Classification and qualitative properties of positive solutions to double-power nonlinear stationary Schrödinger equations

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Abstract

In this paper, we investigate positive radial solutions to double-power nonlinear stationary Schrödinger equations in three space dimensions. It is now known that the non-uniqueness of H^1-positive solutions can occur in three dimensions when the frequency is sufficiently small. Under suitable conditions, in addition to the ground state solution (whose L^∞ norm vanishes as the frequency tends to zero), there exists another positive solution that minimizes a different constrained variational problem, with an L^∞ norm diverging as the frequency tends to zero (see Theorem 1.4). We classify all positive solutions with small frequency into two categories: the ground state and the Aubin–Talenti type solution. As a consequence, we establish the multiplicity of positive solutions. Finally, we also examine the non-degeneracy and Morse index of each positive solution.

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1 Introduction

In this paper, we consider positive solutions to the following double-power nonlinear stationary Schrödinger equations

$$-\Delta u + \omega u - |u|^{p-1}u - |u|^{2^*-2}u = 0 \qquad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $d \geq 3, \omega > 0$, $2^* = \frac{2d}{d-2}$ and $1 < p < 2^* - 1$.

Solutions to (1.1) correspond to *standing wave solutions* $\psi(t,x) = e^{i\omega t}u_\omega(x)$ to the following Schrödinger equations:

$$i\partial_t \psi + \Delta\psi + |\psi|^{p-1}\psi + |\psi|^{2^*-2}\psi = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^d. \tag{1.2}$$

The standing waves play an important role in the global dynamics of (1.2) (see [2, 3, 4, 17, 20, 38] and references therein).

It is known that for any $\psi_0 \in H^1(\mathbb{R}^d)$, there exists a unique solution ψ to (1.2) in $C(I_{\max}; H^1(\mathbb{R}^d))$ with $\psi|_{t=0} = \psi_0$, where $I_{\max} \subset \mathbb{R}$ denotes the maximal existence interval. Moreover, the solution ψ satisfies the following conservation laws of the mass and the energy in this order:

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \qquad \mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0) \qquad \text{for all } t \in I_{\max}. \tag{1.3}$$

Here

$$\mathcal{E}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*}. \tag{1.4}$$

If in addition $\psi|_{t=0} = \psi_0 \in L^2(\mathbb{R}^d, |x|^2\,dx)$, then the solution ψ belongs to $C(I_{\max}; L^2(\mathbb{R}^d), |x|^2\,dx))$ and satisfies the so-called virial identity:

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^2|\psi(t,x)|^2\,dx = \int_{\mathbb{R}^d} |x|^2|\psi_0(x)|^2\,dx + 4t\mathrm{Im}\int_{\mathbb{R}^d} x \cdot \nabla\psi_0(x)\overline{\psi_0(x)}\,dx \\ + 8\int_0^t\int_0^{t'} \mathcal{K}(\psi(t''))\,dt''dt' \qquad \text{for any } t \in I_{\max},\end{aligned} \tag{1.5}$$

where

$$\mathcal{K}(u) := \|\nabla u\|_{L^2}^2 - \frac{d(p-1)}{2(p+1)}\|u\|_{L^{p+1}}^{p+1} - \|u\|_{L^{2^*}}^{2^*}. \tag{1.6}$$

Recently, equation (1.1) has been studied intensively (see [1, 5, 6, 7, 17, 30, 40, 44, 46, 47] and references therein). In this paper, we focus on positive solutions to (1.1). From the result of Gidas, Ni and Nirenberg [24], positive solutions to (1.1) are automatically radially symmetric and monotone decreasing in $|x| > 0$. Moreover, Wei and Wu [47] showed that when $d = 3$, $1 < p < 3$ and the frequency $\omega > 0$ is sufficiently small, equation (1.1) has two distinct positive solutions, implying the non-uniqueness of positive solutions. One of the positive solutions is the ground state and the other one is obtained by a certain minimization problem (see (1.10) and (1.16) below). This result coincides with the numerical computation by Dávila, del Pino and I. Guerra [18]. Our aim is to study the multiplicity, the non-degeneracy and the Morse index of positive solutions to (1.1).

Recall that the ground state Q_ω is a minimizer of the action functional $\mathcal{S}_\omega$ defined on the Nehari manifold

$$\left\{u \in H^1(\mathbb{R}^d) \setminus \{0\}\colon\ \mathcal{N}_\omega(u) = 0\right\}$$

with

$$\mathcal{S}_\omega(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*},$$

$$\mathcal{N}_\omega(u) := \langle \mathcal{S}_\omega'(u), u\rangle = \|\nabla u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 - \|u\|_{L^{p+1}}^{p+1} - \|u\|_{L^{2^*}}^{2^*},$$

where $\mathcal{S}_\omega'$ denotes the Frechét derivative of $\mathcal{S}_\omega$. More precisely, if we consider the following minimization problem

$$m_\omega := \inf\left\{\mathcal{S}_\omega(u)\colon u \in H^1(\mathbb{R}^d) \setminus \{0\},\ \mathcal{N}_\omega(u) = 0\right\}, \tag{1.7}$$

then the ground state Q_ω satisfies $S_\omega(Q_\omega) = m_\omega$ and $\mathcal{N}_\omega(Q_\omega) = 0$. Note that by a standard argument, Q_ω can be taken a positive radial ground state.

Remark 1.1. *Although several definitions of the ground state can be found in the literature, the one adopted above appears to be the most suitable in this paper.*

We can verify that if the ground state Q_ω exists, then it minimizes $\mathcal{S}_\omega$ over all non-trivial H^1-solutions to (1.1), that is, $\mathcal{S}_\omega(Q_\omega) \leq \mathcal{S}_\omega(u)$ for all non-trivial solutions $u \in H^1(\mathbb{R}^d)$ to (1.1). We now recall the results concerning the existence and non-existence of the ground state:

Theorem 1.1 ([5, 47, 48])**.** *(i) Let $d \geq 4$ and $1 < p < 2^* - 1$ or $d = 3$ and $3 < p < 2^* - 1$. For any $\omega > 0$, a ground state Q_ω exists.*

(ii) Let $d = 3$ and $1 < p \leq 3$. There exists $\omega_c > 0$ such that the ground state Q_ω exists for any $\omega \in (0, \omega_c)$ and does not exist for any $\omega \in (\omega_c, \infty)$. Finally, when $\omega = \omega_c$, a ground state Q_{ω_c} exists for $1 < p < 3$ ($p \neq 3$).

Part (i) of Theorem 1.1 was proved by Zhang and Zou [48, Theorem 1.1] while part (ii) was proved independently by [5, Theorem 1.1] and [47, Theorem 1.2.].

Remark 1.2. *(i) In the case of $d = 3$, $p = 3$ and $\omega = \omega_c$, the existence of the ground state is still unknown. We refer to Remark 3.1 of [47] for further discussion.*

(ii) It is known that the limit of the ground state under a certain rescaling is the Aubin-Talenti function

$$W(x) := \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}, \tag{1.8}$$

which solves

$$-\Delta W = W^{2^*-1} \qquad \text{in } \mathbb{R}^d. \tag{1.9}$$

A main reason why the case $d = 3$ is different from $d \geq 4$ arises from the integrability of the Aubin-Talenti function. We can easily check that $W \in L^2(\mathbb{R}^d)$ when $d \geq 5$ but $W \notin L^2(\mathbb{R}^d)$ when $d = 3, 4$ [1]. These are the so-called "resonance" when $d = 3, 4$ (see [5] and [47] for the existence and non-existence of the ground state in the case of $d = 3$ and $1 < p \leq 3$ in detail).

[1] When $d = 4$, $W \in L^{2+\varepsilon}(\mathbb{R}^d)$ for any $\varepsilon > 0$.

Next, we pay our attention to a positive solution which is different from the ground state and characterized by the following minimization problem:

$$E_{\min}(m) := \inf \{\mathcal{E}(u) \colon u \in \mathcal{P}(m)\}, \tag{1.10}$$

where $m > 0$,

$$S(m) := \left\{u \in H^1(\mathbb{R}^d) \colon \|u\|_{L^2}^2 = m\right\}$$

and

$$\mathcal{P}(m) := \{u \in S(m) \colon \mathcal{K}(u) = 0\} = \left\{u \in H^1(\mathbb{R}^d) \colon \|u\|_{L^2}^2 = m,\ \mathcal{K}(u) = 0\right\}.$$

We can easily verify that for each $m > 0$, $\mathcal{P}(m) \neq \emptyset$. Since this kind of the minimization problem is closely related to the global dynamics of solutions to the evolution equations (1.2), it is often used as well as m_ω (see [30, 33, 44, 46] and references therein). Here, we note that for the L^2-constrained minimization problem as (1.10), the L^2-critical exponent $2_* = 1 + 4/d$ plays an important role. To explain this, let us consider the following single power nonlinear Schrödinger equations:

$$i\partial_t \psi + \Delta \psi + |\psi|^{p-1}\psi = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^d, \tag{1.11}$$

where $d \geq 1$ and $1 < p < 2^* - 1$. Equation (1.11) is invariant under the scaling:

$$\psi_\lambda(t,x) := e^{\frac{2}{p-1}\lambda}\psi(e^{2\lambda}t, e^\lambda x) \qquad (\lambda \in \mathbb{R}), \tag{1.12}$$

which preserves the L^2-norm when $2_* = 1 + 4/d$. Thus, the exponent $2_* = 1 + 4/d$ is referred to as "L^2-critical" or "mass critical". The L^2-critical exponent $2_* = 1 + 4/d$ is also known as the threshold for the orbital stability/instability of standing waves of (1.11). The corresponding stationary equation is

$$-\Delta u + \omega u - |u|^{p-1}u = 0 \qquad \text{in } \mathbb{R}^d, \tag{1.13}$$

whose unique positive solution can be written as $U_\omega(\cdot) = \omega^{1/(p-1)}U^\dagger(\sqrt{\omega}\cdot)$, where $U^\dagger$ solves (1.13) with $\omega = 1$. We refer to [10] for the existence and to [35] for the uniqueness. Cazenave and P. L. Lions [16] proved that in the L^2-subcritical case $1 < p < 1 + 4/d$, $U^\dagger$ minimizes the corresponding energy under the L^2-norm constrained, that is,

$$\mathcal{E}^\dagger(U^\dagger) = \inf\left\{\mathcal{E}^\dagger(u) \colon u \in S(\|U^\dagger\|_{L^2}^2)\right\}, \tag{1.14}$$

where

$$\mathcal{E}^\dagger(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

Using the variational characterization (1.14) and the concentration-compactness method, Cazenave and P. L. Lions [16] showed that the standing wave $e^{it}U^\dagger$ is orbitally stable. [2] On the other hand, in the L^2-supercritical case $1 + 4/d < p < 2^* - 1$, the situation becomes different. Let us explain briefly. For each $u \in H^1(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, we put

$$T_\lambda u(\cdot) := e^{\frac{d}{2}\lambda}u(e^\lambda \cdot) \qquad (\lambda \in \mathbb{R}). \tag{1.15}$$

[2] Roughly speaking, we say that the standing wave is orbitally stable if we take the initial data sufficiently close to the standing wave, then the corresponding solution stays close to its orbit. Otherwise, we say that the standing wave is orbitally unstable.

Then, we can easily find that $\|T_\lambda U^\dagger\|_{L^2} = \|U^\dagger\|_{L^2}$, while $\inf_{\lambda\in\mathbb{R}} \mathcal{E}^\dagger(T_\lambda U^\dagger) = -\infty$, that is, the energy $\mathcal{E}^\dagger$ is not bounded from below on the manifold $S(\|U^\dagger\|_{L^2}^2)$ when $1 + 4/d < p < 2^* - 1$. Therefore, (1.14) does not hold in the L^2-supercritical case $1 + 4/d < p < 2^* - 1$. In addition, Berestycki and Cazenave [11] proved that the standing wave $e^{it}U^\dagger$ is orbitally unstable. This explains that the L^2-critical exponent $p = 1 + 4/d$ plays a key role in the L^2-constrained minimization problem.

Now, let us return to our equations (1.1) again. We now summarize the result for the L^2-critical and supercritical cases. Soave [44] obtained the following (see also [39, Theorem 1,3]):

Theorem 1.2 (Theorem 1.1 and Proposition 1.5 of Soave [44])**.** *(i) Let $d \geq 3$ and $1+4/d < p < 2^* - 1$. For each $m > 0$, $E_{\min}(m) > 0$ and there exists a minimizer for $E_{\min}(m)$, which is a positive solution to* (1.1) *with the frequency $\omega = \omega(m)$ for some $\omega(m) > 0$.*

(ii) Let $d \geq 3$ and $p = 1 + 4/d$. For each $m \in (0, \|U^\dagger\|_{L^2}^2)$, the same conclusion as in (i) holds.

Next, we turn our attention to the L^2-subcritical case $1 < p < 1 + 4/d$. In fact, the situation becomes more complex and the functional $\mathcal{E}|_{S_m}$ has a second critical point. To explain this more precisely, we prepare several notations. Jeanjean [29] introduced the following *fiber maps* to study the general L^2-constrained minimization problem:

$$\Psi(u, \lambda) := \mathcal{E}(T_\lambda u) = \frac{e^{2\lambda}}{2}\|\nabla u\|_{L^2}^2 - \frac{e^{\frac{d}{2}(p-1)\lambda}}{p+1}\|u\|_{L^{p+1}}^{p+1} - \frac{e^{2^*\lambda}}{2^*}\|u\|_{L^{2^*}}^{2^*}.$$

We can easily verify that $\mathcal{K}(u) = \frac{d\Psi}{d\lambda}(u, \lambda)\big|_{\lambda=0}$ and $\mathcal{P}(m)$ can be rewritten by

$$\mathcal{P}(m) = \left\{ u \in S(m) \colon \frac{d\Psi}{d\lambda}(u, \lambda)\big|_{\lambda=0} = 0 \right\}.$$

Note that $d(p-1)/2 < 2$ when $1 < p < 1 + 4/d$, so that the graph of $\Psi(u, \lambda)$ may have local maximum and minimum points if $m = \|u\|_{L^2}^2 > 0$ is sufficiently small (see [44, Lemma 4.1]). Taking this into consideration, we decompose $\mathcal{P}(m)$ into the disjoint union $\mathcal{P}(m) = \mathcal{P}_+(m) \cup \mathcal{P}_0(m) \cup \mathcal{P}_-(m)$, where

$$\begin{aligned}
\mathcal{P}_+(m) &:= \left\{ u \in S(m) \colon \frac{d\Psi}{d\lambda}(u, \lambda)\bigg|_{\lambda=0} = 0,\ \frac{d^2\Psi}{d\lambda^2}(u, \lambda)\bigg|_{\lambda=0} > 0 \right\} \\
&= \left\{ u \in \mathcal{P}(m) \colon 2\|\nabla u\|_{L^2}^2 > \frac{d^2(p-1)^2}{4(p+1)}\|u\|_{L^{p+1}}^{p+1} + 2^*\|u\|_{L^{2^*}}^{2^*} \right\}, \\
\mathcal{P}_0(m) &:= \left\{ u \in S(m) \colon \frac{d\Psi}{d\lambda}(u, \lambda)\bigg|_{\lambda=0} = 0,\ \frac{d^2\Psi}{d\lambda^2}(u, \lambda)\bigg|_{\lambda=0} = 0 \right\} \\
&= \left\{ u \in \mathcal{P}(m) \colon 2\|\nabla u\|_{L^2}^2 = \frac{d^2(p-1)^2}{4(p+1)}\|u\|_{L^{p+1}}^{p+1} + 2^*\|u\|_{L^{2^*}}^{2^*} \right\}, \\
\mathcal{P}_-(m) &:= \left\{ u \in S(m) \colon \frac{d\Psi}{d\lambda}(u, \lambda)\big|_{\lambda=0} = 0,\ \frac{d^2\Psi}{d\lambda^2}(u, \lambda)\bigg|_{\lambda=0} < 0 \right\} \\
&= \left\{ u \in \mathcal{P}(m) \colon 2\|\nabla u\|_{L^2}^2 < \frac{d^2(p-1)^2}{4(p+1)}\|u\|_{L^{p+1}}^{p+1} + 2^*\|u\|_{L^{2^*}}^{2^*} \right\}.
\end{aligned}$$

We put

$$E_{\min,\pm}(m) := \inf\left\{\mathcal{E}(u) \colon u \in \mathcal{P}_\pm(m)\right\}. \tag{1.16}$$

Then, Soave [44], Jeanjean and Lu [30], Wei and Wu [46] obtained the following:

Theorem 1.3 ([30, 44, 46]). *Let $d \geq 3$ and $1 < p < 1 + 4/d$.*

(i) There exists $m_* > 0$ such that for each $m \in (0, m_*)$, $E_{\min,-}(m)$ is positive and has a minimizer which is a positive solution to (1.1) with $\omega = \omega_-(m)$ for some $\omega_-(m) > 0$.

(ii) There exists $m_{**} > 0$ such that for each $m \in (0, m_{**})$,

$$E_{\min}(m) = E_{\min,+}(m) < 0.$$

In particular, $E_{\min}(m)$ has a minimizer, which is a positive solution to (1.1) with $\omega = \omega_+(m)$ for some $\omega_+(m) > 0$.

Remark 1.3. *It follows from Theorem 1.3 that $E_{\min,-}(m) > 0 > E_{\min}(m)$.*

Part (i) of Theorem 1.3 was proved by Jeanjean and Lu [30, Theorem 1.6] in the case of $d \geq 4$ and by Wei and Wu [46, Theorem 1.4] in the case of $d = 3$ and Part (ii) was proved by Soave [44, Theorem 1.1 and Proposition 1.5]. In what follows, we denote the minimizers for $E_{\min}(m)$ and $E_{\min,-}(m)$ by $R_{\omega(m)}$ and $R_{\omega_-(m),-}$ which are solutions to (1.1) with $\omega = \omega(m)$ and $\omega = \omega_-(m)$, respectively. [3] Wei and Wu [47] investigated the case where $d = 3$ and $m > 0$ is sufficiently small. The following theorem summarizes their results:

Theorem 1.4 (Theorem 1.2 (3) of Wei and Wu [47]). *(i) Let $d = 3$, $1 + 4/d \leq p < 3$ and $\omega(m) > 0$ be the number given in Theorem 1.2 and 1.3 (i). Then, we have $\lim_{m \to 0} \omega(m) = 0$ and $\lim_{m \to 0} \|R_{\omega(m)}\|_{L^\infty} = \infty$.*

(ii) Let $d = 3$, $1 < p < 1 + 4/d$ and $\omega_-(m) > 0$ be the number given in Theorem 1.3 (ii). Then, we have $\lim_{m \to 0} \omega_-(m) = 0$ and $\lim_{m \to 0} \|R_{\omega_-(m),-}\|_{L^\infty} = \infty$.

Remark 1.4. *(i) Let $Q_{\omega(m)}$ denote the ground state to (1.1) when $\omega = \omega(m)$. We find that $R_{\omega(m)} \neq Q_{\omega(m)}$ for sufficiently small $m > 0$ and $1 + 4/d \leq p < 2^* - 1$. Indeed,*

$$\limsup_{m \to 0} \|Q_{\omega(m)}\|_{L^\infty} < \infty, \qquad \lim_{m \to 0} \|R_{\omega(m)}\|_{L^\infty} = \infty.$$

Hence, the uniqueness of positive solution to (1.1) fails for sufficiently small $\omega > 0$. Similarly, the non-uniqueness also occurs when $1 < p < 1 + 4/d$.

(ii) Contrary to the case where $d = 3$ and $1 + 4/d < p < 3$, in [1, Proposition C.1], it is shown that (1.1) admits a unique positive solution for any $\omega > 0$ when $3 \leq d \leq 6$ and $\frac{4}{d-2} < p < 2^* - 1$.

2 Main results

In this section, we state our main results. One of our aim is to classify all positive solutions to (1.1) for sufficiently small $\omega > 0$ when $d = 3$. Let $\{u_{1,\omega}\}$ and $\{u_{2,\omega}\}$ be two families of positive solutions to (1.1) in $H^1(\mathbb{R}^d)$ satisfying $\limsup_{\omega \to 0} \|u_{1,\omega}\|_{L^\infty} < \infty$ and $\lim_{\omega \to 0} \|u_{2,\omega}\|_{L^\infty} = \infty$, respectively. We shall show that when $d = 3$, $u_{1,\omega}$ must be the ground state Q_ω while $u_{2,\omega}$ be the minimizer R_ω of $E_{\min}(m)$ when $1 + \frac{4}{d} < p < 3$ and

[3] When we would like to stress the dependence on the frequency ω, we denote just by R_ω and $R_{\omega,-}$.

the minimizer $R_{\omega_-,-}$ of $E_{\min,-}$ when $1 < p < 1 + \frac{4}{d}$ for sufficiently small $\omega > 0$. As a result, we can verify (1.1) has exactly two positive solutions.

The second our aim is to study the non-degeneracy and the Morse index of positive solutions to (1.1) for sufficiently small $\omega > 0$. Here, we recall that u_ω is *non-degenerate* in $H^1_{\mathrm{rad}}(\mathbb{R}^d)$ if the linearized equation

$$L_{\omega,+} z = 0 \qquad \text{in } \mathbb{R}^d,$$

has no non-trivial solution in $H^1_{\mathrm{rad}}(\mathbb{R}^d)$, where

$$L_{\omega,+} = -\Delta + \omega - p u_\omega^{p-1} - (2^* - 1) u_\omega^{2^*-2}.$$

The Morse index of the solution u_ω is defined by

$$\max\left\{\dim H \colon H \text{ is a subspace of } H^1(\mathbb{R}^d) \text{ satisfying } \langle L_{\omega,+} h, h\rangle < 0 \text{ for all } h \in H \setminus \{0\}\right\}.$$

The study of the non-degeneracy and the Morse index of solutions to (1.1) becomes important in the analysis of the dynamics of solutions to (1.2). In fact, for each $m > 0$, if the energy $\mathcal{E}$ of an initial data ψ_0 with $\|\psi_0\|_{L^2}^2 = m$ is less than $E_{\min}(m)$, then the corresponding solution ψ to (1.2) blows up or scatters forward in time (see e.g. [2, 3, 33, 44]). On the other hand, when the energy of initial data is equal to or slightly greater than the ground state energy, the dynamics of solutions becomes more complicated. To study this case, the non-degeneracy and the Morse index are needed. We refer to [4, 19, 42] for more details. This is one of our main motivations for the present paper. We note that non-degeneracy of the solution to (1.1) is also key to apply the Lyapunov-Schmidt reduction method (see e.g. [8, 21, 43] and references therein).

We obtain the following classification result of H^1-positive solutions to (1.1):

Theorem 2.1. *Let $d = 3$.*

(i) Let $7/3 \le p < 3$. There exists $\omega_1 > 0$ such that for any $\omega \in (0, \omega_1)$, (1.1) has exactly two positive solutions in $H^1(\mathbb{R}^d)$ up to spatial translations; one is the ground state Q_ω and the other is the minimizer R_ω of $E_{\min}(m)$ for some $m > 0$.

(ii) Let $1 < p < 7/3$. There exists $\omega_2 > 0$ such that for any $\omega \in (0, \omega_2)$, (1.1) has exactly two positive solutions in $H^1(\mathbb{R}^d)$ up to spatial translations; one is the ground state Q_ω and the other is the minimizer $R_{\omega_-(m),-}$ of $E_{\min}(m)$ for some $m > 0$.

To prove Theorem 2.1, we consider two types of families of positive solutions $\{u_{1,\omega}\}$ and $\{u_{2,\omega}\}$ to (1.1) in $H^1(\mathbb{R}^d)$ satisfying $\limsup_{\omega\to 0} \|u_{1,\omega}\|_{L^\infty} < \infty$ and $\lim_{\omega\to 0} \|u_{2,\omega}\|_{L^\infty} = \infty$, respectively.

Concerning the family of positive solutions $\{u_{1,\omega}\}$, we shall show the following:

Theorem 2.2. *Let $d = 3$ and $1 < p < 2^* - 1$. Let $\{u_{1,\omega}\}$ be a family of positive solutions to (1.1) satisfying $\limsup_{\omega\to 0} \|u_{1,\omega}\|_{L^\infty} < \infty$. Putting*

$$\widetilde{u}_{1,\omega}(\cdot) = \omega^{-\frac{1}{p-1}} u_{1,\omega}(\omega^{-\frac{1}{2}} \cdot), \tag{2.1}$$

we have

$$\widetilde{u}_{1,\omega} \to U^\dagger \qquad \textit{strongly in } H^1(\mathbb{R}^d) \textit{ as } \omega \to 0.$$

Theorem 2.3. *Let $d = 3$ and $1 < p < 2^* - 1$.*

(i) There exists $\omega_3 > 0$ with the following properties: let $\{u_{1,\omega}\}$ denote a family of positive solutions to (1.1) *satisfying* $\limsup_{\omega \to 0} \|u_{1,\omega}\|_{L^\infty} < \infty$*. Then the solution is unique for all $\omega \in (0, \omega_3)$, namely, if there exists another family of positive solutions $\{v_{1,\omega}\}$ to* (1.1) *in* $H^1(\mathbb{R}^d)$ *satisfying* $\limsup_{\omega \to 0} \|v_{1,\omega}\|_{L^\infty} < \infty$*, then $u_{1,\omega} = v_{1,\omega}$ for all $\omega \in (0, \omega_3)$.*

(ii) There exists $\omega_4 > 0$ such that for any $\omega \in (0, \omega_4)$, the Morse index of $u_{1,\omega}$ equals 1.

Remark 2.1. *(i) The main difficulty for the proof of Theorem 2.2 is to obtain a boundedness of H^1 norm of $\{\widetilde{u}_{1,\omega}\}$. For example, when Q_ω is the ground state, one can easily obtain a uniform boundedness of $\{\|Q_\omega\|_{H^1}\}$ from the fact that Q_ω is a minimizer for m_ω. However, since we consider a general positive solution to* (1.1)*, we cannot employ a variational argument like the ground state. To overcome this difficulty, we consider another rescaling different from* (2.1) *(see* (3.5) *below). and employ a kind of Pohozaev identity of a limit equation.*

(ii) It follows from Theorem 2.3 (i) that $u_{1,\omega} = Q_\omega$ for sufficiently small $\omega > 0$.

(iii) We do not assert the novelty of Theorem 2.3 (i) and (ii). These assertions can be derived by rather standard arguments, specifically from [4, Proposition 2.0.4] in the case of Theorem 2.3 (i) and [41, Lemma 2.3] in the case of Theorem 2.3 (ii). Nevertheless, we include them here in order to enable a direct comparison with the corresponding results for $u_{2,\omega}$ (see Theorem 2.5 (ii) and (iii) below).

Next, we study a family of positive solutions $\{u_{2,\omega}\}$ to (1.1) in $H^1(\mathbb{R}^d)$ satisfying $\lim_{\omega \to 0} \|u_{2,\omega}\|_{L^\infty} = \infty$. Concerning this, we obtain the following:

Theorem 2.4. *Let $d = 3$ and $1 < p < 3$. Let $\{u_{2,\omega}\}$ be a family of positive solutions to* (1.1) *in* $H^1(\mathbb{R}^d)$ *satisfying* $\lim_{\omega \to 0} \|u_{2,\omega}\|_{L^\infty} = \infty$*. Putting*

$$M_{2,\omega} = \|u_{2,\omega}\|_{L^\infty}, \qquad \widetilde{u}_{2,\omega}(\cdot) = M_{2,\omega}^{-1} u_{2,\omega}(M_{2,\omega}^{-\frac{2}{d-2}} \cdot), \tag{2.2}$$

we have

$$\widetilde{u}_{2,\omega} \to W \qquad \text{strongly in } \dot{H}^1 \cap L^q(\mathbb{R}^d) \ (q > \tfrac{d}{d-2}) \text{ as } \omega \to 0, \tag{2.3}$$

where W is the Aubin-Talenti function defined by (1.8).

Theorem 2.5. *Let $d = 3$ and $1 < p < 3$.*

(i) There exists $\omega_5 > 0$ with the following properties: let $\{u_{2,\omega}\}$ denote a family of positive solutions to (1.1) *satisfying* $\lim_{\omega \to 0} \|u_{2,\omega}\|_{L^\infty} = \infty$*. Then the solution is unique for all $\omega \in (0, \omega_5)$, namely, if there exists another family of positive solutions $\{v_{2,\omega}\}$ to* (1.1) *in* $H^1(\mathbb{R}^d)$ *satisfying* $\lim_{\omega \to 0} \|v_{2,\omega}\|_{L^\infty} = \infty$*, then $u_{2,\omega} = v_{2,\omega}$ for all $\omega \in (0, \omega_5)$.*

(ii) There exists $\omega_6 > 0$ such that when $1 < p < 3$, the solution $u_{2,\omega}$ is non-degenerate for all $\omega \in (0, \omega_6)$.

(iii Let $d = 3$ and $1 + \frac{4}{d} < p < 3$, there exists $\omega_7 > 0$ such that for all $\omega \in (0, \omega_7)$, the Morse index of $u_{2,\omega}$ equals 2.

Remark 2.2. *(i) As in Theorem 2.5 (i), we also meet a difficulty to obtain a boundedness of $\dot{H}^1$ norm of $\{\widetilde{u}_{2,\omega}\}$. Following [22, 34, 47], which is originated in [9], we prove a pointwise estimate of $\widetilde{u}_{2,\omega}$, which is a key role to obtain the boundedness.*

(ii) For the proof of Theorem 2.5 (ii), we follow the arguments in [1] and [7] (see also [28]). In [1], we established the uniqueness and non-degeneracy of the ground state Q_ω of (1.1) when $d \geq 5$ and $\omega > 0$ is sufficiently large. The restriction $d \geq 5$ was required there in order to ensure that the Aubin–Talenti function W belongs to $L^2(\mathbb{R}^d)$. Subsequently, Akahori and Murata [7] extended the result of [1], proving the non-degeneracy of the ground state Q_ω of (1.1) when $d = 3$ and $3 < p < 2^* - 1$ for sufficiently large $\omega > 0$. However, the proof in [7] essentially relies on the condition $p > 3$ and does not apply, at least directly, when $d = 3$ and $1 < p < 3$, so that additional ingredients are required. In the present work, we employ the pointwise estimate of $\widetilde{u}_{2,\omega}$ mentioned in Remark 2.2 (i).

(iii) It follows from Theorem 2.5 (ii), $\lim_{m\to 0}\omega(m) = 0$ and $\lim_{m\to 0}\omega_-(m) = 0$ that $u_{2,\omega(m)} = R_{\omega(m)}$ in the L^2-critical and supercritical case $1 + 4/d \leq p < 3$ and $u_{2,\omega(m)} = R_{\omega_-(m),-}$ in the L^2-subcritical case $1 < p < 1 + 4/d$ for sufficiently small $m > 0$.

(iv) When $d = 3$ and $1 + \frac{4}{d} \leq p < 3$, it follows from the uniqueness result (Theorem 2.5 (ii)) that $u_{2,\omega(m)} = R_{\omega(m)}$. Hence, by the variational characterization (1.10), the Morse index of $u_{2,\omega}$ is either 1 or 2 (see Lemma 10.3). Therefore, it remains to show that the Morse index of $u_{2,\omega}$ is not equal to 1. To this end, we invoke the abstract theory of Grillakis, Shatah, and Strauss [27], which provides sufficient conditions for the stability and instability of the standing wave solution $e^{i\omega t}u_\omega$ under certain assumptions. One of these assumptions is that the Morse index of u_ω equals 1. In fact, applying the result of [27], we obtain a contradiction if we assume that the Morse index of u_ω equals 1. This line of reasoning appears to be new in the study of the Morse index of solutions to nonlinear elliptic equations.

	$1 < p < \frac{7}{3}$	$\frac{7}{3} \leq p < 3$
$u_{1,\omega}$	Q_ω	
$u_{2,\omega}$	$R_{\omega_-,-}$	R_ω

We shall prove Theorem 2.5 by the blowup analysis. Note that $\widetilde{u}_{2,\omega}$ defined by (2.2) satisfies

$$-\Delta\widetilde{u}_{2,\omega} + \alpha_{2,\omega}\widetilde{u}_{2,\omega} - \beta_{2,\omega}\widetilde{u}_{2,\omega}^p - \widetilde{u}_{2,\omega}^5 = 0, \qquad \|\widetilde{u}_{2,\omega}\|_{L^\infty} = \widetilde{u}_{2,\omega}(0) = 1, \tag{2.4}$$

where

$$\alpha_{2,\omega} := \omega M_{2,\omega}^{-4}, \qquad \beta_{2,\omega} := M_{2,\omega}^{p-5}. \tag{2.5}$$

Since $\widetilde{u}_{2,\omega}$ is radially symmetric, (2.4) can be written by the following:

$$\begin{cases} -\widetilde{u}_{2,\omega}'' - \frac{2}{r}\widetilde{u}_{2,\omega}' + \alpha_{2,\omega}\widetilde{u}_{2,\omega} - \beta_{2,\omega}\widetilde{u}_{2,\omega}^p - \widetilde{u}_{2,\omega}^5 = 0 \qquad \text{on } (0,\infty), \\ \widetilde{u}_{2,\omega}(0) = 1, \qquad \widetilde{u}_{2,\omega}'(0) = 0. \end{cases} \tag{2.6}$$

It follows from the definition of (2.5) and $\lim_{\omega\to 0} M_{2,\omega} = \infty$ that

$$\lim_{\omega\to 0}\alpha_{2,\omega} = \lim_{\omega\to 0}\beta_{2,\omega} = 0. \tag{2.7}$$

From this, we can find that the limit equation of (2.4) is (1.9) which the Aubin-Talenti function (1.8) satisfies. However, as in Theorem 2.2, we also face a difficulty obtaining the boundedness of the H^1 norm of $\{\widetilde{u}_{2,\omega}\}$ to prove the convergence (2.3). Following [22, 34, 47], which is originated in [9], we prove a pointwise estimate of $\widetilde{u}_{2,\omega}$ (see (4.1) below), which is a key role in obtaining the boundedness.

Contrary to the solution $U^\dagger$ to (1.13) with $\omega = 1$, the Aubin-Talenti function is degenerate. Thus, we cannot obtain the uniqueness result (Theorem 2.5 (i)) directly. In [1], in order to get over the difficulty, we employed the Pohozaev identity, that is, the solution $\widetilde{u}_{2,\omega}$ to (2.4) satisfy the following:

$$\alpha_{2,\omega}\|\widetilde{u}_{2,\omega}\|_{L^2}^2 = \frac{5-p}{2(p+1)}\beta_{2,\omega}\|\widetilde{u}_{2,\omega}\|_{L^{p+1}}^{p+1}. \tag{2.8}$$

In [1], we considered the case of $d \geq 5$, so that $W \in L^2(\mathbb{R}^d)$. From this and (2.3), we can take a limit of ω in (2.8) directly and showed that

$$\lim_{\omega\to 0}\frac{\beta_{2,\omega}}{\alpha_{2,\omega}} = \frac{2(p+1)}{4-(d-2)(p-1)}\frac{\|W\|_{L^2}^2}{\|W\|_{L^{p+1}}^{p+1}}$$

when $d \geq 5$. This relation between $\alpha_{2,\omega}$ and $\beta_{2,\omega}$ plays an important role for the proof of the uniqueness in [1]. However, when we consider three dimensional case, we see that $W \notin L^2(\mathbb{R}^3)$. Thus, the relation between $\alpha_{2,\omega}$ and $\beta_{2,\omega}$ becomes different. For this reason, the uniqueness result cannot be obtained directly in the case $d = 3$. Moreover, we note that $W \notin L^{p+1}(\mathbb{R}^3)$ when $1 < p \leq 2$. Thus, the situation becomes more complex in the case of $d = 3$ and $1 < p \leq 2$.

Coles and Gustafson [17] obtained the uniqueness of the ground state Q_ω to (1.1) if $\omega > 0$ is sufficiently large when $d = 3$ and $3 < p < 2^* - 1$. They used the resolvent expansion by Jensen and Kato [31]. As Coles and Gustafson [17] did, we use the resolvent expansion and show the following:

$$\lim_{\omega\to 0}\frac{\beta_{2,\omega}}{\sqrt{\alpha_{2,\omega}}} = \frac{12\pi(p+1)}{(5-p)\|W\|_{L^{p+1}}^{p+1}} \qquad \text{when } 2 < p < 3, \tag{2.9}$$

$$\lim_{\omega\to 0}\frac{\beta_{2,\omega}|\log\alpha_{2,\omega}|}{\sqrt{\alpha_{2,\omega}}} = \frac{2}{\sqrt{3}} \qquad \text{when } p = 2. \tag{2.10}$$

See Proposition 5.1 (i) and (ii) below. Since $W \notin L^p(\mathbb{R}^3)$ for $1 < p \leq 2$, it appears natural from (2.3) and (2.8) that $p = 2$ serves as the threshold for the relation between $\alpha_{2,\omega}$ and $\beta_{2,\omega}$. Using (2.9), (2.10) and the pointwise estimate of $\widetilde{u}_{2,\omega}$ (see (4.1) below, we show the uniqueness and non-degeneracy of $\widetilde{u}_{2,\omega}$ for sufficiently large $\omega > 0$.

Next, we pay our attention to the case of $1 < p < 2$. As we mentioned above, since $W \notin L^{p+1}(\mathbb{R}^3)$ when $1 < p \leq 2$, the situation becomes more complex.

Let $\{\omega_n\}$ be a sequence in $(0,\infty)$ with $\lim_{n\to\infty}\omega_n = 0$. We see that there exists a constant $\theta_0 > 0$ such that

$$\lim_{n\to\infty}\frac{\beta_{2,\omega_n}}{\alpha_{2,\omega_n}^{\frac{3-p}{2}}} = \theta_0 \qquad \text{when } 1 < p < 2.$$

See Lemmas 5.15 and 5.18 below. We remark that the limit $\theta_0 > 0$ depends on the sequence $\{\omega_n\}$ at this stage. However, we can find that θ_0 is a universal constant. To prove this, we consider the following rescaling:

$$\widehat{u}_{2,\omega_n}(s) = \alpha_{2,\omega_n}^{-\frac{1}{2}}\widetilde{u}_{2,\omega_n}(r), \qquad s = \alpha_{\omega_n}^{\frac{1}{2}}r.$$

Then, we see that $\widehat{u}_{2,\omega_n}$ satisfies the following:

$$-\frac{d^2\widehat{u}_{2,\omega_n}}{ds^2}-\frac{2}{s}\frac{d\widehat{u}_{2,\omega_n}}{ds}-\widehat{u}_{2,\omega_n}-\frac{\beta_{2,\omega_n}}{\alpha_{2,\omega_n}^{\frac{3-p}{2}}}\widehat{u}_{2,\omega_n}^p-\alpha_{2,\omega_n}\widehat{u}_{2,\omega_n}^5=0 \qquad \text{in } (0,\infty).$$

For any finite interval $I\subset(0,\infty)$, we have

$$\lim_{n\to\infty}\widehat{u}_{2,\omega_n}(s)=U_{\theta_0,\infty}(s) \qquad \text{uniformly in } I,$$

where $U_{\theta_0,\infty}$ is a solution to

$$-\frac{d^2U}{ds^2}-\frac{2}{s}\frac{dU}{ds}+U-\theta_0U^p=0 \qquad \text{in } (0,\infty) \tag{2.11}$$

satisfying

$$\lim_{s\to 0}sU_{\theta_0,\infty}(s)=\sqrt{3}.$$

See Proposition 5.19 below. It is known that the equation (2.11) has infinitely many singular solutions (see [32, Theorem 1]). However, we can obtain a uniqueness of the singular solution $U_{\theta_0,\infty}$. Indeed, putting $\overline{u}_{2,\omega}(s)=s\widehat{u}_{2,\omega}(s)$. we find that $\overline{u}_{2,\omega}$ satisfies

$$\begin{cases} \frac{d^2\overline{u}_{2,\omega}}{ds^2}-\overline{u}_{2,\omega}+\beta_{2,\omega}\alpha_{2,\omega}^{\frac{p-3}{2}}s^{1-p}\overline{u}_{2,\omega}^p+\alpha_{2,\omega}s^{-4}\overline{u}_{2,\omega}^5=0 \qquad \text{in } (0,\infty),\\ \overline{u}_{2,\omega}(s)>0 \qquad \text{in } (0,\infty),\\ \overline{u}_{2,\omega}(0)=0, \qquad \frac{d\overline{u}_{2,\omega}}{ds}(0)=\frac{1}{\sqrt{\alpha_{2,\omega}}}, \qquad \lim_{s\to\infty}\overline{u}_{2,\omega}(s)=0. \end{cases}$$

Then, we can prove that for any finite interval I in $(0,\infty)$, one has $\lim_{n\to\infty}\theta_0^{\frac{1}{p-1}}\overline{u}_{\omega_n}(s)=\overline{v}_0(s)$ for $s\in I$, where $\overline{v}_0(s)$ satisfies

$$\begin{cases} \frac{d^2\overline{v}_0}{ds^2}-\overline{v}_0+s^{1-p}\overline{v}_0^p=0 \qquad \text{on } (0,\infty),\\ \frac{d\overline{v}_0}{ds}(0)=0, \qquad \lim_{s\to\infty}\overline{v}_0(s)=0 \end{cases} \tag{2.12}$$

and

$$\overline{v}_0(0)=\sqrt{3}\theta_0^{\frac{1}{p-1}}, \tag{2.13}$$

See Lemmas 5.21 and 5.23 below. Concerning (2.12), the following result is obtained by Genoud [23] and Toland [45]

Proposition 2.1 (Genoud [23] and Toland [45]). *Let $1<p<2$.*

(i) The equation (2.12) *has a unique positive solution V.*

(ii) The positive solution V to the equation (2.12) *is non-degenerate.*

See Toland [45, Page 262] for the proof of Proposition 2.1 (i) and Genoud [23, Proposition 2.1] for Proposition 2.1 (ii), respectively. It follows from (2.13) and Theorem 2.1 that we obtain $V(0)=\sqrt{3}\theta_0^{\frac{1}{p-1}}$, that is, $\theta_0=3^{-\frac{p-1}{2}}V^{p-1}(0)$, which implies that θ_0 does not depend on the sequence $\{\omega_n\}$ and is a universal constant. As a result, we can show the following:

$$\lim_{\omega\to 0}\frac{\beta_{2,\omega}}{\alpha_{2,\omega}^{\frac{3-p}{2}}}=3^{-\frac{p-1}{2}}V^{p-1}(0) \qquad \text{when } 1<p<2. \tag{2.14}$$

Using (2.14) and Proposition 2.1 (ii), we can prove Theorem 2.5 (i) when $1 < p < 2$. We remark that Wei and Wu [46, Proposition 4.2] have already obtained a result concerning a relation between $\alpha_{2,\omega}$ and $\beta_{2,\omega}$. However, to prove the uniqueness, we need the exact values of the limits as in (2.9), (2.10) and (2.14).

Using the argument in the proof of Theorem 2.4, we can show the non-existence of positive solution to (1.1) for sufficiently large $\omega > 0$.

Theorem 2.6. *Let $d = 3$ and $1 < p < 3$. There exists $\omega_8 > 0$ such that the equation (1.1) does not have any positive solution in $H^1(\mathbb{R}^3)$ for $\omega \in (\omega_8, \infty)$.*

The rest of this paper is organized as follows: The proofs of Theorems 2.2 and 2.3 are presented in Section 3. We prove the convergence result to the Aubin-Talenti function (Theorem 2.4) in Section 4. We study the relation between the coefficients $\alpha_{2,\omega}$ and $\beta_{2,\omega}$ (see (2.5) for the definition of the coefficients) in Section 5. Using the result obtained in Section 5, we give proofs of the uniqueness and non-degeneracy of $u_{2,\omega}$ (Theorem 2.5 (i) and (ii)) in Section 6 and Section 7, respectively. We also give the proof of Theorem 2.1 in Section 6. In Section 8, we show the uniqueness of the minimizer $R_{\omega(m)}$ for each $m > 0$. From the uniqueness of the minimizer $R_{\omega(m)}$, we obtain a regularity of $R_{\omega(m)}$ in Section 9. In Section 10, we shall study the Morse index of $R_{\omega(m)}$ and give a proof of Theorem 2.5 (iii). We give the proof of Theorem 2.6 in Appendix A. In Appendix B, we obtain a boundedness of the L^∞ norm of solutions to some elliptic equations, which is needed in Section 8. In Apendix C, we list a table of symbols and their descriptions used in this papers.

In what follows, we will always fix $d = 3$ unless otherwise noted. Furthermore, we consider only the real-valued functions.

2.1 Notation

(i) We use the symbol $(\cdot, \cdot)_{L^2}$ to denote the inner product in $L^2(\mathbb{R}^3)$, namely, if $f, g \in L^2(\mathbb{R}^3)$, then

$$(f,\, g)_{L^2} = \int_{\mathbb{R}^3} f(x)g(x)\, dx.$$

We also use the same symbol to denote the pairing between $H^1(\mathbb{R}^3)$ and $H^{-1}(\mathbb{R}^3)$, namely if $f \in H^1(\mathbb{R}^3)$ and $g \in H^{-1}(\mathbb{R}^3)$, then

$$\langle f,\, g\rangle = \int_{\mathbb{R}^3} \langle\nabla\rangle f(x)\langle\nabla\rangle^{-1} g(x)\, dx.$$

(ii) Throughout the paper, C denotes a positive constant, that does not depend on the parameters, unless otherwise noted and may change from line to line.

(iii) For given positive quantities a and b, the notation $a \lesssim b$ means the inequality $a \le Cb$ for some positive constant C. We also use the notation $a \sim b$ when $a \lesssim b$ and $b \lesssim a$.

3 Proofs of Theorems 2.2 and 2.3

This section focuses on the family of solutions $\{u_{1,\omega}\}$ satisfying $\limsup_{\omega\to 0} \|u_{1,\omega}\|_{L^\infty} < \infty$. We will prove Theorems 2.2 and 2.3.

3.1 Relation between L^∞-norm of $u_{1,\omega}$ and ω

In this subsection, we shall study the relation between the L^∞-norm of the solution $u_{1,\omega}$ and ω. Put $M_{1,\omega} := \|u_{1,\omega}\|_{L^\infty}$ and

$$\alpha_{1,\omega} := \omega M_{1,\omega}^{-(p-1)}. \tag{3.1}$$

We shall show the following:

Proposition 3.1. *Let $1 < p < 5$ and $\{u_{1,\omega}\}$ be a family of positive solutions to (1.1) satisfying $\limsup_{\omega\to 0} M_{1,\omega} < \infty$. We have*

$$\alpha_{1,\omega} \sim 1 \qquad \text{as } \omega \to 0.$$

We prove Proposition 3.1, we need several preparations. First, we obtain an upper bound of $\alpha_{1,\omega}$:

Lemma 3.2. *Let $1 < p < 5$ and $u_{1,\omega} \in H^1(\mathbb{R}^3)$ be a positive solution to (1.1). We obtain*

$$\alpha_{1,\omega} < \frac{5-p}{2(p+1)}. \tag{3.2}$$

Proof. Multiplying (1.1) by $u_{1,\omega}$ and integrating the resulting equation, we have

$$\|\nabla u_{1,\omega}\|_{L^2}^2 + \omega\|u_{1,\omega}\|_{L^2}^2 = \|u_{1,\omega}\|_{L^{p+1}}^{p+1} + \|u_{1,\omega}\|_{L^6}^6. \tag{3.3}$$

From the Pohozaev identity of (1.1), we obtain

$$\frac{1}{2}\|\nabla u_{1,\omega}\|_{L^2}^2 + \frac{3}{2}\omega\|u_{1,\omega}\|_{L^2}^2 = \frac{3}{p+1}\|u_{1,\omega}\|_{L^{p+1}}^{p+1} + \frac{1}{2}\|u_{1,\omega}\|_{L^6}^6. \tag{3.4}$$

It follows from (3.3), (3.4) and $M_{1,\omega} = \|u_{1,\omega}\|_{L^\infty}$ that

$$\omega\|u_{1,\omega}\|_{L^2}^2 = \frac{5-p}{2(p+1)}\|u_{1,\omega}\|_{L^{p+1}}^{p+1} < \frac{5-p}{2(p+1)} M_{1,\omega}^{p-1}\|u_{1,\omega}\|_{L^2}^2$$

leading to (3.2). $\square$

To obtain a lower bound of $\alpha_{1,\omega}$, we consider the following rescaling:

$$\widehat{u}_{1,\omega}(\cdot) = M_{1,\omega}^{-1} u_{1,\omega}(M_{1,\omega}^{-\frac{p-1}{2}} \cdot). \tag{3.5}$$

Then, we see that $\widehat{u}_{1,\omega}$ satisfies

$$-\Delta\widehat{u}_{1,\omega} + \alpha_{1,\omega}\widehat{u}_{1,\omega} - \widehat{u}_{1,\omega}^p - M_{1,\omega}^{5-p}\widehat{u}_{1,\omega}^5 = 0, \qquad \widehat{u}_{1,\omega}(0) = \|\widehat{u}_{1,\omega}\|_{L^\infty} = 1. \tag{3.6}$$

Recalling that it follows from the result of Gidas, Ni and Nirenberg [24] that $\widehat{u}_{1,\omega}$ is radially symmetric and decreasing in $r = |x|$. Thus, there exists some function $\widehat{v}_{1,\omega}$ on $[0,\infty)$ such that $\widehat{v}_{1,\omega}(r) = \widehat{u}_{1,\omega}(x)$ for $r = |x|$. By abuse of notation, we identify $\widehat{v}_{1,\omega}$ by $\widehat{u}_{1,\omega}$. Since $\widehat{u}_{1,\omega}$ is radially symmetric with $\widehat{u}_{1,\omega}(0) = \|\widehat{u}_{1,\omega}\|_{L^\infty} = 1$, (3.6) can be transformed to the following ordinary differential equations:

$$\begin{cases} -\widehat{u}_{1,\omega}'' - \frac{2}{r}\widehat{u}_{1,\omega}' + \alpha_{1,\omega}\widehat{u}_{1,\omega} - \widehat{u}_{1,\omega}^p - M_{1,\omega}^{5-p}\widehat{u}_{1,\omega}^5 = 0 \qquad \text{on } (0,\infty), \\ \widehat{u}_{1,\omega}(0) = 1, \qquad \widehat{u}_{1,\omega}'(0) = 0, \end{cases} \tag{3.7}$$

where the prime mark denotes the differentiation with respect to r. Here, we remark that throughout this section, we always assume that $\limsup_{\omega\to 0} M_{1,\omega} < +\infty$.

We shall show that $\liminf_{\omega\to 0}\alpha_{1,\omega} > 0$. Suppose to the contrary that there exists a sequence $\{\omega_n\} \subset (0,\infty)$ with $\lim_{n\to\infty}\omega_n = 0$ such that $\lim_{n\to\infty}\alpha_{1,\omega_n} = 0$. Then, there exist a subsequence of $\{\omega_n\}$ (we still denote by the same symbol) and $M_{1,0} \in [0,\infty)$ such that $\lim_{n\to\infty} M_{1,\omega_n} = M_{1,0}$. We introduce the following "limit" equation:

$$\begin{cases} -\widehat{u}_{1,0}'' - \frac{2}{r}\widehat{u}_{1,0}' - \widehat{u}_{1,0}^p - M_{1,0}^{5-p}\widehat{u}_{1,0}^5 = 0 & \text{on } (0,\infty), \\ \widehat{u}_{1,0}(0) = 1, \qquad \widehat{u}_{1,0}'(0) = 0. \end{cases} \tag{3.8}$$

Concerning the solution to (3.8), we shall show the following:

Lemma 3.3. *Let $1 < p < 5$ and $\widehat{u}_{1,0}$ be the unique solution to* (3.8)*. Then, $\widehat{u}_{1,0}$ must change the sign on* $[0,\infty)$.

Proof. We shall prove this lemma by contradiction. Suppose to the contrary that $\widehat{u}_{1,0}(r) > 0$ for any $r > 0$. It follows from the equation in (3.8) that

$$-r^2\widehat{u}_{1,0}'(r) = \int_0^r (\widehat{u}_{1,0}^p(s) + M_{1,0}^{5-p}\widehat{u}_{1,0}^5(s))s^2\,ds > 0. \tag{3.9}$$

Thus, $\widehat{u}_{1,0}(r)$ is monotone decreasing in $r > 0$ and has a limit $\ell_{1,0} \geq 0$, $\ell_{1,0} = \lim_{r\to\infty}\widehat{u}_{1,0}(r)$. We define an energy $E_{1,0}$ by

$$E_{1,0}(r) := \frac{(\widehat{u}_{1,0}'(r))^2}{2} + \frac{\widehat{u}_{1,0}^{p+1}(r)}{p+1} + M_{1,0}^{5-p}\frac{\widehat{u}_{1,0}^6(r)}{6}$$

From (3.8) and (3.9), we can easily verify that

$$E_{1,0}'(r) = \widehat{u}_{1,0}'(r)\widehat{u}_{1,0}''(r) + \widehat{u}_{1,0}^p(r)\widehat{u}_{1,0}'(r) + M_{1,0}^{5-p}\widehat{u}_{1,0}^5(r)\widehat{u}_{1,0}'(r) = -\frac{2}{r}(\widehat{u}_{1,0}'(r))^2 < 0.$$

This together with the initial conditions in (3.8) implies that

$$E_{1,0}(r) \leq E_{1,0}(0) = \frac{1}{p+1} + \frac{M_{1,0}^{5-p}}{6} < \infty.$$

Thus, we see that $\widehat{u}_{1,0}$ and $\widehat{u}_{1,0}'$ are bounded in $r > 0$.

It follows from the boundedness of $\widehat{u}_{1,0}'(r)$, $\ell_{1,0} = \lim_{r\to\infty}\widehat{u}_{1,0}(r)$ and (3.8) that

$$-\lim_{r\to\infty}\widehat{u}_{1,0}''(r) = \ell_{1,0}^p + M_{1,0}\ell_{1,0}^5 (\geq 0).$$

Since $\widehat{u}_{1,0}'(r)$ is bounded, we have $\ell_{1,0} = 0$. Then, from (3.9), the monotonicity and positivity of $\widehat{u}_{1,0}$, we obtain

$$-r^2\widehat{u}_{1,0}'(r) = \int_0^r (\widehat{u}_{1,0}^p(s) + M_{1,0}^{5-p}\widehat{u}_{1,0}^5(s))s^2\,ds \geq \widehat{u}_{1,0}^p(r)\int_0^r s^2\,ds = \frac{r^3\widehat{u}_{1,0}^p(r)}{3}.$$

This implies that

$$\left(\frac{1}{p-1}\widehat{u}_{1,0}^{-(p-1)}(r)\right)' = -\widehat{u}_{1,0}'(r)\widehat{u}_{1,0}^{-p}(r) > \frac{r}{3} = \left(\frac{r^2}{6}\right)'.$$

Thus, integrating the above from 0 to r, we have by the initial condition $\widehat{u}_{1,0}(0) = 1$ that

$$\frac{\widehat{u}_{1,0}^{-(p-1)}(r)}{p-1} > \frac{\widehat{u}_{1,0}^{-(p-1)}(r)}{p-1} - \frac{\widehat{u}_{1,0}^{-(p-1)}(0)}{p-1} > \frac{r^2}{6} - \frac{0^2}{6} = \frac{r^2}{6} > 0. \tag{3.10}$$

From (3.10), we conclude that

$$\widehat{u}_{1,0}(r) \le \left(\frac{6}{p-1}\right)^{\frac{1}{p-1}} r^{-\frac{2}{p-1}}.$$

This together with the condition $p < 5$ implies that

$$\int_0^\infty r^2 \widehat{u}_{1,0}^{p+1}(r)\, dr + \int_0^\infty r^2 \widehat{u}_{1,0}^6(r)\, dr < \infty \tag{3.11}$$

Observe from the equation of (3.8) that

$$\left(\frac{r^3}{2}(\widehat{u}_{1,0}'(r))^2 + \frac{r^3}{p+1}\widehat{u}_{1,0}^{p+1}(r) + \frac{M_{1,0}^{5-p}}{6} r^3 \widehat{u}_{1,0}^6(r)\right)' + \frac{1}{2} r^2 (\widehat{u}_{1,0}'(r))^2 = \frac{3}{p+1} r^2 \widehat{u}_{1,0}^{p+1}(r) + \frac{M_{1,0}^{5-p}}{2} r^2 \widehat{u}_{1,0}^6(r). \tag{3.12}$$

Integrating (3.12) from 0 to r, we have

$$\begin{aligned} &\frac{r^3}{2}(\widehat{u}_{1,0}'(r))^2 + \frac{r^3}{p+1}\widehat{u}_{1,0}^{p+1}(r) + \frac{M_{1,0}^{5-p}}{6} r^3 \widehat{u}_{1,0}^6(r) + \frac{1}{2}\int_0^r s^2 (\widehat{u}_{1,0}'(s))^2\, ds \\ &= \frac{3}{p+1}\int_0^r s^2 \widehat{u}_{1,0}^{p+1}(s)\, ds + \frac{M_{1,0}^{5-p}}{2}\int_0^r s^2 \widehat{u}_{1,0}^6(s)\, ds. \end{aligned} \tag{3.13}$$

This yields that

$$\int_0^r s^2 (\widehat{u}_{1,0}'(s))^2\, ds \le \frac{6}{p+1}\int_0^r s^2 \widehat{u}_{1,0}^{p+1}(s)\, ds + M_{1,0}^{5-p}\int_0^r s^2 \widehat{u}_{1,0}^6(s)\, ds$$

Letting $r \to \infty$, we have by (3.11) that

$$\int_0^\infty s^2 (\widehat{u}_{1,0}'(s))^2\, ds < \infty. \tag{3.14}$$

Thus, it follows from (3.11) and (3.14) that

$$\int_0^\infty s^2 \left\{(\widehat{u}_{1,0}'(s))^2 + \widehat{u}_{1,0}^{p+1}(s) + \widehat{u}_{1,0}^6(s)\right\} ds < \infty. \tag{3.15}$$

We see from (3.15) that there exists a sequence $\{r_n\}$ with $\lim_{n\to\infty} r_n = \infty$ such that

$$r_n^3((\widehat{u}_{1,0}'(r_n))^2 + (\widehat{u}_{1,0}(r_n))^{p+1} + (\widehat{u}_{1,0}(r_n))^6) \to 0 \qquad \text{as } n \to \infty. \tag{3.16}$$

Indeed, if not, there exist $C_0 > 0$ and $r_0 > 0$ such that

$$r^3((\widehat{u}_{1,0}'(r))^2 + (\widehat{u}_{1,0}(r))^{p+1} + (\widehat{u}_{1,0}(r))^6) \ge C_0 \qquad (r \in (r_0, \infty)).$$

This implies that

$$\int_0^\infty s^2 \left\{(\widehat{u}_{1,0}'(s))^2 + \widehat{u}_{1,0}^{p+1}(s) + \widehat{u}_{1,0}^6(s)\right\} ds \ge \int_{r_0}^\infty s^2 \left\{(\widehat{u}_{1,0}'(s))^2 + \widehat{u}_{1,0}^{p+1}(s) + \widehat{u}_{1,0}^6(s)\right\} ds \ge C_0 \int_{r_0}^\infty s^{-1}\, ds = \infty,$$

which contradicts (3.15). Thus, (3.16) holds. Substituting $r = r_n$ in (3.13) and letting n go to infinity, we see from (3.16) that

$$\frac{1}{2}\int_0^\infty s^2 (\widehat{u}_{1,0}'(s))^2\, ds = \frac{3}{p+1}\int_0^\infty s^2 \widehat{u}_{1,0}^{p+1}(s)\, ds + \frac{M_{1,0}^{5-p}}{2}\int_0^\infty s^2 \widehat{u}_{1,0}^6(s)\, ds. \tag{3.17}$$

On the other hand, from the equation in (3.8), we have

$$(r^2 \widehat{u}_{1,0}(r)\widehat{u}_{1,0}'(r))' + r^2 \widehat{u}_{1,0}^{p+1}(r) + M_{1,0}^{5-p} r^2 \widehat{u}_{1,0}^6(r) = r^2 (\widehat{u}_{1,0}'(r))^2.$$

Integrating the above equality from 0 to r, one has

$$r^2\widehat{u}_{1,0}(r)\widehat{u}_{1,0}'(r)+\int_0^r s^2\widehat{u}_{1,0}^{p+1}(s)\,ds+M_{1,0}^{5-p}\int_0^r s^2\widehat{u}_{1,0}^6(s)\,ds=\int_0^r s^2(\widehat{u}_{1,0}'(s))^2\,ds. \tag{3.18}$$

It follows from (3.16) that there exists a constant $C_d>0$ such that $|\widehat{u}_{1,0}'(r_n)|\le C_d r_n^{-\frac{3}{2}}$ and $\widehat{u}_{1,0}(r_n)\le C_d r_n^{-\frac{3}{p+1}}$. This together with the condition $p<5$ implies that $\lim_{n\to\infty} r_n^2\widehat{u}_{1,0}(r_n)\widehat{u}_{1,0}'(r_n)=0$. Thus, substituting $r=r_n$ in (3.18) and letting n go to infinity, we obtain

$$\int_0^\infty s^2(\widehat{u}_{1,0}'(s))\,ds=\int_0^\infty s^2\widehat{u}_{1,0}^{p+1}(s)\,ds+M_{1,0}^{5-p}\int_0^\infty s^2\widehat{u}_{1,0}^6(s)\,ds. \tag{3.19}$$

From (3.17), (3.19) and $p<5$, we obtain

$$\int_0^\infty s^2\widehat{u}_{1,0}^{p+1}(s)\,ds=0,$$

which contradicts the positivity of $\widehat{u}_{1,0}(r)$. This completes the proof. $\square$

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. By Lemma 3.2, it suffices to show that $\liminf_{\omega\to 0}\alpha_{1,\omega}>0$. Suppose to the contrary that there exists a sequence $\{\omega_n\}$ in $(0,\infty)$ with $\lim_{n\to\infty}\omega_n=0$ such that

$$\lim_{n\to\infty}\alpha_{1,\omega_n}=0.$$

From the assumptions $\limsup_{\omega\to 0}\|u_{1,\omega}\|_{L^\infty}<\infty$ and $M_{1,\omega}=\|u_{1,\omega}\|_{L^\infty}$, it follows that up to a subsequence, we may assume that the sequence $\{M_{1,\omega_n}\}$ has a limit $M_{1,0}$, i.e. $\lim_{n\to\infty}M_{1,\omega_n}=M_{1,0}$. Observe from the equation in (3.7) that

$$-r^2\widehat{u}_{1,\omega_n}'(r)=\int_0^r(\widehat{u}_{1,\omega_n}^p(s)+M_{1,\omega_n}^{5-p}\widehat{u}_{1,\omega_n}^5(s)-\alpha_{1,\omega_n}\widehat{u}_{1,\omega_n})s^2\,ds. \tag{3.20}$$

It follows from $\widehat{u}_{1,\omega_n}(0)=\|\widehat{u}_{1,\omega_n}\|_{L^\infty}=1$ for all $n\in\mathbb{N}$ that

$$|r^2\widehat{u}_{1,\omega_n}'(r)|\le(1+M_{1,\omega_n}^{5-p}+\alpha_{1,\omega_n})\int_0^r s^2\,ds=\frac{(1+M_{1,\omega_n}^{5-p}+\alpha_{1,\omega_n})}{3}r^3.$$

Thus, by the assumption $\limsup_{n\to\infty}M_{1,\omega_n}<\infty$ and (3.2), there exists a constant $C_1>0$ independent of $n\in\mathbb{N}$, such that

$$|\widehat{u}_{1,\omega_n}'(r)|\le C_1 r.$$

This together with the equation in (3.7) yields that

$$\sup_{n\in\mathbb{N}}|\widehat{u}_{1,\omega_n}''(r)|\le 2\sup_{n\in\mathbb{N}}\frac{|\widehat{u}_{1,\omega_n}'(r)|}{r}+\sup_{n\in\mathbb{N}}\alpha_{1,\omega_n}|\widehat{u}_{1,\omega_n}(r)|+\sup_{n\in\mathbb{N}}|\widehat{u}_{1,\omega_n}^p(r)|+\sup_{n\in\mathbb{N}}M_{1,\omega_n}^{5-p}|\widehat{u}_{1,\omega_n}^5(r)|<\infty$$

for any $r>0$. Then, by the Ascoli-Arzela theorem, there exists a subsequence of $\{\widehat{u}_{1,\omega_n}\}$ (we denote it by the same symbol) and a function $v_0\in C^1([0,\infty))$ such that for any bounded interval I in $[0,\infty)$, we have

$$\widehat{u}_{1,\omega_n}(r)\to v_0(r),\qquad \widehat{u}_{1,\omega_n}'(r)\to v_0'(r)\qquad \text{uniformly in } r\in I.$$

In addition, taking the limit of (3.20), we see from $\lim_{n\to\infty}\alpha_{1,\omega_n}=0$ that v_0 satisfies

$$-r^2v_0'(r)=\int_0^r(v_0^p(s)+M_{1,0}^{5-p}v_0^5(s))s^2\,ds$$

for any $r \in I$. Then, v_0 satisfies (3.8). It follows from the uniqueness of the solution that $v_0 = \widehat{u}_{1,0}$.

Note that $\widehat{u}_{1,\omega_n}$ is positive and converges to $\widehat{u}_{1,0}$ uniformly on I as n tends to infinity. However, $\widehat{u}_{1,0}$ changes sign (see Lemma 3.3), leading to a contradiction. Thus, we see that $\liminf_{\omega\to 0} \alpha_{1,\omega} > 0$. $\square$

3.2 Conclusion

In this subsection, we complete the proofs of Theorems 2.2 and 2.3. To prove Theorem 2.2, we will obtain a boundedness of $\{\|\widehat{u}_{1,\omega}\|_{H^1}\}$. First, we shall show the following:

Lemma 3.4. *Let $1 < p < 5$ and $\widehat{u}_{1,\omega}$ be the positive solution to (3.7). There exist $C_\omega > 0$ such that*

$$\widehat{u}_{1,\omega}(r),\ |\widehat{u}'_{1,\omega}(r)| \le C_\omega \exp\left[-\sqrt{\frac{\alpha_{1,\omega}}{2}}r\right] \qquad \text{for all } r > 0. \tag{3.21}$$

Proof. Since $\widehat{u}'_{1,\omega}(r) < 0$ $(r > 0)$ and $\lim_{r\to\infty} \widehat{u}_{1,\omega}(r) = 0$, there exists $R_\omega > 0$ such that

$$\begin{aligned}\widehat{u}''_{1,\omega}(r) &= -\frac{2}{r}\widehat{u}'_{1,\omega}(r) + \alpha_{1,\omega}\widehat{u}_{1,\omega}(r) - \widehat{u}^p_{1,\omega}(r) - M^{5-p}_{1,\omega}\widehat{u}^5_{1,\omega}(r)\\ &> (\alpha_{1,\omega} - \widehat{u}^{p-1}_{1,\omega}(r) - M^{5-p}_{1,\omega}\widehat{u}^4_{1,\omega}(r))\widehat{u}_{1,\omega}(r) > \frac{\alpha_{1,\omega}}{2}\widehat{u}_{1,\omega}(r) \qquad \text{for all } r \ge R_\omega.\end{aligned}$$

This yields that

$$\partial_r\left(e^{\sqrt{\frac{\alpha_{1,\omega}}{2}}r}\left(-\partial_r + \sqrt{\frac{\alpha_{1,\omega}}{2}}\right)\widehat{u}_{1,\omega}\right) = e^{\sqrt{\frac{\alpha_{1,\omega}}{2}}r}\left(-\widehat{u}''_{1,\omega} + \frac{\alpha_{1,\omega}}{2}\widehat{u}_{1,\omega}(r)\right) < 0.$$

Thus, we see that $e^{\sqrt{\frac{\alpha_{1,\omega}}{2}}r}\left(-\partial_r + \sqrt{\frac{\alpha_{1,\omega}}{2}}\right)\widehat{u}_{1,\omega}$ is decreasing on $[R_\omega, \infty)$, which implies that

$$e^{\sqrt{\frac{\alpha_{1,\omega}}{2}}r}\left(-\partial_r + \sqrt{\frac{\alpha_{1,\omega}}{2}}\right)\widehat{u}_{1,\omega}(r) \le e^{\sqrt{\frac{\alpha_{1,\omega}}{2}}R_\omega}\left(-\partial_r + \sqrt{\frac{\alpha_{1,\omega}}{2}}\right)\widehat{u}_{1,\omega}(R_\omega) =: \widetilde{C}_\omega \tag{3.22}$$

Since $\widehat{u}'_{1,\omega}(r) < 0$, we have by (3.22) that

$$\widehat{u}_{1,\omega}(r) \le \widetilde{C}_\omega\sqrt{\frac{2}{\alpha_{1,\omega}}}e^{-\sqrt{\frac{\alpha_{1,\omega}}{2}}r}, \qquad |\widehat{u}'_{1,\omega}(r)| = -\widehat{u}'_{1,\omega}(r) \le \widetilde{C}_\omega e^{-\sqrt{\frac{\alpha_{1,\omega}}{2}}r}$$

Thus, putting

$$C_\omega := \widetilde{C}_\omega \max\left\{\sqrt{\frac{2}{\alpha_{1,\omega}}}, 1\right\},$$

we find that (3.21) holds. $\square$

Next, following [22, 34, 47], which are originated in [9], we prove the following pointwise estimate of $\widehat{u}_{1,\omega}$:

Lemma 3.5. *Let $1 < p < 5$ and $\widehat{u}_{1,\omega}$ be the positive solution to (3.7). There exists $\omega_{1,1} > 0$ such that for $\omega \in (0, \omega_{1,1})$, we have*

$$\widehat{u}_{1,\omega}(r) \le \left(1 + \frac{\gamma_{1,\omega}}{3}r^2\right)^{-\frac{1}{2}} \qquad \text{for all } r > 0, \tag{3.23}$$

where

$$\gamma_{1,\omega} := -\alpha_{1,\omega} + 1 + M^{5-p}_{1,\omega}. \tag{3.24}$$

Remark 3.1. *We see from (3.2) that*

$$\gamma_{1,\omega} > \frac{3(p-1)}{2(p+1)}$$

for any $\omega > 0$.

Proof of Lemma 3.5. We put $f_{1,\omega}(s) := \alpha_{1,\omega}s - s^p - M_{1,\omega}^{5-p}s^5$ for $s > 0$. Then, the equation in (3.7) can be written as:

$$\widehat{u}''_{1,\omega} = -2\frac{\widehat{u}'_{1,\omega}}{r} + f_{1,\omega}(\widehat{u}_{1,\omega}). \tag{3.25}$$

Putting

$$\Psi_{1,\omega}(r) := \frac{-\widehat{u}'_{1,\omega}(r)}{r\widehat{u}^3_{1,\omega}(r)}, \tag{3.26}$$

we have by (3.25) that

$$\begin{aligned}\Psi'_{1,\omega}(r) = \frac{-\widehat{u}''_{1,\omega}r\widehat{u}^3_{1,\omega} + \widehat{u}'_{1,\omega}(\widehat{u}^3_{1,\omega} + 3r\widehat{u}^2_{1,\omega}\widehat{u}'_{1,\omega})}{r^2\widehat{u}^6_{1,\omega}} &= \frac{3\widehat{u}^3_{1,\omega}\widehat{u}'_{1,\omega} - f_{1,\omega}(\widehat{u}_{1,\omega})r\widehat{u}^3_{1,\omega} + 3r\widehat{u}^2_{1,\omega}(\widehat{u}'_{1,\omega})^2}{r^2\widehat{u}^6_{1,\omega}} \\ &= 3\frac{H_{1,\omega}(r)}{r^4\widehat{u}^4_{1,\omega}},\end{aligned} \tag{3.27}$$

where

$$H_{1,\omega}(r) := r^3(\widehat{u}'_{1,\omega})^2 + r^2\widehat{u}_{1,\omega}\widehat{u}'_{1,\omega} - \frac{r^3 f_{1,\omega}(\widehat{u}_{1,\omega})\widehat{u}_{1,\omega}}{3}.$$

With a little bit of effort, we see from (3.25) and $f_{1,\omega}(s) = \alpha_{1,\omega}s - s^p - M_{1,\omega}^{5-p}s^5$ that

$$\begin{aligned}H'_{1,\omega}(r) &= 3r^2(\widehat{u}'_{2,\omega})^2 + 2r^3\widehat{u}'_{1,\omega}\widehat{u}''_{1,\omega} + 2r\widehat{u}_{1,\omega}\widehat{u}'_{1,\omega} + r^2(\widehat{u}'_{1,\omega})^2 + r^2\widehat{u}_{1,\omega}\widehat{u}''_{1,\omega} \\ &\quad - r^2\widehat{u}_{1,\omega}f_{1,\omega}(\widehat{u}_{1,\omega}) - \frac{r^3\widehat{u}_{1,\omega}\widehat{u}'_{1,\omega}f'_{1,\omega}(\widehat{u}_{1,\omega})}{3} - \frac{r^3\widehat{u}'_{1,\omega}f_{1,\omega}(\widehat{u}_{1,\omega})}{3} \\ &= \frac{r^3\widehat{u}_{1,\omega}\widehat{u}'_{1,\omega}}{3}(4\alpha_{1,\omega} - (5-p)\widehat{u}^{p-1}_{1,\omega}).\end{aligned} \tag{3.28}$$

By (3.2), $\widehat{u}_{1,\omega}(0) = \|\widehat{u}_{1,\omega}\|_{L^\infty} = 1$ and $p > 1$, one has

$$4\alpha_{1,\omega} - (5-p)\widehat{u}^{p-1}_{1,\omega}(0) < \frac{2(5-p)}{p+1} - (5-p) < 0 \tag{3.29}$$

for any $\omega > 0$. By $\widehat{u}'_{1,\omega}(r) < 0$, $\lim_{r\to\infty}\widehat{u}_{1,\omega}(r) = 0$ (see Lemma 3.4), (3.28) and (3.29), there exists $r_{1,\omega} > 0$ such that

$$H'_{1,\omega}(r)\begin{cases} > 0 & (0 < r < r_{1,\omega}), \\ = 0 & (r = r_{1,\omega}), \\ < 0 & (r_{1,\omega} < r). \end{cases}$$

In addition, since the functions $\widehat{u}_{1,\omega}, -\widehat{u}'_{1,\omega}$ decay exponentially (see Lemma 3.4), we have $\lim_{r\to\infty} H_{1,\omega}(r) = 0$ for each $\omega > 0$. These together with $H_{1,\omega}(0) = 0$ imply that $H_{1,\omega}(r) > 0$ for all $r > 0$. It follows from (3.27) that $\Psi'_{1,\omega}(r) > 0$ for all $r > 0$. This together with the l'Hopital rule, $\widehat{u}_{1,\omega}(0) = 1$ and $\widehat{u}'_{1,\omega}(0) = 0$ yields that

$$\Psi_{1,\omega}(r) > \Psi_{1,\omega}(0) = \lim_{r\to 0}\frac{-\widehat{u}'_{1,\omega}(r)}{r\widehat{u}^3_{1,\omega}(r)} = \lim_{r\to 0}\frac{-\widehat{u}''_{1,\omega}(r)}{\widehat{u}^3_{1,\omega}(r) + 3r\widehat{u}^2_{1,\omega}(r)\widehat{u}'_{1,\omega}(r)} = -\widehat{u}''_{1,\omega}(0).$$

Using the l'Hopital rule again, we have $\lim_{r\to 0}\frac{\widehat{u}'_{1,\omega}(r)}{r} = \widehat{u}''_{1,\omega}(0)$. Then, it follows from (3.25) that

$$\Psi_{1,\omega}(r) > -\widehat{u}''_{1,\omega}(0) = -\frac{f_{1,\omega}(\widehat{u}_{1,\omega}(0))}{3} = -\frac{f_{1,\omega}(1)}{3} = \frac{\gamma_{1,\omega}}{3} \qquad \text{for all } r > 0, \tag{3.30}$$

where $\gamma_{1,\omega}$ is the positive number defined by (3.24). We put

$$Z_{1,\omega}(r) := \left(1 + \frac{\gamma_{1,\omega}}{3} r^2\right)^{-\frac{1}{2}}.$$

We can easily verify that

$$Z^3_{1,\omega}(r) = \left(1 + \frac{\gamma_{1,\omega}}{3} r^2\right)^{-\frac{3}{2}}, \qquad Z'_{1,\omega}(r) = -\frac{\gamma_{1,\omega}}{3}\left(1 + \frac{\gamma_{1,\omega}}{3} r^2\right)^{-\frac{3}{2}} r = -\frac{\gamma_{1,\omega}}{3} Z^3_{1,\omega}(r) r.$$

Thus, we have by (3.30) and (3.26), that

$$(Z^{-2}_{1,\omega}(r))' = -2\frac{Z'_{1,\omega}(r)}{Z^3_{1,\omega}(r)} = \frac{2}{3}\gamma_{1,\omega} r < 2\Psi_{1,\omega}(r) r = -2\frac{\widehat{u}'_{1,\omega}(r)}{\widehat{u}^3_{1,\omega}(r)} = (\widehat{u}^{-2}_{1,\omega}(r))'.$$

Integrating the above inequality from 0 to r, we have by $\widehat{u}_{1,\omega}(0) = Z_{1,\omega}(0) = 1$ that

$$Z^{-2}_{1,\omega}(r) - 1 = Z^{-2}_{1,\omega}(r) - Z^{-2}_{1,\omega}(0) < \widehat{u}^{-2}_{1,\omega}(r) - \widehat{u}^{-2}_{1,\omega}(0) = \widehat{u}^{-2}_{1,\omega}(r) - 1.$$

This yields that $\widehat{u}_{1,\omega}(r) < Z_{1,\omega}(r)$ for all $r > 0$. This concludes the proof. $\square$

The following result gives a decay estimate of the positive solution $\widehat{u}_{1,\omega}$ to (3.6) that is better than the one given by Lemma 3.4:

Lemma 3.6. *Let $1 < p < 5$. There exists $C_0 > 0$, which does not depend on $\omega > 0$, and a sufficiently small $\omega_{1,2} > 0$ such that for $\omega \in (0, \omega_{1,2})$ and positive solution $\widehat{u}_{1,\omega}$ to* (3.6)*, we obtain*

$$\widehat{u}_{1,\omega}(x) \le C_0 |x|^{-1} \exp(-\frac{\sqrt{\alpha_{1,\omega}}}{2} x) \qquad (x \in \mathbb{R}^3). \tag{3.31}$$

Proof of Lemma 3.6. From (3.23) and $\liminf_{\omega\to 0}\alpha_{1,\omega} > 0$ (see Proposition 3.1), there exist $\omega_{1,2}$ and $R_0 > 0$, which are independent of $\omega > 0$, such that for $\omega \in (0, \omega_{1,2})$, we have

$$-\Delta\widehat{u}_{1,\omega} + \frac{\alpha_{1,\omega}}{4}\widehat{u}_{1,\omega} \le 0 \qquad (|x| \ge R_0).$$

For such $R_0 > 0$, by (3.2), we can take a constant $C_0 > 0$ sufficiently large, which does not depend on $\omega > 0$, such that

$$\widehat{u}_{1,\omega}(x) \le \left(1 + \frac{\gamma_{1,2\omega}}{3} R_0^2\right)^{-\frac{1}{2}} \le C_0 R_0^{-1} \exp\left(-\frac{\sqrt{\alpha_{1,\omega}}}{2} R_0\right) \qquad (|x| = R_0).$$

Observe that $Y_\omega(x) := C_0|x|^{-1}\exp(-\frac{\sqrt{\alpha_{1,\omega}}}{2}|x|)$ satisfies $-\Delta Y_\omega + \frac{\alpha_{1,\omega}}{4} Y_\omega = 0$. Then, it follows from the maximum principle that

$$\widehat{u}_{1,\omega}(x) \le Y_\omega(x) = C_0|x|^{-1}\exp(-\frac{\sqrt{\alpha_{1,\omega}}}{2}|x|) \qquad (|x| \ge R_0). \tag{3.32}$$

Taking $C_0 > 0$ sufficiently large, if necessary, we have by (3.23) that

$$\widehat{u}_{1,\omega}(x) \le \left(1 + \frac{\gamma_{1,2\omega}}{3}|x|^2\right)^{-\frac{1}{2}} \le C_0|x|^{-1}\exp(-\frac{\sqrt{\alpha_{1,\omega}}}{2}|x|) \qquad (|x| < R_0). \tag{3.33}$$

Thus, from (3.32) and (3.33), we conclude that (3.31) holds. $\square$

Now, we are in a position to estimate H^1-norm of $\widehat{u}_{1,\omega}$.

Lemma 3.7. *Let* $1 < p < 5$ *and* $\{u_{1,\omega}\}$ *be a family of positive solutions to* (1.1) *satisfying* $\limsup_{\omega\to 0}\|u_{1,\omega}\|_{L^\infty} < \infty$. *We have* $\limsup_{\omega\to 0}\|\widehat{u}_{1,\omega}\|_{H^1} < +\infty$, *where* $\widehat{u}_{1,\omega}$ *is defined by* (3.5).

Proof. We see from Lemmas 3.5 and 3.6 that

$$\limsup_{\omega\to 0}\|\widehat{u}_{1,\omega}\|_{L^q} < +\infty \tag{3.34}$$

for any $q \geq 1$. Multiplying (3.6) by $\widehat{u}_{1,\omega}$ and integrating the resulting equation, one has

$$\|\nabla\widehat{u}_{1,\omega}\|_{L^2}^2 + \alpha_{1,\omega}\|\widehat{u}_{1,\omega}\|_{L^2}^2 = \|\widehat{u}_{1,\omega}\|_{L^{p+1}}^{p+1} + M_{1,\omega}^{5-p}\|\widehat{u}_{1,\omega}\|_{L^6}^6.$$

This together with the assumption $\limsup_{\omega\to 0} M_{1,\omega} < \infty$ (recall that $M_{1,\omega} = \|u_{1,\omega}\|_{L^\infty}$) and (3.34) implies that

$$\limsup_{\omega\to 0}\left\{\|\nabla\widehat{u}_{1,\omega}\|_{L^2} + \alpha_{1,\omega}\|\widehat{u}_{1,\omega}\|_{L^2}^2\right\} \leq \limsup_{\omega\to 0}\left\{\|\widehat{u}_{1,\omega}\|_{L^{p+1}}^{p+1} + M_{1,\omega}^{5-p}\|\widehat{u}_{1,\omega}\|_{L^6}^6\right\} \lesssim 1.$$

Since $\alpha_{1,\omega} \sim 1$ as $\omega \to 0$ (see Proposition 3.1), we obtain the desired result. $\square$

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Note that

$$u_{1,\omega}(r) = M_{1,\omega}\widehat{u}_{1,\omega}(M_{1,\omega}^{\frac{p-1}{2}} r). \tag{3.35}$$

Since $\widetilde{u}_{1,\omega}(\cdot) = \omega^{-\frac{1}{p-1}}u_{1,\omega}(-\omega^{-\frac{1}{2}}\cdot)$ and $\alpha_{1,\omega} = \omega M_{1,\omega}^{1-p}$ (see (2.1) and (3.1)), we have by (3.35) that

$$\widetilde{u}_{1,\omega}(r) = \omega^{-\frac{1}{p-1}}u_{1,\omega}(\omega^{-\frac{1}{2}}r) = \omega^{-\frac{1}{p-1}}M_{1,\omega}\widehat{u}_{1,\omega}(\omega^{-\frac{1}{2}}M_{1,\omega}^{\frac{p-1}{2}} r) = \alpha_{1,\omega}^{-\frac{1}{p-1}}\widehat{u}_{1,\omega}(\alpha_{1,\omega}^{-\frac{1}{2}}r).$$

Since $\alpha_{1,\omega} \sim 1$ as $\omega \to 0$, we see from Lemma 3.7 that

$$\limsup_{\omega\to 0}\|\widetilde{u}_{1,\omega}\|_{H^1} < +\infty.$$

Since $\limsup_{\omega\to 0}\|\widetilde{u}_{1,\omega}\|_{H^1} < \infty$, there exist a subsequence of $\{\widetilde{u}_{1,\omega}\}$ and $u_{1,\infty} \in H^1(\mathbb{R}^3)$ such that $\lim_{\omega\to 0}\widetilde{u}_{1,\omega} = u_{1,\infty}$ weakly in $H^1(\mathbb{R}^3)$. In addition, by Proposition 3.1, we obtain

$$\widetilde{u}_{1,\omega}(0) = \omega^{-\frac{1}{p-1}}M_{1,\omega} = (\omega M_{1,\omega}^{1-p})^{-\frac{1}{p-1}} = \alpha_{1,\omega}^{-\frac{1}{p-1}} \gtrsim 1 \qquad \text{for any sufficiently small } \omega > 0,$$

where the implicit constant does not depend on $\omega > 0$. This yields that $u_{1,\infty} \neq 0$.

Observe that $\widetilde{u}_{1,\omega}$ satisfies

$$-\Delta\widetilde{u}_{1,\omega} + \widetilde{u}_{1,\omega} - \widetilde{u}_{1,\omega}^p - \omega^{\frac{5-p}{p-1}}\widetilde{u}_{1,\omega}^5 = 0.$$

Then, we see that the weak limit $u_{1,\infty}$ is a positive solution to (1.13) with $\omega = 1$. From the uniqueness of the positive radial solution to (1.13), we have $u_{1,\infty} = U^\dagger$. In addition, since $\lim_{\omega\to 0}\widetilde{u}_{1,\omega} = u_{1,\infty} = U^\dagger$ strongly in $L^q(\mathbb{R}^3)$ for $2 < q < 6$ by the radial compactness (see e.g. Berestycki and P. L. Lions [10, Lemma A.2]) and the family $\{\widetilde{u}_{1,\omega}\}$ is bounded in $H^1(\mathbb{R}^3)$, we have

$$\|\widetilde{u}_{1,\omega}\|_{H^1}^2 = \|\widetilde{u}_{1,\omega}\|_{L^{p+1}}^{p+1} + \omega^{\frac{5-p}{p-1}}\|\widetilde{u}_{1,\omega}\|_{L^6}^6 \to \|U^\dagger\|_{L^{p+1}}^{p+1} = \|U^\dagger\|_{H^1}^2$$

This together with $\lim_{\omega\to 0}\widetilde{u}_{1,\omega} = u_{1,\infty} = U^\dagger$ weakly in $H^1(\mathbb{R}^3)$ implies that $\lim_{\omega\to 0}\widetilde{u}_{1,\omega} = u_{1,\infty} = U^\dagger$ strongly in $H^1(\mathbb{R}^3)$. Thus, we see that Theorem 2.2 holds. $\square$

Proof of Theorem 2.3. We can show Theorem 2.3 (i) by using the similar argument of [4, Proposition 2.0.4]. In addition, we can prove Theorem 2.3 (ii) as in the proof of Lemma 2.3 of [41]. □

4 Convergence to the Aubin-Talenti function

In what follows, we consider only the family of positive solutions to (1.1) satisfying condition $\lim_{\omega\to 0}\|u_{2,\omega}\| = \infty$. Thus, when no confusion arises, we denote $u_{2,\omega}, \alpha_{2,\omega}, \beta_{2,\omega}$ simply by $\widetilde{u}_\omega, \alpha_\omega, \beta_\omega$, respectively.

This section is devoted to the proof of Theorem 2.4. Let $\{u_\omega\}$ be a family of positive solutions to (1.1) satisfying $\lim_{\omega\to 0}\|u_\omega\|_{L^\infty} = \infty$. We define

$$M_\omega := \|u_\omega\|_{L^\infty}, \qquad \widetilde{u}_\omega(\cdot) := M_\omega^{-1} u_\omega(M_\omega^{-2}\cdot).$$

Then, we see that for each $\omega > 0$, $\widetilde{u}_\omega$ satisfies (2.4). We can obtain the following decay estimate of $\widetilde{u}_\omega$:

Lemma 4.1. *Let $1 < p < 3$ and $\widetilde{u}_\omega$ be a positive solution to* (2.4). *There exists a sufficiently small $\omega_1 > 0$ such that for $\omega \in (0, \omega_1)$, we have*

$$\widetilde{u}_\omega(r) \le \left(1 + \frac{\gamma_\omega}{3} r^2\right)^{-\frac{1}{2}} \qquad \text{for all } r > 0, \tag{4.1}$$

where

$$\gamma_\omega := -\alpha_\omega + \beta_\omega + 1 \tag{4.2}$$

We can prove Lemma 4.1 by a similar argument to the proof of Lemma 3.5. Thus, we omit the proof. Next, we claim the following identity holds:

Lemma 4.2. *Let $1 < p < 3$ and $\widetilde{u}_\omega$ be a positive solution to* (2.4). *Then, the identity* (2.8) *holds, that is,*

$$\alpha_\omega \|\widetilde{u}_\omega\|_{L^2}^2 = \frac{5-p}{2(p+1)} \beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1}. \tag{4.3}$$

Proof. Multiplying (2.4) by $\widetilde{u}_\omega$ and integrating the resulting equation, we have

$$\|\nabla \widetilde{u}_\omega\|_{L^2}^2 + \alpha_\omega \|\widetilde{u}_\omega\|_{L^2}^2 = \beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} + \|\widetilde{u}_\omega\|_{L^6}^6. \tag{4.4}$$

From the Pohozaev identity of (2.4), we obtain

$$\frac{1}{2}\|\nabla \widetilde{u}_\omega\|_{L^2}^2 + \frac{3}{2}\alpha_\omega \|\widetilde{u}_\omega\|_{L^2}^2 = \frac{3}{p+1} \beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} + \frac{1}{2}\|\widetilde{u}_\omega\|_{L^6}^6. \tag{4.5}$$

From (4.4) and (4.5), we see that (2.8) holds. □

From Lemmas 4.1 and 4.2, we can obtain a uniform boundedness of $\{\|\nabla \widetilde{u}_\omega\|_{L^2}\}$:

Lemma 4.3. *Let $1 < p < 3$ and $\widetilde{u}_\omega$ be a positive solution to* (2.4). *Then, the following hold:*

(i)

$$\limsup_{\omega\to 0} \sqrt{\alpha_\omega}\|\widetilde{u}_\omega\|_{L^2}^2 < \infty. \tag{4.6}$$

(ii)

$$\limsup_{\omega\to 0} \beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} = 0, \tag{4.7}$$

(iii)

$$\limsup_{\omega\to 0}\|\nabla\widetilde{u}_\omega\|_{L^2}<+\infty.$$

Proof. (i) We divide the proof of (i) into 2 steps.

(Step 1). Define

$$L_\omega := \inf\left\{R>0\colon \frac{p+3}{2(p+1)}\beta_\omega\widetilde{u}_\omega^{p-1}(r)\le\alpha_\omega\quad (r\in[R,\infty))\right\}. \tag{4.8}$$

Since $\widetilde{u}_\omega$ is decreasing in $r>0$ and $\lim_{r\to\infty}\widetilde{u}_\omega(r)=0$, we see that

$$\frac{p+3}{2(p+1)}\beta_\omega\widetilde{u}_\omega^{p-1}(r)\begin{cases}>\alpha_\omega & r\in[0,L_\omega),\\ =\alpha_\omega & r=L_\omega,\\ <\alpha_\omega & r\in(L_\omega,\infty).\end{cases}$$

We claim that

$$\sqrt{\alpha_\omega}L_\omega\le\frac{\sqrt{6(p+3)}}{p-1}. \tag{4.9}$$

If $L_\omega=0$, then (4.9) is trivial. Thus, we may assume that $L_\omega>0$. By the definition of L_ω, we have $\frac{p+3}{2(p+1)}\beta_\omega\widetilde{u}_\omega^{p-1}(r)\ge\alpha_\omega$ for $r\le L_\omega$, so that $\alpha_\omega\beta_\omega^{-1}(\widetilde{u}_\omega(r))^{-(p-1)}\le\frac{p+3}{2(p+1)}$. Thus, we obtain

$$\begin{aligned}\beta_\omega\widetilde{u}_\omega^p(r)-\alpha_\omega\widetilde{u}_\omega(r)\ge\beta_\omega\widetilde{u}_\omega^p(r)\left(1-\alpha_\omega\beta_\omega^{-1}(\widetilde{u}_\omega(r))^{-(p-1)}\right)&\ge\beta_\omega\widetilde{u}_\omega^p(r)\left(1-\frac{p+3}{2(p+1)}\right)\\ &=\frac{p-1}{2(p+1)}\beta_\omega\widetilde{u}_\omega^p(r).\end{aligned} \tag{4.10}$$

Then, from (2.4), (4.10), the monotonicity and positivity of $\widetilde{u}_\omega$, one has

$$\begin{aligned}-r^2\widetilde{u}_\omega'(r)=-\int_0^r\frac{d}{ds}(s^2\widetilde{u}'(s))\,ds&=-\int_0^r(s^2\widetilde{u}''(s)+2s\widetilde{u}'(s))\,ds\\ &=\int_0^r s^2(\beta_\omega\widetilde{u}_\omega^p(s)+\widetilde{u}_\omega^5(s)-\alpha_\omega\widetilde{u}_\omega(s))\,ds\\ &\ge\frac{p-1}{2(p+1)}\int_0^r s^2\beta_\omega\widetilde{u}_\omega^p(s)\,ds\\ &\ge\frac{(p-1)}{2(p+1)}\beta_\omega\widetilde{u}_\omega^p(r)\int_0^r s^2\,ds\\ &=\frac{p-1}{6(p+1)}\beta_\omega\widetilde{u}_\omega^p(r)r^3\end{aligned}$$

for $r\le L_\omega$. This implies that

$$\left(\frac{(\widetilde{u}_\omega(r))^{-(p-1)}}{p-1}\right)'=-\widetilde{u}_\omega'(r)(\widetilde{u}_\omega(r))^{-p}\ge\frac{p-1}{6(p+1)}\beta_\omega r=\left(\frac{p-1}{12(p+1)}\beta_\omega r^2\right)'$$

for $r\le L_\omega$. Integrating from 0 to r, we obtain

$$\frac{(\widetilde{u}_\omega(r))^{-(p-1)}}{p-1}-\frac{(\widetilde{u}_\omega(0))^{-(p-1)}}{p-1}\ge\frac{p-1}{12(p+1)}\beta_\omega r^2-\frac{p-1}{12(p+1)}\beta_\omega\times 0^2=\frac{p-1}{12(p+1)}\beta_\omega r^2.$$

It follows from the initial condition $\widetilde{u}_{1,\omega}(0)=1$ (see (2.4)) that

$$\frac{(\widetilde{u}_\omega(r))^{-(p-1)}}{p-1}\ge\frac{p-1}{12(p+1)}\beta_\omega r^2.$$

for $r \le L_\omega$. From this, we conclude that

$$\widetilde{u}_\omega(r) \le \left(\frac{12(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} \beta_\omega^{-\frac{1}{p-1}} r^{-\frac{2}{p-1}}$$

for $r \le L_\omega$. From this inequality and the definition of L_ω, we have

$$\left(\frac{2(p+1)}{p+3}\alpha_\omega\beta_\omega^{-1}\right)^{\frac{1}{p-1}} = \widetilde{u}_\omega(L_\omega) \le \left(\frac{12(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} \beta_\omega^{-\frac{1}{p-1}} L_\omega^{-\frac{2}{p-1}}.$$

Thus, we see that (4.9) holds.

(Step 2). We finish the proof of (i). We put

$$E(r) := \frac{(\widetilde{u}'_\omega(r))^2}{2} + \beta_\omega \frac{(\widetilde{u}_\omega(r))^{p+1}}{p+1} + \frac{(\widetilde{u}_\omega(r))^6}{6} - \alpha_\omega \frac{(\widetilde{u}_\omega(r))^2}{2}.$$

Then, it follows from (2.4) that

$$E'(r) = \widetilde{u}'_\omega(r)\widetilde{u}''_\omega(r) + \beta_\omega \widetilde{u}^p_\omega(r)\widetilde{u}'_\omega(r) + \widetilde{u}^5_\omega(r)\widetilde{u}'_\omega(r) - \alpha_\omega \widetilde{u}_\omega(r)\widetilde{u}'_\omega(r) = -\frac{2}{r}(\widetilde{u}'_\omega(r))^2 < 0.$$

This together with $\lim_{r\to\infty} \widetilde{u}_\omega(r) = 0$, we have

$$E(r) \ge \liminf_{r\to\infty} E(r) = \liminf_{r\to\infty} \frac{(\widetilde{u}'_\omega(r))^2}{2} \ge 0 \qquad (r > 0).$$

Using $E(r) \ge 0$, $\frac{p+3}{2(p+1)}\beta_\omega \widetilde{u}^{p-1}_\omega(r) \le \alpha_\omega$ for $r \ge L_\omega$, (4.1) and (4.9), we obtain

$$\begin{aligned}\frac{(\widetilde{u}'_\omega(r))^2}{2} &\ge \alpha_\omega \frac{(\widetilde{u}_\omega(r))^2}{2} - \beta_\omega \frac{(\widetilde{u}_\omega(r))^{p+1}}{p+1} - \frac{(\widetilde{u}_\omega(r))^6}{6} \\ &\ge \alpha_\omega \frac{(\widetilde{u}_\omega(r))^2}{2} - \frac{2}{p+3}\alpha_\omega (\widetilde{u}_\omega(r))^2 - C\alpha_\omega^4 (\widetilde{u}_\omega(r))^2 \\ &= \frac{p-1}{2(p+3)}\alpha_\omega (\widetilde{u}_\omega(r))^2 - C\alpha_\omega^4 (\widetilde{u}_\omega(r))^2.\end{aligned}$$

for $r \ge \max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}$, where C is a constant independent of ω. Since $\lim_{\omega\to 0}\alpha_\omega = 0$, we have

$$(\widetilde{u}'_\omega(r))^2 \ge 2\left\{\frac{p-1}{2(p+3)}\alpha_\omega (\widetilde{u}_\omega(r))^2 - C\alpha_\omega^4 (\widetilde{u}_\omega(r))^2\right\} \ge \frac{p-1}{2(p+3)}\alpha_\omega (\widetilde{u}_\omega(r))^2. \tag{4.11}$$

Putting

$$C_p := \frac{p-1}{2(p+3)},$$

we have by (4.11) that

$$0 \le (\widetilde{u}'_\omega(r))^2 - C_p\alpha_\omega (\widetilde{u}_\omega(r))^2 = (\widetilde{u}'_\omega(r) + (C_p\alpha_\omega)^{\frac{1}{2}}\widetilde{u}_\omega(r))(\widetilde{u}'_\omega(r) - (C_p\alpha_\omega)^{\frac{1}{2}}\widetilde{u}_\omega(r))$$

for $r \ge \max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}$. This together with $\widetilde{u}'_\omega(r) < 0$ and $\widetilde{u}_\omega(r) > 0$ implies that

$$\widetilde{u}'_\omega(r) + (C_p\alpha_\omega)^{\frac{1}{2}}\widetilde{u}_\omega(r) \le 0$$

for $r \ge \max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}$. It follows that

$$\left(e^{(C_p\alpha_\omega)^{\frac{1}{2}} r}\widetilde{u}_\omega(r)\right)' = e^{(C_p\alpha_\omega)^{\frac{1}{2}} r}(\widetilde{u}'_\omega(r) + (C_p\alpha_\omega)^{\frac{1}{2}}\widetilde{u}_\omega(r)) \le 0$$

for $r \geq \max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}$. Thus, we obtain

$$\widetilde{u}_\omega(r) \leq e^{-(C_p\alpha_\omega)^{\frac{1}{2}} r} e^{(C_p\alpha_\omega)^{\frac{1}{2}} \max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}} \widetilde{u}_\omega(\max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}) \tag{4.12}$$

We see from (4.1), (4.9) and (4.12) that

$$\widetilde{u}_\omega(r) \lesssim e^{-(C_p\alpha_\omega)^{\frac{1}{2}} r} \left(1 + \frac{\gamma_\omega}{3} \max\{L_\omega^2, \alpha_\omega^{-1}\}\right)^{-\frac{1}{2}} \lesssim \sqrt{\alpha_\omega} e^{-(C_p\alpha_\omega)^{\frac{1}{2}} r} \tag{4.13}$$

for $r \geq \max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}$. Using $\limsup_{\omega\to 0} L_\omega \sqrt{\alpha_\omega} < +\infty$, (4.1) and (4.13), we obtain

$$\begin{aligned}
\|\widetilde{u}_\omega\|_{L^2}^2 &\lesssim \int_0^{\max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}} \widetilde{u}_\omega^2(r) r^2\, dr + \int_{\max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}}^\infty \widetilde{u}_\omega^2 r^2\, dr \\
&\lesssim \int_0^{\max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}} r^{-2} r^2\, dr + \int_{\max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\}}^\infty \alpha_\omega e^{-2(C_p\alpha_\omega)^{\frac{1}{2}} r} r^2\, dr \\
&\lesssim \max\{L_\omega, \alpha_\omega^{-\frac{1}{2}}\} + \alpha_\omega^{-\frac{1}{2}} \lesssim \alpha_\omega^{-\frac{1}{2}}.
\end{aligned}$$

Thus, we find that (4.6) holds.

(ii) From the definition of α_ω (see (2.5)), we see that

$$\lim_{\omega\to 0} \alpha_\omega = 0. \tag{4.14}$$

It follows from (4.6) and (4.14) that

$$\limsup_{\omega\to 0} \alpha_\omega \|\widetilde{u}_\omega\|_{L^2}^2 = 0. \tag{4.15}$$

(4.15) together with (4.3) yields that

$$\limsup_{\omega\to 0} \beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} = 0.$$

Thus, we see that (4.7) holds.

(iii) Observe from (4.1) that

$$\limsup_{\omega\to 0} \|\widetilde{u}_\omega\|_{L^6} < +\infty. \tag{4.16}$$

By (4.4), (4.7) and (4.16), we obtain that

$$\limsup_{\omega\to 0} \|\nabla\widetilde{u}_\omega\|_{L^2}^2 \leq \limsup_{\omega\to 0} \left\{\beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} + \|\widetilde{u}_\omega\|_{L^6}^6\right\} < \infty.$$

This completes the proof. □

Here, in order to prove Theorem 2.4, we recall the following compactness result by Brezis and Lieb [14].

Lemma 4.4 (Brezis and Lieb [14])**.** *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^3)$ such that*

$$\lim_{n\to\infty} u_n(x) = u(x) \qquad \textit{almost all } x \in \mathbb{R}^3$$

for some function $u \in H^1(\mathbb{R}^3)$. Then, for any $2 \leq r \leq 6$, up to some subsequence,

$$\lim_{n\to\infty} \{\|u_n\|_{L^r}^r - \|u_n - u\|_{L^r}^r - \|u\|_{L^r}^r\} = 0,$$

$$\lim_{n\to\infty} \left\{\|\nabla u_n\|_{L^2}^2 - \|\nabla(u_n - u)\|_{L^2}^2 - \|\nabla u\|_{L^2}^2\right\} = 0.$$

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. Let $\{\omega_n\}$ be any sequence in $(0, +\infty)$ satisfying $\lim_{n\to\infty} \omega_n = 0$. By the boundedness of $\|\nabla \widetilde{u}_{\omega_n}\|_{L^2}^2$ (see Lemma 4.3 (iii)), $\|\widetilde{u}_{\omega_n}\|_{L^\infty} = 1$, the $W^{2,q}_{loc}$ estimate, and Schauder's estimate (see e.g. [25, Theorems 6.2 and 8.17]), we verify that there exists $\widetilde{u}_\infty \in \dot{H}^1(\mathbb{R}^3)$ such that, passing to some subsequence,

$$\lim_{n\to\infty} \widetilde{u}_{\omega_n} = \widetilde{u}_\infty \quad \text{weakly in } \dot{H}^1(\mathbb{R}^3), \text{ and strongly in } C^2_{\mathrm{loc}}(\mathbb{R}^3). \tag{4.17}$$

In addition, we see from (2.4), (2.7) and (4.17) that $\widetilde{u}_\infty$ is a solution to

$$-\Delta \widetilde{u}_\infty - \widetilde{u}_\infty^5 = 0 \qquad \text{in } \mathbb{R}^3.$$

From (4.17) and $\widetilde{u}_{\omega_n}(0) = \|\widetilde{u}_{\omega_n}\|_{L^\infty} = 1$, one has $1 = \lim_{n\to\infty} \widetilde{u}_{\omega_n}(0) = \widetilde{u}_\infty(0)$. Thus, $\widetilde{u}_\infty$ satisfies

$$\begin{cases} -\Delta \widetilde{u}_\infty = \widetilde{u}_\infty^5 & \text{in } \mathbb{R}^3, \\ \widetilde{u}_\infty(0) = 1. \end{cases}$$

Then, by the results in [15, Corollary 8.2], we infer that $\widetilde{u}_\infty = W$, where W is the Aubin-Talenti function defined by (1.8). We put

$$\mathcal{N}^\ddagger(u) := \|\nabla W\|_{L^2}^2 - \|W\|_{L^6}^6, \qquad \mathcal{E}^\ddagger(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{6}\|u\|_{L^6}^6.$$

It is easy to verify that $\mathcal{N}^\ddagger(W) = 0$. Note that W also satisfies $\|\nabla W\|_{L^2}^2 = \sigma \|W\|_{L^6}^2$, where

$$\sigma := \inf\left\{\|\nabla u\|_{L^2}^2 : u \in \dot{H}^1(\mathbb{R}^3) \text{ with } \|u\|_{L^6} = 1\right\}. \tag{4.18}$$

By $\mathcal{N}^\ddagger(W) = 0$ and $\|\nabla W\|_{L^2}^2 = \sigma \|W\|_{L^6}^2$, we see that $\|W\|_{L^6}^6 = \sigma^{\frac{3}{2}}$ and

$$\mathcal{E}^\ddagger(W) = \frac{1}{2}\|\nabla W\|_{L^2}^2 - \frac{1}{6}\|W\|_{L^6}^6 = \frac{1}{3}\|W\|_{L^6}^6 = \frac{1}{3}\sigma^{\frac{3}{2}}. \tag{4.19}$$

This together with (4.17), $\widetilde{u}_\infty = W$ and (4.1) yields that

$$\lim_{n\to 0} \|\widetilde{u}_{\omega_n}\|_{L^6}^6 = \|W\|_{L^6}^6 = \sigma^{\frac{3}{2}}. \tag{4.20}$$

Observe from (2.4) that for each $n \in \mathbb{N}$, we have $\widetilde{\mathcal{N}}_{\omega_n}(\widetilde{u}_{\omega_n}) = 0$, where

$$\widetilde{\mathcal{N}}_\omega(u) := \|\nabla u\|_{L^2}^2 + \alpha_\omega \|u\|_{L^2}^2 - \beta_\omega \|u\|_{L^{p+1}}^{p+1} - \|u\|_{L^6}^6.$$

It follows from $\widetilde{\mathcal{N}}_{\omega_n}(\widetilde{u}_{\omega_n}) = 0$, (2.7), (4.20), (4.6) and (4.7) that

$$\lim_{n\to\infty} \mathcal{E}^\ddagger(\widetilde{u}_{\omega_n}) = \lim_{n\to\infty} \left\{ \frac{1}{3}\|\widetilde{u}_{\omega_n}\|_{L^6}^6 - \frac{\alpha_{\omega_n}}{2}\|\widetilde{u}_{\omega_n}\|_{L^2}^2 + \frac{\beta_{\omega_n}}{2}\|\widetilde{u}_{\omega_n}\|_{L^{p+1}}^{p+1} \right\} = \frac{1}{3}\sigma^{\frac{3}{2}}. \tag{4.21}$$

Similarly, we can verify that

$$\lim_{n\to\infty} \mathcal{N}^\ddagger(\widetilde{u}_{\omega_n}) = \lim_{n\to\infty} \left\{ -\alpha_{\omega_n}\|\widetilde{u}_{\omega_n}\|_{L^2}^2 + \beta_{\omega_n}\|\widetilde{u}_{\omega_n}\|_{L^{p+1}}^{p+1} \right\} = 0. \tag{4.22}$$

Using the MR699419 lemma (see Lemma 4.4), we have

$$\mathcal{N}^\ddagger(\widetilde{u}_{\omega_n}) - \mathcal{N}^\ddagger(\widetilde{u}_{\omega_n} - W) - \mathcal{N}^\ddagger(W) = o_n(1), \tag{4.23}$$

$$\mathcal{E}^\ddagger(\widetilde{u}_{\omega_n}) - \mathcal{E}^\ddagger(\widetilde{u}_{\omega_n} - W) - \mathcal{E}^\ddagger(W) = o_n(1). \tag{4.24}$$

Putting (4.22), (4.23) and $\mathcal{N}^{\ddagger}(W)=0$ together, we obtain

$$0=\lim_{n\to\infty}\mathcal{N}^{\ddagger}(\widetilde{u}_{\omega_n}-W)=\lim_{n\to\infty}\left\{\ \|\nabla(\widetilde{u}_{\omega_n}-W)\|_{L^2}^2-\|\widetilde{u}_{\omega_n}-W\|_{L^6}^6\ \right\}. \tag{4.25}$$

Furthermore, we see from (4.25) that

$$\mathcal{E}^{\ddagger}(\widetilde{u}_{\omega_n}-W)=\frac{1}{3}\|\nabla(\widetilde{u}_{\omega_n}-\widetilde{u}_{\infty})\|_{L^2}^2+o_n(1). \tag{4.26}$$

Now, we find from (4.24), (4.21) and (4.19) that

$$\mathcal{E}^{\ddagger}(\widetilde{u}_{\omega_n}-W)=\mathcal{E}^{\ddagger}(\widetilde{u}_{\omega_n})-\mathcal{E}^{\ddagger}(W)+o_n(1)=\frac{1}{3}\sigma^{\frac{3}{2}}+o_n(1)-\frac{1}{3}\sigma^{\frac{3}{2}}=o_n(1). \tag{4.27}$$

It follows from (4.26) and (4.27) that

$$\lim_{n\to\infty}\|\nabla(\widetilde{u}_{\omega_n}-W)\|_{L^2}^2=0. \tag{4.28}$$

In addition, we see from (4.1) and (4.28) that

$$\lim_{n\to\infty}\|\widetilde{u}_{\omega_n}-W\|_{L^q}=0 \qquad (q>3).$$

Since $\{\omega_n\}$ is arbitrary sequence in $(0,\infty)$ satisfying $\lim_{n\to\infty}\omega_n=0$, we have obtained the desired result. $\square$

5 Relation between α_ω and β_ω

In this section, in order to prove Theorem 2.5 (i), we shall study the relation between α_ω and β_ω (see (2.5) for the definitions of α_ω and β_ω). More precisely, we shall show the following:

Proposition 5.1. *Let α_ω and β_ω be the number given by* (2.5)*. We have the followings:*

(i) Let $2<p<3$.

$$\lim_{\omega\to0}\frac{\beta_\omega}{\sqrt{\alpha_\omega}}=\frac{12\pi(p+1)}{(5-p)\|W\|_{L^{p+1}}^{p+1}}, \tag{5.1}$$

where W is the Aubin-Talenti function defined by (1.8)*.*

(ii) Let $p=2$.

$$\lim_{\omega\to0}\frac{\beta_\omega|\log\alpha_\omega|}{\sqrt{\alpha_\omega}}=\frac{2}{\sqrt{3}}. \tag{5.2}$$

(iii) Let $1<p<2$.

$$\lim_{\omega\to0}\frac{\beta_\omega}{\alpha_\omega^{\frac{3-p}{2}}}=3^{-\frac{p-1}{2}}V^{p-1}(0), \tag{5.3}$$

where V is the unique positive solution to

$$\begin{cases} v_{ss}-v+s^{1-p}v^p=0, & \text{on } (0,\infty),\\ v_s(0)=0, \qquad \lim_{s\to\infty}v(s)=0. \end{cases} \tag{5.4}$$

Remark 5.1. *(i) Observe that*

$$W \begin{cases} \notin L^{p+1}(\mathbb{R}^3) & \text{when } 1 < p \le 2, \\ \in L^{p+1}(\mathbb{R}^3) & \text{when } p > 2. \end{cases}$$

Therefore, it is natural that the situation becomes different at $p = 2$.

(ii) Note that Wei and Wu [46] already showed the following:

$$\begin{cases} \frac{\beta_\omega}{\sqrt{\alpha_\omega}} \sim 1 & \text{when } 2 < p < 3, \\ \frac{\beta_\omega |\log \alpha_\omega|}{\sqrt{\alpha_\omega}} \sim 1 & \text{when } p = 2, \\ \frac{\beta_\omega}{\alpha_\omega^{\frac{3-p}{2}}} \sim 1 & \text{when } 1 < p < 2. \end{cases}$$

See [46, Proposition 4.2] in detail. However, in order to prove Theorem 2.5 (i), we need to show that the limits (5.1)–(5.3) *exists.*

(iii) The existence, the uniqueness and non-degeneracy of the positive solution to (5.4) *are obtained by Tolland and Genoud.*

5.1 Upper and lower bound of $\widetilde{u}_\omega$

To prove Proposition 5.8, we obtain the following lower bound of the solution $\widetilde{u}_\omega$:

Lemma 5.2 (Lower estimate of $\widetilde{u}_\omega$). *Let $1 < p < 3$ and $Y_\omega(r) := \sqrt{3}\frac{e^{-\sqrt{\alpha_\omega}|r|}}{|r|}$. Then, for any $R_0 > 0$, one has*

$$\widetilde{u}_\omega(r) \ge \frac{\widetilde{u}_\omega(R_0)}{Y_\omega(R_0)} Y_\omega(r) \qquad \text{for all } r \ge R_0. \tag{5.5}$$

Proof. Clearly, we have

$$\widetilde{u}_\omega(R_0) = \frac{\widetilde{u}_\omega(R_0)}{Y_\omega(R_0)} Y_\omega(R_0).$$

Observe that Y_ω satisfies

$$-\Delta Y_\omega + \alpha_\omega Y_\omega = 0 \qquad \text{in } \mathbb{R}^3 \setminus \{0\}. \tag{5.6}$$

In addition, it follows from (2.4) that

$$-\Delta \widetilde{u}_\omega + \alpha_\omega \widetilde{u}_\omega = \beta_\omega \widetilde{u}_\omega^p + \widetilde{u}_\omega^5 > 0. \tag{5.7}$$

Then, we see from (5.6), (5.7) and the maximal principle that (5.5) holds. □

Remark 5.2. *Observe from Theorem 2.4 that for any $R > 0$, we see that*

$$\lim_{\omega \to 0} \widetilde{u}_\omega(r) = W(r) \qquad \text{uniformly in } B(0, R),$$

Thus, there exist $R_2 > 0$ and $\omega_2 > 0$ such that for $0 < \omega < \omega_2$, we obtain

$$\widetilde{u}_\omega(r) \ge \frac{\sqrt{3}}{2r} \ge \frac{Y_\omega(r)}{2} \qquad \text{for } r = R_2.$$

This together with (5.5) *yields that for $1 < p < 3$, we obtain*

$$\widetilde{u}_\omega(r) \ge \frac{Y_\omega(r)}{2}. \qquad \text{for } r \ge R_2. \tag{5.8}$$

We also need the following upper bound of the solution $\widetilde{u}_\omega$:

Lemma 5.3 (Upper estimate of $\widetilde{u}_\omega$)**.** *Let $1 < p < 3$ and $\widetilde{u}_\omega$ be a positive solution to (2.4). For any $\varepsilon > 0$, put*

$$\widetilde{Y}_{\omega,\varepsilon}(r) := \frac{e^{-\sqrt{(1-\varepsilon)\alpha_\omega}|r|}}{|r|}.$$

Take $R_\omega > 0$ such that

$$\varepsilon \geq \frac{\beta_\omega}{\alpha_\omega}\widetilde{u}_\omega^{p-1}(R_\omega) + \frac{1}{\alpha_\omega}\widetilde{u}_\omega^4(R_\omega). \tag{5.9}$$

Then, for any $\omega > 0$, one has

$$\widetilde{u}_\omega(r) \leq \frac{\widetilde{u}_\omega(R_\omega)}{\widetilde{Y}_{\omega,\varepsilon}(R_\omega)}\widetilde{Y}_{\omega,\varepsilon}(r) \qquad \text{for } r \geq R_\omega. \tag{5.10}$$

Proof. Immediately, one has

$$\widetilde{u}_\omega(R_\omega) = \frac{\widetilde{u}_\omega(R_\omega)}{Y_{\omega,\varepsilon}(R_\omega)}Y_{\omega,\varepsilon}(R_\omega).$$

Note that $\widetilde{Y}_{\omega,\varepsilon}$ satisfies

$$-\Delta\widetilde{Y}_{\omega,\varepsilon} + (1-\varepsilon)\alpha_\omega\widetilde{Y}_{\omega,\varepsilon} = 0 \qquad \text{in } \mathbb{R}^3 \setminus \{0\}.$$

Observe from (5.9) that

$$-\Delta\widetilde{u}_\omega + (1-\varepsilon)\alpha_\omega\widetilde{u}_\omega = -\varepsilon\alpha_\omega\widetilde{u}_\omega\beta_\omega\widetilde{u}_\omega^p + \widetilde{u}_\omega^5 = (-\varepsilon\alpha_\omega\beta_\omega\widetilde{u}_\omega^{p-1} + \widetilde{u}_\omega^4)\widetilde{u}_\omega < 0.$$

Then, from the maximum principle theorem, we see that (5.10) holds. □

5.2 (Case 1) $2 < p < 3$ and (Case 2) $p = 2$

5.2.1 (Case 1) $2 < p < 3$

In this subsection, we shall give the proof of Proposition 5.1 (i). Theorem 2.4, which has been proved in Section 4, implies that the limit equation of (2.4) is (1.9). This suggests that the terms $\alpha_\omega\widetilde{u}_\omega$ and $\beta_\omega\widetilde{u}_\omega^p$ involving in (2.4) are error terms. Thus, to study the coefficients α_ω and β_ω, we need to consider the error $\eta_\omega := \widetilde{u}_\omega - W$. Then, we can easily verify that η_ω satisfies

$$(-\Delta - 5W^4 + \alpha_\omega)\eta_\omega = F_\omega(\eta_\omega), \tag{5.11}$$

where

$$F_\omega(\eta_\omega) := -\alpha_\omega W + \beta_\omega W^p + (W+\eta_\omega)^5 - W^5 - 5W^4\eta_\omega + \beta_\omega\big\{(W+\eta_\omega)^p - W^p\big\}.$$

It follows that

$$|F_\omega(\eta_\omega)| \lesssim \alpha_\omega W + \beta_\omega W^p + W^3\eta_\omega^2 + |\eta_\omega|^5 + \beta_\omega W^{p-1}|\eta_\omega| + \beta_\omega|\eta_\omega|^p. \tag{5.12}$$

We can rewrite (5.11) by

$$\eta_\omega = (1 - 5(-\Delta+\alpha_\omega)^{-1}W^4)^{-1}(-\Delta+\alpha_\omega)^{-1}F_\omega(\eta_\omega). \tag{5.13}$$

It follows from Theorem 2.4 that

$$\lim_{\omega\to 0}\eta_\omega = 0 \qquad \text{strongly in } \dot{H}^1 \cap L^q(\mathbb{R}^3)\ (q > 3). \tag{5.14}$$

Put

$$\Lambda W := \frac{1}{2}W + x \cdot \nabla W. \tag{5.15}$$

Then, we see that $\Lambda W \in \dot{H}^1(\mathbb{R}^3)$ satisfies

$$(-\Delta - 5W^4)\Lambda W = 0 \qquad \text{in } \mathbb{R}^3.$$

To prove Proposition 5.1, we recall the following resolvent expansion by Jensen and Kato [31] (see also Remark 2.3 of Coles and Gustafson [17]):

Lemma 5.4 (Lemma 4.3 of Jensen and Kato [31]). *Let s satisfy $3/2 < s < 5/2$. Then, we have the following expansion*

$$(1 - 5(-\Delta + \alpha_\omega)^{-1}W^4)^{-1} = \frac{5}{3\pi}\alpha_\omega^{-\frac{1}{2}}\langle W^4\Lambda W, \cdot\rangle \Lambda W + C_0^1 + o(1) \tag{5.16}$$

as $\omega \to 0$ in $B(H^1_{rad,-s}(\mathbb{R}^3), H^1_{rad,-s}(\mathbb{R}^3))$ for $\frac{3}{2} < s < \frac{5}{2}$. Here, C_0^1 is an explicit operator and $\|u\|_{H^1_{-s}} = \|(1+|x|^2)^{-\frac{s}{2}}u\|_{H^1}$.

It follows from (5.11), (5.13) and Lemma 5.4 that

$$\begin{aligned}\eta_\omega &= (1 - 5(-\Delta + \alpha_\omega)^{-1}W^4)^{-1}(-\Delta + \alpha_\omega)^{-1}F_\omega(\eta_\omega) \\ &= \frac{5}{3\pi}\alpha_\omega^{-\frac{1}{2}}\langle W^4\Lambda W, (-\Delta + \alpha_\omega)^{-1}F_\omega(\eta_\omega)\rangle \Lambda W \\ &\quad + C_0^1(-\Delta + \alpha_\omega)^{-1}F_\omega(\eta_\omega) + o(1) \qquad \text{in } H^1_{\mathrm{rad},-s}(\mathbb{R}^3) \text{ as } \omega \to 0.\end{aligned} \tag{5.17}$$

At least formally, we see from (5.12), (5.14), (5.17) and $\lim_{\omega\to 0}\alpha_\omega^{-\frac{1}{2}} = \infty$ (see (2.7)) that $(-\Delta + \alpha_\omega)^{-1}F_\omega$ satisfies the orthogonal condition

$$\lim_{\omega\to 0}\langle W^4\Lambda W, (-\Delta + \alpha_\omega)^{-1}F_\omega\rangle = 0. \tag{5.18}$$

In fact, we can show this rigorously. To explain this more precisely, we need several notations. For a given $\nu > 0$ and a function f on $\mathbb{R}^3$, we define $S_\nu f$ by

$$S_\nu f := \nu^{-1}f(\nu^{-2}\cdot).$$

Observe that

$$\|\nabla(S_\nu f)\|_{L^2} = \|\nabla f\|_{L^2}, \qquad \|S_\nu f\|_{L^6} = \|f\|_{L^6} \qquad \text{for all } f \in \dot{H}^1(\mathbb{R}^3).$$

We also define $\widetilde{u}_{\omega,\nu} := S_\nu \widetilde{u}_\omega$ for $\nu > 0$. We see from (2.4) that $\widetilde{u}_{\omega,\nu}$ satisfies

$$-\Delta\widetilde{u}_{\omega,\nu} + \alpha_{\omega,\nu}\widetilde{u}_{\omega,\nu} - \beta_{\omega,\nu}\widetilde{u}^p_{\omega,\nu} - \widetilde{u}^5_{\omega,\nu} = 0 \qquad \text{in } \mathbb{R}^3,$$

where $\alpha_{\omega,\nu} = \alpha_\omega\nu^{-4}$ and $\beta_{\omega,\nu} = \beta_\omega\nu^{p-5}$. We decompose $\widetilde{u}_{\omega,\nu}$ as

$$\widetilde{u}_{\omega,\nu} = W + \eta_{\omega,\nu}.$$

Then, we can verify that $\eta_{\omega,\nu}$ satisfies

$$(-\Delta - 5W^4 + \alpha_{\omega,\nu})\eta_{\omega,\nu} = F_{\omega,\nu}, \tag{5.19}$$

where

$$F_{\omega,\nu} := -\alpha_{\omega,\nu} W + \beta_{\omega,\nu} W^p + N_{\omega,\nu},$$

and

$$N_{\omega,\nu} := (W + \eta_{\omega,\nu})^5 - W^5 - 5W^4 \eta_{\omega,\nu} + \beta_{\omega,\nu} \big\{ (W + \eta_{\omega,\nu})^p - W^p \big\}.$$

We can rewrite (5.19) as

$$\eta_{\omega,\nu} = (1 - 5(-\Delta + \alpha_{\omega,\nu})^{-1} W^4)^{-1} (-\Delta + \alpha_{\omega,\nu})^{-1} F_{\omega,\nu}. \tag{5.20}$$

Then, by an argument similar to [17, Lemma 3.8], we can obtain the following result:

Lemma 5.5. *Let $1 < p < 3$. For any $\omega > 0$, there exists $\nu_\omega > 0$ with $\lim_{\omega \to 0} \nu_\omega = 1$ such that*

$$\big\langle W^4 \Lambda W,\ (-\Delta + \alpha_{\omega,\nu_\omega})^{-1} F_{\omega,\nu_\omega} \big\rangle = 0. \tag{5.21}$$

Remark 5.3. *Since $\lim_{\omega \to 0} \nu_\omega = 1$, we see from (5.21) that (5.18) holds.*

Define

$$\widetilde{u}_{2,[\omega]} := \widetilde{u}_{\omega,\nu_\omega}, \qquad \eta_{[\omega]} := \widetilde{u}_{2,[\omega]} - W, \qquad \alpha_{2,[\omega]} := \alpha_{\omega,\nu_\omega}, \qquad \beta_{2,[\omega]} := \beta_{\omega,\nu_\omega}, \qquad F_{[\omega]} := F_{\omega,\nu_\omega}.$$

where $\nu_\omega > 0$ is the number given in Lemma 5.5. Note that since $\lim_{\omega \to 0} \widetilde{u}_\omega = W$ strongly in $\dot{H}^1 \cap L^q(\mathbb{R}^3)$ $(q > 3)$ (see Theorem 2.4) and $\lim_{\omega \to 0} \nu_\omega = 1$, we have

$$\lim_{\omega \to 0} \eta_{[\omega]} = 0 \quad \text{strongly in } \dot{H}^1 \cap L^q(\mathbb{R}^3)\ (q > 3). \tag{5.22}$$

Furthermore, one can see from (5.19) that $\eta_{[\omega]}$ satisfies

$$(-\Delta - 5W^4 + \alpha_{[\omega]}) \eta_{[\omega]} = F_{[\omega]}, \tag{5.23}$$

where $F_{[\omega]} = -\alpha_{[\omega]} W + \beta_{[\omega]} W^p + N_{[\omega]}$ and

$$N_{[\omega]} := (W + \eta_{[\omega]})^5 - W^5 - 5W^4 \eta_{[\omega]} + \beta_{[\omega]} \left\{ (W + \eta_{[\omega]})^p - W^p \right\}. \tag{5.24}$$

We can rewrite (5.21) by

$$\langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} F_{[\omega]} \rangle = 0. \tag{5.25}$$

It follows from (5.25) and $F_{[\omega]} = -\alpha_{[\omega]} W + \beta_{[\omega]} W^p + N_{[\omega]}$ that

$$0 = -\alpha_{[\omega]} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} W \rangle + \beta_{[\omega]} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} W^p \rangle + \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} N_{[\omega]} \rangle. \tag{5.26}$$

Dividing (5.26) by $\sqrt{\alpha_{[\omega]}}$, we obtain

$$\begin{aligned} 0 = &-\sqrt{\alpha_{[\omega]}} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} W \rangle + \beta_{[\omega]} \alpha_{[\omega]}^{-\frac{1}{2}} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} W^p \rangle \\ &+ \alpha_{[\omega]}^{-\frac{1}{2}} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} N_{[\omega]} \rangle. \end{aligned} \tag{5.27}$$

We will use the following result obtained by Coles and Gustafson [17]:

Proposition 5.6 (Lemma 2.7 of Coles and Gustafson [17]). *(i) For any $2 < p < 5$ and any $0 < \delta_1 < p-2$, we have*

$$\langle\, W^4\Lambda W,\, (-\Delta+\alpha_{[\omega]})^{-1}W^p \,\rangle = -\frac{5-p}{10(p+1)}\|W\|_{L^{p+1}}^{p+1} + O(\alpha_{[\omega]}^{\frac{\delta_1}{2}}). \tag{5.28}$$

(ii) For any $\delta_1 > 0$, we have

$$\sqrt{\alpha_{[\omega]}}\,\langle\, W^4\Lambda W,\, (-\Delta+\alpha_{[\omega]})^{-1}W \,\rangle = -\frac{6}{5}\pi + O(\alpha_{[\omega]}^{\frac{\delta_1}{2}}). \tag{5.29}$$

We now estimate the third term of the right-hand side of (5.27). Concerning this, we claim the following:

Lemma 5.7. *Let $2 < p < 3$. Then, the following estimate holds for any sufficiently small $\varepsilon > 0$ and $\omega > 0$:*

$$\|(-\Delta+\alpha_{[\omega]})^{-1}N_{[\omega]}\|_{L^\infty} \lesssim \alpha_{[\omega]}^{\frac{p-1-(p+2)\varepsilon}{2}}. \tag{5.30}$$

The proof of Lemma 5.7 is similar to that of Lemma 2.6 of [5]. However, in [5], we consider the case where $\omega > 0$ is large, so that the details are different. For this reason, we shall give the proof of Lemma 5.7 in Subsection 5.2.4 below. Admitting Lemma 5.7, we will give a proof of Proposition 5.1 (i):

Proof of Proposition 5.1 (i). By the Hölder inequality and Lemma 5.7, one can estimate the last term on the right-hand side of (5.27) as follows: for any sufficiently small $\varepsilon > 0$,

$$\left|\langle W^4\Lambda W,\ (-\Delta+\alpha_{[\omega]})^{-1}N_{[\omega]}\rangle\right| \le \|W^4\Lambda W\|_{L^1}\|(-\Delta+\alpha_{[\omega]})^{-1}N_{[\omega]}\|_{L^\infty} \lesssim \alpha_{[\omega]}^{\frac{p-1-(p+2)\varepsilon}{2}}. \tag{5.31}$$

By (5.31) and $p > 2$, we have

$$\left|\alpha_{[\omega]}^{-\frac{1}{2}}\langle W^4\Lambda W, (-\Delta+\alpha_{[\omega]})^{-1}N_{[\omega]}\rangle\right| \lesssim \alpha_{[\omega]}^{\frac{p-2-(p+2)\varepsilon}{2}} \to 0 \qquad \text{as } \omega \to 0. \tag{5.32}$$

From (5.27), (5.29), (5.28) and (5.32), we see that

$$\begin{aligned}
\lim_{\omega\to 0}\frac{\beta_{[\omega]}}{\sqrt{\alpha_{[\omega]}}} &= \lim_{\omega\to 0}\frac{\sqrt{\alpha_{[\omega]}}\langle(-\Delta+\alpha_{[\omega]})^{-1}W, W^4\Lambda W\rangle - \alpha_{[\omega]}^{-\frac{1}{2}}\langle W^4\Lambda W, (-\Delta+\alpha_{[\omega]})^{-1}N_{[\omega]}\rangle}{\langle(-\Delta+\alpha_{[\omega]})^{-1}W^p, W^4\Lambda W\rangle} \\
&= \lim_{\omega\to 0}\frac{\sqrt{\alpha_{[\omega]}}\langle(-\Delta+\alpha_{[\omega]})^{-1}W, W^4\Lambda W\rangle}{\langle(-\Delta+\alpha_{[\omega]})^{-1}W^p, W^4\Lambda W\rangle} \\
&= -\frac{6}{5}\pi \times \left(-\frac{10(p+1)}{(5-p)\|W\|_{L^{p+1}}^{p+1}}\right) = \frac{12\pi(p+1)}{(5-p)\|W\|_{L^{p+1}}^{p+1}}.
\end{aligned}$$

Since $\lim_{\omega\to 0}\nu_\omega = 1$, we see that (5.1) holds. This completes the proof of Proposition 5.1. □

5.2.2 (Case 2) $p = 2$.

In this subsection, we shall give the proof of Proposition 5.1 (ii). We remark that when $p = 2$, the decay of W^p is slow, so that we cannot regard the term $u_{2,[\omega]}^p$ as a perturbation of W^p. Thus, we rewrite the equation (5.23) by

$$(-\Delta - 5W^4 + \alpha_{[\omega]})\eta_{[\omega]} = F_{[\omega]} = -\alpha_{[\omega]}W + \beta_{[\omega]}\widetilde{u}_{2,[\omega]}^p + \widehat{N}_{[\omega]}, \tag{5.33}$$

where

$$\widehat{N}_{[\omega]} := (W + \eta_{[\omega]})^5 - W^5 - 5W^4 \eta_{[\omega]}. \tag{5.34}$$

It follows from (5.21) and (5.33) that

$$0 = -\alpha_{[\omega]} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} W \rangle + \beta_{[\omega]} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} \widetilde{u}^p_{2,[\omega]} \rangle + \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} \widehat{N}_{[\omega]} \rangle. \tag{5.35}$$

We will use the following result:

Proposition 5.8. *Let $p = 2$. We have*

$$\lim_{\omega \to 0} |\log \alpha_{[\omega]}|^{-1} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} \widetilde{u}^p_{2,[\omega]} \rangle = -\frac{3\sqrt{3}}{5} \pi. \tag{5.36}$$

We will show Proposition 5.8 in Subsection 5.2.3 below. Next, we consider the third term of the right-hand side of (5.33). Concerning this, we claim the following:

Lemma 5.9. *Let $p = 2$. Then, the following estimate holds for any sufficiently small $\varepsilon > 0$ and $\omega > 0$:*

$$\|(-\Delta + \alpha_{[\omega]})^{-1} \widehat{N}_{[\omega]}\|_{L^\infty} \lesssim \alpha_{[\omega]}^{1-2\varepsilon}.$$

We shall give a proof of Lemma 5.9 in Subsection 5.2.4 below. We are now in a position to give the proof of Proposition 5.1 (ii):

Proof of Proposition 5.1 (ii). By the Hölder inequality and Lemma 5.9, one can estimate the last term on the right-hand side of (5.27) as follows: for any sufficiently small $\varepsilon > 0$,

$$\left| \langle W^4 \Lambda W,\ (-\Delta + \alpha_{[\omega]})^{-1} N_{[\omega]} \rangle \right| \le \|W^4 \Lambda W\|_{L^1} \|(-\Delta + \alpha_{[\omega]})^{-1} N_{[\omega]}\|_{L^\infty} \lesssim \alpha_{[\omega]}^{1-p\varepsilon}. \tag{5.37}$$

By (5.37), we have

$$\left| \alpha_{[\omega]}^{-\frac{1}{2}} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} N_{[\omega]} \rangle \right| \lesssim \alpha_{[\omega]}^{\frac{1}{2}-p\varepsilon} \to 0 \qquad \text{as } \omega \to 0. \tag{5.38}$$

From (5.35), (5.29), (5.36) and (5.38), we see that

$$\begin{aligned}
&\lim_{\omega \to 0} \frac{\beta_{[\omega]} |\log \alpha_{[\omega]}|}{\sqrt{\alpha_{[\omega]}}} \\
&= \lim_{\omega \to 0} \frac{|\log \alpha_{[\omega]}|}{\sqrt{\alpha_{[\omega]}}} \left\{ \frac{\alpha_{[\omega]} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} W \rangle - \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} \widehat{N}_{[\omega]} \rangle}{\langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} \widetilde{u}^p_{\omega,\nu_\omega} \rangle} \right\} \\
&= \lim_{\omega \to 0} \frac{\sqrt{\alpha_{[\omega]}} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} W \rangle - \alpha_{[\omega]}^{-\frac{1}{2}} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} \widehat{N}_{[\omega]} \rangle}{|\log \alpha_{[\omega]}|^{-1} \langle W^4 \Lambda W, (-\Delta + \alpha_{[\omega]})^{-1} \widetilde{u}^p_{\omega,\nu_\omega} \rangle} \\
&= \frac{-\frac{6}{5}\pi}{-\frac{3\sqrt{3}}{5}\pi} = \frac{2}{\sqrt{3}}.
\end{aligned}$$

Since $\lim_{\omega \to 0} \nu_\omega = 1$, we see that (5.2) holds. This completes the proof. □

5.2.3 Estimate of the principle term (proof of Proposition 5.8)

In this subsection, we will prove Proposition 5.8. We shall use the following inequalities which we can obtain from the Young inequality (see, e.g., (9) in Section 4.3 of [36])

Lemma 5.10. *Let $\alpha > 0$, and let $1 \le s \le q \le \infty$ and $3(\frac{1}{s} - \frac{1}{q}) < 2$. Then, we have*

$$\|(-\Delta + \alpha)^{-1}\|_{L^s(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} \lesssim \alpha^{\frac{3}{2}(\frac{1}{s} - \frac{1}{q}) - 1}. \tag{5.39}$$

We are now in a position to prove Proposition 5.8.

Proof of Proposition 5.8. We shall divide the proof into 3 steps.

(Step 1). Note that

$$W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}} = \sqrt{3}|x|^{-1}\left(1 + 3|x|^{-2}\right)^{-\frac{1}{2}} = \sqrt{3}|x|^{-1} + O(|x|^{-3}) \qquad \text{as } |x| \to \infty,$$

$$x \cdot \nabla W = -\frac{|x|^2}{3}\left(1 + \frac{|x|^2}{3}\right)^{-\frac{3}{2}} = -\sqrt{3}|x|^{-1}\left(1 + 3|x|^{-2}\right)^{-\frac{3}{2}} = -\sqrt{3}|x|^{-1} + O(|x|^{-3}) \qquad \text{as } |x| \to \infty.$$

Thus, if we put $Z := \Lambda W - \left(-\frac{\sqrt{3}}{2}|x|^{-1}\right)$, we find from $\Lambda W = \frac{1}{2}W + x \cdot \nabla W$ that $Z = O(|x|^{-3})$ as $|x| \to \infty$. In addition, since $-\Delta \Lambda W = 5W^4 \Lambda W$ and $\Lambda W = -\frac{\sqrt{3}}{2}|x|^{-1} + Z$, we have

$$\begin{aligned}\langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, W^4 \Lambda W \rangle &= \frac{1}{5}\langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, -\Delta \Lambda W \rangle \\ &= -\frac{\sqrt{3}}{10}\langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, -\Delta |x|^{-1} \rangle + \frac{1}{5}\langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, -\Delta Z \rangle.\end{aligned} \tag{5.40}$$

Recall (see, e.g., Section 6.23 of [36]) that for any $\alpha > 0$ and any function f on $\mathbb{R}^3$,

$$(-\Delta + \alpha)^{-1} f(x) = \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\alpha}|x-y|}}{4\pi|x-y|} f(y) dy. \tag{5.41}$$

In addition, note that $-\Delta |x|^{-1} = 4\pi\delta_0$ in $\mathcal{D}'(\mathbb{R}^3)$ (see e.g. [36, Page 156]). Thus, it follows that

$$\begin{aligned}-\frac{\sqrt{3}}{10}\langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, (-\Delta)|x|^{-1} \rangle &= -\frac{2\sqrt{3}\pi}{5}\left\{(-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p\right\}(0) \\ &= -\frac{\sqrt{3}}{10}\int_{\mathbb{R}^3} \frac{e^{-\sqrt{\alpha_\omega}|x|}}{|x|} \widetilde{u}_\omega^p(x)\, dx.\end{aligned} \tag{5.42}$$

By the Hölder inequality and Lemma 5.10, $\widetilde{u}_\omega^p = O(|x|^{-p})$ and $Z = O(|x|^{-3})$ as $|x| \to \infty$, we have

$$\begin{aligned}&\left|\langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, \Delta Z \rangle\right| = \left|\langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, \{\alpha_\omega - (-\Delta + \alpha_\omega)\} Z \rangle\right| \\ &\le \alpha_\omega \|(-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p\|_{L^6} \|Z\|_{L^{\frac{6}{5}}} + \|\widetilde{u}_\omega^p\|_{L^6} \|Z\|_{L^{\frac{6}{5}}} \lesssim \alpha_\omega \times \alpha_\omega^{\frac{3}{8}p - \frac{5}{4}} \|\widetilde{u}_\omega^p\|_{L^{\frac{4}{p}}} \|Z\|_{L^{\frac{6}{5}}} + 1 \lesssim 1.\end{aligned} \tag{5.43}$$

From (5.40), (5.42) and (5.43), we obtain

$$\langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, W^4 \Lambda W \rangle = -\frac{\sqrt{3}}{10}\int_{\mathbb{R}^3} \frac{e^{-\sqrt{\alpha_\omega}|x|}}{|x|} \widetilde{u}_\omega^p(x)\, dx + O(1). \tag{5.44}$$

(Step 2). From (5.44) and (5.5), we obtain

$$
\begin{aligned}
&\langle(-\Delta+\alpha_\omega)^{-1}\widetilde{u}_\omega^p, W^4\Lambda W\rangle \\
&= -\frac{\sqrt{3}}{10}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_\omega}|x|}}{|x|}\widetilde{u}_\omega^p(x)\,dx + O(1) \\
&\le -\frac{\sqrt{3}(1-\varepsilon)^p}{10}\times 4\pi\int_{R_\varepsilon}^\infty \frac{e^{-\sqrt{\alpha_\omega}r}}{r}Y_\omega^p(r)\times r^2\,dr + O(1) \\
&= -\frac{2(1-\varepsilon)^p\pi}{5}3^{\frac{p+1}{2}}\int_{R_\varepsilon}^\infty \frac{e^{-(p+1)\sqrt{\alpha_\omega}r}}{r^{p-1}}\,dr + O(1) \\
&= -\frac{2(1-\varepsilon)^p\pi}{5}3^{\frac{p+1}{2}}\left\{\int_{R_\varepsilon}^{\alpha_\omega^{-\frac{1}{2}}}\frac{e^{-(p+1)\sqrt{\alpha_\omega}r}}{r^{p-1}}\,dr + \int_{\alpha_\omega^{-\frac{1}{2}}}^\infty \frac{e^{-(p+1)\sqrt{\alpha_\omega}r}}{r^{p-1}}\,dr\right\} + O(1) \\
&= -\frac{2(1-\varepsilon)^p\pi}{5}3^{\frac{p+1}{2}}\left\{\int_{R_\varepsilon}^{\alpha_\omega^{-\frac{1}{2}}}\frac{1}{r^{p-1}}\,dr + \int_{R_\varepsilon}^{\alpha_\omega^{-\frac{1}{2}}}\frac{e^{-(p+1)\sqrt{\alpha_\omega}r}-1}{r^{p-1}}\,dr + \int_{\alpha_\omega^{-\frac{1}{2}}}^\infty \frac{e^{-(p+1)\sqrt{\alpha_\omega}r}}{r^{p-1}}\,dr\right\} + O(1).
\end{aligned}
$$

Then, since $p=2$, we have

$$
\int_{R_\varepsilon}^{\alpha_\omega^{-\frac{1}{2}}}\frac{dr}{r^{p-1}} = \int_{R_\varepsilon}^{\alpha_\omega^{-\frac{1}{2}}}\frac{dr}{r} = \frac{|\log\alpha_\omega|}{2} - \log R_\varepsilon, \qquad \left|\int_1^{\alpha_\omega^{-\frac{1}{2}}}\frac{e^{-3\sqrt{\alpha_\omega}r}-1}{r^{p-1}}\,dr\right| \le \int_1^{\alpha_\omega^{-\frac{1}{2}}}\frac{\sqrt{\alpha_\omega}r}{r}\,dr \lesssim 1,
$$

where we have used the elementary inequality $e^{-3\sqrt{\alpha_\omega}r} - 1 - \sqrt{\alpha_\omega}r < 0$ for all $r>0$ in the second inequality. In addition, since $p=2$, we obtain

$$
\left|\int_{\alpha_\omega^{-\frac{1}{2}}}^\infty \frac{e^{-(p+1)\sqrt{\alpha_\omega}r}}{r^{p-1}}\,dr\right| \le \int_1^\infty \frac{e^{-3s}}{s}\,ds \lesssim 1.
$$

These together with $p=2$ imply that

$$
\limsup_{\omega\to0}|\log\alpha_\omega|^{-1}\langle(-\Delta+\alpha_\omega)^{-1}\widetilde{u}_\omega^p, W^4\Lambda W\rangle \le -\frac{2(1-\varepsilon)^2\pi}{5}3^{\frac{3}{2}}\times\frac{1}{2} = -\frac{(1-\varepsilon)^2}{5}3^{\frac{3}{2}}\pi. \tag{5.45}
$$

We see from (4.1) that

$$
\widetilde{u}_\omega(x) \le 3^{\frac{1}{2}}\gamma_\omega^{-\frac{1}{2}}r^{-1} \qquad \text{for all } r>1,
$$

where $\gamma_\omega = -\alpha_\omega + \beta_n + 1$. This together with $0 < \widetilde{u}_\omega(x) \le \widetilde{u}_\omega(0) = 1$ and $p=2$ implies that

$$
\begin{aligned}
\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_\omega}|x|}}{|x|}\widetilde{u}_\omega^p\,dx &\le 4\pi\int_0^1 e^{-\sqrt{\alpha_\omega}r}r\,dr + 4\pi\times3^{\frac{p}{2}}\gamma_\omega^{-\frac{p}{2}}\int_1^{\alpha_\omega^{-\frac{1}{2}}}\frac{e^{-\sqrt{\alpha_\omega}r}}{r^{p-1}}\,dr + 4\pi\times3^{\frac{p}{2}}\gamma_\omega^{-\frac{p}{2}}\int_{\alpha_\omega^{-\frac{1}{2}}}^\infty\frac{e^{-\sqrt{\alpha_\omega}r}}{r^{p-1}}\,dr \\
&\le O(1) + 4\pi\times3\gamma_\omega^{-1}\int_1^{\alpha_\omega^{-\frac{1}{2}}}\frac{1}{r}\,dr + 4\pi\times3\gamma_\omega^{-1}\int_{\alpha_\omega^{-\frac{1}{2}}}^\infty\frac{e^{-\sqrt{\alpha_\omega}r}}{r}\,dr \\
&= 2\pi\times3\gamma_\omega^{-1}|\log\alpha_\omega| + O(1).
\end{aligned}
\tag{5.46}
$$

It follows from (5.44), (5.46) and $\lim_{\omega\to0}\gamma_\omega = 1$ that

$$
\liminf_{\omega\to0}|\log\alpha_\omega|^{-1}\langle(-\Delta+\alpha_\omega)^{-1}\widetilde{u}_\omega^p, W^4\Lambda W\rangle \ge -\frac{\sqrt{3}}{10}\times2\pi\times3 = -\frac{3^{\frac{3}{2}}}{5}\pi. \tag{5.47}
$$

Since $\varepsilon > 0$ is arbitrary, we see from (5.45) and (5.47) that

$$\lim_{\omega \to 0} |\log \alpha_\omega|^{-1} \langle (-\Delta + \alpha_\omega)^{-1} \widetilde{u}_\omega^p, W^4 \Lambda W \rangle = -\frac{3^{\frac{3}{2}}}{5}\pi.$$

Thus, (5.36) holds. This concludes the proof. □

5.2.4 Estimate of the remainder term (Proof of Lemmas 5.7 and 5.9)

In this subsection, we give the proof of Lemmas 5.7 and 5.9. To this end, we need several preparations. We will use the following inequalities (see, e.g., [17, (2.10)])

Lemma 5.11. *Let $\alpha > 0$. For any $\varepsilon > 0$, we obtain*

$$\|(-\Delta + \alpha)^{-1}\|_{L^{\frac{3}{2}+\varepsilon}(\mathbb{R}^3) \cap L^{\frac{3}{2}-\varepsilon}(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)} \lesssim 1. \tag{5.48}$$

Lemma 5.10 does not deal with the case where $3(\frac{1}{s} - \frac{1}{q}) = 2$. In that case, we use the following:

Lemma 5.12. *Let $\alpha > 0$, and let $3 < q < \infty$. Then, we have*

$$\|(-\Delta + \alpha)^{-1}\|_{L^{\frac{3q}{3+2q}}(\mathbb{R}^3) \to L^q(\mathbb{R}^3)} \lesssim 1.$$

We also need the following L^∞-estimate of $(-\Delta + \alpha)^{-1}|x|^{-1}$:

Lemma 5.13 (Lemma 3.13 of [28])**.** *The following estimate holds for all $\alpha > 0$:*

$$\|(-\Delta + \alpha)^{-1}|x|^{-1}\|_{L^\infty} \lesssim \alpha^{-\frac{1}{2}},$$

where the implicit constant is independent of α.

From the resolvent expansion in Lemma 5.4, Coles and Gustafson [17, Lemma 2.4] obtained the following resolvent estimate:

Lemma 5.14 (Coles and Gustafson [17])**.** *Let $3 < r \le \infty$.*

$$\|(1 - 5(-\Delta + \alpha_\omega)^{-1} W^4)^{-1} f\|_{L^r} \lesssim \|f\|_{L^r} \tag{5.49}$$

for any $f \in L^r_{rad}(\mathbb{R}^3)$ with $\langle W^4 \Lambda W, f \rangle = 0$.

Remark 5.4. *Note that the resolvent expansion* (5.16) *has a singularity $\alpha_\omega^{-\frac{1}{2}}$. However, we see from Lemma 5.14 that if we have the orthogonal condition $\langle W^4 \Lambda W, f \rangle = 0$, we can remove the singularity in the resolvent estimate* (5.49)*.*

Next, we estimate β_ω by α_ω.

Lemma 5.15. *Let $\widetilde{u}_\omega$ be the positive solution to* (2.4)*.*

(i) Let $2 < p < 3$. There exists a constant $C_1 > 0$ such that

$$\beta_\omega \le C_1 \sqrt{\alpha_\omega}. \tag{5.50}$$

(ii) Let $p=2$. There exists a constant $C_2>0$ such that

$$\beta_\omega \le C_2 \frac{\sqrt{\alpha_\omega}}{|\log \alpha_\omega|}. \tag{5.51}$$

(iii) Let $1<p<2$. There exists a constant $C_3>0$ such that

$$\beta_\omega \le C_3 \alpha_\omega^{\frac{3-p}{2}}. \tag{5.52}$$

We now give the proof of Lemma 5.15.

Proof of Lemma 5.15. (i) We first consider the case of $2<p<3$. It follows from Theorem 2.4 and $W \in L^{p+1}(\mathbb{R}^3)$ that there exists $C_p>0$ such that

$$\|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} \ge C_p$$

for $2<p<3$. In addition, by Lemma 4.3 (i), we have

$$\|\widetilde{u}_\omega\|_{L^2}^2 \le C\alpha_\omega^{-\frac{1}{2}} \tag{5.53}$$

for some $C>0$. These together with (2.8) yields that

$$\frac{5-p}{2(p+1)} C_p \beta_\omega \le \frac{5-p}{2(p+1)} \beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} = \alpha_\omega \|\widetilde{u}_\omega\|_{L^2}^2 \le C\sqrt{\alpha_\omega}.$$

Thus, (5.50) holds.

(ii) We now consider the case of $p=2$. By (5.5) and $p=2$, we have

$$\|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} \ge C\int_{R_2}^{\alpha_\omega^{-\frac{1}{2}}} \frac{e^{-(p+1)\sqrt{\alpha_\omega}r}}{r^{p-1}}\,dr \ge C\int_{R_2}^{\alpha_{2,\omega}^{-\frac{1}{2}}} \frac{1}{r}\,dr \ge C(|\log \alpha_\omega| - \log R_2).$$

It follows from (2.8) and (4.6) that

$$C\beta_\omega |\log \alpha_\omega| \le \frac{\beta_\omega}{2} \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} = \alpha_\omega \|\widetilde{u}_\omega\|_{L^2}^2 \le C\sqrt{\alpha_\omega}$$

This yields (5.51).

(iii) Finally, we consider the case of $1<p<2$. Putting $s=\sqrt{\alpha_\omega}r$, we see from (5.8) that

$$\|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} \ge C\int_{\alpha_\omega^{-\frac{1}{2}}}^{\infty} \frac{e^{-(p+1)\sqrt{\alpha_\omega r}}}{r^{p-1}}\,dr = C\alpha_\omega^{\frac{p-2}{2}} \int_1^\infty \frac{e^{-(p+1)s}}{s^{p-1}}\,ds \ge C\alpha_\omega^{\frac{p-2}{2}}$$

for some $C>0$. By (2.8) and (4.6), one has

$$C\beta_\omega \alpha_\omega^{\frac{p-2}{2}} \le \frac{5-p}{2(p+1)} \beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} = \alpha_\omega \|\widetilde{u}_\omega\|_{L^2}^2 \le C\sqrt{\alpha_\omega}.$$

Therefore, (5.52) holds. □

Next, we obtain the following estimate of $\|\eta_{[\omega]}\|_{L^\infty}$.

Lemma 5.16 (c.f. Lemma 2.5 of [5])**.** *Let $2<p<3$. For any small $\omega>0$, the following estimate holds:*

$$\|\eta_{[\omega]}\|_{L^\infty} \lesssim \sqrt{\alpha_{[\omega]}}. \tag{5.54}$$

Proof. We shall divide the proof into 2 steps.

(Step 1). By (5.20), Lemma 5.14 with $f = (-\Delta + \alpha_{[\omega]})^{-1}F_{[\omega]}$, (5.25) and $F_{[\omega]} = -\alpha_{[\omega]}W + \beta_{[\omega]}W^p + N_{[\omega]}$, one can see that

$$\begin{aligned}\|\eta_{[\omega]}\|_{L^\infty} &= \|(1 - 5(-\Delta + \alpha_{[\omega]})^{-1}W^4)^{-1}(-\Delta + \alpha_{[\omega]})^{-1}F_{[\omega]}\|_{L^\infty} \\ &\le \alpha_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}W\|_{L^\infty} + \beta_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}W^p\|_{L^\infty} + \|(-\Delta + \alpha_{[\omega]})^{-1}N_{[\omega]}\|_{L^\infty}.\end{aligned} \tag{5.55}$$

We consider the first two terms on the right-hand side of (5.55). It follows from Lemmas 5.10 and 5.13, (1.8), (5.50) and $p > 2$ that

$$\begin{aligned}\alpha_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}W\|_{L^\infty} + \beta_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}W^p\|_{L^\infty} &\lesssim \alpha_{[\omega]}\alpha_{[\omega]}^{-\frac{1}{2}} + \sqrt{\alpha_{[\omega]}}\alpha_{[\omega]}^{\frac{p-2}{2}-\frac{p}{2}\varepsilon}\|W^p\|_{L^{\frac{3}{p(1-\varepsilon)}}} \\ &\lesssim \sqrt{\alpha_{[\omega]}} + \alpha_{[\omega]}^{\frac{p-1}{2}-\frac{p}{2}\varepsilon} \lesssim \sqrt{\alpha_{[\omega]}}.\end{aligned} \tag{5.56}$$

Concerning the last term on the right-hand side of (5.55), we claim that

$$\|(-\Delta + \alpha_{[\omega]})^{-1}N_{[\omega]}\|_{L^\infty} \le o_n(\|\eta_{[\omega]}\|_{L^\infty}). \tag{5.57}$$

Once we obtain (5.57), we see that the last term on the right-hand side of (5.55) can be absorbed into the left-hand side. Hence, the claim (5.54) follows from (5.55)–(5.57).

(Step 2). It remains to prove (5.57). It follows from (5.24) and the triangle inequality that

$$\begin{aligned}\|(-\Delta + \alpha_{[\omega]})^{-1}N_{[\omega]}\|_{L^\infty} &\le \|(-\Delta + \alpha_{[\omega]})^{-1}\left\{ (W + \eta_{[\omega]})^5 - W^5 - 5W^4\eta_{[\omega]} \right\}\|_{L^\infty} \\ &\quad + \beta_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}\{(W + \eta_{[\omega]})^p - W^p\}\|_{L^\infty}.\end{aligned} \tag{5.58}$$

We consider the first term on the right-hand side of (5.58). By Lemma 5.11, elementary computations, the Hölder inequality and (5.22), one can see that

$$\begin{aligned}&\|(-\Delta + \alpha_{[\omega]})^{-1}\left\{ (W + \eta_{[\omega]})^5 - W^5 - 5W^4\eta_{[\omega]} \right\}\|_{L^\infty} \\ &\lesssim \left\| (W + \eta_{[\omega]})^5 - W^5 - 5W^4\eta_{[\omega]} \right\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}} \\ &\lesssim \|W^3|\eta_{[\omega]}|^2\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}} + \|\eta_{[\omega]}^5\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}} \\ &\le \|W^3\|_{L^{\frac{6(3+\varepsilon)}{9-\varepsilon}} \cap L^{\frac{6(3-\varepsilon)}{9+\varepsilon}}}\|\eta_{[\omega]}\|_{L^6}\|\eta_{[\omega]}\|_{L^\infty} + \|\eta_{[\omega]}^4\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}}\|\eta_{[\omega]}\|_{L^\infty} \\ &= o_n(1)\|\eta_{[\omega]}\|_{L^\infty}.\end{aligned} \tag{5.59}$$

Next, we consider the second term on the right-hand side of (5.58). It follows from elementary computations and Lemma 5.10 that

$$\begin{aligned}&\beta_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}\{(W + \eta_{[\omega]})^p - W^p\}\|_{L^\infty} \\ &\lesssim \beta_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}\left\{ W^{p-1}|\eta_{[\omega]}| \right\}\|_{L^\infty} + \beta_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}\left\{ |\eta_{[\omega]}|^p \right\}\|_{L^\infty} \\ &\lesssim \beta_{[\omega]}\alpha_{[\omega]}^{\frac{p-3-\varepsilon}{2}}\|W^{p-1}|\eta_{[\omega]}|\|_{L^{\frac{3}{p-1-\varepsilon}}} + \beta_{[\omega]}\alpha_{[\omega]}^{\frac{p-3-\varepsilon}{2}}\||\eta_{[\omega]}|^p\|_{L^{\frac{3}{p-1-\varepsilon}}}.\end{aligned} \tag{5.60}$$

Consider the first term on the right-hand side of (5.60). By the Hölder inequality and (5.50), one can see that

$$\beta_{[\omega]}\alpha_{[\omega]}^{\frac{p-3-\varepsilon}{2}}\|W^{p-1}|\eta_{[\omega]}|\|_{L^{\frac{3}{p-1-\varepsilon}}} \lesssim \alpha_{[\omega]}^{\frac{p-2-\varepsilon}{2}}\|W\|_{L^{\frac{3(p-1)}{p-1-\varepsilon}}}^{p-1}\|\eta_{[\omega]}\|_{L^\infty} \lesssim \alpha_{[\omega]}^{\frac{p-2-\varepsilon}{2}}\|\eta_{[\omega]}\|_{L^\infty}.$$

Similarly, we obtain

$$\beta_{[\omega]}\alpha_{[\omega]}^{\frac{p-3-\varepsilon}{2}}\||\eta_{[\omega]}|^{p}\|_{L^{\frac{3}{p-1-\varepsilon}}} \lesssim \alpha_{[\omega]}^{\frac{p-2-\varepsilon}{2}}\|\eta_{[\omega]}\|_{L^{\frac{3(p-1)}{p-1-\varepsilon}}}^{p-1}\|\eta_{[\omega]}\|_{L^{\infty}} \lesssim \alpha_{[\omega]}^{\frac{p-2-\varepsilon}{2}}\|\eta_{[\omega]}\|_{L^{\infty}}.$$

Therefore, we obtain

$$\beta_{[\omega]}\|(-\Delta+\alpha_{[\omega]})^{-1}\big\{(W+\eta_{[\omega]})^{p}-W^{p}\big\}\|_{L^{\infty}} \lesssim \alpha_{[\omega]}^{\frac{p-2-\varepsilon}{2}}\|\eta_{[\omega]}\|_{L^{\infty}} = o_{n}(1)\|\eta_{[\omega]}\|_{L^{\infty}}. \tag{5.61}$$

Putting the estimates (5.58), (5.59) and (5.61) together, we obtain the desired estimate (5.57). This completes the proof. □

We are now in a position to prove Lemma 5.7.

Proof of Lemma 5.7. As in (5.58), we see that

$$\begin{aligned}\|(-\Delta+\alpha_{[\omega]})^{-1}N_{[\omega]}\|_{L^{\infty}} \le \|(-\Delta+\alpha_{[\omega]})^{-1}\ \big\{\ (W+\eta_{[\omega]})^{5}-W^{5}-5W^{4}\eta_{n}\ \big\}\ \|_{L^{\infty}} \\ +\beta_{[\omega]}\|(-\Delta+\alpha_{[\omega]})^{-1}\big\{|W+\eta_{[\omega]}|^{p-1}(W+\eta_{[\omega]})-W^{p}\big\}\|_{L^{\infty}}.\end{aligned} \tag{5.62}$$

We consider the first term on the right-hand side of (5.62). By Lemma 5.12, (5.39), the Hölder inequality and Lemma 5.16, one can see that

$$\begin{aligned}&\|(-\Delta+\alpha_{[\omega]})^{-1}\ \big\{\ (W+\eta_{[\omega]})^{5}-W^{5}-5W^{4}\eta_{n}\ \big\}\ \|_{L^{\infty}} \\ &\lesssim \|(-\Delta+\alpha_{[\omega]})^{-1}\big\{W^{3}|\eta_{[\omega]}|^{2}\big\}\|_{L^{\infty}} + \|(-\Delta+\alpha_{[\omega]})^{-1}|\eta_{[\omega]}|^{5}\|_{L^{\infty}} \\ &\lesssim \|W^{3}|\eta_{[\omega]}|^{2}\|_{L^{\frac{3}{2}+\varepsilon}\cap L^{\frac{3}{2}-\varepsilon}} + \||\eta_{[\omega]}|^{5}\|_{L^{\frac{3}{2}+\varepsilon}\cap L^{\frac{3}{2}-\varepsilon}} \\ &\le \|W\|_{L^{\frac{9}{2}+3\varepsilon}\cap L^{\frac{9}{2}-3\varepsilon}}^{3}\|\eta_{[\omega]}\|_{L^{\infty}}^{2} + \|\eta_{[\omega]}\|_{L^{\frac{9}{2}+3\varepsilon}\cap L^{\frac{9}{2}-3\varepsilon}}^{3}\|\eta_{[\omega]}\|_{L^{\infty}}^{2} \lesssim \alpha_{[\omega]}.\end{aligned} \tag{5.63}$$

Next, we consider the second term on the right-hand side of (5.62). It follows from (5.39), the Hölder inequality, (5.51) and Lemma 5.16 that if $\varepsilon>0$ is sufficiently small, we have

$$\begin{aligned}&\beta_{[\omega]}\|(-\Delta+\alpha_{[\omega]})^{-1}\ \big\{\ (W+\eta_{[\omega]})^{p}-W^{p}\ \big\}\ \|_{L^{\infty}} \\ &\lesssim \beta_{[\omega]}\|(-\Delta+\alpha_{[\omega]})^{-1}\big\{W^{p-1}\eta_{[\omega]}\big\}\|_{L^{\infty}} + \beta_{[\omega]}\|(-\Delta+\alpha_{[\omega]})^{-1}|\eta_{[\omega]}|^{p}\|_{L^{\infty}} \\ &\lesssim \beta_{[\omega]}\alpha_{[\omega]}^{\frac{p-3-(p+2)\varepsilon}{2}}\|W^{p-1}\eta_{[\omega]}\|_{L^{\frac{3}{p-1-(p-1)\varepsilon}}} + \beta_{[\omega]}\alpha_{[\omega]}^{\frac{p-3-(p+2)\varepsilon}{2}}\||\eta_{[\omega]}|^{p}\|_{L^{\frac{3}{p-1-(p-1)\varepsilon}}} \\ &\lesssim \alpha_{[\omega]}^{\frac{p-2-(p+2)\varepsilon}{2}}\|W\|_{L^{\frac{3(p-1)}{p-1-(p-1)\varepsilon}}}^{p-1}\|\eta_{[\omega]}\|_{L^{\infty}} + \alpha_{[\omega]}^{\frac{p-2-(p+2)\varepsilon}{2}}\|\eta_{[\omega]}\|_{L^{\frac{3(p-1)}{p-1-(p-1)\varepsilon}}}^{p-1}\|\eta_{[\omega]}\|_{L^{\infty}} \\ &\lesssim \alpha_{[\omega]}^{\frac{p-1-(p+2)\varepsilon}{2}}\end{aligned} \tag{5.64}$$

Putting (5.62), (5.63) and (5.64) together, we obtain the desired estimate (5.30). □

Next, we shall give the proof of Lemma 5.9, which is similar to that of Lemma 5.7.

Lemma 5.17. *Let $p=2$. For sufficiently small $\varepsilon>0$ and $\omega>0$, the following estimate holds:*

$$\|\eta_{[\omega]}\|_{L^{\infty}} \lesssim \alpha_{[\omega]}^{\frac{1}{2}-\varepsilon}. \tag{5.65}$$

Proof. Similarly to (5.55), by (5.20), Lemma 5.14 with $f = (-\Delta + \alpha_{[\omega]})^{-1}F_{[\omega]}$, and (5.21), one can see that

$$\begin{aligned}\|\eta_{[\omega]}\|_{L^\infty} &\lesssim \|(-\Delta + \alpha_{[\omega]} - 5W^4)^{-1}F_{[\omega]}\|_{L^\infty}\\ &= \|(1 - 5(-\Delta + \alpha_{[\omega]})^{-1}W^4)^{-1}(-\Delta + \alpha_{[\omega]})^{-1}F_{[\omega]}\|_{L^\infty}\\ &\le \alpha_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}W\|_{L^\infty} + \beta_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}\widetilde{u}^p_{\omega,\nu_\omega}\|_{L^\infty} + \|(-\Delta + \alpha_{[\omega]})^{-1}\widehat{N}_{[\omega]}\|_{L^\infty}.\end{aligned} \tag{5.66}$$

We consider the first two terms on the right-hand side of (5.66). Since $p = 2$, we have by Lemmas 5.10 and 5.13, (5.51) that

$$\begin{aligned}&\alpha_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}W\|_{L^\infty} + \beta_{[\omega]}\|(-\Delta + \alpha_{[\omega]})^{-1}\widetilde{u}^p_{\omega,\nu_\omega}\|_{L^\infty}\\ &\lesssim \alpha_{[\omega]}\alpha_{[\omega]}^{-\frac{1}{2}} + \frac{\sqrt{\alpha_{[\omega]}}}{|\log \alpha_{[\omega]}|}\alpha_{[\omega]}^{-\varepsilon}\|\widetilde{u}^p_{\omega,\nu_\omega}\|_{L^{\frac{3}{2(1-\varepsilon)}}}\\ &\lesssim \sqrt{\alpha_{[\omega]}} + \alpha_{[\omega]}^{\frac{1}{2}-\varepsilon} \lesssim \alpha_{[\omega]}^{\frac{1}{2}-\varepsilon}.\end{aligned} \tag{5.67}$$

Concerning the last term on the right-hand side of (5.66). We claim that

$$\|(-\Delta + \alpha_{[\omega]})^{-1}\widehat{N}_{[\omega]}\|_{L^\infty} \le o_n(\|\eta_{[\omega]}\|_{L^\infty}). \tag{5.68}$$

Indeed, as in (5.59), by (5.34), Lemma 5.11, elementary computations, the Hölder inequality and (5.22), one can see that

$$\begin{aligned}\|(-\Delta + \alpha_{[\omega]})^{-1}\widehat{N}_{[\omega]}\|_{L^\infty} &\le \|(-\Delta + \alpha_{[\omega]})^{-1}\left\{ (W + \eta_{[\omega]})^5 - W^5 - 5W^4\eta_{[\omega]} \right\}\|_{L^\infty}\\ &\lesssim \left\| (W + \eta_{[\omega]})^5 - W^5 - 5W^4\eta_{[\omega]} \right\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}}\\ &\lesssim \|W^3\eta_{[\omega]}^2\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}} + \|\eta_{[\omega]}^5\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}}\\ &\le \|W^3\|_{L^{\frac{6(3+\varepsilon)}{9-\varepsilon}} \cap L^{\frac{6(3-\varepsilon)}{9+\varepsilon}}}\|\eta_{[\omega]}\|_{L^6}\|\eta_{[\omega]}\|_{L^\infty} + \|\eta_{[\omega]}^4\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}}\|\eta_{[\omega]}\|_{L^\infty} = o_n(1)\|\eta_{[\omega]}\|_{L^\infty}.\end{aligned}$$

Thus, we obtain the desired estimate (5.68).

Notice that by (5.68), the last term on the right-hand side of (5.66) can be absorbed into the left-hand side. Hence, (5.65) holds from (5.66), (5.67) and (5.68). □

We now give the proof of Lemma 5.9.

Proof of Lemma 5.9. As in (5.63), we see from (5.65) that

$$\begin{aligned}\|(-\Delta + \alpha_{[\omega]})^{-1}\widehat{N}_{[\omega]}\|_{L^\infty} &\le \|(-\Delta + \alpha_{[\omega]})^{-1}\left\{ (W + \eta_{[\omega]})^5 - W^5 - 5W^4\eta_n \right\}\|_{L^\infty}\\ &\lesssim \|(-\Delta + \alpha_{[\omega]})^{-1}\{W^3|\eta_{[\omega]}|^2\}\|_{L^\infty} + \|(-\Delta + \alpha_{[\omega]})^{-1}|\eta_{[\omega]}|^5\|_{L^\infty}\\ &\lesssim \|W^3|\eta_{[\omega]}|^2\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}} + \|\eta_{[\omega]}^5\|_{L^{\frac{3+\varepsilon}{2}} \cap L^{\frac{3-\varepsilon}{2}}}\\ &\le \|W\|_{L^{\frac{9}{2}+3\varepsilon} \cap L^{\frac{9}{2}-3\varepsilon}}\|\eta_{[\omega]}\|_{L^\infty}^2 + \|\eta_{[\omega]}\|_{L^{\frac{9}{2}+3\varepsilon} \cap L^{\frac{9}{2}-3\varepsilon}}\|\eta_{[\omega]}\|_{L^\infty}^2 \lesssim \alpha_{[\omega]}^{1-2\varepsilon}.\end{aligned}$$

Thus, we have obtained the desired result. □

5.3 (Case 3) $1 < p < 2$.

In this section, we consider the case of $1 < p < 2$. We shall give the proof of Proposition 5.1 (iii).

5.3.1 Convergence to a singular solution

We rescale the solution $\widetilde{u}_\omega$ as follows:

$$\widehat{u}_\omega(s) = \alpha_\omega^{-\frac{1}{2}} \widetilde{u}_\omega(r), \qquad s = \alpha_\omega^{\frac{1}{2}} r. \tag{5.69}$$

Then, $\widehat{u}_{\omega_n}$ satisfies the following:

$$-\frac{d^2\widehat{u}}{ds^2} - \frac{2}{s}\frac{d\widehat{u}}{ds} - \widehat{u} - \frac{\beta_{\omega_n}}{\alpha_{\omega_n}^{\frac{3-p}{2}}}\widehat{u}^p - \alpha_{\omega_n}\widehat{u}^5 = 0 \qquad \text{in } (0,\infty). \tag{5.70}$$

In this subsection, we will show that the rescaled function $\widehat{u}_\omega$ converges to a singular solution $U_{\theta_0,\infty}$ to some scalar field equation as ω tends to zero (see Proposition 5.19 below more precisely). To this end, we prepare the following lemma.

Lemma 5.18. *Let $1 < p < 2$ and $\widetilde{u}_\omega$ be the positive solution to (2.4). There exists a constant $C_3' > 0$ such that*

$$\beta_\omega \geq C_3' \alpha_\omega^{\frac{3-p}{2}}. \tag{5.71}$$

Proof of Lemma 5.18. Putting $s = \sqrt{\alpha_\omega} r$, we see from (5.8) that

$$\|\widetilde{u}_\omega\|_{L^2}^2 \geq C \int_{\alpha_\omega^{-\frac{1}{2}}}^\infty e^{-2\sqrt{\alpha_\omega r}}\, dr = C\alpha_\omega^{-\frac{1}{2}} \int_1^\infty e^{-2s}\, ds \geq C\alpha_\omega^{-\frac{1}{2}} \tag{5.72}$$

for some $C > 0$. Next, we obtain an upper bound of $\widetilde{u}_\omega$ by applying Lemma 5.3. By (5.52) and (4.1), we have

$$\frac{\beta_\omega}{\alpha_\omega}\widetilde{u}_\omega^{p-1}(L\alpha_\omega^{\frac{1}{2}}) + \frac{1}{\alpha_\omega}\widetilde{u}_\omega^4(L\alpha_\omega^{\frac{1}{2}}) \leq \frac{C}{L^{p-1}}\alpha_\omega^{\frac{3-p}{2}-1+\frac{p-1}{2}} + C\frac{\alpha_\omega^3}{L^4} = \frac{C}{L^{p-1}} + C\frac{\alpha_\omega^3}{L^4} < \frac{1}{2}.$$

for sufficiently large $L > 0$. We take $L > 0$ satisfying above and fix it. Then, it follows from Lemma 5.3 with $R_\omega = L\alpha_\omega^{\frac{1}{2}}$ and (4.1) that

$$\widetilde{u}_\omega(r) \leq \frac{\widetilde{u}_\omega(L\alpha_\omega^{\frac{1}{2}})}{\widetilde{Y}_{\omega,1/2}(L\alpha_\omega^{\frac{1}{2}})}\widetilde{Y}_{\omega,1/2}(r) \leq \frac{\frac{C}{L\alpha_\omega^{1/2}}}{\frac{e^{-\sqrt{\frac{\alpha_\omega}{2}}L\alpha_\omega}}{L\alpha_\omega}} \frac{e^{-\sqrt{\frac{\alpha_\omega}{2}}r}}{r} = CLe^{\frac{L}{\sqrt{2}}}\frac{e^{-\sqrt{\frac{\alpha_\omega}{2}}r}}{r} \tag{5.73}$$

for $r \geq L\alpha_\omega^{\frac{1}{2}}$. Observe that

$$\|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} = 4\pi \int_0^{L_{1/2}\alpha_\omega^{-\frac{1}{2}}} \widetilde{u}_\omega^{p+1} r^2\, dr + 4\pi \int_{L_{1/2}\alpha_\omega^{-\frac{1}{2}}}^\infty \widetilde{u}_\omega^{p+1} r^2\, dr.$$

By (4.1), we obtain

$$\int_0^{L_{1/2}\alpha_\omega^{-\frac{1}{2}}} \widetilde{u}_\omega^{p+1} r^2\, dr \leq C \int_0^{L_{1/2}\alpha_\omega^{-\frac{1}{2}}} r^{-p+1}\, dr \leq C\alpha_\omega^{\frac{p-2}{2}}.$$

In addition, by (5.73), we have

$$\int_{L_{1/2}\alpha_\omega^{-\frac{1}{2}}}^\infty \widetilde{u}_\omega^{p+1} r^2\, dr \leq C \int_{L_{1/2}\alpha_\omega^{-\frac{1}{2}}}^\infty e^{-(p+1)(1-\varepsilon)\sqrt{\alpha_\omega}r} r^{-p+1}\, dr \leq C\alpha_\omega^{\frac{p-2}{2}} \int_{L_{1/2}}^\infty e^{-(p+1)(1-\varepsilon)s}\, ds \leq C\alpha_\omega^{\frac{p-2}{2}}.$$

These yields that

$$\|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} \le C\alpha_\omega^{\frac{p-2}{2}}. \tag{5.74}$$

By (2.8), (5.72) and (5.74), one has

$$C\sqrt{\alpha_\omega} \le \alpha_\omega \|\widetilde{u}_\omega\|_{L^2}^2 = \frac{5-p}{2(p+1)}\beta_\omega \|\widetilde{u}_\omega\|_{L^{p+1}}^{p+1} \le C\beta_\omega \alpha_\omega^{\frac{p-2}{2}}.$$

Therefore, (5.71) holds. □

Let $\{\omega_n\}$ be a sequence in $(0,\infty)$ with $\lim_{n\to\infty}\omega_n = 0$. By (5.52) and (5.71), we see that there exists a constant $\theta_0 > 0$ such that

$$\lim_{n\to\infty} \frac{\beta_{\omega_n}}{\alpha_{\omega_n}^{\frac{3-p}{2}}} = \theta_0.$$

Note that the limit θ_0 depends on the sequence $\{\omega_n\}$ in general. However, we will find that θ_0 is a universal number, that is, it does not depend on the choice of the sequence $\{\omega_n\}$. To this end, we first show the following:

Proposition 5.19. *Let $1 < p < 2$ and $\widehat{u}_{\omega_n}$ be a function defined by (5.69). For any finite interval $I \subset (0,\infty)$, we have*

$$\lim_{n\to\infty} \widehat{u}_{\omega_n}(r) = U_{\theta_0,\infty}(r) \qquad \text{uniformly in } I,$$

where $U_{\theta_0,\infty}$ is a solution to

$$-U_{rr} - \frac{2}{r}U_r + U - \theta_0 U^p = 0 \qquad \text{in } (0,\infty) \tag{5.75}$$

satisfying

$$\lim_{r\to 0} rU_{\theta_0,\infty}(r) = \sqrt{3}. \tag{5.76}$$

Remark 5.5. *We see from the result of [32, Theorem 2] that the equation (5.75) has infinitely many positive singular solutions U satisfying*

$$\lim_{s\to 0} sU(s) = C$$

for some $C > 0$. However, we can specify the singular solution $U_{\theta_0,\infty}$ which is obtained in Proposition 5.19 by using the uniqueness of positive solution to (2.11).

Lemma 5.20. *For any $L > 0$ and $\varepsilon > 0$, there exists and $R_\varepsilon > 0$ $n_0 > 0$ such that*

$$\sqrt{3}\frac{1-\varepsilon}{r}e^{-L} < \widetilde{u}_{\omega_n}(r) < \sqrt{3}\frac{1+\varepsilon}{r} \qquad (R_\varepsilon < r < \frac{L}{\sqrt{\alpha_{\omega_n}}}). \tag{5.77}$$

for $n \ge n_0$.

Proof. Note that

$$\widetilde{u}_{\omega_n}(r) \le \left(1 + \frac{\gamma_{\omega_n}}{3}r^2\right)^{-\frac{1}{2}} \qquad \text{for all } r > 0,$$

This together with $\lim_{n\to\infty}\gamma_{\omega_n} = 1$ (see (4.2) for the definition of γ_ω) implies that $\widetilde{u}_{\omega_n}(r) < \sqrt{3}\frac{1+\varepsilon}{r}$ for sufficiently large $n \in \mathbb{N}$.

On the other hand, it follows from (5.5) that for any $\varepsilon > 0$, we obtain

$$\widetilde{u}_{\omega_n}(r) \ge \sqrt{3}(1-\varepsilon)\frac{e^{-L}}{r} \qquad \text{for } R_\varepsilon < r < \tfrac{L}{\sqrt{\alpha_{\omega_n}}}.$$

This completes the proof. □

Putting $\overline{u}_{\omega_n}(s) = s\widehat{u}_{\omega_n}(s)$, we see that $\overline{u}_{\omega_n}$ satisfies the following:

$$\begin{cases} \overline{u}_{ss} - \overline{u} + \beta_{\omega_n}\alpha_{\omega_n}^{\frac{p-3}{2}} s^{1-p}\overline{u}^p + \alpha_{\omega_n} s^{-4}\overline{u}^5 = 0 \qquad \text{in } (0,\infty), \\ \overline{u}(s) > 0 \qquad \text{in } (0,\infty), \\ \overline{u}(0) = 0, \qquad \overline{u}_s(0) = \frac{1}{\sqrt{\alpha_{\omega_n}}}, \qquad \lim_{s\to\infty}\overline{u}(s) = 0. \end{cases} \tag{5.78}$$

We obtain the following:

Lemma 5.21. *For any finite interval I in $(0,\infty)$, one has $\lim_{n\to\infty}\overline{u}_{\omega_n}(s) = \overline{u}_0(s)$ for $s \in I$, where $\overline{u}_0(s)$ satisfies*

$$\begin{cases} \frac{d^2\overline{u}_0}{ds^2} - \overline{u}_0 + \theta_0 s^{1-p}\overline{u}_0^p = 0 \qquad \text{on } (0,\infty), \\ \overline{u}_0(0) = \sqrt{3}, \qquad \lim_{s\to\infty}\overline{u}_0(s) = 0. \end{cases} \tag{5.79}$$

Proof. Observe from Lemma 5.20 that for any $L > 0$ and $\delta > 0$, there exists $C_1 > 0$ and $C_2 > 0$ such that $C_1 < r\widetilde{u}_{\omega_n}(r) < C_2$ for $r \in [\frac{\delta}{\sqrt{\alpha_{\omega_n}}}, \frac{L}{\sqrt{\alpha_{\omega_n}}}]$. This together with $\overline{u}_{\omega_n}(s) = s\widehat{u}_{\omega_n}(s) = \alpha_{\omega_n}^{-\frac{1}{2}} s\widetilde{u}_{\omega_n}(r)$ and $s = \sqrt{\alpha_{\omega_n}}r$ implies that $C_1 < \overline{u}_{\omega_n}(s) < C_2$ for $s \in [\delta, L]$. In addition, we see from (5.78) that

$$\left|\frac{d^2\overline{u}_{\omega_n}}{ds^2}(s)\right| \le \overline{u}_{\omega_n}(s) + \beta_{\omega_n}\alpha_{\omega_n}^{\frac{p-3}{2}} s^{1-p}\overline{u}^p_{\omega(m_n)}(s) + \alpha_{\omega_n}s^{-4}\overline{u}^5_{\omega(m_n)}(s) \lesssim_\delta 1 \qquad \text{for any } s \in [\delta, L].$$

Then, by the Ascoli-Arzela theorem, there exists $\overline{u}_0$ such that

$$\lim_{n\to\infty}\overline{u}_{\omega_n}(s) = \overline{u}_0(s), \qquad \lim_{n\to\infty}\frac{d\overline{u}_{\omega_n}}{ds}(s) = \frac{d\overline{u}_0}{ds}(s) \qquad \text{uniformly in } [\delta, L]. \tag{5.80}$$

Observe that $\overline{u}_{\omega_n}$ satisfies

$$\frac{d\overline{u}_{\omega_n}}{ds}(s) - \frac{d\overline{u}_{\omega_n}}{ds}(\delta) = \int_\delta^s (\overline{u}_{\omega_n}(\sigma) - \beta_{\omega_n}\alpha_{\omega_n}^{\frac{p-3}{2}}\sigma^{1-p}\overline{u}^p_{\omega_n}(\sigma) + \alpha_{\omega_n}\sigma^{-4}\overline{u}^5_{\omega_n}(\sigma))d\sigma. \tag{5.81}$$

It follows from (5.80), (5.81) and $\lim_{n\to\infty}\alpha_{\omega_n} = 0$ that

$$\frac{d\overline{u}_0}{ds}(s) - \frac{d\overline{u}_0}{ds}(\delta) = \int_\delta^s (\overline{u}_0(\sigma) - \theta_0\sigma^{1-p}\overline{u}_0^p(\sigma))d\sigma. \tag{5.82}$$

Since $\delta > 0$ and $L > 0$ are arbitrary, we see from (5.82) that $\overline{u}_0$ satisfies the equation in (5.79).

Next, we shall show that $\overline{u}_0(0) = \lim_{s\to 0}\overline{u}_0(s) = \sqrt{3}$. For any $s_0 > 0$, we see that

$$\begin{aligned} \overline{u}_0(s_0) = \liminf_{n\to\infty}\overline{u}_{\omega_n}(s_0) &= \liminf_{n\to\infty} s_0\widehat{u}_{\omega_n}(s_0) \\ &= \liminf_{n\to\infty}\frac{s_0}{\sqrt{\alpha_{\omega_n}}}\widetilde{u}_{\omega_n}\left(\frac{s_0}{\sqrt{\alpha_{\omega_n}}}\right) \\ &\ge (1-\varepsilon)\liminf_{n\to\infty}\frac{s_0}{\sqrt{\alpha_{\omega_n}}}Y_{\omega_n}\left(\frac{s_0}{\sqrt{\alpha_{\omega_n}}}\right) \\ &= \sqrt{3}(1-\varepsilon)\liminf_{n\to\infty}\frac{s_0}{\sqrt{\alpha_{\omega_n}}} \times \frac{\sqrt{\alpha_{\omega_n}}}{s_0}e^{-\sqrt{\alpha_{\omega_n}}\times\frac{s_0}{\sqrt{\alpha_{\omega_n}}}} \\ &= \sqrt{3}(1-\varepsilon)e^{-s_0}. \end{aligned}$$

This implies that $\overline{u}_0(s_0) \ge \sqrt{3}(1-\varepsilon)e^{-s_0}$.

On the other hand, by (4.1), $\lim_{n\to\infty}\alpha_{\omega_n}=0$ and $\lim_{n\to\infty}\gamma_{\omega_n}=1$ we obtain

$$
\begin{aligned}
\overline{u}_0(s_0)=\limsup_{n\to\infty}\overline{u}_{\omega_n}(s_0)&=\limsup_{n\to\infty}s_0\widehat{u}_{\omega_n}(s_0)\\
&=\limsup_{n\to\infty}\frac{s_0}{\sqrt{\alpha_{\omega_n}}}\widetilde{u}_{\omega_n}\left(\frac{s_0}{\sqrt{\alpha_{\omega_n}}}\right)\\
&\le\limsup_{n\to\infty}\frac{s_0}{\sqrt{\alpha_{\omega_n}}}\left(1+\frac{\gamma_{\omega_n}}{3}\left(\frac{s_0^2}{\alpha_{\omega_n}}\right)\right)^{-\frac{1}{2}}\\
&=\limsup_{n\to\infty}\frac{s_0}{\sqrt{\alpha_{\omega_n}}}\frac{\sqrt{3\alpha_{\omega_n}}}{s_0\sqrt{\gamma_{\omega_n}}}\left(\frac{3}{\gamma_{\omega_n}}\left(\frac{\alpha_{\omega_n}}{s_0^2}\right)+1\right)^{-\frac{1}{2}}\\
&=\limsup_{n\to\infty}\frac{\sqrt{3}}{\sqrt{\gamma_{\omega_n}}}\left(\frac{3}{\gamma_{\omega_n}}\left(\frac{\alpha_{\omega_n}}{s_0^2}\right)+1\right)^{-\frac{1}{2}}=\sqrt{3}.
\end{aligned}
$$

Thus, we have $\sqrt{3}(1-\varepsilon)e^{-s_0}\le\overline{u}_0(s_0)\le\sqrt{3}$ for any $s_0>0$. Since $\varepsilon>0$ is arbitrary, we see that $\overline{u}_0(0)=\lim_{s_0\to0}\overline{u}_0(s_0)=\sqrt{3}$. This completes the proof. □

Proof of Proposition 5.19. We put $U_{\theta_0,\infty}(s):=\overline{u}_0(s)s^{-1}$. Then we see that $U_{\theta_0,\infty}(s)$ is a solution to (5.75) satisfying (5.77). Then, from Lemma 5.21 and $r=\alpha_{\omega_n}^{-\frac{1}{2}}s$, we obtain

$$
\lim_{n\to\infty}\alpha_{\omega_n}^{-\frac{1}{2}}\widetilde{u}_{\omega_n}(\alpha_{\omega_n}^{-\frac{1}{2}}s)=\lim_{n\to\infty}\overline{u}_{\omega_n}(s)s^{-1}=\overline{u}_0(s)s^{-1}=U_{\theta_0,\infty}(s)\qquad\text{uniformly in } s\in I.
$$

This concludes the proof. □

5.3.2 Proof of Proposition 5.1 (iii)

In this subsection, we prove Proposition 5.1 (iii). To this end, we need several preparations.

Lemma 5.22. *Let $1<p<2$. Then, for any $q\in[2,3)$, we obtain*

$$
\lim_{n\to\infty}\alpha_{\omega_n}^{\frac{3-q}{2}}\|\widetilde{u}_{\omega_n}\|_{L^q}^q=\|U_{\theta_0,\infty}\|_{L^q}^q. \tag{5.83}
$$

Proof. For any $\varepsilon>0$, we have

$$
\begin{aligned}
\|\widetilde{u}_{\omega_n}\|_{L^q}^q&=4\pi\int_0^{\varepsilon\alpha_{\omega_n}^{-\frac{1}{2}}}\widetilde{u}_{\omega_n}^q(r)r^2\,dr+4\pi\int_{\varepsilon\alpha_{\omega_n}^{-\frac{1}{2}}}^{\varepsilon^{-1}\alpha_{\omega_n}^{-\frac{1}{2}}}\widetilde{u}_{\omega_n}^q(r)r^2\,dr\\
&\quad+4\pi\int_{\varepsilon^{-1}\alpha_{\omega_n}^{-\frac{1}{2}}}^{\infty}\widetilde{u}_{\omega_n}^q(r)r^2\,dr\\
&=:I_{1,n}+I_{2,n}+I_{3,n}.
\end{aligned} \tag{5.84}
$$

By (4.1), one has

$$
I_{1,n}\lesssim\int_0^{\varepsilon\alpha_{\omega_n}^{-\frac{1}{2}}}r^{-q+2}\,dr\lesssim\varepsilon^{3-q}\alpha_{\omega_n}^{\frac{q-3}{2}}.
$$

By (5.52) and (4.1), we have

$$
\frac{\beta_\omega}{\alpha_\omega}\widetilde{u}_\omega^{p-1}(\varepsilon^{-1}\alpha_\omega^{\frac{1}{2}})+\frac{1}{\alpha_\omega}\widetilde{u}_\omega^4(\varepsilon^{-1}\alpha_\omega^{\frac{1}{2}})\le C\varepsilon^{p-1}\alpha_\omega^{\frac{3-p}{2}-1+\frac{p-1}{2}}+C\varepsilon^4\alpha_\omega^3=C\varepsilon^{p-1}+C\varepsilon^4\alpha_\omega^3<C\varepsilon^{p-1}.
$$

Applying Lemma 5.3 with $R_\omega=\varepsilon^{-1}\alpha_\omega^{\frac{1}{2}}$, we have by (4.1) that

$$
\widetilde{u}_\omega(r)\le\frac{\widetilde{u}_\omega(\varepsilon^{-1}\alpha_\omega^{\frac{1}{2}})}{\widetilde{Y}_{\omega,C\varepsilon^{p-1}}(\varepsilon^{-}1\alpha_\omega^{\frac{1}{2}})}\widetilde{Y}_{\omega,C\varepsilon^{p-1}}(r)\le Ce^{-\sqrt{(1-C\varepsilon^{p-1})}\varepsilon^{-1}}\frac{e^{-\sqrt{\frac{\alpha_\omega}{2}}r}}{r} \tag{5.85}
$$

for $r \geq \varepsilon^{-1}\alpha_{\omega}^{\frac{1}{2}}$. It follows from (5.85) that

$$I_{3,n} \lesssim \int_{\varepsilon^{-1}\alpha_{\omega_n}^{-\frac{1}{2}}}^{\infty} \frac{e^{-q\sqrt{\frac{\alpha_{\omega_n}}{2}}r}}{r^{q-2}}\,dr = \alpha_{\omega_n}^{\frac{q-3}{2}} \int_{\varepsilon^{-1}}^{\infty} \frac{e^{-\frac{q}{\sqrt{2}}s}}{s^{q-2}}\,ds \lesssim \alpha_{\omega_n}^{\frac{q-3}{2}} \int_{\varepsilon^{-1}}^{\infty} e^{-\frac{qs}{2}}\,ds \lesssim \varepsilon\alpha_{\omega_n}^{\frac{q-3}{2}}$$

for any small $\varepsilon > 0$. These imply that

$$\alpha_{\omega_n}^{\frac{3-q}{2}} I_{1,n},\ \alpha_{\omega_n}^{\frac{3-q}{2}} I_{3,n} \lesssim \varepsilon^{3-q}. \tag{5.86}$$

Observe from $\widetilde{u}_{\omega_n}(r) = \alpha_{\omega_n}^{\frac{1}{2}}\widehat{u}_{\omega_n}(\alpha_{\omega_n}^{\frac{1}{2}}r)$ for $r > 0$ that

$$I_{2,n} = 4\pi \int_{\varepsilon\alpha_{\omega_n}^{-\frac{1}{2}}}^{\varepsilon^{-1}\alpha_{\omega_n}^{-\frac{1}{2}}} \widetilde{u}_{\omega_n}^{q}(r)r^2\,dr = 4\pi \int_{\varepsilon\alpha_{\omega_n}^{-\frac{1}{2}}}^{\varepsilon^{-1}\alpha_{\omega_n}^{-\frac{1}{2}}} \alpha_{\omega_n}^{\frac{q}{2}}\widehat{u}_{\omega_n}^{q}(\alpha_{\omega_n}^{\frac{1}{2}}r)r^2\,dr = 4\pi\alpha_{\omega_n}^{\frac{q-3}{2}} \int_{\varepsilon}^{\varepsilon^{-1}} \widehat{u}_{\omega_n}^{q}(s)s^2\,ds.$$

In addition, it follows from Proposition 5.19 that

$$\alpha_{\omega_n}^{\frac{3-q}{2}} I_{2,n} \to 4\pi \int_{\varepsilon}^{\varepsilon^{-1}} U_{\theta_0,\infty}^{q}(s)s^2\,ds \qquad \text{as } n \to \infty.$$

Since $\varepsilon > 0$ is arbitrary, we see from (5.84), (5.86) and (5.55) that (5.83) holds. ☐

Lemma 5.23. *Let $1 < p < 2$ and $\overline{u}_0$ be the solution to* (5.79)*, obtained in Lemma 5.21. Then, we have* $\frac{d\overline{u}_0}{ds}(0) = 0$.

Proof. It follows from (2.8) that

$$\alpha_{\omega_n}\|\widetilde{u}_{\omega_n}\|_{L^2}^2 = \frac{5-p}{2(p+1)}\beta_{\omega_n}\|\widetilde{u}_{\omega_n}\|_{L^{p+1}}^{p+1}.$$

This yields that

$$\alpha_{\omega_n}^{\frac{1}{2}}\|\widetilde{u}_{\omega_n}\|_{L^2}^2 = \frac{5-p}{2(p+1)}\beta_{\omega_n}\alpha_{\omega_n}^{-\frac{1}{2}}\|\widetilde{u}_{\omega_n}\|_{L^{p+1}}^{p+1} = \frac{5-p}{2(p+1)}\beta_{\omega_n}\alpha_{\omega_n}^{\frac{p-3}{2}} \times \alpha_{\omega_n}^{\frac{2-p}{2}}\|\widetilde{u}_{\omega_n}\|_{L^{p+1}}^{p+1}.$$

This together with (5.83) and $U_{\theta_0,\infty}(s) = \overline{u}_0(s)s^{-1}$ yields that

$$\begin{aligned} \theta_0 = \lim_{n\to\infty} \beta_{\omega_n}\alpha_{\omega_n}^{\frac{p-3}{2}} = \lim_{n\to\infty} \frac{2(p+1)}{5-p} \frac{\alpha_{\omega_n}^{\frac{1}{2}}\|\widetilde{u}_{\omega_n}\|_{L^2}^2}{\alpha_{\omega_n}^{\frac{2-p}{2}}\|\widetilde{u}_{\omega_n}\|_{L^{p+1}}^{p+1}} &= \frac{2(p+1)}{5-p}\frac{\|U_{\theta_0,\infty}\|_{L^2}^2}{\|U_{\theta_0,\infty}\|_{L^{p+1}}^{p+1}} \\ &= \frac{2(p+1)}{5-p}\frac{\int_0^\infty \overline{u}_0^2(s)\,ds}{\int_0^\infty \overline{u}_0^{p+1}(s)s^{-p+1}\,ds}. \end{aligned} \tag{5.87}$$

Multiplying the equation in (5.79) by $\overline{u}_0$ and integrating the resulting equation from 0 to ∞, we obtain

$$\frac{d\overline{u}_0}{ds}(0)\overline{u}_0(0) = -\int_0^\infty \left\{ \left(\frac{d\overline{u}_0}{ds}\right)^2 + (\overline{u}_0)^2 - \theta_0(\overline{u}_0)^{p+1}s^{1-p} \right\} ds. \tag{5.88}$$

This implies that $\left|\frac{d\overline{u}_0}{ds}(0)\right| < \infty$. Moreover, observe from (5.79) that

$$\frac{d}{ds}\left(\frac{1}{2}\left(\frac{d\overline{u}_0}{ds}\right)^2 - \frac{(\overline{u}_0)^2}{2} + \theta_0\frac{s^{1-p}(\overline{u}_0)^{p+1}}{p+1} \right) = -\frac{p-1}{p+1}\theta_0 s^{-p}(\overline{u}_0)^{p+1}. \tag{5.89}$$

In addition, multiplying the equation in (5.89) by s and integrating the resulting equation from 0 to ∞, we get

$$\int_0^\infty \left\{ \frac{1}{2}\left(\frac{d\overline{u}_0}{ds}\right)^2 - \frac{(\overline{u}_0)^2}{2} + \theta_0\frac{(\overline{u}_0)^{p+1}s^{1-p}}{p+1} \right\} ds = \frac{p-1}{p+1}\theta_0 \int_0^\infty (\overline{u}_0)^{p+1}s^{1-p}\,ds. \tag{5.90}$$

This together with (5.87) implies that

$$\int_0^\infty \left(\frac{d\overline{u}_0}{ds}\right)^2 ds = \int_0^\infty (\overline{u}_0)^2\, ds + \frac{2(p-2)}{p+1}\theta_0 \int_0^\infty (\overline{u}_0)^{p+1} s^{1-p}\, ds = \frac{3p-3}{5-p}\int_0^\infty (\overline{u}_0)^2\, ds.$$

It follows from (5.88), (5.89) and (5.87) that

$$\begin{aligned} \frac{d\overline{u}_0}{ds}(0)\overline{u}_0(0) &= -\int_0^\infty \left\{ \left(\frac{d\overline{u}_0}{ds}\right)^2 + (\overline{u}_0)^2 - \theta_0(\overline{u}_0)^{p+1}s^{1-p} \right\} ds \\ &= -\left\{ \frac{3p-3}{5-p} + 1 - \frac{2(p+1)}{5-p} \right\} \int_0^\infty \overline{u}_0^2(s) s^2\, ds \\ &= 0. \end{aligned} \tag{5.91}$$

Since $\overline{u}_0(0) = \sqrt{3}$, we see from (5.91) that $\frac{d\overline{u}_0}{ds}(0) = 0$. Thus, we have obtained the desired result. $\square$

We are now in a position to prove Proposition 5.1 (iii).

Proof of Proposition 5.1 (iii). We put $\overline{v}_0 = \theta_0^{\frac{1}{p-1}}\overline{u}_0$. Then, $\overline{v}_0$ satisfies

$$\begin{cases} \frac{d^2\overline{v}}{ds^2} - \overline{v} + s^{1-p}\overline{v}^p = 0 & \text{on } (0,\infty), \\ \overline{v}(0) = \sqrt{3}\theta_0^{\frac{1}{p-1}}, \qquad \frac{d\overline{v}}{ds}(0) = 0, \qquad \lim_{s\to\infty} \overline{v}_0(s) = 0. \end{cases} \tag{5.92}$$

Note that the equation in (5.92) has a unique positive solution V in $H^1(0,\infty)$ (see Proposition 2.1). Thus, we see that $\overline{v}_0 = V$, so that we have $V(0) = \overline{v}_0(0) = \sqrt{3}\theta_0^{\frac{1}{p-1}}$. This implies that $\theta_0 = 3^{-\frac{p-1}{2}} V^{p-1}(0)$. $\square$

6 Uniqueness of the large solution u_ω and proof of Theorem 2.1

In this section, following [1], we shall give the proof of Theorem 2.5 (i) (uniqueness of the positive solution u_ω to (1.1) for sufficiently small $\omega > 0$) by contradiction. Suppose to the contrary that there exists a sequence $\{\omega_n\}$ in $(0,\infty)$ with $\lim_{n\to\infty}\omega_n = 0$ such that for each $n \in \mathbb{N}$, $u_n^{(i)}$ $(i = 1,2)$ is a solution to (1.1) with $\omega = \omega_n$ satisfying $u_n^{(1)} \neq u_n^{(2)}$ and $\lim_{n\to\infty} \|u_n^{(i)}\|_{L^\infty} = \infty$. For $i = 1,2$, we set

$$M_n^{(i)} := \|u_n^{(i)}\|_{L^\infty}, \qquad \widetilde{u}_n^{(i)}(\cdot) = (M_n^{(i)})^{-1} u_n^{(i)}((M_n^{(i)})^{-2}\cdot),$$

$$\alpha_n^{(i)} := \omega_n (M_n^{(i)})^{-4}, \qquad \beta_n^{(i)} := (M_n^{(i)})^{p-5}$$

and

$$\nu_n = \frac{M_n^{(2)}}{M_n^{(1)}}.$$

It follows from (2.5), (5.1), (5.2) and (5.3) that

$$\begin{aligned} &\lim_{n\to\infty} \omega_n^{-\frac{1}{2}} (M_n^{(i)})^{p-3} = \frac{12\pi(p+1)}{(5-p)\|W\|_{L^{p+1}}^{p+1}} && \text{when } 2 < p < 3, \\ &\lim_{n\to\infty} \omega_n^{-\frac{1}{2}} (M_n^{(i)})^{p-3} |\log(\alpha_n^{(i)})| = \frac{2}{\sqrt{3}} && \text{when } p = 2, \\ &\lim_{n\to\infty} \omega_n^{\frac{p-3}{2}} (M_n^{(i)})^{-p+1} = 3^{-\frac{p-1}{2}} V^{p-1}(0) && \text{when } 1 < p < 2 \end{aligned} \tag{6.1}$$

for $i = 1,2$. This implies the following lemma:

Lemma 6.1. *For $1 < p < 3$, up to subsequence, we have*

$$\lim_{n\to\infty} \nu_n = 1. \tag{6.2}$$

Proof. The case of $p \neq 2$ is clear. Thus, we give the proof only for the case of $p = 2$.

Changing the order of $M_n^{(1)}$ and $M_n^{(2)}$ if necessary, we may assume that $\limsup_{n\to\infty} \nu_n \geq 1$. It follows from (6.1), $p = 2$ and $\alpha_n^{(2)} = \omega_n (M_n^{(2)})^{-4} = \omega_n (\nu_n M_n^{(1)})^{-4} = \nu_n^{-4} \alpha_n^{(1)}$ that

$$1 = \lim_{n\to\infty} \nu_n \frac{|\log \alpha_n^{(1)}|}{|\log \alpha_n^{(2)}|} = \lim_{n\to\infty} \nu_n \left| \frac{\log \alpha_n^{(1)}}{-4\log \nu_n + \log \alpha_n^{(1)}} \right|. \tag{6.3}$$

We claim that up to subsequence, one has

$$\lim_{n\to\infty} \frac{\log \nu_n}{(-\log \alpha_n^{(1)})} = 0. \tag{6.4}$$

If (6.4) holds, we have by (6.3) that

$$1 = \lim_{n\to\infty} \nu_n \left| \frac{\log \alpha_n^{(1)}}{-4\log \nu_n + \log \alpha_n^{(1)}} \right| = \lim_{n\to\infty} \nu_n \left| \frac{1}{4\frac{\log \nu_n}{(-\log \alpha_n^{(1)})} + 1} \right| = \lim_{n\to\infty} \nu_n.$$

Thus, we see that (6.2) holds. Suppose to the contrary that (6.4) does not hold. Then, since $\limsup_{n\to\infty} \nu_n \geq 1$, up to subsequence, we see that one of the following occurs

$$\limsup_{n\to\infty} \frac{\log \nu_n}{(-\log \alpha_n^{(1)})} = \kappa \quad \text{for some } \kappa \in (0, \infty].$$

Since $\lim_{n\to\infty}(-\log \alpha_n^{(1)}) = \infty$, we obtain $\lim_{n\to\infty} \nu_n = \infty$, which implies that $\lim_{n\to\infty} \frac{\nu_n}{\log \nu_n} = \infty$. Then, up to subsequence, we see from (6.3) that

$$1 = \lim_{n\to\infty} \frac{\nu_n}{\log \nu_n} \left| \frac{\log \alpha_n^{(1)}}{4 + \frac{(-\log \alpha_n^{(1)})}{\log \nu_n}} \right| = \lim_{n\to\infty} \frac{\nu_n}{\log \nu_n} \frac{|\log \alpha_n^{(1)}|}{4 + \frac{1}{\kappa}} = \infty,$$

which is a contradiction. Thus, we see that (6.4) holds. This completes the proof. $\square$

We see from Theorem 2.4 and (6.2) that

$$\lim_{n\to\infty} \|\widetilde{u}_n^{(1)} - W\|_{\dot{H}^1} = \lim_{n\to\infty} \|\nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot) - W\|_{\dot{H}^1} = 0. \tag{6.5}$$

Next, we define

$$\Psi_n(x) = u_n^{(1)}((M_n^{(1)})^{-2}x) - u_n^{(2)}((M_n^{(2)})^{-2}x) = M_n^{(1)} \left\{ \widetilde{u}_n^{(1)}(x) - \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 x) \right\}$$

and

$$\widetilde{z}_n(x) := \frac{\Psi_n(x)}{\|\nabla \Psi_n\|_{L^2}}.$$

Since $\widetilde{u}_n^{(1)}$ and $\nu_n \widetilde{u}_n^{(2)}(\nu_n^2 x)$ are positive solutions to the following equation:

$$-\Delta \widetilde{u} + \alpha_n^{(1)} \widetilde{u} = \beta_n^{(1)} \widetilde{u}^p + \widetilde{u}^5,$$

we can verify that $\widetilde{z}_n$ satisfies

$$-\Delta \widetilde{z}_n + \alpha_n^{(1)} \widetilde{z}_n = p\beta_n^{(1)} \int_0^1 V_n^{p-1}(x,\theta) d\theta \widetilde{z}_n + 5 \int_0^1 V_n^4(x,\theta) d\theta \widetilde{z}_n, \tag{6.6}$$

where

$$V_n(x,\theta) := \theta \widetilde{u}_n^{(1)}(x) + (1-\theta)\nu_n \widetilde{u}_n^{(2)}(\nu_n^2 x). \tag{6.7}$$

We prepare the following lemma, which is needed later.

Lemma 6.2 (Lemma 4.3 of [1])**.** *Let $1 < p < 3$. Then, there exists $C_1 > 0$ such that for any $x \in \mathbb{R}^3$ and $n \in \mathbb{N}$,*

$$|\widetilde{z}_n(x)| \le C_1 |x|^{-1}. \tag{6.8}$$

Remark 6.1. *Although we assumed the condition $d \ge 5$ in [1, Lemma 4.3], the proof still works for $d = 3$.*

Since $\|\nabla \widetilde{z}_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$, we find that there exists a subsequence of $\{\widetilde{z}_n\}$ (we still denote it by the same letter) and a function $\widetilde{z}_\infty \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \widetilde{z}_n = \widetilde{z}_\infty \qquad \text{weakly in } \dot{H}^1(\mathbb{R}^3) \text{ and almost everywhere in } \mathbb{R}^3.$$

By the elliptic regularity, we can see that

$$\lim_{n\to\infty} \widetilde{z}_n = \widetilde{z}_\infty \qquad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^3).$$

Here, we shall drive a contradiction dividing into 2 cases (case of $2 \le p < 3$ and case of $1 < p < 2$).

6.1 Case of $2 \le p < 3$

In this subsection, we consider the case of $2 \le p < 3$. We first obtain the following:

Lemma 6.3. *Let $2 \le p < 3$. $\widetilde{z}_\infty \neq 0$.*

Proof of Lemma 6.3. Suppose to the contrary that $\widetilde{z}_\infty = 0$. Multiplying (6.6) by $\widetilde{z}_n$ and integrating the resulting equation, we have

$$\|\nabla \widetilde{z}_n\|_{L^2}^2 + \alpha_n^{(1)} \|\widetilde{z}_n\|_{L^2}^2 = p\beta_n^{(1)} \int_{\mathbb{R}^3} \int_0^1 V_n^{p-1}(x,\theta) d\theta \widetilde{z}_n^2 \, dx + 5 \int_{\mathbb{R}^3} \int_0^1 V_n^4(x,\theta) d\theta \widetilde{z}_n^2 \, dx. \tag{6.9}$$

We claim that

$$\lim_{n\to\infty} \beta_n^{(1)} \sup_{\theta\in[0,1]} \int_{\mathbb{R}^3} \int_0^1 V_n^{p-1}(x,\theta) d\theta \widetilde{z}_n^2 \, dx = 0 \tag{6.10}$$

for $2 \le p < 3$. First, we will show that

$$\lim_{n\to\infty} \sup_{\theta\in[0,1]} \int_{\mathbb{R}^3} \int_0^1 V_n^{q-1}(x,\theta) d\theta \, \widetilde{z}_n^2 \, dx = 0. \tag{6.11}$$

for any $q > 2$. It follows from (6.7) that for each $q \ge 2, x \in \mathbb{R}^3$,

$$\sup_{\theta\in[0,1]} |V_n^{q-1}(x,\theta)| \le C_q \left((\widetilde{u}_n^{(1)}(x))^{q-1} + (\widetilde{u}_n^{(2)}(x))^{q-1} \right), \tag{6.12}$$

where $C_q > 0$ is a constant depending only on $q > 2$. For any $R > 0$, using (6.12), (4.1), Lemma 6.2 and $q > 2$, we have

$$\sup_{x\in[0,1]} \int_{|x|\ge R} \int_0^1 V_n^{q-1}(x,\theta) d\theta \, \widetilde{z}_n^2 \, dx \lesssim \int_{|x|\ge R} \left((\widetilde{u}_n^{(1)})^{q-1} + (\widetilde{u}_n^{(2)})^{q-1} \right) \widetilde{z}_n^2 \, dx \lesssim \int_R^\infty r^{-q+1} r^{-2} r^2 \, dr \lesssim R^{-q+2}. \tag{6.13}$$

Since $\lim_{n\to\infty}\widetilde{z}_n = 0$ weakly in $\dot{H}^1(\mathbb{R}^3)$, we see from $\|\widetilde{u}_n^{(i)}\|_{L^\infty} = 1$, the Hölder inequality and the Rellich–Kondrachov theorem that for any sufficiently small $\varepsilon > 0$, one has

$$\begin{aligned}\sup_{\theta\in[0,1]}\int_{|x|<R}\int_0^1 V_n^{q-1}(x,\theta)d\theta\, \widetilde{z}_n^2\, dx &\lesssim \int_{|x|<R}\left((\widetilde{u}_n^{(1)})^{q-1} + (\widetilde{u}_n^{(2)})^{q-1}\right)\, \widetilde{z}_n^2\, dx\\ &\lesssim \left(\|\widetilde{u}_n^{(1)}\|^{q-1}_{L^{\frac{3-\varepsilon}{2-\varepsilon}(q-1)}(B_R)} + \|\widetilde{u}_n^{(2)}\|^{q-1}_{L^{\frac{3-\varepsilon}{2-\varepsilon}(q-1)}(B_R)}\right)\|\widetilde{z}_n\|^2_{L^{6-2\varepsilon}(B_R)}\\ &\to 0 \qquad \text{as } n\to\infty.\end{aligned}\tag{6.14}$$

From (6.13) and (6.14), we deduce that (6.11) holds for $q > 2$.

Next, we obtain the upper bound of $\widetilde{u}_n^{(i)}$. We take $\varepsilon > 0$ arbitrary and fix it. Observe from (4.1), (5.1), (5.2) and $2 \le p < 3$ that

$$\frac{\beta_n^{(i)}}{\alpha_n^{(i)}}(\widetilde{u}_n^{(i)})^{p-1}(\varepsilon(\alpha_n^{(i)})^{-\frac{1}{2}}) + \frac{1}{\alpha_n^{(i)}}(\widetilde{u}_n^{(i)})^4(\varepsilon(\alpha_n^{(i)})^{-\frac{1}{2}}) \le \frac{\sqrt{\alpha_n^{(i)}}}{|\log\alpha_n^{(i)}|}(\alpha_n^{(i)})^{-1}\varepsilon^{-(p-1)}(\alpha_n^{(i)})^{\frac{p-1}{2}} + \varepsilon^{-4}\alpha_n^{(i)} < \varepsilon$$

Then, applying Lemma 5.3, we obtain

$$\widetilde{u}_n^{(i)}(r) \le \frac{\widetilde{u}_n^{(i)}(\varepsilon(\alpha_n^{(i)})^{-\frac{1}{2}})}{\widetilde{Y}_{\omega_n,\varepsilon,i}(\varepsilon(\alpha_n^{(i)})^{-\frac{1}{2}})}\widetilde{Y}_{\omega_n,\varepsilon,i}(r) \le Ce^{\varepsilon\sqrt{1-\varepsilon}}\frac{e^{-\sqrt{(1-\varepsilon)\alpha_n^{(i)}}r}}{r} \qquad \text{for } r \ge \varepsilon(\alpha_n^{(i)})^{-\frac{1}{2}},\tag{6.15}$$

where

$$\widetilde{Y}_{\omega,\varepsilon,i}(r) = \frac{e^{-(1-\varepsilon)\sqrt{\alpha_n^{(i)}}r}}{r}.$$

Now, we pay our attention to the case of $p = 2$. Using $\|\widetilde{u}_n^{(i)}\|_{L^\infty} = 1$, Lemma 6.2 and (4.1) for $1 \le |x| \le (\alpha_n^{(1)})^{-\frac{1}{2}}$, (6.15) for $|x| \ge (\alpha_n^{(1)})^{-\frac{1}{2}}$ and $p = 2$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(\widetilde{u}_n^{(i)})^{p-1}\widetilde{z}_n^2\, dx\right| &= \int_{|x|\le 1}(\widetilde{u}_n^{(i)})^{p-1}\widetilde{z}_n^2\, dx + \int_{1\le|x|\le(\alpha_n^{(1)})^{-\frac{1}{2}}}(\widetilde{u}_n^{(i)})^{p-1}\widetilde{z}_n^2\, dx + \int_{|x|\ge(\alpha_n^{(1)})^{-\frac{1}{2}}}(\widetilde{u}_n^{(i)})^{p-1}\widetilde{z}_n^2\, dx\\ &\lesssim \int_0^1 dr + \int_1^{(\alpha_n^{(1)})^{-\frac{1}{2}}} r^{-(p-1)}r^{-2}r^2\, dr + \int_{(\alpha_n^{(1)})^{-\frac{1}{2}}}^\infty e^{-(1-\varepsilon)(p-1)\sqrt{\alpha_n^{(1)}}r}r^{-(p-1)}r^{-2}r^2\, dr.\\ &\lesssim 1 + \int_1^{(\alpha_n^{(1)})^{-\frac{1}{2}}} r^{-1}\, dr + \int_{(\alpha_n^{(1)})^{-\frac{1}{2}}}^\infty e^{-(1-\varepsilon)(p-1)\sqrt{\alpha_n^{(1)}}r}r^{-1}\, dr\\ &\lesssim 1 + |\log\alpha_n^{(1)}| + \int_1^\infty e^{-(1-\varepsilon)(p-1)s}s^{-1}\, ds\\ &\lesssim |\log\alpha_n^{(1)}|.\end{aligned}\tag{6.16}$$

Then, by (5.2), (6.12) and (6.16), we obtain

$$\beta_n^{(1)}\int_{\mathbb{R}^3}\int_0^1 V_n^{p-1}(x,\theta)d\theta\widetilde{z}_n^2\, dx \lesssim \frac{(\alpha_n^{(1)})^{\frac{1}{2}}}{|\log\alpha_n^{(1)}|}|\log\alpha_n^{(1)}| = (\alpha_n^{(1)})^{\frac{1}{2}} \to 0 \qquad \text{as } n\to\infty\tag{6.17}$$

when $p = 2$. We see from (6.11) and (6.17) that (6.10) holds when $2 \le p < 3$.

We now derive a conclusion. It follows from $\|\nabla\widetilde{z}_n\|_{L^2} = 1$, (6.9), (6.10) and (6.11) with $q = 5$ that

$$\begin{aligned}1 &\le \|\nabla\widetilde{z}_n\|^2_{L^2} + \alpha_n^{(1)}\|\widetilde{z}_n\|^2_{L^2}\\ &= p\beta_n^{(1)}\int_{\mathbb{R}^3}\int_0^1 V_n^{p-1}(x,\theta)d\theta\widetilde{z}_n^2\, dx + 5\int_{\mathbb{R}^3}\int_0^1 V_n^4(x,\theta)d\theta\widetilde{z}_n^2\, dx \to 0 \qquad \text{as } n\to\infty,\end{aligned}$$

which is a contradiction. Thus, we see that $\widetilde{z}_\infty \neq 0$. $\square$

For any $\phi \in C_0^\infty(\mathbb{R}^3)$, we see from (6.6), $\|\nabla \widetilde{z}_n\|_{L^2} = 1$ and (2.3) that

$$\begin{aligned}
&\langle (-\Delta - 5W^4)\widetilde{z}_\infty, \phi\rangle \\
&= \langle (-\Delta - 5W^4)\widetilde{z}_\infty, \phi\rangle - \left\langle \left(-\Delta + \alpha_n^{(1)} - \beta_n^{(1)} \int_0^1 V_n^{p-1}(x,\theta)d\theta - 5\int_0^1 V_n^4(x,\theta)d\theta\right)\widetilde{z}_n, \phi \right\rangle \\
&= \langle (-\Delta - 5W^4)\widetilde{z}_\infty, \phi\rangle - \langle (-\Delta - 5W^4)\widetilde{z}_n, \phi\rangle \\
&\quad + 5\langle (W^4 - \int_0^1 V_n^4(x,\theta)d\theta)\widetilde{z}_n, \phi\rangle + \langle (\alpha_{\omega_n} - \beta_n^{(1)} \int_0^1 V_n^{p-1}(x,\theta)d\theta)\widetilde{z}_n, \phi\rangle \\
&\to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

This implies that

$$(-\Delta - 5W^4)\widetilde{z}_\infty = 0.$$

This together with $\widetilde{z}_\infty \neq 0$ yields that $\widetilde{z}_\infty = \kappa \Lambda W$ for some $\kappa \in \mathbb{R} \setminus \{0\}$. Here, ΛW is the function defined by (5.15). Replacing $\widetilde{z}_n$ by $-\widetilde{z}_n$ if necessary, we may assume that $\kappa > 0$. Similarly to (2.8), we can obtain

$$\omega \|u_n^{(i)}\|_{L^2}^2 = \frac{5-p}{2(p+1)} \|u_n^{(i)}\|_{L^{p+1}}^{p+1} \qquad \text{for } i = 1, 2. \tag{6.18}$$

It follows from (6.18) that

$$\frac{M_n^{(1)}}{\|\nabla \Psi_n\|_{L^2}} \omega_n (\|u_n^{(1)}\|_{L^2}^2 - \|u_n^{(2)}\|_{L^2}^2) = \frac{5-p}{2(p+1)} \frac{M_n^{(1)}}{\|\nabla \Psi_n\|_{L^2}} (\|u_n^{(1)}\|_{L^{p+1}}^{p+1} - \|u_n^{(2)}\|_{L^{p+1}}^{p+1}) \tag{6.19}$$

For the left-hand side of (6.19), we observe that

$$\begin{aligned}
\frac{M_n^{(1)}}{\|\nabla \Psi_n\|_{L^2}} \omega_n (\|u_n^{(1)}\|_{L^2}^2 - \|u_n^{(2)}\|_{L^2}^2) &= M_n^{(1)} \omega_n \int_{\mathbb{R}^3} \frac{u_n^{(1)}(x) - u_n^{(2)}(x)}{\|\nabla \Psi_n\|_{L^2}} (u_n^{(1)}(x) + u_n^{(2)}(x))\, dx \\
&= \alpha_n^{(1)} \langle \widetilde{u}_n^{(1)} + \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot), \widetilde{z}_n\rangle.
\end{aligned}$$

Similarly, for the right-hand side of (6.19), we obtain

$$\begin{aligned}
\frac{5-p}{2(p+1)} \frac{M_n^{(1)}}{\|\nabla \Psi_n\|_{L^2}} (\|u_n^{(1)}\|_{L^{p+1}}^{p+1} - \|u_n^{(2)}\|_{L^{p+1}}^{p+1}) &= \frac{5-p}{2} M_n^{(1)} \int_{\mathbb{R}^3} \left(\int_0^1 \left[\theta u_n^{(1)} + (1-\theta)u_n^{(2)}\right]^p d\theta\right) \frac{u_n^{(1)} - u_n^{(2)}}{\|\nabla \Psi_n\|_{L^2}}\, dx \\
&= \frac{5-p}{2} \beta_n^{(1)} \left\langle \int_0^1 V_n^p(x,\theta)d\theta, \widetilde{z}_n \right\rangle,
\end{aligned} \tag{6.20}$$

where V_n is the function defined by (6.7). These imply that

$$\alpha_n^{(1)} \langle \widetilde{u}_n^{(1)} + \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot), \widetilde{z}_n\rangle = \frac{5-p}{2} \beta_n^{(1)} \left\langle \int_0^1 V_n^p(x,\theta)d\theta, \widetilde{z}_n \right\rangle. \tag{6.21}$$

Here, we claim the following:

Proposition 6.4. *For $2 \leq p < 3$, we have the following:*

$$\liminf_{n\to\infty} \sqrt{\alpha_n^{(1)}} \langle \widetilde{u}_n^{(1)}(x) + \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot), \widetilde{z}_n\rangle \geq -6\pi\kappa. \tag{6.22}$$

Proposition 6.5. *For $2 \leq p < 3$. The following holds:*

$$\limsup_{n\to\infty} \frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}} \left\langle \int_0^1 V_n^p(x,\theta)d\theta, \widetilde{z}_n \right\rangle = -6\pi\kappa. \tag{6.23}$$

Let us admit Propositions 6.4 and 6.5 for a moment. Then, we can give the proof of Theorem 2.5 (i) in the case of $2 \leq p < 3$.

Proof of Theorem 2.5 (i) in the case of $2 \leq p < 3$. It follows from (6.21)–(6.23) that

$$\begin{aligned}-3\pi\kappa(5-p) &= \frac{5-p}{2}\limsup_{n\to\infty} \frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}} \left\langle \int_0^1 V_n^p(x,\theta)d\theta, \widetilde{z}_n \right\rangle \\ &= \liminf_{n\to\infty} \sqrt{\alpha_n^{(1)}} \langle \widetilde{u}_n^{(1)} + \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot), \widetilde{z}_n \rangle \geq -6\pi\kappa.\end{aligned} \tag{6.24}$$

It follows from (6.24) and $\kappa > 0$ that $p \geq 3$, which contradicts the condition $p < 3$. Thus, we conclude that the solution u_ω to (1.1) satisfying $\lim_{\omega\to 0}\|u_\omega\|_{L^\infty} = \infty$ is unique for sufficiently small $\omega > 0$. $\square$

Remark 6.2. *Note that $W \in L^2(\mathbb{R}^d)$ when $d \geq 5$. Thus, in [1], it was shown that*

$$\lim_{n\to\infty} \widetilde{u}_n^{(i)} = W \ (i = 1, 2), \qquad \textit{strongly in } L^2(\mathbb{R}^d)$$

when $d \geq 5$. Namely, we have the L^2-convergence. Then, once we obtain the identity (6.19)*, we can derive a contradiction just by taking a limit in $n \in \mathbb{N}$. On the other hand, since $W, \Lambda W \notin L^2(\mathbb{R}^d)$ when $d = 3$, we cannot do the same way. To overcome the difficulty, we use the resolvent estimate of Lemma 5.14, which comes from the resolvent expansion by Jensen and Kato [31] (see Lemma 5.4).*

6.1.1 Proof of Proposition 6.4

As we see before, we need to obtain Proposition 6.4 in order to show Theorem 2.5 (i). Here, we shall give the proof of Proposition 6.4. To this end, we put

$$U_n := \frac{\widetilde{u}_n^{(1)} + \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot)}{2}.$$

We have

$$\sqrt{\alpha_n^{(1)}} \langle U_n, \widetilde{z}_n \rangle = \frac{\sqrt{\alpha_n}}{2} \langle \widetilde{u}_n^{(1)}(x) + \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot), \widetilde{z}_n \rangle$$

Then, (6.22) can be written by

$$\liminf_{n\to\infty} \sqrt{\alpha_n^{(1)}} \langle U_n, \widetilde{z}_n \rangle \geq -3\pi\kappa. \tag{6.25}$$

Thus, it suffices to show (6.25) holds. In order to prove (6.25), we use the resolvent estimate (5.49) of Lemma 5.14, which comes from the resolvent expansion by Jensen and Kato [31]. Based on the resolvent estimate (5.49), we decompose U_n as follows:

$$U_n = C_n W^4 \Lambda W + g_n, \tag{6.26}$$

where

$$C_n = \frac{\langle (-\Delta + \alpha_n^{(1)})^{-1} U_n, W^4 \Lambda W \rangle}{\langle (-\Delta + \alpha_n^{(1)})^{-1} W^4 \Lambda W, W^4 \Lambda W \rangle}.$$

Observe that

$$\sqrt{\alpha_n^{(1)}}\langle U_n, \widetilde{z}_n\rangle = \sqrt{\alpha_n^{(1)}}C_n\langle W^4\Lambda W, \widetilde{z}_n\rangle + \sqrt{\alpha_n^{(1)}}\langle g_n, \widetilde{z}_n\rangle. \tag{6.27}$$

From the decomposition (6.26), we have

$$\langle(-\Delta + \alpha_n^{(1)})^{-1}g_n, W^4\Lambda W\rangle = 0, \tag{6.28}$$

which implies that we can apply the resolvent estimate (Lemma 5.14) for $(-\Delta + \alpha_n^{(1)})^{-1}g_n$. This enables us to estimate the second term of (6.27). More precisely, we will show the following lemma:

Lemma 6.6. *Let* $2 \le p < 3$. *One has*

$$\lim_{n\to\infty}\sqrt{\alpha_n^{(1)}}\langle g_n, \widetilde{z}_n\rangle = 0. \tag{6.29}$$

To prove Lemma 6.6, we need the following lemma:

Lemma 6.7 (Claim 4 of [7])**.**

$$\lim_{n\to\infty}\langle(-\Delta + \alpha_n^{(1)})^{-1}W^4\Lambda W, W^4\Lambda W\rangle = \frac{1}{5}\langle\Lambda W, W^4\Lambda W\rangle \qquad \textit{as } n \to \infty. \tag{6.30}$$

Using Lemmas 5.13 and 6.7, we give the proof of Lemma 6.6.

Proof of Lemma 6.6. Observe that (6.6) can be rewritten by

$$(-\Delta + \alpha_n^{(1)} - 5W^4)\widetilde{z}_n = R_n\widetilde{z}_n,$$

where

$$R_n := p\beta_n^{(1)}\int_0^1 V_n^{p-1}(x,\theta)d\theta + 5\left(\int_0^1 V_n^4(x,\theta)d\theta - W^4\right). \tag{6.31}$$

Then, we have

$$\begin{aligned}\widetilde{z}_n &= (-\Delta + \alpha_n^{(1)} - 5W^4)^{-1}R_n\widetilde{z}_n\\ &= \left((1 - 5W^4(-\Delta + \alpha_n^{(1)})^{-1})(-\Delta + \alpha_n^{(1)})\right)^{-1}R_n\widetilde{z}_n\\ &= (-\Delta + \alpha_n^{(1)})^{-1}(1 - 5W^4(-\Delta + \alpha_n^{(1)})^{-1})^{-1}R_n\widetilde{z}_n.\end{aligned} \tag{6.32}$$

This yields that

$$\begin{aligned}\langle g_n, \widetilde{z}_n\rangle &= \langle g_n, (-\Delta + \alpha_n^{(1)})^{-1}(1 - 5W^4(-\Delta + \alpha_n^{(1)})^{-1})^{-1}R_n\widetilde{z}_n\rangle\\ &= \langle(1 - 5(-\Delta + \alpha_n^{(1)})^{-1}W^4)^{-1}(-\Delta + \alpha_n^{(1)})^{-1}g_n, R_n\widetilde{z}_n\rangle.\end{aligned} \tag{6.33}$$

Note that $(-\Delta + \alpha_n^{(1)})^{-1}g_n$ satisfies the orthogonal condition (6.28). Then, applying Lemma 5.14 with $f = (-\Delta + \alpha_n^{(1)})^{-1}g_n$, we obtain

$$\begin{aligned}\langle g_n, \widetilde{z}_n\rangle &= \left|\langle(1 - 5(-\Delta + \alpha_n^{(1)})^{-1}W^4)^{-1}(-\Delta + \alpha_n^{(1)})^{-1}g_n, R_n\widetilde{z}_n\rangle\right|\\ &\le \|(1 - 5(-\Delta + \alpha_n^{(1)})^{-1}W^4)^{-1}(-\Delta + \alpha_n^{(1)})^{-1}g_n\|_{L^\infty}\|R_n\widetilde{z}_n\|_{L^1}\\ &\lesssim \|(-\Delta + \alpha_n^{(1)})^{-1}g_n\|_{L^\infty}\|R_n\widetilde{z}_n\|_{L^1}.\end{aligned} \tag{6.34}$$

We shall show that $\lim_{n\to\infty}\|R_n\widetilde{z}_n\|_{L^1}=0$. It follows from $U_n=O(|x|^{-1})$ (see (4.1) and (6.26)), Lemmas 3.5, 6.7 and 5.13 that

$$
\begin{aligned}
|C_n| &\lesssim |\langle(-\Delta+\alpha_n^{(1)})^{-1}U_n, W^4\Lambda W\rangle| = \frac{1}{5}\|(-\Delta+\alpha_n^{(1)})^{-1}U_n\|_{L^\infty}\|W^4\Lambda W\|_{L^1} \\
&\lesssim \|(-\Delta+\alpha_n^{(1)})^{-1}|x|^{-1}\|_{L^\infty}\|W^4\Lambda W\|_{L^1} \lesssim (\alpha_n^{(1)})^{-\frac{1}{2}}.
\end{aligned}
\tag{6.35}
$$

Since $g_n=U_n-C_nW^4\Lambda V$, one has by (6.35), Lemmas 5.11–5.13 and $U_n=O(|x|^{-1})$ (see (4.1)) that

$$
\begin{aligned}
\|(-\Delta+\alpha_n^{(1)})^{-1}g_n\|_{L^\infty} &\le |C_n|\|(-\Delta+\alpha_n^{(1)})^{-1}W^4\Lambda W\|_{L^\infty}+\|(-\Delta+\alpha_n^{(1)})^{-1}U_n\|_{L^\infty} \\
&\lesssim (\alpha_n^{(1)})^{-\frac{1}{2}}\|W^4\Lambda W\|_{L^{\frac{3}{2}+\varepsilon}\cap L^{\frac{3}{2}-\varepsilon}}+\|(-\Delta+\alpha_n^{(1)})^{-1}|x|^{-1}\|_{L^\infty} \\
&\lesssim (\alpha_n^{(1)})^{-\frac{1}{2}}+(\alpha_n^{(1)})^{-\frac{1}{2}}\lesssim(\alpha_n^{(1)})^{-\frac{1}{2}}.
\end{aligned}
\tag{6.36}
$$

Using (4.1) for $1\le r\le(\alpha_n^{(1)})^{-\frac{1}{2}}$ and (6.15) for $r\ge(\alpha_n^{(1)})^{-\frac{1}{2}}$, we have by (6.12) that

$$
\begin{aligned}
\sup_{\theta\in[0,1]}\|V_n^{p-1}(\cdot,\theta)\widetilde{z}_n\|_{L^1} &\lesssim \int_0^1 dr+\int_1^{(\alpha_n^{(1)})^{-\frac{1}{2}}}\frac{1}{r^{p-2}}\,dr+\int_{(\alpha_n^{(1)})^{-\frac{1}{2}}}^\infty\frac{e^{-(p-1)(1-\varepsilon)\sqrt{\alpha_{1,n}}r}}{r^{p-2}}\,dr \\
&\lesssim 1+(\alpha_n^{(1)})^{\frac{p-3}{2}}+(\alpha_n^{(1)})^{\frac{p-3}{2}}\int_1^\infty\frac{e^{-(p-1)(1-\varepsilon)s}}{s^{p-2}}\,ds\lesssim(\alpha_n^{(1)})^{\frac{p-3}{2}}.
\end{aligned}
\tag{6.37}
$$

We put $\eta_n(x,\theta):=V_n(x,\theta)-W$. Then, it follows from (6.5) and (6.7) that

$$
\lim_{n\to\infty}\sup_{\theta\in[0,1]}\|\eta_n(\cdot,\theta)\|_{\dot{H}^1\cap L^q}=0. \tag{6.38}
$$

for any $q>3$. This together with (6.31), the Hölder inequality, (4.1) and (6.37) yields that

$$
\begin{aligned}
\|R_n\widetilde{z}_n\|_{L^1} &\lesssim \sup_{\theta\in[0,1]}\|(V_n^3(\cdot,\theta)+W^3)\eta_n(\cdot,\theta)\widetilde{z}_n\|_{L^1}+\beta_{1,n}\sup_{\theta\in[0,1]}\|V_n^{p-1}(\cdot,\theta)\widetilde{z}_n\|_{L^1} \\
&\lesssim\left(\sup_{\theta\in[0,1]}\|V_n(\cdot,\theta)\|_{L^{\frac{9}{2}}}^3+\|W\|_{L^{\frac{9}{2}}}^3\right)\|\widetilde{z}_n\|_{L^6}\sup_{\theta\in[0,1]}\|\eta_n(\cdot,\theta)\|_{L^6}+\beta_{1,n}\alpha_{1,n}^{\frac{p-3}{2}} \\
&\lesssim \sup_{\theta\in[0,1]}\|\eta_n(\cdot,\theta)\|_{L^6}+\beta_{1,n}\alpha_{1,n}^{\frac{p-3}{2}}.
\end{aligned}
\tag{6.39}
$$

By (5.1) and (5.2), we see that $\lim_{n\to\infty}\beta_{1,n}\alpha_{1,n}^{\frac{p-3}{2}}=0$ for $2\le p<3$. This together with (6.38) and (6.39) yields that $\lim_{n\to\infty}\|R_n\widetilde{z}_n\|_{L^1}=0$. Thus, by (6.34), (6.35), (6.36) and (6.39), we obtain

$$
|(\alpha_{1,n})^{\frac{1}{2}}\langle g_n,\widetilde{z}_n\rangle|\lesssim(\alpha_{1,n})^{\frac{1}{2}}(\alpha_{1,n})^{-\frac{1}{2}}\|R_n\widetilde{z}_n\|_{L^1}=\|R_n\widetilde{z}_n\|_{L^1}\lesssim\sup_{0<\theta<1}\|\eta_n(\cdot,\theta)\|_{L^6}+\beta_{1,n}\alpha_{1,n}^{\frac{p-3}{2}}\to0 \qquad \text{as } n\to\infty.
$$

Thus, we see that (6.29) holds. This completes the proof. □

Next, we pay our attention to the first term of the right-hand side of (6.27). Note that $W^4\Lambda W$ has a good decay, that is, $|W^4\Lambda W|\sim|x|^{-5}$ as $|x|\to\infty$. Using this and $\lim_{n\to\infty}\widetilde{z}_n=\kappa\Lambda W$ weakly in $H^1(\mathbb{R}^d)$, we obtain

$$
\lim_{n\to\infty}\langle W^4\Lambda W,\widetilde{z}_n\rangle=\kappa\langle W^4\Lambda W,\Lambda W\rangle. \tag{6.40}
$$

In order to estimate of the coefficient $\sqrt{\alpha_n^{(1)}}C_n$, we will use the followings:

Lemma 6.8. *Let $2 \le p < 3$. We have*

$$\liminf_{n\to\infty} \sqrt{\alpha_n^{(1)}}\langle(-\Delta+\alpha_n^{(1)})^{-1}U_n, W^4\Lambda W\rangle \ge -\frac{3\pi}{5}. \tag{6.41}$$

Proof of Lemma 6.8. As in (5.44), we obtain

$$\sqrt{\alpha_n^{(1)}}\langle(-\Delta+\alpha_n^{(1)})^{-1}U_n, W^4\Lambda W\rangle = -\frac{\sqrt{3}}{10}\sqrt{\alpha_n^{(1)}}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x|}}{|x|}U_n\,dx + O_n(\sqrt{\alpha_n^{(1)}}).$$

Using (4.1) for $0 \le |x| \le \varepsilon(\alpha_n^{(1)})^{-\frac{1}{2}}$ and (6.15) for $|x| \ge \varepsilon(\alpha_n^{(1)})^{-\frac{1}{2}}$, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \sqrt{\alpha_n^{(1)}}\langle(-\Delta+\alpha_n^{(1)})^{-1}U_n, W^4\Lambda W\rangle \\
&= -\frac{\sqrt{3}}{10}\liminf_{n\to\infty}\sqrt{\alpha_n^{(1)}}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x|}}{|x|}U_n\,dx \\
&\ge -\frac{\sqrt{3}}{10}\times 4\pi\limsup_{n\to\infty}\sqrt{\alpha_n^{(1)}}\left\{2\sqrt{3}\int_0^{\varepsilon(\alpha_n^{(1)})^{-\frac{1}{2}}} e^{-\sqrt{\alpha_n^{(1)}}r}\,dr + \sqrt{3}(1+\varepsilon)\int_{\varepsilon(\alpha_n^{(1)})^{-\frac{1}{2}}}^{\infty} e^{-(2-\varepsilon)\sqrt{\alpha_n^{(1)}}r}\,dr\right\} \\
&= -\frac{2\sqrt{3}\pi}{5}\left\{2\sqrt{3}(1-e^{-\varepsilon}) + \sqrt{3}\frac{1+\varepsilon}{2-\varepsilon}e^{-(2-\varepsilon)\varepsilon}\right\} \\
&= -\frac{12\pi}{5}(1-e^{-\varepsilon}) - \frac{6\pi}{5}\times\frac{1+\varepsilon}{2-\varepsilon}e^{-(2-\varepsilon)\varepsilon} \\
&\ge \frac{6\pi}{5}\varepsilon - \frac{6\pi}{5}\times\frac{1+\varepsilon}{2-\varepsilon}e^{-(2-\varepsilon)\varepsilon}.
\end{aligned} \tag{6.42}$$

for $0 < \varepsilon \ll 1$. Since $\varepsilon > 0$ is arbitrary small, we have

$$\liminf_{n\to\infty} \sqrt{\alpha_n^{(1)}}\langle(-\Delta+\alpha_n^{(1)})^{-1}U_n, W^4\Lambda W\rangle \ge -\frac{3\pi}{5}.$$

This concludes the proof. □

We are now in a position to prove Proposition 6.4.

Proof of Proposition 6.4. It follows from (6.27), (6.29), (6.30), (6.40) and (6.41) that

$$\begin{aligned}
&\liminf_{n\to\infty}\sqrt{\alpha_n^{(1)}}\langle U_n, \widetilde{z}_n\rangle \\
&= \liminf_{n\to\infty}\sqrt{\alpha_n^{(1)}}C_n\langle W^4\Lambda W, \widetilde{z}_n\rangle \\
&= \liminf_{n\to\infty}\sqrt{\alpha_n^{(1)}}\langle(-\Delta+\alpha_n^{(1)})^{-1}U_n, W^4\Lambda W\rangle \times \frac{1}{\langle(-\Delta+\alpha_n^{(1)})^{-1}W^4\Lambda W, W^4\Lambda W\rangle}\times\langle W^4\Lambda W, \widetilde{z}_n\rangle \\
&\ge -\frac{3\pi}{5}\times\frac{5}{\langle\Lambda W, W^4\Lambda W\rangle}\times\kappa\langle W^4\Lambda W, \Lambda W\rangle = -3\pi\kappa,
\end{aligned}$$

which yields (6.25).

□

6.1.2 Proof of Proposition 6.5

In this subsection, we shall prove Proposition 6.5. We first give the proof in case of $2 < p < 3$.

Proof of Proposition 6.5 in the case of $2 < p < 3$. Since

$$\lim_{n\to\infty}\int_0^1 V_n^p(\cdot,\theta)d\theta = W^p \qquad \text{strongly in } L^{\frac{p+1}{p}}(\mathbb{R}^3),$$

$\lim_{n\to\infty}\widetilde{z}_n = \kappa\Lambda W$ weakly in $\dot{H}^1(\mathbb{R}^3)$ and $\Lambda W = \frac{1}{2}W + x\cdot W$, we have

$$\lim_{n\to\infty}\left\langle \int_0^1 V_n^p(\cdot,\theta)d\theta, \widetilde{z}_n\right\rangle = \int_{\mathbb{R}^3} W^p(x)\kappa\Lambda W(x)\,dx = -\kappa\frac{5-p}{2(p+1)}\|W\|_{L^{p+1}}^{p+1}. \tag{6.43}$$

By (5.1) and (6.43), we have

$$\lim_{n\to\infty}\frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}}\left\langle \int_0^1 V_n^p(\cdot,\theta)d\theta, \widetilde{z}_n\right\rangle = \frac{12\pi(p+1)}{(5-p)\|W\|_{L^{p+1}}^{p+1}} \times \left(-\kappa\frac{5-p}{2(p+1)}\|W\|_{L^{p+1}}^{p+1}\right) = -6\pi\kappa.$$

Thus, we see that (6.23) holds. □

Next, we consider the case of $p = 2$. We remark that since $W \notin L^{p+1}(\mathbb{R}^3)$ when $p = 2$, (6.43) does not hold in this case. Thus, it becomes a subtle problem. To overcome the difficulty, we need the following lemma:

Lemma 6.9. *Let $p = 2$. We have the following:*

$$\limsup_{n\to\infty}|\log\alpha_n^{(1)}|^{-1}\int_{\mathbb{R}^3}\left(\int_0^1 V_n^p(x,\theta)d\theta\right)\widetilde{z}_n(x)\,dx = -3\sqrt{3}\pi\kappa \tag{6.44}$$

We will give the proof of Lemma 6.9 later. Admitting it, we shall give the proof of Proposition 6.4.

Proof of Proposition 6.5 in the case of $p = 2$. By (5.2) and (6.44), we have

$$\begin{aligned}\limsup_{n\to\infty}\frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}}\int_{\mathbb{R}^3}\left(\int_0^1 V_n^p(x,\theta)d\theta\right)\widetilde{z}_n(x)\,dx &= \limsup_{n\to\infty}\frac{\beta_n^{(1)}|\log\alpha_n^{(1)}|}{\sqrt{\alpha_n^{(1)}}}|\log\alpha_n^{(1)}|^{-1}\int_{\mathbb{R}^3}\left(\int_0^1 V_n^p(x,\theta)d\theta\right)\widetilde{z}_n(x)\,dx\\ &= \frac{2}{\sqrt{3}}\times(-3\sqrt{3}\pi\kappa) = -6\pi\kappa.\end{aligned}$$

Therefore, we find that (6.23) holds. □

We now give the proof of Lemma 6.9. As in the proof of Proposition 6.4, based on Lemma 5.14, we decompose $\int_0^1 V_n^p(x,\theta)d\theta$ as follows:

$$\int_0^1 V_n^p(x,\theta)d\theta = C_n W^4\Lambda W(x) + g_n(x), \tag{6.45}$$

where

$$C_n := \frac{\langle(-\Delta+\alpha_n)^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle}{\langle(-\Delta+\alpha_n)^{-1}W^4\Lambda W, W^4\Lambda W\rangle}. \tag{6.46}$$

It follows from (6.45) that

$$|\log\alpha_n^{(1)}|^{-1}\int_{\mathbb{R}^3}\int_0^1 V_n^p(x,\theta)d\theta z_n(x)\,dx = C_n|\log\alpha_n^{(1)}|^{-1}\langle W^4\Lambda W, \widetilde{z}_n\rangle + |\log\alpha_n^{(1)}|^{-1}\langle g_n, \widetilde{z}_n\rangle. \tag{6.47}$$

From the decomposition (6.45), we can easily verify that

$$\langle(-\Delta+\alpha_n)^{-1}g_n, W^4\Lambda W\rangle = 0.$$

As in Proposition 5.8, we obtain the following:

Lemma 6.10.

$$\liminf_{n\to\infty} |\log \alpha_n^{(1)}|^{-1}\langle(-\Delta+\alpha_n)^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle = -\frac{3\sqrt{3}}{5}\pi. \tag{6.48}$$

Since the proof of Lemma 6.10 is almost same as that of Proposition 5.8, we omit the proof.

Next, we shall show the following:

Lemma 6.11. *Let $p=2$. Then, we have*

$$\|(-\Delta+\alpha_n^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta\|_{L^\infty} \lesssim |\log \alpha_n^{(1)}|. \tag{6.49}$$

Proof. We use the formula (5.41) and divide the integral as follows:

$$(-\Delta+\alpha_n^{(1)})^{-1}\int_0^1 V_n^p(x,\theta)d\theta = \sum_{i=1}^3 \int_{A_i} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|}\int_0^1 V_n^p(y,\theta)d\theta dy,$$

where

$$A_1 := \left\{y\in\mathbb{R}^3 \colon |x-y|\le \frac{|x|}{2}\right\},$$

$$A_2 := \left\{y\in\mathbb{R}^3 \colon |x-y|> \frac{|x|}{2},\ |y|\le 2|x|\right\},$$

$$A_3 := \left\{y\in\mathbb{R}^3 \colon |x-y|> \frac{|x|}{2},\ |y|> 2|x|\right\}.$$

We first consider the region A_1. Observe from (4.1) and (6.7) that

$$\left|\int_0^1 V_n^p(x,\theta)d\theta\right| \lesssim |x|^{-p}. \tag{6.50}$$

Note that $|y|\ge|x|-|x-y|\ge|x|/2$ for $y\in A_1$. This together with (6.50) and $p=2$ implies that

$$\begin{aligned}
&\int_{A_1}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|}\int_0^1 V_n^p(y,\theta)d\theta dy \lesssim \int_{A_1}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||y|^p}dy = \int_{B(x,|x|/2)}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||y|^p}dy\\
&\lesssim \frac{1}{|x|^p}\int_{B(0,|x|/2)}\frac{e^{-\sqrt{\alpha_n^{(1)}}|y'|}}{|y'|}dy' \lesssim \frac{1}{|x|^p}\int_0^{|x|/2}\frac{e^{-\sqrt{\alpha_n^{(1)}}r}}{r}r^2\,dr \lesssim \frac{1}{|x|^p}\int_0^{|x|/2} e^{-\sqrt{\alpha_n^{(1)}}r}r\,dr\\
&\lesssim \frac{1}{|x|^p}\int_0^{|x|/2} r\,dr \lesssim 1.
\end{aligned}$$

Observe that $|x-y|>|x|/2$ for $y\in A_2$ and the function $\frac{e^{-\sqrt{\alpha_n^{(1)}}r}}{r}$ is decreasing in $r>0$. We have by (6.50) and $p=2$ that

$$\begin{aligned}
&\int_{A_2}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|}\int_0^1 V_n^p(y,\theta)d\theta dy \lesssim \int_{A_2}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||y|^p}dy \lesssim \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}|x|}{2}}}{|x|}\int_{A_2}\frac{1}{|y|^p}dy\\
&\lesssim \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}|x|}{2}}}{|x|}\int_{|y|\le 2|x|}\frac{1}{|y|^p}dy \lesssim \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}|x|}{2}}}{|x|}\int_0^{2|x|}\frac{r^2}{r^p}\,dr \lesssim e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}|x|} \lesssim 1.
\end{aligned}$$

Finally, we shall estimate $\int_{A_3}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|}\int_0^1 V_n^p(y,\theta)d\theta dy$. We first consider the case of $|x|\ge 1$. We note that $|x-y|\ge|y|-|x|\ge|y|-|y|/2\ge|y|/2$ for $y\in A_3$. This together with (6.50), the monotonicity of the function

$\frac{e^{-\sqrt{\alpha_n^{(1)}}r}}{r}$ and $p=2$ yields that

$$\int_{A_3} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 V_n^p(y,\theta)d\theta dy \lesssim \int_{A_3} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||y|^p} dy \lesssim \int_{A_3} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}|y|}}{|y|^{p+1}} dy$$

$$\lesssim \int_{2|x|}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p+1}} r^2\, dr \lesssim \int_{2|x|}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr = \int_{2|x|}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r}\, dr.$$

We estimate $\int_{2|x|}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r}\, dr$ dividing into 3 cases

(Case 1) $1 \le |x| \le (\alpha_n^{(1)})^{-\frac{1}{2}}/2$**.**

One has

$$\int_{2|x|}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r}\, dr \le \int_{2|x|}^{(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r}\, dr + \int_{(\alpha_n^{(1)})^{-\frac{1}{2}}}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r}\, dr \le \int_1^{(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{dr}{r} + \int_1^\infty \frac{e^{-\frac{s}{2}}}{s}\, ds$$

$$\lesssim |\log \alpha_n^{(1)}| + 1 \lesssim |\log \alpha_n^{(1)}|.$$

(Case 2) $|x| \ge (\alpha_n^{(1)})^{-\frac{1}{2}}/2$**.**

We see that

$$\int_{2|x|}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r}\, dr \le \int_{(\alpha_n^{(1)})^{-\frac{1}{2}}}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r}\, dr = \int_1^\infty \frac{e^{-\frac{1}{2}s}}{s}\, ds \lesssim 1.$$

(Case 3) $|x| < 1$**.**

It follows that

$$\int_{A_3} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 V_n^p(y,\theta)d\theta dy = \int_{A_3\cap|x-y|\ge 2} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 V_n^p(y,\theta)d\theta dy$$
$$+ \int_{A_3\cap|x-y|< 2} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 V_n^p(y,\theta)d\theta dy.$$

Since $\left\|\int_0^1 V_n^p(y,\theta)d\theta\right\|_{L^\infty} \le 1$, we have

$$\int_{A_3\cap|x-y|< 2} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 V_n^p(y,\theta)d\theta dy \le \int_{A_3\cap|x-y|< 2} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} dy$$

$$\le \int_{|y'|<2} \frac{e^{-\sqrt{\alpha_n^{(1)}}|y'|}}{|y'|} dy' \le \int_0^2 e^{-\sqrt{\alpha_n^{(1)}}|r|} r\, dr \le \int_0^2 r\, dr \lesssim 1.$$

For $|x-y| \ge 2$ and $|x| < 1$, we have $|y| \ge |x-y| - |x| \ge 1$. Using this together with (6.50), $|x-y| \ge |y| - |x| \ge |y| - |y|/2 \ge |y|/2$ for $y \in A_3$, the monotonicity of the function $\frac{e^{-\sqrt{\alpha_n^{(1)}}r}}{r}$ and $p=2$, we obtain

$$\int_{A_3\cap|x-y|\ge 2} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 V_n^p(y,\theta)d\theta dy \lesssim \int_{A_3\cap|x-y|\ge 2} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||y|^p} dy \lesssim \int_{A_3\cap|x-y|\ge 2} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}|y|}}{|y|^{p+1}} dy$$

$$\lesssim \int_{|y|\ge 1} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}|y|}}{|y|^{p+1}} dy \lesssim \int_1^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr \le \int_1^{(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr + \int_{(\alpha_n^{(1)})^{-\frac{1}{2}}}^\infty \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr$$

$$\lesssim \int_1^{(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{1}{r^{p-1}}\, dr + \int_1^\infty \frac{e^{-\frac{s}{2}}}{s^{p-1}}\, ds \lesssim |\log \alpha_n^{(1)}| + 1 \lesssim |\log \alpha_n^{(1)}|.$$

This completes the proof. $\square$

Next, we study the second term of (6.47). Following Lemma 6.6, we can obtain the following:

Lemma 6.12.

$$\lim_{n\to\infty} |\log \alpha_n^{(1)}|^{-1}\langle g_n, \widetilde{z}_n\rangle = 0. \tag{6.51}$$

Proof. We shall show that

$$\|(-\Delta + \alpha_n^{(1)})g_n\|_{L^\infty} \lesssim |\log \alpha_n^{(1)}|. \tag{6.52}$$

Once (6.52) holds, we can prove Lemma 6.12 by a similar argument in the proof of Lemma 6.6. Observe from (6.45) that

$$\|(-\Delta + \alpha_n^{(1)})^{-1}g_n\|_{L^\infty} \le |C_n|\|(-\Delta + \alpha_n^{(1)})^{-1}W^4\Lambda W\|_{L^\infty} + \|(-\Delta + \alpha_n^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta\|_{L^\infty}. \tag{6.53}$$

As in (5.44), we have

$$\langle(-\Delta + \alpha_\omega)^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle = -\frac{\sqrt{3}}{10}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_\omega}|x|}}{|x|}\int_0^1 V_n^p(x,\theta)d\theta\,dx + O(1).$$

This together with (6.50) and $p=2$ yields that

$$\begin{aligned}
\left|\langle(-\Delta + \alpha_n^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle\right| &= \left|-\frac{\sqrt{3}}{10}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_n^{(1)}}|x|}}{|x|}\int_0^1 V_n^p(x,\theta)d\theta\,dx + O_n(1)\right| \\
&\lesssim \int_0^1 dr + \int_1^{(\alpha_n^{(1)})^{-\frac{1}{2}}}\frac{dr}{r^{p-1}} + \int_{(\alpha_n^{(1)})^{-\frac{1}{2}}}^\infty \frac{e^{-\sqrt{\alpha_n^{(1)}}r}}{r^{p-1}}dr + O_n(1) \\
&\lesssim |\log \alpha_n^{(1)}| + \int_1^\infty \frac{e^{-s}}{s^{p-1}}dr + O_n(1) \\
&\lesssim |\log \alpha_n^{(1)}|.
\end{aligned} \tag{6.54}$$

By (6.46), (6.54) and Lemma 6.7, we have

$$|C_n| \lesssim |\log \alpha_n^{(1)}|. \tag{6.55}$$

Thus, we see from (6.53), (6.55) and (6.49) that

$$\begin{aligned}
\|(-\Delta + \alpha_n^{(1)})^{-1}g_n\|_{L^\infty} &\le |C_n|\|(-\Delta + \alpha_n^{(1)})^{-1}W^4\Lambda W\|_{L^\infty} + \|(-\Delta + \alpha_n^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta\|_{L^\infty} \\
&\lesssim |\log \alpha_n^{(1)}|\|W^4\Lambda W\|_{L^{\frac{3+\varepsilon}{2}}\cap L^{\frac{3-\varepsilon}{2}}} + |\log \alpha_n^{(1)}| \\
&\lesssim |\log \alpha_n^{(1)}|.
\end{aligned}$$

This completes the proof. □

We now give the proof of Lemma 6.9.

Proof of Lemma 6.9. Note that since $\lim_{n\to\infty}\widetilde{z}_n = \kappa\Lambda W$ weakly in $\dot{H}^1(\mathbb{R}^3)$ and $W^4\Lambda W \sim |x|^{-5}$ as $|x|\to\infty$, we see that

$$\lim_{n\to\infty}\langle W^4\Lambda W, \widetilde{z}_n\rangle = \kappa\langle W^4\Lambda W, \Lambda W\rangle. \tag{6.56}$$

Then, it follows from (6.47), (6.51), (6.56), (6.46), (6.30) and (6.48) that

$$\begin{aligned}
&\limsup_{n\to\infty} |\log\alpha_n^{(1)}|^{-1} \int_{\mathbb{R}^3} \int_0^1 V_n^p(x,\theta)d\theta \widetilde{z}_n(x)\,dx \\
&= \liminf_{n\to\infty} \Big(C_n |\log\alpha_n^{(1)}|^{-1} \langle W^4 \Lambda W, \widetilde{z}_n\rangle + |\log\alpha_n^{(1)}|^{-1}\langle g_n, \widetilde{z}_n\rangle \Big) \\
&= \kappa \langle W^4\Lambda W, \Lambda W\rangle \liminf_{n\to\infty} C_n |\log\alpha_n^{(1)}|^{-1} \\
&= \kappa \langle W^4\Lambda W, \Lambda W\rangle \liminf_{n\to\infty} |\log\alpha_n^{(1)}|^{-1} \frac{\langle (-\Delta+\alpha_n)^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle}{\langle (-\Delta+\alpha_n)^{-1} W^4\Lambda W, W^4\Lambda W\rangle} \\
&= 5\kappa \liminf_{n\to\infty} |\log\alpha_n^{(1)}|^{-1} \langle (-\Delta+\alpha_n)^{-1} \int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle \\
&= -3\sqrt{3}\pi\kappa.
\end{aligned}$$

Thus, we see that (6.44) holds. □

6.2 Case of $1 < p < 2$

In this subsection, we consider the case of $1 < p < 2$. Here, we will show the following:

Proposition 6.13. *Let $1 < p < 2$. We obtain the following:*

$$\liminf_{n\to\infty} \sqrt{\alpha_n^{(1)}} \langle \widetilde{u}_n^{(1)}(x) + \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot), \widetilde{z}_n\rangle = 0. \tag{6.57}$$

Proposition 6.14. *Let $1 < p < 2$. We obtain the following:*

$$\limsup_{n\to\infty} \frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}} \left\langle \int_0^1 V_n^p(x,\theta)d\theta, \widetilde{z}_n \right\rangle < -\frac{2\pi}{5} 3^{\frac{p+1}{2}} \int_0^\infty \frac{e^{-(p+1)s}}{s^{p-1}}\,ds. \tag{6.58}$$

From Propositions 6.13 and 6.14, we can prove Theorem (ii) in the case of $1 < p < 2$ immediately.

Proof of Theorem 2.5 (i) in the case of $1 < p < 2$. Suppose to the contrary that there exists $\{\omega_n\}$ in $(0,\infty)$ with $\lim_{n\to\infty}\omega_n = 0$ such that for each $n \in \mathbb{N}$, $u_n^{(i)}$ $(i = 1, 2)$ is a solution to (1.1) with $\omega = \omega_n$ satisfying $u_n^{(1)} \neq u_n^{(2)}$ and $\lim_{n\to\infty} \|u_n^{(i)}\|_{L^\infty} = \infty$. Then, we can derive a contradiction from (6.21), (6.57) and (6.58). This completes the proof. □

6.2.1 Proof of Proposition 6.13

In this subsection, we will prove Proposition 6.13. To this end, we need the following lemma:

Lemma 6.15. *We take δ and $L > 0$ arbitrary and fix it. For any $\varepsilon > 0$, there exists $N_0 = N_0(\delta, L, \varepsilon) \in \mathbb{N}$ such that for $n \geq N_0$, we obtain*

$$|\widetilde{z}_n(r)| < \frac{\varepsilon}{r} \qquad \textit{for any } r \in [\delta(\alpha_n^{(1)})^{-\frac{1}{2}}, L(\alpha_n^{(1)})^{-\frac{1}{2}}]. \tag{6.59}$$

Proof. We put $\widehat{z}_n(s) = \widetilde{z}_n(r)$ and $s = (\alpha_n^{(1)})^{\frac{1}{2}} r$. Then, we see that $\widehat{z}_n$ satisfies

$$-\partial_{ss}\widehat{z}_n - \frac{2}{s}\partial_s \widehat{z}_n + \widehat{z}_n = p\beta_n^{(1)} (\alpha_n^{(1)})^{\frac{p-3}{2}} \int_0^1 \widehat{V}_n^{p-1}(s,\theta)d\theta \overline{z}_n + 5\alpha_n^{(1)} \int_0^1 \widehat{V}_n^4(s,\theta)d\theta \widehat{z}_n. \tag{6.60}$$

where

$$\widehat{V}_n(s,\theta) := \theta\widehat{u}_n^{(1)}(s) + (1-\theta)\nu_n^{-1}\widehat{u}_n^{(2)}(\nu_n^4 s) \tag{6.61}$$

and $\widehat{u}_\omega^{(i)}(s) = (\alpha_n^{(i)})^{-\frac{1}{2}}\widetilde{u}_n^{(i)}(r)$ for $i=1$ and 2. In addition, we put

$$\overline{z}_n(s) = s\widehat{z}_n(s). \tag{6.62}$$

Then, we see from (6.60) that $\overline{z}_n$ satisfies

$$-\partial_{ss}\overline{z}_n + \overline{z}_n = p\beta_n^{(1)}(\alpha_n^{(1)})^{\frac{p-3}{2}} s^{-(p-1)}\int_0^1 \overline{V}_n^{p-1}(s,\theta)d\theta\overline{z}_n + 5\alpha_n^{(1)}s^{-4}\int_0^1 \overline{V}_n^4(s,\theta)d\theta\overline{z}_n, \tag{6.63}$$

where

$$\overline{V}_n(s,\theta) := s\widehat{V}_n(s,\theta) = \theta\overline{u}_n^{(1)}(s) + (1-\theta)\nu_n^{-1}\overline{u}_n^{(2)}(\nu_n^4 s). \tag{6.64}$$

Here $\overline{u}_n^{(i)}(s) := s\widehat{u}_n^{(i)}(s)$ for $i=1$ and 2. From Lemma 6.2, we see that $\sup_{n\in\mathbb{N}}\|\overline{z}_n\|_{L^\infty} < C_1$ for some $C_1>0$. In addition, it follows from (4.1) and Lemma 5.21 that

$$|\overline{u}_n^{(i)}(s)| \lesssim 1 \qquad \text{for all } s>0.$$

and

$$\lim_{n\to\infty}\overline{u}_n^{(i)}(s) = \overline{u}_0(s) \qquad \lim_{n\to\infty}\frac{d\overline{u}_{\omega_n}}{ds}(s) = \frac{d\overline{u}_0}{ds}(s) \qquad \text{uniformly in } s\in[\delta,L],$$

where $\overline{u}_0$ is the unique positive solution to (5.4). Then, as in the similar argument in the proof of Lemma 5.21, we can apply the Ascoli-Arzela theorem and find that there exist a subsequence of $\{\overline{z}_n\}$ (we still denote it by the same letter) and a function $\overline{z}_0$ such that $\lim_{n\to\infty}\overline{z}_n(s) = \overline{z}_0(s), \lim_{n\to\infty}\frac{d\overline{z}_n}{ds}(s) = \frac{d\overline{z}_0}{ds}(s)$ uniformly in $s\in[\delta,L]$. Then, we see that $\overline{z}_0$ satisfies

$$-\partial_{ss}\overline{z}_0 + \overline{z}_0 = p\theta_0\overline{u}_0^{p-1}\overline{z}_0 \qquad \text{in } s\in[\delta,L].$$

Since $\delta>0$ and $L>0$ are arbitrary, we see that $\overline{z}_0$ satisfies

$$-\partial_{ss}\overline{z}_0 + \overline{z}_0 = p\theta_0\overline{u}_0^{p-1}\overline{z}_0 \qquad \text{in } s\in(-\infty,\infty).$$

It follows from non-degeneracy of $\overline{u}_0$ (see Proposition 2.1) that $\overline{z}_0 = 0$. Thus, for any $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that for $n\geq N_\varepsilon$, we have

$$|\overline{z}_n(s)| < \varepsilon \qquad \text{for all } s\in[\delta,L].$$

This together with $\overline{z}_n(s) = r\widetilde{z}_n(r)$ and $s = (\alpha_n^{(1)})^{\frac{1}{2}}r$ yields that

$$|\widetilde{z}_n(r)| = \frac{|\overline{z}_n(s)|}{r} < \frac{\varepsilon}{r} \qquad \text{for all } r\in[\delta(\alpha_n^{(1)})^{-\frac{1}{2}}, L(\alpha_n^{(1)})^{-\frac{1}{2}}].$$

This completes the proof. □

We are now in a position to prove Proposition 6.13.

Proof of Proposition 6.13. We take δ and $L > 0$ arbitrary and fix it. It follows that

$$\begin{aligned}
&|\sqrt{\alpha_n^{(1)}}\langle \widetilde{u}_n^{(1)}(x) + \nu_n \widetilde{u}_n^{(2)}(\nu_n^2 \cdot), \widetilde{z}_n\rangle| \\
&\le \sqrt{\alpha_n^{(1)}} 4\pi \int_0^{\delta(\alpha_n^{(1)})^{-\frac{1}{2}}} (\widetilde{u}_n^{(1)}(r) + \nu_n \widetilde{u}_n^{(2)})|\widetilde{z}_n| r^2\, dr \\
&\quad + \sqrt{\alpha_n^{(1)}} 4\pi \int_{\delta(\alpha_n^{(1)})^{-\frac{1}{2}}}^{L(\alpha_n^{(1)})^{-\frac{1}{2}}} (\widetilde{u}_n^{(1)}(r) + \nu_n \widetilde{u}_n^{(2)})|\widetilde{z}_n| r^2\, dr \\
&\quad + \sqrt{\alpha_n^{(1)}} 4\pi \int_{L(\alpha_n^{(1)})^{-\frac{1}{2}}}^{\infty} (\widetilde{u}_n^{(1)}(r) + \nu_n \widetilde{u}_n^{(2)})|\widetilde{z}_n| r^2\, dr \\
&=: I_n + II_n + III_n
\end{aligned} \tag{6.65}$$

From (4.1), (6.8), (6.15) and (6.59), we see that there exists $L_\delta > 0$ such that if $L > L_\delta$, we obtain

$$I_n \lesssim \sqrt{\alpha_n^{(1)}} \int_0^{\delta(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{1}{r^2} r^2\, dr \lesssim \sqrt{\alpha_n^{(1)}} \times \delta(\alpha_n^{(1)})^{-\frac{1}{2}} = \delta. \tag{6.66}$$

$$III_n \lesssim \sqrt{\alpha_n^{(1)}} \int_{L(\alpha_n^{(1)})^{-\frac{1}{2}}}^{\infty} e^{-\frac{\sqrt{\alpha_\omega^{(1)}}}{2} r}\, dr \lesssim e^{-\frac{L}{2}} < \delta. \tag{6.67}$$

Applying Lemma 6.15 with $\varepsilon = \delta/L$, there exists $N_0 \in \mathbb{N}$ such that for $n \ge N_0$, we have

$$|\widetilde{z}_n(r)| < \frac{\delta}{Lr} \qquad \text{for all } r \in [\delta, L].$$

This together with (4.1) yields that

$$II_n \lesssim \sqrt{\alpha_n^{(1)}} \int_{\delta(\alpha_n^{(1)})^{-\frac{1}{2}}}^{L(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{\delta}{Lr^2} r^2\, dr \le \sqrt{\alpha_n^{(1)}} \int_0^{L(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{\delta}{Lr^2} r^2\, dr = \sqrt{\alpha_n^{(1)}} \times L(\alpha_n^{(1)})^{-\frac{1}{2}} \times \frac{\delta}{L} = \delta. \tag{6.68}$$

Since $\delta > 0$ are arbitrary, we see from (6.65)–(6.68) that (6.57) holds. This completes the proof. $\square$

6.2.2 Proof of Proposition 6.14

Here, we shall show Proposition 6.14. As in the proof of Proposition 6.4, based on Lemma 5.14, we decompose $\int_0^1 V_n^p(x,\theta)d\theta$ as follows:

$$\int_0^1 V_n^p(x,\theta)d\theta = C_n W^4 \Lambda W(x) + g_n(x), \tag{6.69}$$

where

$$C_n := \frac{\langle (-\Delta + \alpha_n^{(1)})^{-1} \int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle}{\langle (-\Delta + \alpha_n^{(1)})^{-1} W^4 \Lambda W, W^4 \Lambda W\rangle}. \tag{6.70}$$

It follows from (6.69) that

$$\frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}} \left\langle \int_0^1 V_n^p(x,\theta)d\theta, z_n(x) \right\rangle = C_n \frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}} \langle W^4 \Lambda W, \widetilde{z}_n\rangle + \frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}} \langle g_n, \widetilde{z}_n\rangle. \tag{6.71}$$

From the decomposition (6.69) and (6.70), we can easily verify that

$$\langle (-\Delta + \alpha_n^{(1)})^{-1} g_n, W^4 \Lambda W\rangle = 0.$$

Using the resolvent estimate (5.49), we shall show the following:

Proposition 6.16. *Let $1 < p < 2$. We obtain*

$$\lim_{n\to\infty} \left(\alpha_n^{(1)}\right)^{\frac{2-p}{2}} |\langle g_n, \widetilde{z}_n\rangle| = 0.$$

To prove Proposition 6.16 , we need the following:

Lemma 6.17. *Let $1 < p < 2$. One has*

$$\|(-\Delta + \alpha_n^{(1)})^{-1} \int_0^1 V_n^p(\cdot, \theta) d\theta\|_{L^\infty} \lesssim (\alpha_n^{(1)})^{-\frac{2-p}{2}}. \tag{6.72}$$

Proof. We use the formula (5.41) and divide the integral as follows:

$$|(-\Delta + \alpha_n^{(1)})^{-1} \int_0^1 V_n^p(\cdot, \theta) d\theta| \le \sum_{i=1}^{3} \int_{A_i} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 |V_n^p(y, \theta)| d\theta dy,$$

where

$$A_1 := \left\{ y \in \mathbb{R}^3 \colon |x-y| \le \frac{|x|}{2} \right\},$$

$$A_2 := \left\{ y \in \mathbb{R}^3 \colon |x-y| > \frac{|x|}{2},\ |y| \le 2|x| \right\},$$

$$A_3 := \left\{ y \in \mathbb{R}^3 \colon |x-y| > \frac{|x|}{2},\ |y| > 2|x| \right\}.$$

Note that

$$|y| \ge |x| - |x-y| \ge |x|/2 \qquad \text{for } y \in A_1. \tag{6.73}$$

For $|x| \le \alpha_n^{-\frac{1}{2}}$, we obtain

$$\begin{aligned}
&\int_{A_1} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 |V_n^p(y, \theta)| d\theta dy \lesssim \int_{A_1} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||y|^p} dy \lesssim \int_{B(x,|x|/2)} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||x|^p} dy \\
&\lesssim \frac{1}{|x|^p} \int_{B(0,|x|/2)} \frac{e^{-\sqrt{\alpha_n^{(1)}}|y'|}}{|y'|} dy' \lesssim \frac{1}{|x|^p} \int_0^{|x|/2} \frac{e^{-\sqrt{\alpha_n^{(1)}}r}}{r} r^2\, dr \lesssim \frac{1}{|x|^p} \int_0^{|x|/2} e^{-\sqrt{\alpha_n^{(1)}}r} r\, dr \\
&\le \frac{1}{|x|^p} \int_0^{|x|/2} r\, dr \lesssim |x|^{2-p} \le \left(\alpha_n^{(1)}\right)^{\frac{p-2}{2}}.
\end{aligned} \tag{6.74}$$

Moreover, since $|x-y| \ge |x|/2$ for $y \in A_2$ and the function $\frac{e^{-\sqrt{\alpha_n} r}}{r}$ is decreasing in $r > 0$, for $|x| \le \alpha_n^{-\frac{1}{2}}$, we have by (6.50) that

$$\begin{aligned}
&\int_{A_2} \frac{e^{-\sqrt{\alpha_n}|x-y|}}{|x-y|} \int_0^1 |V_n^p(y, \theta)| d\theta dy \lesssim \int_{A_2} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||y|^p} dy \le 2 \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}|x|}{2}}}{|x|} \int_{A_2} \frac{1}{|y|^p} dy \\
&\lesssim \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}|x|}{2}}}{|x|} \int_{|y|\le 2|x|} \frac{1}{|y|^p} dy \lesssim \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}|x|}{2}}}{|x|} \int_0^{2|x|} \frac{1}{r^p} r^2\, dr \lesssim e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}|x|} |x|^{2-p} \lesssim \left(\alpha_n^{(1)}\right)^{\frac{p-2}{2}}.
\end{aligned} \tag{6.75}$$

Finally, we shall estimate $\int_{A_3} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 |V_n^p(y, \theta)| d\theta dy$. We note that $|x-y| \ge |y| - |x| \ge |y| - |y|/2 \ge |y|/2$

for $y \in A_3$. This together with (6.50) yields that

$$\begin{aligned}&\int_{A_3} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y|} \int_0^1 |V_n^p(y,\theta)| d\theta dy \lesssim \int_{A_3} \frac{e^{-\sqrt{\alpha_n^{(1)}}|x-y|}}{|x-y||y|^p} dy \lesssim \int_{A_3} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}|y|}}{|y|^{p+1}} dy \\ &\lesssim \int_{2|x|}^{\infty} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p+1}} r^2\, dr \lesssim \int_{2|x|}^{\infty} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr.\end{aligned} \tag{6.76}$$

It follows that

$$\begin{aligned}\int_{2|x|}^{\infty} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr &\le \int_0^{\infty} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr \le \int_0^{(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr + \int_{(\alpha_n^{(1)})^{-\frac{1}{2}}}^{\infty} \frac{e^{-\frac{\sqrt{\alpha_n^{(1)}}}{2}r}}{r^{p-1}}\, dr \\ &\le \int_0^{(\alpha_n^{(1)})^{-\frac{1}{2}}} \frac{dr}{r^{p-1}} + \left(\alpha_n^{(1)}\right)^{\frac{p-2}{2}} \int_1^{\infty} \frac{e^{-\frac{s}{2}}}{s^{p-1}}\, ds \\ &\lesssim \left(\alpha_n^{(1)}\right)^{\frac{p-2}{2}} + \left(\alpha_n^{(1)}\right)^{\frac{p-2}{2}} \lesssim \left(\alpha_n^{(1)}\right)^{\frac{p-2}{2}}.\end{aligned} \tag{6.77}$$

From (6.74)– (6.77), we see that (6.72) holds. □

Using Lemma 6.17, we will show Proposition 6.16.

Proof of Proposition 6.16. The proof is similar to that of (6.29). However, we need several modifications. By (6.33), we have

$$\langle g_n, \widetilde{z}_n \rangle = \langle (1 - 5(-\Delta + \alpha_n^{(1)})^{-1} W^4)^{-1} (-\Delta + \alpha_n^{(1)})^{-1} g_n, R_n \widetilde{z}_n \rangle,$$

where R_n is given by (6.31). As in (6.34), we obtain

$$\left|\langle g_n, \widetilde{z}_n \rangle\right| \lesssim \|(-\Delta + \alpha_n^{(1)})^{-1} g_n\|_{L^\infty} \|R_n \widetilde{z}_n\|_{L^1}. \tag{6.78}$$

Similarly to (6.35), we observe from by Lemmas 3.5, 6.17 and 5.13, that

$$|C_n| \lesssim |\langle (-\Delta + \alpha_n^{(1)})^{-1} \int_0^1 V_n^p(\cdot,\theta) d\theta, W^4 \Lambda W \rangle| \lesssim \left(\alpha_n^{(1)}\right)^{\frac{p-2}{2}}.$$

Using this, by a similar argument to (6.36), we have

$$\begin{aligned}\|(-\Delta + \alpha_n^{(1)})^{-1} g_n\|_{L^\infty} &\le |C_n| \|(-\Delta + \alpha_n^{(1)})^{-1} W^4 \Lambda W\|_{L^\infty} + \|(-\Delta + \alpha_n^{(1)})^{-1} \int_0^1 V_n^p(x,\theta) d\theta\|_{L^\infty} \\ &\lesssim (\alpha_n^{(1)})^{\frac{p-2}{2}} \|W^4 \Lambda W\|_{L^{\frac{3}{2}+\varepsilon} \cap L^{\frac{3}{2}-\varepsilon}} + (\alpha_n^{(1)})^{\frac{p-2}{2}} \\ &\lesssim \left(\alpha_n^{(1)}\right)^{\frac{p-2}{2}}.\end{aligned} \tag{6.79}$$

We shall show that $\lim_{n\to\infty} \|R_n \widetilde{z}_n\|_{L^1} = 0$. Using (4.1) and Lemma 6.2 for $0 < r \le \delta \alpha_{1,n}^{-\frac{1}{2}}$, (4.1) and Lemma 6.15 with $\varepsilon = \frac{\delta}{L^{3-p}}$ for $\delta \left(\alpha_{1,n}^{(1)}\right)^{-\frac{1}{2}} < r < L \left(\alpha_{1,n}^{(1)}\right)^{-\frac{1}{2}}$, and (6.15) and (6.8) for $r \ge L \left(\alpha_{1,n}^{(1)}\right)^{-\frac{1}{2}}$, we obtain

$$\begin{aligned}\|\widetilde{u}_n^{p-1} \widetilde{z}_n\|_{L^1} &\lesssim \int_0^{\delta\left(\alpha_{1,n}^{(1)}\right)^{-\frac{1}{2}}} \frac{1}{r^{p-2}}\, dr + \int_{\delta\left(\alpha_{1,n}^{(1)}\right)^{-\frac{1}{2}}}^{L\left(\alpha_{1,n}^{(1)}\right)^{-\frac{1}{2}}} \frac{\delta}{L^{3-p} r^{p-2}}\, dr + \int_{L\left(\alpha_{1,n}^{(1)}\right)^{-\frac{1}{2}}}^{\infty} \frac{e^{-\frac{(p-1)}{2}\sqrt{\alpha_{1,n}^{(1)}}r}}{r^{p-2}}\, dr \\ &\lesssim \delta^{3-p} \left(\alpha_{1,n}^{(1)}\right)^{\frac{p-3}{2}} + \delta \left(\alpha_{1,n}^{(1)}\right)^{\frac{p-3}{2}} + \left(\alpha_{1,n}^{(1)}\right)^{\frac{p-3}{2}} \int_L^{\infty} \frac{e^{-\frac{(p-1)}{2}s}}{s^{p-2}}\, ds \lesssim \delta \left(\alpha_{1,n}^{(1)}\right)^{\frac{p-3}{2}}.\end{aligned}$$

We put $\eta_n(x,\theta) := V_n(x,\theta) - W$. Then, as in (6.39), we have by (6.31) and the Hölder inequality that

$$\begin{aligned}\|R_n\widetilde{z}_n\|_{L^1} &\lesssim \sup_{0<\theta<1} \|(\widetilde{u}_n^3 + W^3)\eta_n(\cdot,\theta)\widetilde{z}_n\|_{L^1} + \delta\beta_{1,n}^{(1)}\|\widetilde{u}_n^{p-1}\widetilde{z}_n\|_{L^1} \\ &\lesssim \left(\|\widetilde{u}_n\|_{L^{\frac{9}{2}}}^3 + \|W\|_{L^{\frac{9}{2}}}^3\right)\|\widetilde{z}_n\|_{L^6} \sup_{0<\theta<1}\|\eta_n(\cdot,\theta)\|_{L^6} + \delta\beta_{1,n}^{(1)}\left(\alpha_{1,n}^{(1)}\right)^{\frac{p-3}{2}} \\ &\lesssim \sup_{0<\theta<1}\|\eta_n(\cdot,\theta)\|_{L^6} + \delta\beta_{1,n}^{(1)}\left(\alpha_{1,n}^{(1)}\right)^{\frac{p-3}{2}}.\end{aligned} \tag{6.80}$$

This together with (5.3) and (6.38) yields that

$$\limsup_{n\to\infty}\|R_n\widetilde{z}_n\|_{L^1} < \delta. \tag{6.81}$$

Thus, by (6.78), (6.79) and (6.81), we obtain

$$\limsup_{n\to\infty}|(\alpha_{1,n}^{(1)})^{\frac{2-p}{2}}\langle g_n, \widetilde{z}_n\rangle| \lesssim \limsup_{n\to\infty}(\alpha_{1,n}^{(1)})^{\frac{2-p}{2}}(\alpha_{1,n}^{(1)})^{\frac{p-2}{2}}\|R_n\widetilde{z}_n\|_{L^1} = \limsup_{n\to\infty}\|R_n\widetilde{z}_n\|_{L^1} < \delta.$$

Since $\delta > 0$ is arbitrary, we have obtained the desired result. □

Proof of Proposition 6.14. From (6.69) (6.70), Proposition 6.16, (6.30) and $\lim_{n\to\infty}\widetilde{z}_n = \kappa\Lambda W$ weakly in $\dot{H}^1(\mathbb{R}^3)$, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}}\left\langle\int_0^1 V_n^p(x,\theta)d\theta, \widetilde{z}_n\right\rangle \\ &= \limsup_{n\to\infty}\frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}}\left\langle C_nW^4\Lambda W + g_n, \widetilde{z}_n\right\rangle \\ &= \limsup_{n\to\infty}\frac{\beta_n^{(1)}}{\sqrt{\alpha_n^{(1)}}}\frac{\langle(-\Delta+\alpha_n^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle}{\langle(-\Delta+\alpha_n^{(1)})^{-1}W^4\Lambda W, W^4\Lambda W\rangle}\left\langle W^4\Lambda W, \widetilde{z}_n\right\rangle \\ &= 5\kappa\limsup_{n\to\infty}\frac{\beta_n^{(1)}}{(\alpha_n^{(1)})^{\frac{3-p}{2}}} \times (\alpha_n^{(1)})^{\frac{2-p}{2}}\langle(-\Delta+\alpha_n)^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle.\end{aligned} \tag{6.82}$$

We put $Z := \Lambda W - \left(-\frac{\sqrt{3}}{2}|x|^{-1}\right)$. Then, as in the proof of Proposition Proposition 5.8, we have

$$\begin{aligned}\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle &= \frac{1}{5}\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, -\Delta\Lambda W\rangle \\ &= -\frac{\sqrt{3}}{10}\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, -\Delta|x|^{-1}\rangle \\ &\quad + \frac{1}{5}\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, -\Delta Z\rangle.\end{aligned} \tag{6.83}$$

In addition, note that $-\Delta|x|^{-1} = 4\pi\delta_0$ in $\mathcal{D}'(\mathbb{R}^3)$ (see e.g. [36, Page 156]). Thus, it follows from (5.41) that

$$\begin{aligned}&-\frac{\sqrt{3}}{10}\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(x,\theta)d\theta, (-\Delta)|x|^{-1}\rangle \\ &= -\frac{2\sqrt{3}\pi}{5}\left\{(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(x,\theta)d\theta\right\}(0) \\ &= -\frac{\sqrt{3}}{10}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_\omega^{(1)}}|x|}}{|x|}\int_0^1 V_n^p(x,\theta)d\theta\,dx.\end{aligned} \tag{6.84}$$

By the Hölder inequality and Lemma 5.10, $\int_0^1 V_n^p(x,\theta)d\theta = O(|x|^{-p})$ and $Z = O(|x|^{-3})$ as $|x| \to \infty$, we have

$$\begin{aligned}
&\left|\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(x,\theta)d\theta, \Delta Z\rangle\right| = \left|\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(x,\theta)d\theta, \left\{\alpha_\omega - (-\Delta+\alpha_\omega^{(1)})\right\} Z\rangle\right| \\
&\le \alpha_\omega \|(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(x,\theta)d\theta\|_{L^6}\|Z\|_{L^{\frac{6}{5}}} + \left\|\int_0^1 V_n^p(x,\theta)d\theta\right\|_{L^6}\|Z\|_{L^{\frac{6}{5}}} \qquad (6.85) \\
&\lesssim \alpha_\omega^{(1)} \times \left(\alpha_{2,\omega}^{(1)}\right)^{\frac{3}{8}p-\frac{5}{4}} \|\int_0^1 V_n^p(x,\theta)d\theta\|_{L^{\frac{4}{p}}}\|Z\|_{L^{\frac{6}{5}}} + 1 \lesssim 1.
\end{aligned}$$

From (5.40), (6.84) and (6.85), we obtain

$$\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(x,\theta)d\theta, W^4\Lambda W\rangle = -\frac{\sqrt{3}}{10}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_\omega^{(1)}}|x|}}{|x|}\int_0^1 V_n^p(x,\theta)d\theta\, dx + O(1). \qquad (6.86)$$

From (6.86) and (5.5), we obtain

$$\begin{aligned}
&\langle(-\Delta+\alpha_\omega^{(1)})^{-1}\int_0^1 V_n^p(x,\theta)d\theta, W^4\Lambda W\rangle \\
&= -\frac{\sqrt{3}}{10}\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\alpha_\omega^{(1)}}|x|}}{|x|}\int_0^1 V_n^p(x,\theta)d\theta(x)\, dx + O(1) \\
&\le -\frac{\sqrt{3}(1-\varepsilon)^p}{10} \times 4\pi \int_{R_\varepsilon}^\infty \frac{e^{-\sqrt{\alpha_\omega^{(1)}}r}}{r} Y_\omega^p(r) \times r^2\, dr + O(1) \\
&= -\frac{2(1-\varepsilon)^p\pi}{5} 3^{\frac{p+1}{2}} \int_{R_\varepsilon}^\infty \frac{e^{-(p+1)\sqrt{\alpha_\omega^{(1)}}r}}{r^{p-1}}\, dr + O(1) \\
&= -\frac{2(1-\varepsilon)^p\pi}{5} 3^{\frac{p+1}{2}} (\alpha_\omega^{(1)})^{\frac{p-2}{2}} \int_{R_\varepsilon\sqrt{\alpha_\omega^{(1)}}}^\infty \frac{e^{-(p+1)s}}{s^{p-1}}\, ds + O(1).
\end{aligned}$$

$$(\alpha_n^{(1)})^{\frac{2-p}{2}}\langle(-\Delta+\alpha_n^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle \le -\frac{2(1-\varepsilon)^p\pi}{5}3^{\frac{p+1}{2}}\int_{R_\varepsilon\sqrt{\alpha_\omega^{(1)}}}^\infty \frac{e^{-(p+1)s}}{s^{p-1}}\, ds + O((\alpha_\omega^{(1)})^{\frac{2-p}{2}}).$$

Thus, we have

$$\limsup_{n\to\infty}(\alpha_n^{(1)})^{\frac{2-p}{2}}\langle(-\Delta+\alpha_n^{(1)})^{-1}\int_0^1 V_n^p(\cdot,\theta)d\theta, W^4\Lambda W\rangle \le -\frac{2(1-\varepsilon)^p\pi}{5}3^{\frac{p+1}{2}}\int_0^\infty \frac{e^{-(p+1)s}}{s^{p-1}}\, ds.$$

Since $\varepsilon > 0$ is arbitrary, we see that (6.58) holds. This concludes the proof. □

6.3 Proof of Theorem 2.1

By using Theorems 2.3 (i) and 2.5 (i), we shall show Theorem 2.1.

Proof of Theorem 2.1. Suppose to the contrary that there exists $\{\omega_n\}$ in $(0,\infty)$ with $\lim_{n\to\infty}\omega_n = 0$ such that for each $n \in \mathbb{N}$, $u_n^{(i)}$ $(i = 1,2,3)$ is a solution to (1.1) with $\omega = \omega_n$ satisfying $u_n^{(i)} \neq u_n^{(j)}$ $(i \neq j)$. We first consider the following case:

(Case 1). $\limsup_{n\to\infty}\|u_n^{(i)}\|_{L^\infty} < \infty$ for $i = 1,2$.

We can verify from (Case 1) does not occur from Theorem 2.3 (i). Next, we consider the following case:

(Case 2). There exists a subsequence of $\{u_n^{(i)}\}$ $(i = 1, 2)$ (we still denote it by the same symbol) such that $\lim_{n\to\infty} \|u_n^{(i)}\|_{L^\infty} = \infty$ for $i = 1, 2$.

Then, we can deny the possibility of (Case 2) by Theorem 2.5 (i). Finally, we consider the following case:

(Case 3). There exists a subsequence $\{u_{n_k}^{(i)}\}$ of $\{u_n^{(i)}\}$ $(i = 1, 2)$ (we still denote it by the same symbol) such that $\lim_{k\to\infty} \|u_{n_k}^{(1)}\|_{L^\infty} = \infty$ and $\limsup_{k\to\infty} \|u_{n_k}^{(2)}\|_{L^\infty} < \infty$.

Then, we see that either $\lim_{k\to\infty} \|u_{n_k}^{(3)}\|_{L^\infty} = \infty$ or $\limsup_{k\to\infty} \|u_{n_k}^{(3)}\|_{L^\infty} < \infty$ occurs. Then, using Theorems 2.3 (i) and 2.5 (i), we can show that neither of the two cases occurs by a similar argument as above. $\square$

7 Non-degeneracy of the large solution u_ω

In this section, we shall show the non-degeneracy of the positive solution u_ω for sufficiently small $\omega > 0$ (Theorem 2.5 (ii)). The proof is similar to that of the local uniqueness of u_ω (Theorem 2.5 (i)). We shall show the non-degeneracy of the positive solution u_ω by contradiction. Suppose to the contrary that there exist a sequence $\{\omega_n\}$ with $\lim_{n\to\infty} \omega_n = 0$ and $z_n \in H^1_{\mathrm{rad}}(\mathbb{R}^3) \setminus \{0\}$ such that

$$(-\Delta + \omega_n - p u_{\omega_n}^{p-1} - 5u_{\omega_n}^4) z_n = 0 \qquad \text{in } \mathbb{R}^3. \tag{7.1}$$

By the elliptic regularity, we see that $z_n \in C^2(\mathbb{R}^3)$. We put

$$M_n := \|u_{\omega_n}\|_{L^\infty}, \qquad \widetilde{u}_n(x) := M_n^{-1} u_{\omega_n}(M_n^{-2} x), \qquad \widetilde{z}_n := z_n(M_n^{-2}\cdot). \tag{7.2}$$

Then, $\widetilde{z}_n$ satisfies

$$(-\Delta + \alpha_n - p\beta_n \widetilde{u}_n^{p-1} - 5\widetilde{u}_n^4)\widetilde{z}_n = 0 \qquad \text{in } \mathbb{R}^3, \tag{7.3}$$

where

$$\alpha_n := \omega_n M_n^{-4}, \qquad \beta_n := M_n^{p-5}.$$

Since equation (7.3) is linear, we may assume that $\|\nabla \widetilde{z}_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$.

Then, as in [1, Page 22], we obtain the following lemma:

Lemma 7.1. *Let $1 < p < 3$. Then, there exists $C_1 > 0$ such that for any $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$,*

$$|\widetilde{z}_n(x)| \le C_1 |x|^{-1}.$$

It follows from $\|\nabla z_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$ that there exists a subsequence of $\{\widetilde{z}_n\}$ (we still denote it by the same letter) and a function $\widetilde{z}_\infty \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \widetilde{z}_n = \widetilde{z}_\infty \qquad \text{weakly in } \dot{H}^1(\mathbb{R}^3) \text{ and strongly in } L^q(\mathbb{R}^3) \ \ (3 < q < 6).$$

Since $\lim_{n\to\infty} \widetilde{u}_n = W$ strongly in $\dot{H}^1 \cap L^q(\mathbb{R}^3)$ $(3 < q < 6)$ and $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, we see that

$$\begin{aligned}
\langle (-\Delta - 5W^4)\widetilde{z}_\infty, \phi\rangle &= \langle (-\Delta - 5W^4)\widetilde{z}_\infty, \phi\rangle - \langle (-\Delta + \alpha_n - \beta_n \widetilde{u}_n^{p-1} - 5\widetilde{u}_n^4)\widetilde{z}_n, \phi\rangle \\
&= \big(\langle (-\Delta - 5W^4)\widetilde{z}_\infty, \phi\rangle - \langle (-\Delta - 5\widetilde{u}_n^4)\widetilde{z}_n, \phi\rangle\big) + \langle (\alpha_n - \beta_n \widetilde{u}_n^{p-1})\widetilde{z}_n, \phi\rangle \\
&\to 0 \qquad \text{as } n \to \infty
\end{aligned}$$

for any $\phi \in C_0^\infty(\mathbb{R}^3)$. This implies that

$$(-\Delta - 5W^4)\widetilde{z}_\infty = 0 \qquad \text{in } H^{-1}(\mathbb{R}^3). \tag{7.4}$$

Next, similarly to [1], we can derive the following identity:

Lemma 7.2 (c.f. Page 22 of [1])**.** *Let $1 < p < 3$, $\widetilde{u}_n$ and $\widetilde{z}_n$ be the functions defined by (7.2). Then, we have*

$$\frac{\alpha_n}{\beta_n}\int_{\mathbb{R}^3} \widetilde{u}_n \widetilde{z}_n \, dx = \frac{5-p}{4}\int_{\mathbb{R}^3} \widetilde{u}_n^p \widetilde{z}_n \, dx \qquad \text{for all } n \in \mathbb{N}. \tag{7.5}$$

We rewrite (7.5) as follows:

$$\sqrt{\alpha_n}\int_{\mathbb{R}^3} \widetilde{u}_n \widetilde{z}_n \, dx = \frac{5-p}{4}\frac{\beta_n}{\sqrt{\alpha_n}}\int_{\mathbb{R}^3} \widetilde{u}_n^p \widetilde{z}_n \, dx. \tag{7.6}$$

7.1 Case of $2 \le p < 3$

In this subsection, we will give the proof of Theorem 2.5 (ii) in the case of $2 \le p < 3$. By a similar argument in the proof of Lemma 6.3, we can prove the following:

Lemma 7.3. *Let $2 \le p < 3$. $\widetilde{z}_\infty \neq 0$.*

It follows from (7.4) and Lemma 7.3 that $\widetilde{z}_\infty = \kappa \Lambda W$ for some $\kappa \in \mathbb{R} \setminus \{0\}$. Without loss of generality, we may assume that $\kappa > 0$. [4] Then, we shall derive a contradiction if we obtain the following propositions:

Proposition 7.4. *Let $2 \le p < 3$ and $\widetilde{z}_n$ be the functions defined by (7.2). Then, we have*

$$\liminf_{n\to\infty} \sqrt{\alpha_n}\int_{\mathbb{R}^3} \widetilde{u}_n \widetilde{z}_n \, dx \ge -3\pi\kappa. \tag{7.7}$$

Proposition 7.5. *Let $2 \le p < 3$ and $\widetilde{z}_n$ be the functions defined by (7.2). Then, we obtain*

$$\lim_{n\to\infty} \frac{\beta_n}{\sqrt{\alpha_n}}\int_{\mathbb{R}^3} \widetilde{u}_n^p \widetilde{z}_n \, dx = -6\pi\kappa. \tag{7.8}$$

We can prove Propositions 7.4 and 7.5 by similar arguments to those of Propositions 6.4 and 6.5. Thus, we omit the proof. We now prove the non-degeneracy of $\widetilde{u}_n$.

Proof of Theorem 2.5 (ii) in the case of $2 \le p < 3$. It follows from (7.6), (7.7) and (7.8) that

$$-3\pi\kappa \le \lim_{n\to\infty} \sqrt{\alpha_n}\int_{\mathbb{R}^3} \widetilde{u}_n \widetilde{z}_n \, dx = \frac{5-p}{4}\frac{\beta_n}{\sqrt{\alpha_n}}\int_{\mathbb{R}^3} \widetilde{u}_n^p \widetilde{z}_n \, dx = -\frac{3(5-p)}{2}\pi\kappa,$$

which implies $p \ge 3$ because $\kappa > 0$. However, this contradicts our assumption $p < 3$. This completes the proof. □

7.2 Case of $1 < p < 2$

In this subsection, we will prove Theorem 2.5 (ii) in the case of $1 < p < 2$. We can obtain the following propositions:

[4] Otherwise, we replace $\widetilde{z}_n$ by $-\widetilde{z}_n$.

Proposition 7.6. *Let $1 < p < 2$ and $\widetilde{z}_n$ be the functions defined by* (7.2)*.*

$$\liminf_{n\to\infty} \sqrt{\alpha_n}\, \langle \widetilde{u}_n, \widetilde{z}_n \rangle = 0. \tag{7.9}$$

Proposition 7.7. *Let $1 < p < 2$ and $\widetilde{z}_n$ be the functions defined by* (7.2)*. We have*

$$\lim_{n\to\infty} \frac{\beta_n}{\sqrt{\alpha_n}} \langle \widetilde{u}_n^p, \widetilde{z}_n \rangle < -\frac{2\pi}{5} 3^{\frac{p+1}{2}} \int_0^\infty \frac{e^{-(p+1)s}}{s^{p-1}}\, ds. \tag{7.10}$$

Since the proof of Propositions 7.6 and 7.7 are similar to those of Propositions 6.13 and 6.14, we omit them.

Proof of Theorem 2.5 (ii) in the case of $1 < p < 2$. Suppose to the contrary that there exist a sequence $\{\omega_n\}$ with $\lim_{n\to\infty} \omega_n = 0$ and $z_n \in H^1_{\mathrm{rad}}(\mathbb{R}^3) \setminus \{0\}$ satisfying (7.1). Then, we can derive a contradiction from (7.6), (7.9) and (7.10). This completes the proof. $\square$

8 Uniqueness of minimizer for $E_{\min}(m)$ and $E_{\min,-}(m)$

In this section, we shall obtain a uniqueness of minimizer $R_{\omega(m)}$ for $E_{\min}(m)$ and $R_{\omega(m),-}$ for $E_{\min,-}(m)$, where $E_{\min}(m)$ and $E_{\min,-}(m)$ are defined by (1.10) and (1.16), respectively.

The uniqueness is required to study the continuity of $R_{\omega(m)}$ (see Proposition 9.4 below). We shall show the following:

Theorem 8.1. *(i) Let $\frac{7}{3} \le p < 3$. There exists $m_1 > 0$ such that the minimizer $R_{\omega(m)}$ for $E_{\min}(m)$ is unique in $H^1_{rad}(\mathbb{R}^d)$ for $m \in (0, m_1)$.*

(ii) Let $1 < p < \frac{7}{3}$. There exists $m_2 > 0$ such that the minimizer $R_{\omega(m),-}$ for $E_{\min,-}(m)$ is unique in $H^1_{rad}(\mathbb{R}^d)$ for $m \in (0, m_2)$.

Remark 8.1. *It seems that Theorem 8.1 does not follow from Theorem 2.5 (i) because it is non-trivial whether the Lagrange multiplier $\omega(m)$ is determined uniquely or not.*

In what follows, when $1 < p < 1 + \frac{4}{d}$, we write $R_{\omega(m),-}$ and $\widetilde{\omega}(m)$ simply as $R_{\omega(m)}$ and $\omega(m)$ for the sake of clarity.

Remark 8.2. *Since $\lim_{m\to 0} \omega(m) = 0, \lim_{m\to 0} \|R_{\omega(m)}\|_{L^\infty} = \infty$ and $R_{\omega(m)}$ satisfies* (1.1)*, we see from Theorem 2.5 (i) that $R_{\omega(m)} = u_{\omega(m)} = u_{2,\omega(m)}$ for sufficiently small $m > 0$.*

8.1 Case of $2 < p < 3$

8.1.1 Relation between $\|R_{\omega(m)}\|_{L^q}$ and m

In this subsection, we shall study the relation between $\|R_{\omega(m)}\|_{L^q}$ $(q > 3)$ and the parameter m to prove Theorem 8.1 in the case of $2 < p < 3$. More precisely, we shall show the following:

Proposition 8.1. *Let $2 < p < 3$ and $\omega(m) > 0$ be the Lagrange multiplier given in Theorems 1.2 and 1.3 (ii). Putting $M_m = \|R_{\omega(m)}\|_{L^\infty}$, we have the following:*

(i)

$$\lim_{m\to 0} mM_m^{p-1} = 6\pi C_p^{-1}, \tag{8.1}$$

where

$$C_p := \frac{5-p}{12\pi(p+1)}\|W\|_{L^{p+1}}^{p+1}. \tag{8.2}$$

(ii)

$$\lim_{m\to 0} \frac{\omega(m)}{M_m^{2p-6}} = C_p^2. \tag{8.3}$$

(iii) Let $q > 3$.

$$\lim_{m\to 0} \frac{\|R_{\omega(m)}\|_{L^q}}{m^{\frac{6-q}{q(p-1)}}} = (6\pi C_p^{-1})^{-\frac{6-q}{q(p-1)}}\|W\|_{L^q}. \tag{8.4}$$

(iv)

$$\lim_{m\to 0} \frac{\omega(m)}{m^{\frac{6-2p}{p-1}}} = (6\pi)^{\frac{2p-6}{p-1}} C_p^{\frac{4}{p-1}}. \tag{8.5}$$

Proof of Proposition 8.1. (i) We define $\widetilde{R}_m$ by

$$\widetilde{R}_m(\cdot) := M_m^{-1}R_{\omega(m)}(M_m^{-2}\cdot). \tag{8.6}$$

Then, we can verify that $\widetilde{R}_m$ satisfies

$$-\Delta\widetilde{R}_m + \alpha_m\widetilde{R}_m - \beta_m\widetilde{R}_m^p - \widetilde{R}_m^5 = 0, \qquad \|\widetilde{R}_m\|_{L^\infty} = 1, \tag{8.7}$$

where

$$\alpha_m := \alpha_{\omega(m)} = \omega(m)M_m^{-4}, \qquad \beta_m := \beta_{\omega(m)} = M_m^{p-5}. \tag{8.8}$$

We see from Remark 8.2 and Theorem 2.4 that

$$\lim_{m\to 0}\widetilde{R}_m = W \qquad \text{in } \dot{H}\cap L^q(\mathbb{R}^3)\ (q>3). \tag{8.9}$$

Applying (2.8) to $\widetilde{u}_{\omega(m)} = \widetilde{R}_m$ that

$$\|\widetilde{R}_m\|_{L^2}^2 = \frac{5-p}{2(p+1)}\frac{\beta_m}{\alpha_m}\|\widetilde{R}_m\|_{L^{p+1}}^{p+1}. \tag{8.10}$$

This together with $\beta_m = M_m^{p-5}$ and (8.6) yields that

$$m = \|R_{\omega(m)}\|_{L^2}^2 = M_m^{-4}\|\widetilde{R}_m\|_{L^2}^2 = M_m^{-4}\times\frac{5-p}{2(p+1)}\frac{\beta_m}{\alpha_m}\|\widetilde{R}_m\|_{L^{p+1}}^{p+1} = \frac{5-p}{2(p+1)}M_m^{1-p}\frac{\beta_m^2}{\alpha_m}\|\widetilde{R}_m\|_{L^{p+1}}^{p+1}. \tag{8.11}$$

In addition, one has by (5.1) that

$$\lim_{m\to 0}\frac{\beta_m}{\sqrt{\alpha_m}} = C_p^{-1},$$

where C_p is given by (8.2). This together with (5.1), (8.2) and (8.9) yields

$$\lim_{m\to 0} mM_m^{p-1} = \lim_{m\to 0}\frac{5-p}{2(p+1)}\frac{\beta_m^2}{\alpha_m}\|\widetilde{R}_m\|_{L^{p+1}}^{p+1} = \frac{5-p}{2(p+1)}C_p^{-2}\|W\|_{L^{p+1}}^{p+1} = 6\pi C_p^{-1}.$$

Thus, (8.1) holds.

(ii) It follows from (5.1), (8.2), $\alpha_m = \omega(m)M_m^{-4}$ and $\beta_m = M_m^{p-5}$ that

$$C_p = \lim_{m\to 0} \frac{\sqrt{\alpha_m}}{\beta_m} = \lim_{m\to 0} \frac{\sqrt{\omega(m)}M_m^{-2}}{M_m^{p-5}} = \lim_{m\to 0} \frac{\sqrt{\omega(m)}}{M_m^{p-3}}.$$

Thus, (8.3) immediately holds.

(iii) Since $R_{\omega(m)}(x) = M_m \widetilde{R}_m(M_m^2 x)$ for $x \in \mathbb{R}^3$, we have

$$\|R_{\omega(m)}\|_{L^q}^q = M_m^{-(6-q)} \|\widetilde{R}_m\|_{L^q}^q.$$

This together with (8.1) and (8.9) yields that

$$\lim_{m\to 0} \frac{\|R_{\omega(m)}\|_{L^q}}{m^{\frac{6-q}{q(p-1)}}} = \lim_{m\to 0} \frac{M_m^{\frac{-(6-q)}{q}} \|\widetilde{R}_m\|_{L^q}}{m^{\frac{6-q}{q(p-1)}}} = \lim_{m\to 0} (mM_m^{p-1})^{\frac{-(6-q)}{q(p-1)}} \|\widetilde{R}_m\|_{L^q} = (6\pi C_p^{-1})^{-\frac{6-q}{q(p-1)}} \|W\|_{L^q}.$$

(iv) We see from (8.1) and (8.3) that

$$\lim_{m\to 0} \frac{\omega(m)}{m^{\frac{6-2p}{p-1}}} = \lim_{m\to 0} \frac{\omega(m)}{M_m^{2p-6}} \frac{M_m^{2p-6}}{m^{-\frac{2p-6}{p-1}}} = \lim_{m\to 0} \frac{\omega(m)}{M_m^{2p-6}} \lim_{m\to 0} (mM_m^{p-1})^{\frac{2p-6}{p-1}} = C_p^2 \times (6\pi C_p^{-1})^{\frac{2p-6}{p-1}} = (6\pi)^{\frac{2p-6}{p-1}} C_p^{\frac{4}{p-1}}. \tag{8.12}$$

This completes the proof. □

We also show the following:

Lemma 8.2. *Let $q > 1$. Then, we obtain $\|R_{\omega(m_n)}\|_{L^q}^q \lesssim M_{n,j}^{pq-4q-3p+9}$.*

Proof. By (8.3), we obtain

$$\alpha_m = \omega(m)M_m^{-4} \sim M_m^{2p-10} \qquad \text{as } m \to 0. \tag{8.13}$$

Since $R_{\omega(m)}(x) = M_m \widetilde{R}_m(M_m^2 x)$ for $x \in \mathbb{R}^3$, we have

$$\|R_{\omega(m)}\|_{L^q}^q = M_m^{-(6-q)} \|\widetilde{R}_m\|_{L^q}^q. \tag{8.14}$$

Then, it follows from (4.1), (6.15) and (8.13) that

$$\begin{aligned}
\|\widetilde{R}_m\|_{L^q}^q &= 4\pi \int_0^1 \widetilde{R}_m^q(r) r^2\, dr + 4\pi \int_1^{\alpha_m^{-\frac{1}{2}}} \widetilde{R}_m^q(r) r^2\, dr + 4\pi \int_{\alpha_m^{-\frac{1}{2}}}^\infty \widetilde{R}_m^q(r) r^2\, dr \\
&\lesssim 1 + \int_1^{\alpha_m^{-\frac{1}{2}}} r^{2-q}\, dr + \int_{\alpha_m^{-\frac{1}{2}}}^\infty r^{2-q} e^{-q\sqrt{\alpha_m} r}\, dr \\
&\lesssim 1 + \alpha_m^{-\frac{1}{2}(3-q)} + \alpha_m^{-\frac{1}{2}(3-q)} \int_1^\infty s^{2-q} e^{-qs}\, ds \lesssim \alpha_m^{-\frac{1}{2}(3-q)} \sim M_m^{pq-3p-5q+15}.
\end{aligned}$$

Thus, by (8.14), we obtain

$$\|R_{\omega(m)}\|_{L^q}^q = M_m^{-(6-q)} \|\widetilde{R}_m\|_{L^q}^q \lesssim M_m^{pq-4q-3p+9}.$$

This completes the proof. □

8.1.2 Proof of Theorem 8.1 in the case of $2 < p < 3$

This subsection is devoted to the proof of Theorem 8.1. We shall show this by contradiction. Suppose to the contrary that there exists a sequence $\{m_n\}$ in $(0,\infty)$ such that $\lim_{n\to\infty} m_n = 0$ and for each $n \in \mathbb{N}$, $E_{\min}(m_n)$ has two minimizers $R_{\omega_1(m_n)}, R_{\omega_2(m_n)}$ with $R_{\omega_1(m_n)} \neq R_{\omega_2(m_n)}$. For $j = 1, 2$, we set

$$M_{n,j} = R_{\omega_j(m_n)}(0), \qquad \widetilde{R}_{n,j}(x) = M_{n,j}^{-1} R_{\omega_j(m_n)}(M_{n,j}^{-2}x),$$

$$\alpha_{n,j} = \omega_j(m_n) M_{n,j}^{-4}, \qquad \beta_{n,j} = M_{n,j}^{p-5}.$$

By (8.1) and (8.3), we see that

$$\lim_{n\to\infty} \frac{M_{n,1}}{M_{n,2}} = \lim_{n\to\infty} \frac{\omega_1(m_n)}{\omega_2(m_n)} = 1,$$

which implies

$$\lim_{n\to\infty} \frac{\alpha_{n,1}}{\alpha_{n,2}} = \lim_{n\to\infty} \frac{\beta_{n,1}}{\beta_{n,2}} = 1.$$

Let

$$Z_n = \frac{R_{\omega_1(m_n)} - R_{\omega_2(m_n)}}{\|R_{\omega_1(m_n)} - R_{\omega_2(m_n)}\|_{L^\infty} + |\omega_1(m_n) - \omega_2(m_n)|}, \tag{8.15}$$

$$\kappa_n = \frac{\omega_1(m_n) - \omega_2(m_n)}{\|R_{\omega_1(m_n)} - R_{\omega_2(m_n)}\|_{L^\infty} + |\omega_1(m_n) - \omega_2(m_n)|}. \tag{8.16}$$

Clearly, we have

$$1 = \|Z_n\|_{L^\infty} + |\kappa_n| \tag{8.17}$$

We can verify that the pair (Z_n, κ_n) satisfies

$$-\Delta Z_n + \omega_1(m_n) Z_n - p\int_0^1 V_n^{p-1}(x,\tau)\,d\tau Z_n - 5\int_0^1 V_n^4(x,\tau)\,d\tau Z_n = -\kappa_n R_{\omega_2(m_n)}, \tag{8.18}$$

where

$$V_n(x,\tau) = \tau R_{\omega_1(m_n)}(x) + (1-\tau) R_{\omega_2(m_n)}(x) \qquad (\tau \in (0,1),\ x \in \mathbb{R}^3). \tag{8.19}$$

We shall show the following:

Lemma 8.3. *Let $2 < p < 3$. Then, There exists a constant $C > 0$ such that $|\kappa_n| \le C M_n^{p^2 - \frac{11}{2}p + \frac{13}{2}} \|\nabla Z_n\|_{L^2}$ for all $n \in \mathbb{N}$.*

Proof. By a similar argument of (2.8), we see from the Pohozaev identity of $R_{\omega_j(m_n)}$ that

$$\omega_j(m_n) m_n = \omega_j(m_n) \|R_{\omega_j(m_n)}\|_{L^2}^2 = \frac{5-p}{2(p+1)} \|R_{\omega_j(m_n)}\|_{L^{p+1}}^{p+1} \qquad \text{for } j = 1, 2.$$

Taking a difference, we obtain

$$(\omega_1(m_n) - \omega_2(m_n)) m_n = \frac{5-p}{2(p+1)} \left(\|R_{\omega_1(m_n)}\|_{L^{p+1}}^{p+1} - \|R_{\omega_2(m_n)}\|_{L^{p+1}}^{p+1} \right).$$

This together with (8.1) and (8.15) yields that

$$
\begin{aligned}
|\kappa_n| &= \frac{|\omega_1(m_n)-\omega_2(m_n)|}{\|R_{\omega_1(m_n)}-R_{\omega_2(m_n)}\|_{L^\infty}+|\omega_1(m_n)-\omega_2(m_n)|} \\
&\lesssim \frac{1}{m_n}\int_{\mathbb{R}^3}(|R_{\omega_1(m_n)}|^p+|R_{\omega_2(m_n)}|^p)|Z_n|\,dx \\
&\lesssim \frac{1}{m_n}\left(\|R_{\omega_1(m_n)}\|^p_{L^{\frac{6}{5}p}}+\|R_{\omega_2(m_n)}\|^p_{L^{\frac{6}{5}p}}\right)\|Z_n\|_{L^6} \\
&\lesssim M_n^{p-1}M_n^{\frac{5}{6}\left(\frac{6}{5}p^2-\frac{39}{5}p+9\right)}\|\nabla Z_n\|_{L^2} \\
&\lesssim M_n^{p^2-\frac{11}{2}p+\frac{13}{2}}\|\nabla Z_n\|_{L^2}.
\end{aligned}
\tag{8.20}
$$

□

Lemma 8.4. *Let $2<p<3$. Then, there exists a constant $C>0$ such that $\|\nabla Z_n\|^2_{L^2}\le Cm_n^{\frac{2}{p-1}}$ for all $n\in\mathbb{N}$.*

Proof. Multiplying (8.18) by Z_n and integrating the resulting equation on $\mathbb{R}^3$, we have

$$
\begin{aligned}
\|\nabla Z_n\|^2_{L^2}+\omega_1(m_n)\|Z_n\|^2_{L^2} &= p\int_{\mathbb{R}^3}\int_0^1 V_n^{p-1}(x,\tau)d\tau|Z_n|^2\,dx+5\int_{\mathbb{R}^3}\int_0^1 V_n^4(x,\tau)d\tau|Z_n|^2\,dx \\
&\quad -\kappa_n\int_{\mathbb{R}^3}R_{\omega_2(m_n)}Z_n\,dx
\end{aligned}
\tag{8.21}
$$

By the Hölder and Young inequalities, $\|R_{\omega_2(m_n)}\|_{L^2}=m_n^{1/2}$, (8.1), Lemma 8.3 and (8.3), we obtain

$$
\begin{aligned}
\left|\kappa_n\int_{\mathbb{R}^3}R_{\omega_2(m_n)}Z_n\,dx\right| \le |\kappa_n|\|R_{\omega_2(m_n)}\|_{L^2}\|Z_n\|_{L^2} &\le m_n^{\frac{1}{2}}M_n^{p^2-\frac{11}{2}p+\frac{13}{2}}\|\nabla Z_n\|_{L^2}\|Z_n\|_{L^2} \\
&\lesssim M_n^{p^2-6p+7}\|\nabla Z_n\|_{L^2}\|Z_n\|_{L^2} \\
&\lesssim \delta\|\nabla Z_n\|^2_{L^2}+\delta^{-1}M_n^{2p^2-12p+14}\|Z_n\|^2_{L^2} \\
&\lesssim \delta\|\nabla Z_n\|^2_{L^2}+\delta^{-1}\omega(m_n)^{\frac{p^2-6p+7}{p-3}}\|Z_n\|^2_{L^2}.
\end{aligned}
$$

for $2<p<3$. Note that $\frac{p^2-6p+7}{p-3}>1$ for $2<p<3$ and $0<\omega(m_n)\ll 1$ for sufficiently large $n\in\mathbb{N}$. Thus, one has

$$
\left|\kappa_n\int_{\mathbb{R}^3}R_{\omega_2(m_n)}Z_n\,dx\right|\lesssim \delta\|\nabla Z_n\|^2_{L^2}+\delta\omega(m_n)\|Z_n\|^2_{L^2}.
$$

Since $R_{\omega_j(m_n)}(\cdot)=M_{n,j}\widetilde{R}_{n,j}(M^2_{n,j}\cdot)$, we have by (4.1) and (8.19) that

$$
|V_n(x,\tau)|\lesssim \frac{1}{\min\{M_{n,1},M_{n,2}\}r}.
$$

For any $\delta>0$, we observe from (8.3) that

$$
|V_n(x,\tau)|^{p-1}\le \frac{1}{\min\{M_{n,1}^{p-1},M_{n,2}^{p-1}\}}\delta^{p-1}M_{n,1}^{3p-7}\lesssim \delta^{p-1}M_{n,1}^{2p-6}\lesssim 2\delta^{p-1}\omega(m_n)
\tag{8.22}
$$

for $|x| \geq \delta^{-1} M_{n,1}^{\frac{7-3p}{p-1}}$. In addition, by the Hölder inequality and (8.19), we obtain

$$
\begin{aligned}
&\left| p \int_{|x| \leq \delta^{-1} M_{n,1}^{\frac{7-3p}{p-1}}} \int_0^1 V_n^{p-1}(x,\tau) d\tau |Z_n|^2 \, dx \right| \\
&\lesssim \frac{1}{\min\{M_{n,1}^{p-1}, M_{n,2}^{p-1}\}} \int_{|x| \leq \delta^{-1} M_{n,1}^{\frac{7-3p}{p-1}}} |x|^{-(p-1)} |Z_n|^2 \, dx \\
&\lesssim M_{n,1}^{-(p-1)} \left(\int_{|x| \leq \delta^{-1} M_{n,1}^{\frac{7-3p}{p-1}}} |x|^{-\frac{3}{2}(p-1)} \, dx \right)^{\frac{2}{3}} \|\nabla Z_n\|_{L^2}^2 \\
&\lesssim \delta^{p-3} M_{n,1}^{-(p-1)} M_{n,1}^{\frac{(7-3p)(3-p)}{p-1}} \|\nabla Z_n\|_{L^2}^2 \leq C \delta^{p-3} M_{n,1}^{\frac{2}{p-1}(p-2)(p-5)} \|\nabla Z_n\|_{L^2}^2.
\end{aligned}
\tag{8.23}
$$

Therefore, by (8.22) and (8.23), we obtain

$$
\left| p \int_{\mathbb{R}^3} \int_0^1 V_n^{p-1}(x,\tau) d\tau |Z_n|^2 \, dx \right| \leq 4 C_p^2 \delta^{p-1} \omega(m_n) \|Z_n\|_{L^2}^2 + C \delta^{p-3} M_{n,1}^{\frac{2}{p-1}(p-2)(p-5)} \|\nabla Z_n\|_{L^2}^2. \tag{8.24}
$$

For any $\varepsilon > 0$, we have by (8.4) that

$$
\|R_{\omega_j(m_n)}\|_{L^{\frac{24}{4+\varepsilon}}} \lesssim m_n^{\frac{\varepsilon}{4(p-1)}} \qquad \text{for } j = 1, 2.
$$

By the Hölder, Young inequalities and (8.17), we obtain

$$
\begin{aligned}
\left| 5 \int_{\mathbb{R}^3} \int_0^1 V_n^4(x,\tau) d\tau |Z_n|^2 \, dx \right| &\lesssim \int_{\mathbb{R}^3} (|R_{\omega_1(m_n)}|^4 + |R_{\omega_2(m_n)}|^4) |Z_n|^{2-\varepsilon} \, dx \\
&\leq (\|R_{\omega_1(m_n)}\|_{L^{\frac{24}{4+\varepsilon}}}^4 + \|R_{\omega_2(m_n)}\|_{L^{\frac{24}{4+\varepsilon}}}^4) \|Z_n\|_{L^6}^{2-\varepsilon} \\
&\leq C m_n^{\frac{\varepsilon}{p-1}} \|\nabla Z_n\|_{L^2}^{2-\varepsilon} \\
&\leq C m_n^{\frac{2}{p-1}} + \frac{1}{2} \|\nabla Z_n\|_{L^2}^2.
\end{aligned}
\tag{8.25}
$$

From (8.21), (8.23)–(8.25) and $2 < p < 3$, we have obtained the desired result. □

From Lemma 8.4, we have $\lim_{n\to\infty} Z_n = 0$ strongly in $\dot{H}^1(\mathbb{R}^3)$. In addition, we have the following:

Lemma 8.5. *Let $2 < p < 3$. Then, there exists a constant $C > 0$ such that*

$$
|\kappa_n| \leq C M_n^{p^2 - \frac{11}{2}p + \frac{11}{2}} \qquad \textit{for all } n \in \mathbb{N}. \tag{8.26}
$$

Proof. From Lemmas 8.2, 8.4 and (8.1), we have

$$
|\kappa_n| \lesssim M_n^{p^2 - \frac{11}{2}p + \frac{13}{2}} \|\nabla Z_n\|_{L^2} \lesssim M_n^{p^2 - \frac{11}{2}p + \frac{13}{2}} m_n^{\frac{1}{p-1}} \lesssim M_n^{p^2 - \frac{11}{2}p + \frac{11}{2}}.
$$

This completes the proof. □

Remark 8.3. *Note that $p^2 - \frac{11}{2}p + \frac{11}{2} < 0$ for $2 < p < 3$. This together with* (8.26) *yields that*

$$
\lim_{n\to\infty} \kappa_n = 0. \tag{8.27}
$$

From this and (8.17)*, we have* $\lim_{n\to\infty} \|Z_n\|_{L^\infty} = 1$.

We are now in a position to prove the uniqueness of the minimizer $R_{\omega(m)}$.

Proof of Theorem 8.1 in the case of $2 < p < 3$. Since Z_n is radially symmetric, we have by Lemma 8.4 that $|Z_n(x)| \lesssim |x|^{-1/2}$ for $|x| > 0$ (see Berestycki and Lions [10, Lemma A.III]). Using the elliptic regularity argument and $\lim_{n\to\infty} Z_n = 0$ strongly in $\dot{H}^1(\mathbb{R}^3)$ (see Lemma 8.4), we have $\lim_{n\to\infty} Z_n = 0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$, which contradicts $\lim_{n\to\infty} \|Z_n\|_{L^\infty} = 1$. □

8.2 Case of $p = 2$

In this subsection, we will give the proof of Theorem 8.1 in the case of $p = 2$. The strategy is similar to that of the case of $2 < p < 3$. However, since the relation between $\alpha_{\omega(m)}$ and $\beta_{\omega(m)}$ is different (see Proposition 5.1), we need to investigate individually.

8.2.1 Relation between $\|R_{\omega(m)}\|_{L^q}$ and m

Here, similarly to the case of $2 < p < 3$, we shall study the relation between $\|R_{\omega(m)}\|_{L^q}$ $(q > 3)$ and the parameter m to prove Theorem 8.1 in the case of $p = 2$. We will prove the following:

Proposition 8.6. *Let $p = 2$ and $\omega(m) > 0$ be the Lagrange multiplier given in Theorem 1.3 (ii) and $M_m = \|R_{\omega(m)}\|_{L^\infty}$. Then, we have the following:*

(i)

$$\lim_{m\to 0} \frac{\omega(m)}{M_m^{-2}(\log M_m)^2} = 27. \tag{8.28}$$

(ii)

$$m \sim M_m^{-1}(\log M_m)^{-1} \qquad \text{as } m \to 0. \tag{8.29}$$

(iii) Let $q > 3$.

$$\|R_{\omega(m)}\|_{L^q} \sim (m(-\log m))^{\frac{6-q}{q}} \qquad \text{as } m \to 0. \tag{8.30}$$

(iv)

$$\omega(m) \sim m^2(-\log m)^4 \qquad \text{as } m \to 0. \tag{8.31}$$

Proof of Proposition 8.1. (i) First, we shall show that

$$\liminf_{m\to 0} \frac{\omega(m)M_m^2}{(\log M_m)^2} > 0. \tag{8.32}$$

Suppose to the contrary that there exists a sequence $\{m_n\}$ with $\lim_{n\to\infty} m_n = 0$ such that

$$\lim_{n\to\infty} \frac{\omega(m_n)M_{m_n}^2}{(\log M_{m_n})^2} = 0. \tag{8.33}$$

Then, it follows from (5.2), (8.8), $p = 2$, $\log\omega(m_n) < 0$ and (8.33) that

$$\begin{aligned} \frac{2}{\sqrt{3}} = \lim_{n\to\infty} \frac{\beta_{m_n}|\log\alpha_{m_n}|}{\sqrt{\alpha_{m_n}}} &= \lim_{n\to\infty} \frac{|\log\omega(m_n) - 4\log M_{m_n}|}{M_{m_n}\omega^{\frac{1}{2}}(m_n)} \\ &= \lim_{n\to\infty} \frac{(-\log\omega(m_n) + 4\log M_{m_n})}{M_{m_n}\omega^{\frac{1}{2}}(m_n)} \\ &> \liminf_{n\to\infty} 4\frac{\log M_{m_n}}{\omega^{\frac{1}{2}}(m_n)M_{m_n}} \\ &= \infty, \end{aligned} \tag{8.34}$$

which is a contradiction. Thus, (8.32) holds.

Next, we claim that

$$\limsup_{m\to 0}\frac{\omega(m)M_m^2}{(\log M_m)^2}<\infty. \tag{8.35}$$

Suppose to the contrary that there exists a sequence $\{m_n\}$ with $\lim_{n\to\infty} m_n = 0$ such that

$$\lim_{n\to\infty}\frac{\omega(m_n)M_{m_n}^2}{(\log M_{m_n})^2}=\infty. \tag{8.36}$$

Thus, for any $L>1$, there exists $n_L\in\mathbb{N}$ such that for $n\geq n_L$, we have

$$\frac{\omega(m_n)M_{m_n}^2}{(\log M_{m_n})^2}\geq L. \tag{8.37}$$

In addition, we see from (8.36) that $\lim_{n\to\infty}\omega(m_n)M_{m_n}^2=\infty$. These together with (5.2), (8.34) and (8.37) yield that

$$\begin{aligned}\frac{2}{\sqrt{3}}=\lim_{n\to\infty}\frac{\beta_{m_n}|\log\alpha_{m_n}|}{\sqrt{\alpha_{m_n}}}&=\lim_{n\to\infty}\frac{(-\log\omega(m_n)+4\log M_{m_n})}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}\\&<\limsup_{n\to\infty}\frac{\left(-\log\left(LM_{m_n}^{-2}(\log M_{m_n})^2\right)+4\log M_{m_n}\right)}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}\\&=\limsup_{n\to\infty}\left\{6\frac{\log M_{m_n}}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}-2\frac{\log(\log M_{m_n})}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}-\frac{\log L}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}\right\}\\&\leq\frac{6}{\sqrt{L}}<\frac{1}{\sqrt{3}}\end{aligned} \tag{8.38}$$

for sufficiently large $L>0$, which is absurd.

Then, we see from (8.32) and (8.35) that up to subsequence (we still denote it by the same letter), there exists $C_2>0$ such that

$$\lim_{n\to 0}\frac{\omega(m_n)M_{m_n}^2}{(\log M_{m_n})^2}=C_2. \tag{8.39}$$

Putting $\ell_n=\frac{\omega(m_n)M_{m_n}^2}{(\log M_{m_n})^2}$, one has $\lim_{n\to\infty}\ell_n=C_2$ and

$$\frac{-\log\omega(m_n)}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}=-\frac{\log\ell_n}{\sqrt{\ell_n}\log M_{m_n}}+\frac{2}{\sqrt{\ell_n}}-2\frac{\log(\log M_{m_n})}{\sqrt{\ell_n}\log M_{m_n}},$$

which yields that

$$\lim_{n\to\infty}\frac{-\log\omega(m_n)}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}=\frac{2}{\sqrt{C_2}}. \tag{8.40}$$

Then, it follows from (8.34) and (8.40) that

$$\frac{2}{\sqrt{3}}=\lim_{n\to\infty}\frac{(-\log\omega(m_n)+4\log M_{m_n})}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}=\lim_{n\to\infty}\frac{-\log\omega(m_n)}{M_{m_n}\omega^{\frac{1}{2}}(m_n)}+\frac{4}{\sqrt{C_2}}=\frac{6}{\sqrt{C_2}}$$

This implies that $C_2=27$. Thus, (i) holds.

(ii) Observe from (8.10) that

$$\|\widetilde{R}_{\omega(m)}\|_{L^2}^2=\frac{5-p}{2(p+1)}\frac{\beta_m}{\alpha_m}\|\widetilde{R}_{\omega(m)}\|_{L^3}^3. \tag{8.41}$$

From (4.1) and (6.15), we obtain

$$\begin{aligned}\|\widetilde{R}_m\|_{L^3}^3 &= 4\pi\int_0^1 \widetilde{R}_m^3(r)r^2\,dr + 4\pi\int_1^{\alpha_m^{-\frac{1}{2}}} \widetilde{R}_m^3(r)r^2\,dr + 4\pi\int_{\alpha_m^{-\frac{1}{2}}}^\infty \widetilde{R}_m^3(r)r^2\,dr\\ &\lesssim 1 + \int_1^{\alpha_m^{-\frac{1}{2}}} r^{-1}\,dr + \int_{\alpha_m^{-\frac{1}{2}}}^\infty r^{-1}e^{-3\sqrt{\alpha_m}r}\,dr\\ &\lesssim 1 + |\log\alpha_m| + \int_1^\infty s^{-1}e^{-3s}\,ds \lesssim |\log\alpha_m|.\end{aligned} \tag{8.42}$$

Moreover, by (5.8), we have

$$\begin{aligned}\|\widetilde{R}_m\|_{L^3}^3 &= 4\pi\int_0^{R_2} \widetilde{R}_m^3(r)r^2\,dr + 4\pi\int_{R_2}^{\alpha_m^{-\frac{1}{2}}} \widetilde{R}_m^3(r)r^2\,dr + 4\pi\int_{\alpha_m^{-\frac{1}{2}}}^\infty \widetilde{R}_m^3(r)r^2\,dr\\ &\gtrsim \int_{R_2}^{\alpha_m^{-\frac{1}{2}}} \widetilde{R}_m^3(r)r^2\,dr \gtrsim \int_{R_2}^{\alpha_m^{-\frac{1}{2}}} r^{-1}\,dr \gtrsim |\log\alpha_m|.\end{aligned} \tag{8.43}$$

It follows from (8.41), (8.42), (8.43) and (5.2) that

$$\|\widetilde{R}_{\omega(m)}\|_{L^2}^2 \sim \frac{\beta_m|\log\alpha_m|}{\alpha_m} \sim \alpha_m^{-\frac{1}{2}} \qquad \text{as } m\to 0. \tag{8.44}$$

for sufficiently small $m>0$. By (8.28) and (8.44), we obtain

$$m = \|R_{\omega(m)}\|_{L^2}^2 = M_m^{-4}\|\widetilde{R}_{\omega(m)}\|_{L^2}^2 \sim M_m^{-4}\alpha_m^{-\frac{1}{2}} = \omega^{-\frac{1}{2}}(m)M_m^{-2} \sim M_m^{-1}(\log M_m)^{-1}.$$

(iii) Since $R_{\omega(m)}(x) = M_m\widetilde{R}_m(M_m^2x)$ for $x\in\mathbb{R}^3$, we have

$$\|R_{\omega(m)}\|_{L^q}^q = M_m^{-(6-q)}\|\widetilde{R}_m\|_{L^q}^q. \tag{8.45}$$

In addition, it follows from (8.29) that

$$\log m \sim -\log M_m - \log(\log M_m) \sim -\log M_m \qquad \text{as } m\to 0. \tag{8.46}$$

Thus, (8.29) can be written by the following:

$$m(-\log m) \sim M_m^{-1} \qquad \text{as } m\to 0. \tag{8.47}$$

This together with (8.45) yields that

$$\frac{\|R_{\omega(m)}\|_{L^q}}{(m(-\log m))^{\frac{6-q}{q}}} \sim \frac{M_m^{\frac{-(6-q)}{q}}\|\widetilde{R}_m\|_{L^q}}{(m(-\log m))^{\frac{6-q}{q}}} \sim (m(-\log m)M_m)^{-\frac{(6-q)}{q}}\|\widetilde{R}_m\|_{L^q} \sim \|W\|_{L^q}.$$

(iv) We see from (8.28), (8.29) and (8.46) that

$$\begin{aligned}\frac{\omega(m)}{m^2(-\log m)^4} &\sim \frac{\omega(m)}{M_m^{-2}(\log M_m)^2}\frac{M_m^{-2}(\log M_m)^2}{m^2(-\log m)^4} \sim \frac{\omega(m)}{M_m^{-2}(\log M_m)^2}\frac{M_m^{-2}(\log M_m)^{-2}(\log M_m)^4}{m^2(-\log m)^4}\\ &\sim 1 \qquad \text{as } m\to 0.\end{aligned}$$

This completes the proof. □

Lemma 8.7. *Let $p=2$ and $1\le q<3$. Then, there exists a constant $C>0$ such that $\|R_{\omega_j(m)}\|_{L^q}^q \le CM_m^{3-2q}(\log M_m)^{-(3-q)}$ for all $n\in\mathbb{N}$.*

Proof. By (8.28), we obtain

$$\alpha_m \sim M_m^{-6}(\log M_m)^2 \qquad \text{as } n \to \infty. \tag{8.48}$$

Since $R_{\omega(m)}(x) = M_m \widetilde{R}_{\omega(m)}(M_m^2 x)$ for $x \in \mathbb{R}^3$, we have

$$\|R_{\omega(m)}\|_{L^q}^q = M_m^{-(6-q)} \|\widetilde{R}_{\omega(m)}\|_{L^q}^q. \tag{8.49}$$

Then, it follows from (4.1), (6.15) and (8.48) that

$$\begin{aligned}
\|\widetilde{R}_{\omega(m)}\|_{L^q}^q &= 4\pi \int_0^1 \widetilde{R}_{\omega(m)}^q(r) r^2\, dr + 4\pi \int_1^{\alpha_m^{-\frac{1}{2}}} \widetilde{R}_{\omega(m)}^q(r) r^2\, dr + 4\pi \int_{\alpha_m^{-\frac{1}{2}}}^\infty \widetilde{R}_{\omega(m)}^q(r) r^2\, dr \\
&\lesssim 1 + \int_1^{\alpha_m^{-\frac{1}{2}}} r^{2-q}\, dr + \int_{\alpha_m^{-\frac{1}{2}}}^\infty r^{2-q} e^{-\frac{q}{2}\sqrt{\alpha_m} r}\, dr \\
&\lesssim 1 + \alpha_m^{-\frac{1}{2}(3-q)} + \alpha_m^{-\frac{1}{2}(3-q)} \int_1^\infty s^{2-q} e^{-s}\, ds \lesssim \alpha_m^{-\frac{1}{2}(3-q)} \lesssim M_m^{3(3-q)} (\log M_m)^{-(3-q)}.
\end{aligned}$$

Thus, by (8.49), we obtain

$$\|R_{\omega(m)}\|_{L^q}^q = M_m^{-(6-q)} \|\widetilde{R}_{\omega(m)}\|_{L^q}^q \lesssim M_m^{3-2q} (\log M_m)^{-(3-q)}.$$

□

8.2.2 Proof of Theorem 8.1 in the case of $p = 2$

This subsection is devoted to the proof of Theorem 8.1 in the case of $p = 2$. The strategy of the proof is the same as the one in the case of $2 < p < 3$. We shall show this by contradiction. Suppose to the contrary that there exists a sequence $\{m_n\}$ in $(0, \infty)$ such that $\lim_{n\to\infty} m_n = 0$ and for each $n \in \mathbb{N}$, $R_{\omega_1(m_n)}, R_{\omega_2(m_n)}$ and $R_{\omega_1(m_n)} \neq R_{\omega_2(m_n)}$. For $j = 1, 2$, we set

$$M_{n,j} = R_{\omega_j(m_n)}(0), \qquad \widetilde{R}_{n,j}(x) = M_{n,j}^{-1} R_{\omega_j(m_n)}(M_{n,j}^{-2} x),$$

$$\alpha_{n,j} = \omega_j(m_n) M_{n,j}^{-4}, \qquad \beta_{n,j} = M_{n,j}^{p-5}.$$

By (8.1) and (8.3), we see that

$$\lim_{n\to\infty} \frac{M_{n,1}}{M_{n,2}} = \lim_{n\to\infty} \frac{\omega_1(m_n)}{\omega_2(m_n)} = 1, \tag{8.50}$$

which implies

$$\lim_{n\to\infty} \frac{\alpha_{n,1}}{\alpha_{n,2}} = \lim_{n\to\infty} \frac{\beta_{n,1}}{\beta_{n,2}} = 1.$$

Let

$$Z_n = \frac{R_{\omega_1(m_n)} - R_{\omega_2(m_n)}}{\|R_{\omega_1(m_n)} - R_{\omega_2(m_n)}\|_{L^\infty} + |\omega_1(m_n) - \omega_2(m_n)|}, \tag{8.51}$$

$$\kappa_n = \frac{\omega_1(m_n) - \omega_2(m_n)}{\|R_{\omega_1(m_n)} - R_{\omega_2(m_n)}\|_{L^\infty} + |\omega_1(m_n) - \omega_2(m_n)|}. \tag{8.52}$$

Clearly, we have

$$1 = \|Z_n\|_{L^\infty} + |\kappa_n| \tag{8.53}$$

We can verify that the pair (Z_n, κ_n) satisfies

$$-\Delta Z_n + \omega_1(m_n) Z_n - p \int_0^1 V_n^{p-1}(x,\tau) d\tau Z_n - 5 \int_0^1 V_n^4(x,\tau) d\tau Z_n = -\kappa_n R_{\omega_2(m_n)}, \tag{8.54}$$

where

$$V_n(x,\tau) = \tau R_{\omega_1(m_n)}(x) + (1-\tau) R_{\omega_2(m_n)}(x) \qquad (\tau \in (0,1),\ x \in \mathbb{R}^3). \tag{8.55}$$

Lemma 8.8. *Let $p=2$. Then, there exists a constant $C>0$ such that $|\kappa_n| \le C M_{n,1}^{-\frac{1}{2}} (\log M_{n,1})^{\frac{1}{2}} \|\nabla Z_n\|_{L^2}$ for all $n \in \mathbb{N}$.*

Proof. As in (8.20), we have

$$|\kappa_n| \lesssim \frac{1}{m_n} \left(\|R_{\omega_1(m_n)}\|_{L^{\frac{12}{5}}}^2 + \|R_{\omega_2(m_n)}\|_{L^{\frac{12}{5}}}^2 \right) \|Z_n\|_{L^6}.$$

This together with (8.29), (8.50) and Lemma 8.7 yields that

$$|\kappa_n| \lesssim M_{n,1} (\log M_{n,1}) \times M_{n,1}^{-\frac{3}{2}} (\log M_{n,1})^{-\frac{1}{2}} \|\nabla Z_n\|_{L^2} \lesssim M_{n,1}^{-\frac{1}{2}} (\log M_{n,1})^{\frac{1}{2}} \|\nabla Z_n\|_{L^2}.$$

This completes the proof. $\square$

Lemma 8.9. *Let $p=2$. Then, there exists a constant $C>0$ such that $\|\nabla Z_n\|_{L^2}^2 \le C M_{n,1}^{-2}$ for all $n \in \mathbb{N}$.*

Proof. Multiplying (8.54) by Z_n and integrating the resulting equation on $\mathbb{R}^3$, we have

$$\begin{aligned} \|\nabla Z_n\|_{L^2}^2 + \omega_1(m_n) \|Z_n\|_{L^2}^2 = 2 \int_{\mathbb{R}^3} \int_0^1 V_n(x,\tau) d\tau |Z_n|^2 \, dx + 5 \int_{\mathbb{R}^3} \int_0^1 V_n^4(x,\tau) d\tau |Z_n|^2 \, dx \\ - \kappa_n \int_{\mathbb{R}^3} R_{\omega_2(m_n)} Z_n \, dx \end{aligned} \tag{8.56}$$

By the Hölder and Young inequalities, $\|R_{\omega_2(m_n)}\|_{L^2} = m_n^{1/2}$, Lemma 8.7, (8.29) and (8.31), we obtain

$$\begin{aligned} \left| \kappa_n \int_{\mathbb{R}^3} R_{\omega_2(m_n)} Z_n \, dx \right| \le |\kappa_n| \|R_{\omega_2(m_n)}\|_{L^2} \|Z_n\|_{L^2} &\le m_n^{\frac{1}{2}} M_{n,1}^{-\frac{1}{2}} (\log M_{n,1})^{-\frac{1}{2}} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2} \\ &\lesssim M_{n,1}^{-1} (\log M_{n,1})^{-1} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2} \\ &\lesssim \delta \|\nabla Z_n\|_{L^2}^2 + \delta^{-1} M_{n,1}^{-2} (\log M_{n,1})^{-2} \|Z_n\|_{L^2}^2 \\ &\lesssim \delta \|\nabla Z_n\|_{L^2}^2 + \delta^{-1} \frac{\omega(m_n)}{(\log M_{n,1})^4} \|Z_n\|_{L^2}^2. \end{aligned} \tag{8.57}$$

Thus, one has

$$\left| \kappa_n \int_{\mathbb{R}^3} R_{\omega_2(m_n)} Z_n \, dx \right| \lesssim \delta \|\nabla Z_n\|_{L^2}^2 + \frac{\omega(m_n)}{2} \|Z_n\|_{L^2}^2$$

for sufficiently large $n \in \mathbb{N}$. Since $R_{\omega_j(m_n)}(x) = M_{n,j} \widetilde{R}_{n,j}(M_{n,j}^2 x)$ for $x \in \mathbb{R}^3$, we have by (4.1) and (8.55) that

$$|V_n(x,\tau)| \lesssim \frac{1}{M_{n,1} r}.$$

For any $\delta > 0$, we observe from (8.28) that

$$|V_n(x,\tau)| \le \frac{1}{M_{n,1}} \delta M_{n,1}^{-1} (\log M_{n,1})^2 \lesssim \delta M_{n,1}^{-2} (\log M_{n,1})^2 \lesssim \delta \omega(m_n)$$

for $|x| \geq \delta^{-1} M_{n,1}(\log M_{n,1})^{-2}$. In addition, by the Hölder inequality and (8.55), we obtain

$$\begin{aligned}
&\left| 2 \int_{|x|\leq \delta^{-1} M_{n,1}(\log M_{n,1})^{-2}} \int_0^1 V_n(x,\tau) d\tau |Z_n|^2 \, dx \right| \\
&\lesssim \frac{1}{M_{n,1}} \int_{|x|\leq \delta^{-1} M_{n,1}(\log M_{n,1})^{-2}} |x|^{-1} |Z_n|^2 \, dx \\
&\lesssim M_{n,1}^{-1} \left(\int_{|x|\leq \delta^{-1} M_{n,1}(\log M_{n,1})^{-2}} |x|^{-\frac{3}{2}} \, dx \right)^{\frac{2}{3}} \|\nabla Z_n\|_{L^2}^2 \\
&\lesssim \delta^{-1} M_{n,1}^{-1} \times M_{n,1}(\log M_{n,1})^{-2} \|\nabla Z_n\|_{L^2}^2 \leq \delta^{-1} (\log M_{n,1})^{-2} \|\nabla Z_n\|_{L^2}^2.
\end{aligned}$$

Therefore, we obtain

$$\left| 2 \int_{\mathbb{R}^3} \int_0^1 V_n(x,\tau) d\tau |Z_n|^2 \, dx \right| \leq 4C_p^2 \delta \omega(m_n) \|Z_n\|_{L^2}^2 + C\delta^{-1} (\log M_{n,1})^{-2} \|\nabla Z_n\|_{L^2}^2. \tag{8.58}$$

For any $\varepsilon > 0$, we have by (8.30) and (8.47) that

$$\|R_{\omega_j(m_n)}\|_{L^{\frac{24}{4+\varepsilon}}} \lesssim M_{n,j}^{-\frac{\varepsilon}{4}} \qquad \text{for } j = 1, 2.$$

By the Hölder, Young inequalities and (8.17), we obtain

$$\begin{aligned}
\left| 5 \int_{\mathbb{R}^3} \int_0^1 V_n^4(x,\tau) d\tau |Z_n|^2 \, dx \right| &\lesssim \int_{\mathbb{R}^3} (|R_{\omega_1(m_n)}|^4 + |R_{\omega_2(m_n)}|^4) |Z_n|^{2-\varepsilon} \, dx \\
&\leq (\|R_{\omega_1(m_n)}\|_{L^{\frac{24}{4+\varepsilon}}}^4 + \|R_{\omega_2(m_n)}\|_{L^{\frac{24}{4+\varepsilon}}}^4) \|Z_n\|_{L^6}^{2-\varepsilon} \\
&\leq C M_{n,j}^{-\varepsilon} \|\nabla Z_n\|_{L^2}^{2-\varepsilon} \\
&\leq C M_{n,j}^{-2} + \frac{1}{2} \|\nabla Z_n\|_{L^2}^2.
\end{aligned} \tag{8.59}$$

From (8.56)–(8.59), we have obtained the desired result. □

From Lemma 8.4, we have $\lim_{n\to\infty} Z_n = 0$ strongly in $\dot{H}^1(\mathbb{R}^3)$. In addition, we have the following:

Lemma 8.10. *Let $p = 2$.*

$$|\kappa_n| \lesssim M_n^{-\frac{3}{2}} (\log M_n)^{\frac{1}{2}}. \tag{8.60}$$

Proof. From Lemmas 8.8 and 8.9, we have

$$|\kappa_n| \lesssim M_{n,1}^{-\frac{1}{2}} (\log M_{n,1})^{\frac{1}{2}} \|\nabla Z_n\|_{L^2} \lesssim M_{n,1}^{-\frac{1}{2}} (\log M_{n,1})^{\frac{1}{2}} \times M_{n,1}^{-1} \lesssim M_{n,1}^{-\frac{3}{2}} (\log M_{n,1})^{\frac{1}{2}}.$$

This completes the proof. □

From Lemmas 8.8 and 8.10, we can prove the uniqueness of the minimizer $R_{\omega(m)}$ as in the case of $2 < p < 3$. Thus, we omit it.

8.3 Case of $1 < p < 2$

8.3.1 Relation between $\|R_{\omega(m)}\|_{L^q}$ and m

In this subsection, as in the previous cases, we shall study the relation between $\|R_{\omega(m)}\|_{L^q}$ $(2 < q < 3)$ and the parameter m to prove Theorem 8.1 (ii) in the case of $1 < p < 2$. More precisely, we shall show the following:

Proposition 8.11. *Let $1 < p < 2$ and $\omega(m) > 0$ be the Lagrange multiplier given in Theorem 1.3 (i) and $M_m = \|R_{\omega(m)}\|_{L^\infty}$. Then, we have the following:*

(i)

$$\lim_{m\to 0} \omega^{\frac{p-3}{2}}(m) M_m^{1-p} = 3^{-\frac{p-1}{2}} V^{p-1}(0). \tag{8.61}$$

(ii)

$$\omega(m) \sim M_m^{-2\frac{p-1}{3-p}} \qquad \text{as } m \to 0. \tag{8.62}$$

(iii)

$$m \sim \omega^{\frac{7-3p}{2(p-1)}}(m) \qquad \text{as } m \to 0. \tag{8.63}$$

(iv)

$$m \sim M_m^{-\frac{7-3p}{3-p}} \qquad \text{as } m \to 0. \tag{8.64}$$

Proof of Proposition 8.11. (i) It follows from (5.3), $\alpha_m = \omega(m) M_m^{-4}$ and $\beta_m = M_m^{p-5}$ that

$$3^{-\frac{p-1}{2}} V^{p-1}(0) = \lim_{m\to 0} \frac{\beta_m}{\alpha_m^{\frac{3-p}{2}}} = \lim_{m\to 0} \omega^{\frac{p-3}{2}}(m) M_m^{-p+1}.$$

Thus, (8.61) holds.

(ii) We see from (8.61) that (8.62) immediately holds.

(iii) By (5.8), one has

$$\begin{aligned} \|\widetilde{R}_m\|_{L^2}^2 &= 4\pi \int_0^{\alpha_m^{-\frac{1}{2}}} \widetilde{R}_m^2(r) r^2\, dr + 4\pi \int_{\alpha_m^{-\frac{1}{2}}}^\infty \widetilde{R}_m^2(r) r^2\, dr \geq 4\pi \int_{\alpha_m^{-\frac{1}{2}}}^\infty \widetilde{R}_m^2(r) r^2\, dr \gtrsim \int_{\alpha_m^{-\frac{1}{2}}}^\infty e^{-\sqrt{\alpha_m} r}\, dr \\ &= \alpha_m^{-\frac{1}{2}} \int_1^\infty e^{-s}\, ds \gtrsim \alpha_m^{-\frac{1}{2}}. \end{aligned} \tag{8.65}$$

It follows from (8.6), (4.6) and (8.65) that

$$m = \|R_{\omega(m)}\|_{L^2}^2 = M_m^{-4} \|\widetilde{R}_m\|_{L^2}^2 \sim M_m^{-4} \alpha_m^{-\frac{1}{2}} \sim \omega^{-\frac{1}{2}}(m) M_m^{-2}.$$

This together with (8.61) yields that

$$m \sim \omega^{-\frac{1}{2}}(m) M_m^{-2} \sim \omega^{-\frac{1}{2}}(m) \omega^{\frac{3-p}{p-1}}(m) = \omega^{\frac{7-3p}{2(p-1)}}(m).$$

(iv) Using (8.61) and (8.63), one has

$$m \sim \omega^{\frac{7-3p}{2(p-1)}}(m) \sim M_m^{-2\frac{p-1}{3-p} \times \frac{7-3p}{2(p-1)}} \sim M_m^{\frac{3p-7}{3-p}}.$$

This completes the proof. □

Next, we estimate the L^q-norm of $R_{\omega(m)}$. More precisely, we obtain the following:

Lemma 8.12. *Let $q > 1$. Then, we obtain $\|R_{\omega(m)}\|_{L^q}^q \lesssim M_{n,j}^{\frac{3p-2q-3}{3-p}}$.*

Proof. Since $R_{\omega(m)}(x) = M_m \widetilde{R}_{\omega(m)}(M_m^2 x)$, we have

$$\|R_{\omega(m)}\|_{L^q}^q = M_m^{-(6-q)} \|\widetilde{R}_{\omega(m)}\|_{L^q}^q. \tag{8.66}$$

By (8.62), we obtain

$$\alpha_m \sim M_m^{-2\frac{5-p}{3-p}} \qquad \text{as } m \to 0. \tag{8.67}$$

Then, it follows from (4.1), (6.15) and (8.67) that

$$\begin{aligned}
\|\widetilde{R}_{\omega(m)}\|_{L^q}^q &= 4\pi \int_0^1 \widetilde{R}_{\omega(m)}^q(r) r^2\, dr + 4\pi \int_1^{\alpha_m^{-\frac{1}{2}}} \widetilde{R}_{\omega(m)}^q(r) r^2\, dr + 4\pi \int_{\alpha_m^{-\frac{1}{2}}}^\infty \widetilde{R}_{\omega(m)}^q(r) r^2\, dr \\
&\lesssim 1 + \int_1^{\alpha_m^{-\frac{1}{2}}} r^{2-q}\, dr + \int_{\alpha_m^{-\frac{1}{2}}}^\infty r^{2-q} e^{-\frac{q}{2}\sqrt{\alpha_m} r}\, dr \\
&\lesssim 1 + \alpha_m^{-\frac{1}{2}(3-q)} + \alpha_m^{-\frac{1}{2}(3-q)} \int_1^\infty s^{2-q} e^{-\frac{q}{2}s}\, ds \lesssim \alpha_m^{-\frac{1}{2}(3-q)} \sim M_m^{\frac{(5-p)(3-q)}{3-p}}.
\end{aligned} \tag{8.68}$$

Thus, by (8.66), we obtain

$$\|R_{\omega(m)}\|_{L^q}^q = M_m^{-(6-q)} \|\widetilde{R}_{\omega(m)}\|_{L^q}^q = M_m^{\frac{3p-2q-3}{3-p}}.$$

This completes the proof. □

8.3.2 Proof of Theorem 8.1 in the case of $1 < p < 2$

This subsection is devoted to the proof of Theorem 8.1 in the case of $1 < p < 2$. As in the previous cases, we shall show this by contradiction. Suppose to the contrary that there exists a sequence $\{m_n\}$ in $(0, \infty)$ such that $\lim_{n\to\infty} m_n = 0$ and for each $n \in \mathbb{N}$, $R_{\omega_1(m_n)}, R_{\omega_2(m_n)}$ and $R_{\omega_1(m_n)} \neq R_{\omega_2(m_n)}$. For $j = 1, 2$, we set

$$M_{n,j} = R_{\omega_j(m_n)}(0), \qquad \widetilde{R}_{n,j}(x) = M_{n,j}^{-1} R_{\omega_j(m_n)}(M_{n,j}^{-2} x),$$

$$\alpha_{n,j} = \omega_j(m_n) M_{n,j}^{-4}, \qquad \beta_{n,j} = M_{n,j}^{p-5}.$$

By (8.1) and (8.3), we see that

$$\lim_{n\to\infty} \frac{M_{n,1}}{M_{n,2}} = \lim_{n\to\infty} \frac{\omega_1(m_n)}{\omega_2(m_n)} = 1,$$

which implies

$$\lim_{n\to\infty} \frac{\alpha_{n,1}}{\alpha_{n,2}} = \lim_{n\to\infty} \frac{\beta_{n,1}}{\beta_{n,2}} = 1.$$

Let

$$Z_n = \frac{R_{\omega_1(m_n)} - R_{\omega_2(m_n)}}{\|R_{\omega_1(m_n)} - R_{\omega_2(m_n)}\|_{L^\infty} + |\omega_1(m_n) - \omega_2(m_n)|}, \tag{8.69}$$

$$\kappa_n = \frac{\omega_1(m_n) - \omega_2(m_n)}{\|R_{\omega_1(m_n)} - R_{\omega_2(m_n)}\|_{L^\infty} + |\omega_1(m_n) - \omega_2(m_n)|}. \tag{8.70}$$

Clearly, we have

$$1 = \|Z_n\|_{L^\infty} + |\kappa_n| \tag{8.71}$$

We can verify that the pair (Z_n, κ_n) satisfies

$$-\Delta Z_n + \omega_1(m_n) Z_n - p \int_0^1 V_n^{p-1}(x,\tau) d\tau Z_n - 5 \int_0^1 V_n^4(x,\tau) d\tau Z_n = -\kappa_n R_{\omega_2(m_n)}, \tag{8.72}$$

where

$$V_n(x,\tau) = \tau R_{\omega_1(m_n)}(x) + (1-\tau) R_{\omega_1(m_n)}(x) \qquad (\tau \in (0,1),\ x \in \mathbb{R}^3). \tag{8.73}$$

Since $Z_n(r)$ is radially symmetric, (8.72) can be written by the following:

$$\begin{aligned} &-\frac{d^2 Z_n}{dr^2}(r) - \frac{2}{r}\frac{dZ_n}{dr}(r) + \omega_1(m_n) Z_n(r) - p\int_0^1 V_n^{p-1}(r,\tau)d\tau Z_n(r) - 5\int_0^1 V_n^4(r,\tau)d\tau Z_n(r) \\ &= -\kappa_n R_{\omega_2(m_n)}(r), \end{aligned} \tag{8.74}$$

We put $\widetilde{Z}_n(\rho) = Z_n(r)$ and $\rho = M_{n,1}^2 r$. Then, since $R_{\omega_j(m_n)}(r) = M_{n,j}\widetilde{R}_{n,j}(M_{n,j}^2 r) = M_{n,j}\widetilde{R}_{n,j}(\rho)$, $\widetilde{Z}_n(\rho)$ satisfies the following:

$$\begin{aligned} &-\frac{d^2 \widetilde{Z}_n}{d\rho^2}(\rho) - \frac{2}{\rho}\frac{d\widetilde{Z}_n}{d\rho}(\rho) + \alpha_{n,1}(m_n)\widetilde{Z}_n(\rho) - p\beta_{n,1}(m_n)\int_0^1 \widetilde{V}_n^{p-1}(\rho,\tau)d\tau \widetilde{Z}_n(\rho) - 5\int_0^1 \widetilde{V}_n^4(\rho,\tau)d\tau \widetilde{Z}_n(\rho) \\ &= -\kappa_n M_{n,1}^{-3}\widetilde{R}_{\omega_2(m_n)}(\rho), \end{aligned} \tag{8.75}$$

where

$$\widehat{V}_n(\rho,\tau) = \tau \widetilde{R}_{\omega_1(m_n)}(\rho) + (1-\tau)\nu_n \widetilde{R}_{\omega_2(m_n)}(\rho) \qquad (\tau \in (0,1)), \qquad \nu_n = \frac{M_{n,2}}{M_{n,1}}.$$

We will obtain a following uniform decay of $\{\widetilde{Z}_n\}$

Lemma 8.13. *Let $1 < p < 2$. Suppose that $\lim_{n\to\infty}\|\nabla\widetilde{Z}_n\|_{L^2} = \infty$. Then, there exists a constant $C_1 > 0$ such that for any $\rho > 1$ and $n \in \mathbb{N}$, we have*

$$|\widetilde{Z}_n(\rho)| \le C_1 \|\nabla\widetilde{Z_n}\|_{L^2}\rho^{-1} \tag{8.76}$$

Proof of Lemma 8.13. Note that (8.75) can be written by

$$-\Delta\widetilde{Z}_n + \alpha_{n,1}(m_n)\widetilde{Z}_n - p\beta_{n,1}\int_0^1 \widetilde{V}_n^{p-1}(x,\tau)d\tau\widetilde{Z}_n - 5\int_0^1 \widetilde{V}_n^4(x,\tau)d\tau\widetilde{Z}_n = -\kappa_n M_{n,1}^{-3}\widetilde{R}_{\omega_2(m_n)} \qquad \text{in } \mathbb{R}^3. \tag{8.77}$$

We denote by $K[u]$ by the Kelvin transform of a function u on $\mathbb{R}^3$, that is,

$$K[u](x) := |x|^{-1}u(\frac{x}{|x|^2}).$$

Then, we see that $K[\widetilde{Z}_n]$ satisfies

$$\begin{aligned} &-\Delta K[\widetilde{Z}_n] + \frac{\alpha_{n,1}(m_n)}{|x|^4}K[\widetilde{Z}_n] \\ &= |x|^{-4}\left[p\beta_{n,1}\int_0^1 \widetilde{V}_n^{p-1}(\frac{x}{|x|^2},\theta)d\theta + 5\int_0^1 \widetilde{V}_n^4(\frac{x}{|x|^2},\theta)d\theta\right]K[\widetilde{Z}_n] - \kappa_n M_{n,1}^{-3}|x|^{-5}\widetilde{R}_{\omega_2(m_n)}(\frac{x}{|x|^2}) \qquad \text{in } \mathbb{R}^3. \end{aligned} \tag{8.78}$$

Put

$$W_n := \frac{K[\widetilde{Z}_n]}{\|\nabla K[\widetilde{Z}_n]\|_{L^2}}.$$

It follows from (8.78) and $\|\nabla K[\widetilde{Z}_n]\|_{L^2} = \|\nabla\widetilde{Z}_n\|_{L^2}$ that W_n is a solution to the following:

$$\begin{aligned} &-\Delta W_n + \frac{\alpha_{n,1}(m_n)}{|x|^4}W_n \\ &= \frac{1}{|x|^4}\left[p\beta_{n,1}(m_n)\int_0^1 \widetilde{V}_n^{p-1}(\frac{x}{|x|^2},\theta)d\theta + 5\int_0^1 \widetilde{V}_n^4(\frac{x}{|x|^2},\theta)d\theta\right]W_n - \frac{\kappa_n M_{n,1}^{-3}}{\|\nabla\widetilde{Z}_n\|_{L^2}}|x|^{-5}\widetilde{R}_{\omega_2(m_n)}(\frac{x}{|x|^2}) \qquad \text{in } \mathbb{R}^3. \end{aligned} \tag{8.79}$$

In order to Lemma 8.13, we will apply Proposition B.1 to the equation (8.78). We see that

$$\int_{B_4} \frac{\alpha_{n,1}}{|x|^4} |K[\widetilde{Z}_n](x) v(x)|\, dx < \infty \qquad \text{for any } v \in H_0^1(B_4).$$

From the proof of Lemma 4.3 in [1], we find that

$$\sup_{n\in\mathbb{N}} \left\| \frac{\beta_{n,1}(m_n)}{|x|^4} \int_0^1 \widetilde{V}_n^{p-1}\left(\frac{x}{|x|^2}, \theta\right) d\theta \right\|_{L^{q_0}(B_4)} < \infty$$

for any $q_0 > \frac{3}{2}$ and

$$\sup_{n\in\mathbb{N}} \left\| \frac{1}{|x|^4} \int_0^1 \widetilde{V}_n^4\left(\frac{x}{|x|^2}, \theta\right) d\theta \right\|_{L^\infty(B_4)} < \infty.$$

It suffices to show that

$$\sup_{n\in\mathbb{N}} \left\| \frac{\kappa_n M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} |x|^{-5} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right) \right\|_{W^{-1,2q_0}(B_4)} < \infty. \tag{8.80}$$

We take $r_0, s_0 \geq 1$ so that

$$\frac{3}{2} < r_0 < \frac{5-p}{2}, \qquad \frac{1}{r_0} + \frac{1}{s_0} = 1. \tag{8.81}$$

This yields that $s_0 < 3 < 2q_0$. By the Hölder and Hardy inequalities, we obtain

$$\begin{aligned}
&\left\| \frac{\kappa_n M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} |x|^{-5} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right) \right\|_{W^{-1,2q_0}(B_4)} \\
&= \frac{\kappa_n M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \sup_{f \in W^{1,2q_0}(B_4), \|f\|_{W^{1,2q_0}(B_4)}=1} |\langle |x|^{-5} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right), f\rangle| \\
&\leq \frac{M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \sup_{f \in W^{1,2q_0}(B_4), \|f\|_{W^{1,2q_0}(B_4)}=1} \left\| |x|^{-4} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right) \right\|_{L^{r_0}(B_4)} \left\| |x|^{-1} f \right\|_{L^{s_0}(B_4)} \\
&\leq \frac{M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \sup_{f \in W^{1,2q_0}(B_4), \|f\|_{W^{1,2q_0}(B_4)}=1} \left\| |x|^{-4} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right) \right\|_{L^{r_0}(B_4)} \|\nabla f\|_{L^{s_0}(B_4)} \\
&\leq \frac{M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \left\| |x|^{-4} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right) \right\|_{L^{r_0}(B_4)}.
\end{aligned}$$

Then, for sufficiently small $\delta > 0$, we have

$$\begin{aligned}
&\frac{M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \left\| |x|^{-4} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right) \right\|_{L^{r_0}(B_4)} \\
&= \frac{M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \left\| |x|^{-4} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right) \right\|_{L^{r_0}(B_{\delta\sqrt{\alpha_{n,1}}})} + \frac{M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \left\| |x|^{-4} \widetilde{R}_{\omega_2(m_n)}\left(\frac{x}{|x|^2}\right) \right\|_{L^{r_0}(B_4 \setminus B_{\delta\sqrt{\alpha_{n,1}}})} \\
&=: I_n + II_n.
\end{aligned}$$

It follows from (8.67) and (8.81) that

$$
\begin{aligned}
I_n &= \frac{M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \left(\int_{B_{\delta\sqrt{\alpha_{n,1}}}} \left| |x|^{-4} \widetilde{R}_{\omega_2(m_n)}(\frac{x}{|x|^2}) \right|^{r_0} dx \right)^{\frac{1}{r_0}} \\
&\lesssim M_{n,1}^{-3} \left(\int_0^{\delta\sqrt{\alpha_{n,1}}} \left| r^{-4} \widetilde{R}_{\omega_2(m_n)}(\frac{1}{r}) \right|^{r_0} r^2 \, dr \right)^{\frac{1}{r_0}} \\
&\lesssim M_{n,1}^{-3} \left(\int_0^{\sqrt{\alpha_{n,1}}} r^{-3r_0+2} e^{-r_0 \frac{\sqrt{\alpha_{n,1}}}{2r}} \, dr \right)^{\frac{1}{r_0}} \\
&\lesssim M_{n,1}^{-3} \left(\int_0^{\sqrt{\alpha_{n,1}}} r^{-3r_0+2} \, dr \right)^{\frac{1}{r_0}} \\
&\lesssim M_{n,1}^{-3} \alpha_{n,1}^{\frac{3}{2r_0} - \frac{3}{2}} \lesssim M_{n,1}^{\frac{3(2r_0-5+p)}{(3-p)r_0}} \to 0 \qquad (n \to \infty),
\end{aligned}
$$

$$
\begin{aligned}
II_n &= \frac{M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \left(\int_{B_4 \setminus B_{L^{-1}\sqrt{\alpha_{n,1}}}} \left| |x|^{-4} \widetilde{R}_{\omega_2(m_n)}(\frac{x}{|x|^2}) \right|^{r_0} dx \right)^{\frac{1}{r_0}} \\
&\lesssim M_{n,1}^{-3} \left(\int_{L^{-1}\sqrt{\alpha_{n,1}}}^4 \left| r^{-4} \widetilde{R}_{\omega_2(m_n)}(\frac{1}{r}) \right|^{r_0} r^2 \, dr \right)^{\frac{1}{r_0}} \\
&\lesssim M_{n,1}^{-3} \left(\int_0^{L^{-1}\sqrt{\alpha_{n,1}}} r^{-3r_0+2} \, dr \right)^{\frac{1}{r_0}} \\
&\lesssim M_{n,1}^{-3} \left(\int_0^{\sqrt{\alpha_{n,1}}} r^{-3r_0+2} \, dr \right)^{\frac{1}{r_0}} \\
&\lesssim M_{n,1}^{-3} \alpha_{n,1}^{\frac{3}{2r_0} - \frac{3}{2}} \lesssim M_{n,1}^{\frac{3(2r_0-5+p)}{(3-p)r_0}} \to 0 \qquad (n \to \infty).
\end{aligned}
$$

Thus, we see from Proposition B.1 that $\|W_n\|_{L^\infty} \le C$ for some $C > 0$. This yields (8.76). □

Lemma 8.14. *Let $1 < p < 2$. Then, there exists a constant $C > 0$ such that $|\kappa_n| \le C M_{n,1}^{\frac{-p+1}{2(3-p)}} \|\nabla Z_n\|_{L^2}$ for all $n \in \mathbb{N}$.*

Proof. As in (8.20), we have

$$
|\kappa_n| \lesssim \frac{1}{m_n} \left(\|R_{\omega_1(m_n)}\|_{L^{\frac{12}{5}}}^2 + \|R_{\omega_2(m_n)}\|_{L^{\frac{12}{5}}}^2 \right) \|Z_n\|_{L^6}.
$$

This together with (8.63), (8.50) and Lemma 8.12 yields that

$$
|\kappa_n| \lesssim M_{n,1}^{\frac{7-3p}{3-p}} \times M_{n,1}^{\frac{5p-13}{2(3-p)}} \|\nabla Z_n\|_{L^2} \lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}} \|\nabla Z_n\|_{L^2}.
$$

This completes the proof. □

In addition, we put $\overline{Z}_n(s) = \rho \widetilde{Z}_n(\rho)/\|\nabla \widetilde{Z}_n\|_{L^2}$ and $s = \sqrt{\alpha_{n,1}}\rho$. Then, we see that $\overline{Z}_n(s)$ satisfies the following:

$$
\begin{aligned}
&-\frac{d^2 \overline{Z}_n}{ds^2}(s) + \overline{Z}_n(s) - p \frac{\beta_{n,1}}{\alpha_{n,1}^{\frac{3-p}{2}}} s^{-(p-1)} \int_0^1 \overline{V}_n^{p-1}(s,\tau) d\tau \overline{Z}_n(s) - 5\alpha_{n,1} s^{-4} \int_0^1 \overline{V}_n^4(s,\tau) d\tau \overline{Z}_n(s) \\
&= -\frac{\kappa_n \alpha_{n,1}^{-1} M_{n,1}^{-3}}{\|\nabla \widetilde{Z}_n\|_{L^2}} \overline{R}_{\omega_2(m_n)}(s),
\end{aligned}
\tag{8.82}
$$

where

$$\widehat{V}_n(r,\tau) = \tau \overline{R}_{\omega_1(m_n)}(s) + (1-\tau)\nu_n \overline{R}_{\omega_2(m_n)}(s) \qquad \overline{R}_{\omega_i(m_n)}(s) = \rho \widetilde{R}_{\omega_i(m_n)}(\rho) \qquad (\tau \in (0,1),\ s>0).$$

We see from (8.76) that

$$\|\overline{Z}_n\|_{L^\infty} \le C. \tag{8.83}$$

Remark 8.4. *From Lemma 8.13 and $s = \sqrt{\alpha_{n,1}}\rho = \sqrt{\alpha_{n,1}} M_{n,1}^2 r = \omega_1^{\frac{1}{2}}(m_n) r$, we see that*

$$C_1 > \frac{1}{\|\nabla \widetilde{Z}_n\|_{L^2}} |\rho \widetilde{Z}_n(\rho)| = \frac{M_{n,1}^2}{\|\nabla \widetilde{Z}_n\|_{L^2}} r|Z_n(r)| = \frac{M_{n,1}}{\|\nabla Z_n\|_{L^2}} r|Z_n(r)| \qquad \textit{for } r \in (\delta \omega_1^{-\frac{1}{2}}(m_n), L\omega_1^{-\frac{1}{2}}(m_n)). \tag{8.84}$$

Lemma 8.15. *Let $1<p<2$. Suppose that $\lim_{n\to\infty} \|\nabla \widetilde{Z}_n\|_{L^2} = \infty$. Then, we have the following:*

$$\left|\kappa_n \int_{\mathbb{R}^3} R_{\omega_2(m_n)} Z_n \, dx\right| \le M_{n,1}^{-1} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2}, \tag{8.85}$$

$$\left|p \int_{\mathbb{R}^3} \int_0^1 V_n^{p-1}(x,\tau) d\tau |Z_n|^2 \, dx\right| \le C(\delta^{-p+3} + M_{n,1}^{\frac{p-5}{3-p}}(m_n) \|\nabla Z_n\|_{L^2}^2 + \frac{C}{L^{p-1}} \omega_1(m_n) \|Z_n\|_{L^2}^2, \tag{8.86}$$

$$\left|5 \int_{\mathbb{R}^3} \int_0^1 V_n^4(x,\tau) d\tau |Z_n|^2 \, dx\right| \le C M_{n,1}^{\frac{2(p-5)}{3-p}} \|\nabla Z_n\|_{L^2}^2. \tag{8.87}$$

Proof. We first show (8.85). For any $\delta>0$ and $L>0$, we have

$$\begin{aligned}
\left|\kappa_n \int_{\mathbb{R}^3} R_{\omega_2(m_n)} Z_n \, dx\right| &\le |\kappa_n| \int_{\mathbb{R}^3} R_{\omega_2(m_n)} |Z_n| \, dx \\
&\le |\kappa_n| 4\pi \int_0^{L\omega_2^{-\frac{1}{2}}(m_n)} R_{\omega_2(m_n)} |Z_n| r^2 \, dr + |\kappa_n| 4\pi \int_{L\omega_2^{-\frac{1}{2}}(m_n)}^\infty R_{\omega_2(m_n)} |Z_n| r^2 \, dr \\
&=: I_n + II_n.
\end{aligned}$$

First, we estimate I_n. By the Hölder inequality, (4.1), (8.62) and Lemma 8.7, we obtain

$$\begin{aligned}
I_n &\lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}} \|\nabla Z_n\|_{L^2} \left(\int_0^{L\omega_2^{-\frac{1}{2}}(m_n)} R_{\omega_2(m_n)}^2(r) r^2 \, dr \right)^{\frac{1}{2}} \|Z_n\|_{L^2} \\
&\lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}} \|\nabla Z_n\|_{L^2} \left(\int_0^{L\omega_2^{-\frac{1}{2}}(m_n)} M_{n,1}^2 \widetilde{R}_{\omega_2(m_n)}^2(M_{n,2}^2 r) r^2 \, dr \right)^{\frac{1}{2}} \|Z_n\|_{L^2} \\
&\lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}+1} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2} \left(\int_0^L \widetilde{R}_{\omega_2(m_n)}^2(\alpha_{n,2}^{-\frac{1}{2}} s) \omega_2^{-\frac{3}{2}}(m_n) s^2 \, ds \right)^{\frac{1}{2}} \\
&\lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}+1} \omega_2^{-\frac{3}{4}}(m_n) \alpha_{n,2}^{\frac{1}{2}} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2} \left(\int_0^L ds \right)^{\frac{1}{2}} \\
&\lesssim L^{\frac{1}{2}} M_{n,1}^{-1} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2}.
\end{aligned}$$

Similarly, one has by (6.15) that

$$
\begin{aligned}
II_n &\lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}} \|\nabla Z_n\|_{L^2} \left(\int_{L\omega_2^{-\frac{1}{2}}(m_n)}^{\infty} R^2_{\omega_2(m_n)}(r) r^2 \, dr \right)^{\frac{1}{2}} \|Z_n\|_{L^2} \\
&\lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}} \|\nabla Z_n\|_{L^2} \left(\int_{L\omega_2^{-\frac{1}{2}}(m_n)}^{\infty} M_{n,2}^2 \widetilde{R}^2_{\omega_2(m_n)}(M_{n,2}^2 r) r^2 \, dr \right)^{\frac{1}{2}} \|Z_n\|_{L^2} \\
&\lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}+1} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2} \left(\int_{L}^{\infty} \widetilde{R}^2_{\omega_2(m_n)}(\alpha_{n,2}^{-\frac{1}{2}} s) \omega_2^{-\frac{3}{2}}(m_n) s^2 \, ds \right)^{\frac{1}{2}} \\
&\lesssim M_{n,1}^{\frac{-p+1}{2(3-p)}+1} \omega_2^{-\frac{3}{4}}(m_n) \alpha_{n,2}^{\frac{1}{2}} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2} \left(\int_{L}^{\infty} e^{-\frac{s}{2}} \, ds \right)^{\frac{1}{2}} \\
&\lesssim e^{-\frac{L}{2}} M_{n,1}^{-1} \|\nabla Z_n\|_{L^2} \|Z_n\|_{L^2}.
\end{aligned}
$$

Next, we show (8.86). We have

$$
\begin{aligned}
&\left| p \int_{\mathbb{R}^3} \int_0^1 V_n^{p-1}(x,\tau) d\tau |Z_n|^2 \, dx \right| \\
&\lesssim \int_{|x| \leq \delta\omega_1^{-\frac{1}{2}}(m_n)} \int_0^1 |V_n^{p-1}(x,\tau)| d\tau |Z_n|^2 \, dx + \int_{\delta\omega_1^{-\frac{1}{2}}(m_n) \leq |x| \leq L\omega_1^{-\frac{1}{2}}(m_n)} \int_0^1 |V_n^{p-1}(x,\tau)| d\tau |Z_n|^2 \, dx \\
&\quad + \int_{|x| \geq L\omega_1^{-\frac{1}{2}}(m_n)} \int_0^1 |V_n^{p-1}(x,\tau)| d\tau |Z_n|^2 \, dx =: I_n + II_n + III_n.
\end{aligned}
$$

Note that

$$
0 < R_{\omega_j(m_n)}(x) = M_{n,j} \widetilde{R}_{n,j}(M_{n,j}^2 x) \lesssim \frac{M_{n,j}}{M_{n,j}^2 |x|} = \frac{1}{M_{n,j}|x|}. \tag{8.88}
$$

This implies that

$$
0 < \left| \int_0^1 V_n^{p-1}(x,\tau) d\tau \right| \lesssim \left(R^{p-1}_{\omega_1(m_n)}(x) + R^{p-1}_{\omega_2(m_n)}(x) \right) \lesssim (M_{n,1}^{-(p-1)} + M_{n,2}^{-(p-1)}) |x|^{-(p-1)}.
$$

Thus, it follows from (8.62) that there exists $C > 0$, which is independent of $n \in \mathbb{N}$, such that

$$
\left| \int_0^1 V_n^{p-1}(x,\tau) d\tau \right| \lesssim M_{n,1}^{-(p-1)} \frac{\omega_1^{\frac{p-1}{2}}(m_n)}{L^{p-1}} \lesssim \frac{\omega_1(m_n)}{L^{p-1}} \qquad \text{for } |x| \geq L\omega_1^{-\frac{1}{2}}(m_n).
$$

This yields that

$$
III_n = \left| \int_{|x| \geq L\delta\omega_1^{-\frac{1}{2}}(m_n)} \int_0^1 V_n^{p-1}(x,\tau) d\tau |Z_n|^2 \, dx \right| \leq \frac{C}{L^{p-1}} \omega_1(m_n) \|Z_n\|^2_{L^2}.
$$

By the Hölder inequality, (4.1) and (8.62), we obtain

$$
\begin{aligned}
I_n &= \left| \int |x| \leq \delta\omega_1^{-\frac{1}{2}}(m_n) \int_0^1 V_n^{p-1}(x,\tau)d\tau |Z_n|^2 \, dx \right| \\
&\lesssim \left(\int_{|x|\leq \delta\omega_1^{-\frac{1}{2}}(m_n)} \left(R_{\omega_1(m_n)}(x) + R_{\omega_2(m_n)}(x)\right)^{\frac{3}{2}(p-1)} \, dx \right)^{\frac{2}{3}} \|Z_n\|_{L^6}^2 \\
&\lesssim M_{n,1}^{p-1} \left(\int_0^{\delta\omega_1^{-\frac{1}{2}}(m_n)} \left(\widetilde{R}_{\omega_1(m_n)}^{\frac{3}{2}(p-1)}(M_{n,1}^2 r) + \widetilde{R}_{\omega_2(m_n)}^{\frac{3}{2}(p-1)}(M_{n,2}^2 r) \right) r^2 \, dr \right)^{\frac{2}{3}} \|\nabla Z_n\|_{L^2}^2 \\
&\lesssim M_{n,1}^{p-1}\omega_1^{-1}(m_n) \left(\int_0^{\delta} (\widetilde{R}_{\omega_1(m_n)}^{\frac{3}{2}(p-1)}(\alpha_{n,1}^{-\frac{1}{2}}s) + \widetilde{R}_{\omega_2(m_n)}^{\frac{3}{2}(p-1)}(\left(\frac{\omega_2(m_n)}{\omega_1(m_n)}\right)^{\frac{1}{2}} \alpha_{n,2}^{-\frac{1}{2}}s))s^2 \, ds \right)^{\frac{2}{3}} \|\nabla Z_n\|_{L^2}^2 \\
&\lesssim M_{n,1}^{p-1}\omega_1^{-1}(m_n) \left(\int_0^{\delta} \alpha_{n,1}^{\frac{3}{4}(p-1)} s^{-\frac{3}{2}p+\frac{7}{2}} \, ds \right)^{\frac{2}{3}} \|\nabla Z_n\|_{L^2}^2 \\
&\lesssim M_{n,1}^{p-1}\omega_1^{-1}(m_n)\alpha_{n,1}^{\frac{p-1}{2}} \delta^{-p+3} \|\nabla Z_n\|_{L^2}^2 \\
&= \delta^{-p+3}\|\nabla Z_n\|_{L^2}^2.
\end{aligned}
$$

In addition, by (8.88), (8.84) and (8.62), we have

$$
\begin{aligned}
II_n &= \left| \int_{\delta\omega_1^{-\frac{1}{2}}(m_n)\leq|x|\leq L\omega_1^{-\frac{1}{2}}(m_n)} \int_0^1 V_n^{p-1}(x,\tau)d\tau|Z_n|^2 \, dx \right| \\
&\lesssim \int_{\delta\omega_1^{-\frac{1}{2}}(m_n)\leq|x|\leq L\omega_1^{-\frac{1}{2}}(m_n)} \left(R_{\omega_1(m_n)}^{p-1}(x) + R_{\omega_2(m_n)}^{p-1}(x) \right) |Z_n|^2 \, dx \\
&\lesssim \int_{\delta\omega_1^{-\frac{1}{2}}(m_n)}^{L\omega_1^{-\frac{1}{2}}(m_n)} M_{n,1}^{-(p-1)} r^{-(p-1)} M_{n,1}^{-2} \frac{\|\nabla Z_n\|_{L^2}^2}{r^2} r^2 \, dr \\
&\lesssim M_{n,1}^{-p-1}\|\nabla Z_n\|_{L^2}^2 \int_{\delta\omega_1^{-\frac{1}{2}}(m_n)}^{L\omega_1^{-\frac{1}{2}}(m_n)} r^{-p+1} \, dr \\
&\lesssim M_{n,1}^{-p-1}\|\nabla Z_n\|_{L^2}^2 \omega_1^{\frac{p-2}{2}}(m_n) \leq CM_{n,1}^{\frac{p-5}{3-p}}\|\nabla Z_n\|_{L^2}^2.
\end{aligned}
$$

Finally, we give the proof of (8.87). By the Hölder inequality and Lemma 8.12, we obtain

$$
\begin{aligned}
\left| 5\int_{\mathbb{R}^3}\int_0^1 V_n^4(x,\tau)d\tau|Z_n|^2 \, dx \right| &\lesssim \int_{\mathbb{R}^3} (|R_{\omega_1(m_n)}|^4 + |R_{\omega_2(m_n)}|^4)|Z_n|^2 \, dx \\
&\leq (\|R_{\omega_1(m_n)}\|_{L^6}^4 + \|R_{\omega_2(m_n)}\|_{L^6}^4)\|Z_n\|_{L^6}^2 \\
&\leq CM_{n,1}^{\frac{2(p-5)}{3-p}}\|\nabla Z_n\|_{L^2}^2.
\end{aligned}
\tag{8.89}
$$

This completes the proof. □

Lemma 8.16. *Let $1 < p < 2$. We have* $\sup_{n\in\mathbb{N}} \|\nabla \widetilde{Z}_n\|_{L^2} < \infty$.

Proof. Multiplying (8.72) by Z_n and integrating the resulting equation, we obtain

$$
\begin{aligned}
&\|\nabla Z_n\|_{L^2}^2 + \omega_1(m_n)\|Z_n\|_{L^2}^2 - p\int_{\mathbb{R}^3}\int_0^1 V_n^{p-1}(x,\tau)d\tau|Z_n|^2 \, dx - 5\int_{\mathbb{R}^3}\int_0^1 V_n^4(x,\tau)d\tau|Z_n|^2 \, dx \\
&= -\kappa_n \int_{\mathbb{R}^3} R_{\omega_2(m_n)} Z_n \, dx.
\end{aligned}
\tag{8.90}
$$

Suppose that $\lim_{n\to\infty}\|\nabla\widetilde{Z}_n\|_{L^2}=\infty$. From Lemmas 8.15, we obtain

$$\|\nabla Z_n\|_{L^2}^2+\omega_1(m_n)\|Z_n\|_{L^2}^2 \le C(\delta^{-p+3}+\varepsilon^2 M_{n,1}^{\frac{p-5}{3-p}}(m_n)\|\nabla Z_n\|_{L^2}^2+\frac{C}{L^{p-1}}\omega_1(m_n)\|Z_n\|_{L^2}^2+CM_{n,1}^{\frac{2(p-5)}{3-p}}\|\nabla Z_n\|_{L^2}^2 + \varepsilon L M_{n,2}^{\frac{p-5}{2(3-p)}}\|\nabla Z_n\|_{L^2}^2+M_{n,1}^{-1}\|\nabla Z_n\|_{L^2}\|Z_n\|_{L^2}.$$

This implies that $\|\nabla Z_n\|_{L^2}^2+\omega_1(m_n)\|Z_n\|_{L^2}^2<0$, which is absurd. This completes the proof. □

We see from Lemma 8.16, $\widetilde{Z}_n(\rho)=Z_n(r)$ and $\rho=M_{n,1}^2 r$ that

$$\|\nabla Z_n\|_{L^2}\le CM_{n,2}^{-2} \tag{8.91}$$

for some $C>0$, which yields that $\lim_{n\to\infty}Z_n=0$ strongly in $\dot{H}^1(\mathbb{R}^3)$. In addition, we have the following:

Lemma 8.17. *Let* $1<p<2$.

$$\lim_{n\to\infty}\kappa_n=0. \tag{8.92}$$

Proof. It follows from Lemma 8.14 and (8.91) that

$$|\kappa_n|\le CM_{n,1}^{\frac{-p+1}{2(3-p)}}\|\nabla Z_n\|_{L^2}\le CM_{n,1}^{\frac{p-5}{2(3-p)}}\to 0 \qquad \text{as } n\to\infty.$$

This completes the proof. □

Since the rest of the proof is similar to that of the case of $2<p<3$, we omit it.

9 Differentiability of the minimizer R_ω with respect to ω in the L^2-supercritical case

To prove Theorem 2.5 (ii), we apply the abstract theory of Grillakis, Shatah and Strauss [27]. To this end, we need to show that $R_\omega|_{\omega=\omega(m)}$ is differentiable with respect to the frequency $\omega>0$ for sufficiently small $m>0$.

9.1 Continuity of the minimization value $E_{\min}(m)$

In this subsection, we shall obtain the continuity of $E_{\min}(m)$ with respect to m.

Proposition 9.1. *Let* $7/3<p<6$. $E_{\min}(m)$ *is continuous on* $(0,\infty)$.

To prove Proposition 9.1, we collect several basic properties of the minimization problem $E_{\min}(m)$. By an elementary computation, we can obtain the following:

Lemma 9.2. *Let* $7/3<p<6$ *and* $u\in H^1(\mathbb{R}^3)\setminus\{0\}$. *Then, there exists a unique* $\lambda(u)>0$ *such that*

$$\mathcal{K}(\lambda^{\frac{3}{2}}u(\lambda\cdot))\begin{cases}>0 & \text{if } \lambda\in(0,\lambda(u)),\\ =0 & \text{if } \lambda=\lambda(u),\\ <0 & \text{if } \lambda\in(\lambda(u),\infty).\end{cases}$$

Lemma 9.3. *Let $7/3 < p < 6$, $m > 0$ and $u \in H^1(\mathbb{R}^3)$ with $\mathcal{K}(u) = 0$ and $\|u\|_{L^2}^2 = n$ for $0 < n < m$. There exist $\rho = \rho(m,n) \in (0,1)$ and $v \in H^1(\mathbb{R}^3)$ with $\|v\|_{L^2}^2 = m$ and $\mathcal{K}(v) = 0$ such that*

$$\mathcal{E}(v) = \mathcal{E}(u) - \frac{\|\nabla u\|_{L^2}^2}{6(p-1)} \left\{(3p-7)(1-\rho(m,n,u)) + (5-p)(1-\rho^3(m,n,u))\beta(u)\right\}, \tag{9.1}$$

where

$$\beta(u) = \frac{\|u\|_{L^6}^6}{\|\nabla u\|_{L^2}^2} > 0.$$

Remark 9.1. *It follows from (9.1) that $E_{\min}(m) \le \mathcal{E}(v) \le \mathcal{E}(u)$. Taking the infimum over $u \in H^1(\mathbb{R}^3)$ with $\|u\|_{L^2}^2 = n$ and $\mathcal{K}(u) = 0$, we have $E_{\min}(m) \le E_{\min}(n)$. Thus, $E_{\min}(m)$ is non-increasing function of m.*

Proof of Lemma 9.3. It follows from $\mathcal{K}(u) = 0$ that

$$\frac{3(p-1)}{2(p+1)}\|u\|_{L^{p+1}}^{p+1} = \|\nabla u\|_{L^2}^2(1-\beta(u)). \tag{9.2}$$

We see from (9.2) that $0 < \beta(u) < 1$. Putting

$$v(\cdot) = \sqrt{\rho}\lambda^{\frac{1}{2}}u(\lambda\cdot) \qquad \text{for } \lambda > 0 \text{ and } 0 < \rho < 1,$$

we have

$$\begin{aligned} &\|\nabla v\|_{L^2}^2 = \rho\|\nabla u\|_{L^2}^2, \qquad \|v\|_{L^6}^6 = \rho^3\|u\|_{L^6}^6, \\ &\|v\|_{L^2}^2 = \rho\lambda^{-2}\|u\|_{L^2}^2, \qquad \|v\|_{L^{p+1}}^{p+1} = \rho^{\frac{p+1}{2}}\lambda^{-\frac{5-p}{2}}\|u\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{9.3}$$

This together with (9.3) yields that

$$\begin{aligned} \mathcal{K}(v) &= \rho\|\nabla u\|_{L^2}^2 - \frac{3(p-1)}{2(p+1)}\rho^{\frac{p+1}{2}}\lambda^{-\frac{5-p}{2}}\|u\|_{L^{p+1}}^{p+1} - \rho^3\|u\|_{L^6}^6 \\ &= \rho\|\nabla u\|_{L^2}^2\left(1 - \rho^{\frac{p-1}{2}}\lambda^{-\frac{5-p}{2}}(1-\beta(u)) - \rho^2\beta(u)\right). \end{aligned}$$

Note that $0 < \beta(u)$ and $\rho < 1$. Then, choosing

$$\lambda(u,\rho) = \left(\frac{\rho^{\frac{p-1}{2}}(1-\beta(u))}{1-\rho^2\beta(u)}\right)^{\frac{2}{5-p}}, \tag{9.4}$$

we see that $\lambda(u,\rho)$ is positive and satisfies

$$1 - \rho^{\frac{p-1}{2}}\lambda^{-\frac{5-p}{2}}(u,\rho)(1-\beta(u)) - \rho^2\beta(u) = 0.$$

Then, we obtain $\mathcal{K}(v) = 0$. We claim that there exists $\rho(m,n) \in (0,1)$ so that

$$\|v\|_{L^2}^2 = \rho(m,n)\lambda^{-2}(u,\rho)\|u\|_{L^2}^2 = m. \tag{9.5}$$

By (9.4), $\|u\|_{L^2}^2 = n$ and $n < m$, (9.5) is equivalent to

$$\rho^{-\frac{3p-7}{5-p}}(m,n)\left(\frac{1-\rho^2(m,n)\beta(u)}{1-\beta(u)}\right)^{\frac{4}{5-p}} = \frac{m}{n}(>1).$$

We put

$$f(\rho) := \rho^{-\frac{3p-7}{5-p}}\left(\frac{1-\rho^2\beta(u)}{1-\beta(u)}\right)^{\frac{4}{5-p}}.$$

Clearly, we have $f(1) = 1$. In addition, since $p > \frac{7}{3}$, we see that $\lim_{\rho \to 0} f(\rho) = +\infty$. Thus, by the intermediate theorem, there exists $\rho(m,n) \in (0,1)$ such that $f(\rho(m,n)) = \frac{m}{n}$. Thus, (9.5) holds.

Since $\mathcal{K}(v) = 0$, we have by (9.3) that

$$\begin{aligned} \mathcal{E}(v) &= \mathcal{E}(v) - \frac{2}{3(p-1)}\mathcal{K}(v) \\ &= \frac{3p-7}{6(p-1)}\|\nabla v\|_{L^2}^2 + \frac{5-p}{6(p-1)}\|v\|_{L^6}^6 \\ &= \frac{3p-7}{6(p-1)}\rho(m,n)\|\nabla u\|_{L^2}^2 + \frac{5-p}{6(p-1)}\rho^3(m,n)\|u\|_{L^6}^6. \end{aligned} \tag{9.6}$$

Similarly, by $\mathcal{K}(u) = 0$, we obtain

$$\mathcal{E}(u) = \mathcal{E}(u) - \frac{2}{3(p-1)}\mathcal{K}(u) = \frac{3p-7}{6(p-1)}\|\nabla u\|_{L^2}^2 + \frac{5-p}{6(p-1)}\|u\|_{L^6}^6.$$

This together with (9.6) yields that

$$\begin{aligned} \mathcal{E}(v) &= \mathcal{E}(u) - \left\{\frac{3p-7}{6(p-1)}(1-\rho(m,n))\|\nabla u\|_{L^2}^2 + \frac{5-p}{6(p-1)}(1-\rho^3(m,n))\|u\|_{L^6}^6\right\} \\ &= \mathcal{E}(u) - \frac{\|\nabla u\|_{L^2}^2}{6(p-1)}\left\{(3p-7)(1-\rho(m,n)) + (5-p)(1-\rho^3(m,n))\beta(u)\right\}. \end{aligned}$$

Thus, (9.1) holds. □

We are now in a position to prove Proposition 9.1.

Proof of Proposition 9.1. Let $0 < n < m$. It follows from Remark 9.1 that

$$E_{\min}(m) \le E_{\min}(n). \tag{9.7}$$

In addition, note that $\|(n/m)^{\frac{1}{2}} R_{\omega(m)}\|_{L^2}^2 = n$. Since $\mathcal{K}(R_{\omega(m)}) = 0$ and $n/m \in (0,1)$, we see that

$$\begin{aligned} \mathcal{K}((n/m)^{\frac{1}{2}} R_{\omega(m)}) &= \frac{n}{m}\left\{\|\nabla R_{\omega(m)}\|_{L^2}^2 - \frac{3(p-1)}{2(p+1)}\left(\frac{n}{m}\right)^{\frac{p-1}{2}}\|R_{\omega(m)}\|_{L^{p+1}}^{p+1} - \left(\frac{n}{m}\right)^2\|R_{\omega(m)}\|_{L^6}^6\right\} \\ &> \frac{n}{m}\mathcal{K}(R_{\omega(m)}) = 0. \end{aligned}$$

Then, it follows from Lemma 9.2 that there exists $\lambda_{m,n} > 1$ such that

$$\mathcal{K}(\lambda_{m,n}^{\frac{3}{2}}((n/m)^{\frac{1}{2}} R_{\omega(m)})(\lambda_{m,n}\cdot)) = 0.$$

We can easily verify that $\lim_{n \to m}\lambda_{m,n} = 1$ and $\|\lambda_{m,n}^{\frac{3}{2}}((n/m)^{\frac{1}{2}} R_{\omega(m)})(\lambda_{m,n}\cdot)\|_{L^2}^2 = n$. This together with the definition of $E_{\min}(n)$ yields that

$$\begin{aligned} E_{\min}(n) &\le \mathcal{E}(\lambda_{m,n}^{\frac{3}{2}}((n/m)^{\frac{1}{2}} R_{\omega(m)})(\lambda_{m,n}\cdot)) \\ &= \frac{\lambda_{m,n}^2}{2}\frac{n}{m}\|\nabla R_{\omega(m)}\|_{L^2}^2 - \frac{\lambda_{m,n}^{\frac{3}{2}(p-1)}}{p+1}\left(\frac{n}{m}\right)^{\frac{p+1}{2}}\|R_{\omega(m)}\|_{L^{p+1}}^{p+1} - \frac{\lambda_{m,n}^6}{6}\left(\frac{n}{m}\right)^3\|R_{\omega(m)}\|_{L^6}^6 \\ &= E_{\min}(m) + \frac{1}{2}\left(\lambda_{m,n}^2\left(\frac{n}{m}\right) - 1\right)\|\nabla R_{\omega(m)}\|_{L^2}^2 - \frac{1}{p+1}\left(\lambda_{m,n}^{\frac{3}{2}(p-1)}\left(\frac{n}{m}\right)^{\frac{p+1}{2}} - 1\right)\|R_{\omega(m)}\|_{L^{p+1}}^{p+1} \\ &\quad - \frac{1}{6}\left(\lambda_{m,n}^6\left(\frac{n}{m}\right)^3 - 1\right)\|R_{\omega(m)}\|_{L^6}^6. \end{aligned} \tag{9.8}$$

By $\mathcal{K}(R_{\omega(m)}) = 0$ and (9.7), we obtain

$$\begin{aligned} E_{\min}(n) \geq E_{\min}(m) = \mathcal{E}(R_{\omega(m)}) &= \mathcal{E}(R_{\omega(m)}) - \frac{2}{3(p-1)}\mathcal{K}(R_{\omega(m)}) \\ &= \frac{3p-7}{6(p-1)}\|\nabla R_{\omega(m)}\|_{L^2}^2 + \frac{5-p}{6(p-1)}\|R_{\omega(m)}\|_{L^6}^6. \end{aligned} \tag{9.9}$$

This together with (9.8), $\|R_{\omega(m)}\|_{L^2}^2 = m$ and the Hölder inequality yields that there exists a constant $C > 0$ such that

$$\begin{aligned} E_{\min}(n) \leq E_{\min}(m) + CE_{\min}(n)\left|\lambda_{m,n}^2\left(\frac{n}{m}\right) - 1\right| + Cm^{\frac{5-p}{4}}E_{\min}^{\frac{p-1}{4}}(n)\left|\lambda_{m,n}^{\frac{3}{2}(p-1)}\left(\frac{n}{m}\right)^{\frac{p+1}{2}} - 1\right| \\ + CE_{\min}(n)\left|\lambda_{m,n}^6\left(\frac{n}{m}\right)^3 - 1\right| \end{aligned} \tag{9.10}$$

Then, it follows from (9.8) that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $n < m < n + \delta$, we have

$$E_{\min}(n) < E_{\min}(m) + \varepsilon. \tag{9.11}$$

From (9.7) and (9.11), we have obtained the desired result. $\square$

9.2 Continuity of the mapping $m \in (0, \infty) \mapsto R_{\omega(m)} \in H^1_{\mathbf{rad}}(\mathbb{R}^3)$

This subsection is devoted to the following:

Proposition 9.4. *Let $7/3 < p < 6$ and $m_1 > 0$ be the number given in Theorem 8.1. The map $m \in (0, m_1) \mapsto (\omega(m), R_{\omega(m)}) \in (0, \infty) \times H^1_{rad}(\mathbb{R}^3)$ is continuous.*

To prove Proposition 9.4, we need several preparations. First, we obtain the following lower bound of $E_{\min}(m)$:

Lemma 9.5. *Let $7/3 \leq p < 6$. For each $m > 0$, there exists a constant $C = C(m) > 0$ such that $E_{\min}(m) > C$.*

Proof. Let $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ satisfy $\|u\|_{L^2}^2 = m$ and $\mathcal{K}(u) = 0$. Since $\mathcal{K}(u) = 0$, we see that

$$\mathcal{E}(u) = \mathcal{E}(u) - \frac{1}{2}\mathcal{K}(u) = \frac{3p-7}{4(p+1)}\|u\|_{L^{p+1}}^{p+1} + \frac{1}{3}\|u\|_{L^6}^6.$$

In addition, it follows from $\mathcal{K}(u) = 0$, the Gagliardo-Nirenberg and the Sobolev inequalities and $\|u\|_{L^2}^2 = m$ that

$$\|\nabla u\|_{L^2}^2 = \frac{3(p-1)}{2(p+1)}\|u\|_{L^{p+1}}^{p+1} + \|u\|_{L^6}^6 \leq Cm^{\frac{5-p}{4}}\|\nabla u\|_{L^2}^{\frac{3(p-1)}{2}} + C\|\nabla u\|_{L^2}^6 \tag{9.12}$$

for some $C > 0$. Observe that $\frac{3(p-1)}{2} \geq 2$ if $p \geq \frac{7}{3}$. We see from (9.12) that there exists a constant $C = C(m) > 0$ such that $\|\nabla u\|_{L^2} \geq C$ for all $u \in H^1(\mathbb{R}^3)$ satisfying $\|u\|_{L^2}^2 = m$ and $\mathcal{K}(u) = 0$. We have by $\mathcal{K}(u) = 0$ that

$$\mathcal{E}(u) = \mathcal{E}(u) - \frac{\mathcal{K}(u)}{6} = \frac{1}{3}\|\nabla u\|_{L^2}^2 + \frac{3p-7}{6(p+1)}\|u\|_{L^{p+1}}^{p+1} \geq \frac{1}{3}\|\nabla u\|_{L^2}^2 \geq C.$$

This completes the proof. $\square$

Lemma 9.6. *Let $7/3 \le p < 6$. For each $m > 0$, we have*

$$E_{\min}(m) = \inf\left\{\mathcal{J}(u) \colon u \in H^1(\mathbb{R}^3) \setminus \{0\},\ \|u\|_{L^2}^2 = m,\ \mathcal{K}(u) \le 0\right\}, \tag{9.13}$$

where

$$\mathcal{J}(u) = \mathcal{E}(u) - \frac{1}{2}\mathcal{K}(u) = \frac{3p-7}{4(p+1)}\|u\|_{L^{p+1}}^{p+1} + \frac{1}{3}\|u\|_{L^6}^6. \tag{9.14}$$

Proof of Lemma 9.6. We denote the right-hand side of (9.13) by $\widehat{E}_{\min}(m)$, that is,

$$\widehat{E}_{\min}(m) := \inf\left\{\mathcal{J}(u) \colon u \in H^1(\mathbb{R}^3) \setminus \{0\},\ \|u\|_{L^2}^2 = m,\ \mathcal{K}(u) \le 0\right\}$$

From the definitions $E_{\min}(m), \widehat{E}_{\min}(m)$ and (9.14), we have $\widehat{E}_{\min}(m) \le E_{\min}(m)$. Let $u \in H^1(\mathbb{R}^3)$ satisfy $\|u\|_{L^2}^2 = m$ and $\mathcal{K}(u) < 0$. Then, it follows form Lemma 9.2 that there exists $\lambda(u) \in (0,1)$ such that $\mathcal{K}(\lambda^{\frac{3}{2}}(u)u(\lambda(u)\cdot)) = 0$. Clearly, we have $\|\lambda^{\frac{3}{2}}(u)u(\lambda(u)\cdot)\|_{L^2}^2 = \|u\|_{L^2}^2 = m$. From the definition of $E_{\min}$ and $0 < \lambda(u) < 1$, we obtain

$$\begin{aligned}
E_{\min}(m) \le \mathcal{E}(\lambda^{\frac{3}{2}}(u)u(\lambda(u)\cdot)) &= \mathcal{E}(\lambda^{\frac{3}{2}}(u)u(\lambda(u)\cdot)) - \frac{1}{2}\mathcal{K}(\lambda^{\frac{3}{2}}(u)u(\lambda(u)\cdot)) \\
&= \mathcal{J}(\lambda^{\frac{3}{2}}(u)u(\lambda(u)\cdot)) \\
&= \frac{3p-7}{4(p+1)}\lambda^{\frac{3}{2}(p-1)}(u)\|u\|_{L^{p+1}}^{p+1} + \frac{\lambda^6(u)}{3}\|u\|_{L^6}^6 < \mathcal{J}(u).
\end{aligned}$$

Taking an infimum on u, we have $E_{\min}(m) \le \widehat{E}_{\min}(m)$. This completes the proof. □

In order to prove Proposition 9.4, we use the following result obtained by Soave [44]:

Lemma 9.7 (Proposition 1.5 (2) of Soave [44])**.** *Let $7/3 \le p < 6$. For each $m > 0$, we have*

$$E_{\min}(m) < \frac{1}{3}\sigma^{\frac{3}{2}}, \tag{9.15}$$

where σ is the constant defined by (4.18).

We are now in a position to prove Proposition 9.4:

Proof of Proposition 9.4. We divide the proof into 4 steps.

(Step 1). We take $m_* \in (0, m_1)$ arbitrary and fix it. Let $\{m_n\}$ in $(0,\infty)$ be a sequence with $\lim_{n\to\infty} m_n = m_*$. By (9.9), (9.15) and $\|R_{\omega(m_n)}\|_{L^2}^2 = m_n$, we have $\sup_{n\in\mathbb{N}} \|R_{\omega(m_n)}\|_{H^1} < +\infty$. In addition, similarly to (2.8), we see from $\mathcal{N}_{\omega(m)}(R_{\omega(m)}) = 0$ and $\mathcal{K}(R_{\omega(m)}) = 0$ that

$$\omega(m)\|R_{\omega(m)}\|_{L^2}^2 = \frac{5-p}{2(p+1)}\|R_{\omega(m)}\|_{L^{p+1}}^{p+1}. \tag{9.16}$$

From (9.16), $\|R_{\omega(m_n)}\|_{L^2}^2 = m_n, \sup_{n\in\mathbb{N}} \|R_{\omega(m_n)}\|_{H^1} < +\infty$ and $\lim_{n\to\infty} m_n = m_*$, we see that

$$\sup_{n\in\mathbb{N}} |\omega(m_n)| < +\infty.$$

Up to a subsequence, there exists $(\omega_\infty, R_\infty) \in (0,\infty) \times H^1(\mathbb{R}^3)$ such that $\lim_{n\to\infty} \omega(m_n) = \omega_\infty$ and

$$\lim_{n\to\infty} R_{\omega(m_n)} = R_\infty \qquad \text{weakly in } H^1(\mathbb{R}^3) \text{ and strongly in } L^q(\mathbb{R}^3) \text{ for } 2 < q < 6.$$

(Step 2). We claim that $R_\infty \not\equiv 0$. Suppose to the contrary that $R_\infty \equiv 0$. Then, it follows from $\mathcal{K}(R_{\omega(m_n)}) = 0$ that, passing to some subsequence, we have

$$0 = \lim_{n\to\infty} \mathcal{K}(R_{\omega(m_n)}) = \lim_{n\to\infty} \left\|\nabla R_{\omega(m_n)}\right\|_{L^2}^2 - \lim_{n\to\infty} \left\|R_{\omega(m_n)}\right\|_{L^6}^6 . \tag{9.17}$$

Suppose that $\lim_{n\to\infty} \|\nabla R_{\omega(m_n)}\|_{L^2} = 0$. It follows from the boundedness of $\{R_{\omega(m_n)}\}$ in $H^1(\mathbb{R}^3)$ and the Hölder and the Sobolev inequalities that $\lim_{n\to\infty} \|R_{\omega(m_n)}\|_{L^q} = 0$ for all $2 < q \le 6$. This together with Proposition 9.1 and Lemma 9.6 yields that

$$E_{\min}(m_*) = \lim_{n\to\infty} E_{\min}(m_n) = \lim_{n\to\infty} \mathcal{J}(R_{\omega(m_n)}) = 0.$$

However, this contradicts the result of Lemma 9.5. Therefore, by taking a subsequence, we may assume $\lim_{n\to\infty} \|\nabla R_{\omega(m_n)}\|_{L^2} > 0$.

Now, (9.17) with the definition of σ gives us

$$\lim_{n\to\infty} \left\|\nabla R_{\omega(m_n)}\right\|_{L^2}^2 \ge \sigma \lim_{n\to\infty} \left\|R_{\omega(m_n)}\right\|_{L^6}^2 \ge \sigma \lim_{n\to\infty} \left\|\nabla R_{\omega(m_n)}\right\|_{L^2}^{\frac{2}{3}} . \tag{9.18}$$

From this, $\lim_{n\to\infty} \|\nabla R_{\omega(m_n)}\|_{L^2} > 0$ and (9.18), one has

$$\sigma^{\frac{3}{2}} \le \lim_{n\to\infty} \left\|\nabla R_{\omega(m_n)}\right\|_{L^2}^2 \le \lim_{n\to\infty} \left\|R_{\omega(m_n)}\right\|_{L^6}^6 . \tag{9.19}$$

Let $\mathcal{J}$ be the functional defined by (9.14). We see from (9.14) and (9.19) that

$$\begin{aligned}\lim_{n\to\infty} E_{\min}(m_n) &= \lim_{n\to\infty} \mathcal{J}(R_{\omega(m_n)}) \\ &\ge \lim_{n\to\infty} \left\{ \frac{3p-7}{4(p+1)} \|R_{\omega(m_n)}\|_{L^{p+1}}^{p+1} + \frac{1}{3}\|R_{\omega(m_n)}\|_{L^6}^6 \right\} \\ &\ge \frac{1}{3} \lim_{n\to\infty} \left\|R_{\omega(m_n)}\right\|_{L^6}^6 \ge \frac{1}{3}\sigma^{\frac{3}{2}}.\end{aligned}$$

However, this contradicts the fact that $E_{\min}(m) < \frac{1}{3}\sigma^{\frac{3}{2}}$ (see Lemma 9.7). Thus, $R_\infty \not\equiv 0$.

(Step 3). We shall show $R_\infty = R_{\omega(m_*)}$. First, we claim that $\mathcal{K}(R_\infty) \le 0$. It follows from MR699419 lemma (see Lemma 4.4) that

$$\lim_{n\to\infty} \left\{\|R_{\omega(m_n)}\|_{L^2}^2 - \|R_{\omega(m_n)} - R_\infty\|_{L^2}^2 - \|R_\infty\|_{L^2}^2\right\} = 0, \tag{9.20}$$

$$\lim_{n\to\infty} \left\{\mathcal{K}(R_{\omega(m_n)}) - \mathcal{K}(R_{\omega(m_n)} - R_\infty) - \mathcal{K}(R_\infty)\right\} = 0, \tag{9.21}$$

$$\lim_{n\to\infty} \left\{\mathcal{J}(R_{\omega(m_n)}) - \mathcal{J}(R_{\omega(m_n)}) - R_\infty) - \mathcal{J}(R_\infty)\right\} = 0. \tag{9.22}$$

Suppose to the contrary that $\mathcal{K}(R_\infty) > 0$. Then, it follows from (9.21) and $\mathcal{K}(R_{\omega(m_n)}) = 0$ that $\mathcal{K}(R_{\omega(m_n)} - R_\infty) < 0$. We see from (9.20) that $\|R_{\omega(m_n)} - R_\infty\|_{L^2}^2 \le \|R_{\omega(m_n)}\|_{L^2}^2 = m_n$. This together with Remark 9.1 and Lemma 9.6 yields that

$$E_{\min}(m_n) \le E_{\min}\left(\|R_{\omega(m_n)} - R_\infty\|_{L^2}^2\right) \le \mathcal{J}(R_{\omega(m_n)} - R_\infty).$$

Since $\mathcal{J}(R_{\omega(m_n)}) = \mathcal{E}(R_{\omega(m_n)}) = E_{\min}(m_n)$, and $\mathcal{J}(R_{\omega(m_n)}) - R_\infty) \ge E_{\min}(m_n)$, we have by (9.22) that $\mathcal{J}(R_\infty) \le 0$, which contradicts $R_\infty \neq 0$. Thus, we obtain $\mathcal{K}(R_\infty) \le 0$.

By the weak lower semicontinuity and Proposition 9.1, we have

$$\begin{aligned}
&\|R_\infty\|_{L^2}^2 \le \liminf_{n\to\infty} \|R_{\omega(m_n)}\|_{L^2}^2 = m_*, \\
&\mathcal{J}(R_\infty) \le \liminf_{n\to\infty} \mathcal{J}(R_{\omega(m_n)}) = \liminf_{n\to\infty} E_{\min}(m_n) = E_{\min}(m_*).
\end{aligned} \tag{9.23}$$

Then, by (9.23), Remark 9.1, Lemma 9.6 and $\mathcal{K}(R_\infty) \le 0$, we have

$$E_{\min}(m_*) \le E_{\min}(\|R_\infty\|_{L^2}^2) \le \mathcal{J}(R_\infty) \le E_{\min}(m_*).$$

This implies that $\mathcal{J}(R_\infty) = E_{\min}(m_*)$, so that R_∞ is a minimizer for $E_{\min}(m_*)$. In addition, by the uniqueness of the minimizer (see Theorem 8.1), we have $R_\infty = R_{\omega(m_*)}$. By (9.22), $\lim_{n\to\infty}$ and $\mathcal{J}(R_\infty) = \mathcal{J}(R_{\omega(m_*)}) = E_{\min}(m_*)$, we obtain

$$\lim_{n\to\infty} \mathcal{J}(R_{\omega(m_n)} - R_{\omega(m_*)}) = 0,$$

which together with the boundedness of $\{R_{\omega(m_n)}\}$ in $H^1(\mathbb{R}^3)$ implies that $\lim_{n\to\infty} R_{\omega(m_n)} = R_{\omega(m_*)}$ strongly in $L^q(\mathbb{R}^3)$ ($q \in (2,6]$). Moreover, since $\mathcal{K}(R_{\omega(m_n)}) = \mathcal{K}(R_{\omega(m_*)}) = 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} \|\nabla R_{\omega(m_n)}\|_{L^2}^2 &= \lim_{n\to\infty} \left\{ \frac{3(p-1)}{2(p+1)} \|R_{\omega(m_n)}\|_{L^{p+1}}^{p+1} + \|R_{\omega(m_n)}\|_{L^6}^6 \right\} \\
&= \frac{3(p-1)}{2(p+1)} \|R_{\omega(m_*)}\|_{L^{p+1}}^{p+1} + \|R_{\omega(m_*)}\|_{L^6}^6 = \|\nabla R_{\omega(m_*)}\|_{L^2}^2.
\end{aligned}$$

In addition, since $\lim_{n\to\infty} \|R_{\omega(m_n)}\|_{L^2}^2 = \lim_{n\to\infty} m_n = m_* = \|R_{\omega(m_*)}\|_{L^2}^2$, we see that $\lim_{n\to\infty} R_{\omega(m_n)} = R_{\omega_*}$ strongly in $H^1(\mathbb{R}^3)$.

(Step 4). We derive a conclusion. It follows from (9.16) and $\lim_{n\to\infty} R_{\omega(m_n)} = R_{\omega(m_*)}$ strongly in $H^1(\mathbb{R}^3)$ that

$$\omega_\infty = \lim_{n\to\infty} \omega(m_n) = \frac{5-p}{2(p+1)} \lim_{n\to\infty} \frac{\|R_{\omega(m_n)}\|_{L^{p+1}}^{p+1}}{m_n} = \frac{5-p}{2(p+1)} \frac{\|R_{\omega(m_*)}\|_{L^{p+1}}^{p+1}}{m_*} = \omega(m_*).$$

Therefore, we obtain $\lim_{n\to\infty} \omega(m_n) = \omega_\infty = \omega(m_*)$. Since $\{m_n\}$ is an arbitrary sequence with $\lim_{n\to\infty} m_n = m_*$, we obtain the desired result. $\square$

9.3 Differentiability of the mapping $\omega \in (0,\infty) \mapsto R_\omega \in H^1_{\mathbf{rad}}(\mathbb{R}^3)$

In this subsection, we prove that the mapping $\omega \in (0,\infty) \mapsto R_\omega \in H^1_{\text{rad}}(\mathbb{R}^3)$ is differentiable. To prove this, we use the implicit function theorem. Thus, it is convenient that we regard (1.1) as a nonlinear eigenvalue problem, that is, we say that a pair (ω_*, R_{ω_*}) satisfies (1.1) if R_{ω_*} is a solution to (1.1) with $\omega = \omega_*$.

Observe that $\lim_{m\to 0} \omega(m) = 0$. Thus, by Theorem 2.5 (ii), we can take $m_2 \in (0, m_1)$ such that $R_{\omega(m)}$ is non-degenerate for $m \in (0, m_2)$. We first show the following:

Proposition 9.8. *Let $m_* \in (0, m_1)$, where $m_1 > 0$ is given by Theorem 8.1 and $R_{\omega(m_*)}$ be the minimizer for $E_{\min}(m_*)$. There exist $\delta > 0$ and $r > 0$ satisfying the following:*

(i) If $(\omega, u_\omega) \in (\omega(m_*) - \delta, \omega(m_*) + \delta) \times B(u_{\omega(m_*)}, r)$ satisfies (1.1), we have $u_\omega = u_\omega$, where

$$B(u_{\omega(m_*)}, r) = \left\{ u \in H^1_{rad}(\mathbb{R}^3) \colon \|u - u_{\omega(m_*)}\|_{H^1} < r \right\}.$$

(ii) $\omega \in (\omega(m_*) - \delta, \omega(m_*) + \delta) \mapsto u_\omega \in H^1_{rad}(\mathbb{R}^3)$ *is a* C^1 *mapping.*

(iii) There exists $\varepsilon > 0$ *such that if* $|m - m_*| < \varepsilon$*, we have* $\omega(m) \in (\omega(m_*) - \delta, \omega(m_*) + \delta)$*,* $R_{\omega(m)} = u_{\omega(m)}$ *and the mapping* $\omega \in (\omega(m_*) - \delta, \omega(m_*) + \delta) \mapsto R_\omega|_{\omega=\omega(m)} \in H^1_{rad}(\mathbb{R}^3)$ *is* C^1*.*

Proof of Proposition 9.8. We first show (i) and (ii). We put $g(u) = u^p + u^5$ and define a mapping $\mathcal{F} : (0, \infty) \times H^1_{\rm rad}(\mathbb{R}^3) \to H^1_{\rm rad}(\mathbb{R}^3)$ by

$$\mathcal{F}(\omega, u) = u - (-\Delta + \omega)^{-1} g(u) \qquad (\omega > 0, u \in H^1_{\rm rad}(\mathbb{R}^3)).$$

It follows that $(\omega, u_\omega) \in (0, \infty) \times H^1_{\rm rad}(\mathbb{R}^3)$ is a solution to (1.1) if and only if $\mathcal{F}(\omega, u) = 0$. We can easily verify that $\mathcal{F}(\omega(m_*), u_{\omega(m_*)}) = 0$. It follows from Theorem 2.5 (ii) that

$$\frac{\partial \mathcal{F}}{\partial u}(\omega(m_*), u_{\omega(m_*)}) = I + (-\Delta + \omega(m_*))^{-1} g'(u_{\omega(m_*)})$$

is injective from $H^1_{\rm rad}(\mathbb{R}^3)$ to itself. In addition, we observe from $\lim_{|x| \to \infty} u_{\omega(m_*)}(x) = 0$ that $(-\Delta + \omega(m_*))^{-1} g'(u_{\omega(m_*)})$ is a compact operator on $H^1_{\rm rad}(\mathbb{R}^3)$. This together with the Fredholm alternative theorem yields that $R\left(\frac{\partial \mathcal{F}}{\partial u}(\omega(m_*), R_{\omega(m_*)})\right) = H^1_{\rm rad}(\mathbb{R}^3)$, that is, $\frac{\partial \mathcal{F}}{\partial u}(\omega(m_*), R_{\omega(m_*)})$ is surjective. Thus, we can apply the implicit function theorem, which prove (i) and (ii).

Finally, we give the proof of (iii). From Proposition 9.4, there exists sufficiently small $\varepsilon > 0$ such that if $|m - m_*| < \varepsilon$, we have $(\omega(m), R_{\omega(m)}) \in (\omega(m_*) - \delta, \omega(m_*) + \delta) \times B(R_{\omega(m_*)}, r)$. By Proposition 9.8 (i), (ii) and $u_{\omega(m)} = R_{\omega(m)}$ (see Remark 8.2), the mapping $\omega \in (\omega(m_*) - \delta, \omega(m_*) + \delta) \mapsto R_\omega|_{\omega=\omega(m)} \in H^1_{\rm rad}(\mathbb{R}^3)$ is C^1. □

Proposition 9.9. $\omega(m)$ *is a strictly increasing function on* $(0, m_2)$*.*

Proof. We shall show that there is no open interval $I \subset (0, m_2)$ such that $\omega(m)$ is constant on I by contradiction. Suppose to the contrary that $\omega(m)$ is constant on I_0 for some open interval $I_0 \subset (0, m_2)$. Let $m_0 \in I_0$. Then we have

$$\omega(m) = \omega_{m_0} \qquad \text{for all } m \in I_0.$$

Then, by Proposition 9.8 (iii), we have $R_{\omega(m)} = R_{\omega(m_0)}$ for all $m \in I_0$. Note that if $m \in I_0 \setminus \{0\}$, we have $\|R_{\omega(m)}\|^2_{L^2} = m \neq m_0 = \|R_{\omega(m_0)}\|^2_{L^2}$, which is absurd.

Next, we shall derive the conclusion. Suppose to the contrary that $\omega(m)$ is not an increasing function of m. Then, since $\lim_{m \to 0} \omega(m) = 0$, $\omega(m)$ is continuous on m (see Proposition 9.4) and there is no open interval such that $\omega(m)$ is constant, there exists $m_* \in (0, m_2)$ where $\omega(m_*)$ is a strict local maximum. Then, there exists $\mu_1, \mu_2 \in (0, m_2)$ such that

$$\mu_1 < \mu_2, \qquad |\mu_i - m_*| < \varepsilon \quad (i = 1, 2), \qquad \omega(\mu_1) = \omega(\mu_2),$$

where $\varepsilon > 0$ is a positive number given in Proposition 9.8 (iii). Then, using Proposition 9.8 (iii) again, we have $R_{\omega(\mu_1)} = R_{\omega(\mu_2)}$. However, this cannot happen because $\|R_{\omega(\mu_2)}\|^2_{L^2} = \mu_2 > \mu_1 = \|R_{\omega(\mu_2)}\|^2_{L^2}$. This completes the proof. □

10 Morse index of $R_{\omega(m)}$

This section is devoted to the proof of Theorem C (iv). We study the Morse index of $R_{\omega(m)}$.

10.1 The Morse index is either 1 or 2

In this subsection, we shall show for each $m > 0$, the Morse index of $R_{\omega(m)}$ is either 1 or 2. We put

$$g_m := -2\Delta R_{\omega(m)} - \frac{3(p-1)}{2} R^p_{\omega(m)} - 6R^5_{\omega(m)}. \tag{10.1}$$

Note that $\mathcal{K}'(R_{\omega(m)})u = (g_m, u)_{L^2}$. By the standard elliptic regularity (see e.g. Gilberg and Trudinger [26]), we see that $g_m \in H^1(\mathbb{R}^3)$. First, we shall show the following:

Lemma 10.1. *Let $\frac{7}{3} \le p < 3$. For each $m > 0$, $\mathcal{K}'(R_{\omega(m)})$ and $(\|R_{\omega(m)}\|^2_{L^2})'$ are linearly independent.*

Proof. We shall prove this lemma by contradiction. Suppose to the contrary that there exists a constant $C \in \mathbb{R}$ such that $\mathcal{K}'(R_{\omega(m)}) = C(\|R_{\omega(m)}\|^2_{L^2})'/2$. This can be written as

$$-2\Delta R_{\omega(m)} - \frac{3(p-1)}{2} R^p_{\omega(m)} - 6R^5_{\omega(m)} = CR_{\omega(m)} \qquad \text{in } \mathbb{R}^3. \tag{10.2}$$

Multiplying (10.2) by $R_{\omega(m)}$ and integrating the resulting equation, we have

$$2\|\nabla R_{\omega(m)}\|^2_{L^2} - \frac{3(p-1)}{2}\|R_{\omega(m)}\|^{p+1}_{L^{p+1}} - 6\|R_{\omega(m)}\|^6_{L^6} = C\|R_{\omega(m)}\|^2_{L^2}.$$

This together with $\mathcal{K}(R_{\omega(m)}) = 0$ and $p > 1$ yields that

$$\begin{aligned} C\|R_{\omega(m)}\|^2_{L^2} &= \left(\frac{3(p-1)}{p+1} - \frac{3(p-1)}{2}\right)\|R_{\omega(m)}\|^{p+1}_{L^{p+1}} - 4\|R_{\omega(m)}\|^6_{L^6} \\ &= \frac{3(p-1)}{2}\left(\frac{2}{p+1} - 1\right)\|R_{\omega(m)}\|^{p+1}_{L^{p+1}} - 4\|R_{\omega(m)}\|^6_{L^6} < 0. \end{aligned}$$

Thus, we see that $C < 0$. Multiplying (10.2) by $x \cdot \nabla R_{\omega(m)} \in H^1(\mathbb{R}^3)$ and integrating the resulting equation, we obtain

$$\|\nabla R_{\omega(m)}\|^2_{L^2} - \frac{9(p-1)}{2(p+1)}\|R_{\omega(m)}\|^{p+1}_{L^{p+1}} - 3\|R_{\omega(m)}\|^6_{L^6} = \frac{3}{2}C\|R_{\omega(m)}\|^2_{L^2}. \tag{10.3}$$

Multiplying (10.2) by $\frac{3}{2}R_{\omega(m)} \in H^1(\mathbb{R}^3)$ and integrating the resulting equation, we have

$$3\|\nabla R_{\omega(m)}\|^2_{L^2} - \frac{9(p-1)}{4}\|R_{\omega(m)}\|^{p+1}_{L^{p+1}} - 9\|R_{\omega(m)}\|^6_{L^6} = \frac{3}{2}C\|R_{\omega(m)}\|^2_{L^2}. \tag{10.4}$$

Subtracting (10.3) from (10.4) gives us that

$$2\|\nabla R_{\omega(m)}\|^2_{L^2} - \frac{9(p-1)^2}{4(p+1)}\|R_{\omega(m)}\|^{p+1}_{L^{p+1}} - 6\|R_{\omega(m)}\|^6_{L^6} = 0$$

Using $\mathcal{K}(R_{\omega(m)}) = 0$ and $p > \frac{7}{3}$, we obtain

$$0 = -\frac{3(p-1)(3p-7)}{4(p+1)}\|R_{\omega(m)}\|^{p+1}_{L^{p+1}} - 4\|R_{\omega(m)}\|^6_{L^6} < 0,$$

which is absurd. Thus, we see that $\mathcal{K}'(R_{\omega(m)})$ and $(\|R_{\omega(m)}\|^2_{L^2})'$ are linearly independent. ☐

It follows from Lemma 10.1 that g_m and $R_{\omega(m)}$ are linearly independent. We define the linearized operator associated with the equation (1.1) around $R_{\omega(m)}$ on $L^2(\mathbb{R}^3)$ with the domain $H^2(\mathbb{R}^3)$ by

$$L_{m,+} := -\Delta + \omega(m) - pR^{p-1}_{\omega(m)} - 5R^4_{\omega(m)}.$$

We show the following:

Proposition 10.2. *Let $\frac{7}{3} \le p < 3$ and $m > 0$. If $u \in H^1(\mathbb{R}^3)$ satisfies*

$$(u, g_m)_{L^2} = (u, R_{\omega(m)})_{L^2} = 0, \tag{10.5}$$

then we have $\langle L_{m,+}u, u\rangle \ge 0$.

Proof of Proposition 10.2. Let $u \in H^1(\mathbb{R}^3)$ satisfy (10.5). Define a function $Z : \mathbb{R}^3 \to H^1(\mathbb{R}^d)$ by

$$Z(a,b,c) := R_{\omega(m)} + au + bg_m + cR_{\omega(m)}.$$

It is obvious that

$$\frac{\partial Z}{\partial a}(a,b,c) = u, \qquad \frac{\partial Z}{\partial b}(a,b,c) = g_m, \qquad \frac{\partial Z}{\partial c}(a,b,c) = R_{\omega(m)}. \tag{10.6}$$

Define $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\Phi(a,b,c) := \begin{pmatrix} \mathcal{K}(Z(a,b,c)) \\ \|Z(a,b,c)\|_{L^2}^2 - m \end{pmatrix}.$$

Note that

$$\Phi(a,b,c)\Big|_{(a,b,c)=(0,0,0)} = \begin{pmatrix} \mathcal{K}(R_{\omega(m)}) \\ \|R_{\omega(m)}\|_{L^2}^2 - m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By (10.5), (10.6), $g_m = \mathcal{K}'(R_{\omega(m)})$, (10.1) and $\mathcal{K}(R_{\omega(m)}) = 0$, we see that

$$\frac{\partial}{\partial b}\mathcal{K}(Z(a,b,c))\Big|_{(a,b,c)=(0,0,0)} = \langle \mathcal{K}'(R_{\omega(m)}), g_m\rangle = \|g_m\|_{L^2}^2 > 0, \tag{10.7}$$

$$\begin{aligned} \frac{\partial}{\partial c}\mathcal{K}(Z(a,b,c))\Big|_{(a,b,c)=(0,0,0)} &= \langle \mathcal{K}'(R_{\omega(m)}), R_{\omega(m)}\rangle \\ &= \langle g_m, R_{\omega(m)}\rangle \\ &= 2\|\nabla R_{\omega(m)}\|_{L^2}^2 - \frac{3(p-1)}{2}\|R_{\omega(m)}\|_{L^{p+1}}^{p+1} - 6\|R_{\omega(m)}\|_{L^6}^6 \\ &= -\frac{3(p-1)^2}{2(p+1)}\|R_{\omega(m)}\|_{L^{p+1}}^{p+1} - 4\|R_{\omega(m)}\|_{L^{2^*}}^{2^*} < 0. \end{aligned} \tag{10.8}$$

Moreover, by (10.6) and (10.8), we have

$$\begin{aligned} \frac{\partial}{\partial b}\|Z(a,b,c)\|_{L^2}^2\Big|_{(a,b,c)=(0,0,0)} &= 2\langle R_{\omega(m)}, g_m\rangle \\ &= -\frac{3(p-1)^2}{p+1}\|R_{\omega(m)}\|_{L^{p+1}}^{p+1} - 8\|R_{\omega(m)}\|_{L^{2^*}}^{2^*} < 0, \end{aligned} \tag{10.9}$$

$$\frac{\partial}{\partial c}\|Z(a,b,c)\|_{L^2}^2\Big|_{(a,b,c)=(0,0,0)} = 2\langle R_{\omega(m)}, R_{\omega(m)}\rangle = 2\|R_{\omega(m)}\|_{L^2}^2 > 0. \tag{10.10}$$

Put

$$J(a,b,c) := \begin{pmatrix} \frac{\partial}{\partial b}\mathcal{K}(Z(a,b,c)) & \frac{\partial}{\partial c}\mathcal{K}(Z(a,b,c)) \\ \frac{\partial}{\partial b}\|Z(a,b,c))\|_{L^2}^2 & \frac{\partial}{\partial c}\|Z(a,b,c))\|_{L^2}^2 \end{pmatrix}.$$

By (10.7)–(10.10) and the Cauchy-Schwarz inequality , we obtain

$$\det J(0,0,0) = \det \begin{pmatrix} \|g_m\|_{L^2}^2 & \langle g_m, R_{\omega(m)}\rangle \\ 2\langle g_m, R_{\omega(m)}\rangle & 2\|R_{\omega(m)}\|_{L^2}^2 \end{pmatrix}$$

$$= 2\|g_m\|_{L^2}^2 \|R_{\omega(m)}\|_{L^2}^2 - 2\left(\langle g_m, R_{\omega(m)}\rangle\right)^2$$

$$> 2\|g_m\|_{L^2}^2 \|R_{\omega(m)}\|_{L^2}^2 - 2\|g_m\|_{L^2}^2 \|R_{\omega(m)}\|_{L^2}^2 = 0.$$

Note that since g_m and $R_{\omega(m)}$ are linearly independent (see Lemma 10.1), the Cauchy-Schwarz inequality becomes strict. Hence, the implicit function theorem shows that there exist $\varepsilon_0 > 0$ and a C^2-function $\mathbf{h} : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^2$ such that

$$\Phi(a, \mathbf{h}(a)) = \begin{pmatrix} \mathcal{K}(Z(a, \mathbf{h}(a))) \\ \|Z(a, \mathbf{h}(a))\|_{L^2}^2 - m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } a \in (-\varepsilon_0, \varepsilon_0), \qquad \mathbf{h}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.11}$$

Note that we can write

$$Z(a, \mathbf{h}(a)) = R_{\omega(m)} + au + \begin{pmatrix} g_m \\ R_{\omega(m)} \end{pmatrix} \cdot \mathbf{h}(a) \tag{10.12}$$

and

$$J(a, \mathbf{h}(a)) := \begin{pmatrix} \frac{\partial}{\partial b}\mathcal{K}(Z(a, \mathbf{h}(a))) & \frac{\partial}{\partial c}\mathcal{K}(Z(a, \mathbf{h}(a))) \\ \frac{\partial}{\partial b}\|Z(a, \mathbf{h}(a)))\|_{L^2}^2 & \frac{\partial}{\partial c}\|Z(a, \mathbf{h}(a)))\|_{L^2}^2 \end{pmatrix}.$$

$$= \begin{pmatrix} \langle \mathcal{K}'(Z(a, \mathbf{h}(a)), \mathcal{K}'(R_{\omega(m)})\rangle & \langle \mathcal{K}'(Z(a, \mathbf{h}(a))), R_{\omega(m)}\rangle \\ \langle 2Z(a, \mathbf{h}(a)), \mathcal{K}'(R_{\omega(m)})\rangle & 2\langle Z(a, \mathbf{h}(a)), R_{\omega(m)}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \mathcal{K}'(Z(a, \mathbf{h}(a))), g_m\rangle & \langle \mathcal{K}'(Z(a, \mathbf{h}(a))), R_{\omega(m)}\rangle \\ 2\langle Z(a, \mathbf{h}(a)), g_m\rangle & 2\langle Z(a, \mathbf{h}(a)), R_{\omega(m)}\rangle \end{pmatrix}.$$

By (10.11) and (10.12), we see that

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \frac{d}{da}\Phi(a,\mathbf{h}(a)) = \begin{pmatrix} \langle \mathcal{K}'(Z(a,\mathbf{h}(a))), \frac{d}{da}Z(a,\mathbf{h}(a))\rangle \\ \langle 2Z(a,\mathbf{h}(a)), \frac{d}{da}Z(a,\mathbf{h}(a))\rangle \end{pmatrix} \\
&= \begin{pmatrix} \left\langle \mathcal{K}'(Z(a,\mathbf{h}(a))), u + \begin{pmatrix} g_m \\ R_{\omega(m)} \end{pmatrix} \cdot \frac{d\mathbf{h}}{da}(a) \right\rangle \\ \left\langle 2Z(a,\mathbf{h}(a)), u + \begin{pmatrix} g_m \\ R_{\omega(m)} \end{pmatrix} \cdot \frac{d\mathbf{h}}{da}(a) \right\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle \mathcal{K}'(Z(a,\mathbf{h}(a))), u\rangle \\ \langle 2Z(a,\mathbf{h}(a)), u\rangle \end{pmatrix} + \begin{pmatrix} \langle \mathcal{K}'(Z(a,\mathbf{h}(a))), g_m\rangle & \langle \mathcal{K}'(Z(a,\mathbf{h}(a))), R_{\omega(m)}\rangle \\ 2\langle Z(a,\mathbf{h}(a)), g_m\rangle & 2\langle Z(a,\mathbf{h}(a)), R_{\omega(m)}\rangle \end{pmatrix} \frac{d\mathbf{h}}{da}(a) \\
&= \begin{pmatrix} \langle \mathcal{K}'(Z(a,\mathbf{h}(a))), u\rangle \\ \langle 2Z(a,\mathbf{h}(a)), u\rangle \end{pmatrix} + J(a,\mathbf{h}(a))\frac{d\mathbf{h}}{da}(a),
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{d\mathbf{h}}{da}(a) = -J(a,\mathbf{h}(a))^{-1} \begin{pmatrix} \langle \mathcal{K}'(Z(a,\mathbf{h}(a))), u\rangle \\ \langle 2Z(a,\mathbf{h}(a)), u\rangle \end{pmatrix} \\
&= \frac{-1}{\det J(a,\mathbf{h}(a))} \begin{pmatrix} \langle 2Z(a,\mathbf{h}(a)), R_{\omega(m)}\rangle & -\langle \mathcal{K}'(Z(a,\mathbf{h}(a))), R_{\omega(m)}\rangle \\ -\langle 2Z(a,\mathbf{h}(a)), K'(R_{\omega(m)})\rangle & \langle \mathcal{K}'(Z(a,\mathbf{h}(a))), g_m\rangle \end{pmatrix} \begin{pmatrix} \langle \mathcal{K}'(Z(a,\mathbf{h}(a))), u\rangle \\ \langle 2Z(a,\mathbf{h}(a)), u\rangle \end{pmatrix}.
\end{aligned}$$

It follows from $\mathbf{h}(0) = 0$ (see (10.11)), $Z(0,0,0) = R_{\omega(m)}$, $g_m = \mathcal{K}'(R_{\omega(m)})$ and the assumption (10.5) that

$$\frac{d\mathbf{h}}{da}(0) = \frac{-1}{\det J(0,0,0)} \begin{pmatrix} \langle 2R_{\omega(m)}, R_{\omega(m)}\rangle & -\langle g_m, R_{\omega(m)}\rangle \\ -\langle 2R_{\omega(m)}, g_m\rangle & \langle g_m, g_m\rangle \end{pmatrix} \begin{pmatrix} (g_m, u)_{L^2} \\ 2(R_{\omega(m)}, u)_{L^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.13}$$

By (10.12), we have

$$\frac{d}{da}Z(a,\mathbf{h}(a)) = u + \begin{pmatrix} g_m \\ R_{\omega(m)} \end{pmatrix} \cdot \frac{d\mathbf{h}}{da}(a). \tag{10.14}$$

It follows from $\|Z(a),\mathbf{h}(a)\|_{L^2}^2 = m$ (see (10.11)) and (10.14) that

$$\begin{aligned}
\frac{d}{da}\mathcal{E}(Z(a,\mathbf{h}(a))) &= \frac{d}{da}\left(\frac{\omega(m)}{2}\|Z(a,\mathbf{h}(a))\|_{L^2}^2 + \mathcal{E}(Z(a,\mathbf{h}(a)))\right) \\
&= \left\langle \omega(m)Z(a,\mathbf{h}(a)) + \mathcal{E}'(Z(a,\mathbf{h}(a)), u + \begin{pmatrix} g_m \\ R_{\omega(m)} \end{pmatrix} \cdot \frac{d\mathbf{h}}{da}(a) \right\rangle.
\end{aligned} \tag{10.15}$$

Furthermore, by (10.15) and (10.14), one has

$$\begin{aligned}
&\frac{d^2}{da^2}\mathcal{E}(Z(a,\mathbf{h}(a)))\\
&=\frac{d}{da}\left(\left\langle \omega(m)Z(a,\mathbf{h}(a)+\mathcal{E}'(Z(a,\mathbf{h}(a)),u+\begin{pmatrix} g_m \\ R_{\omega(m)}\end{pmatrix}\cdot\frac{d\mathbf{h}}{da}(a)\right\rangle\right)\\
&=\left\langle (\omega(m)+\mathcal{E}''(Z(a,\mathbf{h}(a)))\left\{u+\begin{pmatrix} g_m \\ R_{\omega(m)}\end{pmatrix}\cdot\frac{d\mathbf{h}}{da}(a)\right\},u+\begin{pmatrix} g_m \\ R_{\omega(m)}\end{pmatrix}\cdot\frac{d\mathbf{h}}{da}(a)\right\rangle\\
&\quad+\left\langle \omega(m)Z(a,\mathbf{h}(a))+\mathcal{E}'(Z(a,\mathbf{h}(a)),\begin{pmatrix} g_m \\ R_{\omega(m)}\end{pmatrix}\cdot\frac{d^2\mathbf{h}}{da^2}(a)\right\rangle.
\end{aligned}\tag{10.16}$$

Since $R_{\omega(m)}$ is a minimizer for $E_{\min}(m)$ and

$$Z(0,\mathbf{h}(0))=R_{\omega(m)},\qquad \mathcal{K}(Z(a,\mathbf{h}(a)))=0,\qquad \|Z(a,\mathbf{h}(a))\|_{L^2}^2=m$$

(see (10.11)), $\mathcal{E}(Z(a,\mathbf{h}(a)))$ takes a minimum at $a=0$, which implies

$$\frac{d}{da}\mathcal{E}(Z(a,\mathbf{h}(a)))|_{a=0}=0,\qquad \frac{d^2}{da^2}\mathcal{E}(Z(a,\mathbf{h}(a)))|_{a=0}\geq 0.$$

This together with (10.16), $\mathcal{S}'_{\omega(m)}(R_{\omega(m)})=\omega(m)R_{\omega(m)}+\mathcal{E}'(R_{\omega(m)})=0$ [5] and (10.13) yields that

$$\begin{aligned}
0&\leq\frac{d^2}{da^2}\mathcal{E}(Z(a,\mathbf{h}(a)))|_{a=0}\\
&=\left\langle \left(\omega(m)R_{\omega(m)}+\mathcal{E}''(R_{\omega(m)}\right)\left\{u+\begin{pmatrix} g_m \\ R_{\omega(m)}\end{pmatrix}\cdot\frac{d\mathbf{h}}{da}(0)\right\},\left\{u+\begin{pmatrix} g_m \\ R_{\omega(m)}\end{pmatrix}\cdot\frac{d\mathbf{h}}{da}(0)\right\}\right\rangle\\
&=\langle L_{m,+}u,u\rangle.
\end{aligned}$$

This completes the proof. ☐

Let $I(m)$ be the Morse index of $R_{\omega(m)}$, that is,

$$I(m):=\left\{\dim H\colon H \text{ is a subspace of } H^1(\mathbb{R}^d) \text{ satisfying } \langle L_{m,+}h,h\rangle<0 \text{ for all } h\in H\setminus\{0\}\right\}.$$

Then, we obtain the following:

Lemma 10.3. *Let $7/3<p<5$. Then, for each $m>0$, we have $I(m)=1$ or 2.*

Proof. From the result of Proposition 10.2, we can verify that $I(m)\leq 2$ for each $m>0$. Suppose to the contrary that $I(m_*)\geq 3$ for some $m_*>0$. Then, there exists a function v_* such that $(v_*,g_{m_*})_{L^2}=(v_*,R_{\omega(m_*)})_{L^2}=0$ and $\langle L_{m_*,+}v_*,v_*\rangle<0$, which contradicts the result of Proposition 10.2.

Since $\mathcal{S}'_{\omega(m)}(R_{\omega(m)})=0$, we can easily find that

$$\langle L_{m,+}R_{\omega(m)},R_{\omega(m)}\rangle=-(p-1)\|R_{\omega(m)}\|_{L^{p+1}}^{p+1}-4\|R_{\omega(m)}\|_{L^6}^6<0,\tag{10.17}$$

which implies that $I(m)\geq 1$. This completes the proof. ☐

[5] $R_{\omega(m)}$ satisfies (1.1) with $\omega=\omega(m)$

10.2 Proof of Theorem 2.5 (iii)

In this subsection, we conclude the proof of Theorem 2.5 (iii). We see from Lemma 10.3 that for each $m > 0$, $I(m) = 1$ or 2. Thus, to prove Theorem 2.5 (iii), it suffices to show that $I(m) \neq 1$ for sufficiently small $m > 0$. To this end, we use ideas from Grillakis, Shatah and Strauss [27]. For any $\varepsilon > 0$, we put

$$U_\varepsilon := \left\{ u \in H^1_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{R}) \colon \|u - R_{\omega(m)}\|^2_{H^1} < \varepsilon, \|u\|_{L^2} = \|R_{\omega(m)}\|_{L^2} \right\}. \tag{10.18}$$

Then, we shall show the following:

Lemma 10.4. *Let $7/3 \leq p < 3$, $m \in (0, m_2)$ and $R_{\omega(m)}$ be the minimizer for $E_{\min}(m)$. Suppose that*

(i) $\frac{\partial \|R_\omega\|^2_{L^2}}{\partial \omega}\Big|_{\omega=\omega(m)} > 0$.

(ii) $I(m) = 1$.

Then, there exists constant $C > 0$ and $\varepsilon > 0$ such that

$$\mathcal{E}(u) - \mathcal{E}(R_{\omega(m)}) \geq C\|u - R_{\omega(m)}\|^2_{H^1}$$

for all $u \in U_\varepsilon$.

Proof. We divide the proof into 3 steps.

(Step 1). First, we claim that

$$\langle L_{m,+}u, u\rangle > 0 \tag{10.19}$$

for all $u \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ with $(u, R_{\omega(m)})_{L^2} = 0$. We denote by $\lambda_1(m)$ and $\chi_1(m)$ a negative eigenvalue and eigenfunction of $L_{m,+}$ satisfying $\|\chi_1(m)\|_{L^2} = 1$. We decompose

$$\partial_\omega R_\omega|_{\omega=\omega(m)} = a_0\chi_1(m) + p_0,$$

where $a_0 \in \mathbb{R}$ and $p_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ satisfying $(\chi_1(m), p_0)_{L^2} = 0$. By the Weyl's essential spectrum theorem, we have $\sigma_{\mathrm{ess}}(L_{m,+}) = [\omega(m), \infty)$. It follows from Theorem 2.5 (ii) and Remark 8.2 that Ker $L_{m,+}|_{H^1_{\mathrm{rad}}} = \{0\}$. These together with our assumption $I(m) = 1$ imply that there exists a constant $C_m > 0$ which is independent of u such that

$$\langle L_{m,+}p_0, p_0\rangle \geq C_m\|p_0\|^2_{H^1}.$$

Similarly, for $u \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ with $(u, R_{\omega(m)})_{L^2} = 0$, we decompose

$$u = a\chi_1(m) + p,$$

where $a \in \mathbb{R}$ and $p \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ satisfying $(\chi_1(m), p)_{L^2} = 0$. It follows that

$$\langle L_{m,+}p, p\rangle \geq C_m\|p\|^2_{H^1}.$$

Note that $R_{\omega(m)}$ satisfies (1.1) with $\omega = \omega(m)$. Then, differentiating (1.1) with respect to ω, we have

$$L_{m,+}\partial_\omega R_\omega|_{\omega=\omega(m)} = -R_{\omega(m)}.$$

Thus, from the assumption $\frac{\partial \|R_\omega\|_{L^2}^2}{\partial \omega}\big|_{\omega=\omega(m)} > 0$, we obtain

$$
\begin{aligned}
0 > -\frac{1}{2}\frac{\partial \|R_\omega\|_{L^2}^2}{\partial \omega}\big|_{\omega=\omega(m)} = -\left(R_{\omega(m)}, \partial_\omega R_\omega|_{\omega=\omega(m)}\right)_{L^2} &= 2\langle L_{m,+}\partial_\omega R_\omega|_{\omega=\omega(m)}, \partial_\omega R_\omega|_{\omega=\omega(m)}\rangle \\
&= a_0^2\lambda_1(m) + \langle L_{m,+}p_0, p_0\rangle.
\end{aligned}
\tag{10.20}
$$

In addition, one has

$$
\begin{aligned}
0 = (u, R_{\omega(m)})_{L^2} = -\langle u, L_{m,+}\partial_\omega R_\omega|_{\omega=\omega(m)}\rangle &= -\langle a\chi_1(m) + p, L_{m,+}(a_0\chi_1(m) + p_0)\rangle \\
&= -aa_0\lambda_1(m) - \langle L_{m+}p_0, p\rangle.
\end{aligned}
$$

In addition, we can find that

$$
(\langle L_{m,+}p_0, p\rangle)^2 \leq \langle L_{m,+}p_0, p_0\rangle \langle L_{m,+}p, p\rangle. \tag{10.21}
$$

Therefore, by (10.20)–(10.21), one has

$$
\langle L_{m,+}u, u\rangle = -a^2\lambda_1(m)^2 + \langle L_{m,+}p, p\rangle \geq -a^2\lambda_1(m)^2 + \frac{(\langle L_{m,+}p_0, p\rangle)^2}{\langle L_{m,+}p_0, p_0\rangle} > a^2\lambda_1(m)^2 - \frac{(-a_0 a\lambda_1(m))^2}{a_0^2\lambda_1(m)^2} = 0.
$$

Thus, (10.19) holds.

(Step 2). We claim that there exists $C_0 > 0$ such that

$$
\langle L_{m,+}u, u\rangle \geq C_0\|u\|_{H^1}^2. \tag{10.22}
$$

for all $u \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ with $(u, R_{\omega(m)})_{L^2} = 0$. We shall show (10.22) by contradiction. Suppose to the contrary that (10.22) fails. It follows from (10.19) that there exists a sequence $\{u_n\}$ with $\|u_n\|_{H^1} = 1$ ($n \in \mathbb{N}$) such that $\lim_{n\to\infty}\langle L_{m,+}u_n, u_n\rangle = 0$. Since the sequence $\{u_n\}$ is bounded, there exists a subsequence of $\{u_n\}$ (we still denote it by the same symbol) and $u_\infty \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ such that $\lim_{n\to\infty} u_n = u_\infty$ weakly in $H^1_{\mathrm{rad}}(\mathbb{R}^3)$. We claim that $u_\infty \neq 0$. Suppose to the contrary that $u_\infty = 0$. Since $\lim_{|x|\to\infty} R_{\omega(m)}(x) = 0$, $\|u_n\|_{H^1} = 1$ and $u_\infty = 0$, we have

$$
0 = \lim_{n\to\infty}\langle L_{m,+}u_n, u_n\rangle \gtrsim 1 - \lim_{n\to\infty}\int_{\mathbb{R}^3}(pR_{\omega(m)}^{p-1} + 5R_{\omega(m)}^4)|u_n|^2\,dx \geq 1,
$$

which is a contradiction. Thus, we see that $u_\infty \neq 0$. Then, by the lower semicontinuity, $u_\infty \neq 0$ and (10.19), we have

$$
0 \geq \lim_{n\to\infty}\langle L_{m,+}u_n, u_n\rangle \geq \langle L_{m,+}u_\infty, u_\infty\rangle > 0,
$$

which is a contradiction. Thus, we infer that (10.22) holds.

(Step 3). We conclude the proof. For each $u \in U_\varepsilon$, we put $\eta = u - R_{\omega(m)}$ and decompose

$$
\eta = aR_{\omega(m)} + p,
$$

where $a \in \mathbb{R}$ and $(R_{\omega(m)}, p)_{L^2} = 0$. Then, since $\|u\|_{L^2}^2 = \|R_{\omega(m)}\|_{L^2}^2$ (see (10.18)), we obtain

$$\begin{aligned}
\|R_{\omega(m)}\|_{L^2}^2 &= \|u\|_{L^2}^2 \\
&= \|R_{\omega(m)} + \eta\|_{L^2}^2 \\
&= \|R_{\omega(m)}\|_{L^2}^2 + 2(R_{\omega(m)}, \eta)_{L^2} + \|\eta\|_{L^2}^2 \\
&= \|R_{\omega(m)}\|_{L^2}^2 + 2(R_{\omega(m)}, aR_{\omega(m)} + p)_{L^2} + \|\eta\|_{L^2}^2 \\
&= \|R_{\omega(m)}\|_{L^2}^2 + 2a\|R_{\omega(m)}\|_{L^2}^2 + \|\eta\|_{L^2}^2.
\end{aligned}$$

This implies $a = O(\|\eta\|_{L^2}^2)$. Observe that $\mathcal{S}'_{\omega(m)}(R_{\omega(m)}) = 0$. Thus, the Taylor expansion gives that

$$\begin{aligned}
\mathcal{S}_{\omega(m)}(u) = \mathcal{S}_{\omega(m)}(R_{\omega(m)} + \eta) &= \mathcal{S}_{\omega(m)}(R_{\omega(m)}) + \langle \mathcal{S}'_{\omega(m)}(R_{\omega(m)}), \eta\rangle + \frac{1}{2}\langle \mathcal{S}''_{\omega(m)}(R_{\omega(m)})\eta, \eta\rangle + o(\|\eta\|_{H^1}^2) \\
&= \mathcal{S}_{\omega(m)}(R_{\omega(m)}) + \frac{1}{2}\langle L_{m,+}\eta, \eta\rangle + o(\|\eta\|_{H^1}^2)
\end{aligned}$$

Since $\|u\|_{L^2}^2 = \|R_{\omega(m)}\|_{L^2}^2$, $\mathcal{S}_{\omega(m)}(u) = \mathcal{E}(u) + \omega\|u\|_{L^2}^2/2$, $\eta = aR_{\omega(m)} + p$ and $a = O(\|\eta\|_{L^2}^2)$, one has

$$\begin{aligned}
\mathcal{E}(u) - \mathcal{E}(R_{\omega(m)}) = \frac{1}{2}\langle L_{m,+}\eta, \eta\rangle + o(\|\eta\|_{H^1}^2) &= \frac{1}{2}\langle L_{m,+}p, p\rangle + O(a\|p\|_{H^1}) + O(a^2) + o(\|\eta\|_{H^1}^2) \\
&= \frac{1}{2}\langle L_{m,+}p, p\rangle + o(\|\eta\|_{H^1}^2).
\end{aligned}$$

Then, it follows from (10.22) that

$$\mathcal{E}(u) - \mathcal{E}(R_{\omega(m)}) \geq \frac{C_0}{2}\|p\|_{H^1}^2 + o(\|\eta\|_{H^1}^2). \tag{10.23}$$

In addition, since $a = O(\|\eta\|_{L^2}^2)$ and $\|\eta\|_{L^2} = \varepsilon > 0$ is small, we have

$$\|p\|_{H^1} = \|\eta - aR_{\omega(m)}|_{H^1} \geq \|\eta\|_{H^1} - |a|\|R_{\omega(m)}\|_{H^1} \geq \|\eta\|_{H^1} + O(\|\eta\|_{H^1}^2)$$

This together with (10.23) implies that

$$\mathcal{E}(u) - \mathcal{E}(R_{\omega(m)}) \geq \frac{C_0}{4}\|\eta\|_{H^1}^2.$$

This completes the proof. □

Proposition 10.5. *Let $7/3 \leq p < 3$, $m \in (0, m_2)$ and $R_{\omega(m)}$ be the minimizer for $E_{\min}(m)$. Assume that $\frac{\partial\|R_\omega\|_{L^2}^2}{\partial\omega}\big|_{\omega=\omega(m)} > 0$. Then, we have $I(m) = 2$.*

Proof. It follows from Lemma 10.3 that $I(m) = 1$ or 2 for each $m > 0$. Thus, it suffices to eliminate the possibility that $I(m) = 1$. Suppose to the contrary that $I(m) = 1$. Then, it follows from (10.7) that $R_{\omega(m)}$ is the unique minimizer of the energy E on U_ε. This implies that

$$\frac{d^2}{d\lambda^2}\mathcal{E}(\lambda^{\frac{3}{2}}R_{\omega(m)}(\lambda\cdot))\big|_{\lambda=1} \geq 0. \tag{10.24}$$

On the other hand, observe from Remark 8.2, Theorem 2.4 and $\lim_{m\to 0}\omega(m) = 0$ that

$$\lim_{m\to 0}\|R_{\omega(m)}\|_{L^6}^6 = \lim_{m\to 0}\|u_{\omega(m)}\|_{L^6}^6 = \|W\|_{L^6}^6 = \sigma^{\frac{3}{2}},$$

where $\sigma > 0$ is defined by (4.18). Using $\mathcal{K}(R_{\omega(m)}) = 0$ and $7/3 \le p < 5$, we can compute

$$\begin{aligned}\frac{d^2}{d\lambda^2}\mathcal{E}(\lambda^{\frac{3}{2}}R_{\omega(m)}(\lambda\cdot))\big|_{\lambda=1} &= 2\|\nabla R_{\omega(m)}\|_{L^2}^2 - \frac{9(p-1)^2}{4(p+1)}\|R_{\omega(m)}\|_{L^{p+1}}^{p+1} - 6\|R_{\omega(m)}\|_{L^6}^6 \\ &= -\frac{3(p-1)}{4(p+1)}(3p-7)\|R_{\omega(m)}\|_{L^{p+1}}^{p+1} - 4\|R_{\omega(m)}\|_{L^6}^6 \\ &\le -4\|R_{\omega(m)}\|_{L^6}^6 \le -2\sigma^{\frac{3}{2}} < 0\end{aligned} \tag{10.25}$$

for sufficiently small $m > 0$. Since $\|\lambda^{\frac{3}{2}}R_{\omega(m)}(\lambda\cdot)\|_{L^2} = \|R_{\omega(m)}\|_{L^2}$ for all $\lambda > 0$, (10.25) contradicts (10.24). This completes the proof. $\square$

We see from Proposition 10.5 that $\frac{\partial\|R_\omega\|_{L^2}^2}{\partial\omega}\big|_{\omega=\omega(m)} > 0$ implies that $I(m) = 2$. Thus, we need to compute the sign of $\frac{\partial\|R_\omega\|_{L^2}^2}{\partial\omega}\big|_{\omega=\omega(m)}$. Concerning this, we obtain the following:

Lemma 10.6. *Let $7/3 < p < 3$ and $R_{\omega(m)}$ be the minimizer for $E_{\min}(m)$. We have*

$$\frac{\partial\|R_\omega\|_{L^2}^2}{\partial\omega}\Big|_{\omega=\omega(m)} > 0 \qquad \textit{almost all } m \in (0, m_2). \tag{10.26}$$

Proof. It follows from Proposition 9.9 that $\partial\omega(m)/\partial m > 0$ for almost all $m \in (0, m_2)$. For such a point, we have by $\|R_{\omega(m)}\|_{L^2}^2 = m$ that

$$0 < \frac{\partial\omega(m)}{\partial m} = \frac{1}{\frac{\partial\|R_\omega\|_{L^2}^2}{\partial\omega}\big|_{\omega=\omega(m)}}.$$

Therefore, we see that (10.26) holds. $\square$

We see from (10.17) and the min-max theorem that $\lambda_1(m)$ is written by

$$\lambda_1(m) = \inf_{u\in H^1(\mathbb{R}^3)} \frac{\langle L_{m,+}u, u\rangle}{\|u\|_{L^2}} \tag{10.27}$$

and $\lambda_1(m) < 0$. We shall show the following:

Lemma 10.7. *The maps $\lambda_1(m) : (0,\infty) \to \mathbb{R}$ and $\chi_1(m) : (0,\infty) \to H^1_{rad}(\mathbb{R}^3)$ are continuous on $m > 0$.*

Proof. Let $\{m_n\}$ be a sequence in $(0,\infty)$ with $\lim_{n\to\infty} m_n = m_*$ for some $m_* > 0$. We shall show that $\lim_{n\to\infty}\lambda_1(m_n) = \lambda_1(m_*)$. Note that $\sup_{n\in\mathbb{N}}\|R_{\omega(m_n)}\|_{H^1} < +\infty$. Then, by the elliptic regularity (see e.g. Soave [44, Proposition B.1]), we have $\sup_{n\in\mathbb{N}}\|R_{\omega(m_n)}\|_{L^\infty} < +\infty$. This together with (10.27) and $\|\chi_1(m_n)\|_{L^2} = 1$ yields that

$$\begin{aligned}0 > \lambda_1(m_n) = \langle L_{m_n,+}\chi_1(m_n), \chi_1(m_n)\rangle &\ge \|\nabla\chi_1(m_n)\|_{L^2}^2 + \omega(m_n) - p\|R_{\omega(m_n)}\|_{L^\infty}^{p-1} - 5\|R_{\omega(m_n)}\|_{L^\infty}^4 \\ &\ge \|\nabla\chi_1(m_n)\|_{L^2}^2 - C,\end{aligned}$$

where the constant $C > 0$ is independent of $n \in \mathbb{N}$. This together with $\|\chi_1(m_n)\|_{L^2} = 1$ implies that $\sup_{n\in\mathbb{N}}\|\chi_1(m_n)\|_{H^1} < +\infty$. Since $\lambda_1(m_n) = \langle L_{m_n,+}\chi_1(m_n), \chi_1(m_n)\rangle$, one has $\sup_{n\in\mathbb{N}}|\lambda_1(m_n)| < +\infty$. Then, up to a subsequence, there exist $\chi_{1,\infty} \in H^1(\mathbb{R}^3)$ and $\lambda_{1,\infty} \in \mathbb{R}$ such that $\lim_{n\to\infty}\chi_1(m_n) = \chi_{1,\infty}$ weakly in $H^1(\mathbb{R}^3)$ and $\lim_{n\to\infty}\lambda_1(m_n) = \lambda_{1,\infty}$.

We shall show that $\chi_{1,\infty} \neq 0$. Suppose to the contrary that $\chi_{1,\infty} = 0$. Then, by the weak lower semicontinuity, $\lim_{|x|\to 0} R_{\omega(m_n)}(x) = 0$ and $\lim_{n\to\infty} \omega(m_n) = \omega(m_*)$ (see Proposition 9.4), we see that

$$
\begin{aligned}
0 &\geq \lim_{n\to\infty} \lambda_1(m_n) \\
&\geq \lim_{n\to\infty} \Big\{ \|\nabla \chi_1(m_n)\|_{L^2}^2 + \omega(m_n)\|\chi_1(m_n)\|_{L^2}^2 \\
&\qquad\qquad - p\int_{\mathbb{R}^3} R_{\omega(m_n)}^{p-1} |\chi_1(m_n)|^2 \, dx - 5\int_{\mathbb{R}^3} R_{\omega(m_n)}^4 |\chi_1(m_n)|^2 \, dx \Big\} \\
&\geq \omega(m_*) > 0,
\end{aligned}
$$

which is absurd. Thus, we find that $\chi_{1,\infty} \neq 0$.

By Proposition 9.4, we have $\lim_{n\to\infty} \omega(m_n) = \omega(m_*)$ and $\lim_{n\to\infty} R_{\omega(m_n)} = R_{\omega(m_*)}$ strongly in $H^1(\mathbb{R}^3)$, which yields that

$$
\begin{aligned}
\lambda_{1,\infty} &= \lim_{n\to\infty} \lambda_1(m_n) \\
&\leq \lim_{n\to\infty} \langle L_{m_n,+}\chi_1(m_*), \chi_1(m_*)\rangle \\
&\leq \langle L_{m_*,+}\chi_1(m_*), \chi_1(m_*)\rangle + \lim_{n\to\infty} \Big\{ |\omega(m_n) - \omega(m_*)| \\
&\quad + p\int_{\mathbb{R}^3} |R_{\omega(m_n)}^{p-1} - R_{\omega_*}^{p-1}||\chi_1(m_*)|^2 \, dx + 5\int_{\mathbb{R}^3} ||R_{\omega(m_n)}^4 - R_{\omega_*}^4||\chi_1(m_*)|^2 \, dx \Big\} \\
&= \langle L_{m_*,+}\chi_1(m_*), \chi_1(m_*)\rangle = \lambda_1(m_*).
\end{aligned}
$$

Thus, we have $\lambda_{1,\infty} \leq \lambda_1(m_*)$.

Next, we shall show that $\|\chi_{1,\infty}\|_{L^2} = 1$. By the weak lower semicontinuity, we have

$$
\|\chi_{1,\infty}\|_{L^2} \leq \liminf_{n\to\infty} \|\chi_1(m_n)\|_{L^2} = 1.
$$

Suppose to the contrary that $\|\chi_{1,\infty}\|_{L^2} < 1$. Then, there exists $a > 1$ such that $\|a\chi_{1,\infty}\|_{L^2} = 1$. Then, it follows from (10.27), $\lim_{n\to\infty} \lambda_1(m_n) = \lambda_{1,\infty} \leq \lambda_1(m_*) < 0$ that

$$
\begin{aligned}
\lambda_1(m_*) &\leq a^2 \langle L_{m_*,+}\chi_{1,\infty}, \chi_{1,\infty}\rangle \leq a^2 \liminf_{n\to\infty} \langle L_{m_n,+}\chi_1(m_n), \chi_1(m_n)\rangle \\
&= a^2 \liminf_{n\to\infty} \lambda_1(m_n) = a^2\lambda_{1,\infty} \leq a^2\lambda_1(m_*) < \lambda_1(m_*),
\end{aligned}
$$

which is a contradiction. Thus, we see that $\|\chi_{1,\infty}\|_{L^2} = 1$.

Since $\|\chi_{1,\infty}\|_{L^2} = 1$, we obtain

$$
\lambda_1(m_*) \leq \langle L_{m_*,+}\chi_{1,\infty}, \chi_{1,\infty}\rangle = \liminf_{n\to\infty} \langle L_{m_n,+}\chi_1(m_n), \chi_1(m_n)\rangle = \lim_{n\to\infty} \lambda_1(m_n) = \lambda_{1,\infty} \leq \lambda_1(m_*).
$$

This implies that $\lim_{n\to\infty} \lambda_1(m_n) = \lambda_{1,\infty} = \lambda_1(m_*) = \langle L_{m_*,+}\chi_{1,\infty}, \chi_{1,\infty}\rangle$. Namely, $\chi_{1,\infty}$ is the first eigenfunction of $L_{m_*,+}$. From the simplicity of $\lambda_1(m_*)$, we have $\chi_{1,\infty} = \chi_1(m_*)$.

Finally, we claim that $\lim_{n\to\infty}\chi_1(m_n)=\chi_1(m_*)$ strongly in $H^1(\mathbb{R}^3)$. Since $\lim_{n\to\infty}\chi_1(m_n)=\chi_1(m_*)$ weakly in $H^1(\mathbb{R}^3), \lim_{n\to\infty}\lambda_1(m_n)=\lambda_{1,\infty}=\lambda_1(m_*)$ and $\lim_{n\to\infty}\omega(m_n)=\omega(m_*)$, we obtain

$$
\begin{aligned}
\|\nabla\chi_1(m_*)\|_{L^2}^2 &= \langle L_{m_*,+}\chi_1(m_*),\chi_1(m_*)\rangle - \omega(m_*)\\
&\qquad + p\int_{\mathbb{R}^3} R_{\omega(m_*)}^{p-1}|\chi_1(m_*)|^2\,dx + 5\int_{\mathbb{R}^3} R_{\omega(m_*)}^4|\chi_1(m_*)|^2\,dx\\
&= \lim_{n\to\infty}\left\{\lambda_1(m_n)-\omega(m_n)+p\int_{\mathbb{R}^3} R_{\omega(m_*)}^{p-1}|\chi_1(m_n)|^2\,dx + 5\int_{\mathbb{R}^3} R_{\omega(m_*)}^4|\chi_1(m_n)|^2\,dx\right\}\\
&= \lim_{n\to\infty}\left\{\|\nabla\chi_1(m_n)\|_{L^2}^2 + (\omega(m_n)-\omega(m_*))\right.\\
&\qquad \left. - p\int_{\mathbb{R}^3}(R_{\omega(m_n)}^{p-1}-R_{\omega(m_*)}^{p-1})|\chi_1(m_n)|^2\,dx - 5\int_{\mathbb{R}^3}(R_{\omega(m_n)}^4-R_{\omega(m_*)}^4)|\chi_1(m_n)|^2\,dx\right\}\\
&= \lim_{n\to\infty}\|\nabla\chi_1(m_n)\|_{L^2}^2.
\end{aligned}
$$

Namely, one has $\lim_{n\to\infty}\|\nabla\chi_1(m_n)\|_{L^2}^2=\|\nabla\chi_1(m_*)\|_{L^2}^2$. This together with $\lim_{n\to\infty}\chi_1(m_n)=\chi_1(m_*)$ weakly in $H^1(\mathbb{R}^3)$ concludes the proof. □

We are now in a position to prove Theorem 2.5 (iii).

Proof of Theorem 2.5 (iii). By Proposition 10.5 and Lemma 10.6, we have $I(m)=2$ for almost all $m\in(0,m_2)$. Thus, for any $m_*\in(0,m_2)$, we can take a sequence $\{m_n\}$ in $(0,m_2)$ such that $\lim_{n\to\infty}m_n=m_*$ with $I(m_n)=2$ for all $n\in\mathbb{N}$. Let $\lambda_2(m_n)$ be the second eigenvalue of the operator $L_{m_n,+}$ and $\chi_2(m_n)\in H^1(\mathbb{R}^3)$ be the corresponding eigenfunction with $\|\chi_2(m_n)\|_{L^2}=1$. Thus, it follows that

$$
\langle L_{m_*,+}\chi_2(m_n),\chi_2(m_n)\rangle<0 \qquad \text{for all } n\in\mathbb{N}.
$$

Then, by a similar argument of the proof of Lemma 10.7, we can prove that $\sup_{n\in\mathbb{N}}\|\chi_2(m_n)\|_{H^1}<+\infty$ and $\sup_{n\in\mathbb{N}}|\lambda_2(m_n)|<+\infty$. Let $\varphi_0\in C_0^\infty(\mathbb{R}^3)\setminus\{0\}$ be a cut-off function satisfying $\varphi_0(x)\ge 0$ for all $x\in\mathbb{R}^3$. We put

$$
f_n=\frac{\chi_2(m_n)+a_n\varphi_0}{\|\chi_2(m_n)+a_n\varphi_0\|_{L^2}}, \qquad a_n=-\frac{(\chi_2(m_n),\chi_1(m_*))_{L^2}}{(\varphi_0,\chi_1(m_*))_{L^2}}. \tag{10.28}
$$

Then, we see that

$$
(f_n,\chi_1(m_*))_{L^2}=0, \qquad \|f_n\|_{L^2}=1.
$$

This yields that

$$
\begin{aligned}
\inf_{u\in H^1(\mathbb{R}^3),(u,\chi_1(m_*))_{L^2}=0}\frac{\langle L_{m_*,+}u,u\rangle}{\|u\|_{L^2}^2} &\le \langle L_{m_*,+}f_n,f_n\rangle\\
&\le \frac{\langle L_{m_*,+}\chi_2(m_n),\chi_2(m_n)\rangle}{\|\chi_2(m_n)+a_n\varphi_0\|_{L^2}^2}+2a_n\frac{\langle L_{m_*,+}\chi_2(m_n),\varphi_0\rangle}{\|\chi_2(m_n)+a_n\varphi_0\|_{L^2}^2}\\
&\qquad + a_n^2\frac{\langle L_{m_*,+}\varphi_0,\varphi_0\rangle}{\|\chi_2(m_n)+a_n\varphi_0\|_{L^2}^2}.
\end{aligned} \tag{10.29}
$$

By Lemma 10.7, $\sup_{n\in\mathbb{N}} \|\chi_2(m_n)\|_{H^1} < +\infty$, (10.28), $(\chi_2(m_n), \chi_1(m_n))_{L^2} = 0$ and $\lim_{n\to\infty} \chi_1(m_n) = \chi_1(m_*)$ strongly in $H^1(\mathbb{R}^3)$ (see Lemma 10.7), one has

$$\begin{aligned}\lim_{n\to\infty} a_n &= -\lim_{n\to\infty} \frac{(\chi_2(m_n), \chi_1(m_*))_{L^2}}{(\varphi_0, \chi_1(m_*))_{L^2}} \\ &= -\lim_{n\to\infty} \left\{ \frac{(\chi_2(m_n), \chi_1(m_n))_{L^2}}{(\varphi_0, \chi_1(m_*))_{L^2}} + \frac{(\chi_2(m_n), \chi_1(m_*) - \chi_1(m_n))_{L^2}}{(\varphi_0, \chi_1(m_*))_{L^2}} \right\} \\ &= 0.\end{aligned} \tag{10.30}$$

In addition, we have

$$\begin{aligned}\langle L_{m_*,+}\chi_2(m_n), \chi_2(m_n)\rangle &= \langle L_{m_n,+}\chi_2(m_n), \chi_2(m_n)\rangle + (\omega_* - \omega(m_n)) \\ &\quad - p\int_{\mathbb{R}^3} (R^{p-1}_{\omega(m_*)} - R^{p-1}_{\omega(m_n)})|\chi_2(m_n)|^2\,dx - 5\int_{\mathbb{R}^3} (R^4_{\omega(m_*)} - R^4_{\omega(m_n)})|\chi_2(m_n)|^2\,dx,\end{aligned}$$

which implies

$$\limsup_{n\to\infty} \langle L_{m_*,+}\chi_2(m_n), \chi_2(m_n)\rangle \le 0.$$

In addition, by (10.30), we obtain

$$\begin{aligned}\|\chi_2(m_n) + a_n\varphi_0\|^2_{L^2} = \|\chi_2(m_n)\|^2_{L^2} + 2a_n(\chi_2(m_n), \varphi_0)_{L^2} + a_n^2\|\varphi_0\|^2_{L^2} &= 1 + 2a_n(\chi_2(m_n), \varphi_0)_{L^2} + a_n^2\|\varphi_0\|^2_{L^2} \\ &\to 1 \qquad \text{as } n\to\infty.\end{aligned}$$

These together with (10.29) yield that

$$\inf_{u\in H^1(\mathbb{R}^3), (u,\chi_1(m_*))_{L^2}=0} \frac{\langle L_{m_*,+}u, u\rangle}{\|u\|^2_{L^2}} \le \limsup_{n\to\infty} \langle L_{m_*,+}f_n, f_n\rangle = \limsup_{n\to\infty} \frac{\langle L_{m_*,+}\chi_2(m_n), \chi_2(m_n)\rangle}{\|\chi_2(m_n) + a_n\varphi_0\|^2_{L^2}} \le 0,$$

which implies that $\lambda_2(m_*)$ exists and $\lambda_2(m_*) \le 0$. We see from Theorem 2.5 (ii) and $R_{\omega(m)} = u_{\omega(m)}$ that $\lambda_2(m_*) \ne 0$. Thus, we obtain $\lambda_2(m_*) < 0$, which concludes the proof. □

A Non-existence of positive solution

In this appendix, we shall show the there exists no positive solution to (1.1) for sufficiently large $\omega > 0$ when $1 < p < 3$ by contradiction. Suppose to the contrary that there exists a sequence $\{\omega_n\}$ with $\lim_{n\to\infty} \omega_n = \infty$ such that for each $n \in \mathbb{N}$, (1.1) has a positive solution u_{ω_n} at $\omega = \omega_n$. Then, by the result of [1], we obtain the following:

Lemma A.1 (Lemma 2.3 of [1]). *Let $1 < p < 3$. For each $n \in \mathbb{N}$, we put $M_n := \|u_{\omega_n}\|_{L^\infty}$. Then, we have*

$$\lim_{n\to\infty} M_n = \infty. \tag{A.1}$$

As in Section 4, we put

$$\widetilde{u}_n := M_n^{-1} u_{\omega_n}(M_n^{-2}\cdot). \tag{A.2}$$

Then, we see that $\widetilde{u}_n$ satisfies

$$-\Delta\widetilde{u}_n + \alpha_n\widetilde{u}_n - \beta_n\widetilde{u}_n^p - \widetilde{u}_n^5 = 0 \qquad \text{in } \mathbb{R}^3, \tag{A.3}$$

where $\alpha_n := \omega_n M_n^{-4}$ and $\beta_n := M_n^{p-5}$. Then, by a similar argument to the proof of Theorem 2.4, we obtain the following:

Proposition A.2. *Let $1 < p < 3$ and $\{\widetilde{u}_n\}$ be the sequence of positive solutions to* (A.3) *defined by* (A.2). *we have*

$$\widetilde{u}_n \to W \qquad \text{strongly in } \dot{H}^1 \cap L^q(\mathbb{R}^3)\ (q > 3) \text{ as } n \to \infty. \tag{A.4}$$

Remark A.1. *As we mentioned in Remark 2.2, it seems difficult to obtain the boundedness of $\|\widetilde{u}_n\|_{H^1}$. due to this, we cannot obtain the non-existence of the positive solution to* (1.1) *for large $\omega > 0$ before.*

Once we obtain Proposition A.2, we can show the following lemmas:

Lemma A.3. *Let $1 < p < 3$ and $\{\widetilde{u}_n\}$ be the sequence of positive solutions to* (A.3) *defined by* (A.2). *Then, there exists a constant $C > 0$*

$$\liminf_{n\to\infty} \sqrt{\alpha_n}\|\widetilde{u}_n\|_{L^2}^2 \geq C. \tag{A.5}$$

Proof. We can show that $\widetilde{u}_n$ satisfies (8.65), which yields (A.5). ☐

Lemma A.4. *Let $1 < p < 3$ and $\{\widetilde{u}_n\}$ be the sequence of positive solutions to* (A.3) *defined by* (A.2). *Then, there exists a constant $C_i > 0$ $(i = 1, 2, 3)$ such that*

$$\limsup_{n\to\infty} \|\widetilde{u}_n\|_{L^{p+1}}^{p+1} \leq C_1 \qquad \text{when } 2 < p < 3, \tag{A.6}$$

$$\limsup_{n\to\infty} \|\widetilde{u}_n\|_{L^{p+1}}^{p+1} \leq C_2|\log \alpha_n| \qquad \text{when } p = 2, \tag{A.7}$$

$$\limsup_{n\to\infty} \|\widetilde{u}_n\|_{L^{p+1}}^{p+1} \leq C_3\alpha_n^{\frac{p-2}{2}} \qquad \text{when } 1 < p < 2. \tag{A.8}$$

Proof. We see that the estimate (4.1), (8.42) and (8.68) also hold for $\widetilde{u}_n$. Then, from these, we see that (A.6)–(A.8) holds. ☐

We are now in a position to prove Theorem 2.6.

Proof of Theorem 2.6. As in (2.8), we have

$$\alpha_n\|\widetilde{u}_n\|_{L^2}^2 = \frac{5-p}{2(p+1)}\beta_n\|\widetilde{u}_n\|_{L^{p+1}}^{p+1}. \tag{A.9}$$

When $2 < p < 3$, it follows from Lemmas A.3, A.4 (i) and (A.9) that

$$\sqrt{\alpha_n} \leq C\beta_n \tag{A.10}$$

for some $C > 0$, which does not depend on $n \in \mathbb{N}$. Since $\alpha_n = \omega_n M_n^{-4}$ and $\beta_n = M_n^{p-5}$, we see from (A.10) that $\sqrt{\omega_n} \leq CM_n^{p-3}$, which is absurd because $\lim_{n\to\infty}\omega_n = \infty$, while $\lim_{n\to\infty} M_n^{p-3} = 0$. We can derive a contradiction similarly for the case of $1 < p \leq 2$. ☐

B Boundedness of L^∞ norm of solutions

Here, we obtain a boundedness of L^∞ norm of solution to elliptic equations. Although the following result can be proved by a variant of Brézis–Kato type argument and the Moser iteration ([13, 25]), we give the proof because we don't find any proper references. In fact, we follow the argument of the arxiv version (Version 1) of [1] (see `https://arxiv.org/abs/1801.08696` for the detail).

Proposition B.1. *Let $d \geq 3$, $a(x)$ and $b(x)$ be functions on B_4, and let $u \in H^1(B_4)$ be a weak solution to*

$$-\Delta u + a(x)u = b(x)u + f(x) \quad \text{in } B_4. \tag{B.1}$$

Suppose that $a(x)$ and u satisfy that

$$a(x) \geq 0 \quad \text{for a.a. } x \in B_4, \quad \int_{B_4} a(x)|u(x)v(x)|dx < \infty \quad \text{for each } v \in H_0^1(B_4). \tag{B.2}$$

Let $s > d/2$ and assume that $b \in L^s(B_4)$ and $f \in W^{-1,2s}(B_4)$. Then, there exists a constant $C(d, s, \|b\|_{L^s(B_4)}, \|f\|_{W^{-1,2s}(B_4)})$ such that

$$\|u\|_{L^\infty(B_1)} \leq C\left(d, s, \|b\|_{L^s(B_4)}, \|f\|_{W^{-1,2s}(B_4)}\right) \|u\|_{L^{2^*}(B_4)}.$$

Here, the constant $C(d, s, \|b\|_{L^s(B_4)}, \|f\|_{W^{-1,2s}(B_4)})$ in (i) *and* (ii) *remain bounded as long as t_q and $\|b\|_{L^s(B_4)}$ are bounded.*

Proof. We argue as in [37, Proof of Proposition 2.2]. Let $T > 1$ and set

$$u_T(x) := \begin{cases} T & \text{if } u(x) > T, \\ u(x) & \text{if } |u(x)| \leq T, \\ -T & \text{if } u(x) < -T. \end{cases}$$

For $1 \leq r < R \leq 4$, choose an $\eta_{r,R} \in C_0^\infty(\mathbb{R}^d)$ so that $0 \leq \eta_{r,R} \leq 1$, $\eta_{r,R}(x) = 1$ if $|x| \leq r$ and $\eta_{r,R}(x) = 0$ if $|x| \geq R$. For $q > 0$, we set $\varphi(x) := \eta_{r,R}^2(x)|u_T(x)|^{2q}u(x) \in H_0^1(B_4)$. Since $f \in W^{-1,q}(B_4)$, there exists $f_i \in L^q(B_4)$ for $i = 1, 2, \ldots, d$ such that $f = \sum_{i=1}^d \frac{\partial f}{\partial x_i}$ (see e.g. Brezis [12, Proposition 9.20]). Since $a(x)u(x)\varphi(x) \geq 0$ for a.a. $x \in B_4$ due to (B.2), it follows from (B.1) that

$$\begin{aligned} \int_{B_4} \nabla u \cdot \nabla \varphi dx &\leq \int_{B_4} b(x)\eta_{r,R}^2(x)|u_T(x)|^{2q}|u(x)|^2 dx + \sum_{j=1}^d \int_{B_4} f_j(x)\frac{\partial \varphi}{\partial x_j}(x)dx \\ &\leq \int_{B_4} b(x)\eta_{r,R}^2(x)|u_T(x)|^{2q}|u(x)|^2 dx + \sum_{j=1}^d \int_{B_4} f_j(x)\frac{\partial \varphi}{\partial x_j}(x)dx. \end{aligned} \tag{B.3}$$

Next, it follows from $\varphi(x) = \eta_{r,R}^2(x)|u_T(x)|^{2q}u(x)$ that

$$\frac{\partial \varphi}{\partial x_j} = 2\eta_{r,R}|u_T|^{2q}u\frac{\partial \eta_{r,R}}{\partial x_j} + 2q\eta_{r,R}^2|u_T|^{2q-2}u_T u\frac{\partial u_T}{\partial x_j} + \eta_{r,R}^2|u_T|^{2q}\frac{\partial u}{\partial x_j}, \tag{B.4}$$

$$\nabla \varphi = 2\eta_{r,R}|u_T|^{2q}u\nabla\eta_{r,R} + 2q\eta_{r,R}^2|u_T|^{2q-2}u_T u\nabla u_T + \eta_{r,R}^2|u_T|^{2q}\nabla u \tag{B.5}$$

and that $\nabla u_T = \nabla u$ a.e. $[|u| < T]$ and $\nabla u_T = 0$ a.e. $[|u| \geq T]$. Thus, by (B.5), we get

$$\begin{aligned} \int_{B_4} \nabla u \cdot \nabla \varphi \, dx &= 2\int_{B_4} \eta_{r,R}|u_T|^{2q}u\nabla\eta_{r,R} \cdot \nabla u dx + 2q\int_{B_4} \eta_{r,R}^2|u_T|^{2q-2}u_T u\nabla u_T \cdot \nabla u \, dx \\ &\quad + \int_{B_4} \eta_{r,R}^2|u_T|^{2q}|\nabla u|^2 \, dx \\ &= 2\int_{B_4} \eta_{r,R}|u_T|^{2q}u\nabla\eta_{r,R} \cdot \nabla u dx + 2q\int_{B_4} \eta_{r,R}^2|u_T|^{2q}|\nabla u_T|^2 \, dx \\ &\quad + \int_{B_4} \eta_{r,R}^2|u_T|^{2q}|\nabla u|^2 dx \end{aligned}$$

On the other hand, since

$$\left|2\eta_{r,R}|u_T|^{2q}u\nabla\eta_{r,R}\cdot\nabla u\right|\leq\frac{|u_T|^{2q}\eta_{r,R}^2|\nabla u|^2}{2}+2|u_T|^{2q}u^2|\nabla\eta_{r,R}|^2,$$

$u=u_T$ for $x\in[|u|<T]$ and $\frac{\partial u_T}{\partial x_j}=0$ a.e. $[|u|\geq T]$, we have

$$\begin{aligned}&\int_{B_4}\nabla u\cdot\nabla\varphi\,dx\\&\geq\frac{1}{2}\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|\nabla u|^2dx-2\int_{B_4}|\nabla\eta_{r,R}|^2|u_T|^{2q}u^2dx+2q\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|\nabla u_T|^2\,dx.\end{aligned}\tag{B.6}$$

In addition, observe from (B.4) that

$$\begin{aligned}\sum_{j=1}^d\int_{B_4}f_j\frac{\partial\varphi}{\partial x_j}\,dx&=2\sum_{j=1}^d\int_{B_4}\eta_{r,R}|u_T|^{2q}u\frac{\partial\eta_{r,R}}{\partial x_j}f_jdx+2q\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q-2}u_Tu\frac{\partial u_T}{\partial x_j}f_j\,dx\\&\quad+\int_{B_4}\eta_{r,R}^2|u_T|^{2q}\frac{\partial u}{\partial x_j}f_j\,dx\\&=2\sum_{j=1}^d\int_{B_4}\eta_{r,R}|u_T|^{2q}u\frac{\partial\eta_{r,R}}{\partial x_j}f_jdx+2q\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}\frac{\partial u_T}{\partial x_j}f_j\,dx\\&\quad+\int_{B_4}\eta_{r,R}^2|u_T|^{2q}\frac{\partial u}{\partial x_j}f_j\,dx\end{aligned}$$

We see that

$$\begin{aligned}2\sum_{j=1}^d\int_{B_4}\eta_{r,R}|u_T|^{2q}u\frac{\partial\eta_{r,R}}{\partial x_j}f_jdx&\leq\sum_{j=1}^d\int_{B_4}|u_T|^{2q}u^2\left|\frac{\partial\eta_{r,R}}{\partial x_j}\right|^2dx+\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|f_j|^2\,dx\\&\leq\int_{B_4}|\nabla\eta_{r,R}|^2|u_T|^{2q}u^2dx+\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|f_j|^2\,dx,\end{aligned}$$

$$2q\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}f_j\frac{\partial u_T}{\partial x_j}\,dx\leq q\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|\nabla u_T|^2\,dx+q\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|f_j|^2\,dx$$

$$\begin{aligned}\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}\frac{\partial u}{\partial x_j}f_j\,dx&\leq\frac{1}{4}\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}\left|\frac{\partial u}{\partial x_j}\right|^2dx+2\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|f_j|^2\,dx\\&\leq\frac{1}{4}\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|\nabla u|^2\,dx+2\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|f_j|^2\,dx.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\sum_{j=1}^d\int_{B_4}f_j\frac{\partial\varphi}{\partial x_j}\,dx&\leq\frac{1}{4}\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|\nabla u|^2\,dx+\int_{B_4}|\nabla\eta_{r,R}|^2|u_T|^{2q}u^2\,dx+q\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|\nabla u_T|^2\,dx\\&\quad+(q+3)\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|f_j|^2\,dx.\end{aligned}\tag{B.7}$$

Thus, it follows from (B.3), (B.6) and (B.7) that

$$\begin{aligned}&\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|\nabla u|^2\,dx+2q\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|\nabla u_T|^2\,dx\\&\leq12\int_{B_4}u^2|u_T|^{2q}|\nabla\eta_{r,R}|^2dx+4\int_{B_4}b\eta_{r,R}^2|u_T|^{2q}u^2dx+4(q+3)\sum_{j=1}^d\int_{B_4}\eta_{r,R}^2|u_T|^{2q}|f_j|^2\,dx.\end{aligned}\tag{B.8}$$

Next, observe that

$$\nabla\left(|u_T|^q u\right) = q|u_T|^{q-2}u_T u\nabla u_T + |u_T|^q\nabla u = \begin{cases} |u_T|^q\nabla u & \text{if } |u(x)| \geq T, \\ (q+1)|u_T|^q\nabla u_T & \text{if } |u(x)| < T. \end{cases}$$

Hence, (B.8) gives

$$\begin{aligned}
&\int_{B_4} \eta_{r,R}^2 \left|\nabla\left(|u_T|^q u\right)\right|^2 dx \\
&\leq \frac{(q+1)^2+2q}{2q}\left\{\int_{B_4} \eta_{r,R}^2|u_T|^{2q}|\nabla u|^2\, dx + 2q\int_{B_4} \eta_{r,R}^2|u_T|^{2q}|\nabla u_T|^2\, dx\right\} \\
&\leq \frac{(q+1)^2+2q}{2q}\int_{B_4}\left\{12u^2|u_T|^{2q}|\nabla\eta_{r,R}|^2 + 4b\eta_{r,R}^2|u_T|^{2q}u^2 + 4(q+3)\sum_{j=1}^d \eta_{r,R}^2|u_T|^{2q}|f_j|^2\right\} dx \\
&\leq \frac{2(q+1)^2+4q}{q}(q+3)\int_{B_4}\left\{u^2|u_T|^{2q}|\nabla\eta_{r,R}|^2 + b\eta_{r,R}^2|u_T|^{2q}u^2 + \sum_{j=1}^d \eta_{r,R}^2|u_T|^{2q}|f_j|^2\right\} dx.
\end{aligned}$$

In addition, from $\nabla\left(\eta_{r,R}|u_T|^q u\right) = \eta_{r,R}\nabla\left(|u_T|^q u\right) + |u_T|^q u\nabla\eta_{r,R}$, it follows that

$$\begin{aligned}
\int_{B_4} \left|\nabla\left(\eta_{r,R}|u_T|^q u\right)\right|^2 dx &\leq 2\left\{\int_{B_4} \eta_{r,R}^2|\nabla\left(|u_T|^q u\right)|^2 dx + \int_{B_4} u^2|u_T|^{2q}|\nabla\eta_{r,R}|^2 dx\right\} \\
&\leq \frac{4(q+1)^2+8q}{q}(q+3)\int_{B_4}\left\{u^2|u_T|^{2q}|\nabla\eta_{r,R}|^2 + |b|\eta_{r,R}^2|u_T|^{2q}u^2 + \sum_{j=1}^d \eta_{r,R}^2|u_T|^{2q}|f_j|^2\right\} dx \\
&\quad + 2\int_{B_4} u^2|u_T|^{2q}|\nabla\eta_{r,R}|^2\, dx \\
&\leq C_q\int_{B_4}\left\{u^2|u_T|^{2q}|\nabla\eta_{r,R}|^2 + |b|\eta_{r,R}^2|u_T|^{2q}u^2 + \sum_{j=1}^d \eta_{r,R}^2|u_T|^{2q}|f_j|^2\right\} dx,
\end{aligned}$$

where

$$C_q = \frac{4(q+1)^2+8q}{q}(q+3) + 2. \tag{B.9}$$

Recalling $\eta_{r,R}|u_T|^q u \in H_0^1(B_4)$ and using Sobolev's inequality, we see that

$$\begin{aligned}
\|\eta_{r,R}|u_T|^q u\|_{L^{2^*}}^2 &\leq \sigma^{-1}\left\|\nabla\left(\eta_{r,R}|u_T|^q u\right)\right\|_{L^2}^2 \\
&\leq C_q\sigma^{-1}\int_{B_4}\left\{u^2|u_T|^{2q}|\nabla\eta_{r,R}|^2 + |b|\eta_{r,R}^2|u_T|^{2q}u^2 + \sum_{j=1}^d \eta_{r,R}^2|u_T|^{2q}|f_j|^2\right\} dx.
\end{aligned} \tag{B.10}$$

Letting $T \to \infty$ in (B.10), we see from the monotone convergence theorem that

$$\left\|\eta_{r,R}|u|^{q+1}\right\|_{L^{2^*}}^2 \leq C_q\sigma^{-1}\int_{B_4}\left\{|u|^{2q+2}|\nabla\eta_{r,R}|^2 + |b|\eta_{r,R}^2|u|^{2q+2} + \sum_{j=1}^d \eta_{r,R}^2|u|^{2q}|f_j|^2\right\} dx.$$

Noting $\eta_{r,R} \equiv 1$ on B_r, $\eta_{r,R} \equiv 0$ on B_R^c and $\|\nabla\eta_{r,R}\|_{L^\infty} \leq C_0/(R-r)$ where C_0 is independent of r and R, and using Hölder's inequality, we see that

$$\begin{aligned}
\|u\|_{L^{2^*(q+1)}(B_r)}^{2(q+1)} &\leq C_q\sigma^{-1}\int_{B_4}\left(|\nabla\eta_{r,R}|^2|u|^{2q+2} + |b|\eta_{r,R}^2|u|^{2q+2} + \sum_{j=1}^d \eta_{r,R}^2|u|^{2q}|f_j|^2\right) dx \\
&\leq C_q\sigma^{-1}\left[\frac{C_0^2|B_4|^{1/s}}{(R-r)^2} + \|b\|_{L^s(B_4)}\right]\|u\|_{L^{\frac{(2q+2)s}{s-1}}(B_R)}^{2q+2} + C_q\sigma^{-1}\sum_{j=1}^d \|f_j\|_{L^{2s}(B_4)}^2\|u\|_{L^{\frac{(2q+2)s}{s-1}}(B_R)}^{2q}.
\end{aligned}$$

Writing $\widetilde{q} := q+1$ and $s^* := s/(s-1)$, we have

$$\|u\|_{L^{2^*\widetilde{q}}(B_r)} \leq \left\{ C_q\sigma^{-1}\left[\frac{C_0^2|B_4|^{1/s}}{(R-r)^2} + \|b\|_{L^s(B_4)}\right]\|u\|^{2\widetilde{q}}_{L^{\frac{2\widetilde{q}s}{s-1}}(B_R)} + C_q\sigma^{-1}\sum_{j=1}^d \|f_j\|^2_{L^{2s}(B_4)}\|u\|^{2q}_{L^{\frac{2\widetilde{q}s}{s-1}}(B_R)}\right\}^{\frac{1}{2\widetilde{q}}}. \tag{B.11}$$

For $n \geq 1$, we define

$$\tau_0 := \frac{2^*}{2s^*}, \quad \widetilde{q}_n := \tau_0^n, \quad R_1 = 2,\ r_1 := R_1 - \frac{1}{2}, \quad R_n := r_{n-1} > r_{n-1} - \frac{1}{2^n} =: r_n. \tag{B.12}$$

We remark that since $s > d/2$, we have $\tau_0 > 1$, so that $\widetilde{q}_n \to \infty$ and $2^*\widetilde{q}_{n-1} = 2s^*\widetilde{q}_n$. We claim that

$$\sup_{n\in\mathbb{N}} \|u\|_{L^{2^*\widetilde{q}_n}(B_1)} < +\infty. \tag{B.13}$$

Then, we may assume that $\|u\|_{L^{2^*\widetilde{q}_n}(B_2)} \geq 1$ for any $n \in \mathbb{N}$. Then, setting $q = \widetilde{q}_n$, $R = R_n$ and $r = r_n$ in (B.11), we see that

$$\|u\|_{L^{2^*\widetilde{q}_n}(B_r)} \leq \left\{ 2C_{q_n}\sigma^{-1}\left[\frac{C_0^2|B_4|^{1/s}}{(R_n-r_n)^2} + \|b\|_{L^s(B_4)} + \sum_{j=1}^d \|f_j\|^2_{L^{2s}(B_4)}\right]\right\}^{1/(2\widetilde{q}_n)} \|u\|_{L^{2\widetilde{q}_n s^*}(B_R)}. \tag{B.14}$$

By (B.9), (B.12) and (B.14), we find $C_1 = C_1(d,s) > 0$ such that for each $n \geq 1$,

$$\|u\|_{L^{2^*\widetilde{q}_n}(B_{r_n})} \leq C_1^{1/(2\widetilde{q}_n)}\left[\widetilde{q}_n^2\left(4^n+1\right)\right]^{1/(2\widetilde{q}_n)}\|u\|_{L^{2\widetilde{q}_n s^*}(B_{R_n})},$$

which implies

$$\|u\|_{L^{2^*\widetilde{q}_n}(B_{r_n})} \leq C_1^{\sum_{k=1}^n 1/(2\widetilde{q}_k)}\left(\prod_{k=1}^n \left[\widetilde{q}_k^2\left(4^k+1\right)\right]^{1/(2\widetilde{q}_k)}\right)\|u\|_{L^{2^*}(B_4)}$$

for $n \geq 1$. From the definition of $\widetilde{q}_n$, one sees from $\tau_0 > 1$ that the right hand side is bounded as $n \to \infty$. Therefore, we obtain

$$\|u\|_{L^\infty(B_1)} \leq C_1^{\sum_{k=1}^\infty 1/(2\widetilde{q}_k)}\left(\prod_{k=1}^\infty \left[\widetilde{q}_k^2\left(4^k+1\right)\right]^{1/(2\widetilde{q}_k)}\right)\|u\|_{L^{2^*}(B_4)}.$$

Hence, our assertion holds.

□

C Table of notation

Symbols	Descriptions
$\{u_{1,\omega}\}$	a family of positive solutions to (1.1) with $\limsup_{\omega\to 0}\|u_{1,\omega}\|_{L^\infty}<\infty$
$\widetilde{u}_{1,\omega}$	$\widetilde{u}_{1,\omega}(\cdot)=\omega^{-\frac{1}{p-1}}u_{1,\omega}(\omega^{-\frac{1}{2}}\cdot)$
$M_{1,\omega}$	$M_{1,\omega}=\|u_{1,\omega}\|_{L^\infty}$
$\widehat{u}_{1,\omega}$	$\widehat{u}_{1,\omega}(\cdot)=M_{1,\omega}^{-1}u_{1,\omega}(M_{1,\omega}^{-\frac{p-1}{2}}\cdot)$
$\alpha_{1,\omega}$	$\alpha_{1,\omega}=\omega M_{1,\omega}^{-(p-1)}$
$\gamma_{1,\omega}$	$\gamma_{1,\omega}=-\alpha_{1,\omega}+1+M_{1,\omega}^{5-p}$
$\{u_{2,\omega}\}$	a family of positive solutions to (1.1) with $\limsup_{\omega\to 0}\|u_{2,\omega}\|_{L^\infty}=\infty$
$M_{2,\omega}$	$M_{2,\omega}=\|u_{2,\omega}\|_{L^\infty}$
$\alpha_{2,\omega}$	$\alpha_{2,\omega}=\omega M_{2,\omega}^{-4}$
$\beta_{2,\omega}$	$\beta_{2,\omega}=M_{2,\omega}^{p-5}$
$u_\omega,\ M_\omega,\ \alpha_\omega,\ \beta_\omega$	$u_\omega=u_{2,\omega},\ M_\omega=M_{2,\omega},\ \alpha_\omega=\alpha_{2,\omega},\ \beta_\omega=\beta_{2,\omega}$
W	Aubin-Talenti function $\quad W(x)=\left(1+\frac{\|x\|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}$
$\mathcal{E}(u)$	$\mathcal{E}(u)=\frac{1}{2}\|\nabla u\|_{L^2}^2-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}-\frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*}$
$\mathcal{K}(u)$	$\mathcal{K}(u)=\|\nabla u\|_{L^2}^2-\frac{d(p-1)}{2(p+1)}\|u\|_{L^{p+1}}^{p+1}-\|u\|_{L^{2^*}}^{2^*}$
$\mathcal{S}_\omega(u)$	$\mathcal{S}_\omega(u)=\frac{1}{2}\|\nabla u\|_{L^2}^2+\frac{\omega}{2}\|u\|_{L^2}^2-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}-\frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*}$
$E_{\min}(m)$	$E_{\min}(m)=\inf\{\mathcal{E}(u)\colon u\in\mathcal{P}(m)\}$
$\mathcal{P}(m)$	$\mathcal{P}(m)=\{u\in S(m)\colon \mathcal{K}(u)=0\}=\left\{u\in H^1(\mathbb{R}^d)\colon \|u\|_{L^2}^2=m,\ \mathcal{K}(u)=0\right\}$
$S(m)$	$S(m)=\left\{u\in H^1(\mathbb{R}^d)\colon \|u\|_{L^2}^2=m\right\}$
$E_{\min,\pm}(m)$	$E_{\min,\pm}(m)=\inf\{\mathcal{E}(u)\colon u\in\mathcal{P}_\pm(m)\}$
$\mathcal{P}_+(m)$	$\mathcal{P}_+(m)=\left\{u\in\mathcal{P}(m)\colon 2\|\nabla u\|_{L^2}^2>\frac{d^2(p-1)^2}{4(p+1)}\|u\|_{L^{p+1}}^{p+1}+2^*\|u\|_{L^{2^*}}^{2^*}\right\}$
$\mathcal{P}_-(m)$	$\mathcal{P}_-(m)=\left\{u\in\mathcal{P}(m)\colon 2\|\nabla u\|_{L^2}^2<\frac{d^2(p-1)^2}{4(p+1)}\|u\|_{L^{p+1}}^{p+1}+2^*\|u\|_{L^{2^*}}^{2^*}\right\}$
$Q_{\omega(m)}$	ground state to (1.1) when $\omega=\omega(m)$
$R_{\omega(m)}$	minimizers for $E_{\min}(m)$
$R_{\omega_-(m),-}$	minimizers for $E_{\min,-}(m)$
$L_{\omega,+}$	$L_{\omega,+}=-\Delta+\omega-pu_\omega^{p-1}-(2^*-1)u_\omega^{2^*-2}$

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