TURÁN DENSITIES OF STARS IN UNIFORMLY DENSE HYPERGRAPHS

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Abstract. A 3-uniform hypergraph (or 3-graph) $H = (V, E)$ is $(d, \mu, \therefore)$-*dense* if for any subsets $X, Y, Z \subseteq V$, the number of triples $(x, y, z) \in X \times Y \times Y$ with $\{x, y, z\}$ being an edge of H is at least $d|X||Y||Z| - \mu|V|^3$. Similarly, we say that H is $(d, \mu, \overset{\cdot}{\cdot\!-\!\cdot})$-*dense* if for any subset $X \subseteq V$ and every pair set $P \subseteq V \times V$, the number of pairs $(x, (y, z)) \in X \times P$ with $\{x, y, z\}$ being an edge of H is at least $d|X||P| - \mu|V|^3$. Restricting to $\therefore$-dense 3-graphs and $\overset{\cdot}{\cdot\!-\!\cdot}$-dense 3-graphs, determining the $\therefore$-*uniform Turán density* $\pi_{\therefore}(S_k)$ and the $\overset{\cdot}{\cdot\!-\!\cdot}$-*uniform Turán density* $\pi_{\overset{\cdot}{\cdot-\cdot}}(S_k)$ of the k-star S_k for $k \geq 4$ was proposed by Schacht in ICM 2022. In particular, Reiher, Rödl and Schacht presented that $\pi_{\therefore}(S_k) \geq \pi_{\overset{\cdot}{\cdot-\cdot}}(S_k) \geq \frac{k^2-5k+7}{(k-1)^2}$ for $k \geq 3$ and $\pi_{\therefore}(S_3) = \pi_{\overset{\cdot}{\cdot-\cdot}}(S_3) = 1/4$. Last year, Lamaison and Wu shown that $\pi_{\therefore}(S_k) = \frac{k^2-5k+7}{(k-1)^2}$ for $k \geq 48$.

In this paper, we show that $\pi_{\therefore}(S_k) = \frac{k^2-5k+7}{(k-1)^2}$ for $k \geq 11$. Moreover, we determine the $\overset{\cdot}{\cdot\!-\!\cdot}$-*uniform Turán density* for all S_k except for $k = 4$.

1. Introduction

Turán problems, which ask for the minimum density threshold for the existence of a certain substructure, are one of the most fundamental kinds of problems in extremal combinatorics — and among these, Turán problems of hypergraphs are notoriously difficult. Most known and conjectured extremal constructions for Turán problems of hypergraphs contain large independent sets, i.e., linear-size sets of vertices without edges. This led Erdős and Sós [4] in the 1980s to propose a variant of this problem, restricting attention to F-free 3-graphs that are uniformly dense on large subsets of the vertices. Formally, for $d \in [0, 1]$ and $\mu > 0$, we say that a 3-graph $H = (V, E)$ is $(d, \mu, \therefore)$-*dense* if for any subsets $X, Y, Z \subseteq V$, the number $e_{\therefore}(X, Y, Z)$ of triples $(x, y, z) \in X \times Y \times Z$ with $xyz \in E$ satisfies

$$e_{\therefore}(X, Y, Z) \geq d|X||Y||Z| - \mu|V|^3.$$

Restricting to $\therefore$-dense 3-graphs, the $\therefore$-*uniform Turán density* $\pi_{\therefore}(F)$ for a given 3-graph F is defined as

$$\begin{aligned}\pi_{\therefore}(F) = \sup\{d \in [0, 1] :\ &\text{for every } \mu > 0 \text{ and } n_0 \in \mathbb{N},\ \text{there exists an } F\text{-free},\\ &(d, \mu, \therefore)\text{-dense 3-graph } H \text{ with } |V(H)| \geq n_0\}.\end{aligned}$$

With this notation, Erdős and Sós asked to determine $\pi_{\therefore}(K_4^{(3)-})$ and $\pi_{\therefore}(K_4^{(3)})$, where $K_4^{(3)-}$ denotes the 3-graph consisting of three edges on four vertices and $K_4^{(3)}$ denotes the complete 3-graph on four vertices. However, determining $\pi_{\therefore}(F)$ of a given 3-graph F is also very challenging. For instance, the conjecture that $\pi_{\therefore}(K_4^{(3)}) = 1/2$ has remained a open problem in this area since Rödl [16] proposed a quasi-random construction in 1986. For $\pi_{\therefore}(K_4^{(3)-}) = 1/4$, it was solved by Glebov, Král' and Volec [7] using the flag algebra method of Razborov [11], and independently by Reiher, Rödl and Schacht [15] via the hypergraph regularity method. The latter set of authors also suggested a natural extension of $K_4^{(3)-}$. For $k \geq 3$, a k-*star*, denoted by S_k, is a 3-graph on $(k + 1)$ vertices $u, v_1, \ldots, v_k$ such that $uv_iv_j \in E(S_k)$ for all $1 \leq i < j \leq k$. Clearly, $S_3 = K_4^{(3)-}$. A generalization of the argument in [15] leads to

$$\frac{k^2 - 5k + 7}{(k-1)^2} \leq \pi_{\therefore}(S_k) \leq \left(\frac{k-2}{k-1}\right)^2 \tag{1.1}$$

for all $k \geq 3$. When $k = 3$, the upper and lower bounds coincide; however, for $k \geq 4$, a substantial gap remains. Consequently, Schacht [17] posed the following problem in his lecture at the 2022 International Congress of Mathematicians (ICM):

Problem 1.1 ([17, Problem 3.4])**.** *Determine $\pi_{\therefore}(S_k)$ for $k \geq 4$.*

For the above problem, the best known result was obtained by Lamaison and Wu [9], who proved that the lower bound in (1.1) is sharp for all $k \geq 48$. Using computer-assisted computations, they further remarked that the lower bound in (1.1) appears to be sharp already for $k \geq 40$. In this paper, we extend the approach from [9] to establish the following theorem.

Theorem 1.2. *$\pi_{\therefore}(S_k) = \frac{k^2-5k+7}{(k-1)^2}$ for all $k \geq 11$.*

For further results on $\pi_{\therefore}$-uniform Turán densities of 3-graphs, we refer to [2, 3, 5, 6, 8, 10, 14], and to the excellent survey by Reiher [12] for a more comprehensive treatment of the topic.

Another intriguing problem concerns the $\pi_{\dot\smile}$-uniform Turán densities of 3-graphs. Reiher, Rödl and Schacht [13] considered the following stronger density notion. Given $d \in [0,1]$ and $\mu > 0$, a 3-graph $H = (V,E)$ is said to be *$(d, \mu, \dot\smile)$-dense* if for every subset $X \subseteq V$ and every pair subset $P \subseteq V \times V$, the number $e_{\dot\smile}(X,P)$ of pairs $(x,(y,z)) \in X \times P$ with $xyz \in E$ satisfies

$$e_{\dot\smile}(X,P) \geq d|X||P| - \mu|V|^3.$$

Similarly as above, the *$\dot\smile$-uniform Turán density* $\pi_{\dot\smile}(F)$ of a given 3-graph F is defined as

$$\begin{aligned}\pi_{\dot\smile}(F) = \sup\{d \in [0,1] : &\text{ for every } \mu > 0 \text{ and } n_0 \in \mathbb{N}, \text{ there exists an } F\text{-free,} \\ &(d,\mu,\dot\smile)\text{-dense 3-graph } H \text{ with } |V(H)| \geq n_0\},\end{aligned}$$

and the main result in [13] establishes that $\pi_{\dot\smile}(K_4^{(3)}) = 1/2$. From the definitions, it follows immediately (by setting $P = X \times Y$) that every 3-graph F satisfies

$$\pi_{\therefore}(F) \geq \pi_{\dot\smile}(F).$$

Since the lower bounds for $\pi_{\therefore}(S_k)$ obtained in (1.1) apply to $\pi_{\dot\smile}(S_k)$ as well, Schacht [17] posed the following problem in his lecture at the 2022 International Congress of Mathematicians (ICM).

Problem 1.3 ([17, Problem 4.3])**.** *Determine $\pi_{\dot\smile}(S_k)$ for $k \geq 4$.*

For $\pi_{\dot\smile}(S_k)$, all currently known results are derived from the upper bounds of $\pi_{\therefore}(S_k)$, and we were unable to find any other relevant results addressing Theorem 1.3 in the literature. Interestingly, the approach we employed in the proof of Theorem 1.2 also resolves the above problem completely, except for the case $k = 4$.

Theorem 1.4. *$\pi_{\dot\smile}(S_k) = \frac{k^2-5k+7}{(k-1)^2}$ for all $k \geq 5$.*

Unlike most results in the area, the proofs of Theorem 1.2 and Theorem 1.4 do not rely on the hypergraph regularity lemma or on reduced hypergraphs. Instead, we employ a recently introduced framework due to Lamaison [8], which is based on *palette constructions* for determining the $\therefore$-uniform and $\dot\smile$-uniform Turán densities. In the next section, we collect the necessary notation and known tools, needed for our proofs. In the same section, we reduce Theorem 1.2 and Theorem 1.4 to a key lemma (see Theorem 2.6), whose proof is presented in Section 3. The proof of this key lemma relies on a result concerning digraphs (see Theorem 3.2), whose proof is given in Section 4. Finally, we conclude with some remarks and conjectures in Section 5.

2. Preliminaries

2.1. **Notation and tools.** For a positive integer ℓ, we denote by $[\ell]$ the set $\{1, \ldots, \ell\}$. The key concept in our proofs is the notion of palette, introduced by Reiher [12] as a generalization of a construction of Rödl [16].

Definition 2.1. A *palette* $\mathcal{P}$ is a pair $(\mathcal{C},\mathcal{A})$, where $\mathcal{C}$ is a finite color set and $\mathcal{A}\subseteq\mathcal{C}\times\mathcal{C}\times\mathcal{C}$ is a set of ordered triples of colors. The *density* of $\mathcal{P}$ is

$$d(\mathcal{P}) := |\mathcal{A}|/|\mathcal{C}|^3.$$

Moreover, given a color $a\in\mathcal{C}$ and $i\in[3]$, let $\mathcal{A}_a^i=\{(c_1,c_2,c_3)\in\mathcal{A}: c_i=a\}$. The *minimum degree* of $\mathcal{P}$ is

$$\delta(\mathcal{P}) := \min_{i\in[3],\ a\in\mathcal{C}} |\mathcal{A}_a^i|/|\mathcal{C}|^2.$$

Given a palette $\mathcal{P}=(\mathcal{C},\mathcal{A})$ and a 3-graph F, we say that $\mathcal{P}$ is F*-good* if there exists an order $\preceq$ on $V(F)$ and a function $\varphi:\binom{V(F)}{2}\to\mathcal{C}$ such that for every edge $uvw\in E(F)$ with $u\prec v\prec w$, we have $(\varphi(uv),\varphi(uw),\varphi(vw))\in\mathcal{A}$, otherwise $\mathcal{P}$ is F*-bad*.

In [8], Lamaison established two general results that reduce the problem of determining the uniform Turán densities of given 3-graphs to that of determining their corresponding *palette* Turán densities.

Theorem 2.2 ([8, Theorem 1.1])**.** *For every 3-graph F, let*

$$\pi_{\therefore}^{\mathrm{pal}}(F) := \sup\{d(\mathcal{P}) : \mathcal{P} \textit{ is a } F\textit{-bad palette}\}.$$

Then $\pi_{\therefore}(F)=\pi_{\therefore}^{\mathrm{pal}}(F)$.

Theorem 2.3 ([8, Theorem 5.1])**.** *For every 3-graph F, let*

$$\pi_{\because}^{\mathrm{pal}}(F) := \sup\{\delta(\mathcal{P}) : \mathcal{P} \textit{ is a } F\textit{-bad palette}\}.$$

Then $\pi_{\because}(F)=\pi_{\because}^{\mathrm{pal}}(F)$.

As mentioned above, Reiher, Rödl and Schacht [15] provided a palette construction that shows $\min\{\pi_{\therefore}(S_k),\pi_{\because}(S_k)\}\geq\frac{k^2-5k+7}{(k-1)^2}$ for all $k\geq 4$. Thus, by Theorem 2.2 (or Theorem 2.3), it is sufficient to show that $\pi_{\therefore}^{\mathrm{pal}}(S_k)\leq\frac{k^2-5k+7}{(k-1)^2}$ (or $\pi_{\because}^{\mathrm{pal}}(S_k)\leq\frac{k^2-5k+7}{(k-1)^2}$) for Theorem 1.2 (or Theorem 1.4).

Next, we recall the definition and properties of the *auxiliary digraph* $D_{\mathcal{P}}=(V,E)$ of a palette $\mathcal{P}$. This notion was introduced by Lamaison and Wu [9] to study the upper bound of $\pi_{\therefore}^{\mathrm{pal}}(S_k)$ for $k\geq 48$.

Let $\mathcal{P}=(\mathcal{A},\mathcal{C})$ be a palette. Given colors $a,b\in\mathcal{C}$, and $i\neq j\in[3]$, we say that (a,b) is (i,j)-*admissible* if there exists a triple $(c_1,c_2,c_3)\in\mathcal{A}$ such that $c_i=a$ and $c_j=b$. Let $d_{i,j}(a)$ denote the number of colors b such that (a,b) is (i,j)-admissible and let $e_{i,j}(a)=d_{i,j}(a)/|\mathcal{C}|$. For any $\mathcal{P}=(\mathcal{C},\mathcal{A})$, we define an *auxiliary digraph* $D_{\mathcal{P}}=(V,E)$ as follows:

- $V=\mathcal{C}_1\cup\mathcal{C}_2$ consisting of two disjoint copies of $\mathcal{C}$.
- For each color $a\in\mathcal{C}$, we use a^1 and a^2 to denote its copies in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.
- For every pair of colors $(a,b)\in\mathcal{C}\times\mathcal{C}$, we add arcs by the following rules:
  - add an arc a^1b^1 in $D_{\mathcal{P}}[\mathcal{C}_1]$ if (a,b) is $(2,3)$-admissible,
  - add an arc a^2b^2 in $D_{\mathcal{P}}[\mathcal{C}_2]$ if (a,b) is $(1,2)$-admissible,
  - add the arcs a^1b^2 and b^2a^1 if (a,b) is $(1,3)$-admissible.

Let T_k be a transitive tournament on k vertices, i.e., $T_k=(V,E)$ with $V=[k]$ and $E=\{xy\in[k]\times[k]: x<y\}$. Lamaison and Wu [9] found that the auxiliary digraphs have the following properties.

Lemma 2.4. *For any $k\geq 3$, if a palette $\mathcal{P}$ is S_k-bad, then $D_{\mathcal{P}}$ has no loop and $D_{\mathcal{P}}$ is T_k-free.*

We shall also need the maximum number of arcs in an T_k-free digraph. The following lemma follows directly from a theorem of Brown and Harary [1], by noting that they determined the Turán number of tournaments.

Lemma 2.5. *For any $n\geq k\geq 3$, if D is a n-vertex T_k-free digraph, then the number of arcs of D is at most $\frac{k-2}{k-1}(n^2-\alpha^2)+\binom{\alpha}{2}$, with $\alpha=n \pmod{k-1}$.*

2.2. **Proof by reduction.** Using Theorem 2.2, Theorem 2.3 and the lower bound in (1.1), we now present the proofs of Theorem 1.2 and Theorem 1.4 by reducing to the following a key lemma, whose proofs are given in the next section.

Lemma 2.6. *Let $k \geq 5$ be an integer. For any palette $\mathcal{P}$ with $\delta(\mathcal{P}) \geq \frac{1}{4}$, if $\mathcal{P}$ is S_k-bad, then $d(\mathcal{P}) \leq \frac{k^2-5k+7}{(k-1)^2}$.*

Proof of Theorem 1.2. By Theorem 2.2 and the lower bound in (1.1), it suffices to show that every S_k-bad palette $\mathcal{P}$ satisfies that $d(\mathcal{P}) \leq \frac{k^2-5k+7}{(k-1)^2}$ for all $k \geq 11$. Suppose for the sake of contradiction that there exist S_k-bad palettes with density greater than $\frac{k^2-5k+7}{(k-1)^2}$. Let $\mathcal{P} = (\mathcal{C}, \mathcal{A})$ be one such palette with the minimum number of colors, i.e., removing any color from $\mathcal{C}$ must decrease the density of $\mathcal{P}$. Then we have the following claim.

Claim 2.7. $\delta(\mathcal{P}) \geq 3d(\mathcal{P}) - \frac{2k-3}{k-1}$.

Proof. Let $|\mathcal{C}| = n$ and $D_i := D_{\mathcal{P}}[\mathcal{C}_i]$ be the induced sub-digraph of $D_{\mathcal{P}}$ on $\mathcal{C}_i$ for $i \in [2]$. Combining Theorem 2.4 and Theorem 2.5, we have that $|E(D_i)| \leq \frac{k-2}{k-1}n^2$ for all $i \in [2]$. Note that, for each $a \in \mathcal{C}$, we also have $|\mathcal{A}_a^1| \leq |E(D_1)|$ and $|\mathcal{A}_a^3| \leq |E(D_2)|$. Now assume that $\mathcal{P}' = (\mathcal{C}', \mathcal{A}')$ is the palette obtained by removing the color a from $\mathcal{C}$ such that $|\mathcal{A}_a^i| = \delta(\mathcal{P})n^2$ for some $\ell \in [3]$. Set $\{x, y\} = [3] \setminus \{\ell\}$. Then

$$\begin{aligned}
|\mathcal{A}| &= |\mathcal{A}'| + |\mathcal{A} \setminus \mathcal{A}'| \\
&= |\mathcal{A}'| + |\mathcal{A}_a^\ell| + |\mathcal{A}_a^x \setminus \mathcal{A}_a^\ell| + |\mathcal{A}_a^y \setminus (\mathcal{A}_a^\ell \cup \mathcal{A}_a^x)| \\
&\leq |\mathcal{A}'| + \delta(\mathcal{P})n^2 + \frac{k-2}{k-1}n^2 + (n-1)^2 \\
&\leq (n-1)^3 d(\mathcal{P}) + \delta(\mathcal{P})n^2 + \frac{2k-3}{k-1}n^2 - 2n + 1,
\end{aligned}$$

where the last inequality uses the condition that removing any color from $\mathcal{C}$ decreases the density of $\mathcal{P}$. Hence,

$$\begin{aligned}
\delta(\mathcal{P})n^2 &\geq (n^3 - (n-1)^3)d(\mathcal{P}) - \frac{2k-3}{k-1}n^2 + 2n - 1 \\
&\geq (3d(\mathcal{P}) - \frac{2k-3}{k-1})n^2.
\end{aligned}$$

■

When $k \geq 11$, one can easily check that $\delta(\mathcal{P}) \geq 3d(\mathcal{P}) - \frac{2k-3}{k-1} \geq \frac{1}{4}$. By Theorem 2.6, $d(\mathcal{P}) \leq \frac{k^2-5k+7}{(k-1)^2}$, which is a contradiction. ■

Proof of Theorem 1.4. By Theorem 2.3 and the lower bound in (1.1), it suffices to show that every S_k-bad palette $\mathcal{P}$ satisfies that $\delta(\mathcal{P}) \leq \frac{k^2-5k+7}{(k-1)^2}$ for all $k \geq 5$. By Theorem 2.6, the conclusion follows directly. ■

3. Proof of Theorem 2.6

Given a digraph D, for each vertex $v \in V(D)$, let $d_D^+(v)$ and $d_D^-(v)$ denote the out-degree and in-degree of v, respectively. By the definition of the auxiliary digraphs, we obtain the following observation.

Observation 3.1. *Given a palette $\mathcal{P} = (\mathcal{A}, \mathcal{C})$, let $D := D_{\mathcal{P}}$ be the auxiliary digraph of $\mathcal{P}$, and let $D_i := D[\mathcal{C}_i]$ be the induced sub-digraph on $\mathcal{C}_i$ for $i \in [2]$. Then for each $a \in \mathcal{C}$,*

$$d_{D_1}^+(a^1) = d_{2,3}(a), \qquad d_{D_2}^-(a^2) = d_{2,1}(a),$$

$$d^+_D(a^1) = d_{1,2}(a) + d_{1,3}(a), \qquad d^-_D(a^2) = d_{3,1}(a) + d_{3,2}(a).$$

For the proof of Theorem 2.6, the key step for us is to show that the following conclusion holds for T_k-free digrahps.

Lemma 3.2. *Given $k \geq 4$, let D be a T_k-free digraph on n vertices. For each vertex v, let $m(v) = \max\{d^+_D(v)/n, d^-_D(v)/n\}$. Let $V' = \{v \in V(D) : m(v) \geq \frac{2}{k-1}\}$, then*

$$\sum_{v \in V'} (m(v) - \frac{1}{2})^2 \leq \frac{(k-3)^2}{4(k-1)^2} n.$$

Below we deduce Theorem 2.6 using Theorem 2.4, Theorem 3.1 and Theorem 3.2. We postpone the proof of Theorem 3.2 to the next section.

Proof of Theorem 2.6. Given $k \geq 5$ and an S_k-bad palette $\mathcal{P} = (\mathcal{C}, \mathcal{A})$ with $\delta(\mathcal{P}) \geq \frac{1}{4}$. Let $|\mathcal{C}| = n$ and D be the auxiliary digraph of $\mathcal{P}$, and let $D_i := D[\mathcal{C}_i]$ be the induced sub-digraph on $\mathcal{C}_i$ for $i \in [2]$. By Theorem 2.4, D, D_1 and D_2 are T_k-free. For each $a \in \mathcal{C}$ and $i \in [2]$, set

$$m_D(a^i) := \max\{\frac{d^+_D(a^i)}{2n}, \ \frac{d^-_D(a^i)}{2n}\} \text{ and } m_{D_i}(a^i) = \max\{\frac{d^+_{D_i}(a^i)}{n}, \ \frac{d^-_{D_i}(a^i)}{n}\}.$$

By Theorem 3.1, for each $a \in \mathcal{C}$, we have

$$e_{2,3}(a) = \frac{d_{2,3}(a)}{n} = \frac{d^+_{D_1}(a^1)}{n} \leq m_{D_1}(a^1), \quad e_{2,1}(a) = \frac{d_{2,1}(a)}{n} = \frac{d^-_{D_2}(a^2)}{n} \leq m_{D_2}(a^2), \tag{3.1}$$

$$\frac{1}{2}(e_{1,2}(a) + e_{1,3}(a)) = \frac{d_{1,2}(a) + d_{1,3}(a)}{2n} = \frac{d^+_D(a^1)}{2n} \leq m_D(a^1), \tag{3.2}$$

$$\frac{1}{2}(e_{3,1}(a) + e_{3,2}(a)) = \frac{d_{3,1}(a) + d_{3,2}(a)}{2n} = \frac{d^-_D(a^2)}{2n} \leq m_D(a^2). \tag{3.3}$$

Next, we aim to bound the density of $\mathcal{P}$ by Theorem 3.2. Let

$$X_1 = \{(a,b,c) : (a,b,c) \in \mathcal{C} \times \mathcal{C} \times \mathcal{C}, \text{ for all } d \in \mathcal{C}, (d,b,c) \notin \mathcal{A}\},$$
$$X_2 = \{(a,b,c) : (a,b,c) \in \mathcal{C} \times \mathcal{C} \times \mathcal{C}, \text{ for all } d \in \mathcal{C}, (a,d,c) \notin \mathcal{A}\},$$
$$X_3 = \{(a,b,c) : (a,b,c) \in \mathcal{C} \times \mathcal{C} \times \mathcal{C}, \text{ for all } d \in \mathcal{C}, (a,b,d) \notin \mathcal{A}\}.$$

Note that for any $(a,b,c) \in \mathcal{A}$, we have $(a,b,c) \notin X_1 \cup X_2 \cup X_3$, and thus,

$$\begin{aligned} |\mathcal{A}| &\leq n^3 - |X_1 \cup X_2 \cup X_3| \\ &\leq n^3 - |X_1| - |X_2| - |X_3| + |X_1 \cap X_2| + |X_1 \cap X_3| + |X_2 \cap X_3|. \end{aligned} \tag{3.4}$$

For $a \in \mathcal{C}$, and $i \neq j \in [3]$, let $d'_{i,j}(a) = n - d_{i,j}(a)$. Note that $(a,b,c) \in X_1$ is equivalent to (b,c) not being $(2,3)$-admissible, hence

$$|X_1| = n \sum_{a \in \mathcal{C}} d'_{2,3}(a) = n \sum_{a \in \mathcal{C}} d'_{3,2}(a) = \frac{n}{2} \left(\sum_{a \in \mathcal{C}} d'_{2,3}(a) + \sum_{a \in \mathcal{C}} d'_{3,2}(a) \right).$$

Moreover, $(a,b,c) \in X_1 \cap X_2$ is equivalent to (b,c) not being $(2,3)$-admissible and (a,c) not being $(1,3)$-admissible, hence

$$|X_1 \cap X_2| = \sum_{a \in \mathcal{C}} d'_{3,1}(a) d'_{3,2}(a).$$

Using the same argument for X_2, X_3, we have

$$\begin{aligned} d(\mathcal{P}) &\overset{(3.4)}{\leq} 1 - \frac{|X_1|}{n^3} - \frac{|X_2|}{n^3} - \frac{|X_3|}{n^3} + \frac{|X_1 \cap X_2|}{n^3} + \frac{|X_1 \cap X_3|}{n^3} + \frac{|X_2 \cap X_3|}{n^3} \\ &\leq 1 - \frac{1}{2n^2} \sum_{a \in \mathcal{C}} \sum_{i \neq j} d'_{i,j}(a) + \frac{1}{n^3} \sum_{a \in \mathcal{C}} \left(d'_{1,2}(a) d'_{1,3}(a) + d'_{2,1}(a) d'_{2,3}(a) + d'_{3,1}(a) d'_{3,2}(a) \right) \end{aligned}$$

$$
\begin{aligned}
&= 1 + \frac{1}{n^3}\sum_{a\in\mathcal{C}}\left(d'_{1,2}(a)d'_{1,3}(a) + d'_{2,1}(a)d'_{2,3}(a) + d'_{3,1}(a)d'_{3,2}(a) - \frac{n}{2}\sum_{i\neq j} d'_{i,j}(a)\right)\\
&= 1 + \frac{1}{n^3}\sum_{a\in\mathcal{C}}\left(d_{1,2}(a)d_{1,3}(a) + d_{2,1}(a)d_{2,3}(a) + d_{3,1}(a)d_{3,2}(a) - \frac{n}{2}\sum_{i\neq j} d_{i,j}(a)\right)\\
&= 1 + \frac{1}{n}\sum_{a\in\mathcal{C}}\left(f_1(a) + f_2(a) + f_3(a)\right), \qquad (3.5)
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(a) &:= e_{1,2}(a)e_{1,3}(a) - \frac{1}{2}(e_{1,2}(a) + e_{1,3}(a)),\\
f_2(a) &:= e_{2,1}(a)e_{2,3}(a) - \frac{1}{2}(e_{2,1}(a) + e_{2,3}(a)),\\
f_3(a) &:= e_{3,1}(a)e_{3,2}(a) - \frac{1}{2}(e_{3,1}(a) + e_{3,2}(a)).
\end{aligned}
$$

Next, we will estimate the upper bounds of $f_1(a)$, $f_2(a)$, and$f_3(a)$ in different ways.

For any $a \in \mathcal{C}$, by the basic inequality $xy \le \left(\frac{x+y}{2}\right)^2$, we have

$$
f_1(a) \le \left(\frac{e_{1,2}(a) + e_{1,3}(a)}{2} - \frac{1}{2}\right)^2 - \frac{1}{4} \text{ and } f_3(a) \le \left(\frac{e_{3,1}(a) + e_{3,2}(a)}{2} - \frac{1}{2}\right)^2 - \frac{1}{4}. \qquad (3.6)
$$

Note that we have

$$
\frac{e_{1,2}(a) + e_{1,3}(a)}{2} \ge \sqrt{e_{1,2}(a)e_{1,3}(a)} = \frac{1}{n}\sqrt{d_{1,2}(a)d_{1,3}(a)} \ge \frac{1}{n}\sqrt{|\mathcal{A}_a^1|} \ge \sqrt{\delta(\mathcal{P})} \ge \frac{1}{2}.
$$

Similarly, we also have $\frac{e_{3,1}(a)+e_{3,2}(a)}{2} \ge \frac{1}{2}$. By (3.2) and (3.3), we have

$$
f_1(a) \le \left(m_D(a^1) - \frac{1}{2}\right)^2 - \frac{1}{4} \text{ and } f_3(a) \le \left(m_D(a^2) - \frac{1}{2}\right)^2 - \frac{1}{4}. \qquad (3.7)
$$

For the estimation of $f_2(a)$, we will perform different ways for the following two cases.

Case1: Let $\mathcal{C}' := \{a \in \mathcal{C} : \min\{e_{2,1}(a),\ e_{2,3}(a)\} \ge 1/2$**}.** For each $a \in \mathcal{C}'$, by the basic inequality $xy \le \frac{x^2+y^2}{2}$, we have

$$
\begin{aligned}
f_2(a) &\le \frac{1}{2}\left(e_{2,1}(a) - \frac{1}{2}\right)^2 + \frac{1}{2}\left(e_{2,3}(a) - \frac{1}{2}\right)^2 - \frac{1}{4}\\
&\overset{(3.1)}{\le} \frac{1}{2}\left(m_{D_2}(a^2) - \frac{1}{2}\right)^2 + \frac{1}{2}\left(m_{D_1}(a^1) - \frac{1}{2}\right)^2 - \frac{1}{4}. \qquad (3.8)
\end{aligned}
$$

Case2: Let $\mathcal{C}'' := \{a \in \mathcal{C} : \min\{e_{2,1}(a),\ e_{2,3}(a)\} < 1/2$**}.** For each $a \in \mathcal{C}''$, set $t := e_{2,1}(a)e_{2,3}(a)$. Then $t \ge \delta(\mathcal{P}) \ge 1/4$. Besides, let $x := \min\{e_{2,1}(a),\ e_{2,3}(a)\}$. Then

$$
f_2(a) = t - \frac{1}{2}(x + \frac{t}{x}) \le -\frac{1}{4}. \qquad (3.9)
$$

Combining **Case1** and **Case2**, we have

$$
\begin{aligned}
\sum_{a\in\mathcal{C}} f_2(a) &= \sum_{a\in\mathcal{C}'} f_2(a) + \sum_{a\in\mathcal{C}''} f_2(a)\\
&\overset{(3.9)}{\le} \sum_{a\in\mathcal{C}'} f_2(a) + \sum_{a\in\mathcal{C}''} \frac{-1}{4}
\end{aligned}
$$

$$\overset{(3.8)}{\leq} \frac{1}{2}\sum_{a\in\mathcal{C}'}\left(m_{D_2}(a^2)-\frac{1}{2}\right)^2+\frac{1}{2}\sum_{a\in\mathcal{C}'}\left(m_{D_1}(a^1)-\frac{1}{2}\right)^2+\sum_{a\in\mathcal{C}'}\frac{-1}{4}+\sum_{a\in\mathcal{C}''}\frac{-1}{4}$$
$$\leq \frac{1}{2}\sum_{a\in\mathcal{C}'}\left(m_{D_2}(a^2)-\frac{1}{2}\right)^2+\frac{1}{2}\sum_{a\in\mathcal{C}'}\left(m_{D_1}(a^1)-\frac{1}{2}\right)^2-\frac{n}{4}. \tag{3.10}$$

Let $V = V(D)$ and $V_i' = \mathcal{C}'$ for $i \in [2]$. It follows from (3.5), (3.7) and (3.10) that

$$\begin{aligned} d(\mathcal{P}) &\overset{(3.5)}{\leq} 1+\frac{1}{n}\sum_{a\in\mathcal{C}}(f_1(a)+f_3(a))+\frac{1}{n}\sum_{a\in\mathcal{C}}f_2(a) \\ &\overset{(3.7)}{\leq} 1+\frac{1}{n}\sum_{a\in\mathcal{C}}\left(\left(m_D(a^1)-\frac{1}{2}\right)^2+\left(m_D(a^2)-\frac{1}{2}\right)^2\right)-\frac{1}{2}+\frac{1}{n}\sum_{a\in\mathcal{C}}f_2(a) \\ &\overset{(3.10)}{\leq} \frac{1}{4}+\frac{1}{n}\sum_{v\in V}\left(m_D(v)-\frac{1}{2}\right)^2+\frac{1}{2n}\left(\sum_{v\in V_2'}\left(m_{D_2}(v)-\frac{1}{2}\right)^2+\sum_{v\in V_1'}\left(m_{D_1}(v)-\frac{1}{2}\right)^2\right). \end{aligned}$$

Note that $m_D(a^i) \geq \frac{2}{k-1}$ holds for all $a \in \mathcal{C}$, $i \in [2]$ and $k \geq 5$. Besides, $m_{D_i}(a^i) \geq \frac{2}{k-1}$ holds for all $a \in \mathcal{C}'$, $i \in [2]$ and $k \geq 5$. By Theorem 3.2, we have

$$\begin{aligned} d(\mathcal{P}) &\leq \frac{1}{4}+\frac{1}{n}\cdot\frac{(k-3)^2}{4(k-1)^2}\cdot 2n+\frac{1}{2n}\cdot\left(\frac{(k-3)^2}{4(k-1)^2}\cdot n+\frac{(k-3)^2}{4(k-1)^2}\cdot n\right) \\ &= \frac{k^2-5k+7}{(k-1)^2}. \end{aligned}$$

■

4. Proof of Theorem 3.2

We first introduce a variation of the Caro-Wei theorem in digraphs.

Lemma 4.1. *Given $k \geq 2$, let D be a T_k-free digraph on n vertices. For each vertex v, let $m(v) = \max\{d_D^+(v)/n, d_D^-(v)/n\}$. Then*

$$\sum_{v\in V(H)}\frac{m(v)}{1-m(v)} \leq (k-2)n.$$

The proof of Theorem 4.1 can be obtained by induction on k. Since its proof is similar to the proof of [9, Lemma 6], we omit it here.

Proof of Theorem 3.2. For each vertex $v \in V(D)$, let $M(v) := \frac{m(v)}{1-m(v)}$. Then $M(v) \geq \frac{2}{k-3}$ and

$$\left(m(v)-\frac{1}{2}\right)^2 = \left(\frac{M(v)}{M(v)+1}-\frac{1}{2}\right)^2. \tag{4.1}$$

Set $g(x) = \left(\frac{x}{x+1}-\frac{1}{2}\right)^2$. For $k \geq 4$, we consider the following inequality

$$g(x) \leq \frac{k-3}{(k-1)^3}x+\frac{(k-3)(k^2-8k+11)}{4(k-1)^3}. \tag{4.2}$$

Note that the above inequality is equivalent to $(x-(k-2))^2((k-3)x-2) \geq 0$. Thus, (4.2) holds for all real $x \geq \frac{2}{k-3}$.

Combining (4.1) and (4.2), we obtain that

$$\left(m(v)-\frac{1}{2}\right)^2 \leq \frac{k-3}{(k-1)^3}M(v)+\frac{(k-3)(k^2-8k+11)}{4(k-1)^3}$$

holds for $M(v) \geq \frac{k-1}{k-3}$. Hence, by Theorem 4.1, we have

$$\begin{aligned}\sum_{v\in V'}(m(v)-\frac{1}{2})^2 &\leq \frac{k-3}{(k-1)^3}\sum_{v\in V'}M(v)+\frac{(k-3)(k^2-8k+11)}{4(k-1)^3}n\\ &\leq \frac{k-3}{(k-1)^3}\cdot(k-2)n+\frac{(k-3)(k^2-8k+11)}{4(k-1)^3}n\\ &= \frac{(k-3)^2}{4(k-1)^2}n.\end{aligned}$$

■

5. Concluding remarks

In this paper, we established that $\pi_{\therefore}(S_k)=\frac{k^2-5k+7}{(k-1)^2}$ for $k\geq 11$ (see Theorem 1.2) and $\pi_{\underline{\therefore}}(S_k)=\frac{k^2-5k+7}{(k-1)^2}$ for $k\geq 5$ (see Theorem 1.4), which leaves following two conjectures.

Conjecture 5.1. *$\pi_{\therefore}(S_k)=\frac{k^2-5k+7}{(k-1)^2}$ for all $4\leq k\leq 10$.*

Conjecture 5.2. *$\pi_{\underline{\therefore}}(S_4)=\frac{1}{3}$.*

In view of the proof of Theorem 1.2, we believe that a more refined analysis of the structure of the auxiliary digraph $D_{\mathcal{P}}$ would allow us to relax Theorem 2.7 to $\delta(\mathcal{P})\geq 3d(\mathcal{P})-\frac{2k-2}{k-1}$, which would imply that $\pi_{\therefore}(S_k)=\frac{k^2-5k+7}{(k-1)^2}$ holds for $k\geq 10$.

Recalling the statement of Theorem 2.6, if the condition $k\geq 5$ could be relaxed to $k\geq 4$, then Theorem 5.2 would follow directly from the proof of Theorem 1.4. In the proof of Theorem 2.6, the assumption $k\geq 5$ ensures that $m(v)\geq 1/2\geq 2/(k-1)$, which guarantees that the hypothesis of Theorem 3.2 is satisfied. When $k=4$, however, the condition in Theorem 3.2 requires $m(v)\geq 2/(k-1)=2/3$. One natural approach to proving Theorem 5.2 is to relax the condition in Theorem 3.2. However, the threshold $m(v)\geq 2/3$ is in fact optimal for Theorem 3.2, as the following construction shows.

Fix $\varepsilon>0$ and large $n\in\mathbb{N}$. Consider the digraph $D=(V,E)$ with vertex partition $V=X\cup Y\cup Z$ such that $|X|=|Y|=(1/3-\varepsilon)n$ and $Z=V\setminus(X\cup Y)$. Define

$$E=\{xy,yx:(x,y)\in X\times Y\}\cup\{xz,zx:(x,z)\in X\times Z\}\cup\{yz,zy:(y,z)\in Y\times Z\}.$$

For any $x\in X$, $y\in Y$ and $z\in Z$, we have

$$m(x)=m(y)=(2/3+\varepsilon)n \quad\text{and}\quad m(z)=(2/3-2\varepsilon)n.$$

If we consider $V'=\{v\in V(D):m(v)\geq\frac{2}{3}-2\varepsilon\}$, then

$$\sum_{v\in V'}(m(v)-\frac{1}{2})^2>\frac{1}{16}n\overset{(k=3)}{=}\frac{(k-3)^2}{4(k-1)^2}n.$$

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