LOCAL MOLLIFICATION OF METRICS WITH SMALL CURVATURE CONCENTRATION

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Abstract. In this work, we establish a local smoothing result on metrics with small curvature concentration with respect to Sobolev constants and volume growth. In contrast with all previous works, we remove the Ricci curvature condition and completely localize the smoothing. As an application, we prove the compactness of the space of compact manifolds with bounded curvature concentration under Ahlfors n-regularity and bounded Sobolev constant. In the complete non-compact case, we show that manifolds with Euclidean type Sobolev inequality, Euclidean volume growth, and small curvature concentration are necessarily diffeomorphic to Euclidean spaces.

1. **Introduction**

Given a complete manifold (M, g), one of the general themes in Riemannian geometry is to characterize the topology of M under curvature conditions on the metric g. An important theorem of Cheeger [Che70] shows that there are only finitely many diffeomorphism types for non-collapsed compact manifolds with bounded curvature and diameter. There have been many efforts in generalizing Cheeger's finiteness theorem to weaker curvature assumptions. Relaxing the pointwise control $||\mathrm{Rm}||_{L^\infty}$ to an integral bound $||\mathrm{Rm}||_{L^p}$ has attracted much attention. The case $p = n/2$, which we call *bounded curvature concentration*, is particularly interesting since the integral is scaling invariant without being weighted by the volume. The integral bound of curvature also appears naturally in many geometric problems. Perhaps more importantly, integral curvatures naturally arise in studying singular spaces as a weak formulation of curvature. For instance, it plays an important role in the RCD theory and positive mass theorems on asymptotically flat manifolds with singularities. One naturally asks what geometric and topological results can be extended to integral curvature bounds.

There have been many interesting generalizations of comparison geometry in this direction; for instance, see [PW97, PW01, Car19] and the references therein. An integral curvature bound is however not strong enough to constrain the topology in general. Indeed, it was shown by Yang [Yan92] that the exact analogy of Cheeger's finiteness cannot be true in case $p < +\infty$, due to the failure of controlling the volume locally. There is a gap phenomenon, as already observed in [Gal88, Yan92, PW97, PW01]; see [DW, Section 5] for a survey of more results in this direction.

In this work, we are particularly interested in the gap phenomenon and compactness under the scaling invariant curvature control $||\mathrm{Rm}||_{L^{n/2}(M)} < +\infty$. In Ricci geometry, this condition has been studied extensively in the literature; see [And89, And90, AC91, BKN89, Nak88] for examples. In [Che03], Cheeger systemically studied the singularities of the Gromov–Hausdorff limit of non-collapsed manifolds with Ricci lower bounds and integral curvature controls. Particularly when $p = n/2$, Cheeger showed that the singularities must be isolated and the tangent cone at a singular

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point must be the cone over a quotient of the round sphere. One of the key ingredients is to prove an epsilon regularity result for non-collapsed manifold with small curvature concentration. The method of Cheeger is elliptic, based on harmonic functions.

It is natural to ask if a metric with sufficiently small integral curvature can be perturbed to a metric with pointwise curvature bound in suitable topology, in order to study geometric compactness with only an integral curvature bound. To the best of our knowledge, the smoothing by Ricci flow was first considered by Yang [Yan92]. The Ricci flow is a one-parameter family of metrics $g(t)$ on a manifold such that

$$\partial_t g(t) = -2\,\mathrm{Ric}(g(t)).$$

Viewed as a nonlinear geometric heat equation, the flow has been proven useful to regularize metrics and obtain striking consequences in geometry and topology. When both $||\mathrm{Rm}||_{L^n}$ and $||\,\mathrm{Ric}\,||_{L^p}, p > n/2$, are both small, Yang [Yan92] used the Ricci flow to show that the metric can be perturbed in the C^0 topology to a metric with bounded geometry. However, the C^0 perturbation is not expected in general if the curvature bound is only scaling invariant, which can be seen from the point of view of tangent cones.

In [CHL24], the first named author with Chan and Huang considered a parabolic approach to Cheeger's epsilon regularity theorem. In short, non-collapsed metrics with Ricci lower bounds and small local curvature concentration can be regularized by Ricci flow for a uniform short time with instantaneous bounded geometry. The perturbation is in the (pointed) Gromov–Hausdorff topology, in the presence of Ricci lower bounds by the distance estimate of [TW15]. This improved and sharpened the result [Yan92] of Yang[1]. This is based on constructing distance like functions with critical integral Hessian estimates, using comparison geometry, so as to construct approximating Ricci flow solutions with suitable control. This smoothing result was later utilized by the first named author and Chan [CL24] to prove a metric gap Theorem of manifolds with $\mathrm{Ric} \geq 0$, Euclidean volume growth, and small curvature concentration; see also [Mar25a]. The proof in [CHL24] relies on Schoen–Yau's construction of proper functions which uses Bochner's formula for eigenfunctions. The Ricci curvature lower bound is crucial there in obtaining gradient estimates.

In this work, we further advance the result in [CHL24] and show that the local smoothing holds in a greater generality. In particular, we show that the assumption of Ricci curvature lower bounds in any kind is redundant.

Theorem 1.1. *For $n \geq 3$ and $v_0, C_s > 0$, there exist $\delta_1(n, v_0, C_s), \Lambda_1(n, v_0, C_s), L(n, v_0, C_s)$, $r_1(n, v_0, C_s)$, and $\hat{T}(n, v_0, C_s) > 0$, such that the following holds. Let (M^n, g_0) be a manifold such that $B_{g_0}(x_0, 3 + \bar{r}) \Subset M$ for some $(x_0, \bar{r}) \in M \times [0, +\infty)$. Suppose for all $x \in B_{g_0}(x_0, 2 + \bar{r})$,*

(a) $\mathrm{Vol}_{g_0}(B_{g_0}(x, r)) \leq v_0 r^n$ *for all* $r \in (0, 1]$;
(b) *for all* $u \in W^{1,2}_0(B_{g_0}(x, 1), g_0)$,

$$\left(\int_{B_{g_0}(x,1)} u^{\frac{2n}{n-2}}\,d\mathrm{vol}_{g_0}\right)^{\frac{n-2}{n}} \leq C_s \left(\int_{B_{g_0}(x,1)} |\nabla u|^2 + u^2\,d\mathrm{vol}_{g_0}\right);$$

(c) $||\mathrm{Rm}(g_0)||_{L^{n/2}(B_{g_0}(x,1),g_0)} \leq \varepsilon$ *for some* $\varepsilon \in (0, \delta_1]$.

Then there exists a smooth Ricci flow solution $g(t)$ defined on $B_{g_0}(x_0, 1 + \bar{r}) \times [0, \hat{T}]$ such that $g(0) = g_0$ and

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[1]The methodology in [CHL24] indeed only requires certain small local L^p bounds on Ric_- for $p > n/2$.

(1) $|\mathrm{Rm}(g(t))| \leq \Lambda_1 \varepsilon t^{-1} \leq \frac{1}{4(n-1)} t^{-1}$,
(2) $\mathrm{inj}(g(t)) \geq \sqrt{t}$,
(3) $||\mathrm{Rm}(g(t))||_{L^{n/2}(B_{g(t)}(x,1/2))} \leq \Lambda_1 \varepsilon$, *and*
(4) $\nu\left(B_{g(t)}(x, 2L\sqrt{\hat{T}}, g(t), L^2\hat{T}\right) \geq -\Lambda_1$

for all $(x,t) \in B_{g_0}(x_0, 1+\bar{r}) \times (0, \hat{T}]$. *Moreover, for all* $x, y \in B_{g_0}(x_0, r_1)$ *and* $t \in [0,T]$,

$$\Lambda_1^{-1}\left[d_{g(t)}(x,y)\right]^{\frac{1}{1-2(n-1)\Lambda_1\varepsilon}} \leq d_{g_0}(x,y) \leq \Lambda_1 \left[d_{g(t)}(x,y)\right]^{\frac{1}{1+2(n-1)\Lambda_1\varepsilon}}.$$

Remark 1.2. The entropy lower bound measures the optimal constant of the log-Sobolev inequality. In terms of boundedness, this is equivalent to a (weighted) Sobolev constant. See [CCL22, Ye15] for example.

The smoothing is purely local in nature and does not require any local curvature bound. We use it to consider Ahlfors n-regular compact manifolds with bounded diameters, Sobolev constants and curvature concentration. We use the local smoothing to show that the Gromov–Hausdorff limit of a sequence of manifolds in this collection must be a topological manifold away from finitely many points.

Theorem 1.3. *Let* $n \geq 3$ *and* $C_s, v_0, \Lambda_0, D_0 > 0$. *Suppose* (M_i^n, g_i) *is a sequence of compact manifolds such that*

(a) *for all* $u \in W^{1,2}(M_i)$,

$$\left(\int_{M_i} u^{\frac{2n}{n-2}}\, d\mathrm{vol}_{g_i}\right)^{\frac{n-2}{n}} \leq C_s \int_{M_i} \left(|\nabla u|^2 + u^2\right) d\mathrm{vol}_{g_i};$$

(b) $\mathrm{Vol}_{g_i}\left(B_{g_i}(x,r)\right) \leq v_0 r^n$ *for all* $(x,r) \in M_i \times (0,1]$;
(c) $||\mathrm{Rm}(g_i)||_{L^{n/2}(M_i)} \leq \Lambda_0$;
(d) $\mathrm{diam}(M_i) \leq D_0$.

Then there exists a compact metric space (X, d_∞) *and* $\mathcal{S} = \{p_i\}_{i=1}^N$ *such that* (M_i, g_i) *converges subsequentially to* (X, d_∞) *in the Gromov–Hausdorff topology and* $X \setminus \mathcal{S}$ *is a topological manifold.*

In the presence of Ricci lower bounds, the Sobolev constant can be ensured by volume non-collapsing. Thus, this theorem can be viewed as a generalization of results in Ricci geometry; see the compactness results in [And89, And90, BKN89]. Particularly, we extend the related compactness to case without Ricci control. When Λ_0 is sufficiently small, the singular set $\mathcal{S}$ is empty and the limit X admits a smooth structure by the Ricci flow smoothing in Theorem 1.1, see [CCL22, Theorem 1.4]. It is, however, unclear to us if this is the case in general; see Remark 4.6. In the presence of Ricci curvature lower bounds, it is also known that the tangent cone at $p \in \mathcal{S}$ is given by $C(\mathbb{S}^{n-1}/\Gamma)$ by the deep work of Cheeger [Che03]. This plays an important role in the recent work of Jiang–Wei [JW25] on the finiteness of diffeomorphisms. It will be interesting to know if the finite diffeomorphism theorem in [JW25] can be generalized to the case without the Ricci curvature assumption. This is related to identifying the asymptotic behavior near a singular point $p \in \mathcal{S}$.

Another application of Theorem 1.1 is about complete non-compact geometry and is related to the case without singularities. We consider the topological gap phenomenon of complete non-compact manifolds with small curvature concentration. The problem has been studied by many authors; for example, see [CC97, Led99, Xia99, Xia01, Che03, Che22, CHL24] and the references therein. We show that a manifold with small curvature concentration is diffeomorphic to the Euclidean space provided it supports a Euclidean-type Sobolev inequality and has Euclidean volume growth. This does not rely on the Ricci curvature and thus extends the results in [CHL24, Che03].

Theorem 1.4. *For $n \geq 3$ and $C_s, v_0 > 0$, there exists $\delta_0(n, C_s, v_0) > 0$ such that the following is true. Suppose (M^n, g_0) is a complete non-compact manifold such that*

(a) *for all $u \in W_0^{1,2}(M)$,*

$$\left(\int_M u^{\frac{2n}{n-2}}\, d\mathrm{vol}_{g_0}\right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla u|^2\, d\mathrm{vol}_{g_0};$$

(b) $\mathrm{Vol}_{g_0}(B_{g_0}(x, r)) \leq v_0 r^n$ *for all $(x, r) \in M \times \mathbb{R}_+$;*

(c) $||\mathrm{Rm}(g_0)||_{L^{n/2}(M)} \leq \delta_0(n, C_s, v_0)$.

Then M is diffeomorphic to $\mathbb{R}^n$.

The Sobolev condition plays a strong role in non-collapsing. It is also easily seen from the Eguchi-Hanson metric that smallness on $||\mathrm{Rm}||_{L^{n/2}}$ is necessary. It is, however, unclear if this is still necessary when n is odd.

When the negative part of Ricci curvature has a stronger decay, the volume growth condition holds automatically by the work of Carron [Car19, Theorem B]. It will be interesting to know if it is necessary to assume (b) a-priori in Theorem 1.4.

A natural collection to consider is the class of Euclidean submanifolds. Based on the Sobolev inequality of Michael–Simon [MS73], a complete Euclidean submanifold with small total curvature satisfies the Euclidean-type Sobolev inequality. Thus, Theorem 1.4 implies that such a submanifold is diffeomorphic to $\mathbb{R}^n$ if its volume growth is Euclidean. Moreover, when the submanifold is a convex hypersurface in $\mathbb{R}^{n+1}$, one can remove the volume growth assumption and use the main theorem in [CL24] to show that the hypersurface is isometric to $\mathbb{R}^n$ when it has small total curvature. In general, in Theorem 1.4, when the scalar curvature is, furthermore, assumed to be non-negative, we expect to get a stronger rigidity result as in the Eucludean case. See Remark 5.4.

We organize the paper as follows. Section 2 provides the necessary a priori estimates that we will need in Section 3, in which the main local smoothing result (Theorem 1.1) is proven. We prove the compactness theorem (Theorem 1.3) in Section 4 and the topological gap theorem (Theorem 1.4) in Section 5.

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2. A priori estimates of Ricci flow

In this section, we will derive some a-priori estimates of the Ricci flow, which is related to the curvature concentration.

Along the Ricci flow, it is more common to consider the log-Sobolev constant, which is measured by the ν-entropy. It is well-known that the ν-entropy is monotone along a complete Ricci flow with bounded curvature thanks to the celebrated work of Perelman [Per02]; see also the complete non-compact case by Chau–Tam–Yu [CTY11]. We need a localized version by Wang [Wan18]. Following [Wan18], for a connected domain Ω with possibly empty boundary in M, we define

$$\begin{cases} D_g(\Omega) := \left\{ u \in W_0^{1,2}(\Omega) : u \geq 0 \text{ and } \|u\|_{L^2(\Omega)} = 1 \right\}, \\ W(\Omega, g, u, \tau) := \displaystyle\int_\Omega \left(\tau(\mathcal{R} \cdot u^2 + 4|\nabla u|^2) - 2u^2 \log u \right) \, d\mathrm{vol}_g - \frac{n}{2} \log(4\pi\tau) - n, \\ \nu(\Omega, g, \tau) := \displaystyle\inf_{u \in D_g(\Omega), s \in (0,\tau]} W(\Omega, g, u, s) \end{cases}$$

where $\mathcal{R}$ denotes the scalar curvature.

The local entropy $\nu(\Omega, g, \tau)$ measures the optimal (weighed) log-Sobolev constant of Ω in scale of τ with respect to g. This also controls the Sobolev constant in some suitable scale by [Mar25a, Lemma 3.3] which is built on [Ye15]. If $g(t)$ is a complete Ricci flow on Ω with bounded curvature, it is well-known that $\nu(\Omega, g(t), \tau - t)$ is monotonically non-decreasing in t by the celebrated work of Perelman [Per02].

Although the (almost) monotonicity of (localized) ν-entropy does not necessarily hold for incomplete local Ricci flow solutions, we will construct local solutions in a way that the entropy is still under control.

In this section, we will derive a priori estimates, assuming a bound on the localized entropy. We first need a local persistence of local curvature concentration. This is a localized version of [CL24, Lemma 2.2].

Lemma 2.1. *For all $A, \alpha > 0$, there exist $\delta_1(n, A) \in (0,1)$, $\tilde{T}(n, \alpha), C_1(n) > 0$ such that the following is true. Suppose $g(t)$ is a smooth solution to the Ricci flow on $M \times [0,T]$ and $x_0 \in M$ such that for all $t \in [0,T]$,*

(a) $B_{g(t)}(x_0, 2) \Subset M$;
(b) $||\mathrm{Rm}(g(t))||_{L^{n/2}(B_{g(t)}(x_0,2), g(t))} < \delta_0$ *for some* $\delta_0 \leq \delta_1$;
(c) $\nu(B_{g(t)}(x_0, 2), g(t), 1) \geq -A$;
(d) $\mathrm{Ric}(g(t)) \leq \alpha(n-1)t^{-1}$.

Then for all $t \in [0, T \wedge \tilde{T}]$, we have

$$\int_{B_{g(t)}(x_0,1)} |\mathrm{Rm}(g(t))|^{n/2} \, d\mathrm{vol}_{g(t)} \leq \int_{B_{g_0}(x_0,2)} |\mathrm{Rm}(g_0)|^{n/2} \, d\mathrm{vol}_{g_0} + C_1 \delta_0^{n/2} t.$$

Proof. Let $\varphi(s)$ be a smooth non-increasing function on $[0, +\infty)$ such that $\varphi \equiv 1$ on $[0,1]$, vanishes outside $[0,2]$, and satisfies $|\varphi'|^2 \leq 10^3 \varphi$ and $\varphi'' \geq -10^3 \varphi$. Let

$$\eta(x,t) := d_{g(t)}(x, x_0) + 2(\alpha+1)(n-1)\sqrt{t}$$

and define $\phi(x,t) := e^{-10^3 t}\varphi(\eta(x,t))$ for $x \in B_{g(t)}(x_0, 2)$ and $t \in [0,T]$. It follows from [Per02, Lemma 8.3] that there exists $\tilde{T} = \tilde{T}(n,\alpha) > 0$ such that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\phi \leq 0$$

for $t \in [0, T \wedge \tilde{T}]$, in the barrier sense; see also [ST22, Lemma 7.1] for a detailed discussion. The inequality also holds in the distribution sense by [MMU14, Appendix A].

Furthermore, $|\nabla\phi|^2 \leq 10^3\phi$, so that [CCL22, Lemma 2.1] implies

$$\begin{aligned}\frac{d}{dt}\int_M \phi^2|\mathrm{Rm}|^{n/2}\,d\mathrm{vol}_{g(t)} &\leq -C_0^{-1}\int_M |\nabla\left(\phi|\mathrm{Rm}|^{n/4}\right)|^2 d\mathrm{vol}_{g(t)} \\ &\quad + C_0\int_M \phi^2|\mathrm{Rm}|^{n/2+1}\,d\mathrm{vol}_{g(t)} \\ &\quad + C_0\int_M \phi|\mathrm{Rm}|^{n/2}\,d\mathrm{vol}_{g(t)} \\ &:= -\mathbf{I} + \mathbf{II} + \mathbf{III}\end{aligned}$$

for some dimensional constant $C_0(n) > 0$. We note that the evolution equation in [CCL22, Lemma 2.1] is purely local, and thus completeness is not necessary.

Since $\phi(\cdot,t)$ is compactly supported on $B_{g(t)}(x_0, 2)$ for each $t \in [0,T]$,

$$\mathbf{III} \leq C_0\delta_0^{n/2}.$$

Similarly, together with Hölder's inequality, we have

$$\begin{aligned}\mathbf{II} &\leq C_0||\mathrm{Rm}||_{L^{n/2}(B_{g(t)}(x_0,2))}\cdot\left(\int_M \phi^{\frac{2n}{n-2}}|\mathrm{Rm}|^{\frac{n^2}{2(n-2)}}\,d\mathrm{vol}_{g(t)}\right)^{\frac{n-2}{n}} \\ &\leq C_0\delta_0\cdot\left(\int_M \phi^{\frac{2n}{n-2}}|\mathrm{Rm}|^{\frac{n^2}{2(n-2)}}\,d\mathrm{vol}_{g(t)}\right)^{\frac{n-2}{n}}.\end{aligned}$$

It remains to handle $\mathbf{I}$. For any $t \in [0, T\wedge\tilde{T}]$, we apply [Mar25a, Lemma 3.3(2)] with $\Omega = B_{g(t)}(x_0, 2)$, $g = g(t)$ and $\tau = 1$ to deduce that for some $\Lambda_0(n, A) > 0$, we have

$$\left(\int_{B_{g(t)}(x_0,2)} |u|^{\frac{2n}{n-2}}\,d\mathrm{vol}_{g(t)}\right)^{\frac{n-2}{n}} \leq \Lambda_0\int_{B_{g(t)}(x_0,2)}\left(|\nabla u|^2 + u^2\right)\,d\mathrm{vol}_{g(t)} \tag{2.2}$$

for all $u \in W_0^{1,2}(B_{g(t)}(x_0, 2))$, provided that $\delta_0 \leq \delta_1$ and δ_1 is small enough depending on A. Applying (2.2) to $u := \phi|\mathrm{Rm}|^{n/4}$ yields

$$\mathbf{I} \geq (C_0\Lambda_0)^{-1}\left(\int_M \phi^{\frac{2n}{n-2}}|\mathrm{Rm}|^{\frac{n^2}{2(n-2)}}\,d\mathrm{vol}_{g(t)}\right)^{\frac{n-2}{n}} - C_0^{-1}\int_M \phi^2|\mathrm{Rm}|^{n/2}\,d\mathrm{vol}_{g(t)}.$$

Combining all, we conclude that if $\delta_0 \le \delta_1 \le (C_0^2\Lambda_0)^{-1}$, then

$$\begin{aligned}
&\frac{d}{dt}\int_M \phi^2 |\mathrm{Rm}|^{n/2}\, d\mathrm{vol}_{g(t)} \\
&\le (C_0 + C_0^{-1})\delta_0^{n/2} + \left(C_0\delta_0 - (C_0\Lambda_0)^{-1}\right)\cdot \left(\int_M \phi^{\frac{2n}{n-2}} |\mathrm{Rm}|^{\frac{n^2}{2(n-2)}}\, d\mathrm{vol}_{g(t)}\right)^{\frac{n-2}{n}} \\
&\le C_1(n)\delta_0^{n/2}.
\end{aligned}$$

The assertion follows from integrating it from the initial time slice. This completes the proof. □

Remark 2.3. Here we implicitly assume δ_1 is small enough so that assumption in [Mar25a, (2), Lemma 3.3] is fulfilled.

We now show that if we have a-priori scaling invariant curvature decay, then the assumption (b) in Lemma 2.1 will be satisfied, provided that the initial metric is quantitatively Ahlfors n-regular.

Proposition 2.4. *For all $A, v_0 > 1$, there exist $\tilde{T}(n)$, $\delta_0(n, v_0, A) > 0$ and $\alpha_0(n, v_0, A) \in (0, 1)$ such that the following holds. Suppose $g(t)$ is a smooth solution to the Ricci flow on $M \times [0, T]$ and $x_0 \in M$ such that for all $t \in [0, T]$, we have*

(a) $B_{g(t)}(x_0, 2) \Subset M$;
(b) $|\mathrm{Rm}(g(t))| \le \alpha_0 t^{-1}$ *on* $B_{g(t)}(x_0, 2)$;
(c) $\nu\left(B_{g(t)}(x_0, 2), g(t), 1\right) \ge -A$;
(d) $\mathrm{Vol}_{g_0}\left(B_{g_0}(x, r)\right) \le v_0 r^n$ *for* $x \in B_{g_0}(x_0, 1)$ *and* $r \in (0, 1]$;
(e) $||\mathrm{Rm}(g_0)||_{L^{n/2}(B_{g_0}(x_0,2))} < \delta_0$.

Then for all $t \in [0, T \wedge \tilde{T}]$,

$$\left(\int_{B_{g(t)}(x_0,1)} |\mathrm{Rm}(g(t))|^{n/2}\, d\mathrm{vol}_{g(t)}\right)^{2/n} \le \delta_1$$

where $\delta_1(n, A)$ is the constant from Lemma 2.1.

Proof. In the proof, we will use C_i to denote any dimensional constant. For $t \in [0, T]$ and $x \in B_{g(t)}(x_0, 1)$, we define $\rho(x, t) := \frac{1}{2}\left(1 - d_{g(t)}(x, x_0)\right)$, which is half of the $g(t)$-distance from x to the boundary of $B_{g(t)}(x_0, 1)$.

Let $\delta_0 \in (0, \delta_1)$ be a constant to be specified. We claim that

$$\left(\int_{B_{g(t)}(x,\rho(x,t))} |\mathrm{Rm}(g(t))|^{n/2}\, d\mathrm{vol}_{g(t)}\right)^{2/n} < \delta_1 \tag{2.5}$$

for all $x \in B_{g(t)}(x_0, 1)$ and $t \in [0, T]$.

By [ST22, Lemma 4.1], for some $\tilde{T}(n) > 0$, $B_{g(t)}(x_0, 1) \Subset B_{g_0}(x_0, 2)$ for $t \in [0, T \wedge \tilde{T}]$ and $\rho(x, t) \to 0$ as $x \to \partial B_{g(t)}(x_0, 1)$. Therefore, if the claim (2.5) is not true, then there exists $t_1 \in (0, T \wedge \tilde{T}]$ such that

(i) $||\mathrm{Rm}(g(t))||_{L^{n/2}(B_{g(t)}(x,\rho(x,t)))} < \delta_1$ for all $t \in [0, t_1)$ and $x \in B_{g(t)}(x_0, 1)$;
(ii) $||\mathrm{Rm}(g(t_1))||_{L^{n/2}(B_{g(t_1)}(x_1,\rho(x_1,t_1)))} = \delta_1$ for some $x_1 \in B_{g(t_1)}(x_0, 1)$.

We denote $\rho_1 := \rho(x_1, t_1) \le \frac{1}{2}$ for convenience.

Claim 2.6. *There exists $C_0(n) > 1$ such that for any $\beta > 1$, if $\alpha_0 \le C_0^{-2}\beta^{-2}\delta_1$, then $\rho_1 \ge \beta\sqrt{t_1}$.*

Proof of Claim 2.6. Given $\beta > 1$, we show that $\rho_1 \ge \beta\sqrt{t_1}$ if α is sufficiently small. Suppose this is not the case, that is, $\rho_1 \le \beta\sqrt{t_1}$. By (ii) and the assumption on curvature estimates,

$$\begin{aligned}\delta_1^{n/2} &\le (\alpha t_1^{-1})^{n/2} \cdot \mathrm{Vol}_{g(t_1)}\left(B_{g(t_1)}(x_1, \rho_1)\right)\\ &\le (\alpha t_1^{-1})^{n/2} \cdot \mathrm{Vol}_{g(t_1)}\left(B_{g(t_1)}(x_1, \beta\sqrt{t_1})\right).\end{aligned}$$

By the volume comparison using $\mathrm{Ric}(g(t)) \ge -(n-1)\alpha t^{-1}$ and a simple scaling argument, we can estimate the volume from above as long as $\alpha_0\beta^2 \le 1$, so that

$$\delta_1^{n/2} \le \left(\frac{1}{2}C_0\alpha_0^{1/2}\beta\right)^n,$$

which is impossible if $\alpha_0 \le C_0^{-2}\beta^{-2}\delta_1$. □

On the other hand, for all $x \in B_{g(t_1)}(x_1, \rho_1)$ and $t \in [0, t_1]$,

$$\begin{aligned}2\rho(x,t) &= 1 - d_{g(t)}(x_0, x)\\ &\ge 1 - d_{g(t)}(x_0, x_1) - d_{g(t)}(x_1, x)\\ &\ge 1 - d_{g(t_1)}(x_0, x_1) - d_{g(t_1)}(x_1, x) - C_1\sqrt{\alpha_0 t_1}\\ &\ge \rho_1 - C_1\sqrt{t_1}\end{aligned}$$

where we have used [ST22, Lemma 4.1]. Hence, by Claim 2.6, we might assume $\rho(x,t) \ge \frac{1}{4}\rho_1$ for all $x \in B_{g(t_1)}(x_1, \rho_1)$ and $t \in [0, t_1]$, if α_0 is small enough. Therefore, (i) implies

$$\int_{B_{g(t)}\left(x, \frac{1}{4}\rho_1\right)} |\mathrm{Rm}(g(t))|^{n/2}\, d\mathrm{vol}_{g(t)} \le \delta_1^{n/2} \tag{2.7}$$

for all $t \in [0, t_1]$ and $x \in B_{g(t_1)}(x_1, \rho_1)$.

Fix $z \in B_{g(t_1)}(x_1, \rho_1)$ and consider the rescaled Ricci flow $\tilde{g}(t) := \lambda^2 g(\lambda^{-2}t), t \in [0, \lambda^2 t_1]$ where $\lambda = 8\rho_1^{-1}$. By (2.7) and the assumptions, for $t \in [0, \lambda^2 t_1]$, the rescaled flow satisfies

- $B_{\tilde{g}(t)}(z, 2) = B_{g(\lambda^{-2}t)}(z, \frac{1}{4}\rho_1) \Subset M$;
- $||\mathrm{Rm}(\tilde{g}(t))||_{L^{n/2}(B_{\tilde{g}(t)}(z,2))} < \delta_1$;
- $\nu\left(B_{\tilde{g}(t)}(z, 2), \tilde{g}(t), 1\right) \ge -A$;
- $|\mathrm{Rm}(\tilde{g}(t))| \le \alpha_0 t^{-1}$.

Here we have used the monotonicity on the scale τ in ν entropy from [Wan18, Proposition 3.2] and $\lambda^{-2} = 8^{-2}\rho_1^2 \le 1$.

We apply Lemma 2.1 to $\tilde{g}(t)$ to conclude that

$$\begin{aligned}\int_{B_{\tilde{g}(t)}(z,1)} |\mathrm{Rm}(\tilde{g}(t))|^{n/2}\, d\mathrm{vol}_{\tilde{g}(t)} &\le \int_{B_{\tilde{g}_0}(z,2)} |\mathrm{Rm}(\tilde{g}_0)|^{n/2}\, d\mathrm{vol}_{\tilde{g}_0} + C_2\delta_1^{n/2}t\\ &\le \delta_0^{n/2} + C_2\delta_1^{n/2}t\end{aligned}$$

for all $t \in [0, \lambda^2 t_1 \wedge \tilde{T}(n)]$. Rescaling it back, we conclude that

$$\begin{aligned}\int_{B_{g(s)}(z,\frac{1}{8}\rho_1)} |\mathrm{Rm}(g(s))|^{n/2}\, d\mathrm{vol}_{g(s)} &\le \delta_0^{n/2} + 64C_2\delta_1^{n/2}\rho_1^{-2}s \\ &=: \delta_0^{n/2} + C_3\delta_1^{n/2}\rho_1^{-2}s\end{aligned} \tag{2.8}$$

for all $s \in [0, t_1 \wedge 8^{-2}\tilde{T}\rho_1^2]$ and $z \in B_{g(t_1)}(x_1, \rho_1)$. By Claim 2.6, if

$$\alpha_0 \le \min\left\{(8C_0)^{-2}\tilde{T}\delta_1, C_0^{-2}(4C_1/3)^{-2}\delta_2\right\},$$

then (2.8) holds at $s = t_1$.

We now use (2.8) to contradict (ii), the equality at (x_1, t_1). We choose $\{z_i\}_{i=1}^N \subset B_{g(t_1)}(x_1, \rho_1)$ such that

$$\coprod_{i=1}^N B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right) \subseteq B_{g(t_1)}(x_1,\rho_1) \subseteq \bigcup_{i=1}^N B_{g(t_1)}\left(z_i, \frac{1}{8}\rho_1\right). \tag{2.9}$$

Then (ii) and (2.8) imply

$$\delta_1^{n/2} \le N\left(\delta_0^{n/2} + C_3\delta_1^{n/2}\rho_1^{-2}t_1\right). \tag{2.10}$$

Claim 2.11. *There exist $\gamma_1(n, A), N_0(n, A, v_0) > 0$ such that if $\alpha_0 \le \gamma_1\delta_1$, then $N \le N_0$.*

Proof of Claim 2.11. By [ST22, Lemma 4.1] and Claim 2.6, we might assume $B_{g(t_1)}(x_1, \rho_1) \subseteq B_{g_0}(x_1, 2\rho_1)$ and $B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right) \subseteq B_{g(s)}(x, \rho_1/4)$ for $s \in [0, t_1]$. From (2.9) and the assumption of Ahlfors n-regularity, we have

$$\begin{aligned}\sum_{i=1}^N \mathrm{Vol}_{g_0}\left(B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right)\right) &\le \mathrm{Vol}_{g_0}\left(B_{g(t_1)}(x_1,\rho_1)\right) \\ &\le \mathrm{Vol}_{g_0}\left(B_{g_0}(x_1, 2\rho_1)\right) \le v_0(2\rho_1)^n.\end{aligned} \tag{2.12}$$

It remains to control each $\mathrm{Vol}_{g_0}\left(B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right)\right)$ from below. We compare it with volume at $t = t_1$ as follows. For $s \in [0, t_1]$ and any i,

$$\begin{aligned}&\frac{d}{ds}\mathrm{Vol}_{g(s)}\left(B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right)\right) \\ &= -\int_{B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right)} \mathcal{R}(g(s))\, d\mathrm{vol}_{g(s)} \\ &\le \left(\int_{B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right)} |\mathrm{Rm}(g(s))|^{\frac{n}{2}}\, d\mathrm{vol}_{g(s)}\right)^{2/n} \left[\mathrm{Vol}_{g(s)}\left(B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right)\right)\right]^{1-\frac{2}{n}} \\ &\le \delta_1 \left[\mathrm{Vol}_{g(s)}\left(B_{g(t_1)}\left(z_i, \frac{1}{40}\rho_1\right)\right)\right]^{1-\frac{2}{n}}\end{aligned}$$

where we have used (2.7) and $z_i \in B_{g(t_1)}(x_1,\rho_1)$. By integrating it from $s=0$ to $s=t_1$, we conclude that

$$\left[\mathrm{Vol}_{g(t_1)}\left(B_{g(t_1)}\left(z_i,\frac{1}{40}\rho_1\right)\right)\right]^{2/n} \le \left[\mathrm{Vol}_{g_0}\left(B_{g(t_1)}\left(z_i,\frac{1}{40}\rho_1\right)\right)\right]^{2/n} + \frac{2\delta_1}{n}t_1. \tag{2.13}$$

On the other hand, by assumption (c) and monotonicity of ν-entropy [Wan18, Propositions 2.1 & 3.2], $\nu(B_{g(t_1)}(z_i,\frac{1}{40}\rho_1,g(t_1),1) \ge -A$ and thus (2.7), [Mar25a, Lemmas 3.1 & 3.3], and Remark 2.3 imply

$$\mathrm{Vol}_{g(t_1)}\left(B_{g(t_1)}\left(z_i,\frac{1}{40}\rho_1\right)\right) \ge \gamma_0\cdot\rho_1^n \tag{2.14}$$

for some $\gamma_0(n,A)>0$.

By Claim 2.6 with $\beta = 2\gamma_0^{-1/n}$, then we have

$$\gamma_0^{2/n}\rho_1^2 \ge 4t_1 > \frac{4}{n}\delta_1 t_1$$

provided that $\alpha_0 \le C_0^{-2}\delta_1\cdot\min\left\{2\gamma_0^{-1/n}, 8^{-2}\tilde{T}\right\}^{-2} =: \gamma_1(n,A)\cdot\delta_1$. Using this with (2.14) and (2.13), we estimate the left hand side in (2.12) by

$$\begin{aligned}\sum_{i=1}^N \mathrm{Vol}_{g_0}\left(B_{g(t_1)}\left(z_i,\frac{1}{40}\rho_1\right)\right) &\ge \sum_{i=1}^N\left(\gamma_0^{2/n}\rho_1^2 - \frac{2\delta_1}{n}t_1\right)^{n/2}\\ &\ge N2^{-n/2}\gamma_0\rho_1^n\end{aligned}$$

so that $N \le 4^n\gamma_0^{-1}v_0 =: N_0(n,A,v_0)$ from (2.12). □

We can finish the proof of Proposition 2.4. We now choose $\delta_0(n,v_0,A) := (4N_0)^{2/n}\delta_1$ so that from Claim 2.11, we can deduce from (2.10) that

$$\rho_1^2 < 2C_3N_0t_1.$$

By Claim 2.6, this is impossible if we choose $\alpha_0 := \delta_1\cdot\min\{\gamma_1, C_0^{-2}(2C_3N_0)^{-1}\}$. This proves the claim (2.5) and the proposition. □

The upshot of Proposition 2.4 is that it allows us to re-apply Lemma 2.1 to obtain an improved estimate of curvature.

Proposition 2.15. *For $n,A>0$, there exists $\Lambda_0(n,A)>0$ such that the following holds. Under the assumptions in Proposition 2.4, if, in addition, we have $||\mathrm{Rm}(g_0)||_{L^{n/2}(B_{g_0}(x_0,1))} \le \varepsilon$, then we have*

$$|\mathrm{Rm}(x_0,t)| \le \Lambda_0 t^{-1}(\varepsilon + t^{2/n})$$

for $t\in(0,T\wedge\tilde{T}]$.

Proof. By Proposition 2.4, we have $||\mathrm{Rm}(g(t))||_{L^{n/2}(B_{g(t)}(x_0,1))} \le \delta_1$ for $t\in[0,T\wedge\tilde{T}]$, where $\delta_1(n,A)$ is the constant from Lemma 2.1. Applying Lemma 2.1 to the rescaled Ricci flow $\tilde{g}(t) := 4g(4^{-1}t), t \in$

$[0, 4(T \wedge \tilde{T})]$ and then rescaling it back to $g(t)$, we see that

$$\begin{aligned}\int_{B_{g(t)}(x_0,\frac{1}{2})} |\mathrm{Rm}(g(t))|^{n/2}\, d\mathrm{vol}_{g(t)} &\leq \int_{B_{g_0}(x_0,1)} |\mathrm{Rm}(g_0)|^{n/2}\, d\mathrm{vol}_{g(0)} + 4C_1\delta_1^{n/2}t \\ &\leq \varepsilon^{n/2} + 4C_1\delta_1^{n/2}t\end{aligned}$$

for all $t \in [0, T \wedge \tilde{T}]$. Thanks to the curvature assumption and the volume non-collapsing from [Wan18, Theorem 5.4], standard Moser iteration yields

$$t|\mathrm{Rm}(x_0, t)| \leq \Lambda_0(\varepsilon + t^{2/n})$$

for some $\Lambda_0(n, A) > 0$. This completes the proof. $\square$

In particular, Proposition 2.15 asserts that the Ricci flow has an improved estimate from the smallness of the initial local curvature concentration.

The next Lemma states that for a *complete bounded curvature* Ricci flow, the local ν-functional is (almost) preserved. This was first considered by Wang [Wan18], localizing the work of Perelman [Per02]. The following refined version is by Cheng [Che25].

Lemma 2.16 (Lemma 2.5 in [Che25])**.** *Fix $0 \leq s_1 \leq s_2 \leq 1$. Suppose $g(t)$ is a smooth solution to the Ricci flow on $M \times [s_1, s_2]$ with bounded curvature such that for all $t \in [s_1, s_2]$ and $x \in B_{g(t)}(x_0, \sqrt{t})$,*

$$\mathrm{Ric}(g(t)) \leq (n-1)\hat{\alpha}t^{-1}.$$

Then for any $\tau > 0$, $\beta \in (0, \frac{1}{10})$, and $\gamma \in [0, 8 - 20\beta)$, we have

$$\begin{aligned}&\nu(\Omega_2, g(s_2), \tau + 1 - s_2) - \nu(\Omega_1, g(s_1), \tau + 1 - s_1) \\ &\geq -\left[\frac{1+\tau}{10\hat{\alpha}^2\beta^2} + \frac{1}{e}\right] \cdot \left[\exp\left(\frac{s_2 - s_1}{10\hat{\alpha}^2\beta^2}\right) - 1\right]\end{aligned}$$

where

$$\begin{cases} \Omega_1 := B_{g(s_1)}\left(x_0, 10\hat{\alpha} - 2\hat{\alpha}\sqrt{s_1} - \hat{\alpha}\gamma\right); \\ \Omega_2 := B_{g(s_2)}\left(x_0, 10\hat{\alpha}(1-2\beta) - 2\hat{\alpha}\sqrt{s_2} - \hat{\alpha}\gamma\right). \end{cases}$$

3. Local smoothing using Ricci flow

In this section, we will construct local solutions to the Ricci flow which locally regularize metrics with small local curvature concentration, i.e. Theorem 1.1. We adapt the idea of Cheng [Che25] with the estimates in Section 2 to construct local Ricci flow, building on the idea of Hochard [Hoc19]. To prove Theorem 1.1, we need to specify some constants precisely. We need two more ingredients. The first is a result of Hochard that allows us to construct a local Ricci flow on regions with bounded curvature.

Proposition 3.1 (Proposition 4.2 in [LT22], based on [Hoc19])**.** *For $n \geq 2$ there exist constants $\gamma(n) \in (0, 1]$ and $\Lambda(n) > 1$ so that the following is true. Suppose (N^n, h_0) is a smooth manifold (not necessarily complete) that satisfies $|\mathrm{Rm}(h_0)| \leq \rho^{-2}$ throughout, for some $\rho > 0$. Then there exists a smooth complete Ricci flow $h(t)$ on N for $t \in [0, \gamma\rho^2]$, with the properties that*

(i) $h(0) = h_0$ *on* $N_\rho = \{x \in N : B_{h_0}(x, \rho) \Subset N\}$*;*
(ii) $|\mathrm{Rm}(h(t))| \leq \Lambda\rho^{-2}$ *throughout* $N \times [0, \gamma\rho^2]$.

The Ricci flow obtained in [LT22, Proposition 4.2] is constructed by modifying the incomplete metric at its extremities in order to make it complete, and followed by running the classical Shi's solution. Thus, the solution $h(t)$ is by construction a complete solution on the whole N.

We also recall the shrinking balls lemma, which is one of the local ball inclusion results based on the distance distortion estimates of Hamilton and Perelman from [Per02, Lemma 8.3].

Lemma 3.2 ([ST22, Corollary 3.3])**.** *For $n \geq 2$ there exists a constant $\beta(n) \geq 1$ such that the following is true. Suppose $(N^n, g(t))$ is a Ricci flow for $t \in [0, S]$ and $x_0 \in N$ with $B_{g(0)}(x_0, r) \Subset N$ for some $r > 0$, and $\mathrm{Ric}_{g(t)} \leq a(n-1)/t$ on $B_{g_0}(x_0, r)$ for each $t \in (0, S]$. Then for each $t \in (0, S]$,*

$$B_{g(t)}\left(x_0, r - \beta\sqrt{at}\right) \subseteq B_{g_0}(x_0, r).$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By [Mar25a, Lemma 3.2], we might assume $\nu\left(B_{g_0}(x,1), g_0, 2\right) \geq -A_0$ for some $A_0(n, C_s)$, by working on a slightly smaller ball if necessary. We need to specify how the positive constants are chosen, since the precise forms of the constants will play a crucial role in the construction.

- $\gamma(n)$ and $\Lambda(n)$ from Proposition 3.1;
- $\beta(n)$ from Lemma 3.2;
- $A := A_0 + 1$;
- $\tilde{T}(n), \delta_0(n, v_0, A)$ and $\alpha_0(n, v_0, A)$ from Proposition 2.4;
- $\Lambda_0(n, A)$ from Proposition 2.15;
- $\varepsilon_1(n, v_0, A) := \min\left\{\delta_0, \alpha_0(2\Lambda_0)^{-1}, \alpha_0(4\Lambda\Lambda_0)^{-1}\right\}$;
- $\mu(n, v_0, A) := \min\left\{10^{-2}, \gamma(2\varepsilon_1\Lambda_0)^{-1}\right\}$;
- $\sigma(n, v_0, A) := 10^{-3}(1 - (1+\mu)^{-1/5})$;
- $L(n, v_0, A) := 10^4\alpha_0^{-1/2} \cdot \max\left\{\beta, \tilde{T}^{-1/2}, \varepsilon_1^{-n}, n, \mu\sigma^{-2}, \sigma^{-2}\left[1 - (1+\mu)^{-1/5}\right]^{-1}\right\}$;
- $\hat{T}(n, v_0, A) := \min\left\{10^{-1}\alpha_0^{-1}L^{-2}, \left(\frac{1}{4}L^2\alpha_0 + \mu + 1\right)^{-1}\right\}$.

Let $\rho_0 \in (0, \frac{1}{2})$ be small such that $|\mathrm{Rm}(g_0)| \leq \rho_0^{-2}$ on $N_0 := B_{g_0}(x_0, 2 + \bar{r})$. Then from Proposition 3.1, there exists a smooth complete Ricci flow $h(t)$ on $N_0 \times [0, \gamma\rho_0^2]$ such that

- $h(0) = g_0$ on $(N_0)_{\rho_0} := B_{g_0}(x_0, 2 + \bar{r} - \rho_0)$;
- $|\mathrm{Rm}(h(t))| \leq \Lambda\rho_0^{-2}$ on $N_0 \times [0, \gamma\rho_0^2]$.

We choose $t_0 := \rho_0^2 \cdot \min\{\gamma, \Lambda^{-1}\alpha_0\}$ so that

$$|\mathrm{Rm}(h(t))| \leq \Lambda\rho_0^{-2} \leq \alpha_0 t_0^{-1} \leq \alpha_0 t^{-1}$$

on $N_0 \times (0, t_0]$. We also denote $t_{-1} = 0$ for notional consistence. We define a sequence of times t_k and radii r_k inductively as follows.

(i) $r_0 := 2 + \bar{r} - \rho_0$;
(ii) $t_{k+1} := (1+\mu)t_k$ for $k \geq 0$;
(iii) $r_{k+1} = r_k - 8L\sqrt{\alpha_0 t_k}$ for $k \geq 0$.

We consider the statement $\mathcal{P}(k)$:

There exists a smooth Ricci flow $g(t)$ defined on $B_{g_0}(x_0, r_k) \times [0, t_k]$ such that

(I) $g(0) = g_0$,
(II) $|\mathrm{Rm}(g(t))| \le \alpha_0 t^{-1}$ for $t \in (0, t_k]$, and
(III) $\nu\left(B_{g(t)}\left(x, L\sqrt{\alpha_0 t_k}\right), g(t), \frac{1}{4}L^2\alpha_0 t_k\right) \ge -A$ for all $(x, t) \in B_{g_0}(x_0, r_k) \times (0, t_k]$.

From the discussion above, since $h(t)$ is smooth and $A > A_0$, we might assume $\mathcal{P}(0)$ is true by shrinking ρ_0 if necessary, for instances using the continuity of entropy [Wan18, Corollary 2.5]. We want to show that $\mathcal{P}(k)$ is true if $r_k > 0$ and $t_k < \hat{T}$, both of which hold when $k = 0$.

We prove this by induction on k. $\mathcal{P}(0)$ is true by the construction above. Suppose that for some $k \ge 0$ with $r_{k+1} > 0$ and $t_{k+1} < \hat{T}$, $\mathcal{P}(i)$ is true for $0 \le i \le k$. We let $g(t)$ be the Ricci flow on $B_{g_0}(x_0, r_k) \times [0, t_k]$ from $\mathcal{P}(k)$. We first improve the curvature estimate on a smaller domain.

Claim 3.3. *We have*
$$|\mathrm{Rm}(g(t))| \le 2\varepsilon_1 \Lambda_0 t^{-1}$$
on $B_{g_0}(x_0, r_k - 2L\sqrt{\alpha_0 t_k}) \times [0, t_k]$.

Proof of Claim 3.3. Let $z \in B_{g_0}(x_0, r_k - 2L\sqrt{\alpha_0 t_k})$ so that Lemma 3.2 implies
$$B_{g(t)}(z, L\sqrt{\alpha_0 t_k}) \subseteq B_{g_0}(z, 2L\sqrt{\alpha_0 t_k}) \subseteq B_{g_0}(x_0, r_k)$$
for all $t \in [0, t_k]$. We consider the rescaled Ricci flow $\tilde{g}(t) := \lambda^{-2} g(\lambda^2 t)$ for $t \in [0, \lambda^{-2} t_k]$ where $\lambda = \frac{1}{2}L\sqrt{\alpha_0 t_k}$ and denote $\tilde{g}_0 := \tilde{g}(0)$. Then for all $t \in [0, 4L^{-2}\alpha_0^{-1}]$, we have

(A) $B_{\tilde{g}(t)}(z, 2) = B_{g(\lambda^2 t)}(z, 2\lambda) \Subset M$;
(B) $|\mathrm{Rm}(\tilde{g}(t))| \le \alpha_0 t^{-1}$ on $B_{\tilde{g}(t)}(z, 2)$;
(C) $\nu\left(B_{\tilde{g}(t)}(z, 2), \tilde{g}(t), 1\right) \ge -A$;
(D) $\mathrm{Vol}_{\tilde{g}_0}\left(B_{\tilde{g}_0}(x, r)\right) \le v_0 r^n$ for $x \in B_{\tilde{g}_0}(z, 1)$ and $r \in (0, 1]$;
(E) $||\mathrm{Rm}(\tilde{g}_0)||_{L^{n/2}(B_{\tilde{g}_0}(z,2), \tilde{g}_0)} < \varepsilon_1$.

Here we have used $2\lambda = L\sqrt{\alpha_0 t_k} < 1$ so that $B_{\tilde{g}_0}(x, 2) \subseteq B_{g_0}(x, 1)$ where the initial estimates hold. The entropy lower bound (C) follows from (III):
$$\nu\left(B_{\tilde{g}(t)}(z, 2), \tilde{g}(t), 1\right) = \nu\left(B_{g(s)}\left(z, L\sqrt{\alpha_0 t_k}\right), g(s), \frac{1}{4}L^2\alpha_0 t_k\right) \ge -A$$
for $s = \lambda^2 t \in [0, t_k]$. Thus, the rescaled flow $\tilde{g}(t)$ satisfies all the conditions of Proposition 2.4. It follows from Proposition 2.15 and our choice of L and ε_1 that for $t \in (0, t_k]$,
$$|\mathrm{Rm}(z, t)| \le 2\varepsilon_1 \Lambda_0 t^{-1}.$$
This proves the Claim. ☐

We now use the improved estimate from Claim 3.3 to extend the solution.

Claim 3.4. *The Ricci flow $g(t)$ can be extended to $[0, t_{k+1}]$ on $B_{g_0}(x_0, r_k - 4L\sqrt{\alpha_0 t_k})$ such that (I)-(II) in $\mathcal{P}(k+1)$ hold. Moreover, there is a complete bounded curvature Ricci flow $h_k(t)$ on $B_{g_0}(x_0, r_k - 2L\sqrt{\alpha_0 t_k}) \times [t_k, t_{k+1}]$ such that $|\mathrm{Rm}(h_k(t))| \le \alpha_0 t^{-1}$ and $h_k(t)|_{B_{g_0}(x_0, r_k - 4L\sqrt{\alpha_0 t_k})} = g(t)$.*

Proof of Claim 3.4. Denote $\alpha_1 := 2\varepsilon_1 \Lambda_0 \le \alpha_0$ for convenience. We apply Proposition 3.1 with a translation in time, on $N = N_k := B_{g_0}(x_0, r_k - 2L\sqrt{\alpha_0 t_k})$ and $h_{0,k} = g(t_k)$ with $\rho = \rho_k := \sqrt{\alpha_1^{-1} t_k}$ to obtain a complete Ricci flow $h_k(t)$ on $N_k \times [t_k, t_k + \gamma \alpha_1^{-1} t_k]$ such that

- $h_k(t_k) = h_{0,k}$ on $(N_k)_{\rho_k} = \{x \in N_k : B_{g(t_k)}(x, \rho_k) \Subset N_k\}$;
- $|\mathrm{Rm}(h_k(t))| \leq \Lambda \rho_k^{-2} = \Lambda \alpha_1 t_k^{-1}$ on $N_k \times [t_k, (1+\gamma\alpha_1^{-1})t_k]$.

It remains to show that $(N_k)_{\rho_k} \supseteq B_{g_0}(x_0, r_k - 4L\sqrt{\alpha_0 t_k})$. Let $x \in B_{g_0}(x_0, r_k - 4L\sqrt{\alpha_0 t_k})$, and hence we have $B_{g_0}(x, 2L\sqrt{\alpha_1 t_k}) \subseteq N_k$. Thanks to (II) in the induction hypothesis $\mathcal{P}(k)$, Lemma 3.2 implies

$$B_{g(t_k)}(x, L\sqrt{\alpha_0 t_k}) \Subset B_{g_0}(x, 2L\sqrt{\alpha_0 t_k}) \subseteq N_k$$

so that $x \in (N_k)_{\rho_k}$. Thus, $h_k(t_k) = h_{0,k} = g(t_k)$ on $B_{g_0}(x_0, r_k - 4L\sqrt{\alpha_0 t_k})$.

By our choice of μ, we know $t_{k+1} - t_k = \mu t_k \leq \gamma\alpha_1^{-1} t_k$. Therefore, we might patch the Ricci flow together on $B_{g_0}(x_0, r_k - 4L\sqrt{\alpha_0 t_k}) \times [0, t_{k+1}]$ by taking

$$g(t) := \begin{cases} g(t) & \text{if } t \in [0, t_k] \\ h_k(t) & \text{if } t \in [t_k, t_{k+1}] \end{cases}.$$

By our choice of ε_1, we see that for $t \in [t_k, t_{k+1}]$,

$$\begin{aligned} |\mathrm{Rm}(g_k(t))| \leq \Lambda\alpha_1 t_k^{-1} &= 2\varepsilon_1 \Lambda_0 \Lambda (1+\mu) t_{k+1}^{-1} \\ &\leq 4\varepsilon_1 \Lambda_0 \Lambda t^{-1} \leq \alpha_0 t^{-1}. \end{aligned}$$

This finishes the proof of the claim. □

We will assume $g(t)$ for $t \leq t_k$, is constructed by inductively using Claim 3.4 so that on each sub-interval $[t_i, t_{i+1}]$, the Ricci flow $g(t)$ is restriction of some complete bounded curvature Ricci flow.

It remains to show that (III) in $\mathcal{P}(k+1)$ holds. For each i, from Claim 3.4 and the induction hypothesis, there is a complete bounded curvature Ricci flow $h_i(t)$ on $B_{g_0}(x_0, r_i - 2L\sqrt{\alpha_0 t_i}) \times [t_i, t_{i+1}]$ such that

- $h_i(t)|_{B_{g_0}(x_0, r_i - 4L\sqrt{\alpha_0 t_i})} = g(t)$;
- $|\mathrm{Rm}(h_i(t))| \leq \alpha_0 t^{-1}$ where $\alpha_0 < 1 < 10^3 n$.

We now consider the rescaled Ricci flow $\tilde{g}(t) := t_k^{-1} g(t_k t)$ for $t \in [0, 1+\mu]$. For each i, we also define the rescaled Ricci flow $\tilde{h}_i(t) := t_k^{-1} h_i(t_k t)$ for $t \in [\tilde{t}_i, \tilde{t}_{i+1}]$ where $\tilde{t}_i := t_k^{-1} t_i$. For $z \in B_{g_0}(x_0, r_{k+1})$, we apply Lemma 2.16 to deduce

$$\begin{aligned} &\nu\left(B_{\tilde{h}_i(\tilde{t}_{i+1})}\left(z, \left(10(1-2\hat{\beta}) - \gamma - 2\sqrt{\tilde{t}_{i+1}}\right)\hat{\alpha}\right), \tilde{h}_i(\tilde{t}_{i+1}), \tau + 1 - \tilde{t}_{i+1}\right) \\ &\geq \nu\left(B_{\tilde{h}_i(\tilde{t}_i)}\left(z, \left(10 - \gamma - 2\sqrt{\tilde{t}_i}\right)\hat{\alpha}\right), \tilde{h}_i(\tilde{t}_i), \tau + 1 - \tilde{t}_i\right) \\ &\quad - \left(\frac{1+\tau}{10\hat{\alpha}^2\hat{\beta}^2} + \frac{1}{e}\right) \cdot \left[\exp\left(\frac{\mu\tilde{t}_i}{10\hat{\alpha}^2\hat{\beta}^2}\right) - 1\right] \end{aligned} \tag{3.5}$$

for any $\hat{\alpha} \geq 10^3 n$, $\tau > 0$, $\hat{\beta} \in (0, \frac{1}{10})$, and $\gamma \in [0, 8 - 20\hat{\beta})$. The analogous estimate also holds if $\tilde{t}_{i+1}$ is replaced by any $t \in [\tilde{t}_i, \tilde{t}_{i+1}]$.

We start to prove (III) in $\mathcal{P}(k+1)$. Suppose $\tilde{t} \in [\tilde{t}_\ell, \tilde{t}_{\ell+1}]$ for some $-1 \leq \ell \leq k$. We fix

$$\hat{\alpha} := 5^{-1} L\sqrt{\alpha_0} \geq 10^3 n \quad \text{and} \quad \tau := \mu + \frac{1}{4}L^2\alpha_0.$$

For $-1 \le i \le \ell$, we define

$$
\begin{aligned}
&\hat\beta_i := \sigma \cdot \tilde t_i^{1/5},\\
&\gamma_i := \gamma_{i-1} + 20\hat\beta_{i-1}, \text{ and}\\
&\nu_i := \nu\left(B_{\tilde h_i(\tilde t_i)}\left(z, \left(10-\gamma_i - 2\sqrt{\tilde t_i}\right)\hat\alpha\right), \tilde h_i(\tilde t_i), \tau + 1 - \tilde t_i\right),
\end{aligned}
$$

where γ_i is defined inductively starting from the final data below. Here σ is the constant defined in the beginning of the proof. Recall that we also set $t_{-1} := 0$. We also define the "initial data":

$$
\begin{aligned}
&\hat\beta_{\tilde t} := \sigma\, \tilde t^{1/5},\\
&\gamma_{\tilde t} := 10(1-2\hat\beta_{\tilde t}) - 2\sqrt{\tilde t} - L\hat\alpha^{-1}\sqrt{\alpha_0},\\
&\gamma_\ell := \gamma_{\tilde t} + 20\hat\beta_{\tilde t}, \text{ and}\\
&\nu_{\tilde t} := \nu\left(B_{\tilde h_\ell(\tilde t)}\left(z, \left(10-\gamma_{\tilde t} - 2\sqrt{\tilde t}\right)\hat\alpha\right), \tilde h_\ell(\tilde t), \tau + 1 - \tilde t\right).
\end{aligned}
$$

The definition of $\gamma_{\tilde t}$ implies

$$
10\left(1-2\hat\beta_{\tilde t}\right) - \gamma_{\tilde t} - 2\sqrt{\tilde t} = L\hat\alpha^{-1}\sqrt{\alpha_0} = 5. \tag{3.6}
$$

Since $B_{g_0}(z, 4L\sqrt{\alpha_0 t_k}) \Subset B_{g_0}(x_0, r_k - 4L\sqrt{\alpha_0 t_k})$ where $g(t) = h_i(t)$ for $t \in [t_i, t_{i+1}]$ from Claim 3.4 and the remark after it, Lemma 3.2 implies that for $t \in [t_i, t_{i+1}]$,

$$
B_{g(t)}\left(z, 3L\sqrt{\alpha_0 t_k}\right) \Subset B_{g_0}(x_0, r_i - 4L\sqrt{\alpha_0 t_i})
$$

and thus,

$$
\nu_i = \nu\left(B_{\tilde g(\tilde t_i)}\left(z, \left(10-\gamma_i - 2\sqrt{\tilde t_i}\right)\hat\alpha\right), \tilde g(\tilde t_i), \tau + 1 - \tilde t_i\right).
$$

By the monotonicity of ν-entropy [Wan18, Propositions 2.1], thanks to $t_k \le \hat T$ and our choice of $\hat T$, we have

$$
\begin{aligned}
\nu_{-1} &\ge \nu\left(B_{\tilde g_0}\left(z, 10\hat\alpha\right), \tilde g_0, \tau+1\right)\\
&= \nu\left(B_{g_0}\left(z, 2L\sqrt{\alpha_0 t_k}\right), g_0, \left(\frac14 L^2\alpha_0 + \mu + 1\right)t_k\right)\\
&\ge \nu\left(B_{g_0}(x,1), g_0, 2)\right) \ge -A_0.
\end{aligned} \tag{3.7}
$$

We show that σ is chosen so that the assumptions on $\hat\beta_i$ and γ_i in (3.5) are justified. Clearly, $\hat\beta_i \le \sigma(1+\mu)^{1/5} \le 10^{-2}$ if $\sigma \le 10^{-3}$. We now estimate γ_i. The upper bound is trivial from decreasing property so that $\gamma_i \le 5 < 8 - 20\hat\beta_i$. To estimate the lower bound, we claim that $\gamma_{-1} > 0$.

$$
\begin{aligned}
\gamma_{\tilde t} - \gamma_{-1} &= \gamma_{\tilde t} - \gamma_\ell + \sum_{i=0}^{\ell}(\gamma_i - \gamma_{i-1})\\
&\le 20\sigma \cdot \sum_{i=0}^{k}(\tilde t_{k-i})^{1/5}\\
&\le 20\sigma\sum_{i=0}^{\infty}(1+\mu)^{-i/5} = \frac{20\sigma}{1-(1+\mu)^{-1/5}}.
\end{aligned} \tag{3.8}
$$

From (3.6), we have

$$5 \le 2(1+\mu)^{1/2} + 20\sigma(1+\mu)^{1/5} + \gamma_{\tilde{t}} \tag{3.9}$$

so that (3.8) implies

$$\gamma_i \ge \gamma_{-1} \ge 5 - 2(1+\mu)^{1/2} - 20\sigma(1+\mu)^{1/5} - \frac{20\sigma}{1-(1+\mu)^{-1/5}} > 1 \tag{3.10}$$

for all $-1 \le i \le \ell$, by our choice of μ.

Without loss of generality, we might assume $\tilde{t} = \tilde{t}_{\ell+1}$. By our choices of $\hat{\beta}_i$ and γ_i, (3.5) implies that for all $-1 \le i \le \ell$,

$$\nu_{i+1} - \nu_i \ge -\left(\frac{(1+\tau)}{10\sigma^2\hat{\alpha}^2(\tilde{t}_i)^{2/5}} + \frac{1}{e}\right)\cdot\left[\exp\left(\frac{\mu(\tilde{t}_i)^{3/5}}{10\sigma^2\hat{\alpha}^2}\right) - 1\right].$$

Since $10^{-1}\mu\hat{\alpha}^{-2}\sigma^{-2}\tilde{t}_i^{3/5} \le \frac{25}{10}\mu L^{-2}\alpha_0^{-1}\sigma^{-2} \le 1$, using $e^z - 1 \le 2z$ for $z \in (0,1)$, we can simplify it as

$$\begin{aligned}\nu_{i+1} - \nu_i &\ge -2\left(\frac{10(1+\mu+\frac{1}{4}L^2\alpha_0)}{\sigma^2L^2\alpha_0(\tilde{t}_i)^{2/5}} + \frac{1}{e}\right)\cdot\left(\frac{10\mu(\tilde{t}_i)^{3/5}}{\sigma^2L^2\alpha_0}\right)\\ &\ge -10^4\sigma^{-2}L^{-2}\alpha_0^{-1}(1+\mu)^{(i-\ell)/5}\end{aligned}$$

Combining it with (3.7), we conclude that

$$\begin{aligned}\nu_{\tilde{t}} &\ge \nu_{-1} - 10^4\sigma^{-2}L^{-2}\alpha_0^{-1}\sum_{j=0}^{\ell+1}(1+\mu)^{-j/5}\\ &\ge -A_0 - \frac{10^4\sigma^{-2}}{1-(1+\mu)^{-1/5}}\cdot L^{-2}\alpha_0^{-1}\\ &\ge -A_0 - 1 = -A.\end{aligned}$$

By (3.6) and the monotonicity of ν-entropy [Wan18, Propositions 2.1] again, this proves (III) in $\mathcal{P}(k+1)$ holds. By induction, this shows that $\mathcal{P}(k)$ is true whenever $r_k > 0$ and $t_k \le \hat{T}$.

We let k_0 be the maximal k such that $r_k \ge 1+\bar{r}$, i.e., $r_{k_0+1} < 1+\bar{r} \le r_{k_0}$. We claim that t_{k_0} is uniformly bounded from below. To see this, by the construction, we have

$$\begin{aligned}1+\bar{r} > r_{k_0+1} &= r_0 + \sum_{i=0}^{k_0}(r_{i+1}-r_i)\\ &= 2+\bar{r}-\rho_0 - 8L\sqrt{\alpha_0}\cdot\sum_{i=0}^{k_0}\sqrt{t_{k_0-i}}\\ &\ge \frac{3}{2}+\bar{r} - 8L\sqrt{\alpha_0 t_{k_0}}\cdot\sum_{i=0}^{\infty}(1+\mu)^{-i/2}.\end{aligned}$$

Thus, $t_{k_0} \ge \tilde{T}(n, v_0, A) > 0$. Replacing $\hat{T}$ with $\hat{T}\wedge\tilde{T}$, we prove the existence with (1), (3), and (4) on $B_{g_0}(x_0, 1+\bar{r})$ using $\mathcal{P}(k_0)$ and the monotonicity of entropy (on scales and domains) after relabelling the constants. The injectivity radius lower bound (2) follows from (4) and the result of

Cheeger–Gromov–Taylor [CGT82]. The Hölder equivalence of distances follows from the proof of [Mar25b, Claim 3.2]; see also the proof of [CHL24, Lemma 5.1]. ☐

By letting $\bar{r} \to +\infty$, we obtain a short-time solution to the Ricci flow on $M \times [0, \hat{T}]$ with the desired estimates.

Theorem 3.11. *For $n \geq 3$ and $v_0, C_s > 0$, there exist $\delta_1(n, v_0, C_s), \Lambda_1(n, v_0, C_s), L(n, v_0, C_s)$, and $\hat{T}(n, v_0, C_s) > 0$, such that the following holds. Suppose (M^n, g_0) is a complete non-compact manifold such that for all $x \in M$,*

(a) $\mathrm{Vol}_{g_0}(B_{g_0}(x, r)) \leq v_0 r^n$ *for all* $r \in (0, 1]$;
(b) *for all* $u \in W_0^{1,2}(B_{g_0}(x, 1), g_0)$,

$$\left(\int_{B_{g_0}(x,1)} u^{\frac{2n}{n-2}} \, d\mathrm{vol}_{g_0} \right)^{\frac{n-2}{n}} \leq C_s \int_{B_{g_0}(x,1)} \left(|\nabla u|^2 + u^2 \right) \, d\mathrm{vol}_{g_0};$$

(c) $||\mathrm{Rm}(g_0)||_{L^{n/2}(B_{g_0}(x,1), g_0)} \leq \varepsilon$ *for some* $\varepsilon \in (0, \delta_1]$.

Then there exists a smooth complete Ricci flow solution $g(t)$ on $M \times [0, \hat{T}]$ such that $g(0) = g_0$ and

(1) $|\mathrm{Rm}(g(t))| \leq \Lambda_1 \varepsilon t^{-1}$;
(2) $\mathrm{inj}(g(t)) \geq \sqrt{t}$;
(3) $||\mathrm{Rm}(g(t))||_{L^{n/2}(B_{g(t)}(x, 1/2))} \leq \Lambda_1 \varepsilon$;
(4) $\nu \left(B_{g(t)}(x, 2L\sqrt{\hat{T}}, g(t), L^2 \hat{T} \right) \geq -\Lambda_1$

for all $(x, t) \in M \times (0, \hat{T}]$.

Proof. Let $x_0 \in M$ and $\bar{r}_i \to +\infty$. By Theorem 1.1, there exists a sequence of Ricci flows $g_i(t), t \in [0, \hat{T}]$ such that (1)-(4) hold for all $x \in B_{g_0}(x_0, 1 + \bar{r}_i) \times (0, \hat{T}]$. By [Che09, Corollary 3.2] and [CCG+08, Theorem 14.16], $g_i(t)$ sub-converges to a Ricci flow $g(t)$ on $M \times [0, \hat{T}]$ in the local smooth topology as $i \to +\infty$. ☐

Remark 3.12. In case $\bar{r} = +\infty$, we only manage to control the small local curvature concentration in the suitable scale; see Proposition 2.4. In contrast with the bounded curvature case, the small global curvature concentration is preserved under bounded curvature Ricci flow. It is unclear to us whether there is any jumping phenomenon of the curvature concentration in the case the curvature is initially unbounded.

4. Compactness of manifolds with bounded curvature concentration

In this section, we use the local smoothing result from Theorem 1.1 to discuss the compactness of the space of compact manifolds with bounded curvature concentration, i.e., Theorem 1.3. We start with a simple observation that the collection of manifolds satisfying the assumptions in Theorem 1.3 is pre-compact with respect to the Gromov–Hausdorff topology.

Lemma 4.1. *Suppose (M, g) is a compact manifold such that*

(1) *there exists* $C_s > 0$ *so that for all* $u \in W^{1,2}(M)$,

$$\left(\int_M u^{\frac{2n}{n-2}}\, d\mathrm{vol}_g\right)^{\frac{n-2}{n}} \le C_s \int_M \left(|\nabla u|^2 + u^2\right)\, d\mathrm{vol}_g;$$

(2) $\mathrm{diam}(M,g) \le D_0$.

Then there exists $v_1(n, C_s, D_0) > 0$ *such that for all* $0 < r \le \mathrm{diam}(M,g)$ *and* $x \in M$,

$$\mathrm{Vol}_g\left(B_g(x,r)\right) \ge v_1 r^n.$$

Proof. This follows from [Ye15, Lemma 6.1]. □

Lemma 4.2. *The collection of compact manifolds satisfying (a), (b), and (d) in Theorem 1.3 is pre-compact in the Gromov–Hausdorff topology.*

Proof. By Lemma 4.1 and (b) in Theorem 1.3, a manifold (M,g) in the collection is uniformly volume doubling. The conclusion then follows from Gromov's compactness Theorem [Gro99]. □

The basic idea in proving the compactness without Ricci geometry is based on finding a local homeomorphism from an open ball around a regular point to an open ball in a smooth manifold. This is based on mollifying the initial metrics locally using Theorem 1.1 to a metric with regularity.

Proof of Theorem 1.3. We will use $g_{i,0}$ to denote g_i to emphasize that it is the initial metric whenever smoothing is used. We closely follow the argument in [Nak88] with modifications using ideas from [ST21] and the new smoothing result, i.e., Theorem 1.1. It is more convenient to first obtain a convergent sequence in metric geometry. By Lemma 4.2, we may assume $(M_i, g_{i,0})$ converges to a metric space (X, d_∞) in the Gromov–Hausdorff topology after passing to subsequence. We also let $\varphi_i : (M_i, d_{g_{i,0}}) \to (X, d_\infty)$ be a sequence of ε_i-Gromov–Hausdorff approximations for some $\varepsilon_i \to 0$ as $i \to +\infty$.

We let $\delta_1(n, v_0, C_s)$ be the constant from Theorem 1.1. We define the singular set $\mathcal{S}_r$ of scale $r > 0$ by

$$\mathcal{S}_r := \left\{ x \in X : \liminf_{i\to\infty} ||\mathrm{Rm}(g_{i,0})||_{L^{n/2}(B_{g_{i,0}}(x_i,r))} \ge \delta_1 \text{ for any } x_i \in M_i \text{ such that } d_\infty(\varphi_i(x_i), x) \to 0 \right\}$$

and we consider

$$\mathcal{S} := \bigcap_{r>0} \mathcal{S}_r.$$

Claim 4.3. *The set* $\mathcal{S}$ *is a finite set. That is,* $\mathcal{S} = \{p_i\}_{i=1}^N$ *for some* $N \le (2\Lambda_0 \delta_1^{-1})^{n/2}$. *In particular,* $\mathcal{S}$ *is empty if* $\Lambda_0 < \delta_1/2$.

Proof of Claim 4.3. For any fixed $r > 0$, we take a collection $\{x^j\}_{j\in J} \subseteq \mathcal{S}$ such that $\left\{B_{d_\infty}(x^j, 2r) : j \in J\right\}$ is a disjoint collection of balls satisfying

$$\mathcal{S} \subseteq \bigcup_{j\in J} B_{d_\infty}\left(x^j, 3r\right).$$

For each $j \in J$, $x^j \in \mathcal{S}_r$ for $r > 0$. We can find an approximating sequence $x_i^j \in M_i$ of x^j such that $d_\infty(\varphi_i(x_i), x) \to 0$ and for large enough i, we have

$$\left(\int_{B_{g_i}(x_i^j, r)} |\mathrm{Rm}(g_{i,0})|^{n/2} d\mathrm{vol}_{g_{i,0}}\right)^{2/n} > \frac{1}{2}\delta_1$$

By the disjoint property in X and the approximating property of x_i^j's, we know that for large i, $\left\{B_{g_i}(x_i^j, r) : j \in J\right\}$ is a disjoint collection in M_i. Hence, in particular, we derive

$$|J| \leq (2\Lambda_0\delta_1^{-1})^{n/2}$$

where Λ_0 is from the assumption of Theorem 1.3. This bound is independent of r and it particularly implies

$$|\mathcal{S}| \leq (2\Lambda_0\delta_1^{-1})^{n/2}.$$

This proves the claim. $\square$

To show that the topological space M induced by $(X \setminus \mathcal{S}, d_\infty)$ is in fact a manifold with bi-Hölder charts, we must show that given an arbitrary $x \in X \setminus \mathcal{S}$, there is a neighborhood of x that is bi-Hölder homeomorphic to an open subset of a complete Riemannian manifold. We follow the treatment in [ST21].

Let $x \in X \setminus \mathcal{S}$. Then there exist $r > 0$ and $x_i \in M_i$ such that $d_\infty(\varphi_i(x_i), x) \to 0$ and

$$||\mathrm{Rm}(g_{i,0})||_{L^{n/2}(B_{g_{i,0}}(x_i,r))} < \delta_1$$

for large enough i. By considering $\tilde{g}_{i,0} := (3r)^{-2}g_{i,0}$, we might assume $r = 3$. By Theorem 1.1 with $\bar{r} = 0$, there exists a Ricci flow $g_i(t)$ on $B_{g_{i,0}}(x_i, 1) \times [0, \hat{T}]$ such that $g_i(0) = g_{i,0}$ and

(1) $|\mathrm{Rm}(g_i(t))| \leq \Lambda_1\delta_1 t^{-1}$;
(2) $\mathrm{inj}(g_i(t)) \geq \sqrt{t}$;
(3) $||\mathrm{Rm}(g_i(t))||_{L^{n/2}(B_{g_i(t)}(x_i,1/2))} \leq \Lambda_1\varepsilon$;
(4) $\nu\left(B_{g_i(t)}(x_i, 2L\sqrt{\hat{T}}, g_i(t), L^2\hat{T}\right) \geq -\Lambda_1$

for all $(x,t) \in B_{g_{i,0}}(x_i, 1) \times (0, \hat{T}]$. Moreover, we might assume that for some $C_1(n, v_0, C_s) > 0, \gamma(n, v_0, C_s) \in (0,1)$, we have

$$C_1^{-1}\left(d_{g_i(t)}(x,y)\right)^{1/\gamma} \leq d_{g_{i,0}}(x,y) \leq C_1\left(d_{g_i(t)}(x,y)\right)^{\gamma} \tag{4.4}$$

for all $x, y \in B_{g_{i,0}}(x_i, 1)$ and $t \in [0, \hat{T}]$.

We now have all necessary ingredients to carry out the proof in [ST21, Theorem 1.4 & Corollary 1.5]. We only give a sketch. We use Shi's estimates and apply smooth compactness on $g_i(\hat{T})$ to obtain a family of local diffeomorphisms F_i's such that $F_i^* g_i(t)$ sub-converges to $g_\infty(t)$ on $B_{g_\infty(\hat{T})}(x_\infty, \sigma)$ for $\sigma \leq C_1^{-1}\sigma^{-\gamma}$. Using F_i, we might use the argument in the proof of [CCL22, Theorem 1.4] to show that $F_i^* d_{g_{i,0}}$ sub-converges to a distance function $\tilde{d}_\infty$ on $B_{g_\infty(\hat{T})}(x_\infty, \sigma)$ which is locally bi-Hölder to the distance $d_{g_\infty(\hat{T})}$ thanks to (4.4). The topological manifold structure nearby x follows from the uniqueness of the pointed Gromov–Hausdorff limits, which identifies $\tilde{d}_\infty$ nearby x_∞ with d_∞ nearby x. $\square$

Remark 4.5. From Claim 4.3, when Λ_0 is sufficiently small, then $\mathcal{S}$ is empty. In this case from the proof of Theorem 1.3, $g_i(t)$ exists globally on $M_i \times [0, \hat{T}]$. In particular, we can construct X by applying smooth compactness on $(M_i, g_i(\hat{T}))$ so that X admits a smooth structure.

Remark 4.6. Using the Hölder equivalence (4.4), it should not be difficult to show that $X \setminus \mathcal{S}$ in Theorem 1.3 is a C^α manifold for some $\alpha \in (0,1)$. In general, we conjecture that $X \setminus \mathcal{S}$ is always a smooth manifold.

5. **Application to topological gap theorems**

In this section, we will use Theorem 1.1 to prove the gap theorem in differentiable structure of manifolds with small curvature concentration, i.e., Theorem 1.4. We start with the long-time existence. If the initial metric is assumed to have bounded curvature, this was obtained earlier by [CM24]; see also [Che22].

Theorem 5.1. *For $n \geq 3$ and $C_s > 0$, there exist $\delta_0(n, C_s), \Lambda(n, C_s) > 0$ such that if (M^n, g_0) is a complete non-compact manifold satisfying (a)-(c) in Theorem 1.4, then there exists a unique complete Ricci flow $g(t)$ on $M \times [0, +\infty)$ such that $g(0) = g_0$ and*

(i) $|\mathrm{Rm}(g(t))| \leq \Lambda \varepsilon t^{-1}$;
(ii) $\mathrm{inj}(g(t)) \geq \sqrt{t}$;
(iii) $||\mathrm{Rm}(g(t))||_{L^{n/2}(M, g(t))} \leq \varepsilon$;
(iv) $\nu(M, g(t)) \geq -\Lambda$.

Furthermore, we have $\sup_M |\mathrm{Rm}(g(t))| = o(1)/t$ *as* $t \to +\infty$.

Proof. By [Mar25a, Lemma 3.3], the global entropy is bounded from below. By applying Theorem 1.1 on $g_{i,0} := R_i^{-2} g_0$ for some $R_i \to +\infty$ and rescaling it back, we obtain a sequence of Ricci flows $g_i(t), t \in [0, R_i^2 \hat{T}]$ satisfying the desired properties. By [Lee25], $g_i(t) = g_{i+k}(t)$ for all $t \in [0, \hat{T} R_i]$ and $k \in \mathbb{N}$. Thus, we obtain a long-time solution $g(t)$ with (i), (ii), and (iv). It also satisfies a rough form of (iii) that the upper bound is $\Lambda \varepsilon$. By Lemma 2.1, we recover the sharp upper bound ε. The curvature estimate as $t \to +\infty$ follows from [CM24, Theorem 1.2]. □

Remark 5.2. In contrast to the bounded curvature case studied in [CM24], we require volume growth to be at most Euclidean. It is unclear whether the volume growth assumption is necessary.

Using Theorem 5.1, the diffeomorphic gap theorem is immediate from the Ricci flow existence and its estimates.

Proof of Theorem 1.4. This follows from Theorem 5.1 and [HP25, Theorem 1.1]. See also [HL21, Theorem 1.1] and [Wan20, Theorem 5.7]. □

Basic examples of manifolds satisfying the assumptions in Theorem 1.4 are compact perturbations of Euclidean metrics. By the rigidity case in positive mass Theorems, we observe that the additional condition on scalar curvature $\mathcal{R} \geq 0$ is strong enough to rule them out. Motivated by this, we conjecture:

Conjecture 5.3. *For any $C_s, v_0 > 0$ and $n \geq 3$, there exists $\delta_0(n, C_s, v_0) > 0$ such that the following holds. Suppose (M^n, g_0) is a complete non-compact manifold such that*

(a) $C_{sob}(M) \geq C_s^{-1}$;
(b) $\mathrm{Vol}_{g_0}(B_{g_0}(x,r)) \leq v_0 r^n$ *for all* $(x,r) \in M \times \mathbb{R}_+$;
(c) $||\mathrm{Rm}(g_0)||_{L^{n/2}(M)} \leq \delta_0(n, C_s, v_0)$;
(d) $\mathcal{R}(g_0) \geq 0$.

Then (M, g_0) *is isometric to* $\mathbb{R}^n$.

Remark 5.4. When (d) is strengthened to $\mathrm{Ric}(g_0) \geq 0$, the conjecture was proved by Chan and the first named author [CL24]; see also [Mar25a]. This is based on using the long-time behavior of the Ricci flow to show that the tangent cone of M is isometric to Euclidean space and thus M must be Euclidean space by Colding's volume continuity [Col97]. Using the same circle of ideas, one should still be able to show that a tangent cone of M exists by [CCL22, Theorem 1.4] and the blow-down of (M, g_0) sub-converges to $\mathbb{R}^n$ in the d_p-sense, introduced in [LNN23], using the method in [LNN23, Theorem 1.7]. It is however unclear how this asymptotic information constrains the original manifold M.

References

[And89] M. T. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc. **2** (1989), no. 3, 455–490.

[And90] ———, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), no. 2, 429–445.

[AC91] M. T. Anderson and Jeff Cheeger, *Diffeomorphism finiteness for manifolds with Ricci curvature and $L^{n/2}$-norm of curvature bounded*, Geom. Funct. Anal. **1** (1991), no. 3, 231–252.

[BKN89] S. Bando, A. Kasue, and H. Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. **97** (1989), no. 2, 313–349.

[Car19] G. Carron, *Geometric inequalities for manifolds with Ricci curvature in the Kato class*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 7, 3095–3167 (English, with English and French summaries).

[CCL22] P.-Y. Chan, E. Chen, and M.-C. Lee, *Small curvature concentration and Ricci flow smoothing*, J. Funct. Anal. **282** (2022), no. 10, Paper No. 109420, 29.

[CHL24] P.-Y. Chan, S. Huang, and M.-C. Lee, *Manifolds with small curvature concentration*, Ann. PDE **10** (2024), no. 2, Paper No. 23, 31.

[CL24] P.-Y. Chan and M.-C. Lee, *Gap theorem on manifolds with small curvature concentration*, Math. Z. **308** (2024), no. 1, Paper No. 9, 10.

[CM24] A. Chau and A. Martens, *Long-time Ricci flow existence and topological rigidity from manifolds with pinched scale-invariant integral curvature* (2024), available at `arXiv:2403.02564`.

[CTY11] A. Chau, L.-F. Tam, and C. Yu, *Pseudolocality for the Ricci flow and applications*, Canad. J. Math. **63** (2011), no. 1, 55–85.

[Che70] J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. **92** (1970), 61–74.

[Che03] ———, *Integral bounds on curvature elliptic estimates and rectifiability of singular sets*, Geom. Funct. Anal. **13** (2003), no. 1, 20–72.

[CC97] J. Cheeger and T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480. MR1484888

[CGT82] J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982), 15–53.

[Che09] B.-L. Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom. **82** (2009), no. 2, 363–382.

[Che22] E. Chen, *Convergence of the Ricci flow on asymptotically flat manifolds with integral curvature pinching*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **23** (2022), no. 1, 15–48.

[Che25] L. Cheng, *Pseudolocality theorems of Ricci flows on incomplete manifolds*, Adv. Math. **463** (2025), Paper No. 110127, 22.

[CCG+08] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *Ricci Flow: Techniques and Applications. Part II: Analytic Aspects*, Mathematical Surveys and Monographs, vol. 144, Amer. Math. Soc., Providence, RI, 2008.

[Col97] T. H. Colding, *Ricci curvature and volume convergence*, Ann. Math. **145** (1997), 477–501.

[DW] X. Dai and G. Wei, *Comparison Geometry for Ricci Curvature*. Book available at web.math.ucsb.edu/ dai/Ricci-book.pdf.

[Gal88] S. Gallot, *Isoperimetric inequalities based on integral norms of Ricci curvature*, Astérisque **157-158** (1988), 191–216. Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987).

[Gro99] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original; with appendices by M. Katz, P. Pansu, and S. Semmes; translated by Sean Michael Bates.

[HL21] F. He and M.-C. Lee, *Weakly PIC1 manifolds with maximal volume growth*, J. Geom. Anal. **31** (2021), no. 11, 10868–10885.

[Hoc19] R. Hochard, *Théorèmes d'existence en temps court du flot de Ricci pour des variétés non-complètes, non-effondrées, à courbure minorée*, Ph.D. Thesis, Université de Bordeaux, 2019.

[HP25] S. Huang and Z. Peng, *A note on a diffeomorphism criterion via long-time Ricci flow* (2025), available at `arXiv:2509.05802`.

[JW25] W. Jiang and G. Wei, *Finite diffeomorphism theorem for manifolds with lower Ricci curvature and bounded energy*, Math. Z. **309** (2025), no. 1, Paper No. 16, 17.

[Led99] M. Ledoux, *On manifolds with non-negative Ricci curvature and Sobolev inequalities*, Comm. Anal. Geom. **7** (1999), no. 2, 347–353.

[Lee25] M.-C. Lee, *Uniqueness of Ricci flow with scaling invariant estimates* (2025), available at `arXiv:2503.20292`.

[LNN23] M.-C. Lee, A. Naber, and R. Neumayer, *d_p-convergence and ϵ-regularity theorems for entropy and scalar curvature lower bounds*, Geom. Topol. **27** (2023), no. 1, 227–350.

[LT22] M.-C. Lee and P. M. Topping, *Three-manifolds with non-negatively pinched Ricci curvature* (2022), available at `arXiv:2204.00504`. J. Differential Geom. to appear.

[MMU14] C. Mantegazza, G. Mascellani, and G. Uraltsev, *On the distributional Hessian of the distance function*, Pacific J. Math. **270** (2014), no. 1, 151–166.

[Mar25a] A. Martens, *Sharpening a gap theorem: nonnegative Ricci and small curvature concentration*, Calc. Var. Partial Differential Equations **64** (2025), no. 3, Paper No. 79, 30.

[Mar25b] ———, *Removing scalar curvature assumption for Ricci flow smoothing*, Bull. Lond. Math. Soc. **57** (2025), no. 7, 1968–1989.

[MS73] J. H. Michael and L. M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of R^n*, Comm. Pure Appl. Math. **26** (1973), 361–379.

[Nak88] H. Nakajima, *Hausdorff convergence of Einstein 4-manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), no. 2, 411–424.

[Per02] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications* (2002), available at `arXiv:math.DG/0211159`.

[PW97] P. Petersen and G. Wei, *Relative volume comparison with integral curvature bounds*, Geom. Funct. Anal. **7** (1997), no. 6, 1031–1045.

[PW01] ———, *Analysis and geometry on manifolds with integral Ricci curvature bounds. II*, Trans. Amer. Math. Soc. **353** (2001), no. 2, 457–478.

[ST21] M. Simon and P. M. Topping, *Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces*, Geom. Topol. **25** (2021), no. 2, 913–948.

[ST22] ———, *Local control on the geometry in 3D Ricci flow*, J. Differential Geom. **122** (2022), no. 3, 467–518.

[TW15] G. Tian and B. Wang, *On the structure of almost Einstein manifolds*, J. Amer. Math. Soc. **28** (2015), no. 4, 1169–1209.

[Wan18] B. Wang, *The local entropy along Ricci flow. Part A: the no-local-collapsing theorems*, Camb. J. Math. **6** (2018), no. 3, 267–346.

[Wan20] ———, *The local entropy along Ricci flow—Part B: the pseudo-locality theorems* (2020), available at `arXiv:2010.09981`.

[Xia99] C. Xia, *Open manifolds with nonnegative Ricci curvature and large volume growth*, Comment. Math. Helv. **74** (1999), no. 3, 456–466.

[Xia01] ______, *Complete manifolds with nonnegative Ricci curvature and almost best Sobolev constant*, Illinois J. Math. **45** (2001), no. 4, 1253–1259.

[Yan92] D. Yang, *Convergence of Riemannian manifolds with integral bounds on curvature. I*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 1, 77–105.

[Ye15] R. Ye, *The logarithmic Sobolev and Sobolev inequalities along the Ricci flow*, Commun. Math. Stat. **3** (2015), no. 1, 1–36.

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