Dominating Hadwiger's Conjecture holds for all $2K_2$-free graphs

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Abstract

A dominating K_t minor in a graph G is a sequence $(T_1, \ldots, T_t)$ of pairwise disjoint non-empty connected subgraphs of G, such that for $1 \le i < j \le t$, every vertex in T_j has a neighbor in T_i. Replacing "every vertex in T_j" by "some vertex in T_j" retrieves the standard definition of a K_t minor. The strengthened notion was introduced by Illingworth and Wood [arXiv:2405.14299], who asked whether every graph with chromatic number t contains a dominating K_t minor. This is a substantial strengthening of the celebrated Hadwiger's Conjecture, which asserts that every graph with chromatic number t contains a K_t minor. At the "New Perspectives in Colouring and Structure" workshop held at the Banff International Research Station from September 29 - October 4, 2024, Norin referred to this question as the "Dominating Hadwiger's Conjecture" and believes it is likely false. In this paper we prove that the Dominating Hadwiger's Conjecture holds for all $2K_2$-free graphs. A key component of our proof is the clever use of the existence of an induced banner, obtained by adding a vertex adjacent to exactly one vertex on a cycle of length four.

1 Introduction

All graphs in this paper are finite and simple. For a graph G, we use $\alpha(G)$ and $\chi(G)$ to denote the independence number and chromatic number of G, respectively. A *clique* in a graph G is a set of pairwise adjacent vertices. For a vertex $v \in V(G)$, we use $N(v)$ to denote the set of vertices in G which are all adjacent to v. If $X, Y \subseteq V(G)$ are disjoint, we say that X is *complete* to Y if every vertex in X is adjacent to all vertices in Y, and X is *anticomplete* to Y if no vertex in X is adjacent to any vertex in Y. If $X = \{x\}$, we simply say x is complete to Y or x is anticomplete to Y. The subgraph of G induced by X is denoted as $G[X]$. We denote by $X \setminus Y$ the set $X - Y$ and $G \setminus X$ the subgraph of G induced on $V(G) \setminus X$. If $X = \{x\}$, we simply write $X \setminus x$ and $G \setminus x$, respectively. Throughout, we use the notation "$A :=$" to indicate that A is defined to be the expression on the right-hand side. For any positive integer k, we write $[k] := \{1, \ldots, k\}$.

Given a graph H, we say that a graph G is *H-free* if G has no induced subgraph isomorphic to H. A graph G contains a K_t *minor* if there exist pairwise disjoint non-empty connected subgraphs $T_1, \ldots, T_t$ of G, such that for $1 \le i < j \le t$, some vertex in T_j has a neighbor in T_i. Our work is motivated by the celebrated Hadwiger's Conjecture [Had43].

Conjecture 1.1 (Hadwiger's Conjecture [Had43]). *For every integer $t \ge 1$, every graph with no K_t minor is $(t-1)$-colorable.*

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Hadwiger's Conjecture is widely considered among the most important problems in graph theory and has motivated numerous developments in graph coloring and graph minor theory. We refer the reader to recent surveys [Tof96, Sey16, Nor23] for further background on Hadwiger's Conjecture. Hadwiger [Had43] and Dirac [Dir52] independently showed that Conjecture 1.1 holds for $t \leq 4$. However, for $t \geq 5$, Hadwiger's Conjecture implies the Four Color Theorem [AH77, AHK77]. Wagner [Wag37] proved that the case $t = 5$ of Hadwiger's Conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $t = 6$ by Robertson, Seymour and Thomas [RST93]. Despite receiving considerable attention over the years, Hadwiger's Conjecture remains open for $t \geq 7$. In the 1980s, Kostochka [Kos82, Kos84] and Thomason [Tho84] proved that every graph with chromatic number t contains a clique minor of order $\Omega(t/\sqrt{\log t})$, which was improved to $\Omega(t/(\log t)^{1/4+\epsilon})$ by Norin, Postle, and the first author [NPS23], and very recently, to $\Omega(t/\log\log t)$ by Delcourt and Postle [DP25].

Recently, Illingworth and Wood [IW] introduced the notion of a "dominating K_t minor": a graph G contains a *dominating K_t minor* if there exists a sequence $(T_1, \ldots, T_t)$ of pairwise disjoint non-empty connected subgraphs of G, such that for $1 \leq i < j \leq t$, every vertex in T_j has a neighbor in T_i. Replacing "every vertex in T_j" by "some vertex in T_j" recovers the standard definition of a K_t minor of G. Illingworth and Wood explored in what sense dominating K_t minors behave like (non-dominating) K_t minors: they observed that the two notions are equivalent for $t \leq 3$, but are already very different for $t = 4$, for instance, the 1-subdivision of K_n contains a K_n minor for all $n \geq 4$, yet lacks a dominating K_4 minor. This led them to propose a natural strengthening of Hadwiger's Conjecture: *is every graph with no dominating K_t minor $(t-1)$-colorable?* Norin [Nor] referred to this as the "Dominating Hadwiger's Conjecture" and suspects the conjecture is false.

Conjecture 1.2 (Dominating Hadwiger's Conjecture [IW]). *For every integer $t \geq 1$, every graph with no dominating K_t minor is $(t-1)$-colorable.*

Conjecture 1.2 is trivially true for $t \leq 3$. Illingworth and Wood [IW] verified the case $t = 4$ by showing that every graph with no dominating K_4 minor is 2-degenerate and 3-colorable. Norin [Nor] announced a proof that every graph with no dominating K_5 minor is 5-degenerate. More generally, Illingworth and Wood [IW] proved that every graph with no dominating K_t minor is 2^{t-2}-colorable. They further provided two pieces of evidence for the Dominating Hadwiger's Conjecture: (1) It is true for almost every graph, (2) Every graph G with no dominating K_t minor has a $(t-1)$-colorable induced subgraph on at least half the vertices, which implies there is an independent set of size at least $\lceil |V(G)|/(2t-2) \rceil$.

The *dominating Hadwiger number* $h_d(G)$ of a graph G is the largest integer t such that G contains a dominating K_t minor. In our search for potential counterexamples, we prove that the Dominating Hadwiger's Conjecture holds for all $2K_2$-free graphs, where $2K_2$ denotes the complement of the cycle of length four.

Theorem 1.3. *Every $2K_2$-free graph G satisfies $h_d(G) \geq \chi(G)$.*

We prove Theorem 1.3 in Section 2. Our novel proof cleverly relies on the existence of an induced *banner*, obtained by adding a vertex adjacent to exactly one vertex on a cycle of length four. It also uses Theorem 1.4 that follows directly from the proof of Theorem 2.2 of B. Thomas and the first author [ST17]. For brevity, we omit its proof here.

Theorem 1.4. *Every $\{C_4, C_5, C_6, \ldots, C_{2\alpha(G)}\}$-free graph G satisfies $h_d(G) \geq \chi(G)$.*

Theorem 1.3 was used by Scully and the first author [SS] to provider further evidence for the Dominating Hadwiger's Conjecture in the case of graphs G with $\alpha(G) \leq 2$. A significantly stronger result than Theorem 1.5 was obtained in [SS].

Theorem 1.5 (Scully and Song [SS]). *Let G be a graph on n vertices with $\alpha(G) \le 2$. Let $H \ne K_2 \cup K_3$ be a graph with $|V(H)| \le 5$ and $\alpha(H) \le 2$. If G is H-free, then $h_d(G) \ge \chi(G)$.*

It is worth noting that Micu [Mic05] proved that Hadwiger's Conjecture holds for all $2K_2$-free graphs. For completeness, we recall the proof here, as it is remarkably short and relies on the Strong Perfect Graph Theorem [CRST06]. However, Micu's argument does not extend to establishing a dominating $K_{\chi(G)}$ minor.

Theorem 1.6 (Micu [Mic05]). *Every $2K_2$-free graph G contains a $K_{\chi(G)}$ minor.*

Proof. Let G be a $2K_2$-free graph. Suppose G contains no $K_{\chi(G)}$ minor. We choose G with $|V(G)|$ minimum. Then G is not perfect. By the Strong Perfect Graph Theorem [CRST06], G contains an induced path of length three, say with vertices v_1, v_2, v_3, v_4 in order. Let

$$A := \{v \in V(G) \setminus \{v_1, v_2, v_3, v_4\} \mid v \text{ is anticomplete to } \{v_1, v_2\}\} \text{ and}$$

$$B := \{v \in V(G) \setminus (\{v_1, v_2, v_3, v_4\} \cup A) \mid v \text{ is anticomplete to } \{v_3, v_4\}\}.$$

Since G is $2K_2$-free, we see that both A and B are independent sets in G, v_4 is anticomplete to A and v_1 is anticomplete to B. Let $C := \{v_1, v_2, v_3, v_4\} \cup A \cup B$. By the choice of A and B, each vertex in $V(G) \setminus C$ is adjacent to at least one vertex in both $\{v_1, v_2\}$ and $\{v_3, v_4\}$. Moreover, $A \cup \{v_2, v_4\}$ and $B \cup \{v_1, v_3\}$ are independent sets in G and so $G[C]$ is 2-colorable. Thus $\chi(G \setminus C) \ge \chi(G) - 2$. By the minimality of G, $G \setminus C$ contains a $K_{\chi(G)-2}$ minor. This, together with $G[\{v_1, v_2\}]$ and $G[\{v_3, v_4\}]$, yields a $K_{\chi(G)}$ minor in G, a contradiction. $\square$

2 Proof of Theorem 1.3

A *banner* is the graph obtained from the cycle C_4 by adding a new vertex adjacent to exactly one vertex on C_4. We use $B = (b_1, b_2, b_3, b; b')$ to denote a banner, where $V(B) = \{b_1, b_2, b_3, b, b'\}$ and $E(B) = \{b_1b_2, b_2b_3, b_3b, bb_1, bb'\}$. For instance, $T[\{b_1, b_2, b_3, b, b'\}] = B$ is a banner, where T is depicted in Figure 1.

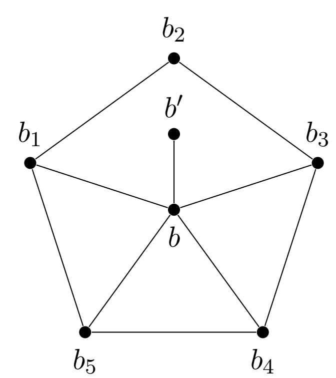

Figure 1: The graph T in which $T[\{b_1, b_2, b_3, b, b'\}] = B$ is a banner.

In this section we prove Theorem 1.3 which we restate below for convenience.

Theorem 1.3. *Every $2K_2$-free graph G satisfies $h_d(G) \ge \chi(G)$.*

Proof. Suppose the statement is false. Let G be a counterexample such that $|V(G)|$ is minimum. Then G is connected and $h_d(G) < \chi(G)$. We next prove several claims.

Claim 1. If G contains an induced banner $B = (b_1, b_2, b_3, b; b')$, then there exist two adjacent vertices $b_4, b_5 \in V(G)$ such that b_4 is complete to $\{b, b_3\}$ and anticomplete to $\{b', b_1, b_2\}$; b_5 is complete to $\{b, b_1\}$ and anticomplete to $\{b', b_2, b_3\}$. That is, G contains an induced T such that T contains B as an induced subgraph, where T is depicted in Figure 1.

Proof. Suppose G contains an induced banner $B = (b_1, b_2, b_3, b; b')$. Let $D_1 := \{b_1, b, b'\}]$ and $D_2 := \{b_2, b_3\}$. Let

$$A_1 := \{v \in V(G) \setminus V(B) \mid v \text{ is anticomplete to } D_1\} \text{ and}$$

$$A_2 := \{v \in V(G) \setminus (V(B) \cup A_1) \mid v \text{ is anticomplete to } D_2\}.$$

Since G is $2K_2$-free, we see that b_2 is anticomplete to A_1 and b' is anticomplete to A_2. We next show that some vertex in A_2 is complete to $\{b, b_1\}$. Suppose no vertex in A_2 is complete to $\{b, b_1\}$. Then A_2 can be partitioned into A_2' and A_2'' such that b is anticomplete to A_2' and complete to A_2''. Then b_1 is anticomplete to A_2''. Thus $A_1 \cup A_2'$ is anticomplete to $\{b, b', b_2\}$ and A_2'' is anticomplete to $\{b', b_1, b_2, b_3\}$. Moreover, $A_1 \cup A_2' \cup \{b, b_2\}$ and $A_2'' \cup \{b', b_1, b_3\}$ are independent sets in G because G is $2K_2$-free. Let $U := V(B) \cup A_1 \cup A_2$. Then $\chi(G[U]) = 2$ and $\chi(G \setminus U) \geq \chi(G) - 2$. Note that each vertex in $G \setminus U$ is adjacent to at least one vertex in both D_1 and D_2. By the minimality of G, $G \setminus U$ contains a dominating clique minor of order $\chi(G) - 2$, say $(T_3, \ldots, T_{\chi(G)})$. Then $(G[D_1], G[D_2], T_3, \ldots, T_{\chi(G)})$ yields a dominating clique minor of order $\chi(G)$ in G, a contradiction. Thus there exists a vertex, say b_5, in A_2 such that b_5 is complete to $\{b, b_1\}$ and anticomplete to $\{b', b_2, b_3\}$. By symmetry, there exists a vertex, say b_4, in $G \setminus V(B)$ such that b_4 is complete to $\{b, b_3\}$ and anticomplete to $\{b', b_1, b_2\}$. Then $b_4 \neq b_5$ and so $b_4 b_5 \in E(G)$, else $G[\{b_1, b_5, b_3, b_4\}] = 2K_2$. It follows that $G[V(B) \cup \{b_4, b_5\}] = T$. $\square$

Claim 2. G contains C_5, the cycle of length five, as an induced subgraph.

Proof. Suppose G is C_5-free. Since G is $2K_2$-free, we see that G is C_ℓ-free for each $\ell \geq 6$. By Theorem 1.4 and the fact that $h_d(G) < \chi(G)$, G contains an induced C_4, say with vertices v_1, v_2, v_3, v_4 in order. Define

$$I := \{v \in V(G) \mid v \text{ is anticomplete to } V(C_4)\}.$$

Then I is an independent set since G is $2K_2$-free. Let $H := G \setminus (I \cup V(C_4))$. By the choice of I, each vertex in H is adjacent to at least one vertex on C_4. Note that $\chi(H) \geq \chi(G) - 2$. By the minimality of G, H contains a dominating clique minor of order $\chi(G) - 2$, say $(T_3, \ldots, T_{\chi(G)})$. By Claim 1, G is banner-free because the graph T in Figure 1 contains C_5 as an induced subgraph. Thus each vertex in H is adjacent to at least two vertices on C_4. Let $D_1 := \{v_1, v_2\}$ and $D_2 := \{v_3, v_4\}$. If each vertex in H is adjacent to at least one vertex in both D_1 and D_2, then $(G[D_1], G[D_2], T_3, \ldots, T_{\chi(G)})$ yields a dominating clique minor of order $\chi(G)$ in G, a contradiction. Thus there exists a vertex x in H such that x is anticomplete to D_1 or D_2, say D_2. Then x is complete to $D_1 = \{v_1, v_2\}$. Let $D_1' := \{v_1, v_4\}$ and $D_2' := \{v_2, v_3\}$. Using a similar argument, there exists a vertex y in H such that y is anticomplete to D_1' or D_2', say D_2'. Then y is complete to $D_1' = \{v_1, v_4\}$. Thus $x \neq y$ and so $xy \in E(G)$, else $G[\{x, v_2, y, v_4\}] = 2K_2$. It follows that $G[\{x, v_2, v_3, v_4, y\}] = C_5$, a contradiction. $\square$

By Claim 2, G contains an induced C_5, say with vertices v_1, v_2, v_3, v_4, v_5 in order. Define

$$I := \{v \in V(G) \mid v \text{ is anticomplete to } V(C_5)\}.$$

Then I is an independent set. Let $H := G \setminus (I \cup V(C_5))$. By the choice of I, each vertex in H is adjacent to at least one vertex on C_5.

Claim 3. Each vertex in H is adjacent to at least four vertices on C_5.

Proof. Suppose there exists a vertex $x \in V(H)$ such that x is adjacent to at most three vertices on C_5. We choose C_5 and x such that $|N(x) \cap V(C_5)|$ is minimum. We may assume that $xv_1 \in E(G)$ because $x \notin I$. Suppose x is anticomplete to $\{v_3, v_4\}$. Then $G[\{x, v_1, v_3, v_4\}] = 2K_2$, a contradiction. Thus $xv_3 \in E(G)$ or $xv_4 \in E(G)$. We may assume that $xv_3 \in E(G)$. Then $xv_2 \notin E(G)$, else $G[\{x, v_2, v_4, v_5\}] = 2K_2$ because x is adjacent to at most three vertices on C_5. We may further assume that $xv_4 \notin E(G)$ because x is adjacent to at most three vertices on C_5. Then $(x, v_1, v_2, v_3; v_4)$ is an induced banner in G. By Claim 1, there exist $y, z \in V(H)$ such that $yz \in E(G)$, y is complete to $\{x, v_3\}$ and anticomplete to $\{v_1, v_2, v_4\}$, and z is complete to $\{v_2, v_3\}$ and anticomplete to $\{x, v_1, v_4\}$. Note that $N(y) \cap V(C_5) \subseteq \{v_3, v_5\}$. By the choice of x, we have $xv_5 \notin E(G)$. Then $(x, v_3, v_2, v_1; v_5)$ is an induced banner in G. By Claim 1, there exist $u, w \in V(H)$ such that $uw \in E(G)$, u is complete to $\{v_1, v_2\}$ and anticomplete to $\{x, v_3, v_5\}$, and w is complete to $\{v_1, x\}$ and anticomplete to $\{v_2, v_3, v_5\}$. Since $G[\{y, z, v_3, v_4, u, w, v_1, v_5\}]$ is $2K_2$-free, we see that $G[\{y, z, u, w\}] = K_4$, v_4 is complete to $\{u, w\}$ and v_5 is complete to $\{y, z\}$. Let $D_1 := \{x, v_1, v_5\}$, $D_2 := \{v_2, v_3, v_4\}$, $D_3 := \{z, u\}$, and $D_4 := \{y, w\}$. It can be easily checked that $(G[D_1], G[D_2], G[D_3], G[D_4])$ yields a dominating K_4 minor. Let $S := \{x, y, z, u, w\}$ and

$$A_1 := \{v \in V(H) \setminus S \mid v \text{ is anticomplete to } D_1\},$$

$$A_i := \{v \in V(H) \setminus (S \cup (\cup_{k=1}^{i-1} A_k)) \mid v \text{ is anticomplete to } D_i\} \text{ for each } i \in \{2, 3, 4\}.$$

Let $H^* := G \setminus (I \cup V(C_5) \cup S \cup (\bigcup_{k=1}^{4} A_k)$. Then $A_1, \ldots, A_4$ are pairwise disjoint and each vertex in H^* is adjacent to at least one vertex in D_j for each $j \in [4]$. We next show that $\chi(H^*) \geq \chi(G) - 4$. Note that $A_1 \cup \{z, v_4\}$ is anticomplete to $\{x, v_1\}$, $A_2 \cup \{w, v_5\}$ is anticomplete to $\{v_2, v_3\}$, $A_3 \cup \{x\}$ is anticomplete to $\{z, u\}$, $A_4 \cup \{v_2\}$ is anticomplete to $\{y, w\}$, and x is anticomplete to I because $\{x\} \cup I$ is anticomplete to $\{v_4, v_5\}$. Thus z is anticomplete to I because $\{z\} \cup I$ is anticomplete to $\{x, v_1\}$. It follows that $A_1 \cup I \cup \{z, v_4\}$, $A_2 \cup \{w, v_5\}$, $A_3 \cup \{x\}$ and $A_4 \cup \{v_2\}$ are pairwise disjoint independent sets in G because G is $2K_2$-free. Thus $A_1 \cup I \cup \{v_1, v_4, z\}$, $A_2 \cup \{v_3, v_5, w\}$, $A_3 \cup \{x, u\}$ and $A_4 \cup \{v_2, y\}$ are pairwise disjoint independent sets in G, and so $G[I \cup V(C_5) \cup S \cup (\bigcup_{k=1}^{4} A_k)]$ is 4-colorable. This proves that $\chi(H^*) \geq \chi(G) - 4$. By the minimality of G, H^* contains a dominating clique minor of order $\chi(G) - 4$, say $(T_5, \ldots, T_{\chi(G)})$. Then $(G[D_1], G[D_2], G[D_3], G[D_4], T_5, \ldots, T_{\chi(G)})$ yields a dominating clique minor of order $\chi(G)$ in G, a contradiction. $\square$

By Claim 3, every vertex in H is adjacent to at least four vertices on C_5. By the arbitrary choice of C_5, we see that for any induced cycle C of length five in G, every vertex in $V(G) \setminus V(C)$ is either anticomplete to $V(C)$ or adjacent to at least four vertices on C. Let

$$J := \{v \in V(H) \mid v \text{ is complete to } V(C_5)\} \text{ and}$$

$$Y_i := \{v \in V(H) \mid vv_i \notin E(G)\} \text{ for each } i \in [5].$$

Then $\{J, Y_1, \ldots, Y_5\}$ partitions $V(H)$. Let $Y := \bigcup_{k=1}^{5} Y_k$.

Claim 4. I is anticomplete to Y.

Proof. Suppose there exist $u \in I$ and $v \in Y$ such that $uv \in E(G)$. We may assume that $v \in Y_1$. Then $(v_5, v_1, v_2, v; u)$ induces a banner in G. By Claim 1, there exists a vertex $w \in V(H)$ such that w is complete to $\{v, v_2\}$ and anticomplete to $\{u, v_1, v_5\}$. Thus w is adjacent to at most three vertices on C_5, which contradicts Claim 3. $\square$

Claim 5. For each $i \in [5]$, Y_i is a clique of order at least two.

Proof. Suppose Y_i is not a clique for some $i \in [5]$. We may assume that $i = 1$. Let $u, v \in Y_1$ such that $uv \notin E(G)$. Then $(u, v_4, v, v_2; v_1)$ induces a banner in G. By Claim 1, there exists a vertex $w \in V(H)$ such that w is complete to $\{u, v_2\}$ and anticomplete to $\{v, v_1, v_4\}$. Then w is adjacent to at most three vertices on C_5, which contradicts Claim 3. Thus Y_i is a clique for all $i \in [5]$. We next show that $|Y_i| \geq 2$ for each $i \in [5]$.

Suppose $Y = \emptyset$. Then $\{V(C_5), I, J\}$ partitions $V(G)$ and $\chi(G[J]) \geq \chi(G) - 3$. By the minimality of G, $G[J]$ contains a dominating clique minor of order $\chi(G) - 3$, say $(T_4, \ldots, T_{\chi(G)})$. Then $(G[\{v_1, v_2, v_3\}], G[\{v_4\}], G[\{v_5\}], T_4, \ldots, T_{\chi(G)})$ yields a dominating clique minor of order $\chi(G)$ in G, a contradiction. Thus $Y \neq \emptyset$. We may assume that $Y_1 \neq \emptyset$. Let $y_1 \in Y_1$. By Claim 4, $G[V(C_5) \cup I \cup \{y_1\}]$ is 3-colorable. Thus $\chi(H \setminus y_1) \geq \chi(G) - 3$. By the minimality of G, $H \setminus y_1$ contains a dominating clique minor of order $\chi(G) - 3$, say $(T_4, \ldots, T_{\chi(G)})$. Suppose $Y_1 = \{y_1\}$ or $Y_2 = \emptyset$. Then v_1 is complete to $V(H) \setminus \{y_1\}$ if $Y_1 = \{y_1\}$ and v_2 is complete to $V(H)$ if $Y_2 = \emptyset$. Let

$$D_1 := \{y_1, v_2, v_3\}, D_2 := \{v_4, v_5\}, D_3 := \{v_1\} \text{ if } Y_1 = \{y_1\},$$

$$D_1 := \{y_1, v_1, v_5\}, D_2 := \{v_3, v_4\}, D_3 := \{v_2\} \text{ if } Y_2 = \emptyset.$$

It is easy to check that $(G[D_1], G[D_2], G[D_3])$ yields a dominating K_3 minor, and each vertex in $H \setminus y_1$ is adjacent to at least one vertex in D_j for all $j \in [3]$. Thus $(G[D_1], G[D_2], G[D_3], T_4, \ldots, T_{\chi(G)})$ yields a dominating clique minor of order $\chi(G)$ in G, a contradiction. This proves that $|Y_1| \geq 2$ and $Y_2 \neq \emptyset$. By symmetry, Y_i is a clique of order at least two for each $i \in [5]$. $\square$

Claim 6. Every vertex in J is adjacent to at least $|Y_i| - 1 \geq 1$ vertices in Y_i for all $i \in [5]$.

Proof. By Claim 5, $|Y_i| - 1 \geq 1$ for all $i \in [5]$. Suppose there exists a vertex $v \in J$ such that v is not adjacent to two vertices, say y_i, y'_i, in Y_i for some $i \in [5]$. Since J is complete to $V(C_5)$, we see that $G[\{v, v_i, y_i, y'_i\}] = 2K_2$, a contradiction. $\square$

For the remainder of the proof, all arithmetic on indices is done modulo 5.

Claim 7. For each $i \in [5]$, Y_i is complete to $Y_{i+2} \cup Y_{i-2}$. Furthermore, every vertex in Y_i is adjacent to at least $|Y_{i+1}| - 1$ vertices in Y_{i+1} and at least $|Y_{i-1}| - 1$ vertices in Y_{i-1}.

Proof. Let $y_i \in Y_i$. Suppose there exists a vertex $y_{i+2} \in Y_{i+2}$ such that $y_i y_{i+2} \notin E(G)$. Then $G[\{y_i, v_{i+2}, y_{i+2}, v_i\}] = 2K_2$, a contradiction. Thus Y_i is complete to Y_{i+2}. By symmetry, Y_i is complete to Y_{i-2}. Next, suppose there exist $y_{i+1}, y'_{i+1} \in Y_{i+1}$ such that $y_i y_{i+1}, y_i y'_{i+1} \notin E(G)$. Recall that $y_i v_{i+1} \in E(G)$. Then $G[\{y_i, v_{i+1}, y_{i+1}, y'_{i+1}\}] = 2K_2$, a contradiction. Thus each vertex in Y_i is adjacent to at least $|Y_{i+1}| - 1$ vertices in Y_{i+1}. By symmetry, each vertex in Y_i is adjacent to at least $|Y_{i-1}| - 1$ vertices in Y_{i-1}. $\square$

For each $i, j \in [5]$ with $i \neq j$, we use $G[Y_i, Y_j]$ to denoted the bipartite graph with bipartition $\{Y_i, Y_j\}$ such that $E(G[Y_i, Y_j])$ consists of all edges in G with one end in Y_i and the other in Y_j.

Claim 8. For each $i \in [5]$, $|Y_i| = |Y_{i+1}|$ and $G[Y_i, Y_{i+1}]$ is $(|Y_i| - 1)$-regular.

Proof. Let $y_i \in Y_i$. By Claim 7, y_i is adjacent to at least $|Y_{i+1}| - 1$ vertices in Y_{i+1}. Suppose y_i is complete to Y_{i+1}. Let $y_{i+1} \in Y_{i+1}$ and $y_{i-2} \in Y_{i-2}$. Then $y_{i+1}y_{i-2} \in E(G)$ by Claim 7. Let $S := \{v_i, v_{i+1}, v_{i-2}, y_i, y_{i+1}, y_{i-2}\}$. Note that I is anticomplete to S by Claim 4 and the choice of I. Then $G[S \cup I]$ is 3-colorable and so $\chi(G \setminus (S \cup I)) \geq \chi(G) - 3$. By the minimality of G, $G \setminus (S \cup I)$ contains a dominating clique minor of order $\chi(G) - 3$, say $(T_4, \ldots, T_{\chi(G)})$. Let $D_1 := \{y_i, v_{i+1}\}$, $D_2 := \{y_{i+1}, v_{i-2}\}$, and $D_3 := \{y_{i-2}, v_i\}$. Since y_i is complete to Y_{i+1}, we see that $(G[D_1], G[D_2], G[D_3])$ yields a dominating K_3 minor and each vertex in $Y_{i+1} \setminus y_{i+1}$ is adjacent to y_i in D_1. By Claim 7 and Claim 5, it can be easily checked that each vertex in $G \setminus (S \cup I)$ is adjacent to at least one vertex in D_ℓ for each $\ell \in [3]$. Thus $(G[D_1], G[D_2], G[D_3], T_4, \ldots, T_{\chi(G)})$ yields a dominating clique minor of order $\chi(G)$ in G, a contradiction. This proves that no vertex in Y_i is complete to Y_{i+1}. By symmetry, no vertex in Y_{i+1} is complete to Y_i. By Claim 7, $|Y_i| = |Y_{i+1}|$ and $G[Y_i, Y_{i+1}]$ is $(|Y_i| - 1)$-regular for each $i \in [5]$. $\square$

By Claim 8, $|Y_1| = \cdots = |Y_5|$. Let $m := |Y_1|$. Then $m \geq 2$ by Claim 5. For each $i \in [5]$, by Claim 8, $G[Y_i, Y_{i-1}]$ and $G[Y_i, Y_{i+1}]$ are both $(m-1)$-regular, that is, every vertex in Y_i has a unique non-neighbor in Y_{i-1} and a unique non-neighbor in Y_{i+1}.

Claim 9. For each $i \in [5]$, let $y_i \in Y_i$, $y_{i-1} \in Y_{i-1}$ and $y_{i+1} \in Y_{i+1}$ such that $y_iy_{i-1}, y_iy_{i+1} \notin E(G)$. Then no vertex in J is anticomplete to $\{y_{i-1}, y_{i+1}\}$.

Proof. Suppose there exists a vertex $v \in J$ such that v is anticomplete to $\{y_{i-1}, y_{i+1}\}$. By Claim 7, $y_{i-1}y_{i+1} \in E(G)$. Let $C := \{y_{i-1}, y_{i+1}, v_{i-1}, y_i, v_{i+1}\}$. Then $G[C] = C_5$ and v is adjacent to at most three vertices on $G[C]$, contrary to Claim 3. $\square$

Let $R := \{v_1, v_2, v_3, v_4\} \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. Then I is anticomplete to R by Claim 4. We first show that $G[R \cup I]$ is $(2m+2)$-colorable. By Claim 7, $G[Y_1 \cup Y_4 \cup \{v_2, v_3\}] = K_{2m+2}$ and so $\chi(G[R \cup I]) \geq 2m + 2$. By Claim 7 again, $G[Y_1 \cup Y_2]$ is m-colorable and $G[Y_3 \cup Y_4]$ is m-colorable. Note that $G[\{v_1, v_2, v_3, v_4\}]$ is 2-colorable. It follows that $\chi(G[R \cup I]) = 2m + 2$ and so $\chi(G \setminus (R \cup I)) \geq \chi(G) - (2m+2)$. By the minimality of G, $G \setminus (R \cup I)$ contains a dominating clique minor of order $\chi(G) - (2m+2)$, say $(T_{2m+3}, \ldots, T_{\chi(G)})$. For each $i \in [4]$, let $Y_i := \{y^i_1, \ldots, y^i_m\}$ such that y^2_ℓ is anticomplete to $\{y^1_\ell, y^3_\ell\}$ for each $\ell \in [m]$. This is possible because $G[Y_2, Y_1]$ and $G[Y_2, Y_3]$ are $(m-1)$-regular bipartite graphs by Claim 8. By Claim 9, no vertex in J is anticomplete to $\{y^1_\ell, y^3_\ell\}$ for each $\ell \in [m]$.

Let $m := 2p + r$, where p is a positive integer and $r \in \{0, 1\}$. Let $D_1 := \{v_2, v_3, v_4\}$ when m is even, and let $D_1 := \{v_2, v_3, y^4_m\}$ and $D_2 := \{v_4, y^2_m\}$ when m is odd. For each $j \in [p]$, let $D_{1+r+j} := \{y^2_{2j-1}, y^2_{2j}\}$ and $D_{p+1+r+j} := \{y^4_{2j-1}, y^4_{2j}\}$. For each $\ell \in [m]$, let $D_{m+1+\ell} := \{y^1_\ell, y^3_\ell\}$. Finally, let $D_{2m+2} := \{v_1\}$. Then $R = \bigcup_{k=1}^{2m+2} D_k$ and $D_1, \ldots, D_{2m+2}$ are pairwise disjoint. By Claim 7 and Claim 6, we see that $(G[D_1], \ldots, G[D_{2m+2}])$ yields a dominating K_{2m+2} minor. Note that $V(G \setminus (R \cup I)) = \{v_5\} \cup Y_5 \cup J$. By Claim 9, Claim 7 and Claim 6, each vertex in $G \setminus (R \cup I)$ is adjacent to at least one vertex in D_i for each $i \in [2m+2]$. Thus $(G[D_1], \ldots, G[D_{2m+2}], T_{2m+3}, \ldots, T_{\chi(G)})$ yields a dominating clique minor of order $\chi(G)$ in G, a contradiction.

This completes the proof of Theorem 1.3. $\square$

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