = \sum_\lambda p(\lambda)D_A(a|x\lambda)D_B(b|ay\lambda).$$

where $D_A(a|x\lambda)$ and $D_B(b|ay\lambda)$ are deterministic. Let us define

$$\langle a_x^\lambda\rangle = \sum_a aD_A(a|x\lambda); \tag{19a}$$

$$\langle b_{y,1}^\lambda\rangle = \sum_b bD_B(b|1y\lambda); \tag{19b}$$

$$\langle b_{y,-1}^\lambda\rangle = \sum_b bD_B(b|-1y\lambda). \tag{19c}$$

It follows that $\langle a_x^\lambda\rangle, \langle b_{y,1}^\lambda\rangle, \langle b_{y,-1}^\lambda\rangle \in \{\pm 1\}$. Moreover, we also have that $D_A(a|x\lambda) = \delta(a, \langle a_x^\lambda\rangle)$, which implies

$$D_A(a|x\lambda)D_B(b|ay\lambda) = D_A(a|x\lambda)D_B(b|\langle a_x^\lambda\rangle y\lambda). \tag{20}$$

Thus,

$$\begin{aligned}&\sum_\lambda p(\lambda)\langle a_x^\lambda\rangle\langle b_{y,\langle a_x^\lambda\rangle}^\lambda\rangle\\ &= \sum_{a,b} ab\sum_\lambda p(\lambda)D_A(a|x\lambda)D_B(b|\langle a_x^\lambda\rangle y\lambda)\\ &= \sum_{a,b} ab\sum_\lambda p(\lambda)D_A(a|x\lambda)D_B(b|ay\lambda)\\ &= \langle a_x b_y\rangle.\end{aligned} \tag{21}$$

Moreover, it is also true that

$$\langle a_x\rangle = \sum_a ap(a|x) = \sum_\lambda p(\lambda)\sum_a aD_A(a|x\lambda) \tag{22a}$$

$$= \sum_\lambda p(\lambda)\langle a_x^\lambda\rangle. \tag{22b}$$

Finally, we also have

$$
\begin{aligned}
\langle b_y\rangle &= \sum_b bp(b|y) = \sum_{a,b} bp(ab|xy) \\
&= \sum_\lambda p(\lambda) \sum_{a,b} bD_A(a|x\lambda)D_B(b|ay\lambda) \\
&= \sum_\lambda p(\lambda) \sum_{a,b} bD_A(a|x\lambda)D_B(b|\langle a_x^\lambda\rangle y\lambda) \\
&= \sum_\lambda p(\lambda) \left(\sum_a D_A(a|x\lambda)\right) \left(\sum_b bD_B(b|\langle a_x^\lambda\rangle y\lambda)\right) \\
&= \sum_\lambda p(\lambda) \langle b^\lambda_{y,\langle a_x^\lambda\rangle}\rangle,
\end{aligned} \tag{23}
$$

where the nonsignalling property was used. Altogether, Eqs. (18a) to (18c) are satisfied.

To prove the "$\Leftarrow$" direction, let us first define:

$$
p_A(a|x\lambda) = \frac{1 + a\langle a_x^\lambda\rangle}{2}, \tag{24a}
$$

$$
p_B(b|\pm 1y\lambda) = \frac{1 + b\langle b^\lambda_{y,\pm 1}\rangle}{2}. \tag{24b}
$$

It follows that $p_A(a|x\lambda)$ and $p_B(b|\pm 1y\lambda)$ are valid probability distributions. Moreover, $a\langle a_x^\lambda\rangle = 2p_A(a|x\lambda) - 1$, and $b\langle b^\lambda_{y,\pm 1}\rangle = 2p_B(b|\pm 1y\lambda) - 1$. Together with Eq. (18), these can be used to construct an LHV+Out model for $p(ab|xy)$ by:

$$
\begin{aligned}
p(ab|xy) &= \frac{1}{4}\left[1 + a\langle a_x\rangle + b\langle b_y\rangle + ab\langle a_x b_y\rangle\right] \\
&= \frac{1}{4}\sum_\lambda p(\lambda)\left[1 + a\langle a_x^\lambda\rangle + b\langle b^\lambda_{y,\langle a_x^\lambda\rangle}\rangle + ab\langle a_x^\lambda\rangle\langle b^\lambda_{y,\langle a_x^\lambda\rangle}\rangle\right] \\
&= \sum_\lambda p(\lambda) p_A(a|x\lambda) p_B(b|\langle a_x^\lambda\rangle y\lambda) \\
&= \sum_\lambda p(\lambda) p_A(a|x\lambda) p_B(b|ay\lambda).
\end{aligned} \tag{25}
$$

Thus, the result follows. □

We now employ the correlator representation to construct an LHV+Out model for the Werner state under all projective measurements lying on the upper hemisphere of the Bloch sphere. Nevertheless, the same ideas could be adapted to any other two-qubit quantum state. Specifically, let Alice and Bob share the Werner state $W(v) = v\,|\psi^-\rangle\!\langle\psi^-| + (1-v)\frac{\mathbb{1}}{4}$, with $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, and suppose that each party performs a local projective measurement. Since both parties hold two-dimensional systems, their projective measurements can be represented by sets of normalised Bloch vectors $\{\vec{u}_x\}_{x=1}^{n_x}$ and $\{\vec{u}_y\}_{y=1}^{n_y}$, respectively. Each vector $\vec{u}$ is then associated with a measurement composed of the two effects $\Pi_{\vec{u},c} = \frac{1}{2}\left(\mathbb{1} + c\,\vec{u}\cdot\vec{\sigma}\right)$, where $c \in \{\pm 1\}$ and $\vec{\sigma} = (X, Y, Z)$ denotes the Pauli matrices. A straightforward calculation shows that, when applied to $W(v)$, these measurements yield the expectation values

$$
\langle a_x\rangle \equiv \sum_a a\,p_A(a|x) = \sum_a a\,\mathrm{tr}\left[W(v)\cdot(\Pi_{\vec{u}_x,a}\otimes\mathbb{1})\right] = 0, \tag{26a}
$$

$$
\langle b_y\rangle \equiv \sum_b b\,p_B(b|y) = \sum_b b\,\mathrm{tr}\left[W(v)\cdot(\mathbb{1}\otimes\Pi_{\vec{u}_y,b})\right] = 0, \tag{26b}
$$

$$
\langle a_x b_y\rangle \equiv \sum_{a,b} ab\,p(ab|xy) = \sum_{a,b} ab\,\mathrm{tr}\left[W(v)\cdot(\Pi_{\vec{u}_x,a}\otimes\Pi_{\vec{u}_y,b})\right] = -v\vec{u}_x\cdot\vec{u}_y. \tag{26c}
$$

Recent advances in conditional gradient algorithms [37, 49] and an efficient implementation have enabled the numerical search for LHV models for behaviours with up to hundreds of measurements [30]. Building on this progress, we have extended their implementation to the LHV+Out case [47], and used it to find an exact model valid for a particular set of 401 measurements per party, all restricted to the upper hemisphere of the Bloch sphere (i.e., $\vec{u}_x, \vec{u}_y$ all lying on the upper hemisphere). For this configuration, we found a model that very closely approximates the statistics of the Werner state with visibility $v = 0.7071$. Indeed, the Euclidean distance between the model's statistics and the quantum ones was $\epsilon \approx 2 \times 10^{-4}$. From this, one can guarantee that by slightly reducing the visibility to $v = \frac{1}{1+\epsilon} \cdot 0.7071 \approx 0.70695$, the new quantum statistics admit an exact LHV+Out model (see Lemma I in Ref. [30]). The measurement set and the resulting model can be found in Ref. [39].

Lastly, we must still argue that the model remains valid beyond the behaviours of $W(v)$ under the particular choice of measurements considered. A well known method in the LHV case is to take the convex hull of the measurement vectors for each party, compute the radii η_A and η_B of the largest spheres that can be inscribed in each, then observe that the state $W(\eta_A \eta_B v)$ admits the model for all rank-1 measurements [27, 28]. This argument extends directly to the LHV+Out case.

To enlarge the inscribed radius, a common trick in the LHV case is to double the measurement set by including all antipodal points on the Bloch sphere: given a measurement $\vec{u}$, one also adds its antipode $-\vec{u}$. It is straightforward to show that if an LHV model exists for the sets $\{u_x\}_x$ and $\{u_y\}_y$, then it also exists for the enlarged sets $\{u_x\}_x \cup \{-u_x\}_x$ and $\{u_y\}_y \cup \{-u_y\}_y$. However, in the LHV+Out scenario this doubling trick does not apply symmetrically. While it can be implemented for Bob's measurements, it cannot be applied to Alice's, since Alice communicates her input to Bob, which restricts the pre-processing she can perform without altering Bob's strategy.

The 401 measurements were chosen so that their convex hull provides a good approximation to the upper hemisphere of the Bloch sphere. Indeed, the convex hull of this set contains a half-sphere of radius $\cos(\pi/40)^2 \approx 0.99384$. By further applying the doubling trick only to Bob's measurements, we conclude that the Werner state $W(v)$ admits an LHV+Out model for all $v \leqslant 0.70695 \cos(\pi/40)^4 \approx 0.69828$, valid for arbitrary projective measurements on Bob's side and for projective measurements restricted to the upper hemisphere on Alice's side. Since $W(v_{\text{NL}})$ is known to be nonlocal for any $v_{\text{NL}} \geqslant 0.69604$ [35], this establishes Result 3. An executable script reproducing this argument using the measurements and model files is provided in Ref. [39] and reproduced below for completeness.

EXPLICIT CONSTRUCTION OF THE LHV+OUT MODEL FOR A NONLOCAL WERNER STATE

The code below, together with the files `werner_nomarg_sqrt2.dat` and `HQV20_v.dat` can be used to construct an LHV+Out model for a nonlocal two qubit Werner state under all projective measurements on a hemisphere of Alice's Bloch ball and all projective measurements for Bob (see Result 3 in the main text). Our construction is done with an explicit LHV+Out model (`werner_nomarg_sqrt2.dat`) and polyhedral approximations of the measurement sphere (`HQV20_v.dat`).

Constructing the quantum behaviour

Let us choose the same set of n_m measurements for both Alice and Bob, represented by the unit vectors $\vec{u}_x$ where $x \in \{1, \ldots, n_m\}$ and correspondingly by $\vec{u}_y$ for Bob. A case with 401 measurements is provided in the file `HQV20_v.dat`, which we open by executing

[2]:
```
meas = deserialize("./HQV20_v.dat");
nx = size(meas, 1); ny = nx;
```

They are represented as a matrix with one vector per row:

[3]:
```
meas
```

```
401×3 Matrix{Float64}:
 1.0         0.0         0.0
 0.987688    0.156434    0.0
 0.951057    0.309017    0.0
 0.891007    0.45399     0.0
 0.809017    0.587785    0.0
 0.707107    0.707107    0.0
```

```
0.587785   0.809017  0.0
0.45399    0.891007  0.0
0.309017   0.951057  0.0
0.156434   0.987688  0.0
⋮
0.309017  -0.951057  0.0
0.45399   -0.891007  0.0
0.587785  -0.809017  0.0
0.707107  -0.707107  0.0
0.809017  -0.587785  0.0
0.891007  -0.45399   0.0
0.951057  -0.309017  0.0
0.987688  -0.156434  0.0
0.0        0.0       1.0
```

To be valid measurements, they must all be normalised, and we also want them to be on the same hemisphere:

[14]:
```
@assert all(norm(meas[i, :]) == 1.0 for i in size(meas, 1))
@assert all(meas[i, 3] >= 0.0 for i in size(meas, 1))
```

As previously explained, on a Werner state $W(v)$ these measurements yield the expectation values

$$\langle a_x \rangle = 0, \quad \langle b_y \rangle = 0, \quad \langle a_x b_y \rangle = -v\ \vec{u}_x \cdot \vec{u}_y.$$

Using a visibility $v = 0.7071$, let us construct a matrix with the correlators $\langle a_x b_y \rangle$:

[5]:
```
vis = 0.7071;
p = -vis .* meas * meas';
```

Constructing a model for the behaviour

We will now check whether the quantum behaviour $\boldsymbol{p}$ defined above has an LHV+Out model. This amounts to finding a probability distribution $p(\lambda)$, a set of deterministic expectations $\{\langle a_x^\lambda \rangle\}_{x,\lambda} \subseteq \{-1, 1\}$ for Alice, and for Bob the sets $\{\langle b_{y,1}^\lambda \rangle\}_{y,\lambda}, \{\langle b_{y,-1}^\lambda \rangle\}_{y,\lambda} \subseteq \{-1, 1\}$, reproducing the quantum behaviour through $\langle a_x b_y \rangle = \sum_\lambda p(\lambda) \langle a_x^\lambda \rangle \langle b_{y,\langle a_x^\lambda \rangle}^\lambda \rangle$ [see Eq. (26)].

Remark 2. *Since for the Werner state the marginal expectations vanish, it suffices to construct a model that reproduces only the correlator terms $\langle a_x b_y \rangle$. This holds because any model that reproduces $\langle a_x b_y \rangle$ can be extended to a model with vanishing marginals and the same correlators. Explicitly, one can duplicate all deterministic strategies and:*

1. *Divide the weight $p(\lambda)$ of each strategy by two.*
2. *In the duplicated strategies, flip the signs of all outcomes $\langle a_x^\lambda \rangle$, $\langle b_{y,-1}^\lambda \rangle$, and $\langle b_{y,+1}^\lambda \rangle$.*

Our model is stored in the file `werner_nomarg_sqrt2.dat` and has four components: `as`, `bms`, `bps`, and `weights`. These correspond, respectively, to Alice's and Bob's deterministic assignments, and the probability distribution over the hidden variables. Let us load the model and check it is valid:

[11]:
```
model = deserialize("./werner_nomarg_sqrt2.dat");
weights = model["weights"];
as = model["as"];
bms = model["bms"];
bps = model["bps"];

@assert all(w >= 0 for w in weights) && isapprox(sum(weights), 1.0) # normalized weights?
@assert all(x -> x == 0 || x == 1, [as; bms; bps]) # all outcomes in {0, 1}?
@assert size(as, 2) == nx && size(bms, 2) == ny # nof. columns equal to the nof. settings?
```

```
@assert size(bps) == size(bms) # equal nof. "+" and "-" measurements and strategies for Bob
@assert size(bps, 1) == length(weights) && size(as, 1) == length(weights)

# the outcomes are represented as 0/1; these lines convert them to -1/+1:
as = (2 * as .- 1);
bms = (2 * bms .- 1);
bps = (2 * bps .- 1);
```

With this data, we will now construct a matrix with elements $\sum_\lambda w_\lambda \langle a_x^\lambda \rangle \langle b^\lambda_{y,\langle a_x^\lambda \rangle} \rangle$. First let us make a helper function that takes a single deterministic strategy (that is, a single term in the sum above) represented by `weight`, `a`, `bm` and `bp`, and adds the corresponding correlations matrix $w_\lambda \langle a_x^\lambda \rangle \langle b^\lambda_{y,\langle a_x \rangle} \rangle$ to a given matrix `q`:

[7]:
```
function add_weighted_outcome_matrix!(q, weight, a, bm, bp)
    for x in axes(q, 1)
        if a[x] == 1
            for y in axes(q, 2)
                q[x, y] += weight * bp[y] # When a_x = 1 then a_x * b_y = b_{y,1}
            end
        elseif a[x] == -1
            for y in axes(q, 2)
                q[x, y] -= weight * bm[y] # When a_x = -1 then a_x * b_y = -b_{y,-1}
            end
        else
            @error "Invalid element in a matrix."
        end
    end
    return q
end;
```

Starting from $\boldsymbol{q} = 0$, we run the function above for each deterministic strategy:

[8]:
```
q = zeros(eltype(weights), size(p));
for i in 1:length(weights) # for each deterministic strategy add its matrix to q:
    @views add_weighted_outcome_matrix!(q, weights[i], as[i, :], bms[i, :], bps[i, :])
end;
```

Exact decomposition

By construction, $\boldsymbol{q}$ is an LHV+Out behaviour. How well does it reproduce the quantum behaviour in $\boldsymbol{p}$? To measure that, we define $\epsilon = \|\boldsymbol{p} - \boldsymbol{q}\|_2$:

[9]:
```
epsilon = norm(p - q) # this is the Euclidean norm
print(epsilon)
```

```
0.00019999656135527604
```

As we can see, $\boldsymbol{q}$ closely reproduces $\boldsymbol{p}$, but not exactly. To get an exact model for a quantum behaviour, consider instead the behaviour given by

$$\nu \boldsymbol{p} = \nu \boldsymbol{q} + (1 - \nu)\boldsymbol{y},$$

where ν is a scalar and $\boldsymbol{y}$ is any LHV+Out behaviour. If we find any ν and $\boldsymbol{y}$ satisfying this equation, we can guarantee that $\nu\boldsymbol{p}$ has an exact LHV+Out representation. Lemma 1 in Ref. [30] shows that any correlations matrix $\boldsymbol{y}$ with $\|\boldsymbol{y}\|_2 \leqslant 1$ is an LHV behaviour. As $\mathrm{LHV}(n_x, n_y, n_a, n_b) \subset \mathrm{LHV{+}Out}(n_x, n_y, n_a, n_b)$, the same $\boldsymbol{y}$ is also LHV+Out. This condition on $\boldsymbol{y}$ can be satisfied by choosing $\nu = 1/(1 + \epsilon)$, which leads to an exact model for the behaviour of the state $W(\nu v)$, under the same set of measurements.

Converting into a model for the state

Lastly, we must argue that this model is valid not only for the behaviours of $W(\nu v)$ under our particular choice of measurements, but rather, for some state $W(\widetilde{\nu} v)$ under *all* possible projective measurements by Bob, and all projective measurements by Alice on a hemisphere of the Bloch ball. As explained in the previous section, one can consider the convex hull of the measurement vectors, compute the radius η of the largest hemisphere that can be inscribed in it, then observe that the state $W(\eta^2 \nu v)$ admits the model for all projective measurements on the hemisphere [27, 28]. For our particular set of measurements, η is known to be $\cos(\pi/40)^2$ [29]. Altogether, this leads to the following visibility for which the Werner state has an LHV+Out model for all measurements in the hemisphere:

[10]:
```
nu = 1 / (1 + epsilon)
eta_squared = cos(pi/40)^4
final_visibility = nu * eta_squared * vis
```

```
0.6982815667392431
```

Because there exists a Bell inequality which is violated by the Werner state with visibility 0.69604 [35], this LHV+Out state is nonlocal, thus proving there exist nonlocal states which are LHV+Out for every projective measurement on a hemisphere.