# GRADIENT EINSTEIN-TYPE SASAKIAN MANIFOLDS WITH $\alpha = 0$ ARE TRIVIAL

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ABSTRACT. Catino, Mastrolia, Monticelli, and Rigoli have launched an ambitious program to study known geometric solitons from a unified perspective, termed Einstein-type manifolds. This framework allows one to treat Ricci solitons, Yamabe solitons, and all of their generalizations simultaneously. Einstein-type manifolds are characterized by four constants $\alpha, \beta, \mu$ and $\rho$. In this paper, we show that when $\alpha = 0$ gradient Einstein-type Sasakian manifolds are trivial. As a consequence, many geometric solitons on Sasakian manifolds turn out to be trivial.

## 1. INTRODUCTION

Geometric flows have become one of the most powerful tools for understanding Riemannian manifolds (cf. [Brendle05], [Brendle07], [BS08], [Chow92], [Perelman02], [SS03], [Ye94]). To analyze geometric flows, it is important to study their self-similar solutions, which are known as geometric solitons. Geometric solitons have become central to geometry (see [BR13], [CSZ12], [CC12], [CC13], [CCCMM14], [CD08], [CMM12], [CMMR17], [CMM16], [CMM17], [Chowetal07], [CLN06], [DS13], [FOW09], [Hamilton89], [HL14], [Maeta21], [Maeta23], [Maeta24], [PBS23], [Perelman02]). There are various types of geometric solitons, including Ricci solitons, Yamabe solitons, $k$-Yamabe solitons, quasi-Yamabe solitons, almost Yamabe solitons, and conformal solitons. In particular, it has recently been recognized that quasi-type solitons are highly effective for resolving problems arising in other areas of research (cf. [CMR24]; see also [CL24]).

An ambitious project was initiated by Catino, Mastrolia, Monticelli, and Rigoli (cf. [CMMR17]). They introduced the notion of Einstein-type manifolds, a unifying framework that encompasses previously known geometric solitons and which now appears to clarify why analogous results recur across different soliton classes.

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Let $(M, g)$ be a Riemannian manifold. If there exist $X \in \mathfrak{X}(M)$ and $\lambda \in C^\infty(M)$ such that

$$\alpha \operatorname{Ric} + \frac{\beta}{2} L_X g + \mu X^\flat \otimes X^\flat = (\rho R + \lambda) g,$$

for some constants $\alpha, \beta, \mu, \rho \in \mathbb{R}$ with $(\alpha, \beta, \mu) \neq (0,0,0)$, then $(M, g, X)$ is called an *Einstein-type manifold*. Here, Ric denotes the Ricci tensor of $M$, $L_X$ denotes the Lie derivative along $X$, and $R$ is the scalar curvature of $M$. If $X = \nabla F$ for some $F \in C^\infty(M)$, then $(M, g, F)$ is called a *gradient* Einstein-type manifold:

$$\alpha \operatorname{Ric} + \beta \nabla\nabla F + \mu \nabla F \otimes \nabla F = (\rho R + \lambda) g, \tag{1.1}$$

where $\nabla\nabla F$ is the Hessian of $F$, and $F$ is called the potential function. If $F$ is constant, then $(M, g, F)$ is called *trivial*.

As pointed out in [CMMR17], gradient Einstein-type manifolds include gradient Yamabe solitons, gradient almost Yamabe solitons, gradient $k$-Yamabe solitons, gradient conformal solitons, and quasi-Yamabe gradient solitons (see, for example, [Hamilton82], [BR13], [CMM12], [HL14]).

In this paper, we consider gradient Einstein-type manifolds with $\alpha = 0$, which include all the classes above. In [CMMR17], Catino, Mastrolia, Monticelli, and Rigoli provided a structure theorem for complete $n$-dimensional ($n \geq 3$) gradient Einstein-type manifolds with $\alpha = 0$ (see Theorem 1.4 in [CMMR17]). Theorem 1.4 of [CMMR17] generalizes results in [CSZ12], [CMM12], [Tashiro65].

As is well known, Sasakian manifolds provide the odd-dimensional counterpart of Kähler geometry via their Kähler cones and Reeb foliations. In particular, Sasaki-Einstein manifolds appear prominently in high-energy physics and string theory, for example as internal spaces in supersymmetric AdS/CFT compactifications, where they model the geometric structure underlying certain superconformal field theories (see, for example, [BG08], [Sparks11], [GMSW05]).

Therefore, in this paper, we consider gradient Einstein-type Sasakian manifolds with $\alpha = 0$. Interestingly, we can show that they are trivial.

**Theorem 1.1.** *Any gradient Einstein-type Sasakian manifold with $\alpha = 0$ is trivial.*

## 2. Preliminaries

In this section, we present the definitions needed to introduce Sasakian manifolds. After defining Sasakian manifolds, we recall standard consequences and identities. For details, see [YK84], [BG08].

**Definition 2.1** (Almost contact manifold)**.** An *almost contact structure* on a smooth $(2n+1)$-dimensional manifold $M$ is a triple $(\phi, \xi, \eta)$ with $\phi \in \Gamma(T^1_1 M)$, $\xi \in \mathfrak{X}(M)$, and $\eta \in \Omega^1(M)$ such that

$$\eta(\xi) = 1, \qquad \phi^2 = -I + \eta \otimes \xi.$$

A manifold endowed with such a triple is called an *almost contact manifold*.

**Remark 2.2.** *For any almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$, the following identities hold:*

$$\phi\,\xi = 0,\ \ \eta(\phi X) = 0 \quad \textit{for any vector field } X.$$

**Definition 2.3** (Almost contact metric manifolds)**.** An *almost contact metric structure* on $M^{2n+1}$ is a quadruple $(\phi, \xi, \eta, g)$ where $(\phi, \xi, \eta)$ is an almost contact structure and $g$ is a Riemannian metric satisfying

$$\eta(X) = g(X, \xi), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.1}$$

A manifold endowed with such a quadruple is called an *almost contact metric manifold.* The associated *fundamental* 2*-form* is $\Phi(X, Y) := g(X, \phi Y)$.

**Definition 2.4** (Contact metric manifold)**.** A *contact metric manifold* is an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ such that

$$d\eta = \Phi.$$

**Definition 2.5** (Nijenhuis tensor)**.** Given an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$, define an almost complex structure $J$ on $M \times \mathbb{R}$ by

$$J\big(X, f\,\tfrac{\partial}{\partial t}\big) := \big(\phi X - f\,\xi,\ \eta(X)\tfrac{\partial}{\partial t}\big).$$

The *Nijenhuis tensor* of $J$ is the $(1, 2)$-tensor $N_J$ defined by

$$N_J(U, V) = [JU, JV] - J[JU, V] - J[U, JV] + J^2[U, V],$$

for vector fields $U, V$. The structure $(\phi, \xi, \eta)$ is called *normal* if $N_J \equiv 0$.

**Definition 2.6** (Sasakian manifold)**.** A *Sasakian manifold* is a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ that is normal.

We will use the following known results later.

**Theorem 2.7.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost contact metric manifold. Then $M$ is Sasakian if and only if*

$$(\nabla_X \phi)Y = g(X, Y)\,\xi - \eta(Y)\,X \quad \textit{for any vector fields } X, Y.$$

**Lemma 2.8.** *If $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold, then*

$$\mathrm{Ric}(X, \xi) = 2n\,\eta(X),$$
$$\mathrm{Ric}(\phi X, \phi Y) = \mathrm{Ric}(X, Y) - 2n\,\eta(X)\eta(Y).$$

**Definition 2.9** ($K$-contact)**.** A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called *$K$-contact* if and only if $\xi$ is a Killing vector field.

**Proposition 2.10.** *On a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the following are equivalent.*

(1) *The structure is $K$-contact.*
(2) $\nabla_X \xi = -\,\phi X$ *holds for any vector field $X$.*

## 3. Proof of Theorem 1.1

In this section, we show Theorem 1.1.

**Theorem 3.1.** *Any gradient Einstein-type Sasakian manifold* $(M^{2n+1},\phi,\xi,\eta,g,F)$ *with* $\alpha=0$ *is trivial.*

*Proof.* Since $\lambda$ is a smooth function, the equation (1.1) can be reduced to

$$\beta\nabla\nabla F+\mu\nabla F\otimes\nabla F=\varphi g, \tag{3.1}$$

for some $\varphi\in C^\infty(M)$ and constants $(\beta,\mu)\neq(0,0)$. If $\beta\neq 0$, without loss of generality, (3.1) can be written as

$$\nabla\nabla F+c\nabla F\otimes\nabla F=\varphi g,$$

for some $c\in\mathbb{R}$. If $\beta=0$, without loss of generality, (3.1) can be written as

$$\nabla F\otimes\nabla F=\varphi g.$$

Case 1: $\beta\neq 0$.

Since $M$ is a gradient Einstein-type manifold with $\alpha=0$, we have

$$\nabla_X\nabla F=\varphi X+cg(X,\nabla F)\nabla F. \tag{3.2}$$

By the equation, we have

$$\begin{aligned}
\mathrm{Rm}(X,Y)\nabla F&=\nabla_X\nabla_Y\nabla F-\nabla_Y\nabla_X\nabla F-\nabla_{[X,Y]}\nabla F\\
&=\nabla_X(\varphi Y+cg(Y,\nabla F)\nabla F)\\
&\quad-\nabla_Y(\varphi X+cg(X,\nabla F)\nabla F)\\
&\quad-\{\varphi[X,Y]+cg([X,Y],\nabla F)\nabla F\}\\
&=(X\varphi)Y+\varphi\nabla_XY+cg(\nabla_XY,\nabla F)\nabla F\\
&\quad+cg(Y,\nabla_X\nabla F)\nabla F+cg(Y,\nabla F)\nabla_X\nabla F\\
&\quad-\{(Y\varphi)X+\varphi\nabla_YX+cg(\nabla_YX,\nabla F)\nabla F\\
&\quad+cg(X,\nabla_Y\nabla F)\nabla F+cg(X,\nabla F)\nabla_Y\nabla F\}\\
&\quad-\varphi[X,Y]-cg([X,Y],\nabla F)\nabla F\\
&=(X\varphi)Y+cg(Y,\varphi X+cg(X,\nabla F)\nabla F)\nabla F\\
&\quad+cg(Y,\nabla F)\{\varphi X+cg(X,\nabla F)\nabla F\}\\
&\quad-\{(Y\varphi)X+cg(X,\varphi Y+cg(Y,\nabla F)\nabla F)\nabla F\\
&\quad+cg(X,\nabla F)\{\varphi Y+cg(Y,\nabla F)\nabla F\}\}\\
&=(X\varphi-c\varphi XF)Y-(Y\varphi-c\varphi YF)X.
\end{aligned} \tag{3.3}$$

By taking the trace, one has

$$\mathrm{Ric}(Y,\nabla F)=g(Y,-2n(\nabla\varphi-c\varphi\nabla F)). \tag{3.4}$$

Since $M$ is a Sasakian manifold, we have

$$\begin{aligned}\mathrm{Rm}(X,Y)\phi\nabla F = \phi&\mathrm{Rm}(X,Y)\nabla F\\ &+ g(Y,\nabla F)\nabla_X\xi - (X(\eta(\nabla F)))Y\\ &- \{g(X,\nabla F)\nabla_Y\xi - (Y(\eta(\nabla F)))X\}.\end{aligned}$$

By taking the trace, one has

$$\begin{aligned}\mathrm{Ric}(Y,\phi\nabla F) &= g(\phi\mathrm{Rm}(e_i,Y)\nabla F, e_i)\\ &\quad + g(Y,\nabla F)g(\nabla_{e_i}\xi, e_i) - (e_i(\eta(\nabla F)))g(Y,e_i)\\ &\quad - \{g(e_i,\nabla F)g(\nabla_Y\xi, e_i) - (2n+1)Y(\eta(\nabla F))\}\\ &= g(Y, -\phi(\nabla\varphi - c\varphi\nabla F))\\ &\quad + g(Y,\nabla F)g(\nabla_{e_i}\xi, e_i) - (e_i(\eta(\nabla F)))g(Y,e_i)\\ &\quad - \{g(e_i,\nabla F)g(\nabla_Y\xi, e_i) - (2n+1)Y(\eta(\nabla F))\},\end{aligned} \tag{3.5}$$

where the last equality follows from (3.3). Take $Y = \phi X$. By Lemma 2.8, one has

$$\begin{aligned}\mathrm{Ric}(\phi X,\phi\nabla F) &= \mathrm{Ric}(X,\nabla F) - 2n\eta(X)\eta(\nabla F)\\ &= g(X, -2n(\nabla\varphi - c\varphi\nabla F)) - 2n\eta(X)\eta(\nabla F),\end{aligned}$$

where the last equality follows from (3.4). On the other hand,

$$\begin{aligned}\mathrm{Ric}(\phi X,\phi\nabla F) &= g(\phi X, -\phi(\nabla\varphi - c\varphi\nabla F))\\ &\quad + g(\phi X,\nabla F)g(\nabla_{e_i}\xi, e_i) - (e_i(\eta(\nabla F)))g(\phi X,e_i)\\ &\quad - \{g(e_i,\nabla F)g(\nabla_{\phi X}\xi, e_i) - (2n+1)(\phi X)(\eta(\nabla F))\}\\ &= g(X, -(\nabla\varphi - c\varphi\nabla F)) - \eta(X)\eta(-(\nabla\varphi - c\varphi\nabla F))\\ &\quad + g(\phi X,\nabla F)g(\nabla_{e_i}\xi, e_i) - (e_i(\eta(\nabla F)))g(\phi X,e_i)\\ &\quad - \{g(e_i,\nabla F)g(\nabla_{\phi X}\xi, e_i) - (2n+1)(\phi X)(\eta(\nabla F))\},\end{aligned}$$

where the last equality follows from (2.1). Combining the above two equations, we have

$$\begin{aligned}&g(X, -(2n-1)(\nabla\varphi - c\varphi\nabla F)) - 2n\eta(X)\eta(\nabla F) - \eta(X)\eta(\nabla\varphi - c\varphi\nabla F)\\ &- g(\phi X,\nabla F)g(\nabla_{e_i}\xi, e_i) + (e_i(\eta(\nabla F)))g(\phi X, e_i)\\ &+ g(e_i,\nabla F)g(\nabla_{\phi X}\xi, e_i) - (2n+1)(\phi(X))(\eta(\nabla F)) = 0.\end{aligned} \tag{3.6}$$

We consider the left-hand side of (3.6).

We consider the second and third terms of the left-hand side of (3.6).

$$-2n\eta(X)\eta(\nabla F) - \eta(X)\eta(\nabla\varphi - c\varphi\nabla F) = g(X, -2n\eta(\nabla F)\xi - \eta(\nabla\varphi - c\varphi\nabla F)\xi).$$

We consider the fourth term of the left-hand side of (3.6). It is known that a Sasakian manifold is a K-contact manifold (cf. [YK84]). Hence, we have

$$\nabla_W\xi = -\phi W,$$

for any vector field $W$ on $M$. Therefore, the fourth term of the left-hand side of (3.6) vanishes. In fact,

$$g(\nabla_{e_i}\xi, e_i) = g(-\phi e_i, e_i) = 0.$$

We consider the fifth term of the left-hand side of (3.6).

$$\begin{aligned}(e_i(\eta(\nabla F)))g(\phi X, e_i) &= g(\phi X, \nabla(\eta(\nabla F)))\\ &= g(X, -\phi\nabla(\eta(\nabla F))).\end{aligned}$$

We consider the sixth term of the left-hand side of (3.6). Since

$$\nabla_{\phi X}\xi = -\phi(\phi X) = -\phi^2 X = -(-X + \eta(X)\xi) = X - \eta(X)\xi,$$

we have

$$\begin{aligned}g(e_i, \nabla F)g(\nabla_{\phi X}\xi, e_i) &= g(e_i, \nabla F)g(X - \eta(X)\xi, e_i)\\ &= g(X - \eta(X)\xi, \nabla F)\\ &= g(X, \nabla F - \eta(\nabla F)\xi).\end{aligned}$$

We consider the seventh term of the left-hand side of (3.6).

$$-(2n+1)(\phi(X))(\eta(\nabla F)) = g(\phi X, -(2n+1)\nabla(\eta(\nabla F))) = g(X, (2n+1)\phi(\nabla(\eta(\nabla F)))).$$

Combining these equations with (3.6), we have

$$\begin{aligned}&g(X, -(2n-1)(\nabla\varphi - c\varphi\nabla F)) + g(X, -2n\eta(\nabla F)\xi - \eta(\nabla\varphi - c\varphi\nabla F)\xi)\\ &+ g(X, -\phi\nabla(\eta(\nabla F))) + g(X, \nabla F - \eta(\nabla F)\xi)\\ &+ g(X, (2n+1)\phi(\nabla(\eta(\nabla F)))) = 0.\end{aligned}$$

Since $X$ is arbitrary, one has

$$\begin{aligned}&-(2n-1)(\nabla\varphi - c\varphi\nabla F) - 2n\eta(\nabla F)\xi - \eta(\nabla\varphi - c\varphi\nabla F)\xi\\ &+ 2n\phi(\nabla(\eta(\nabla F))) + \nabla F - \eta(\nabla F)\xi = 0.\end{aligned} \tag{3.7}$$

By taking $Y = \xi$ in (3.4) and using Lemma 2.8, one has

$$2n\eta(\nabla F) = \mathrm{Ric}(\xi, \nabla F) = g(\xi, -2n(\nabla\varphi - c\phi\nabla F)).$$

Hence, one has

$$\begin{aligned}\eta(\nabla F) &= -(\xi\varphi - c\varphi\xi F)\\ &= -\eta(\nabla\varphi) + c\varphi\eta(\nabla F).\end{aligned}$$

We also have

$$\eta(\nabla\varphi) = (c\varphi - 1)\eta(\nabla F), \tag{3.8}$$

and

$$\eta(\nabla\varphi - c\varphi\nabla F) = -\eta(\nabla F). \tag{3.9}$$

By the above argument and (3.5), we have

$$\begin{aligned}\mathrm{Ric}(Y, \phi\nabla F) = g(&Y, -\phi(\nabla\varphi - c\varphi\nabla F)) \\ &+ g(Y, -\nabla(\eta(\nabla F))) + g(\phi Y, \nabla F) + (2n+1)Y(\eta(\nabla F)).\end{aligned} \tag{3.10}$$

Take $Y = \xi$ in (3.10). By Lemma 2.8, we have

$$\begin{aligned}2n\eta(\phi\nabla F) &= g(\xi, -\phi(\nabla\varphi - c\varphi\nabla F)) + g(\xi, -\nabla(\eta(\nabla F))) + (2n+1)\xi(\eta(\nabla F)) \\ &= 2n\xi(\eta(\nabla F)).\end{aligned}$$

Therefore, by Remark 2.2, we have

$$\xi(\eta(\nabla F)) = 0. \tag{3.11}$$

We consider $\nabla(\eta(\nabla F))$.

$$\begin{aligned}\nabla(\eta(\nabla F)) &= e_i g(\nabla F, \xi)e_i \\ &= g(\nabla_{e_i}\nabla F, \xi)e_i + g(\nabla F, \nabla_{e_i}\xi)e_i \\ &= \varphi\xi + c\eta(\nabla F)\nabla F + \phi\nabla F,\end{aligned} \tag{3.12}$$

where the last equality follows from (3.2) and from the fact that $M$ is Sasakian. By (3.11) and (3.12), we have

$$\begin{aligned}0 &= \xi(\eta(\nabla F)) \\ &= g(\nabla(\eta(\nabla F)), \xi) \\ &= g(\varphi\xi + c\eta(\nabla F)\nabla F + \phi\nabla F, \xi) \\ &= \varphi + c(\eta(\nabla F))^2.\end{aligned}$$

Therefore, we have

$$\varphi = -c(\eta(\nabla F))^2. \tag{3.13}$$

By (3.13), we have

$$\eta(\nabla\varphi) = g(\nabla\varphi, \xi) = \xi\varphi = \xi(-c(\eta(\nabla F))^2) = -2c\eta(\nabla F)\xi(\eta(\nabla F)) = 0,$$

where the last equality follows from (3.11). On the other hand, by (3.8) and (3.13), we have

$$\eta(\nabla\varphi) = (c\varphi - 1)\eta(\nabla F) = -(c^2\{\eta(\nabla F)\}^2 + 1)\eta(\nabla F).$$

Combining these equations, one has

$$\eta(\nabla F) = 0,$$

because $c^2\{\eta(\nabla F)\}^2 + 1 > 0$. Substituting this into (3.13) and (3.9), we have

$$\varphi = \eta(\nabla\varphi - c\varphi\nabla F) = 0.$$

Substituting these into (3.7), we have

$$\nabla F = 0.$$

Therefore, $F$ is constant and $M$ is trivial.

Case 2: $\beta = 0$.

In this case, the equation for a gradient Einstein-type manifold is as follows.

$$\nabla F \otimes \nabla F = \varphi g. \tag{3.14}$$

By taking the trace, we have

$$|\nabla F|^2 = (2n+1)\varphi. \tag{3.15}$$

Assume that $\nabla F(p) \neq 0$ at some point $p \in M$. Since $\dim M \geq 2$, one can take a unit vector $e$ such that $g(\nabla F, e) = 0$ at $p$. By (3.14), we have

$$0 = (g(\nabla F, e))^2 = \varphi(p).$$

Hence, one has

$$\varphi(p) = 0.$$

However, by (3.15), we have

$$|\nabla F|^2(p) = (2n+1)\varphi(p) = 0,$$

which is a contradiction. Therefore, $F$ is constant and $M$ is trivial. □

In Case 2 of the proof of the theorem, we did not use the assumption that $M$ is a Sasakian manifold. Therefore, we also have the following.

**Corollary 3.2.** *Any $n$-dimensional ($n \geq 2$) gradient Einstein-type manifold with $\alpha = \beta = 0$ is trivial.*

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