On real functions with graphs either connected or locally connected

Gerald Kuba

Abstract. Let $\mathcal{G}$ denote the family of all subspaces G of the plane $\mathbb{R}^2$ such that G is the graph of a function from $\mathbb{R}$ to $\mathbb{R}$. We prove that $\mathcal{G}$ has two subfamilies $\mathcal{G}_1, \mathcal{G}_2$ of *connected* spaces such that the cardinality of $\mathcal{G}_1$ is $\mathbf{c} := 2^{\aleph_0}$ and the cardinality of $\mathcal{G}_2$ is $2^{\mathbf{c}}$, every space in $\mathcal{G}_1$ is *completely metrizable*, each $G \in \mathcal{G}_2$ is a dense subset of $\mathbb{R}^2$, and if $X_1, X_2 \in \mathcal{G}_1 \cup \mathcal{G}_2$ are distinct then the space X_1 is neither homeomorphic to a subspace of X_2 nor homeomorphic to a proper subspace of X_1. On the other hand, the family $\mathcal{G}$ contains precisely $\aleph_0$ *locally connected* spaces up to homeomorphism, and if X, Y are such spaces (including the case $X = Y$) then X is homeomorphic to some proper subspace of Y. Furthermore, if τ is a topology on the set $\mathbb{R}$ finer than the Euclidean topology and the space $(\mathbb{R}, \tau)$ is separable and locally connected then the space is locally compact and homeomorphic to some space in $\mathcal{G}$. In a very natural way we establish a complete classification of all these refinements τ of the real line.

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1. Introduction

Let $|S|$ denote the cardinal number (the *size*) of any set S and let $\mathbf{c} := |\mathbb{R}| = 2^{\aleph_0}$ denote the cardinality of the continuum. A *real function* is a function from $\mathbb{R}$ to $\mathbb{R}$. Let $\mathcal{R}$ denote the family of all real functions. Naturally, $|\mathcal{R}| = 2^{\mathbf{c}}$. As usual, any function is regarded as identical with its graph, whence F is a *subset* of $\mathbb{R}^2$ for every $F \in \mathcal{R}$. In the following, we consider $F \in \mathcal{R}$ as a *subspace* of the Euclidean plane $\mathbb{R}^2$. In doing so every $F \in \mathcal{R}$ induces an interesting topology on the real line, see Section 2.

It is evident that a real function is continuous if and only if it is a pathwise connected subset of the plane. While it is clear that all pathwise connected spaces $F \in \mathcal{R}$ are homeomorphic, this is not true for connected spaces $F \in \mathcal{R}$. Actually, our first goal is to prove the following theorem about connected real functions. For abbreviation, let us call topological spaces X_i $(i \in I)$ *incomparable* if and only if for $i, j \in I$ the space X_i is homeomorphic to a subspace S_j of X_j only in the trivial case $X_i = S_j = X_j$. In other words, spaces are incomparable when they are mutually non-homeomorphic and no space is homeomorphic to a proper subspace of itself or of any other space.

Theorem 1. *There exist $2^{\mathbf{c}}$ real functions which are incomparable, connected, dense subspaces of $\mathbb{R}^2$.*

It is plain that if a real function $F \in \mathcal{R}$ is a *dense* subset of $\mathbb{R}^2$ (as in Theorem 1) then the function F is *everywhere discontinuous.* It is well-known (see [1]) that a subspace X of $\mathbb{R}^2$ is *completely metrizable* if and only if X is a G_δ-subset of $\mathbb{R}^2$. Since it is clear that $\mathbb{R}^2$ has precisely $\mathbf{c}$ G_δ-subsets, at most $\mathbf{c}$ functions provided by Theorem 1 are completely metrizable spaces. In fact, no space provided by Theorem 1 is completely metrizable due to the following theorem.

Theorem 2. *If $F \in \mathcal{R}$ is a connected, completely metrizable space and S is the set of all $x \in \mathbb{R}$ such that the function F is not continuous at the point x then S is a meager subset of $\mathbb{R}$.*

Note that the set S in Theorem 2 is a F_σ-subset of $\mathbb{R}$, see [7] 7.1. In view of Theorem 1 and Theorem 2 there arises a natural question which is answered by the following theorem.

Theorem 3. *There exist* $\mathbf{c}$ *real functions which are incomparable, connected, completely metrizable spaces.*

2. Refinements of the real line

If τ is the topology of a space and x is a point then let $\mathcal{U}_\tau(x)$ denote the filter of all neighborhoods of x. Let η denote the natural Euclidean topology on the real line $\mathbb{R}$. Let $\mathcal{T}$ be the family of all topologies τ on $\mathbb{R}$ which are finer than η or, equivalently, where $\mathcal{U}_\eta(x) \subset \mathcal{U}_\tau(x)$ for every $x \in \mathbb{R}$. (Naturally, $(\mathbb{R}, \tau)$ is a Hausdorff space for every $\tau \in \mathcal{T}$.) For $\tau \in \mathcal{T}$ put $\Gamma(\tau) := \{\, x \in \mathbb{R} \mid \mathcal{U}_\eta(x) \neq \mathcal{U}_\tau(x) \,\}$. So we can say that τ *is strictly finer than* η *at* x if and only if $x \in \Gamma(\tau)$.

By considering the functions $F \in \mathcal{R}$ as subspaces of the Euclidean plane, in a very natural way we obtain topologies in the family $\mathcal{T}$. For $F \in \mathcal{R}$ define a topology $\tau[F]$ on $\mathbb{R}$ by declaring $U \subset \mathbb{R}$ open if and only if $U = \{\, x \in \mathbb{R} \mid (x, F(x)) \in V \,\}$ for some open subset V of the plane $\mathbb{R}^2$. In other words, $\tau[F]$ is the unique topology τ on $\mathbb{R}$ such that $x \mapsto (x, F(x))$ is a homeomorphism from the space $(\mathbb{R}, \tau)$ onto the subspace F of $\mathbb{R}^2$. In particular, the space $(\mathbb{R}, \tau[F])$ is separable and metrizable for every $F \in \mathcal{R}$. Furthermore, $(\mathbb{R}, \tau[F])$ is connected if and only if F is connected, and $(\mathbb{R}, \tau[F])$ is completely metrizable if and only if F is a G_δ-subset of $\mathbb{R}^2$.

A proof of the following important observation is straightforward.

(2.1) *If* $F \in \mathcal{R}$ *and* $\xi \in \mathbb{R}$ *then* $\xi \in \Gamma(\tau[F])$ *if and only if* F *is not continuous at* ξ.

In particular, $\tau[F] = \eta$ if and only if the real function F is continuous. In view of (2.1), from Theorem 1 and Theorem 3 we derive the following two corollaries.

Corollary 1. *There exist* $2^{\mathbf{c}}$ *topologies* $\tau \in \mathcal{T}$ *with* $\Gamma(\tau) = \mathbb{R}$ *such that the corresponding spaces* $(\mathbb{R}, \tau)$ *are incomparable, connected, separable, and metrizable.*

Corollary 2. *There exist* $\mathbf{c}$ *topologies* $\tau \in \mathcal{T}$ *such that the corresponding spaces* $(\mathbb{R}, \tau)$ *are incomparable, connected, separable, and completely metrizable.*

The cardinality $2^{\mathbf{c}}$ in Corollary 1 is maximal since a metric on $\mathbb{R}$ is a function from $\mathbb{R} \times \mathbb{R}$ into $[0, \infty[$ and, trivially, there are precisely $2^{\mathbf{c}}$ mappings from X to Y whenever $|X| = |Y| = \mathbf{c}$. Also the cardinality $\mathbf{c}$ in Corollary 2 is maximal because incomparable spaces are mutually non-homeomorphic and there exist precisely $\mathbf{c}$ separable, completely metrizable spaces up to homeomorphism (see [4]).

Remark. By (2.1) and [7] 7.1, for every $F \in \mathcal{R}$ the set $\Gamma(\tau[F])$ is a F_σ-set of reals. More generally, it is a nice exercise to verify that $\Gamma(\tau)$ is a F_σ-set for every *metrizable* topology $\tau \in \mathcal{T}$. This is not true for an arbitrary topology $\tau \in \mathcal{T}$. Actually, for *every* set $S \subset \mathbb{R}$ we can easily define a topology $\tau_S \in \mathcal{T}$ with $\Gamma(\tau_S) = S$ by declaring a set U τ_S-open if and only if $U = V \cup T$ for some Euclidean open $V \subset \mathbb{R}$ and some $T \subset S$. (The space $(\mathbb{R}, \tau_S)$ is obviously paracompact. In view of the metrization theorem [1] 4.4.7 it is evident that the space $(\mathbb{R}, \tau_S)$ is metrizable if S is a F_σ-set.)

Clearly, if $\tau \in \mathcal{T}$ and the space $(\mathbb{R}, \tau)$ is *not separable* or *not metrizable* then $\tau = \tau[F]$ is impossible for any $F \in \mathcal{R}$. A natural example of a topology $\tau \in \mathcal{T}$ which is metrizable but not separable is the discrete topology. In Section 9 we will apply Theorem 1 in order to construct extremely many incomparable, separable but not metrizable, connected refinements of the real line. On the other hand, in the following we investigate an important subfamily $\mathcal{L}$ of $\mathcal{T}$ where every topology $\tau \in \mathcal{L}$ coincides with $\tau[F]$ for some $F \in \mathcal{R}$. This family $\mathcal{L}$ is the family of all *locally connected* topologies in $\mathcal{T}$.

Naturally, the property *locally connected* is quite different from the property *connected.* This difference is extreme in the realm of real functions or refinements of the real line, respectively. Indeed, the following proposition shows that a discontinuous real function and, more generally, a strict refinement of the real line can never be both connected and locally connected.

Proposition 1. *If $\tau \in \mathcal{T}$ and and the space $(\mathbb{R}, \tau)$ is connected and locally connected then τ is the Euclidean topology η. Moreover, if $\xi \in \mathbb{R}$ and $\tau \in \mathcal{T}$ is strictly finer than η at ξ and the space $(\mathbb{R}, \tau)$ is connected then the point ξ does not have a local basis of connected sets in the space $(\mathbb{R}, \tau)$.*

Proof. Let $a \in \mathbb{R}$ and let $\tau \in \mathcal{T}$ and assume that $\mathcal{U}_\eta(a) \neq \mathcal{U}_\tau(a)$. Let $\mathcal{B}_a$ be a local basis at a which consists of τ-connected, τ-open sets. A fortiori, any τ-connected set is η-connected and hence an interval. Thus $\mathcal{B}_a$ contains an interval which is not η-open. If, for example, $[x, y[$ is such an interval, then $[x, \infty[= [x, y[\cup]x, \infty[$ and $]-\infty, x[$ are disjoint τ-open sets whose union equals $\mathbb{R}$. If $[x, y]$ with $x < y$ or $x = y$ is such an interval then $[x, y]$ is η-closed and hence both τ-open and τ-closed. Generally, in any case we can be sure that the space $(\mathbb{R}, \tau)$ is not connected, *q.e.d.*

Remark. In view of (2.1) and by virtue of Proposition 1, all spaces depicted in Theorem 1 and in Corollary 1 are *nowhere locally connected.*

The difference between the properties *connected* and *locally connected* in the realm $\mathcal{R}$ or in the realm $\{ (\mathbb{R}, \tau) \mid \tau \in \mathcal{T} \}$ can also be quantified by transfinite cardinal numbers when we compare Theorems 1 and 3 and Corollaries 1 and 2 with the following theorem.

Theorem 4. *There are only countably many mutually non-homeomorphic topologies τ in $\mathcal{T}$ such that the space $(\mathbb{R}, \tau)$ is locally connected. If such a space $(\mathbb{R}, \tau)$ is separable then $\tau = \tau[F]$ for some real function F and the set $\Gamma(\tau)$ is countable and Euclidean closed.*

It is very easy to find for each countably infinite closed set $S \subset \mathbb{R}$ a sequence $F_1, F_2, F_3, ...$ of mutually non-homeomorphic locally connected spaces in $\mathcal{R}$ with $\Gamma(\tau[F_k]) = S$ for every $k \in \mathbb{N}$. Indeed, since S is scattered, we can define mutually disjoint compact intervals $[a_k, b_k]$ $(k \in \mathbb{N})$ with $a_k < b_k$ and $S \cap [a_k, b_k] = \{a_k, b_k\}$. Choose a bijection φ from S onto $\mathbb{N} \setminus \{1\}$ and define for every $k \in \mathbb{N}$ a function $F_k \in \mathcal{R}$ by $F_k(x) = 1$ if $x \in [a_1, b_1] \cup \cdots \cup [a_k, b_k]$ and $F_k(x) = \varphi(x)$ if $x \in S \setminus \{ a_1, b_1, ..., a_k, b_k\}$ and $F_k(x) = 0$ if $x \in \mathbb{R} \setminus (S \cup [a_1, b_1] \cup \cdots \cup [a_k, b_k])$. While the spaces F_n and F_m are never homeomorphic for distinct $n, m \in \mathbb{N}$ (because the space F_k has precisely k infinite compact components), the spaces F_k $(k \in \mathbb{N})$ are not incomparable. On the contrary, it is plain that F_n is homeomorphic to a proper subspace of F_m whenever $n, m \in \mathbb{N}$. (If $n \geq m$ then embed the subspace $([a_1, b_1] \cup \cdots \cup [a_n, b_n]) \times \{1\}$ of F_n into one component $[a, b] \times \{1\}$ of F_m.) Actually, locally connected spaces $F \in \mathcal{R}$ are never incomparable. Moreover, in Section 8 we will prove that every locally connected refinement of $\mathbb{R}$ is homeomorphic to some proper subspace of itself and that if $\mathcal{F}$ is a family of locally connected spaces $(\mathbb{R}, \tau)$ with $\tau \in \mathcal{T}$ such that $|\mathcal{F}| \geq 2$ and no $X \in \mathcal{F}$ is homeomorphic to a subspace of any $Y \in \mathcal{F} \setminus \{X\}$ then $|\mathcal{F}| = 2$ and one space in $\mathcal{F}$ is discrete and the other space in $\mathcal{F}$ is separable.

3. Massive connected functions

Naturally, a real function F is a Lebesgue measurable subset of $\mathbb{R}^2$ if and only if F is a null set in $\mathbb{R}^2$. In particular, if F is a *Borel subset* of $\mathbb{R}^2$ then F is null. Of course, all spaces F provided by Theorem 3 are Borel sets. Since there are only $\mathbf{c}$ Borel subsets of $\mathbb{R}^2$, it is impossible that all spaces depicted in Theorem 1 are Borel sets. We will prove Theorem 1 by constructing functions which all are far from being Lebesgue measurable.

Following [3], a set $A \subset \mathbb{R}^2$ is *massive* if and only if $A \cap L \neq \emptyset$ for every set $L \subset \mathbb{R}^2$ of positive Lebesgue measure (or, equivalently, for every non-null Lebesgue measurable set $L \subset \mathbb{R}^2$). In particular, every massive subset of $\mathbb{R}^2$ is *dense* in $\mathbb{R}^2$. Clearly, if $A \subset \mathbb{R}^2$ is Lebesgue measurable and massive then $\mathbb{R}^2 \setminus A$ is a null set. Consequently, if $F \in \mathcal{R}$ is massive then F is not Lebesgue measurable. Our goal is to prove a stronger version of Theorem 1 where the property *dense* is replaced with the property *massive*.

A natural way to accomplish this is to combine the main ideas of three different proofs using transfinite induction, namely the proofs of [6, §35, V. Theorem 2] and [3, Ch. 8, Theorem 5] and [5, Theorem 1]. Since this has to be done in a bit tricky way, for the sake of better reading we conclude this section with four lemmas pivotal for the considerations which prove our sharper version of Theorem 1. In the next section we carry out the remaining part of the proof of Theorem 1 in detail. In the following let π denote the projection $(x, y) \mapsto x$ from $\mathbb{R}^2$ onto $\mathbb{R}$.

Lemma 1. *If $B \subset \mathbb{R}^2$ is a Borel set which is not null in $\mathbb{R}^2$ then $|\pi(B)| = \mathbf{c}$.*

Proof. Let $B \subset \mathbb{R}^2$ be a Borel set. Assume that $|\pi(B)| < \mathbf{c}$. Then $\pi(B)$ must be countable because $\pi(B)$ is an *analytic* set and (see [2] 11.20) every uncountable analytic set is of size $\mathbf{c}$. Consequently, $\pi(B) \times \mathbb{R}$ is a countable union of straight lines and hence a null set in the plane. Since B is a subset of $\pi(B) \times \mathbb{R}$, we conclude that B is null, *q.e.d.*

Lemma 2. *If $F \in \mathcal{R}$ and $A \cap F \neq \emptyset$ whenever A is a closed subset of $\mathbb{R}^2$ and $\pi(A)$ contains a nondegenerate interval then F is connected.*

Proof. Assume indirectly that F is not connected. Then we can find open sets $U, V \subset \mathbb{R}^2$ such that $U \cap F$ and $V \cap F$ are nonempty and disjoint and $U \cup V \supset F$. Consequently, $U \cap V = \emptyset$ since F is a dense subset of $\mathbb{R}^2$. Since the projection π is an open mapping, the two sets $U' := \pi(U)$ and $V' := \pi(V)$ are open subsets of $\mathbb{R}$. Since $F \subset U \cup V$ and F is a function from $\mathbb{R}$ to $\mathbb{R}$, we have $\mathbb{R} \subset U' \cup V'$. Thus $U' \cap V' \neq \emptyset$ because $\mathbb{R}$ is connected and (since $U, V \neq \emptyset$) both sets U', V' are nonempty. Hence we can find $a, b, c \in \mathbb{R}$ with $(a, b) \in U$ and $(a, c) \in V$. Since $U, V \subset \mathbb{R}^2$ are open, there are real numbers u, v with $u < a < v$ such that $[u, v] \times \{b\} \subset U$ and $[u, v] \times \{c\} \subset V$. Now consider the closed set $A := [u, v] \times \mathbb{R} \setminus (U \cup V)$. We claim that $[u, v] \subset \pi(A)$ and this yields to the desired contradiction because if the interval $[u, v]$ is contained in $\pi(A)$ then $A \cap F \neq \emptyset$ contrarily to $A \cap F \subset A \cap (U \cup V) = \emptyset$. In order to verify $[u, v] \subset \pi(A)$ let $x \in [u, v]$. Then $(x, b) \in U$ and $(x, c) \in V$ and hence the connected set $\{x\} \times \mathbb{R}$ meets the two disjoint open sets U, V and therefore $\{x\} \times \mathbb{R}$ cannot be a subset of $U \cup V$. Thus we can find a real number y such that $(x, y) \in (\{x\} \times \mathbb{R}) \setminus (U \cup V) \subset A$ and we arrive at $x \in \pi(A)$, *q.e.d.*

Lemma 3. *If Y is an infinite set then there exists a family $\mathcal{Y}$ of subsets of Y such that $|\mathcal{Y}| = 2^{|Y|}$ and $|A \setminus B| = |Y|$ whenever $A, B \in \mathcal{Y}$ and $A \neq B$.*

Proof. Write $Y = R \cup S$ with $|R| = |S| = |Y|$ and $R \cap S = \emptyset$ and let f be a bijection from R onto S. Put $\mathcal{W} := \{\, T \cup (S \setminus f(T)) \mid T \subset R \,\}$. Obviously, $|\mathcal{W}| = 2^{|Y|}$ and $A \not\subset B$ whenever $A, B \in \mathcal{W}$ and $A \neq B$. Consequently, the set $(A \times Y) \setminus (B \times Y) = (A \setminus B) \times Y$ is nonempty and hence equipollent with Y whenever $A, B \in \mathcal{W}$ and $A \neq B$. Therefore, if g is a bijection from $Y \times Y$ onto Y then $\mathcal{Y} := \{\, g(S \times Y) \mid S \in \mathcal{W} \,\}$ does the job, *q.e.d.*

Lemma 4. *If A is a subspace of $\mathbb{R}^2$ and φ is a homeomorphism from A onto a subspace of $\mathbb{R}^2$ then there is a G_δ-set $G \subset \mathbb{R}^2$ and a homeomorphism $\hat{\varphi}$ from G onto a subspace of $\mathbb{R}^2$ such that $A \subset G$ and $\hat{\varphi}(x) = \varphi(x)$ for every $x \in A$.*

Lemma 4 is an immediate consequence of the Lavrentieff expansion theorem [1] 4.3.20.

4. Proof of Theorem 1

Let $\mathcal{G}$ denote the family of all nonempty G_δ-subsets of $\mathbb{R}^2$. Again, let π be the projection $(x, y) \mapsto x$ from $\mathbb{R}^2$ onto $\mathbb{R}$. Let $\mathcal{B}$ denote the family of all Borel sets $B \subset \mathbb{R}^2$ such that $|\pi(B)| = \mathbf{c}$. Then $|\mathcal{B}| = |\mathcal{G}| = \mathbf{c}$ since $\mathbb{R}^2$ has precisely $\mathbf{c}$ Borel subsets and $[0, t] \times \mathbb{R} \in \mathcal{G} \subset \mathcal{B}$ for every $t > 0$. Let $\mathcal{F}$ denote the family of all homeomorphisms from sets $G \in \mathcal{G}$ onto subspaces of $\mathbb{R}^2$. (Note that $f(G) \in \mathcal{G}$ since $X \subset \mathbb{R}^2$ is completely metrizable if and only if $X \in \mathcal{G}$.) Since $|\mathcal{G}| = \mathbf{c}$ and since there are only $\mathbf{c}$ continuous functions from any $G \in \mathcal{G}$ into $\mathbb{R}^2$, we have $|\mathcal{F}| = \mathbf{c}$. Define a bijection $x \mapsto f_x$ from $\mathbb{R}$ onto $\mathcal{F}$ and a bijection $x \mapsto B_x$ from $[0, \infty[$ onto $\mathcal{B}$. (The domain $[0, \infty[$ is chosen for the only reason that $[0, \infty[$ is a set $J \subset \mathbb{R}$ with $|J| = |\mathbb{R} \setminus J| = \mathbf{c}$.)

Let $\preceq$ be a well-ordering of $\mathbb{R}$ such that $|\{\, x \mid x \prec y \,\}| < \mathbf{c}$ for every $y \in \mathbb{R}$. Via induction we define for every $x \in \mathbb{R}$ points $y_x, z_x \in \mathbb{R}^2$ with $\pi(y_x) = \pi(z_x)$ as follows. Assume that for $\xi \in \mathbb{R}$ points $y_x, z_x \in \mathbb{R}^2$ with $\pi(y_x) = \pi(z_x)$ are already defined for all $x \prec \xi$. Then the definition of y_ξ and z_ξ is carried out as follows. We distinguish the two cases that either $\xi \in [0, \infty[$ or $\xi \notin [0, \infty[$. (Only for $\xi \geq 0$ the set B_ξ is defined!)

Put $Y(\xi) := \{\, f_u(y_v) \mid u, v \prec \xi \,\}$ and $Z(\xi) := \{\, f_u(z_v) \mid u, v \prec \xi \,\}$ and $U(\xi) := \{\, \pi(y_v) \mid v \prec \xi \,\}$. Clearly, the sizes of these three sets are smaller than $\mathbf{c}$.

If $\xi \geq 0$ then choose a point a in the nonempty set

$$B_\xi \setminus \big(Y(\xi) \cup Z(\xi) \cup (U(\xi) \times \mathbb{R})\big)$$

and define $y_\xi = z_\xi = a$. (This set is nonempty since $|Y(\xi) \cup Z(\xi)| < \mathbf{c}$ and the size of $\pi(U(\xi) \times \mathbb{R}) = U(\xi)$ is smaller than $\mathbf{c}$, whereas $|\pi(B_\xi)| = \mathbf{c}$.)

If $\xi < 0$ then let μ be the minimum of the nonempty set $\mathbb{R} \setminus U(\xi)$ with respect to the well-ordering $\preceq$ and choose distinct points y_ξ, z_ξ in the vertical line $\{\mu\} \times \mathbb{R}$ such that $\{y_\xi, z_\xi\}$ is disjoint from $Y(\xi) \cup Z(\xi)$. (This choice is possible since the size of $Y(\xi) \cup Z(\xi)$ is smaller than $\mathbf{c}$ while $|\{\mu\} \times \mathbb{R}| = \mathbf{c}$.)

Now, having defined points $y_x, z_x \in \mathbb{R}^2$ for every index $x \in \mathbb{R}$, put $S_y := \{\, y_x \mid x \in \mathbb{R} \,\}$ and $S_z := \{\, z_x \mid x \in \mathbb{R} \,\}$ and $S := S_y \cup S_z$. By definition, $y_u \neq y_v$ and $z_u \neq z_v$ whenever $u \neq v$, and $y_x \neq z_x$ and $\pi(y_x) = \pi(z_x)$ for every index $x < 0$, and $y_x = z_x$ for every index $x \geq 0$. In particular, $|S_y| = |S_z| = |S| = \mathbf{c}$. Furthermore, $\pi(S) = \mathbb{R}$ and hence $1 \leq |S \cap (\{x\} \times \mathbb{R})| \leq 2$ for every $x \in \mathbb{R}$. (It cannot happen by accident that $\pi(S) \neq \mathbb{R}$ due to the $\preceq$-minimality of the index μ and the fact that $]-\infty, 0[\not\subset \{\, x \mid x \prec y \,\}$ for every $y \in \mathbb{R}$.) Consequently, for every set T of negative real indices we can define a real function $F[T] \subset S$ by $F[T] := \{ z_x \mid x \in T \} \cup \{ y_x \mid x \in \mathbb{R} \setminus T \}$. We claim that the following statement is true.

(4.1) *If* $A, B \subset S$ *and* $|A \setminus B| = \mathbf{c}$ *then the subspace* A *of* $\mathbb{R}^2$ *cannot be embedded into the subspace* B *of* $\mathbb{R}^2$ *.*

In order to prove (4.1) let $A, B \subset S$ and let φ be a homeomorphism from A onto a subspace of B. By definition, the following two conditions hold.

(4.2) *If* $a, b, x \in \mathbb{R}$ *and* $a, b \prec x$ *and* y_b *lies in the domain of* f_a *then* $y_x \neq f_a(y_b)$ *.*

(4.3) *If* $a, b, x \in \mathbb{R}$ *and* $a, b \prec x$ *and* z_b *lies in the domain of* f_a *then* $z_x \neq f_a(z_b)$ *.*

By Lemma 4 we can expand φ to a homeomorphic embedding f from G into $\mathbb{R}^2$ for some $G \in \mathcal{G}$ with $A \subset G$. The inverse f^{-1} is a homeomorphism from $f(G) \in \mathcal{G}$ onto G. Thus $f, f^{-1} \in \mathcal{F}$ and $f(A) \subset B$.

Define $\Omega := \{ x \in \mathbb{R} \mid y_x \in A \backslash B \}$ and choose $\gamma, \alpha \in \mathbb{R}$ such that $f = f_\gamma$ and $f^{-1} = f_\alpha$ and put $\Omega_\alpha := \{ x \in \Omega \mid x \succ \alpha \}$. Let $\xi \in \Omega_\alpha$ and $\beta \in \mathbb{R}$ such that $f_\alpha(y_\beta) = y_\xi$. Then $\beta \succeq \xi$ in view of (4.2) since $\alpha \prec \xi$. Moreover, $\beta \succ \xi$ because $y_\beta = f_\gamma(y_\xi)$ lies in B while $y_\xi \notin B$. From $\xi \prec \beta$ and $f_\gamma(y_\xi) = y_\beta$ and (4.2) we conclude $\beta \preceq \gamma$, whence $\xi \preceq \gamma$. Thus $\Omega_\alpha \subset \{ \xi \mid \xi \preceq \gamma \}$ and hence $|\Omega_\alpha| < \mathbf{c}$. Trivially, $|\Omega \setminus \Omega_\alpha| \leq |\{ x \mid x \preceq \alpha \}| < \mathbf{c}$. Thus $|(A \setminus B) \cap S_y| = |\Omega| < \mathbf{c}$. Similarly, in view of (4.3) we conclude $|(A \setminus B) \cap S_z| < \mathbf{c}$. Consequently, from $S_x \cup S_y = S$ we derive $|A \setminus B| < \mathbf{c}$ and this proves (4.1).

Now in order to conclude the proof of Theorem 1, by applying Lemma 3 we can define a family $\mathcal{V}$ of subsets of $]-\infty, 0[$ such that $|\mathcal{V}| = 2^c$ and $|V \setminus W| = \mathbf{c}$ whenever $V, W \in \mathcal{V}$ and $V \neq W$. Clearly, if V, W are sets of negative reals then $|F[V] \setminus F[W]| = |(V \setminus W) \cup (W \setminus V)|$. Therefore, $|F[V] \setminus F[W]| = \mathbf{c}$ whenever $V, W \in \mathcal{V}$ and $V \neq W$. Consequently by virtue of (4.1), for each $T \in \mathcal{V}$ the space $F[T]$ cannot be homeomorphic to a subspace of $F[T']$ whenever $T \neq T' \in \mathcal{V}$. Furthermore, for each $T \in \mathcal{V}$ the space $F = F[T]$ is not homeomorphic to any proper subspace of itself. This is indeed true in view of (4.1) because if G is a connected subspace of F and $G \neq F$ then $G = F \cap (I \times \mathbb{R})$ for some interval $I \neq \mathbb{R}$ and hence $|F \setminus G| = \mathbf{c}$. Thus we conclude that the $2^{\mathbf{c}}$ spaces $F[T]$ $(T \in \mathcal{V})$ are incomparable subspaces of $\mathbb{R}^2$.

Finally, since $F[T] \cap B \neq \emptyset$ for every $B \in \mathcal{B}$, by virtue of Lemma 1 we can be sure that $F[T]$ intersects every non-null Borel set. A fortiori, $F[T]$ is massive. And since every closed subset of $\mathbb{R}^2$ is a Borel set, for each $T \in \mathcal{V}$ the space $F[T]$ is connected by virtue of Lemma 2. This concludes the proof of Theorem 1 with the property *dense* sharpened to *massive*.

Remark. The spaces $F[T]$ are not only massive with respect to the Lebesgue measure on $\mathbb{R}^2$ but also with respect to the Baire property, meaning that $F[T]$ intersects every non-meager subset of $\mathbb{R}^2$ satisfying the Baire property. (Because, in view of [2] 11.16, Lemma 1 remains true if the property *not null* is replaced with *not meager*.) Furthermore, all spaces $F[T]$ are *totally imperfect* (see [3]), i.e. no space $F[T]$ contains an uncountable closed set. (Suppose that some uncountable closed subset A of $\mathbb{R}^2$ lies in $F[T]$. Then $|A| = \mathbf{c}$ and hence $|\pi(A)| = \mathbf{c}$ since $F[T]$ is a real function. Then $B = \{ (x, y+1) \mid (x, y) \in A \}$ is a closed set disjoint from $F[T]$ with $\pi(B) = \pi(A)$. This is impossible since $F[T]$ meets the Borel set B due to $|\pi(B)| = \mathbf{c}$.)

5. Proof of Theorem 2

The following lemma is essential for the proofs of Theorems 2 and 3.

Lemma 5. *If $\tau \in \mathcal{T}$ and the space $(\mathbb{R}, \tau)$ is connected then every interval is τ-connected. If $F \in \mathcal{R}$ is connected then $F \cap (I \times \mathbb{R})$ is connected for every interval I.*

Proof. Let $\tau \in \mathcal{T}$ and assume that the space $(\mathbb{R}, \tau)$ is connected. Let $a < b$. Firstly we verify that $I = [a, b]$ is τ-connected. Assume indirectly that I is not connected. Then there are τ-open sets $U, V \subset \mathbb{R}$ such that $U \cap I \neq \emptyset$ and $V \cap I \neq \emptyset$ and $U \cup V \supset I$ and $U \cap V \cap I = \emptyset$. Assume $a \in U$ and $b \in V$ (whence $a \notin V$ and $b \notin U$). Then $(U \setminus [b, \infty[) \cup]{-\infty}, a[$ and $(V \setminus]{-\infty}, a]) \cup]b, \infty[$ are disjoint nonempty τ-open sets which cover the whole set $\mathbb{R}$. This contradicts the connectedness of the space $(\mathbb{R}, \tau)$. In a similar way a contradiction can be derived if $\{a, b\}$ is a subset of U or of V. So we have proved that $[a, b]$ is τ-connected whenever $a < b$. Since every nondegenerate interval is a union of intervals $[a_n, b_n]$ $(n \in \mathbb{N})$ with $a_m \leq a_n < b_n \leq b_m$ whenever $n < m$, we conclude that every every interval is τ-connected. This proves the first statement of Lemma 5. The second statement can be verified in a similar way. Alternatively, by considering the space $(\mathbb{R}, \tau[F])$ for $F \in \mathcal{R}$, the second statement is a consequence of the first, *q.e.d.*

For the proof of Theorem 2 we also need the following noteworthy lemma.

Lemma 6. *If $F \in \mathcal{R}$ is completely metrizable then F is a nowhere dense subset of $\mathbb{R}^2$.*

Proof. Since $\mathbb{R}^2$ is a Baire space, it is enough to verify that the closure $\overline{F}$ of F in $\mathbb{R}^2$ is a meager subset of $\mathbb{R}^2$. Since F is a G_δ-subset of $\mathbb{R}^2$, F is a *dense* G_δ-subset of the subspace $\overline{F}$ of $\mathbb{R}^2$. Hence $\overline{F} \setminus F$ is a *meager* F_σ-subset of the space $\overline{F}$. Therefore, $\overline{F} \setminus F$ is a meager subset of the plane. (For if A is a nowhere dense subset of a closed subspace of $\mathbb{R}^2$ then A is clearly nowhere dense in $\mathbb{R}^2$.) Since F is a G_δ-set and $|F \cap (\{t\} \times \mathbb{R})| = 1$ for every $t \in \mathbb{R}$, by the supplementary Kuratowski-Ulam theorem [7] 15.4 also F is a meager subset of $\mathbb{R}^2$. Thus, $\overline{F} = (\overline{F} \setminus F) \cup F$ is a meager subset of $\mathbb{R}^2$, *q.e.d.*

Now in order to prove Theorem 2 let $F \in \mathcal{R}$ be a connected G_δ-subset of the Euclidean plane. We have to show that the discontinuity points of F form a meager subset of $\mathbb{R}$. Let π_2 denote the projection $(x, y) \mapsto y$ from $\mathbb{R}^2$ onto $\mathbb{R}$.

Let $a \in \mathbb{R}$ and assume that F is not continuous at $a \in \mathbb{R}$. Then we may choose $t_a > 0$ and a sequence x_n converging to a such that either $F(x_n) > F(a) + t_a$ for every $n \in \mathbb{N}$ or $F(x_n) < F(a) - t_a$ for every $n \in \mathbb{N}$. Put $I_n := [\min\{a, x_n\}, \max\{a, x_n\}]$ for every $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ the set $F \cap (I_n \times \mathbb{R})$ is connected and hence $\pi_2(F \cap (I_n \times \mathbb{R}))$ is an interval which contains $F(a)$ and $F(x_n)$. Consequently, either $\{a\} \times [F(a), F(a) + t_a] \subset \overline{F}$ or $\{a\} \times [F(a) - t_a, F(a)] \subset \overline{F}$.

So for every discontinuity point a of F there is a nondegenerate closed interval J_a such that $\{a\} \times J_a \subset \overline{F}$. Of course, J_a is of second category in $\mathbb{R}$. By Lemma 6, F and hence $\overline{F}$ is a nowhere dense subset of $\mathbb{R}^2$. Consequently, in view of the Kuratowski-Ulam theorem [7] 15.1, the set of all discontinuity points of F must be a meager subset of $\mathbb{R}$. This concludes the proof of Theorem 2.

6. Proof of Theorem 3

Let Ω denote the family of all mappings from $\mathbb{N}$ into $\{1, 2\}$ such that $g(n) = 2$ for infinitely many $n \in \mathbb{N}$. Naturally, $|\Omega| = \mathbf{c}$. Transform $g \in \Omega$ into the double sequence

$$\ldots, 1, 1, 1, 1, 1, 2, g(1), g(2), g(3), g(4), g(5), \ldots$$

and then replace each comma with the digit 0 in order to obtain a string

$$\cdots 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,2\,0\,g(1)\,0\,g(2)\,0\,g(3)\,0\,g(4)\,0\,g(5)\ \cdots$$

which consists of digits from $\{0,1,2\}$. Let $S(g)$ denote this string of digits. Since in the string $S(g)$ there is clearly a unique first position of the digit 2, we can distinguish the string $S(g_1)$ from the string $S(g_2)$ whenever $g_1, g_2 \in \Omega$ are distinct. Moreover, it is clear that every sequence $g \in \Omega$ can be reconstructed from the string $S(g)$. Our clue in proving Theorem 3 is to define for every $g \in \Omega$ a completely metrizable, connected space $F[g] \in \mathcal{R}$ such that the string $S(g)$ (and hence the sequence g) can be obtained directly from the topology of the space $F[g]$ by investigating the path-components of $F[g]$. There will be three topological types of path-components P corresponding with the total number n of *noncut points* of P where $n \in \{0,1,2\}$. (A point a in a connected set A is a noncut point of A if and only if $A \setminus \{a\}$ is connected.)

Consider the function

$$\sigma(u,v) \ := \ \{\, (x, \sin \tfrac{1}{(x-u)(v-x)}) \mid u < x < v \,\}$$

defined on $]u,v[$ for $u < v$. Clearly, the subspace $\{(u,0)\} \cup \sigma(u,v) \cup \{(v,0)\}$ of $\mathbb{R}^2$ is connected and has precisely three path-components, namely $\{(u,0)\}$ and $\{(v,0)\}$ and $\sigma[u,v]$.

For $g \in \Omega$ define $F[g] \in \mathcal{R}$ via

$$F[g] \ := \ \{\, (-n,0) \mid n \in \mathbb{N} \,\} \cup ([0,1] \times \{0\}) \cup \bigcup_{k=0}^{\infty} \sigma(-k-1,-k) \cup \bigcup_{k=1}^{\infty} \Sigma[g,k]$$

where $\Sigma[g,k] \subset\,]k,k+1] \times [-1,1]$ is defined by

$$\Sigma[g,k] \ = \ \sigma(k,k+1) \cup \{(k+1,0)\}$$

if $g(k) = 1$ and

$$\Sigma[g,k] \ = \ \sigma(k, k+\tfrac{1}{2}) \cup ([k+\tfrac{1}{2}, k+1] \times \{0\})$$

if $g(k) = 2$.

The set $F[g]$ is connected because it is clearly possible to write $F[g] = \bigcup_{n=1}^{\infty} C_n$ where C_n are connected subsets of $\mathbb{R}^2$ for all $n \in \mathbb{N}$ and C_{n+1} is not separated from C_n whenever $n \in \mathbb{N}$. With $\overline{F[g]}$ denoting the closure of $F[g]$ in $\mathbb{R}^2$ we have $F[g] = (S \times \{0\}) \cup (\overline{S} \setminus (S \times [-1,1]))$ for some set $S \subset \frac{1}{2}\mathbb{Z}$ and hence $F[g]$ is a G_δ-subset of $\mathbb{R}^2$. We are done by proving that for distinct sequences $g, h \in \Omega$ the space $F[g]$ cannot be homeomorphic either to a subspace of $F[h]$ or to a proper subspace of $F[g]$.

For $g \in \Omega$ let $\mathcal{P}[g]$ denote the family of all path-components of $F[g]$. Let $\mathcal{P}_1[g]$ denote the set of all *singleton* path-components of $F[g]$. Thus if $P \in \mathcal{P}_1$ then $P = \{(x,0)\}$ for some $x \in \mathbb{Z}$. Let $\mathcal{P}_2[g]$ denote the set of all *infinite and compact* path-components of $F[g]$ and let $\mathcal{P}_0[g]$ denote the set of all *not compact* path-components of $F[g]$. Obviously, if $P \in \mathcal{P}_2$ then $P = [a,b] \times \{0\}$ for some $a < b$, and if $P \in \mathcal{P}_0$ then $P = \sigma(u,v)$ for some $u < v$. Consequently, $\mathcal{P}[g]$ is the union of the mutually disjoint *infinite* sets $\mathcal{P}_0[g]$ and $\mathcal{P}_1[g]$ and $\mathcal{P}_2[g]$. The notation $\mathcal{P}_n[g]$ with $n \in \{0,1,2\}$ corresponds to the fact that every space $P \in \mathcal{P}_n[g]$ has precisely n *noncut points.* (A singleton has precisely one noncut point since the empty set is connected vacuously.)

The family $\mathcal{P}[g]$ is naturally ordered by defining $P_1 \prec P_2$ for $P_1, P_2 \in \mathcal{P}[g]$ if and only if $x_1 < x_2$ for some $(x_1, y_1) \in P_1$ and some $(x_2, y_2) \in P_2$. Obviously the linearly ordered set $(\mathcal{P}[g], \prec)$ is isomorphic with the naturally ordered set $\mathbb{Z}$. We claim that the ordering $\prec$ is completely determined by the topology of the space $F[g]$.

Obviously there are precisely two linear orderings $<_1$ and $<_2$ of $\mathcal{P}[g]$ such that $P, Q \in \mathcal{P}[g]$ are consecutive points with respect to $<_1$ and to $<_2$ if and only if $P \neq Q$ and $P \cup Q$ is a connected subset of $F[g]$. One of these linear orderings coincides with $\prec$ and we can find out which one in a purely topological way. Indeed, by definition, for every $P \in \mathcal{P}[g]$ the set $\{\, Q \in \mathcal{P}_2[g] \mid Q \prec P \,\}$ is finite while the set $\{\, Q \in \mathcal{P}_2[g] \mid P \prec Q \,\}$ is infinite. This proves the claim.

Since the ordering $\prec$ is completely determined by the topology of the space $F[g]$, also the sequence $g \in \Omega$ is completely determined by the topology of the space $F[g]$. To verify this, write the linearly ordered countable set $\mathcal{P}[g]$ as a string

$$\cdots\ P_{-5}\ P_{-4}\ P_{-3}\ P_{-2}\ P_{-1}\ P_0\ P_1\ P_2\ P_3\ P_4\ P_5\ \cdots$$

where $P_n \prec P_m$ if and only if $n < m$. Then replace each P_i with the total number n of the noncut points of P_i in order to obtain a string of digits $n \in \{0, 1, 2\}$. Obviously, this string of digits coincides with the string $S(g)$.

Due to the previous considerations, we can be sure that for distinct $g, h \in \Omega$ the spaces $F[g]$ and $F[h]$ cannot be homeomorphic. Therefore, the proof is finished by verifying that, whenever $g, h \in \Omega$, the space $F[h]$ cannot be homeomorphic to a *proper* subspace of $F[g]$.

Let $g, h \in \Omega$ and assume that φ is a homeomorphism from $F[h]$ onto a proper subspace G of $F[g]$. By Lemma 5, any connected subspace of $F[g]$ equals $(I \times \mathbb{R}) \cap F[g]$ for some interval I. Therefore, $G = (I \times \mathbb{R}) \cap F[g]$ for some interval $I \neq \mathbb{R}$. Let $\mathcal{P}(G)$ denote the family of the path-components of the space G. Naturally, $\mathcal{P}(G) = \{\, \varphi(P) \mid P \in \mathcal{P}[h] \,\}$. Let $\prec_h$ be the linear ordering of $\mathcal{P}[h]$ and define $A \prec_G B$ for $A, B \in \mathcal{P}(G)$ if and only if $\varphi^{-1}(A) \prec_h \varphi^{-1}(B)$. Trivially, the linearly ordered sets $(\mathcal{P}[h], \prec_h)$ and $(\mathcal{P}(G), \prec_G)$ are isomorphic. This, however, is not possible. Indeed, if $<$ is any linear ordering of $\mathcal{P}(G)$ such that $\bigcup\{\, A \in \mathcal{P}(G) \mid A \leq B \,\}$ is a connected subspace of G for every $B \in \mathcal{P}(G)$ then the linearly ordered set $(\mathcal{P}(G), <)$ has a maximum or a minimum since $I \neq \mathbb{R}$. On the other hand, the linearly ordered set $(\mathcal{P}[h], \prec_h)$ is isomorphic with $\mathbb{Z}$ and hence it has neither a maximum nor a minimum. This concludes the proof of Theorem 3.

Remark. Concerning a statement in the abstract, if F_1 and F_3 is any function provided by the proof of Theorem 1 and Theorem 3, respectively, we can be sure that F_1 and F_3 are incomparable. (Because F_1 is connected and totally pathwise disconnected, whereas F_3 has infinite path-components and only countably many singleton path-components and every connected subset of a path-component of F_3 is pathwise connected.)

7. Proof of Theorem 4

In the following, an *interval partition* is any partition $\mathcal{P}$ of the set $\mathbb{R}$ which consists of intervals. (Note that $\{a\} = [a, a]$ is an interval for every $a \in \mathbb{R}$.) For every topology $\tau \in \mathcal{T}$ let $\mathcal{C}_\tau$ denote the family of all components of the space $(\mathbb{R}, \tau)$. Furthermore, let $\mathcal{T}^*$ denote the family of all topologies $\tau \in \mathcal{T}$ such that $(\mathbb{R}, \tau)$ is locally connected.

First of all note, again, that if a topology τ on $\mathbb{R}$ is finer than η then every τ-connected set of reals is η-connected a fortiori. Naturally a set of reals is η-connected if and only if it is an interval. Consequently, if $\tau \in \mathcal{T}$ then the family $\mathcal{C}_\tau$ is an interval partition.

Now assume that $\tau \in \mathcal{T}^*$. If $V \in \mathcal{C}_\tau$ then the relative topologies of τ and η coincide on the set V because otherwise, by similar arguments as in the proof of Proposition 1, V would not be τ-connected. Naturally, the elements of $\mathcal{C}_\tau$ are τ-closed sets. Moreover, since $(\mathbb{R}, \tau)$ is locally connected, all members of $\mathcal{C}_\tau$ are τ-open as well. So the space $(\mathbb{R}, \tau)$ is the topological sum of all $A \in \mathcal{C}_\tau$, and each $A \in \mathcal{C}_\tau$ is both a subspace of the space $(\mathbb{R}, \tau)$ and the Euclidean space $\mathbb{R}$. Consequently, each topology $\tau \in \mathcal{T}^*$ is completely determined by the family $\mathcal{C}_\tau$ and hence $\tau \mapsto \mathcal{C}_\tau$ is an injective mapping from $\mathcal{T}^*$ into the family of all interval partitions.

If $\mathcal{P}$ is an interval partition then let us call a quadruple $(\alpha, \beta, \gamma, \delta)$ of cardinal numbers the *type* of $\mathcal{P}$ if and only if $\mathcal{P}$ contains exactly α singletons and exactly β compact intervals of positive length and exactly γ half-open intervals and exactly δ open intervals. Thereby, an interval is *half-open* when it is of the form $]-v, -u]$ or $[u, v[$ with $u < v$ including the possibility $v = \infty$. (So we do not care that half-open intervals $[t, \infty[$ and $]-\infty, t]$ are Euclidean closed.)

It goes without saying that for topologies $\tau, \sigma \in \mathcal{T}^*$ the spaces $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \sigma)$ are homeomorphic if and only if the types of the interval partitions $\mathcal{C}_\tau$ and $\mathcal{C}_\sigma$ coincide. Consequently, the first statement in Theorem 4 is verified by proving that there are only countably many quadruples $(\alpha, \beta, \gamma, \delta)$ which occur as types of interval partitions.

If $(\alpha, \beta, \gamma, \delta)$ is the type of an interval partition then β, γ, δ are countable cardinals or, equivalently, $\beta, \gamma, \delta \in \mathbb{N} \cup \{0, \aleph_0\}$. (Because every nondegenerate interval contains a rational number.) Consequently, for every cardinal number α there are not more than countably many triples (β, γ, δ) such that the quadruple $(\alpha, \beta, \gamma, \delta)$ is the type of an interval partition. So if there occur only countably many cardinals α then there occur only countably many quadruples $(\alpha, \beta, \gamma, \delta)$ as possible types. For the cardinal number α we have $\alpha \leq \mathbf{c}$ since there are precisely $\mathbf{c}$ singleton subsets of $\mathbb{R}$. Unfortunately we cannot rule out that there are uncountably many cardinals below $\mathbf{c}$. Actually, it is consistent with ZFC set theory that there are $\mathbf{c}$ cardinals below $\mathbf{c}$. (This can easily be established by routine forcing. In Gödel's universe L let θ be the smallest ordinal number of uncountable cofinality satisfying $\aleph_\theta = \theta$. While $2^{\aleph_0} = \aleph_1$ holds in L, there is a generic extension $V[\mathrm{L}]$ of L such that $2^{\aleph_0} = \aleph_\theta$ holds in $V[\mathrm{L}]$, see [2] p. 226.) Fortunately, we can prove that if $(\alpha, \beta, \gamma, \delta)$ is the type of an interval partition $\mathcal{P}$ then α lies in the countable set $\mathbb{N} \cup \{0, \aleph_0, \mathbf{c}\}$.

Indeed, let $\mathcal{A}$ be the family of all singletons $\{a\}$ in $\mathcal{P}$, whence $\alpha = |\mathcal{A}|$. Assume that $\mathcal{A}$ is uncountable. Trivially, $|\mathcal{A}| = |\bigcup \mathcal{A}|$. Now, $\bigcup \mathcal{A}$ is a G_δ-subset of $\mathbb{R}$ because $\bigcup \mathcal{A} = \mathbb{R} \setminus \bigcup(\mathcal{P} \setminus \mathcal{A})$ and every nondegenerate interval is a F_σ-subset of $\mathbb{R}$. Well, any uncountable G_δ-subset of $\mathbb{R}$ is an uncountable Borel set and hence of size $\mathbf{c}$ (see [2] 11.20). Consequently, $\alpha = |\mathcal{A}| = \mathbf{c}$. This concludes the proof of the first statement of Theorem 4.

Clearly, for $\tau \in \mathcal{T}^*$ the locally connected space $(\mathbb{R}, \tau)$ is separable if and only if the family $\mathcal{C}_\tau$ is countable. (In other words, $\alpha \neq \mathbf{c}$ for the type $(\alpha, \beta, \gamma, \delta)$ of the interval partition $\mathcal{C}_\tau$.) In order to prove the second statement of Theorem 4 let $\tau \in \mathcal{T}^*$ and assume that $(\mathbb{R}, \tau)$ is separable and let φ be an injective mapping from the countable set $\mathcal{C}_\tau$ into $\mathbb{N}$. Then define $F \in \mathcal{R}$ by $F(x) = \varphi(V)$ for $x \in V$ and $V \in \mathcal{C}_\tau$. Clearly, $\tau[F] = \tau$.

Furthermore, the set $\Gamma(\tau)$ is countable because $\mathcal{C}_\tau$ is countable and the Euclidean interior of any set in the family $\mathcal{C}_\tau$ is disjoint from $\Gamma(\tau)$.

To conclude the proof of Theorem 4, we claim that $\Gamma(\tau)$ is a closed subset of $\mathbb{R}$ if $\tau \in \mathcal{T}^*$. (We do not assume separability and hence we show a bit more than stated in Theorem 4.)

Assume indirectly that an Euclidean limit point a of $\Gamma(\tau)$ does not lie in $\Gamma(\tau)$. Then $\mathcal{U}_\eta(a) = \mathcal{U}_\tau(a)$ and $\mathcal{U}_\tau(a)$ has a basis $\mathcal{B}(a)$ consisting of τ-connected and Euclidean open neighborhoods of a. Let $B \in \mathcal{B}(a)$. Then $B =]u,v[$ for $u < a < v$ and $\Gamma(\tau) \cap B \neq \emptyset$. Let $x \in \Gamma(\tau) \cap B$. Then we can find a τ-connected, τ-open neighborhood V of x such that $x \in V \subset B$ and V is not η-open, whence $V = \{x\}$ or $V = [a,b[$ or $V =]a,b]$ or $V = [a,b]$ for $a < b$. This enables us to write B as a disjoint union of nonempty τ-open subsets and hence we arrive at the contradiction that B is not τ-connected.

This finishes the proof of Theorem 4. Furthermore, since any interval is clearly a locally compact and completely metrizable subspace of the real line, we additionally have proved the following noteworthy proposition.

Proposition 2. *If $\tau \in \mathcal{T}$ and $(\mathbb{R},\tau)$ is locally connected then $(\mathbb{R},\tau)$ is locally compact and completely metrizable and the set $\Gamma(\tau)$ is Euclidean closed. (And $\Gamma(\tau)$ is countable if and only if the locally compact space $(\mathbb{R},\tau)$ is second countable.)*

Remark. There exists a topology $\tau \in \mathcal{T}$ such that the space $(\mathbb{R},\tau)$ is locally compact and separable but neither locally connected nor metrizable, see [8] No. 65. If $F \in \mathcal{R}$ and $a \in F$ and the space F has a local basis of connected sets at a then, despite Proposition 2, one cannot conclude that the point a has a compact neighborhood. For a counterexample consider the completely metrizable, connected space $F \in \mathcal{R}$ defined in the following way. With $\sigma(u,v)$ as in Section 6 put $g_n = \sigma(2^{-n}, 2^{-n+1})$ for every $n \in \mathbb{N}$ and define $F \in \mathcal{R}$ by $F(x) = 2^{-n} g_n(x)$ if $n \in \mathbb{N}$ and $2^{-n} < x < 2^{-n+1}$ and $F(x) = 0$ otherwise.

8. Partitions of the real line

In a very natural way the considerations in the proof of Theorem 4 lead to a complete classification of all locally connected refinements of the real line. First of all we point out that the injective mapping $\tau \mapsto \mathcal{C}_\tau$ from $\mathcal{T}^*$ into the family of all interval partitions is bijective. Because if $\mathcal{P}$ is any interval partition then there is a unique topology τ in the family $\mathcal{T}^*$ with $\mathcal{C}_\tau = \mathcal{P}$. This topology τ is generated by the basis $\mathcal{B}$ where $B \in \mathcal{B}$ if and only if $B = J \cap A$ for some open interval J and some $A \in \mathcal{P}$. (Alternatively, τ is generated by the metric d defined by $d(x,y) = \arctan|x-y|$ when x,y lie in a common set $A \in \mathcal{P}$ and $d(x,y) = 2$ otherwise.) Since the mapping $\tau \mapsto \mathcal{C}_\tau$ is bijective and since each topology $\tau \in \mathcal{T}^*$ is completely determined up to homeomorphism by the type of the interval partition $\mathcal{C}_\tau$, in order to establish a classification of all topologies $\tau \in \mathcal{T}^*$ we have to find out which quadruples of cardinal numbers actually occur as types of interval partitions. It is reasonable to distinguish beteen countable and uncountable interval partitions because for $\tau \in \mathcal{T}^*$ the space $(\mathbb{R},\tau)$ is separable if and only if the interval partition $\mathcal{C}_\tau$ is countable.

Let $\Omega_1 := \mathbb{N} \cup \{0, \aleph_0\}$ denote the set of all countable cardinal numbers. Note that $\aleph_0 + \aleph_0 = \aleph_0$ and $n + \aleph_0 = n \cdot \aleph_0 = \aleph_0$ for every $n \in \mathbb{N}$. Define sets of quadruples of cardinal numbers by $\mathcal{Q}_2 := \{\mathbf{c}\} \times \Omega_1^3$ and

$$\mathcal{Q}_1 := \left\{ (\alpha,\beta,\gamma,\delta) \in \Omega_1^4 \;\middle|\; \alpha+\beta+1 = \delta \;\vee\; (\alpha+\beta+1 > \delta \;\wedge\; \gamma = \aleph_0) \right\}.$$

Theorem 5. *If $\mathcal{P}$ is an interval partition of type ϑ then $\vartheta \in \mathcal{Q}_1$ when $\mathcal{P}$ is countable and $\vartheta \in \mathcal{Q}_2$ when $\mathcal{P}$ is uncountable. For every $\vartheta \in \mathcal{Q}_1 \cup \mathcal{Q}_2$ there exists an interval partition of type ϑ.*

In order to prove Theorem 5 let $\mathcal{P}$ be an interval partition of type $(\alpha, \beta, \gamma, \delta)$. As already verified, $\beta, \gamma, \delta \in \Omega_1$ and $\alpha = \mathbf{c}$ if and only if $\mathcal{P}$ is uncountable. In particular, if $\mathcal{P}$ is uncountable then $(\alpha, \beta, \gamma, \delta) \in \mathcal{Q}_2$. Conversely, if $(\alpha, \beta, \gamma, \delta) \in \mathcal{Q}_2$ (and hence $\alpha = \mathbf{c}$) then we can easily find an (uncountable) interval partition of type $(\alpha, \beta, \gamma, \delta)$. Indeed, let β, γ, δ be countable cardinals and put $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ where

$$\begin{aligned}
\mathcal{P}_1 &= \{\, [6n, 6n{+}1] \mid n \in \mathbb{Z} \,\wedge\, 0 \leq n < \beta \,\} \qquad \text{and} \\
\mathcal{P}_2 &= \{\, [6n{+}2, 6n{+}3[\mid n \in \mathbb{Z} \,\wedge\, 0 \leq n < \gamma \,\} \qquad \text{and} \\
\mathcal{P}_3 &= \{\,]6n{+}4, 6n{+}5[\mid n \in \mathbb{Z} \,\wedge\, 0 \leq n < \delta \,\} \qquad \text{and} \\
\mathcal{P}_0 &= \{\, \{x\} \mid x \in \mathbb{R} \setminus \bigcup(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3) \,\}\,.
\end{aligned}$$

Thus the proof of Theorem 5 is finished by verifying the following two statements.

(8.1) *Every quadruple in $\mathcal{Q}_1$ is the type of some countable interval partition.*

(8.2) *The type of each countable interval partition lies in $\mathcal{Q}_1$.*

In the following, $\alpha, \beta, \gamma, \delta$ are always countable cardinals and $(\alpha, \beta, \gamma, \delta)$ is *admissible* if and only if $(\alpha, \beta, \gamma, \delta)$ is the type of some countable interval partition. First of all we point out that if $\alpha + \beta + 1 = \delta$ then $(\alpha, \beta, 0, \delta)$ is admissible. This is true because

$$W := \bigcup_{0 \leq n < \alpha} \{3n\} \;\cup\; \bigcup_{0 \leq n < \beta} [3n{+}1, 3n{+}2]$$

is Euclidean closed and the open set $\mathbb{R} \setminus W$ is a union of precisely δ mutually disjoint nonempty open intervals.

If J is an infinite interval and κ is a cardinal number with $\kappa \leq \aleph_0$ then we clearly can find a subinterval $\tilde{J}$ of J homeomorphic with J such that $J \setminus \tilde{J}$ can be written as a disjoint union of precisely κ half-open intervals. In case that $\kappa = \aleph_0$ and J is open, the interval J itself is such a disjoint union. Therefore and since it is clear that $\beta + \gamma + \delta > 0$, the following statement is true.

(8.3) *If $(\alpha, \beta, \gamma, \delta)$ is admissible then $(\alpha, \beta, \gamma', \delta')$ is admissible for all pairs (γ', δ') of countable cardinals satisfying either $\gamma' \geq \gamma$ and $\delta' = \delta$, or $\gamma' = \aleph_0$ and $\delta' < \delta$.*

Now in order to prove (8.1) assume that $\vartheta = (\alpha, \beta, \gamma, \delta)$ lies in $\mathcal{Q}_1$. We already know that ϑ is admissible if $\alpha + \beta + 1 = \delta$ and $\gamma = 0$. Thus by (8.3) ϑ is admissible if $\alpha + \beta + 1 = \delta$ and γ is arbitrary. In particular, $(\alpha, \beta, \aleph_0, \alpha + \beta + 1)$ is admissible. Thus by (8.3) ϑ is admissible whenever $\alpha + \beta + 1 > \delta$ and $\gamma = \aleph_0$. This concludes the proof of (8.1).

Before we prove (8.2) we verify the following two statements.

(8.4) *If $(\alpha, \beta, \gamma, \delta)$ is admissible then $(\alpha + 2\beta + \gamma, 0, 0, \beta + \gamma + \delta)$ is admissible.*

(8.5) *If $(\alpha, \beta, \gamma, \delta)$ is admissible and α, β, γ are finite then $\delta = \alpha + \beta + 1$.*

To verify (8.4) let $\mathcal{P}$ be an interval partition of type $(\alpha, \beta, \gamma, \delta)$ and define $B \subset \mathbb{R}$ such that $x \in B$ if and only if x is a noncut point of the Euclidean space I for some $I \in \mathcal{P}$. Clearly, $|B| = \alpha + 2\beta + \gamma$. Then $(\{\, I \setminus B \mid I \in \mathcal{P} \,\} \setminus \{\emptyset\}) \cup \{\, \{b\} \mid b \in B \,\}$ is an interval partition which consists of precisely $|B|$ singletons and precisely $\beta + \gamma + \delta$ open intervals.

To verify (8.5) let $(\alpha, \beta, \gamma, \delta)$ be admissible. Clearly, if $E \subset \mathbb{R}$ is finite then $\mathbb{R} \setminus E$ is a disjoint union of precisely $|E| + 1$ open intervals. Therefore and by applying (8.4) we derive $\alpha + 2\beta + \gamma + 1 = \beta + \gamma + \delta$ and hence $\delta = \alpha + \beta + 1$ if α, β, γ are finite.

Now in order to prove (8.2) assume that $\vartheta = (\alpha, \beta, \gamma, \delta)$ is admissable. By definition, $\vartheta \in \mathcal{Q}_1$ if $\alpha + \beta + 1 = \delta$. In the following we will check the two remaining cases $\alpha + \beta + 1 < \delta$ and $\alpha + \beta + 1 > \delta$.

Assume firstly that $\alpha + \beta + 1 > \delta$. Then $\alpha + \beta + \gamma = \aleph_0$ because $\alpha + \beta + \gamma < \aleph_0$ implies $\alpha + \beta + 1 = \delta$ by (8.5). If $\gamma < \aleph_0$ then $\alpha + \beta = \aleph_0$ (since $\alpha + \beta + \gamma = \aleph_0$) and $\delta < \aleph_0$ (since $\alpha + \beta + 1 > \delta$) and hence $\gamma + \delta < \aleph_0$. But $\alpha + \beta = \aleph_0 > \gamma + \delta$ is not possible because if $K_1, K_2, K_3 ...$ is a sequence of mutually disjoint nonempty compact intervals (of positive length or of length 0) then $\mathbb{R} \setminus \bigcup_{n=1}^{\infty} K_n$ is nonempty and cannot be written as a union of finitely many intervals. (This fact is not so evident as it might seem at first sight. To prove this fact note that, by applying Sierpiński's theorem [1] 6.1.27, if $u \in K_1$ and $v \in K_2$ and $u < v$ then $[u, v] \neq [u, v] \cap \bigcup_{n=1}^{\infty} K_n$.) So we can be sure that if $\alpha + \beta + 1 > \delta$ then $\gamma = \aleph_0$ and hence $\vartheta \in \mathcal{Q}_1$. Furthermore, (8.2) is settled by verifying that the case $\alpha + \beta + 1 < \delta$ can be ruled out.

In order to verify this, assume $\alpha + \beta + 1 < \delta$. Then $\alpha + \beta = m$ for some $m \in \mathbb{N}$. Let $\mathcal{P}$ be an interval partition of type ϑ and let K be the union of the m compact elements of $\mathcal{P}$. Then K is a compact subspace of $\mathbb{R}$ with precisely m components. Let $\mathcal{U}$ be the family of the components of the open subspace $\mathbb{R} \setminus K$ of $\mathbb{R}$. Naturally, each set $U \in \mathcal{U}$ is a nonempty open interval and $|\mathcal{U}| = m+1$. Furthermore, each interval $U \in \mathcal{U}$ is a disjoint union of non-compact elements of $\mathcal{P}$. Therefore, from $|\mathcal{U}| = \alpha + \beta + 1 < \delta$ we conclude that some open interval $V \in \mathcal{U}$ must contain two disjoint *open* intervals $I_1, I_2 \in \mathcal{P}$ with $\sup I_1 = a \leq b = \inf I_2$. Then $[a, b] \subset V$ and $[a, b]$ is disjoint from K. In particular, $a < b$. If $\mathcal{J}$ is the family of all *open* $J \in \mathcal{P}$ with $J \subset [a, b]$ then we further conclude that each component $[u, v]$ of the non-empty compact set $[a, b] \setminus \bigcup \mathcal{J}$ is a disjoint union of half-open intervals. But it is straightforward to verify that a compact interval cannot be decomposed into half-open intervals. This finishes the proof of (8.2) and of Theorem 5.

The classification of the locally connected refinements of the real line leads to the following interesting proposition about incomparability and local connectedness.

Proposition 3. *Let $\tau_1, \tau_2 \in \mathcal{T}$ be such that $(\mathbb{R}, \tau_i)$ is locally connected for $i \in \{1, 2\}$. If $(\mathbb{R}, \tau_1)$ is separable and $(\mathbb{R}, \tau_2)$ is not discrete then $(\mathbb{R}, \tau_1)$ is homemorphic to a subspace of $(\mathbb{R}, \tau_2)$. If $(\mathbb{R}, \tau_2)$ is not separable and not discrete then $(\mathbb{R}, \tau_1)$ is homemorphic to a subspace of $(\mathbb{R}, \tau_2)$.*

Proof. Let $\tau \in \mathcal{T}$ such that $(\mathbb{R}, \tau)$ is locally connected and separable and $(\alpha, \beta, \gamma, \delta)$ is the type of the family $\mathcal{C}_\tau$ of all components of the space $(\mathbb{R}, \tau)$. With $n \in \mathbb{N} \cup \{0\}$ define a subspace X of the real line by

$$X := \bigcup_{n<\alpha} \{7n\} \cup \bigcup_{n<\beta} [7n+1, 7n+2] \cup \bigcup_{n<\gamma} [7n+3, 7n+4[\cup \bigcup_{n<\delta}]7n+5, 7n+6[.$$

It goes without saying that the spaces $(\mathbb{R}, \tau)$ and X are homeomorphic. If $\tau_2 \in \mathcal{T}$ such that $(\mathbb{R}, \tau_2)$ is locally connected and not discrete then some component C of the space is not a singleton and hence the real line $\mathbb{R}$ is homeomorphic to a subspace of C. Consequently, by transitivity, $(\mathbb{R}, \tau)$ is homeomorphic to a subspace of $(\mathbb{R}, \tau_2)$.

Now let $\tau_2 \in \mathcal{T}$ such that $(\mathbb{R}, \tau_2)$ is not separable and not discrete then $(\mathbb{R}, \tau_2)$ has an open-closed discrete subset D of size $\mathbf{c}$ and the τ_2-open-closed set $\mathbb{R} \setminus D$ contains a copy of the real line. If $\tau_1 \in \mathcal{T}$ such that $(\mathbb{R}, \tau_1)$ is separable then we already know that $(\mathbb{R}, \tau_1)$ is homeomorphic to a subspace of the real line and hence $(\mathbb{R}, \tau_1)$ is homeomorphic to a subspace of the subspace $\mathbb{R} \setminus D$ of $(\mathbb{R}, \tau_2)$. If $\tau_1 \in \mathcal{T}$ such that $(\mathbb{R}, \tau_1)$ is not separable then $(\mathbb{R}, \tau_1)$ is the topological sum of a discrete space of size $\mathbf{c}$ and a locally connected subspace of $\mathbb{R}$. Hence $(\mathbb{R}, \tau_1)$ is homeomorphic to a subspace of the space $(\mathbb{R}, \tau_2)$, *q.e.d.*

9. An application of Theorem 1

A trivial consequence of Theorem 1 is the following corollary whose direct proof would not be significantly easier than our proof of Theorem 1.

Corollary 3. *There exists a real function F such that F is a connected, dense subspace of $\mathbb{R}^2$ and F is not homeomorphic to a proper subspace of F.*

In this final section we apply Corollary 3 in order to prove a noteworthy theorem about separable and connected refinements of the real line which are not of the form $(\mathbb{R}, \tau[F])$ for a real function F. In doing so we realize that in Corollary 1 the cardinality $2^{\mathbf{c}}$ can be increased to the greatest possible cardinality if the property *metrizable* is omitted.

Theorem 6. *There exist $2^{2^{\mathbf{c}}}$ topologies $\tau \in \mathcal{T}$ with $\Gamma(\tau) = \mathbb{R}$ such that the corresponding Hausdorff spaces $(\mathbb{R}, \tau)$ are incomparable, separable and connected.*

Proof. In the following, call a set S of reals $\mathbf{c}$-dense if and only if $|S \cap [u, v]| = \mathbf{c}$ whenever $u < v$. For the sake of simply writing we use terminology from linear algebra and regard $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$. Fix a set Y of irrational numbers such that $\{1\} \cup Y$ is a basis of the vector space $\mathbb{R}$ over $\mathbb{Q}$. Naturally, $|Y| = \mathbf{c}$. For $A \subset Y$ let $L(A)$ denote the linear subspace of $\mathbb{R}$ generated by the vectors in the set $\{1\} \cup A$. Obviously, $\mathbb{Q} \subset L(A)$ for every $A \subset Y$ and if $A_1, A_2 \subset Y$ are disjoint then $L(A_1) \cap L(A_2) = \mathbb{Q}$.

Let $\mathbf{U}$ denote the family of all ultrafilters $\mathcal{U}$ on Y such that $|A| = \mathbf{c}$ for every $A \in \mathcal{U}$. Then $|\mathbf{U}| = 2^{2^{\mathbf{c}}}$ by [2] Theorem 7.6. It is plain that $L(A)$ is $\mathbf{c}$-dense whenever $A \subset Y$ and $|A| = \mathbf{c}$. In particular, if $\mathcal{U} \in \mathbf{U}$ then $L(A)$ is $\mathbf{c}$-dense for every $A \in \mathcal{U}$. We claim that the following is true for every $\mathcal{U} \in \mathbf{U}$.

(9.1) *For every $x \in \mathbb{R}$ there exists a set $A \in \mathcal{U}$ such that $x \in L(A)$ and $[u, v] \setminus L(A)$ is an infinite set whenever $u < v$.*

In order to prove (9.1) fix a countably infinite set $Y_0 \subset Y$. Then $Y_0 \notin \mathcal{U}$ and hence $Y \setminus Y_0 \in \mathcal{U}$. Let $x \in \mathbb{R}$. Then $x \in L(T)$ for some finite $T \subset Y$. Put $A = (Y \setminus Y_0) \cup T$. Then $A \in \mathcal{U}$ and $x \in L(A)$. Furthermore, $Y \setminus A \neq \emptyset$ since $|T| < \aleph_0 = |Y_0|$. Thus the set $L(Y \setminus A) \setminus \mathbb{Q}$ is a dense subset of $\mathbb{R}$. And, clearly, $L(Y \setminus A) \setminus \mathbb{Q} \subset \mathbb{R} \setminus L(A)$.

Put $J := [-1, 1]$ and let $\eta(J) = \{\, U \cap J \mid U \in \eta \,\}$ denote the relative topology of η on the interval J. For every ultrafilter $\mathcal{U} \in \mathbf{U}$ let $\eta[\mathcal{U}]$ denote the topology on the set J generated by the subbasis $\eta(J) \cup \{\, L(A) \cap J \mid A \in \mathcal{U} \,\}$.

Put $\tau = \eta[\mathcal{U}]$. Since $L(A)$ is $\mathbf{c}$-dense and $\mathbb{Q} \subset L(A)$ for every $A \in \mathcal{U}$, it is clear that every nonempty open subset of the space (J, τ) is a set of size $\mathbf{c}$ which intersects the countable set $\mathbb{Q} \cap J$. Consequently, the space (J, τ) is separable. From (9.1) we conclude that every $x \in J$ has a τ-neighborhood which is not an $\eta(J)$-neighborhood. Thus τ is strictly finer than $\eta(J)$ at every point in J. A moment's reflection suffices to see that every subinterval of J is connected in the space (J,τ). In view of (9.1) it is straightforward that a nondegenerate subinterval of J is never a regular subspace of (J,τ).

Assume that $\mathcal{U}_1, \mathcal{U}_2 \in \mathbf{U}$ are distinct. Then we can choose $A \subset \mathbb{R}$ such that A lies in $\mathcal{U}_1$ and $Y \setminus A$ lies in $\mathcal{U}_2$. Hence $W_1 := J \cap L(A)$ is $\eta[\mathcal{U}_1]$-open and $W_2 = J \cap L(Y \setminus A)$ is $\eta[\mathcal{U}_2]$-open. Therefore, the topologies $\eta[\mathcal{U}_1]$ and $\eta[\mathcal{U}_2]$ are distinct as well because $\eta[\mathcal{U}_1] = \eta[\mathcal{U}_2] = \tau$ implies that W_1 and W_2 are τ-open sets and hence $W_1 \cap W_2 = J \cap \mathbb{Q}$ is a nonempty τ-open set of size smaller than $\mathbf{c}$ whose existence has already been ruled out. Consequently, the family

$$\mathcal{G} := \{ \eta[\mathcal{U}] \mid \mathcal{U} \in \mathbf{U} \}$$

consists of $2^{2^{\mathbf{c}}}$ topologies τ on $J = [-1,1]$ everywhere strictly finer than $\eta(J)$ such that the spaces (J,τ) are separable and connected and not regular but, a fortiori, Hausdorff.

Define an equivalence relation $\sim$ on $\mathcal{G}$ such that $\tau_1 \sim \tau_2$ if and only if the spaces $([-1,1],\tau_1)$ and $([-1,1],\tau_2)$ are homeomorphic. Since there are only $2^{\mathbf{c}}$ permutations of the set $[-1,1]$, the size of an equivalence class cannot exceed $2^{\mathbf{c}}$. Therefore, since $|\mathcal{G}| = 2^{2^{\mathbf{c}}} > 2^{\mathbf{c}}$, there are $2^{2^{\mathbf{c}}}$ equivalence classes. Hence by choosing one topology in each equivalence class we can select a family $\mathcal{E} \subset \mathcal{G}$ such that $|\mathcal{E}| = |\mathcal{G}| = 2^{2^{\mathbf{c}}}$ and all spaces $([-1,1],\tau)$ $(\tau \in \mathcal{E})$ are *mutually non-homeomorphic.*

Now we are ready to verify Theorem 6. Put $I_{-1} :=]-\infty,-1[$ and $I_1 :=]1,\infty[$. Let F be a real function as depicted in Corollary 3 and consider the separable, connected, metrizable topology $\tau[F] \in \mathcal{T}$. Since both intervals can easily be mapped onto $\mathbb{R}$ by strictly increasing functions, with the help of the one topology $\tau[F]$ on $\mathbb{R}$ for both indices $k = \pm 1$ we can define a topology μ_k on I_k strictly finer than the restriction of η to I_k such that (I_k,μ_k) is homeomorphic to the space F. For every topology $\tau \in \mathcal{E}$ define a separable Hausdorff topology $\hat{\tau}$ on the set $\mathbb{R}$ in the following way. Firstly, for $k \in \{-1,1\}$ the subspace I_k of $(\mathbb{R},\hat{\tau})$ coincides with the space (I_k,μ_k) and the subspace $[-1,1]$ of $(\mathbb{R},\hat{\tau})$ coincides with the space $([-1,1],\tau)$. Secondly, for $k \in \{-1,1\}$ the family $\{]k-t,k+t[\cap V \mid t > 0,\ k \in V \in \tau \}$ is a local basis at the point k. Obviously, $\hat{\tau}$ is strictly finer than η at every $x \in \mathbb{R}$. In view of Lemma 5 it is clear that the separable Hausdorff space $(\mathbb{R},\hat{\tau})$ is connected and, moreover, the connected sets are precisely the intervals.

Let τ_1,τ_2 be topologies in $\mathcal{E}$ and assume that f is a homeomorphism from the connected space $(\mathbb{R},\hat{\tau}_1)$ onto a subspace S of $(\mathbb{R},\hat{\tau}_2)$. Then f maps every interval onto some interval and hence f is a *monotonic* bijection from $\mathbb{R}$ onto the interval S. Therefore, since $x \in \mathbb{R}$ has a metrizable neighborhood in the space $(\mathbb{R},\hat{\tau}_i)$ if and only if $x \notin [-1,1]$ and since the subspace $S \setminus f([-1,1])$ of $(\mathbb{R},\hat{\tau}_2)$ has precisely two components and both components are metrizable spaces, we conclude that $f([-1,1]) = [-1,1]$. This implies firstly $\tau_1 = \tau_2$ and, secondly, $S \cap I_k = I_k$ for $k = \pm 1$ in view of Corollary 3 and since the two spaces $(I_{\pm 1},\mu_{\pm 1})$, are homeomorphic with F. Hence we obtain $\hat{\tau}_1 = \hat{\tau}_2$ and $S = \mathbb{R}$ and arrive at the conclusion that the $2^{2^{\mathbf{c}}}$ spaces $(\mathbb{R},\hat{\tau})$ $(\tau \in \mathcal{E})$ are incomparable, *q.e.d.*

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Author's address: Institute of Mathematics,
University of Natural Resources and Life Sciences, Vienna, Austria.

E-mail: `gerald.kuba(at)boku.ac.at`