

ON IRREDUCIBILITY OF CERTAIN LOW DIMENSIONAL AUTOMORPHIC GALOIS REPRESENTATIONS

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ABSTRACT. We study irreducibility of Galois representations $\rho_{\pi,\lambda}$ associated to a $n = 7$ or 8-dimensional regular algebraic essentially self-dual cuspidal automorphic representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$. We show $\rho_{\pi,\lambda}$ is irreducible for all but finitely many λ under the following extra conditions.

- (i) If $n = 7$, and there exists no λ such that the Lie type of $\rho_{\pi,\lambda}$ is the standard representation of exceptional group $\mathbf{G}_2$.
- (ii) If $n = 8$, and when there exist infinitely many λ such that the Lie type of $\rho_{\pi,\lambda}$ is the spin representation of SO_7 , we assume there exist no three distinct Hodge-Tate weights form a 3-term arithmetic progression.

1. INTRODUCTION

It is a folklore conjecture (see [Ra08]) that the Galois representations associated to algebraic cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ of a number field F are irreducible. For classical modular forms this was proved in [Ri77], and the proof was extended to Hilbert modular forms in [Ta95]. For $n = 3$, F is CM and π is essentially self-dual, the result was proved in [BR92, Theorem 2.2.1]. For $n = 3$, F is totally real and without essentially self-dual condition, the result was proved in [BH25]. For $n = 4$, F being totally real and π is essentially self-dual, the irreducibility for almost all ℓ was proved in [Ra13].

For general dimension, due to the work of many people, one can attach a strictly compatible system (see Definition 2.4) $\{\rho_{\pi,\lambda}\}$ to an algebraic, regular, cuspidal, essentially self-dual automorphic representation π of $GL_n(\mathbb{A}_F)$ where F is a CM or a totally real field (see Theorem 2.10). The irreducibility for almost all λ when $n \leq 6$ was proved in [Hu23b]. For general n , the irreducibility for a positive density set of λ was proved in [PT15]. When $4 \nmid n$ and $7 \nmid n$, the irreducibility for a density one set of λ was proved in [FW25]. When F is totally real and some irreducible ρ_{π,λ_0} is of certain A_1 type, the irreducibility of all $\rho_{\pi,\lambda}$ was proved in [HL24], [HL25].

The present paper continues to investigate the irreducibility of low dimensional automorphic Galois representations. We focus on $n = 7$ and 8. For a λ -adic semisimple Galois representation $\rho : \text{Gal}_{\mathbb{Q}} \rightarrow GL_n(\overline{E}_{\lambda})$, the Zariski closure $\mathbf{G}$ of its image inside $GL_{n,\overline{E}_{\lambda}}$ is a reductive group. Denote by $\mathbf{G}^{\text{der}} = [\mathbf{G}^{\circ}, \mathbf{G}^{\circ}]$ its derived subgroup, which is semisimple. Our main result is:

Theorem 1.1. *Let $\{\rho_{\pi,\lambda} : \text{Gal}_{\mathbb{Q}} \rightarrow GL_n(\overline{E}_{\lambda})\}_{\lambda}$ be the E -rational strictly compatible system of $\mathbb{Q}$ associated to a regular algebraic essentially self-dual cuspidal automorphic representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$ where $n = 7$ or 8. Moreover we require:*

- (i) *If $n = 7$, there exists no λ such that tautological representation of $\mathbf{G}_{\lambda}^{\text{der}}$ is the standard representation of exceptional group $\mathbf{G}_2$.*
- (ii) *If $n = 8$, and when there exist infinitely many λ such that tautological representation of $\mathbf{G}_{\lambda}^{\text{der}}$ is the spin representation of SO_7 , we assume for any three distinct Hodge-Tate weights $\{a, b, c\}$ one has $a + b \neq 2c$.*

Then $\rho_{\pi,\lambda}$ is irreducible for all but finitely many λ .

We organize the paper as following. In section 2 we give preliminaries including ℓ -independence properties of compatible systems, potential automorphy theorem, big image results and some p -adic Hodge theoretic lift results. In section 3 we follow the treatment in [Xi19] to assume there exist infinitely many λ such that $\rho_{\pi,\lambda}$ is Lie-irreducible. Finally in section 4 we prove the main theorem.

2. PRELIMINARIES

2.1. Compatible systems and ℓ -independence.

Definition 2.1. Let K be a number field. An n -dimensional E -rational Serre compatible system of Gal_K is the datum

$$\mathcal{M} = (E, S, \{p_v(T)\}, \{\rho_\lambda\})$$

where:

- E is a number field.
- S is a finite set of primes of K called exceptional set.
- $P_v(T) \in E[T]$ is a degree n monic polynomial for each prime $v \notin S$ of K .
- $\rho_\lambda : \text{Gal}_K \rightarrow \text{GL}_n(E_\lambda)$ is an n -dimensional continuous Galois representation.

such that:

- (i) ρ_λ is unramified outside $S \cup S_\lambda$, where S_λ is the primes of K that divide the same rational prime as λ .
- (ii) For each $v \notin S \cup S_\lambda$, the characteristic polynomial of $\rho_\lambda(\text{Frob}_v)$ is $P_v(T)$.

A Serre compatible system is called semisimple if each ρ_λ is semisimple.

Consider a semisimple λ -adic Galois representation $\rho : \text{Gal}_K \rightarrow \text{GL}_n(E_\lambda)$. Denote by $\mathbf{G}$ the Zariski closure of the image inside algebraic group GL_{n, E_λ} , which is called the algebraic monodromy group of ρ . As ρ is semisimple, the identity component $\mathbf{G}^\circ$ is a reductive group. We write $\mathbf{G}^{\text{der}} = [\mathbf{G}^\circ, \mathbf{G}^\circ]$ to be the derived subgroup of $\mathbf{G}^\circ$, which is semisimple. To describe ℓ -independence properties of compatible systems, we need the following concepts:

Definition 2.2. Let F be a field and $\mathbf{G} \subseteq \text{GL}_{n, F}$ be a reductive subgroup. Denote by $\overline{F}$ a fixed algebraic closure of F .

- (i) Denote by $\mathbf{T}$ a maximal torus of $\mathbf{G} \times \overline{F}$ and by $\mathbf{T}'$ a maximal torus of $\mathbf{G}^{\text{der}} \times \overline{F}$. Then the formal character (resp. formal bi-character) of $\mathbf{G}$ is the conjugacy class of $\mathbf{T}$ in $\text{GL}_{n, \overline{F}}$ (resp. conjugacy class of the chain $\mathbf{T}' \subseteq \mathbf{T}$ in $\text{GL}_{n, \overline{F}}$).
- (ii) Given two fields F_1, F_2 and two reductive groups $\mathbf{G}_i \subseteq \text{GL}_{n, F_i}, i = 1, 2$. We say they have same formal character (resp. formal bi-character), if $n_1 = n_2 = n$ and there exists a split $\mathbb{Z}$ -subtorus $\mathbf{T}_{\mathbb{Z}} \subseteq \text{GL}_{n, \mathbb{Z}}$ (resp. a chain of split $\mathbb{Z}$ -subtori $\mathbf{T}'_{\mathbb{Z}} \subseteq \mathbf{T}_{\mathbb{Z}} \subseteq \text{GL}_{n, \mathbb{Z}}$) such that $\mathbf{T}_{\mathbb{Z}} \times \overline{F}_i$ (resp. $\mathbf{T}'_{\mathbb{Z}} \times \overline{F}_i \subseteq \mathbf{T}_{\mathbb{Z}} \times \overline{F}_i$) is contained in formal character (resp. formal bi-character) of $\mathbf{G}_i$ for each i . This defines an equivalence relation on formal characters (resp. formal bi-characters) of reductive groups over different fields.
- (iii) Let $\{F_i\}$ be a family of fields and $\{\mathbf{G}_i \subseteq \text{GL}_{n, F_i}\}$ be a family of reductive groups. We say they have same formal character (resp. same formal bi-character) if they belong to the same class under the equivalence relation in (ii). We say they have bounded formal characters (resp. bounded formal bi-characters) if they belong to finitely many classes under the equivalence relation in (ii).

The following results are standard λ -independence properties on algebraic monodromy groups.

Theorem 2.3. [Se81], [Se84], [Hu13, Theorem 3.19]. Given an E -rational semisimple Serre compatible system $\{\rho_\lambda : \text{Gal}_K \rightarrow \text{GL}_n(E_\lambda)\}$. Denote by $\mathbf{G}_\lambda$ the algebraic monodromy group of $\rho_\lambda \otimes \overline{E}_\lambda$

- (i) The component group $\pi_0(\mathbf{G}_\lambda) = \mathbf{G}_\lambda / \mathbf{G}_\lambda^\circ$ is independent of λ . In particular the connectedness of $\mathbf{G}_\lambda$ is independent of λ and one has a smallest extension K'/K such that when restricting the compatible system to $\text{Gal}_{K'}$, each algebraic monodromy group is connected.
- (ii) The formal bi-character of the tautological representation $\mathbf{G}_\lambda \hookrightarrow \text{GL}_{n, \overline{E}_\lambda}$ and hence the rank and semisimple rank of $\mathbf{G}_\lambda$ are independent of λ .

The following definition of compatible systems has extra conditions than Serre compatible systems which makes them more treatable.

Definition 2.4. Let K be a number field. An n -dimensional E -rational strictly compatible system of Gal_K is the datum

$$\mathcal{M} = (E, S, \{P_v(T)\}, \{\rho_\lambda\}, \{HT_\tau\}, \{WD_v\})$$

where:

- E is a number field.
- S is a finite set of primes of K called exceptional set.
- $P_v(T) \in E[T]$ is a degree n monic polynomial for each prime $v \notin S$ of K .
- $\rho_\lambda : \text{Gal}_K \rightarrow \text{GL}_n(\overline{E}_\lambda)$ is an n -dimensional continuous semisimple Galois representation.
- HT_τ is a multiset of n integers for each embedding $\tau : K \hookrightarrow \overline{E}$.
- WD_v is a semisimple Weil-Deligne representation of K_v for each prime v .

such that:

- (i) Each ρ_λ is a geometric representation in the sense of Fontaine-Mazur with exceptional set S , this means
 - ρ_λ is unramified outside $S \cup S_\lambda$, where S_λ is the primes of K that divide the same rational prime as λ .
 - If $v \in S_\lambda$ then $\rho_\lambda|_{\text{Gal}_{K_v}}$ is de Rham.
 Moreover, $\rho_\lambda|_{\text{Gal}_{K_v}}$ is crystalline if $v \in S_\lambda$ and $v \notin S$.
- (ii) For each $v \notin S \cup S_\lambda$, the characteristic polynomial of $\rho_\lambda(\text{Frob}_v)$ is $P_v(T)$.
- (iii) For each embedding $\tau : K \hookrightarrow \overline{E}$ and each E -embedding $\overline{E} \hookrightarrow \overline{E}_\lambda$, the Hodge-Tate weights of ρ_λ is HT_τ .
- (iv) For each $v \notin S_\lambda$ and each isomorphism $\iota : \overline{E}_\lambda \cong \mathbb{C}$, the Frobenius semisimplified Weil-Deligne representation $\iota WD(\rho_\lambda|_{\text{Gal}_{K_v}})^{F-ss}$ is isomorphic to WD_v .

A Hodge-Tate representation is called regular, if its Hodge-Tate weights are distinct. Under this condition, the following result shows that one can descend the coefficients of a strictly compatible system to E_λ after enlarging E , which makes it a Serre compatible system.

Lemma 2.5. [BLGGT14, Lemma 5.3.1.(3)] Let $\{\rho_\lambda\}$ be an E -rational strictly compatible system of K . Suppose $\mathcal{M}$ is regular, then after replacing E with a finite extension, we may assume that for any open subgroup H of Gal_K , any λ and any H -subrepresentation σ of ρ_λ , the representation σ is defined over E_λ .

2.2. Rectangular representations and ℓ -independence. Let $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ be a finite dimensional representation of a complex Lie algebra $\mathfrak{g}$. Denote by Λ a weight lattice (with respect to some fixed Cartan subalgebra $\mathfrak{t}$), by Ξ the (multi)set of weights. For non-negative integer d , denote by $Z_d = \{-d, -d+2, -d+4, \dots, d-2, d\}$. Let r be the rank of $\mathfrak{g}$.

Definition 2.6. [HL25, Section 1.1]

- (i) ρ is called rectangular if every weight in Ξ is of multiplicity one and there exist an $\mathbb{R}$ -isomorphism $\iota : \Lambda \otimes \mathbb{R} \rightarrow \mathbb{R}^n$ and non-negative integers $d_1, d_2, \dots, d_r$ such that

$$\iota(\Xi) = Z_{d_1} \times Z_{d_2} \times \dots \times Z_{d_r}$$

The (multi)set $\{d_i + 1, 1 \leq i \leq r\}$ is called the set of lengths of ρ . The representation ρ is called hypercubic if $d_1 = d_2 = \dots = d_r$. A rectangular representation is called indecomposable if it is not equivalent to an external tensor product of two rectangular representations.

- (ii) Let $\tilde{\rho} : \text{Gal}_K \rightarrow \text{GL}_n(E_\lambda)$ be a semisimple λ -adic Galois representation of a number field K . Denote by $\mathbf{G}$ the algebraic monodromy group. Fix some embedding $E_\lambda \hookrightarrow \mathbb{C}$ and consider the complex base change $\mathbf{G}_{\mathbb{C}} \rightarrow \text{GL}_n(\mathbb{C})$. Consider the associated complex Lie algebra representation $\text{Lie}(\mathbf{G}_{\mathbb{C}}) \rightarrow \text{GL}_n(\mathbb{C})$ and denote by ρ its restriction to the semisimple part $\text{Lie}(\mathbf{G})^{ss}$. We call $\tilde{\rho}$ rectangular or hypercubic or indecomposable if ρ is so.

One of the main results in [HL25] gave a complete classification of rectangular representations of complex Lie algebras.

Theorem 2.7. [HL25, Theorem 1.1] Let $(\mathfrak{g}, \rho)$ be a faithful rectangular Lie algebra representation of a complex semisimple Lie algebra $\mathfrak{g}$. Fix a decomposition $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \times \dots \times \mathfrak{g}_k$ where $\mathfrak{g}_1$

denotes the product of A_1 -factors and $\mathfrak{g}_2, \dots, \mathfrak{g}_k$ denote other simple factors. Then the following assertions hold.

- (i) There exist faithful rectangular representation $(\mathfrak{g}_1, \rho_1)$ and faithful indecomposable hypercubic representations $(\mathfrak{g}_i, \rho_i)$ for $2 \leq i \leq k$ such that

$$(\mathfrak{g}, \rho) = \bigotimes_{i=1}^k (\mathfrak{g}_i, \rho_i)$$

- (ii) The rectangular representation $(\mathfrak{g}_1, \rho_1)$ admits an external tensor product of indecomposable hypercubic representations

$$(\mathfrak{g}_1, \rho_1) = \bigotimes_{j=1}^s (\mathfrak{g}_{1,j}, \rho_{1,j})$$

such that $\mathfrak{g}_1 = \prod_{j=1}^s \mathfrak{g}_{1,j}$ and each $\rho_{1,j}$ is one of the following.

- (a) $(A_1, \text{Sym}^r(\text{Std})), r \in \mathbb{N}$.
 - (b) $(A_1, \text{Sym}^{r_1}(\text{Std}) \otimes \text{Sym}^{r_2}(\text{Std})), r_1, r_2 \in \mathbb{Z}_{\geq 0}, |r_1 - r_2| = 1$.
 - (c) $(A_1 \times A_1, (\text{Std} \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes \text{Std})) = (D_2, \text{Spin})$.
- (iii) The hypercubic representation $(\mathfrak{g}_i, \rho_i)$ for $2 \leq i \leq k$ is one of the following.
- (a) $(B_2, \text{Std} \oplus \text{Spin})$.
 - (b) $(B_m, \text{Spin}), m \geq 2$.
 - (c) $(A_3, \text{Std} \oplus \text{Std}^\vee)$.
 - (d) $(D_m, \text{Spin}), m \geq 4$.
- (iv) The external tensor products in (i) and (ii) are unique up to permutations of the A_1 -factors and the non- A_1 factors.

Due to Theorem 2.3.(ii), for a semisimple Serre compatible system $\{\rho_\lambda\}$, if one ρ_{λ_0} is rectangular, then all ρ_λ are rectangular with same (multi)set of lengths. Hence it makes sense to call a compatible system rectangular and define its (multi)set of lengths. In the proof we use the following direct consequence.

Proposition 2.8. *Let $\{\rho_\lambda\}$ be an 8-dimensional semisimple rectangular Serre compatible system such that some ρ_{λ_0} is Lie-irreducible. Denote by $\mathbf{G}_\lambda^{\text{der}}$ the derived subgroup of algebraic monodromy group of ρ_λ . Denote by $\mathcal{L}$ the (multi)set of lengths. Then exactly one of the following happens.*

- (i) $\mathcal{L} = \{8\}$ and all $\mathbf{G}_\lambda^{\text{der}}$ equal to

$$(\text{SL}_2, \text{Sym}^7(\text{Std}))$$

- (ii) $\mathcal{L} = \{2, 4\}$ and all $\mathbf{G}_\lambda^{\text{der}}$ equal to

$$(\text{SL}_2, \text{Std}) \otimes (\text{SL}_2, \text{Sym}^3(\text{Std}))$$

- (iii) $\mathcal{L} = \{2, 2, 2\}$ and $\mathbf{G}_\lambda^{\text{der}}$ equals to one of the following.

- (a) $(\text{SL}_2, \text{Std}) \otimes (\text{SL}_2, \text{Std}) \otimes (\text{SL}_2, \text{Std})$.
- (b) $(\text{SL}_2, \text{Std}) \otimes (\text{SL}_2 \times \text{SL}_2, (\text{Std} \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes \text{Std}))$.
- (c) $(\text{SL}_4, \text{Std} \oplus \text{Std}^\vee)$.
- (d) $(\text{SL}_2, \text{Std}) \otimes (\text{Sp}_4, \text{Std})$.
- (e) $(\text{SO}_7, \text{Spin})$.

2.3. Automorphic Galois representations.

Definition/Proposition 2.9. [BLGGT14, Section 2.1] *Let F be a CM or a totally real field. Denote by F^+ the maximal totally real subfield of F . Let E be a number field and λ be a prime of E .*

- (i) *A λ -adic Galois representation $\rho : \text{Gal}_F \rightarrow \text{GL}_n(\overline{E}_\lambda)$ is called essentially self-dual, if there exists a character $\chi : \text{Gal}_{F^+} \rightarrow \overline{E}_\lambda^*$ such that for some (hence all) infinite place v of F^+ , there exists $\varepsilon_v \in \{\pm 1\}$ and a non-degenerate pairing $\langle -, - \rangle_v$ on $\overline{E}_\lambda^n$ such that*

$$\langle x, y \rangle_v = \varepsilon_v \langle y, x \rangle_v$$

$$\langle \rho(g)x, \rho(c_v g c_v)y \rangle_v = \chi(g) \langle x, y \rangle_v$$

for all $x, y \in \overline{E}_\lambda^n$ and all $g \in \text{Gal}_F$. Here c_v is the complex conjugation associated to v . For F being a CM field we further assume $\varepsilon_v = -\chi(c_v)$.

- (ii) Moreover, ρ is called totally odd if $\varepsilon_v = 1$ for all infinite place v .
- (iii) If F is totally real, then ρ is essentially self-dual, if and only if it either factors through $\text{GSp}_n(\overline{E}_\lambda)$ (if $\chi(c_v) = -\varepsilon_v$) or $\text{GO}_n(\overline{E}_\lambda)$ (if $\chi(c_v) = \varepsilon_v$). In particular there exists some continuous character $\chi : \text{Gal}_K \rightarrow \overline{E}_\lambda^*$ called similitude character, such that $\rho \cong \rho^\vee \otimes \chi$. In such case, ρ is totally odd if and only if it either factors through GSp_n with totally odd similitude character (i.e. for any complex conjugation c one has $\chi(c) = -1$) or it factors through GO_n with totally even similitude character (i.e. for any complex conjugation c one has $\chi(c) = 1$).

Automorphic Galois representations refer to the ones arising from following result.

Theorem 2.10. [BLGGT14, Theorem 2.1.1] *Let F be a CM or a totally real field. Suppose that (π, χ) is a regular algebraic cuspidal polarized automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. Then there exists a CM field E and an E -rational Serre compatible system*

$$\{\rho_{\pi, \lambda} : \text{Gal}_F \rightarrow \text{GL}_n(\overline{E}_\lambda)\}$$

such that

- (i) $(\rho_{\pi, \lambda}, \varepsilon_\ell^{1-n} \rho_{\chi, \lambda})$ is essentially self-dual and totally odd, where ε_ℓ is the ℓ -adic cyclotomic character and $\lambda \nmid \ell$.
- (ii) Fix an embedding $\iota : E_\lambda \hookrightarrow \mathbb{C}$. For $v \nmid \ell$, the semisimplified Weil-Deligne representation is independent of λ and satisfies:

$$\iota \text{WD}(\rho_{\pi, \lambda} |_{\text{Gal}_{F_v}})^{\text{F-ss}} \cong \text{rec} \left(\pi_v \otimes |\det|_v^{(1-n)/2} \right)$$

and these Weil-Deligne representations are pure of weight ω .

- (iii) $\rho_{\pi, \lambda}$ is de Rham, has pure of weight ω and distinct τ -Hodge-Tate weights for all $\tau : F \hookrightarrow \overline{E}$.
- (iv) If $v \mid \ell$ and π_v has an Iwahori fixed vector then

$$\iota \text{WD}(\rho_{\pi, \lambda} |_{\text{Gal}_{F_v}})^{\text{F-ss}} \cong \text{rec} \left(\pi_v \otimes |\det|_v^{(1-n)/2} \right)$$

In particular $\rho_{\pi, \lambda}$ is semi-stable at v , and if π_v is unramified then it is crystalline.

One has following criterion on automorphic Galois representations of totally real fields.

Theorem 2.11. [BLGGT14, Theorem C]. *Suppose K is a totally real field. Let n be an integer and $\ell \geq 2(n+1)$ be a prime. Let*

$$\rho : \text{Gal}_K \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_\ell)$$

be a continuous representation. Suppose that the following conditions are satisfies.

- (i) (Unramified almost everywhere) ρ is unramified at all but finitely many primes.
- (ii) (Odd essential self-duality) Either ρ maps to GSp_n with totally odd similitude character or it maps to GO_n with totally even similitude character.
- (iii) (Potential diagonalizability and regularity) ρ is potentially diagonalizable (and hence potentially crystalline) at each prime v of K above ℓ and regular, i.e. for each $\tau : K \hookrightarrow \overline{\mathbb{Q}}_\ell$ it has n distinct τ -Hodge-Tate weights.
- (iv) (Irreducibility) $\rho|_{\text{Gal}_{K(\zeta_\ell)}}$ is residually irreducible.

Then we can find a finite Galois totally real extension K'/K such that $\rho|_{\text{Gal}_{K'}}$ is automorphic. Moreover ρ is part of a strictly pure compatible system of K .

Condition (iii) can be checked by following.

Lemma 2.12. *When $K = \mathbb{Q}$, condition (iii) is satisfied when ρ is crystalline and regular, and the Hodge-Tate numbers $\text{Ht}(\rho) \subseteq [a, a + \ell - 2]$ for some integer a .*

Proof. One takes $K = \mathbb{Q}$ in [BLGGT14, Lemma 1.4.3.(2)]. □

The following result shows that certain low dimensional subrepresentations of strictly compatible systems fit into strictly compatible systems.

Proposition 2.13. [Hu23a, Proposition 2.12]. *Given an E -rational strictly compatible system $\{\rho_\lambda\}$ of some totally real field, then for all but finitely many λ ,*

- (i) *If σ is a 2-dimensional irreducible regular subrepresentation of ρ_λ , then σ can be extended to a 2-dimensional regular irreducible strictly compatible system.*
- (ii) *If σ is a 3-dimensional irreducible regular essentially self-dual subrepresentation of ρ_λ , then σ can be extended to a 3-dimensional regular irreducible strictly compatible system.*

One has the following criterion to check an irreducible subrepresentation of $\rho_{\pi,\lambda}$ is essentially self-dual and totally odd.

Theorem 2.14. [CG13, Theorem 2.3] *Let (π, χ) be a regular algebraic cuspidal polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ where F is a totally real field. Denote by $(\rho_{\pi,\lambda}, \rho_{\chi,\lambda})$ the corresponding compatible system of Galois representations. If for some λ we have an irreducible subrepresentation r of $\rho_{\pi,\lambda}$ such that $r \cong r^\vee \otimes \varepsilon_\ell^{1-n} \rho_{\chi,\lambda}$, where $\lambda|\ell$ and ε_ℓ is the ℓ -adic cyclotomic character, then $(r, \varepsilon_\ell^{1-n} \rho_{\chi,\lambda})$ is essentially self-dual and totally odd.*

In particular, for dimensional reason one has the following consequence.

Corollary 2.15. *Under the above setting, if the irreducible components of some $\rho_{\pi,\lambda}$ have distinct dimensions, then each of them is essentially self-dual and totally odd.*

2.4. Big images and irreducibility. In the sequel we denote by $(\bar{\rho}^{\mathrm{ss}}, \bar{V}^{\mathrm{ss}})$ the semisimple reduction of a λ -adic Galois representation (ρ, V) , by ε_ℓ the ℓ -adic cyclotomic character of some number field.

Definition/Theorem 2.16. [Hu23b, Theorem 3.1], [Hu23a, Theorem 2.10]. *Given an n -dimensional regular E -rational semisimple Serre compatible system $\{(\rho_\lambda, V_\lambda)\}$ of number field K . Write $d = [E : \mathbb{Q}]$. By restriction of scalars, we have an nd -dimensional $\mathbb{Q}$ -rational compatible system:*

$$\left\{ \rho_\ell := \bigoplus_{\lambda|\ell} \rho_\lambda : \mathrm{Gal}_K \rightarrow (\mathrm{Res}_{E/\mathbb{Q}})(\mathbb{Q}_\ell) \subseteq \mathrm{GL}_{nd}(\mathbb{Q}_\ell) \right\}_\ell$$

Suppose that there exist integers $N_1, N_2 \geq 0$ and a finite extension K'/K such that the following conditions hold.

- (a) (Bounded tame inertia weights): *For all $\ell \gg 0$ and each finite place v of K above ℓ , the tame inertia weights of the local representation $(\bar{\rho}_\ell^{\mathrm{ss}} \otimes \bar{\varepsilon}_\ell^{N_1})|_{\mathrm{Gal}_{K_v}}$ belong to $[0, N_2]$.*
- (b) (Potential semistability): *For all $\ell \gg 0$ and each finite place w of K' not above ℓ , the semisimplification of the local representation $\bar{\rho}^{\mathrm{ss}}$.*

Then there exists a finite Galois extension L/K such that, up to isomorphism there exists a unique connected reductive group

$$\underline{G}_\ell \subseteq \mathrm{GL}_{nd, \mathbb{F}_\ell}$$

for each sufficiently large ℓ called algebraic envelope, such that:

- (i) $\bar{\rho}_\ell^{\mathrm{ss}}(\mathrm{Gal}_L)$ is a subgroup of $\underline{G}_\ell(\mathbb{F}_\ell)$ with index uniformly bounded when ℓ varies.
- (ii) $\underline{G}_\ell$ acts on the ambient space semisimply.
- (iii) The formal characters of $\underline{G}_\ell \hookrightarrow \mathrm{GL}_{nd, \mathbb{F}}$ for all λ are bounded.

For all but finitely many ℓ such that the algebraic envelope $\underline{G}_\ell$ exists, let $\lambda \in \Sigma_E$ be any finite place of E that divides ℓ and (σ, W) be a subrepresentation of $\rho_\lambda \otimes \mathbb{Q}_\ell$. Denote by $\underline{G}_W$ the image of $\underline{G}_\ell$ in $\mathrm{GL}_{W^{\mathrm{ss}}}$, which is called algebraic envelope of W .

Theorem 2.17. [Hu23b, Theorem 3.12]. *Given an n -dimensional E -rational semisimple Serre compatible system $\{\rho_\lambda\}$ of number field K . Assume the conditions (a) and (b) in Definition/Theorem 2.16 hold. Then except for finitely many λ , for any subrepresentation (σ, W) of $\rho_\lambda \otimes \bar{E}_\lambda$ one has:*

- (i) *The algebraic envelope $\underline{G}_W$ and algebraic monodromy $\mathbf{G}_W$ of σ have the same formal bi-characters.*

- (ii) There exists a finite Galois extension L/K , independent of W , such that the commutants of $\bar{\sigma}_\lambda^{ss}(\text{Gal}_L)$ and $\underline{G}_W$ (resp. $[\bar{\sigma}_\lambda^{ss}(\text{Gal}_L), \bar{\sigma}_\lambda^{ss}(\text{Gal}_L)]$ and $\underline{G}_W^{ss}$) in $\text{End}(\bar{W})^{ss}$ are equal. In particular, $\bar{\sigma}_\lambda^{ss}(\text{Gal}_L)$ (resp. $[\bar{\sigma}_\lambda^{ss}(\text{Gal}_L), \bar{\sigma}_\lambda^{ss}(\text{Gal}_L)]$) is irreducible on $\bar{W}^{ss}$ if and only if $\underline{G}_W$ (resp. $\underline{G}_W^{der}$) is irreducible on $\bar{W}^{ss}$.
- (iii) If $\mathbf{G}_W$ is of type A and $\mathbf{G}_W^\circ \rightarrow \text{GL}_W$ is irreducible (in particular for Lie-irreducible dimension ≤ 3 ones), then $\underline{G}_W$ and thus Gal_K (resp. $\text{Gal}_{K^{ab}}$) are irreducible on $\bar{W}^{ss}$.
- (iv) If σ is irreducible and of type A, then it is residually irreducible.

Given an E -rational regular strictly compatible system. By Lemma 2.5, after enlarging E , one regards the system as a Serre compatible system $\{\rho_\lambda : \text{Gal}_K \rightarrow \text{GL}_n(E_\lambda)\}$.

Theorem 2.18. [Hu23b, Theorem 4.1] *The conclusions in Theorem 2.17 hold for E -rational regular strictly compatible systems, in particular for compatible systems arising from Theorem 2.10.*

We need following result in the proof.

Proposition 2.19. [Da25, Proposition 2.25] *Let $\{\rho_\lambda : \text{Gal}_\mathbb{Q} \rightarrow \text{GL}_n(\bar{E}_\lambda)\}$ be an E -rational strictly compatible system of $\mathbb{Q}$. Consider its modulo λ compatible system $\{\bar{\rho}_\lambda^{ss}\}$ by taking semisimple reductions. Suppose for infinitely many λ one has a 2-dimensional odd irreducible subrepresentation*

$$\bar{\sigma}_\lambda \subseteq \bar{\rho}_\lambda^{ss}$$

Then after replacing E with a finite extension, there exists a 2-dimensional E -rational strictly compatible system $\{\sigma_\lambda\}$ such that for infinitely many λ one has $\bar{\sigma}_\lambda$ is the semisimple reduction of σ_λ .

2.5. p -adic Hodge theoretic lift. Let $\tilde{H} \twoheadrightarrow H$ be a central torus quotient of algebraic groups. Let F be a global or local field. The following results lift an ℓ -adic representation $\rho : \text{Gal}_F \rightarrow H(\bar{\mathbb{Q}}_\ell)$ to $\tilde{\rho} : \text{Gal}_F \rightarrow \tilde{H}(\bar{\mathbb{Q}}_\ell)$ that preserve certain p -adic Hodge properties.

$$\begin{array}{ccc} & & \tilde{H}(\bar{\mathbb{Q}}_\ell) \\ & \nearrow \tilde{\rho} & \downarrow \\ \text{Gal}_F & \xrightarrow{\rho} & H(\bar{\mathbb{Q}}_\ell) \end{array} \tag{1}$$

Theorem 2.20. [Pa19, Corollary 3.2.12] *Let $\tilde{H} \twoheadrightarrow H$ be a central torus quotient of algebraic groups, and let $\rho : \text{Gal}_F \rightarrow H(\bar{\mathbb{Q}}_\ell)$ be a Hodge-Tate representation of a local field F . Then there exists a Hodge-Tate representation $\tilde{\rho} : \text{Gal}_F \rightarrow \tilde{H}(\bar{\mathbb{Q}}_\ell)$ such that (1) commutes.*

Theorem 2.21. [DWW24, Theorem 2.13] *Let $\tilde{H} \twoheadrightarrow H$ be a central torus quotient of algebraic groups, and let $\rho : \text{Gal}_\mathbb{Q} \rightarrow H(\bar{\mathbb{Q}}_\ell)$ be an ℓ -adic representation of $F = \mathbb{Q}$ that is unramified almost everywhere and the restriction to $\text{Gal}_{\mathbb{Q}_\ell}$ is crystalline. Then there exists a representation $\tilde{\rho} : \text{Gal}_{F_v} \rightarrow \tilde{H}(\bar{\mathbb{Q}}_\ell)$ that is unramified almost everywhere and the restriction to $\text{Gal}_{\mathbb{Q}_\ell}$ is crystalline, such that (1) commutes.*

Proof. By [Pa15, Proposition 5.5] under the setting one has a geometric lift $\tilde{\rho}'$ of ρ . Hence one just needs to modify this lift such that its restriction to $\text{Gal}_{\mathbb{Q}_\ell}$ is moreover crystalline. This can be done by [Pa19, Corollary 3.2.13], which says locally one has a lift $\tau : \text{Gal}_{\mathbb{Q}_\ell} \rightarrow \tilde{H}(\bar{\mathbb{Q}}_\ell)$ of ρ that is crystalline. As both being a lift of $\rho|_{\text{Gal}_{\mathbb{Q}_\ell}}$, one has $\tau \cong \tilde{\rho}'|_{\text{Gal}_{\mathbb{Q}_\ell}} \otimes \chi$ for some Hodge-Tate character χ . One can twist τ with suitable power of ε_ℓ making the Hodge-Tate weight of χ to be zero. In particular the restriction to inertia subgroup $\chi|_{I_\ell}$ has finite image. Hence one can choose some global character $\chi' : \text{Gal}_\mathbb{Q} \rightarrow \bar{\mathbb{Q}}_\ell^*$ such that $\chi'|_{I_\ell} = \chi|_{I_\ell}$. Then $\tilde{\rho} = \tilde{\rho}' \otimes \chi'$ is the desired lift of ρ . $\square$

3. A STEP OF XIA

It is known that at least for infinitely many λ , one has $\rho_{\pi, \lambda}$ is irreducible.

Theorem 3.1. [PT15, Theorem 1.7] *Let F be a CM field and π is a polarisable, regular algebraic, cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$. Denote by E_π the number field that each $\rho_{\pi,\lambda}$ is defined over, whose existence is guaranteed by Lemma 2.5. Then there is a finite CM extension E/E_π and a Dirichlet density 1 set $\mathcal{L}$ of rational primes, such that for all conjugation-invariant primes λ of E dividing an $\ell \in \mathcal{L}$, $\rho_{\pi,\lambda}|_{E_\pi}$ is irreducible. In particular, there is a positive Dirichlet density set $\mathcal{L}'$ of rational primes such that if a prime λ of E_π divides some $\ell \in \mathcal{L}'$, then $\rho_{\pi,\lambda}$ is irreducible.*

In this section we follow the treatment in [Xi19] to prove the following result.

Proposition 3.2. *To prove Theorem 1.1 it is enough to assume there exist infinitely many places λ such that $\rho_{\pi,\lambda}$ is Lie-irreducible.*

Notice that when $n = 7$, this is easy to verify. Pick one irreducible ρ_{π,λ_0} . If this is not Lie-irreducible, then it is induced by a character χ_{λ_0} of a 7-degree number field K . By class field theory, after possibly enlarging the coefficients, this χ_{λ_0} extends to a strictly compatible system $\{\chi_\lambda\}$. Then by semisimplicity of $\{\rho_{\pi,\lambda}\}$, one has $\rho_{\pi,\lambda} = \text{Ind}_K^{\mathbb{Q}} \chi_\lambda$. Then one uses regularity to check conditions in Mackey's irreducibility criterion, hence in this case **all** $\rho_{\pi,\lambda}$ are irreducible.

We focus on $n = 8$. One has the following result which is a totally real analogy to [Xi19, Proposition 2].

Proposition 3.3. [Hu23b, Proposition 4.14] *Let F^+ be a totally real field, $\{\rho_{\pi,\lambda}\}$ the associated compatible system of F^+ defined over E as in Lemma 2.5. Let F be a CM field containing F^+ as maximal totally real subfield. Let $F_{1,\pi}$ be the minimal extension of F^+ such that the compatible system*

$$\left\{ \left(\text{Ind}_F^{F^+} \text{Res}_F^{F^+} \rho_{\pi,\lambda} \right) \oplus \rho_{\chi,\lambda} \right\}$$

is connected. Let F_2 be the maximal CM subextension of $F_{1,\pi}/F^+$. After enlarging the CM field E if necessary, there exist a family of Galois representations $\{r_{1,\lambda}\}_\lambda$ of a subextension F_4 of F_2/F^+ and a regular algebraic polarized cuspidal automorphic representation π_1 of $GL_m(\mathbb{A}_{F_3})$ where F_3 is a finite CM extension of F_2 such that

$$\{\text{Ind}_{F_4}^{F^+} r_{1,\lambda}\}_\lambda \cong \{\rho_{\pi,\lambda}\}_\lambda \text{ and } \{\text{Res}_{F_3}^{F_4} r_{1,\lambda}\}_\lambda \cong \{\rho_{\pi_1,\lambda}\}_\lambda$$

and this F_3 is the maximal CM subextension of $F_{1,\pi}/F^+$.

If $F_4 \neq F^+$, then $m \leq 4$. Hence $\rho_{\pi_1,\lambda}$ is irreducible for all but finitely many λ due to [Hu23b, Theorem 1.4]. Then by regularity and Mackey's irreducibility criterion we have $\{\rho_{\pi,\lambda}\}$ is irreducible for all but finitely many λ . Hence we can assume $F_4 = F^+$. Then $r_{1,\lambda} \cong \rho_{\pi,\lambda}$ by semisimplicity and Proposition 3.2 follows from below.

Proposition 3.4. [Xi19, Corollary 1] *Let F be a CM field and $\{\rho_{\pi,\lambda}\}_\lambda$ be the compatible system associated to π . If F is maximal CM subextension of $F_{1,\pi}/F^+$. Then there exist infinitely many places λ such that $\rho_{\pi,\lambda}$ is Lie-irreducible.*

4. PROOF

For simplicity we omit π in each associated representation $\rho_{\pi,\lambda}$. By Proposition 3.2 we assume there are infinitely many λ_0 such that ρ_{λ_0} is Lie-irreducible. Then the restriction to its derived subgroup $\mathbf{G}_{\lambda_0}^{\text{der}} = [\mathbf{G}_{\lambda_0}^\circ, \mathbf{G}_{\lambda_0}^\circ]$, which we denote by $\rho_{\lambda_0}^{\text{der}}$, is irreducible.

Proposition 4.1. *The list item 4 gives all the isomorphism classes of connected semisimple subgroups $G \subseteq GL_V$ that are irreducible on $V = \mathbb{Q}_\ell^n$ for $n = 7$ and 8 .*

Proof. The tautological representation ρ of G admits an exterior tensor decomposition

$$(G, \rho) \cong (G_1 G_2 \cdots G_m, \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_m)$$

where each G_i is an almost simple factor of G and (G_i, ρ_i) is an irreducible representation. Then one tracks down the low dimensional irreducible representations of almost simple lie algebra gives the complete list.

- (i) $n = 7$ and $m = 1$: cases (1), (2), (3), (4).
- (ii) $n = 8$ and $m = 1$: cases (5), (7), (10), (12), (13), (14), (15).
- (iii) $n = 8$ and $m = 2$: cases (6), (9), (11).
- (iv) $n = 8$ and $m = 3$: case (8) only.

□

Given a Lie-irreducible Galois representation ρ , by Lie type of ρ , we mean the isomorphism class of the tautological representation of ρ^{der} . As there are only finitely many Lie types for ρ_{λ_0} , we can assume there exists an infinite set of places $L \subseteq \Sigma_E$ such that all $\rho_{\lambda_0}^{\text{der}}$ for $\lambda_0 \in L$ have the same Lie type listed in item 4. As $\text{Im} \rho_\lambda$ must factor through GO_n or GSp_n due to Theorem 2.10.(i) and Definition/Proposition 2.9.(iii), we rule out cases (4), (11) and (15). In the sequel we assume there exist infinitely many λ_1 such that ρ_{λ_1} is reducible.

	Types	(G,V)	dim	rank	formal character
(1)	$7A_1$	$(\text{SL}_2, \text{Sym}^6(\text{Std}))$	7	1	$\{x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, x^3\}$
(2)	$7G_2$	(G_2, Std)	7	2	$\{x, x^{-1}, y, y^{-1}, xy, (xy)^{-1}, 1\}$
(3)	$7B_3$	$(\text{SO}_7, \text{Std})$	7	3	$\{x, x^{-1}, y, y^{-1}, z, z^{-1}, 1\}$
(4)	$7A_6$	$(\text{SL}_7, \text{Std})$	7	6	omitted
(5)	$8A_1$	$(\text{SL}_2, \text{Sym}^7(\text{Std}))$	8	1	$\{x^{-7}, x^{-5}, x^{-3}, x^{-1}, x, x^3, x^5, x^7\}$
(6)	$2A_1 \times 4A_1$	$(\text{SL}_2 \times \text{SL}_2, \text{Std} \otimes \text{Sym}^3(\text{Std}))$	8	2	omitted
(7)	$8A_2$	$(\text{SL}_3, \text{adjoint representation})$	8	2	omitted
(8)	$2A_1 \times 2A_1 \times 2A_1$	$(\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2, \text{Std} \otimes \text{Std} \otimes \text{Std})$	8	3	$\{x^{\pm 1} y^{\pm 1} z^{\pm 1}\}$
(9)	$2A_1 \times 4C_2$	$(\text{SL}_2 \times \text{Sp}_4, \text{Std} \otimes \text{Std})$	8	3	$\{x^{\pm 1} y^{\pm 1} z^{\pm 1}\}$
(10)	$8B_3$	$(\text{SO}_7, \text{Spin representation})$	8	3	$\{x^{\pm 1} y^{\pm 1} z^{\pm 1}\}$
(11)	$2A_1 \times 4A_3$	$(\text{SL}_2 \times \text{SL}_4, \text{Std} \otimes \text{Std})$	8	4	omitted
(12)	$8C_4$	$(\text{Sp}_8, \text{Std})$	8	4	$\{x, x^{-1}, y, y^{-1}, z, z^{-1}, w, w^{-1}\}$
(13)	$8D_4$	$(\text{SO}_8, \text{Std})$	8	4	$\{x, x^{-1}, y, y^{-1}, z, z^{-1}, w, w^{-1}\}$
(14)	$8D_4$	$(\text{SO}_8, \text{two half - spin representations})$	8	4	omitted
(15)	$8A_7$	$(\text{SL}_8, \text{Std})$	8	7	omitted

4.1. **Case (1).** Assume $\rho_{\lambda_0}^{\text{der}}$ is of type $7A_1$. As the formal bi-character of ρ_{λ_1} is the same as that of ρ_{λ_0} due to Theorem 2.3.(ii), the decompositions of $\rho_{\lambda_1}^{\text{der}}$ can only be

$$\rho_{\lambda_1}^{\text{der}} = (\text{SL}_2, \text{Sym}^2(\text{Std})) \oplus (\text{SL}_2, \text{Sym}^3(\text{Std}))$$

We denote by $\rho_{\lambda_1} = \sigma_{\lambda_1,1} \oplus \sigma_{\lambda_1,2}$ the irreducible decomposition. We check conditions in Theorem 2.11 to show both $\sigma_{\lambda_1,1}$ and $\sigma_{\lambda_1,2}$ extend to a compatible system for some λ_1 . Then the semisimplicity of $\{\rho_\lambda\}$ would give ρ_{λ_0} is reducible hence a contradiction. Condition (i) is obvious. As ρ_{λ_1} is essential self-dual and odd, and $\sigma_{\lambda_1,1}, \sigma_{\lambda_1,2}$ have different dimensions, (ii) can be checked by Corollary 2.15. (iii) can be checked by Lemma 2.12 after taking λ_1 sufficiently large. Finally, as both $\sigma_{\lambda_1,i}$ are of type A, by Theorem 2.17.(iii), the last condition (iv) is satisfied for λ_1 sufficiently large. Hence in such case ρ_λ is irreducible for all but finitely many λ .

4.2. **Case (3).** Assume $\rho_{\lambda_0}^{\text{der}}$ is of type $7B_3$ with standard representation. By Theorem 2.3.(ii), the formal character of each $\rho_\lambda^{\text{der}}$ is $\{x, x^{-1}, y, y^{-1}, z, z^{-1}, 1\}$. Since there exists only one zero weight, there cannot be more than one character in the decomposition of ρ_{λ_1} . Hence there exist infinitely many λ_1 such that ρ_{λ_1} all have dimension type one of the following cases:

- (i) $6 + 1$.
- (ii) ρ_{λ_1} has a 2 or 3-dimensional component.

We first consider case (ii). The 2 or 3-dimensional component φ_{λ_1} of ρ_{λ_1} must be Lie-irreducible. Since otherwise the derived subgroup of its algebraic monodromy group is trivial, which contradicts with the fact that the formal character of $\rho_\lambda^{\text{der}}$ has no repeated zero weights.

As the formal character of $\rho_{\lambda_0}^{\text{der}}$ contains no three nonzero weights such that their sum is zero, the 3-dimensional Lie-irreducible component of ρ_{λ_1} (if exists) must be of type SO_3 . Then by Proposition 2.13 for some λ_1 our φ_{λ_1} fits into a strictly compatible system $\{\varphi_\lambda\}$. The derived

subgroup of algebraic monodromy group of φ_{λ_1} is either SL_2 or SO_3 . Consider compatible system $\{\rho_\lambda \oplus \varphi_\lambda\}$. At place λ_1 the semisimple rank of $\rho_{\lambda_1} \oplus \varphi_{\lambda_1}$ is 3 since φ_{λ_1} is a subrepresentation of ρ_{λ_1} . At place λ_0 , by Goursat's lemma, the derived subgroup of monodromy group of $\rho_{\lambda_0} \oplus \varphi_{\lambda_0}$ is either $\mathrm{SO}_7 \times \mathrm{SL}_2$ or $\mathrm{SO}_7 \times \mathrm{SO}_3$, which has rank 4. This contradicts with Theorem 2.3.(ii).

For case (i), denote by $\varphi_{\lambda_1} \oplus \chi_{\lambda_1}$ the irreducible decomposition of ρ_{λ_1} , where φ_{λ_1} is a 6-dimensional component and χ_{λ_1} is a character. Since each ρ_λ factors through GO_7 and $\mathrm{Sp}_6 \times \{1\}$ is not contained in GO_7 , the component φ_{λ_1} must factor through GO_6 . Since ρ_{λ_1} is essentially self-dual and odd, and the components of ρ_{λ_1} have different dimensions, by Corollary 2.15, φ_{λ_1} is essentially self-dual and odd. Since SO_6 is of type A, Theorem 2.17.(iii) shows that there exists some λ_1 such that (iv) of Theorem 2.11 is true. Other conditions of the theorem are easy to verify. Hence this φ_{λ_1} fits into a strictly compatible system. The character χ_{λ_1} naturally fits into a compatible system (after possibly enlarging the coefficients) due to class field theory. Hence semisimplicity of $\{\rho_\lambda\}$ implies ρ_{λ_0} is reducible, a contradiction.

4.3. Case (5). Assume $\rho_{\lambda_0}^{\mathrm{der}}$ is of type $8A_1$. Since the formal character of $\rho_{\lambda_0}^{\mathrm{der}}$ does not contain zero weight, it cannot be decomposed as the union of two formal characters of some representations. Hence by Theorem 2.3.(ii) all ρ_λ are irreducible with same Lie type.

4.4. Case (7). Assume $\rho_{\lambda_0}^{\mathrm{der}}$ is the adjoint representation of SL_3 . We show in this case ρ_{λ_0} is irregular hence rule out this situation.

Denote by K the smallest extension of $\mathbb{Q}$ as in Theorem 2.3.(i). We restrict the compatible system to Gal_K . Then all algebraic monodromy groups are connected. Since the image of adjoint representation is PGL_3 , the algebraic monodromy group $\mathbf{G}_{\lambda_0}$ is either PGL_3 or $\mathbb{G}_m \cdot \mathrm{PGL}_3$. The $\mathbb{G}_m$ part corresponds to a character which is a weakly abelian direct summand of ρ_{λ_0} , hence by [BH25, Theorem 1.1] this character is Hodge-Tate. Hence after twisting a compatible system of Hodge-Tate characters to $\{\rho_\lambda\}$, we may assume $\mathbf{G}_{\lambda_0} = \mathrm{PGL}_3$.

Consider the surjection $\mathrm{GL}_3 \rightarrow \mathrm{PGL}_3$ whose kernel is a central torus. By Theorem 2.20 the restriction of ρ_{λ_0} to $\mathrm{Gal}_{\mathbb{Q}_\ell}$ can lift to some Hodge-Tate representation σ :

$$\begin{array}{ccc} & & \mathrm{GL}_3(\overline{E}_\lambda) \\ & \nearrow \sigma & \downarrow \\ \mathrm{Gal}_{\mathbb{Q}_\ell} & \xrightarrow{\rho_{\lambda_0}|_{\mathrm{Gal}_{\mathbb{Q}_\ell}}} & \mathrm{PGL}_3(\overline{E}_\lambda) \end{array}$$

Then there exists some characters χ_1 and χ_2 such that

$$\sigma \otimes (\sigma^\vee \otimes \chi_1) \cong \chi_2 \oplus \rho_{\lambda_0}|_{\mathrm{Gal}_{\mathbb{Q}_\ell}}$$

In particular ρ_{λ_0} has repeated Hodge-Tate weights.

4.5. Cases (10), (12), (13), (14). As the formal character of $\rho_{\lambda_0}^{\mathrm{der}}$ contains no zero weights and no three weights whose sum is zero, ρ_{λ_1} cannot contain 1 or 3 or 5-dimensional components. We first show the following lemma.

Lemma 4.2. *Given an 8-dimensional compatible system $\{\rho_\lambda\}$ which is attached to a regular algebraic essentially self-dual cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ such that at least one ρ_{λ_0} is Lie-irreducible. If some ρ_{λ_1} is reducible with dimensional type $4 + 4$, then exactly one of the following happens.*

- (i) *The two 4-dimensional components are essentially self-dual and odd.*
- (ii) *Both components are not essentially self-dual. In such case one has*

$$\mathbf{G}_{\lambda_1}^{\mathrm{der}} = (\mathrm{SL}_4, \mathrm{Std}) \oplus (\mathrm{SL}_4, \mathrm{Std}^\vee) \quad (2)$$

Proof. We assume the irreducible decomposition of ρ_{λ_1} is $W_1 \oplus W_2$ where $\dim W_1 = \dim W_2 = 4$. Denote by $\chi = \varepsilon_\ell^7 \rho_{\chi, \lambda_1}^{-1}$. Then by Theorem 2.10.(i) one has $\rho_{\lambda_1}^\vee \cong \rho_{\lambda_1} \otimes \chi$. Then for dimensional reason we have two cases.

- (a) $W_i^\vee \cong W_i \otimes \chi$ for $i = 1, 2$.

(b) $W_1^\vee \cong W_2 \otimes \chi$ and $W_2^\vee \cong W_1 \otimes \chi$.

In case (a), due to Theorem 2.14, we have both W_1 and W_2 are essentially self-dual and odd. Hence this is in case (i). In case (b), if one of them is not Lie-irreducible, say W_1 , one writes it as

$$W_1 = \text{Ind}_K^{\mathbb{Q}} \sigma$$

for some Lie-irreducible representation σ of K . Then W_2 would also not be Lie-irreducible.

$$W_2 = \text{Ind}_K^{\mathbb{Q}} (\sigma^\vee \otimes \chi|_{\text{Gal}_K}^{-1})$$

Here $\sigma^\vee \otimes \chi|_{\text{Gal}_K}^{-1}$ also is Lie-irreducible. Hence we have some ρ_{λ_0} is Lie-irreducible yet ρ_{λ_1} is induced by some representation of a nontrivial field extension. This contradicts with the compatibility of Frobenii due to [Pa19, Proposition 3.4.9] (see also [Da25, Proposition 2.15]).

Hence both W_1 and W_2 are Lie-irreducible, then their Lie type are either $(\text{SL}_2, \text{Sym}^3(\text{Std}))$, $(\text{SO}_4, \text{Std})$, $(\text{Sp}_4, \text{Std})$ or $(\text{SL}_4, \text{Std})$. We rule out case SL_2 since the semisimple rank of algebraic monodromy group of ρ_{λ_1} is 1, hence the Lie-irreducible ρ_{λ_0} must have Lie type (5) as in item 4. But we have shown in this case all ρ_λ are irreducible.

If the Lie type is SL_4 , then both W_1 and W_2 are not essentially self-dual and we are in case (ii).

If the Lie type is Sp_4 or SO_4 , then both W_1 and W_2 are essentially self-dual. We write η a similitude character of W_1 . Then $W_1^\vee \cong W_1 \otimes \eta$. Hence $W_2 \cong W_1 \otimes \eta\chi^{-1}$. However this would implies each weight in the formal character of $\{\rho_\lambda^{\text{der}}\}$ has multiplicity more than one. This contradicts with item 4. $\square$

We separate the following three cases:

- (a) There exist infinite many λ_1 such that each ρ_{λ_1} contains a 2-dimensional component.
- (b) There exist infinite many λ_1 such that the decomposition of ρ_{λ_1} has dimensional type $4 + 4$ with each component essentially self-dual and odd.
- (c) There exist infinite many λ_1 such that (2) hold.

In case (a), by Proposition 2.13.(i) for some λ_1 this 2-dimensional component φ_{λ_1} fits into a strictly compatible system $\{\varphi_\lambda\}$. Then consider compatible system $\{\rho_\lambda \oplus \varphi_\lambda\}$. The semisimple rank at place λ_1 is the same as that of ρ_{λ_0} . However Goursat's lemma guarantees the semisimple rank at place λ_0 is strictly large by 1.

In case (b), as the formal characters of $\rho_{\lambda_0}^{\text{der}}$ in the cases we consider have no repeated weights, the two components are both Lie-irreducible. We show for some λ_1 , one of the component φ_{λ_1} fits into a strictly compatible system $\{\varphi_\lambda\}$. Then the semisimple rank of compatible system $\{\rho_\lambda \oplus \varphi_\lambda\}$ at places λ_0 and λ_1 do not match due to Goursat's lemma again.

To do so again we check Theorem 2.11. Only condition Theorem 2.11.(iv) requires explanation. If the Lie type of any component is of type A, then Theorem 2.17.(iii) gives the conclusion. The only remaining case is both components have Lie type $(\text{Sp}_4, \text{Std})$ for sufficiently large λ_1 . We first show they are residually irreducible for sufficiently large λ_1 . If there are infinitely many λ_1 such that one of the component φ_{λ_1} is residually reducible. By Theorem 2.17.(i), one must have

$$\overline{\varphi}_{\lambda_1}^{\text{ss}} = \overline{\sigma}_{\lambda_1,1} \oplus \overline{\sigma}_{\lambda_1,2}$$

where $\overline{\sigma}_{\lambda_1,i}$ are 2-dimensional irreducible representations. As φ_{λ_1} has an odd similitude character χ_{λ_1} . Either one has

$$\overline{\sigma}_{\lambda_1,i}^\vee \cong \overline{\sigma}_{\lambda_1,i} \otimes \overline{\chi}_{\lambda_1}, i = 1, 2$$

in which case both $\overline{\sigma}_{\lambda_1,i}$ are odd, or one has

$$\overline{\sigma}_{\lambda_1,1}^\vee \cong \overline{\sigma}_{\lambda_1,2} \otimes \overline{\chi}_{\lambda_1}, \overline{\sigma}_{\lambda_1,2}^\vee \cong \overline{\sigma}_{\lambda_1,1} \otimes \overline{\chi}_{\lambda_1}$$

in which case the derived subgroup $\underline{G}_{\lambda_1}^{\text{der}}$ of algebraic envelope at λ_1 would be SL_2 and this contradicts with Theorem 2.17.(i). We denote by $\overline{\sigma}_{\lambda_1}$ any odd component of $\overline{\varphi}_{\lambda_1}^{\text{ss}}$. Then by Proposition 2.19, after possibly enlarging E , there exists a 2-dimensional strictly compatible system $\{\sigma_\lambda\}$ such that the semisimple reduction of σ_{λ_1} is $\overline{\sigma}_{\lambda_1}$ for sufficiently large λ_1 . But consider compatible system $\{\alpha_\lambda = \rho_\lambda \oplus \sigma_\lambda\}$. Denote by s the semisimple rank of algebraic envelope at place λ_1 . We have $s = 3$ in case (10) and $s = 4$ in cases (12), (13) and (14). Then by Theorem 2.17.(i)

the semisimple rank of α_{λ_1} is s . However Goursat's lemma asserts the semisimple rank of α_{λ_0} is $s + 1$. This contradicts with Theorem 2.3.(ii).

Now if one of the component in (b) is residually irreducible after restricting to $\text{Gal}_{\mathbb{Q}(\zeta)_{\ell_1}}$ for $\lambda_1 | \ell_1$, we are done. Hence we assume both components, which we denote by φ_{λ_1} and φ'_{λ_1} , are residually reducible after restricting to $\mathbb{Q}(\zeta_{\ell_1})$ for λ_1 sufficiently large. Then their semisimple reductions are induced by two-dimensional Lie-irreducible representations over quadratic fields K_{λ_1} and K'_{λ_1} inside number field L in Theorem 2.17.(ii).

$$\bar{\varphi}_{\lambda_1} = \text{Ind}_{K_{\lambda_1}}^{\mathbb{Q}} \bar{\sigma}, \quad \bar{\varphi}'_{\lambda_1} = \text{Ind}_{K'_{\lambda_1}}^{\mathbb{Q}} \bar{\sigma}'$$

Hence there exists infinitely many λ_1 such that K_{λ_1} coincide and K'_{λ_1} coincide. However, then the density of trace zero primes under ρ_{λ_1} is not zero. This contradicts with the fact ρ_{λ_0} is Lie-irreducible.

Finally in case (c). As the semisimple rank is 3, the only possible Lie type of ρ_{λ_0} would be case (10), i.e. SO_7 with spin representation. Our proof needs the extra Hodge-Tate condition in the statement in Theorem 1.1. We write irreducible decomposition $\rho_{\lambda_1} = \sigma_{\lambda_1,1} \oplus \sigma_{\lambda_1,2}$. Now consider the compatible system $\{\varphi_{\lambda} = \rho_{\lambda} \otimes \rho_{\lambda}\}$. At place λ_0 the irreducible decomposition is:

$$\rho_{\lambda_0} \otimes \rho_{\lambda_0} = \sigma_0 \oplus \sigma_1 \oplus \sigma_2 \oplus \sigma_3$$

where the Lie type of σ_i is $\wedge^i(\text{SO}_7, \text{Std})$. At place λ_1 the irreducible decomposition is (for simplicity we omit index λ_1 in $\sigma_{\lambda_1,i}$):

$$\begin{aligned} \rho_{\lambda_1} \otimes \rho_{\lambda_1} &= (\sigma_1 \otimes \sigma_1) \oplus (\sigma_2 \otimes \sigma_2) \oplus (\sigma_1 \otimes \sigma_2)^2 \\ &= (\text{Sym}^2(\sigma_1) \oplus \wedge^2(\sigma_1)) \oplus (\text{Sym}^2(\sigma_2) \oplus \wedge^2(\sigma_2)) \oplus (\chi \oplus \tau)^2 \end{aligned}$$

where χ is a character and τ is a 15-dimensional irreducible representation. Now consider its subrepresentation:

$$\alpha = \text{Sym}^2(\sigma_1) \oplus \text{Sym}^2(\sigma_2)$$

The Lie type of each component is SO_6 . Moreover one has

$$\text{Sym}^2(\text{SL}_4, \text{Std}) \oplus \text{Sym}^2(\text{SL}_4, \text{Std}^{\vee}) \cong \wedge^3(\text{SO}_6, \text{Std})$$

Hence the derived subgroup $\mathbf{G}_{\alpha}^{\text{der}}$ of α is $\text{SO}_6/\{\pm E_6\}$. One twists a compatible system of (Hodge-Tate) characters to $\{\varphi_{\lambda}\}$ so that the algebraic monodromy group $\mathbf{G}_{\alpha} = \mathbb{G}_m \mathbf{G}_{\alpha}^{\text{der}}$. One has the following isomorphism:

$$\pi : \text{GO}_6 \rightarrow \mathbf{G}_{\alpha}$$

$$g \mapsto \det(g)^{-1/3} \wedge^3(g)$$

Hence one writes $\wedge^3 \psi = \beta \otimes \alpha$ where β is a (Hodge-Tate) character and $\psi : \text{Gal}_{\mathbb{Q}} \rightarrow \text{GO}_6(\overline{E}_{\lambda_1})$ is unramified almost everywhere and its restriction to $\text{Gal}_{\mathbb{Q}_{\ell}}$ is crystalline except for a finite set of λ_1 , where $\lambda_1 | \ell$.

We show for suitable λ_1 this ψ fits into a compatible system. To do so we check conditions in Theorem 2.11. (i) is obvious. To check (ii), as ψ is essentially self-dual, we denote by χ its similitude character. Then $\alpha^{\vee} \cong \alpha \otimes \chi^3$. We want to show $\chi(c) = 1$ for some complex conjugation c . Suppose otherwise $\chi(c) = -1$. Since $\rho_{\lambda_1} \otimes \rho_{\lambda_1}$ has a similitude character η with $\eta(c) = 1$, and there is no other 10-dimensional components than $\text{Sym}^2(\sigma_1)$ and $\text{Sym}^2(\sigma_2)$ in the decomposition, one must have $\alpha^{\vee} \cong \alpha \otimes \eta$. Hence one has $\alpha \cong \alpha \otimes \chi^3 \eta^{-1}$. As $\chi^3(c) \eta^{-1}(c) = -1$, this shows the set of eigenvalues of $\alpha(c)$ is symmetric under multiplying -1 . Hence there are exactly ten eigenvalues -1 and ten eigenvalues 1 of $\alpha(c)$. However one has the list Table 1. As one cannot choose σ_1 and σ_2 in the list such that the union of eigenvalues of $\text{Sym}^2(\sigma_1)$ and $\text{Sym}^2(\sigma_2)$ satisfies this, one must have $\chi(c) = 1$. Hence ψ is odd.

To check (iii), by Lemma 2.12, after enlarging λ_1 , it is enough to show ψ is regular under our assumption on Hodge-Tate weights. Denote by $\text{Ht}(\sigma_1) = \{a_1, a_2, a_3, a_4\}$ the set of Hodge-Tate weights of σ_1 , then $\text{Ht}(\sigma_2) = \{n - a_1, n - a_2, n - a_3, n - a_4\}$ for some integer n . We know

TABLE 1.

eigenvalues of c on 4-dim σ	no. of eigenvalues -1 in $\text{Sym}^2(\sigma)$	no. of eigenvalues 1 in $\text{Sym}^2(\sigma)$
$\{1, 1, 1, 1\}$	0	10
$\{1, 1, 1, -1\}$	3	7
$\{1, 1, -1, -1\}$	4	6
$\{1, -1, -1, -1\}$	3	7
$\{-1, -1, -1, -1\}$	0	10

$\{a_1, a_2, a_3, a_4, n - a_1, n - a_2, n - a_3, n - a_4\}$ are distinct. Denote by $\text{Ht}(\psi) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Then $\text{Ht}(\beta \otimes \alpha)$ is the multiset:

$$A = \{a_i + a_j + m, 2n - (a_i + a_j) + m, 1 \leq i < j \leq 4\}$$

And $\text{Ht}(\wedge^3(\psi))$ is the multiset

$$B = \{x_i + x_j + x_k, 1 \leq i < j < k \leq 6\}$$

We have $A = B$. Consider following condition.

(P): $\{a_1, a_2, a_3, a_4, n - a_1, n - a_2, n - a_3, n - a_4\}$ are distinct and there exist no three distinct elements of them form a 3-term arithmetic progression.

Lemma 4.3. *Under condition (P), we have $\{x_1, x_2, \dots, x_6\}$ are distinct.*

Proof. Since we only care about distinctness of $\{x_1, x_2, \dots, x_6\}$, we can replace each x_i with $x_i - m/3$, each a_i with $a_i - m/2$ and n with $n - m$. This does not change the fact $\{a_1, a_2, a_3, a_4, n - a_1, n - a_2, n - a_3, n - a_4\}$ satisfies condition (P). Hence we can assume $m = 0$. Similiarly, replace a_i with $a_i + n$ and x_i with $x_i + 2n/3$ we can assume $n = 0$.

If say $x_5 = x_6$, we can find six pairs of elements in B that coincide:

$$x_5 + x_i + x_j = x_6 + x_i + x_j, 1 \leq i < j \leq 4$$

Write B_2 the multiset consist of them and B_1 to be its compliment multiset in B . We also divide A into the disjoint union of two multisets A_1 and A_2 .

$$A_1 = \{2a_1, 2a_2, 2a_3, 2a_4, -2a_1, -2a_2, -2a_3, -2a_4\}$$

$$A_2 = \{\pm(a_i + a_j), 1 \leq i < j \leq 4\}$$

Due to condition (P), we see elements in A_1 are distinct with any element in A . Hence we have $A_2 = B_2$ consist of six pairs of elements with equal values. Suppose one has indexes $i \neq j$ and $s \neq t$ such that $(a_i + a_j) = \varepsilon(a_s + a_t)$ where $\varepsilon = \pm 1$. Then $\{i, j\}$ and $\{s, t\}$ must be disjoint otherwise would contradict with condition (P). Now we write $\{s, t\}$ to be the compliment set of $\{i, j\}$ in $\{1, 2, 3, 4\}$. Then we have six equations:

$$(a_i + a_j) = \varepsilon_{i,j}(a_s + a_t), 1 \leq i < j \leq 4$$

where $\varepsilon_{i,j} = \pm 1$. Then one must have all $\varepsilon_{i,j} = -1$ since other situations all contradict with (P).

(i) If all $\varepsilon_{i,j} = 1$ then $a_1 = a_2 = a_3 = a_4$.

(ii) If some $\varepsilon = -1$ which implies $a_1 + a_2 + a_3 + a_4 = 0$, and some $\varepsilon_{i,j} = 1$, then $a_i = -a_j$.

Hence the only possible partition of $A_2 = B_2$ is:

$$(a_i + a_j) = -(a_s + a_t), \forall 1 \leq i < j \leq 4$$

with $\{s, t\}$ being the compliment set of $\{i, j\}$ in $\{1, 2, 3, 4\}$.

Next we claim we can rearrange the order of $\{a_1, a_2, a_3, a_4\}$ so that one has

$$x_5 + x_i + x_j = a_i + a_j, \forall 1 \leq i < j \leq 4 \quad (3)$$

To do so we define S to be the set of all 2-element subset of $\{1, 2, 3, 4\}$. Then the corresponding from A_2 to B_2 gives a bijection f on S . Namely if $f(\{i, j\}) = \{s, t\}$ then $x_5 + x_i + x_j = a_s + a_t$. The claim is equivalent to show if $p, q \in S$ have intersection then $f(p)$ and $f(q)$ have intersection. Say on the contrary we have $\{i, j, k, l\} = \{s, t, u, v\} = \{1, 2, 3, 4\}$ and

$$x_5 + x_i + x_j = a_s + a_t$$

$$x_5 + x_i + x_k = a_u + a_v$$

Then take summation of the two equations gives $x_i = x_l$. However this would produce more pairs with equal values in B , say $x_i + x_j + x_k = x_l + x_j + x_k$. But elements in multiset A_1 are distinct hence a contradiction.

By (3) one gets $a_i = x_i + x_5/2$ for $1 \leq i \leq 4$. Write $u = 3x_5/2$. We divide B_1 into four pairs such that in each pair the summation of two elements is zero.

$$A_{1,i} = \left\{ \left(\sum_{k \neq i} a_k - u \right), (a_i + u) \right\}, 1 \leq i \leq 4 \quad (4)$$

As $A_1 = B_1$, we have a bijection g on $\{1, 2, 3, 4\}$ so that $2a_i \in A_{1,g(i)}$. This gives four equations when $1 \leq i \leq 4$. We say the equation given by index i is of type 1 or 2, if $2a_i$ is the first or second term in (4) for $A_{1,g(i)}$. Denote by r the number of type 1 equations.

We first rule out $r \geq 3$. Say we have three distinct indexes i, j, k whose corresponding equations are of type 1. Then $\{i, j, k\}$ and $\{g(i), g(j), g(k)\}$ have intersection, say $g(i) = j$. Then

$$\begin{aligned} \sum_{1 \leq l \leq 4} a_l - u &= 2a_i + a_j \\ \sum_{1 \leq l \leq 4} a_l - u &= 2a_j + a_{g(j)} \end{aligned}$$

Hence $2a_i = a_j + a_{g(j)}$, this contradicts with condition (P) no matter which index $g(j)$ is.

Now for $r \leq 2$. We show the coefficient matrix M of the system of linear equations given by the four index $\{1, 2, 3, 4\}$ is invertible. Hence the obvious solution $a_1 = a_2 = a_3 = a_4 = u$ is the unique solution and this contradicts with condition (P) again.

Let $U = \text{diag}\{d_1, d_2, d_3, d_4\}$ be a diagonal matrix with $d_i = \pm 1$ and $\omega = (w_1, w_2, w_3, w_4)^T$ be a vector such that if $d_i = 1$ then $w_i = 1$ and if $d_i = -1$ then $w_i = 0$. Denote by $\mathbf{1}$ the vector $(1, 1, 1, 1)^T$, by I_4 the identity matrix. Denote by P the permutation matrix corresponding to the action of g on $\{1, 2, 3, 4\}$. Then one has

$$M = 2I_4 + UP - \mathbf{1}\omega^T$$

The i -th row of M corresponds to type 1 equation if $d_i = 1$ and type 2 equation if $d_i = -1$. And $r = \omega^T \mathbf{1}$. Denote by $K = 2I_4 + UP$. As UP is an orthogonal matrix, K has no zero eigenvalues. Hence K is invertible. Then one has

$$\det(M) = (1 - \omega^T K^{-1} \mathbf{1}) \det(K)$$

Hence we only need to show $h = \omega^T K^{-1} \mathbf{1} \neq 1$ under the condition $r \leq 2$. As $\|UP/2\| = 1/2 < 1$, one has the following power series expansion.

$$\begin{aligned} h &= \frac{1}{2} \omega^T \left(I_4 + \frac{UP}{2} \right)^{-1} \mathbf{1} \\ &= \frac{1}{2} \sum_{m \geq 0} \left(-\frac{1}{2} \right)^m \omega^T (UP)^m \mathbf{1} \end{aligned}$$

We write $a_m = \omega^T (UP)^m \mathbf{1}$. As UP is a permutation matrix with possible negative signs on the entries, we have a smallest postive integer N such that $(UP)^N = \varepsilon I_4$, where $\varepsilon = \pm 1$. Then $a_{m+N} = \varepsilon a_m$. Hence we have

$$h = \frac{\sum_{m=0}^{N-1} \left(-\frac{1}{2} \right)^m a_m}{2 - 2\varepsilon \left(-\frac{1}{2} \right)^N}$$

If $h = 1$, one has:

$$\sum_{m=0}^{N-1} (-1)^m 2^{N-1-m} a_m = 2^N - (-1)^N \varepsilon 2 \quad (5)$$

Here $a_0 = \sum_{1 \leq i \leq 4} w_i = r$, $a_1 = \sum_{1 \leq i \leq 4} w_i d_i = r$ and in general $|a_m| \leq r$. One has following estimate on the two sides.

$$\text{LHS} \leq 2^{N-2}r + r(1 + 2 + \cdots + 2^{N-3}) \leq 2^N - 2 \leq \text{RHS}$$

Hence for the equation holds, one must have $r = 2$ and $a_2 = 2$, $a_3 = -2$. Say $w_i = w_j = 1$ and $w_s = w_t = 0$ where $\{i, j, s, t\} = \{1, 2, 3, 4\}$. Then one has

$$a_2 = \sum_{1 \leq l \leq 4} w_l d_l d_{g(l)} = 2$$

This means $g(\{i, j\}) = \{i, j\}$. However, this would implies $a_3 = \sum_{1 \leq l \leq 4} w_l d_l d_{g(l)} d_{g^2(i)} = 2$ again. This is a contradiction. $\square$

We check Theorem 2.11.(iv). If the semisimple reduction of ψ is reducible after restricting to $\text{Gal}_{\mathbb{Q}(\zeta_\ell)}$, one of the semisimple reductions of $\text{Sym}^2(\sigma_1)$ and $\text{Sym}^2(\sigma_2)$ would also be reducible after restricting to $\text{Gal}_{\mathbb{Q}(\zeta_\ell)}$. However those symmetric squares are Lie-irreducible and of type A, hence by Theorem 2.17.(iii) this cannot happen.

Finally, as ψ fits into a compatible system for some λ_1 , the representation $\alpha_{\lambda_1} = \alpha = \beta^{-1} \wedge^3 \psi$ also fits into a compatible system $\{\alpha_\lambda\}$. Consider compatible system $\{\varphi_\lambda \oplus \alpha_\lambda\}$. At place λ_1 the semisimple rank is the same as that of $\{\varphi_\lambda\}$. However, at place λ_0 , since the derived subgroup of algebraic monodromy groups of σ_i for $i \geq 1$ is some quotient of SO_7 , by Goursat's lemma, the semisimple rank of $\varphi_{\lambda_0} \oplus \alpha_{\lambda_0}$ is strictly larger than that of $\varphi_{\lambda_1} \oplus \alpha_{\lambda_1}$. This contradicts with Theorem 2.3.(ii).

4.6. Cases (6), (8), (9). The three cases we consider are rectangular representations, hence $\mathbf{G}_{\lambda_1}^{\text{der}}$ can only be one of the six cases in Proposition 2.8.(ii) and (iii) We may assume Proposition 2.8.(iii).(e) does not happen for sufficiently large λ_1 since this case has been taken care of before. We can also rule out Proposition 2.8.(iii).(d) since in such case ρ_{λ_1} would be irreducible. Hence we may assume each ρ_{λ_1} is of type A.

Due to same reason as in the cases (10), (12), (13), (14), ρ_{λ_1} cannot contain 1 or 3 or 5-dimensional components. By Lemma 4.2, we separated following two cases.

- (a) There are infinitely many λ_1 such that the decompositions of ρ_{λ_1} have dimensional type $2 + 2 + 2 + 2$, $6 + 2$, $4 + 2 + 2$ or $4 + 4$ such that the two 4-dimensional components are essentially self-dual and odd.
- (b) There are infinitely many λ_1 such that (2) is true.

In case (a), we show for some λ_1 , each component of ρ_{λ_1} fits into a strictly compatible system, then this contradicts with the irreducibility of ρ_{λ_0} . For sufficiently large λ_1 , the 2-dimensional component fits into a strictly compatible system due to Proposition 2.13.(i). For other components we apply Theorem 2.11. Conditions Theorem 2.11.(i) and Theorem 2.11.(iii) are obvious. When the dimensional type is $6 + 2$ or $4 + 2 + 2$, the 4 or 6-dimensional component is essentially self-dual and odd due to Theorem 2.14. Hence condition Theorem 2.11.(ii) holds. Finally, since the formal character of $\rho_{\lambda_0}^{\text{der}}$ has no repeated weights in the cases we consider, the 4 or 6-dimensional component is Lie-irreducible. Also as explained at the beginning, these ρ_{λ_1} , hence the irreducible components, are of type A. Hence by Theorem 2.17.(iii), condition Theorem 2.11.(iv) holds.

In case (b). One writes $\rho_{\lambda_0} = f \otimes g$ where f is a 2-dimensional irreducible representation. Then there exists a 3-dimensional irreducible subrepresentation φ_{λ_0} of $\rho_{\lambda_0} \otimes \rho_{\lambda_0}^\vee$ such that the restriction of φ_{λ_0} to $\text{Gal}_{\mathbb{Q}_\ell}$ is the trace zero subrepresentation of $f \otimes f^\vee$. Since ρ_{λ_0} is regular, so is f . Hence φ_{λ_0} is regular. The Lie type of φ_{λ_0} is SO_3 . Recall our λ_0 runs through an infinite set $\mathcal{L}$. We choose λ_0 large enough so that Proposition 2.13.(ii) is satisfied for compatible system $\{\rho_\lambda \otimes \rho_\lambda^\vee\}$. Hence it extends to a compatible system $\{\varphi_\lambda\}$.

Consider the 11-dimensional strictly compatible system $\{\rho_\lambda \oplus \varphi_\lambda\}$. At place λ_0 the semisimple rank is the same as that of ρ_{λ_0} . However at place λ_1 , by Goursat's lemma the derived subgroup of algebraic monodromy group has strictly larger rank. This is a contradiction.

ACKNOWLEDGEMENT

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