Path decompositions of Eulerian graphs

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Abstract

Gallai's conjecture asserts that every connected graph on n vertices can be decomposed into $\frac{n+1}{2}$ paths. For general graphs (possibly disconnected), it was proved that every graph on n vertices can be decomposed into $\frac{2n}{3}$ paths. This is also best possible (consider the graphs consisting of vertex-disjoint triangles). Lovász showed that every n-vertex graph with at most one vertex of even degree can be decomposed into $\frac{n}{2}$ paths. However, Gallai's conjecture is difficult for graphs with many vertices of even degrees. Favaron and Kouider verified Gallai's conjecture for all Eulerian graphs with maximum degree at most 4. In this paper, we show if G is an Eulerian graph on $n \geq 4$ vertices and the distance between any two triangles in G is at least 3, then G can be decomposed into at most $\frac{3n}{5}$ paths.

Keywords: Path decomposition, Eulerian graph, Gallai's conjecture

1 Introduction

All graphs considered in this paper are finite, undirected and simple. A *path decomposition* of a graph G is a set of edge-disjoint paths whose union contains all the edges of G. Let $p(G)$ denote the minimum number of paths needed in a path decomposition of G. Gallai proposed the following well-known conjecture on path decomposition (see [20]).

Conjecture 1.1. *If G is a connected graph on n vertices, then $p(G) \leq \frac{n+1}{2}$.*

Conjecture 1.1 has been studied extensively. For example, the following contribution is due to Lovász [20].

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Theorem 1.2 (Lovász [20])**.** *Let G be a graph (possibly disconnected) on n vertices. If G contains at most one vertex of even degree, then $p(G) \leq \frac{n}{2}$.*

Note that the bound $\frac{n+1}{2}$ of Conjecture 1.1 is sharp for connected graphs (consider the complete graphs of odd order). For disconnected graphs, Donald [11] extended Theorem 1.2 by showing that $p(G) \leq \frac{3n}{4}$ for any graph G on n vertices. Furthermore, Dean and Kouider [10], and Yan [22] independently showed that $\frac{2n}{3}$ is best possible since the graph consisting of k vertex-disjoint triangles needs at least $2k$ paths in any path decomposition. The *girth* of a graph G is the minimum length of a cycle in G. Harding and McGuinness [18] considered graphs with girth $g \geq 4$ and showed that $\frac{(g+1)n}{2g}$ is sufficient. Chu, Fan and Zhou [8] proved that $p(G) \leq \frac{3n}{5}$ for triangle-free graphs. For sufficiently large graph G with linear minimum degree, Girão, Granet, Kühn and Osthus [16] showed that $p(G) \leq \frac{n}{2} + o(n)$.

As a consequence of Theorem 1.2, Conjecture 1.1 holds for graphs with all vertices of odd degrees. The main difficulty in studying Conjecture 1.1 is to handle graphs with many vertices of even degrees. Favaron and Kouider [15] verified Conjecture 1.1 for all Eulerian graphs with maximum degree at most 4. For $2k$-regular graphs ($k \geq 3$), Botler and Jiménez [2] showed that if G has girth at least $2k-2$ and a pair of disjoint perfect matchings, then $p(G) \leq \frac{n}{2}$. The *E-subgraph* of a graph G is the subgraph induced by the vertices of even degree in G. Pyber [21] extended Theorem 1.2 by showing that $p(G) \leq \frac{n}{2}$ if the E-subgraph of G is a forest. A forest can be regarded as a graph in which each block is a single edge or an isolated vertex. Fan [13] generalized Pyber's result by proving that if each block of the E-subgraph of G is a triangle-free graph with maximum degree at most 3, then $p(G) \leq \frac{n}{2}$. Botler and Sambinelli [4] further extended this by allowing the components of the E-subgraph to contain any number of blocks with triangles as long as they are subgraphs of a family of special graphs. Other progress on Conjecture 1.1 can be found in [1, 3, 5, 14, 23].

The *distance* between two triangles T_1 and T_2 in a graph is the length of the shortest path between any vertex of T_1 and any vertex of T_2. In this paper, we prove the following.

Theorem 1.3. *Let G be an Eulerian graph on $n \geq 4$ vertices. If the distance between any two triangles in G is at least 3, then $p(G) \leq \frac{3n}{5}$.*

Note that any path decomposition of a triangle requires at least two paths. So we need $n \geq 4$ in the theorem above.

A *triangle removal set* of a graph G is a set of vertices $R = \{v_1, \ldots, v_k\}$ such that $G - R$ is triangle-free and R has the following properties: for any distinct $i, j \in [k]$, (i) $d_G(v_i) \geq 4$; (ii) all the vertices in $V(G) \backslash R$ have even degree in G; and (iii) $(N_G(v_i) \cup \{v_i\}) \cap (N_G(v_j) \cup \{v_j\}) = \emptyset$.

In fact, we show the following theorem which is slightly stronger than Theorem 1.3.

Theorem 1.4. *Let G be a graph on n vertices with no component isomorphic to a triangle. If G has a triangle removal set, then $p(G) \leq \frac{3n}{5}$.*

Note that Theorem 1.3 is related to the cycle decompositions of Eulerian graphs. In 1960s, Erdős and Gallai conjectured that every n-vertex Eulerian graph can be decomposed into $O(n)$ cycles (see [12]). An equivalent and more well-known form is that every n-vertex graph can be decomposed into $O(n)$ cycles and edges. This conjecture was established for typical random graphs in [9, 17, 19]. Conlon, Fox and Sudakov [9] proved the conjecture for graphs with linear minimum degree while the asymptotically sharp bound of $(\frac{3}{2} + o(1))n$ was shown in [16]. For general graphs, Erdős and Gallai proved that every n-vertex graph can be decomposed into $O(n \log n)$ cycles and edges by iteratively removing a longest cycle. Conlon, Fox and Sudakov [9] obtained a major breakthrough by showing that $O(n \log \log n)$ cycles and edges suffice. Recently, Bucić and Montgomery [6] further improved the bound to $O(n \log^\star n)$ where $\log^\star$ is the iterated logarithm function.

The organization of this paper is as follows. In the next section, we show several useful lemmas. We prove Theorems 1.3 and 1.4 in Section 3.

We conclude this section with some useful notations. Let G be a graph. The set of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. Let $u, v \in V(G)$. The edge with ends u and v is denoted by uv. We use $N_G(v)$ and $d_G(v)$ to denote the set and the number of neighbors of v in G, respectively. For $S \subseteq V(G)$, $G - S$ denotes the induced subgraph of G on $V(G) \setminus S$. Let F be a set of edges (resp. non-edges) of G. We denote the graph obtained from G by deleting (resp. adding) all the edges of F by $G - F$ (resp. $G + F$). When $F = \{uv\}$, we simply write $G - uv$ (resp. $G + uv$) instead of $G - \{uv\}$ (resp. $G + \{uv\}$). Let P be a path and x, y be two vertices on P. The subpath of P with end vertices x and y is denoted by xPy. Note if $x = y$, then xPy is just the vertex x. We say that a graph G can be *decomposed into* k paths if G has a path decomposition $\mathcal{P}$ such that $|\mathcal{P}| = k$. Given a path decomposition $\mathcal{P}$ of G and a vertex $v \in V(G)$, we denote by $\mathcal{P}(v)$ the number of paths in $\mathcal{P}$ with v as an end vertex. Note that if v is a vertex of G with odd degree, then $\mathcal{P}(v) \geq 1$.

2 Decomposition lemmas

We need the following lemma in [7].

Lemma 2.1 (Lemma 2.1 in [7])**.** *Let G be a connected graph consisting of edge-disjoint path P and cycle C. If $|V(P) \cap V(C)| \leq 5$, then G can be decomposed into two paths, unless G is isomorphic to the exceptional graph (see Figure 1).*

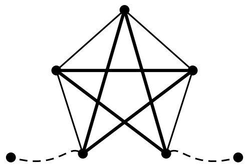

Figure 1: The exceptional graph with $|V(C)| = 5$ and $|V(P) \cap V(C)| = 5$ (C is depicted by thick solid lines; P is depicted by thin lines, with dashed lines used to represent its subpaths).

We obtain the following lemma, which is an extension of Lemma 2.1. Note that the condition $|V(P)| \le 4$ in Lemma 2.2 is necessary because of the exceptional graph in Figure 1 (note $|V(P)| = 5$ and $|\mathcal{C}| = 1$ in the exceptional graph). Let $\mathcal{C}$ be a set of edge-disjoint subgraphs of G. Write $V(\mathcal{C}) = \bigcup_{C\in\mathcal{C}} V(C)$ and $E(\mathcal{C}) = \bigcup_{C\in\mathcal{C}} E(C)$.

Lemma 2.2. *Let $\mathcal{C}$ be a set of edge-disjoint cycles and P be a path such that $E(P) \cap E(C) = \emptyset$ and $V(P) \cap V(C) \neq \emptyset$ for every cycle $C \in \mathcal{C}$. If $|V(P)| + |\mathcal{C}| \le 6$ and $|V(P)| \le 4$, then $E(P) \cup E(\mathcal{C})$ can be decomposed into $|\mathcal{C}| + 1$ paths.*

Proof. The proof follows by induction on $|\mathcal{C}|$. It is easy to see that the case when $|\mathcal{C}| = 0$ holds. If $|\mathcal{C}| = 1$, let C be the only cycle in $\mathcal{C}$, then $|V(P) \cap V(C)| \le 4$ since $|V(P)| \le 4$. So $E(P) \cup E(C)$ is not isomorphic to the exceptional graph. Thus by Lemma 2.1, $E(P) \cup E(C)$ can be decomposed into two paths.

Now suppose that $|\mathcal{C}| \ge 2$ and the lemma holds for smaller $|\mathcal{C}|$. Let $P = v_1 v_2 \cdots v_k$, where $k \le 4$. We may assume that v_i ($1 \le i \le k$) is the vertex of P with smallest index that belongs to $V(\mathcal{C})$, say $v_i \in V(C_1)$ for some $C_1 \in \mathcal{C}$. Let x_1 and x_2 be two neighbors of v_i on C_1. If one of x_1 and x_2, say x_1, is not in $V(P)$, let $P' = P + x_1 v_i - E(v_1 P v_i)$, $P_1 = C_1 + E(v_1 P v_i) - x_1 v_i$, and $\mathcal{C}' = \mathcal{C} \backslash \{C_1\}$. Clearly, $|V(P')| + |\mathcal{C}'| \le 6$. If $|V(P')| \le 4$, then by inductive hypothesis, $E(P') \cup E(\mathcal{C}')$ can be decomposed into $|\mathcal{C}'| + 1$ paths. Together with P_1, we obtain $|\mathcal{C}| + 1$ paths, as desired. Thus, assume that either $|V(P')| = 5$, or $x_1, x_2 \in V(P)$. In both cases, it follows that $k = 4$, $i = 1$ and $|\mathcal{C}| = 2$. Let $\mathcal{C} = \{C_1, C_2\}$.

Next, we show that $E(P) \cup E(C_1) \cup E(C_2)$ can be decomposed into three paths. If $|V(P')| = 5$, then $P' \cup C_2$ is not isomorphic to the exceptional graph (For otherwise, C_2 is a cycle of length 5 with $C_2 = v_1 v_3 x_1 v_2 v_4 v_1$ (see Figure 2(a)), it follows that $x_2 \notin V(P) \cup V(C_2)$, then we can redefine $P' = P + x_2 v_1$, $P_1 = C_1 - x_2 v_1$.). By Lemma 2.1, $E(P') \cup E(C_2)$ can be decomposed into two paths P_2 and P_3, thus $\{P_1, P_2, P_3\}$ is the desired decomposition. If $x_1, x_2 \in V(P)$, then $\{x_1, x_2\} = \{v_3, v_4\}$ (see Figure 2 (b)). Let x be the other neighbor of v_3 on C_1. It is easy to check that $x \notin V(P)$. Let $P'' = x v_3 v_2 v_1 v_4$, $P_4 = C_1 - x v_3 - v_1 v_4 + v_3 v_4$. Then $P'' \cup C_2$ is not isomorphic to the exceptional graph, for otherwise, $v_3 v_4 \in E(C_2)$, which is a contradiction to the fact that $v_3 v_4 \in E(P)$.

Thus, by Lemma 2.1, $E(P'') \cup E(C_2)$ can be decomposed into two paths P_5, P_6, and then $\{P_4, P_5, P_6\}$ is as desired. $\square$

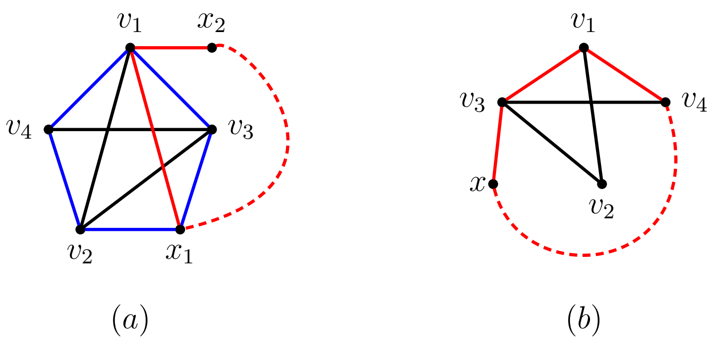

Figure 2: C_1, C_2 and P are in red, blue and black, respectively.

The following lemma finds a path decomposition when all the cycles have a common vertex.

Lemma 2.3. *Let $\mathcal{C}$ be a set of edge-disjoint cycles such that all elements in $\mathcal{C}$ contain a common vertex. If $|\mathcal{C}| \leq 6$, then $E(\mathcal{C})$ can be decomposed into $|\mathcal{C}| + 1$ paths.*

Proof. If $|\mathcal{C}| \leq 5$, we apply Lemma 2.2 to $\mathcal{C}$ and $P = v$ (v is a common vertex on every cycle in $\mathcal{C}$) and conclude that $E(\mathcal{C})$ can be decomposed into $|\mathcal{C}| + 1$ paths. So we only need to consider the case when $|\mathcal{C}| = 6$, and aim to show that $E(\mathcal{C})$ can be decomposed into seven paths. Let $\mathcal{C} = \{C_1, \dots, C_6\}$.

We first claim that $V(C_i) = V(C_j)$ for every $i \neq j$. For otherwise, let u and v be two vertices such that v is contained in every cycle in $\mathcal{C}$ and u is a neighbor of v on C_i that is not on C_j. Let $P_1 = C_j - zv + vu$ where $zv \in E(C_j)$, $P_2 = C_i - vu$, $\mathcal{C}' = \mathcal{C} \backslash \{C_i, C_j\}$ and $P' = zv$. Then $|V(P')| + |\mathcal{C}'| = 2 + 4 = 6$. By Lemma 2.2, $E(P') \cup E(\mathcal{C}')$ can be decomposed into five paths, which together with P_1 and P_2, is a desired path decomposition.

Pick $v_1v_2 \in E(C_1)$, $v_3 \in N_{C_2}(v_2)$ and $v_4 \in N_{C_3}(v_3) \backslash \{v_1\}$. Clearly, $v_1v_2v_3v_4$ is a path. Let $v_5 \in N_{C_4 \cup C_5 \cup C_6}(v_4) \backslash \{v_1, v_2\}$. Then $v_1v_2v_3v_4v_5$ is also a path. Without loss of generality, assume that $v_4v_5 \in E(C_4)$. Choose $v_6 \in N_{C_5 \cup C_6}(v_5) \backslash \{v_1, v_2, v_3\}$. It is clear that $v_1v_2v_3v_4v_5v_6$ is a path. Similarly, we may assume that $v_5v_6 \in E(C_5)$. Let x_1, x_2 be two neighbors of v_6 on C_6 and y_1, y_2 be two neighbors of v_1 on C_6, respectively. If $x_1 \notin \{v_1, v_2, v_3, v_4\}$, set $P'' = v_1v_2v_3v_4v_5v_6x_1$, $P'_6 = C_6 - v_6x_1$ and $P'_i = C_i - v_iv_{i+1}$ for $i = 1, \dots, 5$. Then $\{P'', P'_1, \dots, P'_6\}$ is a desired path decomposition of $E(\mathcal{C})$. So by symmetry, we may assume that $\{x_1, x_2, y_1, y_2\} \subseteq \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

If $v_6 \in \{y_1, y_2\}$, that is $v_1v_6 \in E(C_6)$, then without loss of generality, assume that $y_1 = v_6$ and $x_1 = v_1$. Let $C = v_1v_2v_3v_4v_5v_6v_1$. For $i = 2, \ldots, 5$, let $v_i' \in N_{C_{i-1}}(v_i)\backslash\{v_{i-1}\}$ and $v_i'' \in N_{C_i}(v_i)\backslash\{v_{i+1}\}$. Then there exists some i such that $v_i' \notin V(C)$ or $v_i'' \notin V(C)$. For otherwise, it is easy to check that for all index $i \in \{2, 3, 4, 5\}$, v_iv_i', v_iv_i'' are distinct chords of C and $y_2v_1, x_2v_6 \in E(C_6)$ are two chords of C which are different from v_iv_i' and v_iv_i''. This contradicts to the fact that a cycle of length 6 has at most nine chords. If $v_i' \notin V(C)$, let $P'' = C - v_{i-1}v_i + v_iv_i'$, $P_{i-1}' = C_{i-1} - v_iv_i'$, $P_6' = C_6 - v_1v_6$, and $P_j' = C_j - v_jv_{j+1}$ for $j \in \{1, 2, 3, 4, 5\}\backslash\{i-1\}$; If $v_i'' \notin V(C)$, let $P'' = C - v_iv_{i+1} + v_iv_i''$, $P_i' = C_i - v_iv_i''$, $P_6' = C_6 - v_1v_6$, and $P_j' = C_j - v_jv_{j+1}$ for $j \in \{1, 2, 3, 4, 5\}\backslash\{i\}$. In both cases, $\{P'', P_1', \ldots, P_6'\}$ is a path decomposition of $E(\mathcal{C})$, as desired. Thus, we may assume that $v_6 \notin \{y_1, y_2\}$. By symmetry, we have $\{y_1, y_2\} \subseteq \{v_3, v_4, v_5\}$ and $\{x_1, x_2\} \subseteq \{v_2, v_3, v_4\}$.

Now without loss of generality, suppose that $v_i = y_1$ and $v_j = y_2$ with $3 \le i < j \le 5$. Let $w \in N_{C_{i-1}\cup C_{j-1}}(v_6)\backslash\{v_1, v_2, v_3, v_4\}$ (such vertex w must exist since $\{x_1, x_2\} \subseteq \{v_2, v_3, v_4\}$). For $s \in \{i, j\}$, set $P'' = wv_6 \ldots v_sv_1v_2 \ldots v_{s-1}$, $P_{s-1}' = C_{s-1} - wv_6$, $P_6' = C_6 - v_1v_s$, and $P_t' = C_t - v_tv_{t+1}$ for $t \in \{1, 2, 3, 4, 5\}\backslash\{s-1\}$. Then $\{P'', P_1', \ldots, P_6'\}$ is a path decomposition of $E(\mathcal{C})$, as desired. □

The following lemma is obtained by grouping cycles in families of six cycles.

Lemma 2.4. *Let G be a connected graph consisting of q edge-disjoint cycles such that all cycles contain a common vertex.*

(1) If $q \equiv 0 \pmod 6$, then G can be decomposed into $\frac{7q}{6}$ paths;

(2) If $q \equiv \delta \pmod 6$ for $\delta = 1, 2, 3, 4, 5$, then G can be decomposed into $\frac{7q}{6} + 1 - \frac{\delta}{6}$ paths.

Proof. We put 6 cycles into a group. By applying Lemma 2.3 to every group of 6 cycles, we can decompose G into at most $\frac{7q}{6}$ paths if $q \equiv 0 \pmod 6$. If $q \equiv \delta \pmod 6$ for $\delta = 1, 2, 3, 4, 5$, then the remaining δ cycles can be decomposed into $\delta + 1$ paths by applying Lemma 2.3 one more time. Thus, G has a path decomposition with at most $\frac{7(q-\delta)}{6} + \delta + 1 = \frac{7q}{6} + 1 - \frac{\delta}{6}$ paths for $\delta = 1, 2, 3, 4, 5$. □

Let $\alpha(G)$ and $\beta(G)$ denote the number of vertices of odd degree and the number of non-isolated vertices of even degree in G, respectively. We also need the following result in [8].

Theorem 2.5 (Theorem 3.1 in [8])**.** *If G is a triangle-free graph, then $p(G) \le \frac{\alpha(G)}{2} + \lfloor \frac{3\beta(G)}{5} \rfloor$.*

3 Proof of main theorems

In this section, we prove Theorems 1.3 and 1.4. First we show Theorem 1.4.

Proof of Theorem 1.4. Let $R = \{v_1, v_2, \ldots, v_k\}$ be a triangle removal set of G. Let $G' = G - R$. Since all vertices in $V(G)\backslash R$ have even degree in G and $(N_G(v_i) \cup \{v_i\}) \cap (N_G(v_j) \cup \{v_j\}) = \emptyset$ for any $i, j \in [k]$ by definition of R, all the neighbors of $v_1, v_2, \ldots, v_k$ have odd degree in G', and thus $\alpha(G') = d_1 + \ldots + d_k$, $\beta(G') = n - k - d_1 - \ldots - d_k$ where $d_i = d_G(v_i)$. Since G' is a triangle-free graph, by Theorem 2.5, G' has a path decomposition $\mathcal{P}'$ such that

$$\begin{aligned}
|\mathcal{P}'| &\leq \frac{1}{2}(d_1 + \ldots + d_k) + \left\lfloor \frac{3}{5}(n - k - d_1 - \ldots - d_k) \right\rfloor \\
&\leq \frac{1}{2}(d_1 + \ldots + d_k) + \frac{3}{5}(n - k - d_1 - \ldots - d_k) \\
&= \frac{3n}{5} + (\frac{d_1}{2} - \frac{3}{5}(1 + d_1)) + \ldots + (\frac{d_k}{2} - \frac{3}{5}(1 + d_k)).
\end{aligned}$$

First consider v_1. Let $N_G(v_1) = \{v_{11}, v_{12}, \ldots, v_{1d_1}\}$. For $j = 1, 2, \ldots, d_1$, we may assume that P_j is a path in $\mathcal{P}'$ containing v_{1j} as one of end vertices since v_{1j} has odd degree in G'. (Note that it is possible that $P_j = P_t$ for some $t \neq j \in \{1, 2, \ldots, d_1\}$.) For $j = 1, 2, \ldots, d_1$, set $P'_j := P_j + v_1v_{1j} + v_1v_{1t}$ if $P_j = P_t$ with $j \neq t$; or otherwise, $P'_j := P_j + v_1v_{1j}$. Note that P'_j is a cycle in the first case, or a path in the second case. If every P'_j is a path, let $\mathcal{P} = (\mathcal{P}'\backslash\{P_1, \ldots, P_{d_1}\}) \cup \{P'_1, \ldots, P'_{d_1}\}$, then $\mathcal{P}$ is a path decomposition of $G' + v_1$ with $|\mathcal{P}| = |\mathcal{P}'|$. Otherwise, let $\{P'_1, \ldots, P'_q\}$ be the set of all cycles that appear after the above operation. Then $1 \leq q \leq \frac{d_1}{2}$. By Lemma 2.4, $E(P'_1) \cup E(P'_2) \cup \ldots \cup E(P'_q)$ can be decomposed into at most t paths $P''_1, \ldots, P''_t$, where $t = \frac{7q}{6}$ if $q \equiv 0 \pmod 6$, or $t = \frac{7q}{6} + 1 - \frac{\delta}{6}$ if $q \equiv \delta \pmod 6$ for $\delta = 1, 2, 3, 4, 5$. Let $\mathcal{P} = (\mathcal{P}'\backslash\{P_1, \ldots, P_{d_1}\}) \cup \{P''_1, \ldots, P''_t\} \cup \{P'_j : P'_j \text{ is a path}, j \in \{1, \ldots, d_1\}\}$. Then $\mathcal{P}$ is a path decomposition of $G_1 = G' + v_1 = G - \{v_2, \ldots, v_k\}$ with

$$|\mathcal{P}| = \begin{cases} |\mathcal{P}'| - q + \dfrac{7q}{6}, \text{ if } q \equiv 0 \pmod 6; \\ |\mathcal{P}'| - q + \dfrac{7q}{6} + 1 - \dfrac{\delta}{6}, \text{ if } q \equiv \delta \pmod 6 \text{ for } \delta = 1, 2, 3, 4, 5. \end{cases}$$

Since $q \leq \frac{d_1}{2}$, we have

$$|\mathcal{P}| \leq \frac{3n}{5} + (\frac{d_2}{2} - \frac{3}{5}(1 + d_2)) + \ldots + (\frac{d_k}{2} - \frac{3}{5}(1 + d_k))$$

unless $q \equiv 1, 2 \pmod 6$.

When $q \equiv 2 \pmod 6$, since $d_1 \geq 4$, we have

$$\begin{aligned}
|\mathcal{P}| &\leq \frac{3n}{5} + \frac{d_1}{12} + 1 - \frac{2}{6} + (\frac{d_1}{2} - \frac{3}{5}(1+d_1)) + (\frac{d_2}{2} - \frac{3}{5}(1+d_2)) + \ldots + (\frac{d_k}{2} - \frac{3}{5}(1+d_k)) \\
&\leq \frac{3n}{5} - \frac{d_1}{60} + \frac{1}{15} + (\frac{d_2}{2} - \frac{3}{5}(1+d_2)) + \ldots + (\frac{d_k}{2} - \frac{3}{5}(1+d_k)) \\
&\leq \frac{3n}{5} + (\frac{d_2}{2} - \frac{3}{5}(1+d_2)) + \ldots + (\frac{d_k}{2} - \frac{3}{5}(1+d_k)).
\end{aligned}$$

When $q \equiv 1 \pmod 6$, we have

$$\begin{aligned}
|\mathcal{P}| &\leq \frac{3n}{5} + \frac{q}{6} + 1 - \frac{1}{6} + (\frac{d_1}{2} - \frac{3}{5}(1+d_1)) + (\frac{d_2}{2} - \frac{3}{5}(1+d_2)) + \ldots + (\frac{d_k}{2} - \frac{3}{5}(1+d_k)) \\
&\leq \frac{3n}{5} + \frac{q}{6} - \frac{d_1}{10} + \frac{7}{30} + (\frac{d_2}{2} - \frac{3}{5}(1+d_2)) + \ldots + (\frac{d_k}{2} - \frac{3}{5}(1+d_k)).
\end{aligned}$$

Since $q \leq \frac{d_1}{2}$, we have that $\frac{q}{6} - \frac{d_1}{10} + \frac{7}{30} \leq \frac{d_1}{12} - \frac{d_1}{10} + \frac{7}{30} = \frac{14-d_1}{60} \leq 0$ if $d_1 \geq 14$. If $d_1 \leq 13$, then $q \leq 6$. Since $q \equiv 1 \pmod 6$, $q = 1$. Thus, $\frac{q}{6} - \frac{d_1}{10} + \frac{7}{30} = \frac{1}{6} - \frac{d_1}{10} + \frac{7}{30} \leq 0$ since $d_1 \geq 4$. Therefore, we have

$$|\mathcal{P}| \leq \frac{3n}{5} + (\frac{d_2}{2} - \frac{3}{5}(1+d_2)) + \ldots + (\frac{d_k}{2} - \frac{3}{5}(1+d_k)).$$

Performing the same operations on every vertex in $\{v_2, v_3, \ldots, v_k\}$, we obtain that G has a path decomposition with at most $\frac{3n}{5}$ paths. This completes the proof. $\square$

Now Theorem 1.3 is a direct consequence of Theorem 1.4.

Proof of Theorem 1.3. Let k denote the number of triangles in G. If $k = 0$, it follows from Theorem 2.5. Now suppose $k \geq 1$. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ be the set of triangles in G, and for each $i = 1, 2, \ldots, k$, let v_i denote a vertex of T_i with maximum $d_G(v_i)$. Since G is Eulerian on $n \geq 4$ vertices, every triangle T_i has at least one vertex of degree at least 4. So $d_G(v_i) \geq 4$. Since the distance between any two triangles is at least 3, $(N_G(v_i) \cup \{v_i\}) \cap (N_G(v_j) \cup \{v_j\}) = \emptyset$ for any $i, j \in [k]$. So $\{v_1, \ldots, v_k\}$ is a triangle removal set of G. It follows from Theorem 1.4 that $p(G) \leq \frac{3n}{5}$. This completes the proof. $\square$

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