

REPRESENTING FINE SHAPE OF LOCAL COMPACTA BY HOMOTOPY CLASSES OF ORDINARY MAPS

VLADISLAV ZEMLYANOV

Doctoral School in Mathematics, Higher School of Economics (HSE University)

Moscow, Russia

tagambit@yandex.ru

ORCID 0000-0001-7450-5325

Abstract

Fine shape, as defined by Melikhov, is an extension of the strong shape category of compacta (compact metrizable topological spaces) to all metrizable spaces, notable for being compatible with both Čech cohomology and Steenrod-Sitnikov homology. In this work we study fine shape of local compacta (locally compact separable metrizable spaces), and construct, for every local compactum X , a space $|X|$ unique up to a homotopy equivalence and such that fine shape classes from any locally compact metrizable space Y to X bijectively correspond to homotopy classes of ordinary maps from Y to $|X|$. This correspondence is (contravariantly) functorial in Y , thus giving a representation of Y -dependent contravariant functor for a fixed X ; the universal class corresponding to the identity map of X is the homotopy class of a specific embedding of X into $|X|$ that is a fine shape equivalence.

Keywords: Metrizable topological spaces; shape theory; fine shape; homotopy representations.

AMS Subject Classification: 54C56, 54E45, 55P55

1 Introduction

The strong shape category of compact metrizable topological spaces (compacta) is well-known, defined in multiple ways that all turn out to be equivalent [3, 7, 4, 5, 6]; its theory and applications are extensive, and still being researched (for example, [11, 1]). In the case of noncompact spaces, strong shape is less extensively studied or applied. Strong shape category even of all topological spaces can be defined [2, 8] and has interesting properties of its own, though the complexity inherent in the definition and computations seems to have limited its usage in practice.

Fine shape, as defined by Melikhov [9], is a different extension of strong shape from compacta to all metrizable topological spaces. Notably, as [9] proves, a (co)homology theory is fine shape invariant if and only if it satisfies the map excision axiom (that is, for any closed map $f: (X, A) \rightarrow (Y, B)$ between pairs of spaces, if f restricts to an homeomorphism of $X \setminus A$ onto $Y \setminus B$, then f induces an isomorphism $H_n(X, A) \simeq H_n(Y, B)$ for all n); this includes both Čech cohomology and Steenrod-Sitnikov homology (defined as the direct limit of Steenrod homology of compacta; see footnote 3 in [9]). These two theories, in turn, seem to be a natural choice of homology and cohomology for metrizable spaces, and especially for local compacta — that is, metrizable topological spaces that are locally compact and separable (equivalently, second-countable); see [12]. This makes fine shape a strong candidate for a good homotopy theory of

metrizable spaces; what makes it potentially useful, however, is the simplicity of its definition and application compared to the established noncompact strong shape.

The definition of fine shape is based on the following. Every metrizable space X can be embedded as a closed subset in a metrizable space M that is an absolute retract (equivalently, absolute extensor) with respect to metrizable spaces: for every closed subset $A \subseteq Z$ (with Z , and then also A , metrizable), every continuous map $f: A \rightarrow M$ can be extended to a continuous map $\bar{f}: Z \rightarrow M$. Moreover, it can be done so that X is *homotopy negligible* in M , meaning that there exists a homotopy $H: M \times [0, 1] \rightarrow M$ with $H_0 = id_M$ and $H(M \times (0, 1]) \subseteq M \setminus X$. Now we have the following

Definition 1.1. Let two metrizable spaces X and Y be embedded as closed homotopy negligible subsets in absolute retracts M and N respectively. A continuous map $\phi: M \setminus X \rightarrow N \setminus Y$ is called *$X - Y$ -approaching* if for every sequence $\{p_n\}$ of points of $M \setminus X$ that has an accumulation point in X , the sequence $\{\phi(p_n)\}$ (of points of $N \setminus Y$) has an accumulation point in Y .

This definition clearly looks like an extension of the usual notion of continuous map, and approaching maps can be obtained from ordinary maps from X to Y :

Proposition 1.2. (Lemma 2.9) *Let $f: X \rightarrow Y$ be a continuous map between metrizable spaces X and Y , and let the those be embedded as closed homotopy negligible subsets in absolute retracts M and N respectively. Then*

- (1) *There exists an $X - Y$ -approaching map $\phi: M \setminus X \rightarrow N \setminus Y$ such that ϕ and f combine in a continuous map from M to N ;*
- (2) *For any two homotopic maps $f, f': X \rightarrow Y$, any two corresponding maps ϕ, ϕ' given by (1) are homotopic through maps of the same class.*

This means that every map (homotopy class of maps, actually) from X to Y specifies a homotopy class of $X - Y$ -approaching maps for any choice of M and N . The set of those latter homotopy classes, too, turns out to be independent of that choice, depending instead only on X and Y ; this is why we can call every such homotopy class a *fine shape class* from X to Y . Further, we can define composition of fine shape classes, identity fine shape classes, and so on. From this rough description, it can be seen why fine shape is relatively straightforward to work with; the following sections will necessarily give examples of actual usage of the definition.

Our goal for the present work is to represent fine shape morphisms via (homotopy classes of) ordinary maps. For a restricted class of spaces, it turns out to be possible, if notationally cumbersome, to give a representation of fine shape morphisms into a given space that is functorial in the domain. To elaborate, we eventually restrict our attention to an arbitrary local compactum (that is, a locally compact separable metrizable topological space) X , and construct a metrizable space $|X|$ (which we show to be unique up to a homotopy equivalence) such that for all locally compact metrizable spaces Y , there is a bijection between the set of homotopy classes of continuous maps $[Y, |X|]$ and the set of fine shape classes $[Y, X]_{fSh}$, with the bijection contravariantly functorial in Y .

If we assume X to be a compactum, the construction can be simplified (although it is still homotopy equivalent to the general one); moreover, Y can then be assumed to be any metrizable space at all. As strong shape of compacta is well-studied, there is effectively such a construction already given by Cathey [5], except that it is introduced and used there in a different way. On the other hand, our general construction of $|X|$ is better demonstrated by first defining the simpler compact case, and then modifying it for the broader class of spaces.

This dictates the structure of the present work: after stating the previously known definitions and theorems we rely on in Section 2, we devote Section 3 to the compact case; there we also

explain how our construction of $|X|$ relates to the statements of [5]. Section 4 diverts from the fine shape discussion to establish a particular class of functions we use in the general case; nothing in it is mathematically challenging, but we feel that explaining the notion fully makes the actual general construction clearer, given the somewhat convoluted notation involved. Finally, in Section 5 we define the space $|X|$ for any local compactum X , and prove the combination of properties we claim it to have. The end result of the present work is the following

Theorem 1.3. *(Theorem 5.8, Corollary 5.9, Remark 5.11) For any local compactum X , there exists a metrizable space $|X|$ unique up to a homotopy equivalence and such that for every locally compact metrizable space Y , there is a bijection $[Y, X]_{fsh} \simeq [Y, |X|]$, contravariantly functorial in Y .*

Other results of further interest include the structure of the space $|X|$ itself, showing how to apply this representation in practice, as well as the expected universal homotopy class in $[X, |X|]$, which corresponds under the bijection to the identity class of $[X, X]_{fsh}$. This class is best represented by a particular closed embedding of X into $|X|$ that is also a fine shape equivalence; from our work in [13], we call such maps FDR-embeddings (see Proposition 2.18).

Finally, it should be noted that while we do achieve the goal of using ordinary maps to represent fine shape classes, our construction is not category-theoretical perfect (see Remark 5.11). This is one reason to raise a question of achieving the same result for a different class of spaces; another reason, of course, is to try to find a generalization of $|X|$ to any metrizable space X .

2 Preliminaries

In this section we introduce definitions and conventions we shall use, as well as various previously established results on which we rely.

2.1 Spaces

We work exclusively with metrizable topological spaces and continuous maps between them, so we adopt the following

Convention. By a space, we shall always mean a metrizable topological space, unless specified otherwise. By a map between spaces, we shall always mean a continuous map.

Remark 2.1. It is often convenient to understand the topology of a metrizable space in terms of point convergence; this is applicable due to the following property: on a given set, any two metrizable topologies with the same point convergence (that is, a sequence of points $\{x_n\}_{n \in \mathbb{N}}$ converges to a point x in one topology if and only if it converges to the same point in the other) must coincide. Be reminded that this is not true if even one of these topologies is not metrizable. Similarly, a map $f: X \rightarrow Y$ between metrizable spaces is continuous if and only if for every sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of X converging to a point x , we have $f(x_n) \rightarrow f(x)$ in Y ; we shall make use of this way to prove map continuity.

Notation. We denote the homotopy class of a map f by $[f]$, and we write $f \simeq g$ to mean that maps f and g are homotopic. We denote the set of all homotopy classes from X to Y by $[X, Y]$. For every space X , we denote the identity map of X by id_X . Restriction of a map $f: X \rightarrow Y$ to a subspace $A \subseteq X$, we denote by $f|_A$.

2.2 Approaching maps

Following [9], we construct fine shape using approaching maps, which we define here.

Definition 2.2. Let X be a closed subset of a space M . We say that X is *homotopy negligible* in M to mean that there exists a homotopy $H: M \times [0, 1] \rightarrow M$ such that $H_0 = id_M$ and $H(M \times (0, 1]) \subseteq M \setminus X$. In other words, there is a deformation of M into itself that never crosses X except at the initial (identity) map.

When speaking of homotopy negligible subsets, we will also use a more specific kind of homotopy:

Definition 2.3. Given any homotopy $H: X \times [0, 1] \rightarrow Y$, we say H is *additive* whenever $H_s \circ H_t = H_{\min\{s+t, 1\}}$ for all $s, t \in [0, 1]$. For a space M and a closed subset X of M , we say that X is *additive homotopy negligible* in M whenever there is an additive homotopy $H: M \times [0, 1] \rightarrow M$ such that $H_0 = id_M$ and $H(M \times (0, 1]) \subseteq M \setminus X$.

Definition 2.4. Let X and Y be closed homotopy negligible in spaces M and N respectively, and let $\phi: M \setminus X \rightarrow N \setminus Y$ be a map (continuous on its domain, as per our convention). We say that ϕ is *$X - Y$ -approaching* to mean that for any sequence $\{m_i\} \subset M \setminus X$ converging (in M) to a point of X , the sequence $\{\phi(m_i)\} \subset N \setminus Y$ contains a subsequence that converges (in N) to a point of Y . Equivalently, ϕ being $X - Y$ -approaching means that whenever a sequence in $M \setminus X$ has an accumulation point in X , the sequence's image in $N \setminus Y$ has an accumulation point in Y .

Helpful equivalent definitions can be given from

Proposition 2.5. ([9, Proposition 2.5]) *For a map $\phi: M \setminus X \rightarrow N \setminus Y$, the following are equivalent:*

- (1) *For every open subset U of N , $\phi^{-1}(U \setminus Y)$ is open in M (equivalently, $\phi^{-1}(U \setminus Y)$ is of the form $V \setminus X$ for some open subset V of M);*
- (2) *For every compact subset C of M , $\phi(C \setminus X)$ is contained in a compact subset of N (equivalently, $\phi(C \setminus X)$ is of the form $D \setminus Y$ for some compact subset D of N);*
- (3) *ϕ is $X - Y$ -approaching.*

It is clear that a composition of approaching maps is an approaching map, and that the identity map $id_{M \setminus X}$ is $X - X$ -approaching.

For the rest of the present work, we adopt the following

Convention. Whenever we speak of an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$, we mean that X and Y are closed homotopy negligible subsets of M and N respectively, and ϕ is continuous (on its domain) and $X - Y$ -approaching.

Homotopies of approaching maps are readily defined:

Definition 2.6. Given two approaching maps $\phi, \psi: M \setminus X \rightarrow N \setminus Y$, by an *approaching homotopy* between ϕ and ψ we mean an approaching map $\Phi: (M \times [0, 1]) \setminus (X \times [0, 1]) \rightarrow N \setminus Y$ such that $\Phi_0 = \phi$ and $\Phi_1 = \psi$. Whenever such approaching homotopy exists, we say that ϕ and ψ are *approaching homotopic*, which is an equivalence relation.

2.3 Absolute retracts

The second notion we need to construct fine shape, along with approaching maps, is that of an absolute retract. This, of course, goes back to the Borsuk's definition of shape using absolute neighborhood retracts; here, though, we will not need the latter.

Definition 2.7. By an *absolute retract (AR)*, we shall mean a space M that is, in fact, an absolute extensor for all metrizable spaces in the following sense: given any space X and a closed subset A of X , any map $f: A \rightarrow M$ can be extended to a map $\bar{f}: X \rightarrow M$ such that $\bar{f}|_A = f$:

$$\begin{array}{ccc} A & \xrightarrow{f} & M \\ \downarrow & \nearrow \bar{f} & \\ X & & \end{array}$$

It is known [10, Corollary 18.3] that (at least with respect to metrizable spaces) absolute retracts and absolute extensors (AEs) are exactly the same spaces; by convention, we call them all ARs.

Aside from the definition, we make use of the following facts about ARs:

Proposition 2.8. [10, Chapter 18] (1) Every contractible polyhedron is an AR, including, in particular, the unit interval $[0, 1]$;

(2) A direct product of any countable set of ARs is an AR;

(3) For any AR M and compact metrizable space C , the mapping space M^C , in the compact convergence (equivalently — given that C is compact — uniform convergence) topology, is an AR;

(4) A retract of an AR is an AR;

(5) [10, Theorem 19.3] If X is closed and homotopy negligible in M , then M is an AR if and only if $M \setminus X$ is an AR;

(6) For any (metrizable) space X , there exists an AR M containing X as a closed subset (therefore, $M \times [0, 1]$ contains $X = X \times \{0\}$ as a closed additive homotopy negligible subset, homotopy given by $H_t(m, s) := (m, \min\{s + t, 1\})$).

2.4 Fine shape

The notions of an approaching map and of an absolute retract are combined into the notion of fine shape based on the following fact [9, Lemma 2.13]:

Lemma 2.9. Let X be a closed subset of any space M , and Y be closed homotopy negligible in an AR N . Then

(1) Every map $f: X \rightarrow Y$ extends to a map $\bar{f}: M \rightarrow N$ such that $\bar{f}^{-1}(Y) = X$ (and therefore $\bar{f}|_{M \setminus X}$ is $X - Y$ -approaching);

(2) For any homotopy $F: X \times [0, 1] \rightarrow Y$ and any extensions $\bar{F}_0, \bar{F}_1: M \rightarrow N$ of F_0 and F_1 such that $\bar{F}_0^{-1}(Y) = \bar{F}_1^{-1}(Y) = X$, there is an extension $\bar{F}: M \times [0, 1] \rightarrow N$ such that $\bar{F}(-, 0) = \bar{F}_0$, $\bar{F}(-, 1) = \bar{F}_1$, and $\bar{F}^{-1}(Y) = X$.

Proof. (1) First we can extend f to any map $f': M \rightarrow N$, since N is an AR. Then choose any homotopy $H: N \times [0, 1] \rightarrow N$ such that $H_0 = id_N$ and $H(N \times (0, 1]) \subseteq N \setminus Y$, and any continuous function $h: M \rightarrow [0, 1]$ such that $h^{-1}(0) = X$. From those we define $\bar{f}(m) := H_{h(m)} \circ f'(m)$. Then $\bar{f}(M \setminus X) \subseteq N \setminus Y$, and $\bar{f}|_{M \setminus X}$ is $X - Y$ -approaching (as it actually extends on X by a map into Y), as needed.

(2) $M \times \{0, 1\} \cup X \times [0, 1]$ is a closed subset of $M \times [0, 1]$; we combine the maps $\bar{F}_0$, $\bar{F}_1$, and F into a map of $M \times \{0, 1\} \cup X \times [0, 1]$ into N , then extend it to a map $\bar{F}': M \times [0, 1] \rightarrow N$. Now same as in 1), choose any homotopy $H: N \times [0, 1] \rightarrow N$ such that $H_0 = id_N$ and $H(N \times (0, 1]) \subseteq N \setminus Y$,

and any continuous function $h: M \times [0, 1] \rightarrow [0, 1]$ such that $h^{-1}(0) = M \times \{0, 1\} \cup X \times [0, 1]$, and define $\bar{F}(m, t) := H_{h(m)} \circ \bar{F}'(m, t)$. $\square$

Now fine shape is constructed from the following, which is clearly an equivalence relation:

Definition 2.10. Let X and Y be any two spaces. Let M and M' be ARs, each containing X as a closed homotopy negligible subset, whereas N and N' are ARs each containing Y as such. For any approaching maps $\phi: M \setminus X \rightarrow N \setminus Y$ and $\psi: M' \setminus X \rightarrow N' \setminus Y$, we say that ϕ and ψ are of the same fine shape class (from X to Y) whenever there are some maps $\bar{id}_X: M \rightarrow M'$ and $\bar{id}_Y: N \rightarrow N'$, extending id_X and id_Y , such that $\bar{id}_X^{-1}(X) = X$, $\bar{id}_Y^{-1}(Y) = Y$, and that there is an approaching homotopy (from $(M \setminus X) \times [0, 1]$ to $N' \setminus Y$) between $\bar{id}_Y \circ \phi$ and $\psi \circ \bar{id}_X$, so that the following diagram commutes in approaching homotopy:

$$\begin{array}{ccc} M \setminus X & \xrightarrow{\phi} & N \setminus Y \\ \bar{id}_X|_{M \setminus X} \downarrow & & \downarrow \bar{id}_Y|_{N \setminus Y} \\ M' \setminus X & \xrightarrow{\psi} & N' \setminus Y \end{array}$$

By using Lemma 2.9(1), we can construct fine shape classes from ordinary maps:

Definition 2.11. Given a map $f: X \rightarrow Y$, the fine shape class (from X to Y) induced by f , denoted $[f]_{fsh}$, is defined as follows: take any ARs M and N containing X and Y respectively as closed homotopy negligible subsets, extend f to a map $\bar{f}: M \setminus X \rightarrow N \setminus Y$ such that $\bar{f}^{-1}(Y) = X$, and take the fine shape class of $\bar{f}|_{M \setminus X}$.

There are, however, fine shape classes that are not induced by any ordinary maps.

By definition, every fine shape from X to Y is represented by some approaching map $\phi: M \setminus X \rightarrow N \setminus Y$ for some ARs M and N (containing X and Y respectively as closed homotopy negligible subsets). In fact, however, it can be represented for *any* such choice of ARs:

Lemma 2.12. Let X and Y be any two spaces, M and N be ARs containing X and Y respectively as closed homotopy negligible subsets, and $\phi: M \setminus X \rightarrow N \setminus Y$ be an approaching map. For any two other ARs M' and N' containing X and Y respectively as closed homotopy negligible subsets, there is an approaching map $\psi: M' \setminus X \rightarrow N' \setminus Y$ that is of the same fine shape class as ϕ .

Proof. Take some maps $\bar{id}_X: M' \rightarrow M$ and $\bar{id}_Y: N \rightarrow N'$ extending id_X and id_Y respectively and such that $\bar{id}_X^{-1}(X) = X$ and $\bar{id}_Y^{-1}(Y) = Y$. Then the map $\bar{id}_Y \circ \phi \circ \bar{id}_X: M' \setminus X \rightarrow N' \setminus Y$ is of the same fine shape class as ϕ . $\square$

Corollary 2.13. Fine shape classes are composable: a fine shape class from X to Y , represented by $\phi: M \setminus X \rightarrow N \setminus Y$, and a fine shape class from Y to Z , represented by $\psi: N' \setminus Y \rightarrow L' \setminus Z$, compose through taking any $\phi': M \setminus X \rightarrow N' \setminus Y$ of the same fine shape class as ϕ and taking the fine shape class of $\psi \circ \phi'$, or by taking any map $\psi': N \setminus Y \rightarrow L' \setminus Z$ of the same fine shape class as ψ and taking the fine shape class of $\psi' \circ \phi$ — the two compositions are of the same fine shape class from X to Z .

Thus fine shape can be defined from X to Y without relying on any specific choice of spaces containing them; this is what differentiates fine shape from approaching maps.

By calling onto Lemma 2.9 again, we easily see that fine shape is weaker than homotopy:

Proposition 2.14. *The fine shape class $[f]_{fSh}$ depends only on the homotopy class $[f]$. Composition of maps, or of homotopy classes, induces composition of the corresponding fine shape classes.*

We shall use the following

Notation. For an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$, we shall use $[\phi]$ to denote the fine shape class from X to Y defined by ϕ . For an actual map $f: X \rightarrow Y$, we shall use $[f]_{fSh}$ to denote the fine shape class (again from X to Y) defined by f , or by its homotopy class $[f]$. The set of all fine shape classes from X to Y , we denote by $[X, Y]_{fSh}$.

Some maps induce invertible fine shape classes:

Definition 2.15. A map $f: X \rightarrow Y$ is called a *fine shape equivalence* whenever $[f]_{fSh}$ has an inverse in fine shape — that is, whenever there exists a (necessarily unique) fine shape class in $[Y, X]_{fSh}$, denoted by $[f]_{fSh}^{-1}$, such that $[f]_{fSh}[f]_{fSh}^{-1} = [id_Y]_{fSh}$ and $[f]_{fSh}^{-1}[f]_{fSh} = [id_X]_{fSh}$.

2.5 FDR-embeddings and fine shape cylinders

We shall make use of several constructions that have been introduced for fine shape in [13]. All material in this subsection is quoted directly from there:

Definition 2.16. ([13, Definition 3.1]) Let A be a closed subset of a space X , and assume that:

- there exists an AR M containing X as a closed homotopy negligible subset;
- there exists a closed subset L of M such that L is an AR, $L \cap X = A$, and A is homotopy negligible in L ;
- there exists an $(X \times [0, 1]) - X$ -approaching map $\Phi: (M \setminus X) \times [0, 1] \rightarrow M \setminus X$ such that $\Phi_0 = id_{M \setminus X}$, $\Phi_1(M \setminus X) = L \setminus A$, and $\Phi_t|_{L \setminus A} = id_{L \setminus A}$ for all $t \in [0, 1]$.

In this case we say that the fine shape class $[\Phi]$ is a *fine shape strong deformation retraction* of X on A . We also say that A is a *fine shape strong deformation retract* of X , the inclusion $A \subseteq X$ is an *FDR-embedding*, and Φ is an *approaching strong deformation retraction* representing $[\Phi]$.

In the notation of the definition, Φ_1 can be considered as an approaching map from $M \setminus X$ to $L \setminus A$, and then $[\Phi_1] \in [X, A]_{fSh}$ is the inverse to the fine shape class of the embedding $A \subseteq X$.

Moreover, if $A \subseteq X$ is an FDR-embedding, we can find some choice of M and Φ for any particular choice of L :

Lemma 2.17. ([13, Lemma 3.7]) *Assume the inclusion $A \subseteq X$ is an FDR-embedding, with M , L , and Φ as in Definition 2.16. For any other AR L' containing A as a closed homotopy negligible subset, we can always construct an AR M' containing X and L' as closed subsets with $L' \cap X = A$ and X homotopy negligible in M' , along with an approaching strong deformation retraction Φ' of $M' \setminus X$ onto $L' \setminus A$.*

FDR-embeddings can be characterized in a simple way:

Proposition 2.18. ([13, Theorem 3.13]) *A map is an FDR-embedding if and only if it is a closed embedding and a fine shape equivalence (see Definition 2.15).*

One property we will want to use is given by

Proposition 2.19. ([13, Corollary 3.11]) *If $A \subseteq W$ is an FDR-embedding, then so is $A \times [0, 1] \cup W \times \{0, 1\} \subseteq W \times [0, 1]$.*

Finally, we condense an important result [13, Section 6] in the following form suitable for our purposes:

Theorem 2.20. *For any fine shape class $[\phi] \in [X, Y]_{fSh}$, there exists a space F (called the fine shape cylinder of $[\phi]$), along with closed embeddings $u: X \rightarrow F$ and $i: Y \rightarrow F$, such that i is an FDR-embedding, and $[i]_{fSh}^{-1}[u]_{fSh} = [\phi]$. Moreover, such a space F is unique up to a fine shape equivalence.*

Note that here $[i]_{fSh}^{-1}$ is well-defined, as per Proposition 2.18 and Definition 2.15.

3 Representing fine shape from arbitrary metrizable spaces into a compactum

As it turns out, we want to have the intended construction for compact metrizable spaces, so that we can reference it in defining the more general one; thus this section shall be devoted to describing the more specific case. Fortunately, Cathey’s work [5] already gives the whole construction, although stated in terms we will not need; moreover, Cathey there is almost entirely focused on compact spaces, and the resulting statements will not be directly sufficient for our purposes. As such, here we provide what is effectively the same construction, but restated and explored the way we shall use it — both removing the references to other terms that Cathey needed but we do not, and expanding the properties to include the non-compact spaces.

One difference we should mention explicitly is that Cathey’s definition of an SSDR-map (Definition (1.1) of [5]), given for compacta, has been subsumed by our definition of an FDR-embedding of arbitrary metrizable spaces (see 2.16). With this in mind, we can explain our intent: for a compact metrizable space X , Cathey defines a metrizable (but generally noncompact) space $|X|$ (Theorem (2.5) there) that has two properties. One is that X is FDR-embedded in $|X|$, albeit Cathey does not use that terminology — nor is that called an SSDR-map, as it was not defined for cases where $|X|$ is not compact; we, of course, shall simply state it to be an FDR-embedding. The second property is that for any SSDR-map $i: A \rightarrow W$ and any map $f: A \rightarrow |X|$, there is a map $f': W \rightarrow |X|$ such that $f'i = f$ — that is, any map into $|X|$ extends onto any space containing its original domain via an SSDR-map (Cathey reasonably calls such spaces “fibrant”, using that name — Definition (2.1) — for all metrizable spaces rather than only for compact ones). For our purposes, this is insufficient strictly as stated: we want to use the same property, but with FDR-embeddings instead of only SSDR-maps, meaning spaces A and W may be noncompact, and that is precisely the way we state it after we define the space $|X|$ in suitable terms. We also provide the proofs for everything we state; while most of those could in principle be inferred from Cathey’s work, the wide difference in approaches between there and here would be quite demanding. As such, we greatly prefer to have all proofs in the form compatible with our methodology.

Having stated our goal for this section, we can start with the constructions we shall use:

Definition 3.1. Let M be a metrizable topological space, X a compact subset of M , and d a metric giving the topology of M . We denote by $d_{M/X}$ the metric on the quotient space M/X defined by $d_{M/X}(m, \{X\}) := d(m, X)$ and $d_{M/X}(m, m') := \min\{d(m, m'), d(m, \{X\}) + d(m', \{X\})\}$ for all $m, m' \in M$.

Remark 3.2. Clearly $d_{M/X}$ is a metric, and gives the quotient topology on M/X . The latter is not true if X is not compact; we return to that in Definition 5.1. Also, the set $M \setminus X$ has the same topology whether considered as the subset of M or the subset of M/X .

Definition 3.3. (cf. [5, Theorem (2.5)]) Let X be a compact metrizable space, and choose a compact AR M containing X as a closed additive homotopy negligible subset. In the path space $(M/X)^{[0,1]}$, denote by $|M|$ the subset of all paths $\gamma: [0, 1] \rightarrow M/X$ such that $\gamma((0, 1]) \subseteq M \setminus X$. Also denote by $|X|$ the subset of $|M|$ consisting of all γ with $\gamma(0) = \{X\}$.

Remark 3.4. (1) Note how a map $\phi: [0, 1] \setminus \{0\} \rightarrow M \setminus X$ is an approaching map if and only if the obvious corresponding map from $[0, 1]$ to M/X is continuous, therefore a path in M/X , thus a point of $|X|$;

(2) In accordance with Remark 2.1, it is convenient to understand the topology of $|M|$ in terms of point convergence: $|M| \setminus |X|$ is homeomorphic to the path space of $M \setminus X$, and has the same point convergence; and for a point $\gamma \in |X|$, a sequence $\{\gamma_n\}$ of points of $|M|$ converges to γ if and only if (a) for every $s \in (0, 1]$, every open neighborhood of $\gamma(s)$ in M (or even in $M \setminus X$) contains $\gamma_n([s - \sigma, s + \sigma])$ for some $\sigma > 0$ and all n large enough, and (b) every open neighborhood of X in M contains $\gamma_n((0, \sigma])$ for some $\sigma > 0$ and all n large enough. Be warned that condition (b) cannot be replaced by requiring that $\gamma_n(0) \rightarrow \{X\}$, $n \rightarrow \infty$, in M/X .

(3) Combining (1) and (2), we can see that a path $\Gamma: [0, 1] \rightarrow |X|$ is precisely an approaching map $\Gamma': ([0, 1] \times [0, 1]) \setminus ([0, 1] \times \{0\}) \rightarrow M \setminus X$, where $[\Gamma(t)](s) = \Gamma'(t, s)$.

Proposition 3.5. *In the notation of the preceding definition,*

- (1) $|M|$ is an AR containing $|X|$ as a closed additive homotopy negligible subset;
- (2) There is an FDR-embedding of X in $|X|$, with an approaching strong deformation retraction of $|M| \setminus |X|$ onto $M \setminus X$;
- (3) For any FDR-embedding $i: A \rightarrow W$ and any map $f: A \rightarrow |X|$, there is a map $f': W \rightarrow |X|$ extending f (that is, $f'i = f$).

Remark 3.6. By the universal property given by (2) and (3), for any space X , $|X|$ is unique up to a homotopy equivalence that is constant on X ; we elaborate on this for the general case in Corollary 5.9(2).

Proof (of Proposition 3.5). (1) Clearly $|X|$ is closed in $|M|$, and also additive homotopy negligible via the homotopy $\Gamma: |M| \times [0, 1] \rightarrow |M|$ defined by $[\Gamma(\gamma, s)](t) := \gamma(\max\{s, t\})$. Now $|M| \setminus |X|$ is just $(M \setminus X)^{[0,1]}$. But $M \setminus X$ is an AR by Proposition 2.8(5), and then by Proposition 2.8(3) so is the space of paths in it; therefore $|M| \setminus |X|$ is an AR, and by Proposition 2.8(5) again so is $|M|$.

(2) Let $H: M \times [0, 1] \rightarrow M$ be any additive homotopy with $H_0 = id_M$ and $H(M \times (0, 1]) \subseteq M \setminus X$. Embed X into $|X|$ by sending $x \in X$ to $\gamma_x: [0, 1] \rightarrow M/X$ for which $\gamma_x(t) := H(x, t) \in M \setminus X$ for $t \in (0, 1]$, and $\gamma_x(0) := \{X\}$ (note that this embedding is injective due to Hausdorff property of M). Extend this to the embedding of M into $|M|$ by $\gamma_m(t) := H(m, t)$ for $m \in M \setminus X$. Now the approaching strong deformation retraction $R: (|M| \setminus |X|) \times [0, 1] \rightarrow |M| \setminus |X|$ can be defined by

$$[R(\gamma, s)](t) := \begin{cases} \gamma(t), & t \leq s; \\ H(\gamma(s), t - s), & t > s. \end{cases}$$

R is continuous, and is constant on the copy of $M \setminus X$ in $|M| \setminus |X|$ because H was chosen additive; now assume a sequence $\{\gamma_n\}$ in $|M| \setminus |X|$ converges to a point $\gamma \in |X|$, and also $s_n \rightarrow s$. For

$s \neq 0$, $\{R(\gamma_n, s_n)\}$ simply converges to the path $\gamma^{(s)} \in |X|$ defined by

$$\gamma^{(s)}(t) := \begin{cases} \gamma(t), & t \leq s; \\ H(\gamma(s), t-s), & t > s \end{cases}$$

(in other words, Γ extends continuously onto $|X| \times (0, 1]$). For $s = 0$, the points $\gamma_n(s_n)$, converge to X in M (and then to $\{X\}$ in M/X), and thus have an accumulation point $x \in X$ in it by compactness (this is where the construction fails if X is noncompact). Therefore also the paths $R(\gamma_n, s_n)$ have the path γ_x as an accumulation point in $|M|$; note that the sequence $\{R(\gamma_n, s_n)\}$ satisfies condition (b) of Remark 3.4(2) because the sequence $\{\gamma_n\}$ must do so to converge to a point of $|X|$.

(3) Assume we are given a map $f: A \rightarrow |X|$ and an FDR-embedding $i: A \rightarrow W$. We need to extend f to a map $f': W \rightarrow |X|$. Let L' be any AR containing A as a closed homotopy negligible subset and take $L := L' \times [0, 1]$, an AR containing $A = A \times \{0\}$ in the same way, with homotopy $H: L \times [0, 1] \rightarrow L$ defined by $H((l, t), s) := (l, \max\{t, s\})$, where $(l, t) \in L' \times [0, 1] = L$. Now f extends to some map $F: L \rightarrow |M|$ such that $F^{-1}(|X|) = A$ and also for all $(a, s) \in A \times [0, 1] \subset L$, $F(a, s) = F(H((a, s))) = [f(a)](s)$ (note that replacing L' by $L' \times [0, 1]$ was done to ensure that H is injective on $A \times [0, 1]$). Using Lemma 2.17, we can choose some N , $\bar{H}$, and R' , where N is an AR containing W and L as closed subsets, with $L \cap W = A$, $\bar{H}: N \times [0, 1] \rightarrow N$ is a homotopy extending H in such way that $\bar{H}_0 = id_N$ and $\bar{H}(N \times (0, 1]) \subseteq N \setminus W$, and $R': (N \setminus W) \times [0, 1] \rightarrow N \setminus W$ is an approaching strong deformation retraction of $N \setminus W$ onto $L \setminus A$. Then we define $f': W \rightarrow |X|$ via $[f'(p)](t) := RFR'H(p, t)$, $t \in (0, 1]$; now f' restricts to f by the construction of F , and by R' and R being constant on $L \setminus A$ and $M \setminus X$ respectively. Moreover, f' is a continuous map, and it is important to understand why: by Remark 2.1, assume we have $w_n \rightarrow w$ in W , and see how $f'(w_n) \rightarrow f'(w)$. The condition (a) of Remark 3.4(2) is satisfied because H , R' , F , and R are all continuous on their respective domains; for the condition (b), note that if we take an open neighborhood U of X in M , then the preimage $(RFR')^{-1}(U \cap (M \setminus X)) \subseteq N \setminus W$ must contain $V \cap (N \setminus W)$ for some open neighborhood V of w in N (otherwise, there is a sequence v_n of points of $N \setminus W$ such that $v_n \rightarrow w$, but $\{RFR'(v_n)\}_{n \in \mathbb{N}}$ has no accumulation points in X , contradicting the fact that R , F , and R' are all approaching); but by $w_n \rightarrow w$ and H being continuous, V contains $H(\{w_n\} \times [0, \sigma])$ for some $\sigma > 0$ and all n large enough, thus U also contains $[f'(w_n)]((0, \sigma])$ for all such n . $\square$

Remark 3.7. For noncompact X , the construction of $|X|$ fails in multiple ways. The most obvious problem is that M/X is not metrizable, but that can be redefined (see Definition 5.1). Another, mentioned in the preceding proof, is that the retraction R_1 is not $|X| - X$ -approaching anymore: there must exist a sequence $\{m_n\}$ of points of M converging to X yet without any accumulation points in it (or indeed in all of M). Choose a sequence of open neighborhoods U_n of X in M such that $m_n \in U_n$ and $U_{n+1} \subseteq U_n$ for all n , and moreover $\bigcap_{n \in \mathbb{N}} U_n = X$ (we can do so because X is a closed subset of a metrizable — therefore perfectly normal — space). Now take a path $\gamma \in |X|$, and define a sequence of paths $\{\gamma_n\}$ in $|M| \setminus |X|$ so that $\gamma_n|_{[1/n, 1]} = \gamma|_{[1/n, 1]}$ and $\gamma_n([0, 1/n]) \subseteq U_n$ for all n (and therefore $\gamma_n \rightarrow \gamma$ in $|X|$), but $\gamma_n(0) = m_n$; then the sequence $\{R(\gamma_n, 1)\}$ does not have an accumulation point in the copy of X in $|X|$. It is easy, using Definition 5.1, to construct such an example with $M = (0, 1) \times [0, 1]$, $X = (0, 1) \times \{0\}$, $m_n = (1/n, 1/n)$, $\gamma(s) = (1/2, s)$ ($\gamma(0) = \{X\}$, in the set M/X with the defined metrizable topology), and $\gamma_n|_{[0, 1/n]}(s) = (1/n + s(1/2 - 1/n), 1/n)$, assuming the metric is chosen in the obvious way so that we can take $U_n = (0, 1) \times [0, 1/n]$. There is, in fact, yet another problem: it might not be possible to choose a metric topology for M/X such that $|X|$ is homotopy negligible in $|M|$.

Avoiding all of these obstacles at once seems to require not only some changes in the construction, but also placing additional restrictions on X ; this is what Section 5 explores.

One more consideration is that $|X|$ is not compact despite X being so; this is to be expected, as $|X|$ is effectively the strong shape version of a space of paths (from one-point space to X). Due to this, Cathey had to use this construction in a roundabout way, not being able to utilize the whole space $|X|$ directly, and this is also what prevents building the model structure while working with compact spaces only (because model structure implies existence of path spaces). We, however, seeking specifically to work also with noncompact spaces, will not be hindered by $|X|$ not being compact in the present work.

Thus by the end of this section, we have the space $|X|$ for every compact metrizable space X defined and studied in terms we need. Our goal from now on is to give an equivalent construction for all locally compact metrizable spaces; for this, we will first explore a particular class of function on such spaces that will be used in our construction.

As a note, we did not yet actually show that the space $|X|$ is in a sense unique for any given X ; we shall prove this in the general case.

4 Exhaustion functions

This section describes the construction of functions of a specific kind on metrizable topological spaces. The statements and proofs given here are easy to understand, but still necessary for the following section; thus, we provide them in full for clarity.

We begin by stating a standard form of some definitions we shall use:

Definition 4.1. For a topological space X , we say that X is locally compact whenever every point of X has an open neighborhood with compact closure; we say that X is separable whenever X is the closure of a countable subset of X ; by a local compactum, we mean a locally compact separable metrizable space; and if X is locally compact and Hausdorff, we use $X^* := X \cup \infty$ to denote the one-point closure of X .

Now we define what we shall call an exhaustion function:

Definition 4.2. Let X be a metrizable topological space. A continuous function $u: X \rightarrow (0, 1]$ will be called an *exhaustion function* on X if $u(x_n) \rightarrow 0$, $n \rightarrow \infty$, for every sequence x_n of points of X that has no accumulation points in X .

Clearly, such a function $u: X \rightarrow (0, 1]$ gives an exhaustion of X by the non-decreasing sequence of compact sets $u^{-1}([\frac{1}{n}, 1])$, $n = 1, 2, \dots$. In fact, such a function is effectively a continuous equivalent of an exhaustion by sequence, assigning a compact set $u^{-1}([\varepsilon, 1])$ to every $\varepsilon \in (0, 1]$; this, of course, is the reason for the name. For completeness, we give a thorough description of when exhaustion functions exist:

Proposition 4.3. For a noncompact metrizable space X , the following are equivalent:

- (1) There exists an exhaustion function on X ;
- (2) X is the direct limit of a sequence of its compact subspaces $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq \dots$ with the inclusion maps (and therefore, $X = \bigcup_{n \in \mathbb{N}} X_n$) such that X_n is contained in the interior (in X) of X_{n+1} for all n ;
- (3) X is locally compact and separable (that is, a local compactum);

- (4) X is locally compact, and the point ∞ of X^* has a countable basis of open neighborhoods;
- (5) X is locally compact, and X^* is first-countable;
- (6) X is locally compact, and X^* is perfectly normal;
- (7) X is locally compact, and X^* is metrizable.

Proof. (1) $\Rightarrow$ (2) Given an exhaustion function u , take $X_n := u^{-1}([1/n, 1])$; it is compact from the defining property of an exhaustion function. Moreover, $u^{-1}((1/(n+1), 1])$ is open, so $X_n \subseteq u^{-1}((1/(n+1), 1]) \subseteq X_{n+1}$ shows that X_n is contained in the interior of X_{n+1} .

(2) $\Rightarrow$ (3) Each X_n is separable, being compact and metrizable; thus X , a countable union of all X_n , is also separable. Now every point of x is contained in X_n for some n , and thus has an open neighborhood U contained in the interior of X_{n+1} ; the closure of U is then compact (contained in X_{n+1} , in fact).

(3) $\Rightarrow$ (4) Let Y be a countable dense subset of X . Call an open subset U of X *marked* whenever it satisfies any (and then every) of the following equivalent conditions:

- U has compact closure in X ;
- the closure of U in X^* does not contain ∞ ;
- ∞ has an open neighborhood in X^* disjoint from U .

Every point of X has a marked neighborhood, and every open subset of a marked set is marked; choose a metric d giving the topology of X , and let F be the collection of all marked open d -balls $B(y, 1/n)$ of radius $1/n$, $n = 1, 2, \dots$, with center $y \in Y$. Then F is countable, and the sets of F cover X : for a point $x \in X$, some open ball $B(x, 2\varepsilon)$ has compact closure (in X), and $B(x, \varepsilon)$ contains some $y \in Y$ (as Y is dense in X), so $B(y, \varepsilon)$ is marked (thus in F) and contains x . Now enumerate the sets in F in some way as $U_0, U_1, \dots, U_n, \dots$; for every $n \in \mathbb{N}$, take V_n to be an open neighborhood of ∞ in X^* disjoint from U_n , and define $W_n := \bigcap_{k=0}^n V_k$. Then W_n are open neighborhoods of ∞ with $W_{n+1} \subseteq W_n$ for all n . To see that W_n form a basis of the system of open neighborhoods of ∞ , take an arbitrary open neighborhood V of ∞ ; the compact set $X^* \setminus V$ is covered by a finite subcollection of F , say by the sets $U_0, \dots, U_n$. Then W_n is a open neighborhood of ∞ disjoint from each of these, therefore disjoint from $X^* \setminus V$, thus contained in V .

(4) $\Rightarrow$ (5) X , being metrizable, is first countable, and thus every point $x \in X$ also has a basis of the system of its open neighborhoods in X^* , by first restricting to an open neighborhood with compact closure in X . Since ∞ also has such a basis by the assumption, this makes X^* first countable.

(5) $\Rightarrow$ (6) Let $\{W_n\}$ be a countable basis of the system of open neighborhoods of ∞ in X^* ; then $\bigcap_{n \in \mathbb{N}} W_n = \{\infty\}$ (as X^* is Hausdorff). Given a closed set Z in X^* , $Z \setminus \{\infty\}$ is closed in X , thus (as X is metrizable and so perfectly normal) an intersection of some countable collection of sets U_n that are open in X — and thus also in X^* . Now if $\infty \notin Z$ or, equivalently, $Z = Z \setminus \{\infty\}$, then $Z = \bigcap_{n \in \mathbb{N}} U_n$ also proves Z to be a G_δ set in X^* ; otherwise, this is proved by $Z = (Z \setminus \{\infty\}) \cup \{\infty\} = \bigcap_{n \in \mathbb{N}} (U_n \cup W_n)$.

(6) $\Rightarrow$ (7) It is a well-known consequence of the Urysohn's construction that for any normal topological space X and any closed G_δ subset Z of X , there is a continuous function $f: X \rightarrow [0, 1]$ such that $Z = f^{-1}(0)$ (whereas for Z not G_δ , we can only ensure that $Z \subseteq f^{-1}(0)$). Thus let $u: X^* \rightarrow [0, 1]$ be continuous with $u^{-1}(0) = \{\infty\}$. Now $d^u(x, x') := |u(x) - u(x')|$ is a pseudometric on X ; if the topology of X is given by a metric d , then $d + d^u$ is also a metric giving the same topology. Define a metric d^* on X^* by $d^*(\infty, x) := u(x)$ and $d^*(x, x') := \min\{u(x) + u(x'), d(x, x') + |u(x) - u(x')|\}$ for all $x, x' \in X$. After checking that the triangle inequality holds, we see that d^* is in fact a metric, and it gives the topology of X^* : the d^* -open

balls $B_{d^*}(\infty, c)$ are open in X^* as their complements $u^{-1}([c, 1])$ are compact for $c > 0$, and form the basis of the system of open neighborhoods of ∞ , since for any open neighborhood V of ∞ , u has a nonzero minimum c on the compact set $X^* \setminus V$, so V contains $B_{d^*}(\infty, c)$.

(7) $\Rightarrow$ (1) For any metric d^* providing the topology of X^* , $d^*(\infty, -): X \ni x \mapsto d^*(\infty, x) \in (0, 1]$ is an exhaustion function on X . $\square$

Remark 4.4. As we can see now, all local compacta — and only local compacta — have exhaustion functions. For a compactum, the constant function equal to 1 everywhere is an exhaustion function; for a non-compact local compactum, the image of any exhaustion function must contain, more or less by definition, values arbitrarily close to 0.

5 Representing fine shape from locally compact spaces into a local compactum

Now we are in position to achieve our aim: while Section 3 was effectively a restatement of Cathey's relevant results of [5] in terms we use, here we shall extend the same to the case of non-compact spaces. Our approach relies on exhaustion functions, which exist precisely on locally compact separable metrizable spaces (local compacta), and therefore these will constitute the class of spaces we study.

We proceed in the same way as in Section 3, modifying our expanded definitions according to the differences caused by the loss of compactness:

Definition 5.1. Let X be a closed subset of a metrizable space M . For any metric d giving the topology of M , define a metric $d_{M/X}$ on the set M/X by $d_{M/X}(m, \{X\}) := d(m, X)$ and $d_{M/X}(m, m') := \min\{d(m, m'), d(m, X) + d(m', X)\}$ for all $m, m' \in M \setminus X$.

Remark 5.2. As noted in Remark 3.2, for noncompact X , different choices of d will result in topologically different metrics $d_{M/X}$. However, for any closed subset M' of M with $M' \cap X$ compact, the inherited topology on $M'/(M' \cap X)$ is the quotient topology (although the metric $d_{M/X}$ restricted to $M'/(M' \cap X)$ is not the same as $d_{M'/(M' \cap X)}$).

Lemma 5.3. *For every local compactum X , there exist a locally compact separable AR M that contains X as a closed additive homotopy negligible subset, an exhaustion function u on M (and then $u|_X$ is an exhaustion function on X), and an additive homotopy $H: M \times [0, 1] \rightarrow M$ such that $H_0 = \text{id}_M$, $H(M \times (0, 1]) \subseteq M \setminus X$, and also $u(H(m, s)) \geq u(m)$ for all $m \in M$ and all $s \in [0, 1]$. In fact, it is always possible to have $u(H(m, s)) = u(m)$ for all m and all s .*

Proof. By [9, Lemma 3.18], we can choose M that is locally compact and separable, and contains X as a closed subset; now for any exhaustion function u on M , replace M by $M \times [0, 1]$ (with $X = X \times \{0\}$), u by $(m, s) \mapsto u(m)$, and take $H((m, s), t) := (m, \max\{s, t\})$. $\square$

Definition 5.4. Let X be a local compactum, and choose M , H , and u as given by the previous lemma. For every $\varepsilon \in (0, 1]$, define compact subspaces $M_\varepsilon := u^{-1}([\varepsilon, 1])$ and $X_\varepsilon := M_\varepsilon \cap X$. Definition 3.3 gives us spaces $|M_\varepsilon|$ and $|X_\varepsilon|$; any metric d giving the topology of M metrizes all these via restricting $d_{M/X}$, as per Remark 5.2 and Definition 3.3; thus we have an isometric embedding of $|M_\varepsilon|$ in $|M'_\varepsilon|$ (as metric spaces) whenever $\varepsilon > \varepsilon'$. Finally, define $|M|$ to be the union $\bigcup_{\varepsilon \in (0, 1]} \{\varepsilon\} \times |M_\varepsilon|$, metrizable by $d_{|M|}((\varepsilon, \gamma), (\varepsilon', \gamma')) := |\varepsilon - \varepsilon'| + d_{|M_{\min\{\varepsilon, \varepsilon'\}}|}(\gamma, \gamma')$; here, we rely on the isometric embeddings for the second term to be well-defined.

Remark 5.5. For compact X , we can choose M compact, and u everywhere equal to 1; then $|M|$ will be the same as $|M|$ of Definition 3.3 multiplied by $(0, 1]$. As stated later in Corollary 5.9, $|M|$ is in general unique up to a homotopy equivalence; thus, up to the same, Definition 5.4 restricts to Definition 3.3 on compacta.

For this new definition, we want to extend the properties of Proposition 3.5:

Proposition 5.6. *In the notation of the preceding definition,*

- (1) $|M|$ is an AR containing $|X|$ as a closed additive homotopy negligible subset;
- (2) There is an FDR-embedding of X into $|X|$, extending to a closed embedding of M into $|M|$ with $X = |X| \cap M$ and an approaching strong deformation retraction of $|M| \setminus |X|$ onto $M \setminus X$.

Proof. (1) As in the proof of Proposition 3.5(1), for all $\varepsilon \in (0, 1]$, $|M_\varepsilon|$ is an AR containing $|X_\varepsilon|$ as a closed additive homotopy negligible subset; the same homotopy Γ proves that $|X|$ is additive homotopy negligible in $|M|$: using Remark 2.1, assume that $(\varepsilon_n, \gamma_n) \rightarrow (\varepsilon, \gamma)$, and see that for some $\varepsilon' \in (0, \varepsilon]$, $|M_{\varepsilon'}|$ contains γ_n for all large enough n , and then for any converging sequence $s_n \rightarrow s$ in $[0, 1]$, $\Gamma(\gamma_n, s_n) \rightarrow \Gamma(\gamma, s)$ in $|M_{\varepsilon'}|$ (be warned that the existence of such ε' is important for continuity here). Now the space $(M \setminus X)^{[0, 1]}$ of paths in $M \setminus X$ is an AR by Proposition 2.8(3), and $|M| \setminus |X|$ is a retract of $(0, 1] \times (M \setminus X)^{[0, 1]}$ via $(q, \gamma) \mapsto (\min\{q, \inf_{s \in [0, 1]} u(\gamma(s))\}, \gamma)$; therefore by Proposition 2.8(5) and (2) and (4), $|M| \setminus |X|$ is an AR, and again by (5) so is $|M|$.

(2) Embed M into $|M|$ via $m \mapsto (u(m), \gamma_m)$, where $\gamma_m(s) := H(m, s)$; here for $m \in X$, $\gamma_m(0) := \{X_{u(m)}\} \in M_{u(m)}/X_{u(m)}$, and also note that $u(\gamma_m(s)) \geq u(m)$ for all $s \in (0, 1]$, since u and H were chosen via Lemma 5.3. Also denote by M' the subset $\{(q, \gamma_m) \mid m \in M, u(m) \geq q\}$ of $|M|$ and take $X' := |X| \cap M'$; clearly $M \subseteq M' \subseteq |M|$, and there is a strong deformation retraction of M' onto M that restricts to a strong deformation retraction of X' onto X . Thus it suffices to construct an approaching strong deformation retraction of $|M| \setminus |X|$ onto $M' \setminus X'$.

Now, similar to Proposition 3.5(2), we define that approaching strong deformation retraction $R: (|M| \setminus |X|) \times [0, 1] \rightarrow |M| \setminus |X|$ by $R((q, \gamma), s) := (q, \gamma_s)$, where

$$\gamma_s(t) := \begin{cases} \gamma(t), & t \leq s; \\ H(\gamma(s), t - s), & t > s; \end{cases}$$

note that for $t \in [s, 1]$, $u(\gamma_s(t)) \geq u(\gamma(s)) \geq q$ due to how u and H were chosen (also, $u(\gamma(0))$ is always defined and a point of $M \setminus X$, as $\gamma \in |M| \setminus |X|$), so that $R_s(q, \gamma)$ always stays in $|M| \setminus |X|$. Then R is continuous, the image of R_1 is precisely $M' \setminus X'$, and R stays constant on $M' \setminus X'$ (again, because H was chosen additive), so it remains to show that R is approaching. This is shown as in Proposition 3.5(2), with one difference: if a sequence $\{(q_n, \gamma_n)\}$ in $|M| \setminus |X|$ converges to $(q, \gamma) \in |X|$ and $s_n \rightarrow 0$ in $[0, 1]$, then $u(\gamma_n(s_n)) \geq q_n$, so the sequence $\{\gamma_n(s_n)\}$ (in M) has an accumulation point in the compact subset $u^{-1}([q, 1]) \cap X = X_q$ of M . $\square$

Remark 5.7. We know that the topological quotient space M/X may not be metrizable for a noncompact X , and that different choices of a metric d on M lead to different topologies on the set M/X . Even then, it would be possible to extend $|X|$ of Definition 3.3 by defining it to be the space of all approaching maps $\gamma: [0, 1] \setminus \{0\} \rightarrow M \setminus X$ with the compact convergence topology; in the compact case, this is homeomorphic to the path space of Definition 3.3. Even in the noncompact case, the proof of Proposition 3.5(3) works without changes. Finally, X clearly embeds in that space in much the same way as in Definition 5.4. The last step of the preceding proof, however, shows why that embedding is not an FDR-embedding, thus invalidating the whole approach: the sequence $\gamma_n(s_n)$ might not have an accumulation point in the noncompact

space M , which is why we first introduce some exhaustion of M by compact subspaces and define a topology that relies on it.

At this point, we obviously want to state the equivalent of Proposition 3.5(3) to complete what we claim to be an extension of the whole Section 3. This is the primary result of the present work, and it will be different in two notable ways. First, we will show extension of maps only to within a homotopy; we believe this to be reasonable, because the space $|X|$, as we are about to show, is unique (for a given X) up to a homotopy equivalence — not up to a homeomorphism, not even for a compact X . As to the second difference, the proof will, of course, require the existence of exhaustion functions on X , meaning X must be a local compactum — a separable locally compact metrizable space. The spaces A and W (as in the compact version), on the other hand, need not be separable, and yet it is telling that the following proof does require those to be locally compact:

Theorem 5.8. *For a local compactum X , $|X|$ has the following property: for every map $f: A \rightarrow |X|$ and every FDR-embedding $i: A \rightarrow W$ with W (and then also A) locally compact, there is a map $f': W \rightarrow |X|$ such that $f'|_A \simeq f$. Moreover, any two such maps are homotopic.*

Corollary 5.9. (1) *For any local compactum X and every locally compact metrizable space Y , there is a bijection $[Y, X]_{fsh} \cong [Y, |X|]$;*

(2) *For any local compactum X , the space $|X|$ is unique up to a homotopy equivalence.*

Remark 5.10. The first part of the corollary states that $|X|$ is the classifying space for fine shape classes from locally compact spaces into X . The second part shows why we can talk about *the* space $|X|$ for a given X , regardless of the choices made in the construction of Definition 5.4. This was first mentioned in Remark 3.6, and now we prove it in the general case.

Proof (of Corollary 5.9). (1) For a class $[\phi] \in [Y, X]_{fsh}$ represented by any $Y - X$ -approaching map ϕ , apply the theorem with $A = X$, W the fine shape cylinder of ϕ (see Theorem 2.20), and f the embedding (of X in W); restricting to the copy of Y in W singles out exactly one class in $[Y, |X|]$. Conversely, an element of $[Y, |X|]$ can be concatenated with the fine shape inverse to the embedding (of X into $|X|$) to obtain an element of $[Y, X]_{fsh}$. Uniqueness up to a homotopy implies that the two constructions are inverse to each other, thus defining the bijection.

(2) Routine proof: $|X|$ is the universal space characterized by Proposition 5.6(2) and the theorem. $\square$

Proof (of Theorem 5.8). Since every point of $|X|$ is a pair (q, γ) , we have the first coordinate map $f_1: A \rightarrow (0, 1]$, and, for every point $a \in A$, a point γ_a of every space $|X_\varepsilon|$ with $\varepsilon \in (0, f_1(a)]$. Following the proof of Proposition 3.5(3), we can extend the latter to assign to every point $w \in W$ a point γ_w of every space $|X_\varepsilon|$ for all ε small enough; this works because (in the notation of that proof) H is continuous, and RFR' is a composition of approaching maps. This assignment is continuous wherever it is defined, same as in that proof.

Now if we construct some continuous map $f'_1: W \rightarrow (0, 1]$ such that $\gamma_w \in |X_{f'_1(w)}|$ for all $w \in W$, then this immediately gives us the map f' we need; the homotopy $f'|_A \simeq f$ can be taken linear on the first coordinate of each pair, and constant on the second. To that end, in the product $W \times [0, 1]$, consider the set C of all pairs (w, q) such that $\gamma_w((0, 1]) \subseteq u^{-1}([q, 1])$; we claim that C contains some open neighborhood V of $W \times \{0\}$. Then by [10, Lemma 18.15(a)], there must be some continuous map $p: W \rightarrow [0, 1]$ such that $p^{-1}(0)$ is actually empty, and the graph of p is contained in V . From that we will be able to define $f'_1 := p$, finishing the proof.

To verify the claim, let $w \in W$. By local compactness, w has an open neighborhood U in W with compact closure $\bar{U}$, and $H(\bar{U} \times [0, 1])$ is then compact in N (again, using the notation

from the proof of Proposition 3.5(3)). Then by Proposition 2.5(2), the image of $H(\bar{U} \times (0, 1])$, a subset of $N \setminus W$, under the approaching map RFR' is contained in a compact subset of M , and thus in $u^{-1}[\varepsilon, 1]$ for some $\varepsilon > 0$. Therefore $U \times [0, \varepsilon] \subseteq C$; clearly $W \times \{0\}$ is covered by products of such form in $W \times [0, 1]$, the union of which can be taken as V . This ends the construction of f' .

Finally, assume we have two maps f' and f'' from W to $|X|$ such that $f'|_A \simeq f \simeq f''|_A$. By using those homotopies, we can construct a map $g: W \times \{0, 1\} \cup A \times [0, 1] \rightarrow |X|$; by Proposition 2.19 and the construction of the current proof, we can obtain a map $g': W \times [0, 1] \rightarrow |X|$ such that its restriction to the domain of g is homotopic to g . Denoting such a homotopy by G , we have a concatenation of homotopies $G|_{W \times \{0\}} \circ g' \circ G|_{W \times \{1\}}$ proving that $f' \simeq f''$. $\square$

Remark 5.11. It can be seen from the proofs that if we write i_X for the FDR-embedding of X into $|X|$, the bijection $[Y, X]_{fSh} \cong [Y, |X|]$ works by sending a homotopy class $[f] \in [Y, |X|]$ to $[i_X]_{fSh}^{-1}[f]_{fSh}$, and for a fine shape class $[\phi] \in [Y, X]_{fSh}$, the corresponding homotopy class in $[Y, |X|]$ turns into the fine shape class $[\phi][i_X]_{fSh}$ in $[Y, |X|]_{fSh}$. It is now easy to see that, for fixed X , the bijection is contravariantly functorial in Y — or simply functorial, if Y is taken as an object of the (opposed) category hLCM^* of locally compact metrizable spaces and *reversed* homotopy classes of continuous maps. Moreover, the homotopy class $[i_X]$ corresponds to the identity fine shape class $[id_X]_{fSh} \in [X, X]_{fSh}$. This, of course, is simply a case of the Yoneda lemma, as we represent the functor $[-, X]_{fSh}: \text{hLCM}^* \ni Y \mapsto [Y, X]_{fSh} \in \text{Set}$ into the usual set category by $[-, |X|]$; this, too, is why we have called our construction a representation.

Still, it is not precisely a representation in the strictest category-theoretical sense, because $|X|$ is not locally compact, so $[Y, |X|]$ is not a morphism set in the same category as Y . This suggests that a further search should be conducted to find a good class of spaces such that, at least, for any space X of that class, we can construct $|X|$ belonging to a (possibly different, although preferably the same) class from which we can also choose any space Y and still obtain the same bijection.

References

- [1] V. Baladze, A. Beridze, and R. Tsinaridze. Strong shape theory of continuous maps. 01 2020. URL <https://www.researchgate.net/publication/338898450>.
- [2] M. Batanin. Categorical strong shape theory. *Cahiers Topol. Géom. Diff. Cat.*, volume 38:pages 3–66, 1997. ISSN 1245-530X. URL https://www.researchgate.net/publication/2659900_Categorical_Strong_Shape_Theory.
- [3] F. W. Bauer. Some relations between shape constructions. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, volume 19(4):pages 337–367, 1978. ISSN 1245-530X. URL <https://eudml.org/doc/91206>.
- [4] A. Calder and H. M. Hastings. Realizing strong shape equivalences. *J. Pure and Appl. Alg.*, volume 20:pages 129–156, 02 1981. URL [http://dx.doi.org/10.1016/0022-4049\(81\)90088-8](http://dx.doi.org/10.1016/0022-4049(81)90088-8).
- [5] F. W. Cathey. Strong shape theory. In S. Mardesic and J. Segal (editors), *Shape theory and geometrical topology*, volume 870 of *Lecture Notes in Math.*, pages 215–238. Springer, 1981. URL <https://doi.org/10.1007/BFb0089717>.

- [6] J. Dydak and J. Segal. *Strong shape theory*. Instytut Matematyczny Polskiej Akademii Nauk, 1981. URL <https://eudml.org/doc/268436>.
- [7] Y. Kodama and J. Ono. On fine shape theory. *Fund. Math.*, volume 105(1):pages 29–39, 1979. ISSN 0016-2736. URL <https://eudml.org/doc/211079>.
- [8] S. Mardešić. *Strong Shape and Homology*. Springer-Verlag Berlin Heidelberg, 2000. ISBN 978-3-642-08546-8.
- [9] S. Melikhov. Fine shape I, 2022. URL <https://arxiv.org/abs/1808.10228v2>.
- [10] S. Melikhov. *Topology of Metric Spaces*. 11 2022. URL <https://www.researchgate.net/publication/365476532>.
- [11] P. Mroziński. Strong excision and strong shape invariants are equivalent for homology theories on the category of compact metric pairs. *Homol. Homot. Appl.*, volume 10:pages 345–351, 2008. URL <http://intlpress.com/HHA/v10/n1/a15>.
- [12] E. Sklyarenko. On homology theory associated with the aleksandrov–Čech cohomology. *Russ. Math. Surv.*, volume 34:pages 103–137, 1979. URL <http://dx.doi.org/https://doi.org/10.1070/RM1979v034n06ABEH003307>.
- [13] V. Zemlyanoy. Fine shape of metrizable spaces as a left fraction localization. URL <https://arxiv.org/abs/2311.11624>.