

TRIANGULATED CATEGORIES ADMITTING LINEAR GENERATORS

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ABSTRACT. The main result of this paper is that there is an additive equivalence between $\overline{\mathcal{C}}_n$, the Paquette-Yıldırım completion of the discrete cluster categories of Dynkin type A_∞ , and the perfect derived category of a certain DG algebra. This additive equivalence preserves some of the triangulated structure: it commutes with the suspension functor and preserves triangles with at least two indecomposable terms. In the process, we introduce the notion of a linear generator G in a Krull-Schmidt, Hom-finite triangulated category. It turns out that the existence of a linear generator affords a large amount of control over $\mathcal{T}$. For example, it allows us to describe all indecomposable objects in $\mathcal{T}$ in terms of G , to determine all triangles of $\mathcal{T}$ with at least two indecomposable objects, and to show that the Rouquier dimension of $\mathcal{T}$ is at most one. Moreover, we prove that there is an additive equivalence (which preserves some of the triangulated structure) between $\mathcal{T}$ and the perfect derived category of a certain DG algebra. Finally, we show that any triangulated category with a linear generator is additively equivalent to a thick subcategory of $\overline{\mathcal{C}}_n$.

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1. Introduction

Cluster categories have been studied extensively as examples of triangulated categories which exhibit remarkable algebraic and combinatorial structure, and which have connections to other areas of mathematics, such as singularity categories and cluster algebras (see, for example, [Ami09; BKL08; Bua+06; CCS05; HJ10; Lam15; Tod07]). Indeed, cluster

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categories have often been studied through the lens of triangulations of marked surfaces; see [BZ11; Sch08; QZ17] for finite rank cases, and [cKP; Fis14; Fra25b; Fra25a; GHJ19; GZ23; IT15; LP17] for infinite rank cases.

In this paper, we are interested in a family of infinite rank cluster categories $\bar{\mathcal{C}}_n$ for $n \geq 1$, constructed in [PY21] called *Paquette-Yildirim completions of discrete cluster categories of Dynkin type A_∞* . These categories have exceptionally powerful combinatorial properties, and can be used to develop combinatorial techniques in the study of representation theory of triangulated categories containing infinitely many indecomposable objects up to isomorphism. Moreover, these categories are *algebraic*, so that, given a compact generator $G \in \bar{\mathcal{C}}_n$, there is a DG algebra $\text{End}^{\text{DG}}(G)$ and a triangle equivalence $\bar{\mathcal{C}}_n \rightarrow \text{perf } \text{End}^{\text{DG}}(G)$ by a result of Keller [Kel94].

In [Aug+23], the authors construct a compact generator $G \in \bar{\mathcal{C}}_1$ and find that $\text{End}^{\text{DG}}(G)$ is the DG algebra with underlying graded algebra $k[x]$, where x concentrated in degree -1 , and with trivial differential. Furthermore, in [Mur25a], the second named author classifies the compact generators G of $\bar{\mathcal{C}}_n$ for $n \geq 1$. Thus, a natural question arises: can we describe the DG algebra $\text{End}^{\text{DG}}(G)$ such that $\bar{\mathcal{C}}_n$ is triangle equivalent to the perfect derived category of $\text{End}^{\text{DG}}(G)$?

With this question in mind, we introduce in Section 2 the concept of triangulated categories with *linear generators*. Given a triangulated category $\mathcal{T}$, and an object $G \in \mathcal{T}$, let $\langle G \rangle_1 \subset \mathcal{T}$ denote the smallest full subcategory containing G and closed under direct sums, direct summands, and suspension. Roughly, a linear generator G of $\mathcal{T}$ is a classical generator $G \in \mathcal{T}$ such that the set $\text{ind}(\langle G \rangle_1)$ of indecomposable objects in $\langle G \rangle_1$ admits a total order where $Q < P \in \text{ind}(\langle G \rangle_1)$ implies that $\mathcal{T}(P, Q) \cong k$ and $\mathcal{T}(Q, P) = 0$. Moreover, we place some conditions on the total ordering concerning the factorisation of morphisms.

Throughout, we work within the following setup.

Setup 1. *Let k be a field and let $\mathcal{T}$ be a k -linear Hom-finite, Krull-Schmidt triangulated category.*

It turns out that the existence of a linear generator provides us a large amount of control over $\mathcal{T}$. For example, in Section 2, we describe all indecomposable objects of $\mathcal{T}$ in terms of G , we describe all triangles in $\mathcal{T}$ with at least two indecomposable terms, and we compute the Rouquier dimension of $\mathcal{T}$. Most importantly, the properties of linear generators allow us to prove our main result in Section 2.

Theorem 2 (Theorem 2.29). *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories satisfying Setup 1 and admitting linear generators G and G' , respectively. If there is a fully faithful additive functor $F: \langle G \rangle_1 \rightarrow \langle G' \rangle_1$ that commutes with the respective suspension functors, then there exists a fully faithful additive functor $\mathcal{F}: \mathcal{T} \rightarrow \mathcal{S}$ which commutes with suspension and preserves triangles with at least two indecomposable terms. Further, $\mathcal{F}$ is an additive equivalence if and only if F is.*

Moreover, using the results of Section 2, we are able to compute the graded endomorphism algebra of minimal linear generators. By a minimal generator G , we mean that there does not exist a proper direct summand H of G , such that H is a generator of $\mathcal{T}$.

Proposition 3 (Proposition 3.2). *Let $\mathcal{T}$ be a triangulated category satisfying Setup 1 and admitting a minimal linear generator $G = \bigoplus_{i=1}^m G_i$ with m indecomposable summands. Then, $\text{End}_{\mathcal{T}}^*(G)$ is isomorphic to the upper triangular $m \times m$ -matrix algebra χ^G , with*

$$\chi_{i,j}^G := \begin{cases} k[x] & \text{if } i = j \text{ and } G_i < G_i[1] \\ 0 & \text{if } j < i \\ k[x^\pm] & \text{otherwise.} \end{cases}$$

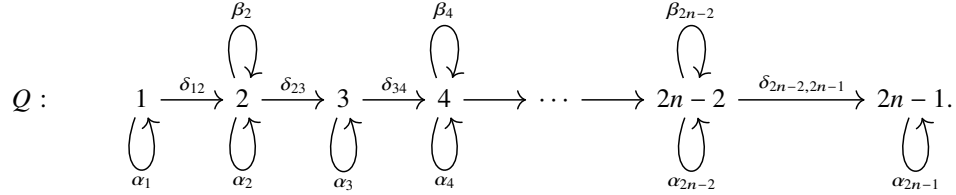
where x is concentrated in degree -1 .

We may view χ^G as a DG algebra with zero differential. Let us denote such DG algebra as Λ^G . Then, as an application of Theorem 2, we obtain the main result of Section 3.

Theorem 4 (Theorem 3.5). *Let $G \in \mathcal{T}$ be a minimal linear generator of a triangulated category satisfying Setup 2.1. Then, there is an additive equivalence $\mathcal{F} : \mathcal{T} \rightarrow \text{perf}(\Lambda^G)$ which commutes with the respective suspension functors and preserves triangles with at least two indecomposable terms.*

In Section 3.2, we focus on triangulated categories satisfying Setup 1 and admitting a minimal linear generator $H = \bigoplus_{i=1}^{2n-1} H_i$ with $H_i \cong H_i[1]$ if and only if i is even. Let Λ_n denote the graded endomorphism algebra of such a generator. Then, Section 3.2 shows that Λ_n is isomorphic to the path algebra of a graded quiver with relations.

More precisely, consider the quiver Q



Let Q_0 denote the set of vertices of Q and for each $a \in Q_0$ write ι_a for the trivial idempotent path in Q at vertex a . Moreover, let $B \subset Q_0$ be the set of vertices such that there is no loop β_b for $b \in B$. Let I be the ideal

$$I := \langle \alpha_a \beta_a - \iota_a, \beta_a \alpha_a - \iota_a, \alpha_b \delta_{b,b+1} - \delta_{b,b+1} \alpha_{b+1}, \beta_a \delta_{a,a+1} \delta_{a+1,a+2} - \delta_{a,a+1} \delta_{a+1,a+2} \beta_{a+2} \rangle,$$

for $1 \leq a, b \leq 2n-1$, and $a \notin B$.

Theorem 5 (Theorem 3.7). *Let $\mathcal{T}$ be a triangulated category satisfying Setup 1, and let $H \in \mathcal{T}$ be a minimal linear generator with $H \cong \bigoplus_{i=1}^{2n-1} P_i$, where $P_i \cong P_i[1]$ if and only if $i \equiv 0 \pmod{2}$. Then there is an isomorphism of graded k -algebras,*

$$\Lambda_n \xrightarrow{\sim} kQ/I,$$

Our interest in Λ_n is motivated by [Mur23], which proves that there exists a classical generator $E \in \overline{\mathcal{C}}_n$, called a *fan generator*, satisfying the assumptions placed on H in Theorem 5. The construction of E and the fact that it does indeed satisfy the assumptions of Theorem 5 are discussed in Section 4. This allows us to combine Theorem 5 and Theorem 4 to partially answer our motivating question and prove the main result of Section 4.2.

Theorem 6 (Theorem 4.8). *There is an additive equivalence $\mathcal{F} : \overline{\mathcal{C}}_n \xrightarrow{\sim} \text{perf} \Lambda_n$ which commutes with the respective suspension functors and preserves triangles with at most two indecomposable terms.*

Moreover, let $\mathcal{S}$ be a triangulated category satisfying Setup 1 and admitting a linear generator G . Then $\mathcal{S}$ is additively equivalent to a thick subcategory of $\overline{\mathcal{C}}_n$ for some $n \in \mathbb{Z}$.

It is possible that the additive equivalence $\mathcal{F}$ is in fact a triangulated equivalence, however our techniques are not currently able to prove this. The interested reader is encouraged to investigate if it is possible to lift $\mathcal{F}$ to a triangulated equivalence.

Conventions

Throughout, we let k be an algebraically closed field. All modules will be considered as right modules, unless otherwise stated, and we will compose paths from left to right. That is, if a and b are paths, then we write ab to mean first apply a then b . However, morphisms will be composed right to left, i.e. if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then $gf : X \rightarrow Z$. For a category $\mathcal{C}$ and two objects $X, Y \in \mathcal{C}$, we write $\mathcal{C}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$.

We will assume that the suspension functor $[1]_{\mathcal{T}}$ for a triangulated category $\mathcal{T}$ has a natural inverse, $[-1]_{\mathcal{T}}$, and we will simply denote the suspension functor by $[1]$ if the category is clear from context. We will always take triangle to mean a distinguished triangle in a triangulated category.

For a graded/differential graded algebra A , we denote the degree $m \in \mathbb{Z}$ component of A by A^m .

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2. Triangulated Categories with a Linear Generator

This section introduces the concept of triangulated categories with linear generators. It turns out that the existence of such generators imply very strong and interesting properties on the triangulated category and its morphisms. We are interested in these types of categories as the main examples of categories in this paper satisfy this setup and, moreover, the abstract properties of this setup that are proved in this section will play a key role in the remainder of the paper.

Throughout, we will be interested in triangulated categories satisfying the following setup.

Setup 2.1. *Let k be a field and let $\mathcal{T}$ be a k -linear Hom-finite, Krull-Schmidt triangulated category.*

2.1. Linear Generators

We begin by recalling some standard notation and terminology concerning generators of triangulated categories.

Let $\mathcal{T}$ be a triangulated category and consider a subcategory $\mathcal{S}$ of $\mathcal{T}$. Recall $\mathcal{S}$ is *thick* if it is a triangulated subcategory of $\mathcal{T}$ closed under direct summands. Moreover, given an object $X \in \mathcal{T}$, we are interested in the following subcategories associated to X .

- (1) The *thick closure* $\langle X \rangle$ of X is the smallest thick subcategory of $\mathcal{T}$ containing X .
- (2) The category $\langle X \rangle_1$ is the smallest additive full subcategory of $\mathcal{T}$ which contains X , and is closed under direct summands and the suspension functor $[1]$ of $\mathcal{T}$.
- (3) The category $\langle X \rangle_m$ is defined inductively as the full subcategory of $\mathcal{T}$ consisting of objects $A \in \langle X \rangle_m$ which fit into a triangle

$$B \rightarrow A \rightarrow C \rightarrow B[1],$$

where $B \in \langle X \rangle_1$ and $C \in \langle X \rangle_{m-1}$. In other words, $\langle X \rangle_m := \langle X \rangle_1 \star \langle X \rangle_{m-1}$.

A *classical generator* of $\mathcal{T}$ is an object $G \in \mathcal{T}$ such that $\langle G \rangle = \mathcal{T}$. A classical generator is, moreover, a *strong generator* if $\langle G \rangle_m = \mathcal{T}$ for some integer $m \geq 1$. Given a strong generator G , let n be the smallest positive integer such that $\langle G \rangle_n = \mathcal{T}$. Then, the *generation time* of G is $n - 1$.

This section investigates the properties of Krull-Schmidt, Hom-finite, k -linear, triangulated categories equipped with a classical generator G such that $\langle G \rangle_1$ satisfies a certain property.

Definition 2.2. Let $\mathcal{T}$ be a triangulated category satisfying [Setup 2.1](#) and suppose that $G \in \mathcal{T}$ is a classical generator. Then, we say that G is a *linear generator* if it satisfies the properties:

- (G1) The set of classes of indecomposable objects in $\langle G \rangle_1$ admits a total ordering defined as:

For indecomposable objects $Q, P \in \langle G \rangle_1$, then $Q \leq P$ if and only if $\mathcal{T}(P, Q) \cong k$. Moreover, if $Q \not\leq P$, then $\mathcal{T}(Q, P) = 0$.

- (G2) Given an indecomposable object $P \in \langle G \rangle_1$, then $P \leq P[1]$.
- (G3) For indecomposable objects $P, Q \in \langle G \rangle_1$, then $P \leq Q \leq P[1]$ implies that Q is isomorphic to at least one of P and $P[1]$.
- (G4) For indecomposable objects $R \leq Q \leq P$ in $\langle G \rangle_1$, then any morphism from P to R factors non-trivially through Q .

Lemma 2.3. *Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1 and admitting a linear generator G . Suppose that $f: P \rightarrow Q$ is a non-zero morphism between non-isomorphic, indecomposable objects in $\langle G \rangle_1$. Then, cone f is indecomposable in $\mathcal{T}$.*

Proof. We will prove the statement by proving that cone f has a local endomorphism ring.

Given that f is non-zero and G is a linear generator, then it must be that $Q \leq P$ by property (G1) in Definition 2.2. If $Q \cong P$, then, since $\mathcal{T}(P, Q) \cong k$, it follows that f is an isomorphism and we may conclude that cone $f = 0$. Hence, it suffices to consider the case where P and Q are not isomorphic.

Consider the triangle

$$P \xrightarrow{f} Q \xrightarrow{g} \text{cone } f \rightarrow P[1].$$

Applying the covariant functor $\mathcal{T}(P[1], -)$ induces the long exact sequence

$$\mathcal{T}(P[1], P) \xrightarrow{\mathcal{T}(P[1], f)} \mathcal{T}(P[1], Q) \rightarrow \mathcal{T}(P[1], \text{cone } f) \rightarrow \mathcal{T}(P[1], P[1]) \xrightarrow{\mathcal{T}(P[1], f[1])} \mathcal{T}(P[1], Q[1])$$

It follows from properties (G1) and (G2) in Definition 2.2 that $\mathcal{T}(P[1], P) \cong k$ and $\mathcal{T}(P[1], Q) \cong k$. Further, observe that $\mathcal{T}(P[1], f)$ and $\mathcal{T}(P[1], f[1])$ are not the zero map since f is nonzero, and therefore, by dimension arguments, they must be isomorphisms. As a result, we may conclude from the long exact sequence that $\mathcal{T}(P[1], \text{cone } f) = 0$. Similarly, applying $\mathcal{T}(P, -)$ to the triangle defining cone f shows that $\mathcal{T}(P, \text{cone } f)$ vanishes.

Moreover, since $P \not\cong Q$, then $\mathcal{T}(Q, P) = 0$ by Definition 2.2 (G1). Therefore, applying the functor $\mathcal{T}(Q, -)$ to the triangle defining cone f shows that $\mathcal{T}(Q, \text{cone } f) \cong k$.

Finally, the contravariant functor $\mathcal{T}(-, \text{cone } f)$ applied to the triangle defining cone f induces the exact sequence

$$\mathcal{T}(P[1], \text{cone } f) \rightarrow \mathcal{T}(\text{cone } f, \text{cone } f) \rightarrow \mathcal{T}(Q, \text{cone } f) \rightarrow \mathcal{T}(P, \text{cone } f).$$

Since $\mathcal{T}(P, \text{cone } f)$, $\mathcal{T}(P[1], \text{cone } f)$ vanish, and $\mathcal{T}(Q, \text{cone } f) \cong k$, then it must be that $\mathcal{T}(\text{cone } f, \text{cone } f) \cong k$. That is, cone f has a local endomorphism ring, whence cone f is indecomposable. $\square$

Given indecomposable objects $P, Q \in \langle G \rangle_1$, it is worth noting that the cones cone f and cone g of any two nonzero morphisms $f: P \rightarrow Q$ and $g: P \rightarrow Q$ are isomorphic. To see this simply observe that, since there exists nonzero morphisms $P \rightarrow Q$, then $\mathcal{T}(P, Q) \cong k$. Therefore, there exists a scalar $\lambda \in k$ such that $f = \lambda \cdot g$. In other words, since composition is k -linear, $f = g \circ \lambda \cdot \text{id}_P$. We may thus construct the following commutative diagram

$$\begin{array}{ccccccc} P & \xrightarrow{f} & Q & \longrightarrow & \text{cone } f & \longrightarrow & P[1] \\ \lambda \cdot \text{id}_P \downarrow \wr & & \parallel & & \downarrow & & \downarrow \\ P & \xrightarrow{g} & Q & \longrightarrow & \text{cone } g & \longrightarrow & P[1] \end{array}$$

showing that cone $f \cong$ cone g .

Notation 2.4. In view of the remark above, given any non-zero morphism $f: P \rightarrow Q$, the indecomposable object cone f will be denoted $X_{P,Q}$.

2.2. Morphism Spaces

Given a triangulated category $\mathcal{T}$ as in [Setup 2.1](#) and a linear generator G in $\mathcal{T}$, this subsection aims to describe morphisms in $\mathcal{T}(X, Y)$ where X and Y are indecomposable objects in $\langle G \rangle_2$. In order to do so, we first describe morphisms whose domain or codomain object is $X_{P,Q}$.

Lemma 2.5. *With notation as above, then $\mathcal{T}(X_{P,Q}, R) \cong k$ if and only if $Q[1] < R \leq P[1]$. Otherwise, $\mathcal{T}(X_{P,Q}, R) = 0$.*

Proof. Applying the functor $\mathcal{T}(-, R)$ to the triangle

$$P \xrightarrow{f} Q \rightarrow X_{P,Q} \rightarrow P[1]$$

defining $X_{P,Q}$ induces the long exact sequence

$$\mathcal{T}(Q[1], R) \rightarrow \mathcal{T}(P[1], R) \rightarrow \mathcal{T}(X_{P,Q}, R) \rightarrow \mathcal{T}(Q, R) \rightarrow \mathcal{T}(P, R).$$

Observe that $Q \leq P$ since $X_{P,Q}$ is the cone of a non-zero morphism $P \rightarrow Q$ by assumption.

Suppose first that $Q[1] < R \leq P[1]$, then we will show that $\mathcal{T}(X_{P,Q}, R) \cong k$. Since $R \leq P[1]$, $\mathcal{T}(P[1], R) \cong k$. Moreover, since $Q \leq Q[1] < R$, then $\mathcal{T}(Q, R)$ and $\mathcal{T}(Q[1], R)$ vanish and so we may conclude that $\mathcal{T}(X_{P,Q}, R) \cong k$.

For the converse, suppose that $Q[1] < R \leq P[1]$ does not hold. Then, either $R = Q[1]$, $R < Q[1]$ or $P[1] < R$. Consider the latter case where $Q \leq P \leq P[1] < R$. Then, $\mathcal{T}(P, R)$, $\mathcal{T}(Q, R)$, and $\mathcal{T}(P[1], R)$ all vanish, which, by the long exact sequence, forces $\mathcal{T}(X_{P,Q}, R)$ to vanish and proves the claim for the $P[1] < R$ case.

Assume next that $R < Q[1] \leq P[1]$. Then, the Hom spaces $\mathcal{T}(Q, R)$, $\mathcal{T}(P, R)$, $\mathcal{T}(Q[1], R)$, and $\mathcal{T}(P[1], R)$ are all one-dimensional. Hence, if $\mathcal{T}(f, R)$ and $\mathcal{T}(f[1], R)$ are nonzero, then they are isomorphisms, and so it must be that $\mathcal{T}(X_{P,Q}, R) = 0$ by the long exact sequence.

To see that $\mathcal{T}(f, R)$ is non-zero, observe that since $\mathcal{T}(P, R)$ is one dimensional, then we may choose a non-zero morphism $\alpha_{P,R} \in \mathcal{T}(P, R)$. By property (G4) of [Definition 2.2](#), there are morphisms $\beta \in \mathcal{T}(Q, R)$ and $\gamma \in \mathcal{T}(P, Q)$ such that $\alpha_{P,R} = \beta \cdot \gamma$. Moreover, since f is nonzero and $\mathcal{T}(P, Q)$ is one-dimensional, there is a non-zero $c \in k$ such that $f = c \cdot \gamma$. Since composition is k -linear, this means that

$$\mathcal{T}(f, R) \left(\frac{1}{c} \beta \right) = \frac{1}{c} \beta \cdot f = \beta \cdot \gamma = \alpha_{P,R} \neq 0$$

as required. A similar proof works to show that $\mathcal{T}(f[1], R) \neq 0$.

Finally, suppose that $R = Q[1]$. If $Q < Q[1]$, then, $\mathcal{T}(Q, R)$ vanishes, and $\mathcal{T}(P[1], R)$, $\mathcal{T}(Q[1], R)$ are one-dimensional since $R \leq Q[1]$, $P[1]$. Thus, if $\mathcal{T}(f[1], R) \neq 0$, then it also must be that $\mathcal{T}(X_{P,Q}, R) = 0$. The fact that $\mathcal{T}(f[1], R)$ is nonzero follows from similar arguments as in the $R < Q[1]$ case.

On the other hand, if $R = Q[1]$ and $Q = Q[1]$, then the Hom spaces $\mathcal{T}(Q, R)$, $\mathcal{T}(P, R)$, $\mathcal{T}(Q[1], R)$, and $\mathcal{T}(P[1], R)$ are all one-dimensional. That is, similarly to before, since $\mathcal{T}(f, R)$ and $\mathcal{T}(f[1], R)$ are nonzero, then they are isomorphisms, and we may conclude that $\mathcal{T}(X_{P,Q}, R)$ vanishes. $\square$

Lemma 2.6. *Let $\mathcal{T}$ be a triangulated category satisfying [Setup 2.1](#) and equipped with a linear generator G . Then, $\mathcal{T}(R, X_{P,Q}) \cong k$ if and only if $Q \leq R < P$. Else, $\mathcal{T}(R, X_{P,Q}) = 0$.*

Proof. The proof is analogous to that of [Lemma 2.5](#) by considering the functor $\mathcal{T}(R, -)$. $\square$

Corollary 2.7. *Let $\mathcal{T}$ be a triangulated category satisfying [Setup 2.1](#) and equipped with a linear generator G . Then, Q and P are the unique (up to isomorphism) indecomposable objects in $\langle G \rangle_1$ which sit in a triangle*

$$P \rightarrow Q \rightarrow X_{P,Q} \rightarrow P[1].$$

In other words, $X_{P,Q} \cong X_{P',Q'}$ if and only if $P \cong P'$ and $Q \cong Q'$.

Proof. Suppose that there is a nonzero morphism $g: P' \rightarrow Q'$ such that $\text{cone } g = X_{P',Q'} \cong X_{P,Q}$. We will show that $P' \cong P$ and $Q' \cong Q$. Well, since $X_{P',Q'} \cong X_{P,Q}$ by assumption, there are nonzero morphisms $Q' \rightarrow X_{P,Q}$ and $Q \rightarrow X_{P',Q'}$. By Lemma 2.6, it must be that $Q \leq Q'$ and $Q' \leq Q$. That is, $Q \cong Q'$. Hence, we may construct the following commutative diagram

$$\begin{array}{ccccccc} P & \xrightarrow{f} & Q & \longrightarrow & X_{P,Q} & \longrightarrow & P[1] \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ P' & \xrightarrow{g} & Q' & \longrightarrow & X_{P,Q} & \longrightarrow & P[1] \end{array}$$

showing that $P \cong P'$. □

Lemma 2.8. *Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1 and equipped with a linear generator G . Suppose that P, Q, R, S are indecomposables in $\langle G \rangle_1$ such that $Q \leq S \leq R < P$. Then, a morphism $f: R \rightarrow X_{P,Q}$ factors through S .*

Proof. Applying the functor $\mathcal{T}(R, -)$ to the triangle

$$P \rightarrow Q \xrightarrow{g} X_{P,Q} \rightarrow P[1],$$

induces the exact sequence

$$\mathcal{T}(R, P) \rightarrow \mathcal{T}(R, Q) \xrightarrow{\mathcal{T}(R, g)} \mathcal{T}(R, X_{P,Q}) \rightarrow \mathcal{T}(R, P[1]).$$

Since $R < P \leq P[1]$, it follows from the definition of a linear generator that $\mathcal{T}(R, P) = \mathcal{T}(R, P[1]) = 0$, hence $\mathcal{T}(R, g)$ is an isomorphism. Therefore, any morphism $f: R \rightarrow X_{P,Q}$ factors through a morphism $q: R \rightarrow Q$. Moreover, because $Q \leq S \leq R$, then property (G4) in Definition 2.2 implies that q (and thus f) also factors through S . □

An analogous proof to Lemma 2.8, with the application of the functor $\mathcal{T}(-, S)$ proves the following.

Lemma 2.9. *Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1 and equipped with a linear generator G . Suppose that P, Q, R, S are indecomposables in $\langle G \rangle_1$ such that $Q[1] < S \leq R \leq P[1]$. Then, a morphism $f: X_{P,Q} \rightarrow S$ factors through R .*

These lemmas allow us to completely describe the indecomposable objects in $\langle G \rangle_2$ in terms of indecomposable objects $\langle G \rangle_1$ and the cones $X_{P,Q}$.

Proposition 2.10. *Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1 and equipped with a linear generator G . Suppose that $X \in \langle G \rangle_2$ is an indecomposable object. Then, X is either isomorphic to an object $P \in \langle G \rangle_1$ or to $X_{P,Q}$, where P, Q are indecomposable objects in $\langle G \rangle_1$.*

Proof. It suffices to show that any morphism $f: P \rightarrow Q \in \langle G \rangle_1$ is such that $\text{cone } f \cong A \oplus B$, where $A \in \langle G \rangle_1$ and $B \cong \bigoplus_{(P',Q') \in I} X_{P',Q'}$ where I is a set containing tuples (P', Q') of indecomposable objects in $\langle G \rangle_1$. We will do so by considering three cases.

- (1) Consider first a morphism $f: P \rightarrow \bigoplus_{i=1}^n P_i$, where P and P_i are indecomposable objects in $\langle G \rangle_1$. We may write $f = (f_i)_{i=1}^n$. If there is a j where $f_j: P \rightarrow P_j$ is the zero morphism, then $\text{cone } f \cong \text{cone } (f_i)_{i \neq j} \oplus P_j$. Hence, we may reduce to the case where none of the f_i are the zero morphism. Thus, by reindexing if necessary, we may assume that $P_i \leq P_1 \leq P$ for all $1 \leq i \leq n$. By property (G4) in Definition 2.2, it follows that $f_i: P \rightarrow P_i$ factors through a morphism $g_i: P \rightarrow P_1$ for all $2 \leq i \leq n$. By rescaling the morphisms g_i if

necessary, we may assume $f_i = g_i \circ f_1$. We may thus construct the following morphism of triangles.

$$\begin{array}{ccccccc}
 P & \xrightarrow{\begin{pmatrix} f_1 \\ 0 \end{pmatrix}} & \bigoplus_{i=1}^n P_i & \longrightarrow & X_{P,P_1} \oplus \bigoplus_{i=2}^n P_i & \longrightarrow & P_1[1] \\
 \downarrow \wr & & \downarrow h & & \downarrow & & \downarrow \\
 P & \xrightarrow{f} & \bigoplus_{i=1}^n P_i & \longrightarrow & \text{cone } f & \longrightarrow & P[1].
 \end{array}$$

Here, $h = \begin{pmatrix} \text{id}_P & 0 \\ g & \text{id}_{\bigoplus_{i=2}^n P_i} \end{pmatrix}$ and $g = (g_i)_{i=2}^n : P \rightarrow \bigoplus_{i=2}^n P_i$. Observe that h is an isomorphism, as it admits the inverse $h^{-1} = \begin{pmatrix} \text{id}_P & 0 \\ -g & \text{id}_{\bigoplus_{i=2}^n P_i} \end{pmatrix}$. We may thus conclude that $\text{cone } f \cong X_{P,P_1} \oplus \bigoplus_{i=2}^n P_i$.

- (2) Next, consider a morphism $f = (f_{ij})_{i=1,j=1}^{m,n} : \bigoplus_{j=1}^m Q_j \rightarrow \bigoplus_{i=1}^n P_i$, where $f_{ij} : Q_j \rightarrow P_i$ is a morphism between indecomposable objects in $\langle G \rangle_1$ for all i, j . We will show by induction on m that $\text{cone } f \cong A \oplus B$ where $A \in \langle G \rangle_1$ and $B \cong \bigoplus_{(P',Q') \in I} X_{P',Q'}$. Observe that (1) proves the statement for $m = 1$, and suppose by induction that it holds for $m - 1$.

By reindexing if necessary, assume that $Q_j \leq Q_1$ for all $1 \leq j \leq n$. If there is an i_0 such that $P_{i_0} > Q_1$, then $f_{i_0,j} = 0$ for all j and $\text{cone } f \cong \text{cone } (f_{ij})_{i \neq i_0, j} \oplus P_{i_0}$. Hence, we may reduce to the case where $P_i, Q_j \leq Q_1$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Consider the diagram

$$\begin{array}{ccccccc}
 Q_1 & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \bigoplus_{j=1}^m Q_j & \longrightarrow & \bigoplus_{j=2}^n Q_j & \longrightarrow & Q_1[1] \\
 \downarrow \wr & & \downarrow f & & \downarrow w \oplus v & & \downarrow \\
 Q_1 & \xrightarrow{(f_{i1})_{i=1}^n} & \bigoplus_{i=1}^n P_i & \longrightarrow & X_{Q_1,P_1} \oplus \bigoplus_{i=2}^n P_i & \longrightarrow & Q_1[1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \dashrightarrow & \text{cone } f & \dashrightarrow & \text{cone } (w \oplus v) & \dashrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow u & & \downarrow \\
 Q_1[1] & \longrightarrow & \bigoplus_{j=1}^m Q_j[1] & \longrightarrow & \bigoplus_{j=2}^m Q_j & \longrightarrow & Q_1[2].
 \end{array}$$

The top left square of this diagram clearly commutes. The bottom two rows in this diagram, as well as the dashed morphisms, are obtained by the octahedral axiom. In fact, it follows from this axiom that the diagram commutes and all four columns and rows are distinguished triangles. The fact that the $\text{cone } (f_{i1})_{i=1}^n = X_{Q_1,P_1} \oplus \bigoplus_{i=2}^n P_i$ follows from (1) above.

Furthermore, observe that w can be written as $w = (w_j : Q_j \rightarrow X_{Q_1,P_1})_{j=1}^m$ and, by Lemma 2.6, $\mathcal{T}(Q_j, X_{Q_1,P_1}) = 0$ since $Q_j \leq Q_1$. Thus, $w = 0$, and so there is an isomorphism $\text{cone } f \cong \text{cone } v \oplus X_{Q_1,P_1}$. However, by the inductive hypothesis, the cone of the morphism $v : \bigoplus_{j=2}^m Q_j \rightarrow \bigoplus_{i=2}^n P_i$ is of the required form. Hence, our claim follows by induction. $\square$

In view of Proposition 2.10, it is possible to separate nonzero morphisms between indecomposable objects in $\langle G \rangle_2$ into two classes, which will play an important role in controlling the behaviour of the Hom spaces.

Definition 2.11. Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1 and equipped with a linear generator G . Let $Y, Z \in \langle G \rangle_2$ be indecomposable objects, and let $f : Y \rightarrow Z$ be a non-zero morphism. Then, f is a *forwards morphism*

if there exists a pair of triangles

$$\begin{array}{ccccccc} P & \longrightarrow & Q & \longrightarrow & Y & \longrightarrow & P[1] \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \downarrow \\ P' & \longrightarrow & Q' & \longrightarrow & Z & \longrightarrow & P'[1], \end{array}$$

where $P, P', Q, Q' \in \langle G \rangle_1$ and a pair of morphisms f_1 and $f_0 \neq 0$ making the above diagram a morphism of triangles. A morphism which is not a forwards morphism is called a *backwards morphism*.

Lemma 2.12. *Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1 and equipped with a linear generator G . Suppose that $P \in \langle G \rangle_1$ and $X \in \langle G \rangle_2$ are indecomposable objects. Then,*

- (1) *a non-zero morphism $f: P \rightarrow X$ is a forwards morphism,*
- (2) *a non-zero morphism $g: X \rightarrow P$ is a backwards morphism if and only if $X \notin \langle G \rangle_1$.*

Proof. We begin by proving (1). By Lemma 2.9, it suffices to consider the cases $f: P \rightarrow Q$ or $f: P \rightarrow X_{R,Q}$ for P, Q, R indecomposable objects in $\langle G \rangle_1$. Suppose $X \cong Q$, then the natural morphism of triangles

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{1} & P & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & Q & \xrightarrow{1} & Q & \longrightarrow & 0, \end{array}$$

proves that f is a forwards morphism.

Suppose next that $X \cong X_{R,Q}$. Since $f: P \rightarrow X_{R,Q}$ is a nonzero morphism, then $R < P \leq Q$ by Lemma 2.6. Observe that by Lemma 2.8, there are nonzero morphisms $g: P \rightarrow Q$ and $h: Q \rightarrow X_{R,Q}$ such that $f = h \circ g$. Moreover, by rescaling if necessary, we may assume that $h: Q \rightarrow X_{R,Q}$ is the morphism in the triangle that defines $X_{R,Q}$. Hence, we may construct the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{1} & P & \longrightarrow & 0 \\ \downarrow & & \downarrow g & & \downarrow f & & \downarrow \\ R & \longrightarrow & Q & \xrightarrow{h} & X_{R,Q} & \longrightarrow & R[1], \end{array}$$

which is a morphism of triangles, proving that f is a forward morphism.

To prove (2), consider a nonzero morphism $g: X \rightarrow P$. If $X \in \langle G \rangle_1$, then, since X is indecomposable, we are in the situation of (1), and we have already proved that in such cases g is a forward morphism. For the converse, suppose that $X \notin \langle G \rangle_1$. Then, $X \cong X_{R,Q}$ for indecomposables $R, Q \in \langle G \rangle_1$ by Lemma 2.9. By Corollary 2.7, it suffices to check that there does not exist a morphism of triangles as in the diagram

$$\begin{array}{ccccccc} R & \longrightarrow & Q & \longrightarrow & X_{R,Q} & \longrightarrow & R[1] \\ \downarrow & & \downarrow h & & \downarrow g & & \downarrow \\ 0 & \longrightarrow & P & \xrightarrow{1} & P & \longrightarrow & 0. \end{array}$$

with $h \neq 0$. Suppose for a contradiction that such diagram exists. Because g is nonzero, then $Q[1] < P \leq R[1]$ by Lemma 2.5, and so $Q < P$. That is, $h = 0$, since there does not exist a nonzero morphism $Q \rightarrow P$. This contradicts the assumption that g is a forwards morphism, whence it is a backwards morphism. $\square$

Due to Lemma 2.12, it is possible to complete the classification of Hom-spaces between indecomposable objects in $\mathcal{T}$.

Lemma 2.13. *Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1 and equipped with a linear generator G . Let P, Q, R, S be indecomposable objects in $\langle G \rangle_1$. Then, $\mathcal{T}(X_{P,Q}, X_{R,S}) \cong k$ if and only if one of the following holds;*

- (1) $S \leq Q < R \leq P$,
- (2) $Q[1] < S \leq P[1] < R$.

Else $\mathcal{T}(X_{P,Q}, X_{R,S}) = 0$. Moreover, a morphism of type (1) is a forwards morphism, and a morphism of type (2) is a backwards morphism.

Proof. Applying the functor $\mathcal{T}(-, X_{R,S})$ to the triangle

$$P \xrightarrow{\alpha} Q \xrightarrow{\beta} X_{P,Q} \rightarrow P[1]$$

defining $X_{P,Q}$ induces the long exact sequence

$$\mathcal{T}(Q[1], X_{R,S}) \rightarrow \mathcal{T}(P[1], X_{R,S}) \rightarrow \mathcal{T}(X_{P,Q}, X_{R,S}) \rightarrow \mathcal{T}(Q, X_{R,S}) \rightarrow \mathcal{T}(P, X_{R,S}). \quad (2.A)$$

- (1) Consider first the case $S \leq Q < R \leq P$. Then, we will show that $\mathcal{T}(X_{P,Q}, X_{R,S}) \cong k$ and, moreover, any morphism $f \in \mathcal{T}(X_{P,Q}, X_{R,S})$ is a forwards morphism. Observe that by Lemma 2.6, since $S \leq Q < R$, then $\mathcal{T}(Q, X_{R,S}) \cong k$. Moreover, because $R \leq P \leq P[1]$ and by Lemma 2.6, then $\mathcal{T}(P, X_{R,S})$ and $\mathcal{T}(P[1], X_{R,S})$ both vanish. Thus, it follows from the exact sequence Equation (2.A) that $\mathcal{T}(X_{P,Q}, X_{R,S}) \cong k$, as required. Moreover, given $f \in \mathcal{T}(X_{P,Q}, X_{R,S})$, it is possible to construct the morphism of triangles

$$\begin{array}{ccccccc} P & \longrightarrow & Q & \xrightarrow{g} & X_{P,Q} & \longrightarrow & P[1] \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \downarrow \\ R & \longrightarrow & S & \longrightarrow & X_{R,S} & \longrightarrow & R[1] \end{array} \quad (2.B)$$

in $\langle G \rangle_2$. To see this, observe that, by Lemma 2.9, since $S \leq Q < R$, then the morphism $f \circ g: Q \rightarrow X_{R,S}$ factors through a nonzero morphism $f_0: Q \rightarrow S$. Moreover, by the axioms of triangulated categories, there exists a map $f_1: P \rightarrow R$ completing the diagram into a morphism of triangles. Thus, f is a forwards morphism.

- (2) Next, suppose that $Q[1] < S \leq P[1] < R$. Then, since $Q < Q[1] < S$, Lemma 2.6 implies that $\mathcal{T}(Q, X_{R,S})$ and $\mathcal{T}(Q[1], X_{R,S})$ both vanish. Moreover, since $S \leq P[1] < R$, then Lemma 2.6 implies that $\mathcal{T}(P[1], X_{R,S}) \cong k$. It thus follows from Equation (2.A) that $\mathcal{T}(X_{P,Q}, X_{R,S}) \cong k$, as required. To see that a morphism $f \in \mathcal{T}(X_{P,Q}, X_{R,S})$ is a backwards morphism, it suffices to check that there does not exist a morphism of triangles as in 2.B with $f_0 \neq 0$. However, by assumption, $Q < S$ and so there do not exist nonzero morphisms $Q \rightarrow S$.
- (3) We now begin to prove the converse statements. Suppose that neither $S \leq Q < R \leq P$ nor $Q[1] < S \leq P[1] < R$ hold. Then, we are in one of the following cases: $S = Q[1] \leq R \leq P[1]$, $S = Q[1] \leq P[1] < R$, $S \leq Q \leq P < R = P[1]$, $P[1] < S$, $R \leq Q$, $Q < S \leq R < P[1]$ and $S < Q[1] \leq P[1] \leq R$. We will show that in any of these cases, $\mathcal{T}(X_{P,Q}, X_{R,S})$ vanishes.

Well, note that Lemma 2.6 implies that $\mathcal{T}(P[1], X_{R,S})$ vanishes if $P[1] < S$ or $R \leq P[1]$, and $\mathcal{T}(Q, X_{R,S})$ vanishes if $Q < S$ or $R \leq Q$. Therefore, if $S = Q[1] \leq R \leq P[1]$, $P[1] < S$, $Q < S \leq R \leq P[1]$ or $R \leq Q$, then both $\mathcal{T}(P[1], X_{R,S})$ and $\mathcal{T}(Q, X_{R,S})$ vanish, so that $\mathcal{T}(X_{P,Q}, X_{R,S})$ must also vanish by Equation (2.A).

Consider the case $S \leq Q \leq P < R = P[1]$. Then, Lemma 2.6 implies that $\mathcal{T}(P[1], X_{R,S}) = 0$ and that $\mathcal{T}(Q, X_{R,S})$ and $\mathcal{T}(P, X_{R,S})$ are one dimensional. Hence, if $\mathcal{T}(\alpha, X_{R,S}) \neq 0$, then it follows from Equation (2.A) that $\mathcal{T}(X_{P,Q}, X_{R,S})$ vanishes. The fact that $\mathcal{T}(\alpha, X_{R,S})$ is nonzero follows from the observation that, by Lemma 2.8, any nonzero morphism $P \rightarrow X_{R,S}$ factors through a morphism $P \rightarrow Q$. Since the Hom-spaces are all one dimensional and composition is k -linear, then any nonzero morphism $P \rightarrow X_{R,S}$ factors through α . That is, any nonzero morphism is in the image of $\mathcal{T}(\alpha, X_{R,S})$.

Similarly, suppose that $S = Q[1] \leq P[1] < R$. Then, by Lemma 2.6, $\mathcal{T}(Q, X_{R,S})$ vanishes, and $\mathcal{T}(P[1], X_{R,S})$ and $\mathcal{T}(Q[1], X_{R,S})$ are both one dimensional. Observe, moreover, that $\mathcal{T}(\alpha[1], X_{R,S})$ is nonzero for similar reasons as in the $S \leq Q \leq P < R = P[1]$ case above. Whence, Equation (2.A) implies that $\mathcal{T}(X_{P,Q}, X_{R,S})$ vanishes.

Finally, consider the case that $S < Q[1] \leq P[1] < R$. Then, by [Lemma 2.6](#), the Hom spaces $\mathcal{T}(P, X_{R,S})$, $\mathcal{T}(Q, X_{R,S})$, $\mathcal{T}(P[1], X_{R,S})$, and $\mathcal{T}(Q[1], X_{R,S})$ are all one dimensional. Moreover, the morphisms $\mathcal{T}(\alpha, X_{R,S})$ and $\mathcal{T}(\alpha[1], X_{R,S})$ are nonzero, by the same argument as in the $S \leq Q \leq P < R = P[1]$ case above. Hence, it follows from [Equation \(2.A\)](#) that $\mathcal{T}(X_{P,Q}, X_{R,S})$ vanishes, as required. $\square$

Remark 2.14. As a corollary of [Lemma 2.13](#), we see that there is a canonical isomorphism between cones of a morphism $P \rightarrow Q \in \langle G \rangle_1$, when P, Q are indecomposable. Consider the morphism of triangles

$$\begin{array}{ccccccc} P & \xrightarrow{f} & Q & \xrightarrow{g} & X & \xrightarrow{h} & P[1] \\ \parallel & & \parallel & & \downarrow u \downarrow v & & \\ P & \xrightarrow{f} & Q & \xrightarrow{g'} & X' & \xrightarrow{h'} & P[1]. \end{array}$$

Since $\mathcal{T}(X, X')$ is one dimensional, there is a $\lambda \in k^\times$ such that $v = \lambda u$. However, commutativity of the diagram implies that $\lambda = 1$.

2.3. Rouquier Dimension

This section uses the properties of linear generators established in the previous sections, in order to compute the Rouquier dimension of a triangulated category admitting a linear generator.

Lemma 2.15. *Let $\mathcal{T}$ be a triangulated category satisfying [Setup 2.1](#) and equipped with a linear generator G . Let P be an indecomposable object in $\langle G \rangle_1$ and let $f: P \rightarrow Z$ be a nonzero morphism with $Z \in \langle G \rangle_2$. Then, $\text{cone } f \in \langle G \rangle_2$.*

Proof. Since $\mathcal{T}$ is Krull-Schmidt, we may write $Z = \bigoplus_{i=1}^n Z_i$ where Z_i are indecomposable objects in $\langle G \rangle_2$. Moreover, we may write $f = (f_i: P \rightarrow Z_i)_{i=1}^n$, and we may assume without loss of generality that $f_i: P \rightarrow Z_i$ is nonzero for all i .

By [Proposition 2.10](#), $Z_i \cong Q \in \langle G \rangle_1$ or $Z_i \cong X_{R,S}$ for $R, S \in \langle G \rangle_1$. For each $1 \leq i \leq n$, let $Q_i = Z_i$ if $Z_i \in \langle G \rangle_1$. Otherwise, let $Q_i = S$ if $Z_i \cong X_{R,S}$. Similarly, for each $1 \leq i \leq n$, let $P_i = 0$ if $Z_i \in \langle G \rangle_1$ and let $P_i = R$ if $Z_i \cong X_{R,S}$. Observe, then, that here is a natural morphism $h = (h_i)_{i=1}^n: \bigoplus_{i=1}^n Q_i \rightarrow Z$ whose cone is $\bigoplus_{i=1}^n P_i[1]$.

Furthermore, for all $1 \leq i \leq n$, we claim that there are nonzero morphisms $g_i: P \rightarrow Q_i$ such that $h \circ (g_i)_{i=1}^n = f$. To construct such g_i , if $Z_i \in \langle G \rangle_1$, let $g_i = f_i$, which is nonzero by assumption. On the other hand, if $Z_i = X_{R,S}$, then, by [Lemma 2.8](#), there is a nonzero morphism $g_i: P \rightarrow S$ such that $f = h_i \circ g_i$. Thus, an application of the octahedral axiom induces the commutative diagram

$$\begin{array}{ccccccc} P & \xrightarrow{g=(g_i)_{i=1}^n} & \bigoplus_{i=1}^n Q_i & \longrightarrow & X_{P,Q_1} \oplus \bigoplus_{i=2}^n Q_i & \longrightarrow & P[1] \\ \downarrow \wr & & \downarrow h & & \downarrow & & \downarrow \\ P & \xrightarrow{f} & Z & \longrightarrow & \text{cone } f & \longrightarrow & P[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{i=1}^n P_i[1] & \longrightarrow & \bigoplus_{i=1}^n P_i[1] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow (u,v) & & \downarrow \\ P[1] & \longrightarrow & \bigoplus_{i=1}^n Q_i[1] & \longrightarrow & X_{P,Q_1}[1] \oplus \bigoplus_{i=2}^n Q_i[1] & \longrightarrow & P[2]. \end{array}$$

which has exact rows and columns. The fact that $\text{cone } g \cong X_{P,Q_1} \oplus \bigoplus_{i=2}^n Q_i$ follows from [Proposition 2.10 1](#). If $u = 0$, then $\text{cone } f \cong X_{P,Q_1} \oplus \text{cone } v$, where $v: \bigoplus_{i=1}^n P_i[1] \rightarrow \bigoplus_{i=2}^n Q_i[2]$ is a morphism in $\langle G \rangle_1$, whence $\text{cone } v \in \langle G \rangle_2$ by definition of $\langle G \rangle_2$. That is, if $u = 0$, then $\text{cone } f \cong X_{P,Q_1} \oplus \text{cone } v \in \langle G \rangle_2$.

To see that $u = (u_i : P_i[1] \rightarrow X_{P,Q_1})_{i=1}^n = 0$, observe that, if $Z_i \in \langle G \rangle_1$, then $P_i = 0$ by construction, and so $u_i = 0$. Otherwise, $P_i = R$, and the assumption that $f_i : P \rightarrow X_{R,S}$ is nonzero means that $S \leq P < R = P_i$ by Lemma 2.6. By the same lemma and since $P < P_i$ implies that $P[1] \leq P_i$, then $\mathcal{T}(P_i[1], X_{P,Q_1}) = 0$. Whence, $u_i = 0$. $\square$

Theorem 2.16. *Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1, and equipped with a linear generator G . Then, category $\langle G \rangle_2$ is triangulated, and hence equivalent to $\mathcal{T}$. In other words, the generation time of G is 1.*

Proof. To show that $\langle G \rangle_2$ is triangulated, it suffices to show that $\langle G \rangle_3 = \langle G \rangle_2$. That it is, it is enough to show that the cone of any morphism $f : P \rightarrow Z$ with $P \in \langle G \rangle_1$ and $Z \in \langle G \rangle_2$ is in $\langle G \rangle_2$. Since $\mathcal{T}$ is Krull-Schmidt, we may write $P = \bigoplus_{i=1}^n P_i$, where P_i are indecomposable objects, and $f = (f_i : P_i \rightarrow Z)_{i=1}^n$. We will show that $\text{cone } f \in \langle G \rangle_2$ by induction on n . The $n = 1$ case is Lemma 2.15. Suppose by induction that $\text{cone } f' \in \langle G \rangle_2$ for all $f' : Q \rightarrow Z$ where $Q = \bigoplus_{i=1}^{n-1} Q_i$ for indecomposable objects $Q_i \in \langle G \rangle_1$.

By reindexing if necessary, assume $P_i \leq P_1$ for all $1 \leq i \leq n$. By applying the octahedral axiom, it is possible to construct the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 P_1 & \xrightarrow{(1,0,\dots)} & P & \longrightarrow & P/P_1 & \longrightarrow & P_1[1] \\
 \downarrow \wr & & \downarrow f & & \downarrow h & & \downarrow \\
 P_1 & \xrightarrow{g=(f_1,0,\dots)} & Z & \longrightarrow & \text{cone } g & \longrightarrow & P[1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{cone } f & \xrightarrow{\sim} & \text{cone } h & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P_1[1] & \longrightarrow & P[1] & \longrightarrow & P/P_1[1] & \longrightarrow & P_1[2].
 \end{array}$$

By Lemma 2.15, $\text{cone } g \in \langle G \rangle_2$. By the inductive hypothesis, since P/P_1 has exactly $n - 1$ indecomposable summands, $\text{cone } h \in \langle G \rangle_2$. Since $\text{cone } f \cong \text{cone } h$, it follows that $\text{cone } f \in \langle G \rangle_2$.

Therefore $\langle G \rangle_2$ is the smallest triangulated subcategory of $\mathcal{T}$ containing G . Hence $\langle G \rangle_2 = \mathcal{T}$. $\square$

As an immediate corollary of Theorem 2.16, we are able to compute the Rouquier dimension of certain triangulated categories equipped with a linear generator. Recall that the *Orlov spectrum* $\mathcal{O}(\mathcal{S})$ of a triangulated category $\mathcal{S}$ is the collection of generation times across all classical generators in $\mathcal{S}$ [Orl09]. Further, the *Rouquier dimension* of a triangulated category is the infimum of $\mathcal{O}(\mathcal{S})$ [Rou08].

Corollary 2.17. *Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1, and equipped with a linear generator G . Then, $\mathcal{T}$ has Rouquier dimension 1 if and only if $G \not\cong G[1]$.*

Proof. Suppose that $G \not\cong G[1]$. In view of Theorem 2.16, it suffices to show that the Rouquier dimension of $\mathcal{T}$ is greater than zero. That is, it suffices to show that there does not exist an object $G' \in \mathcal{T}$ such that $\langle G' \rangle_1 = \mathcal{T}$.

Observe first that, since $\mathcal{T}$ is Krull-Schmidt, G' is a finite direct sum of indecomposable objects in $\mathcal{T}$, and so $\langle G' \rangle_1$ has finitely many indecomposable objects up to suspension. However, when $G \not\cong G[1]$, then we can fix $P \in \langle G \rangle_1$ such that $P \not\cong P[1]$, which in turn, implies that $P[i - 1] < P[i]$ for all $i \in \mathbb{Z}$. Thus, by Corollary 2.7 the collection of objects $\{X_{P,P[i]}\}_{i < 0}$ is infinite. Again, by Corollary 2.7, since P is fixed, then $X_{P,P[i]} \not\cong X_{P,P[i']}[j]$, for all $i, i' < 0$ and $j \in \mathbb{Z}$. Hence, the collection $\{X_{P,P[i]}\}_{i < 0}$ is an infinite collection of indecomposable objects which are not isomorphic and are not suspensions of each other. It thus follows that this collection is not in $\langle G' \rangle_1$, from which we conclude that $\langle G' \rangle_1 \subsetneq \mathcal{T}$ for all objects $G' \in \mathcal{T}$, and so the Rouquier dimension of $\mathcal{T}$ is greater than zero.

Now consider the case $G \cong G[1]$. Then, $\langle G \rangle_1$ has finitely many isomorphism classes of indecomposable objects. Hence, it follows from Proposition 2.10 and Theorem 2.16 that $\mathcal{T}$ has finitely many isomorphism classes of indecomposable objects $\{P_i\}_{i=1}^n$. Hence, by letting $G' = \bigoplus_{i=1}^n P_i$, then $\mathcal{T} = \langle G' \rangle_1$ and we may conclude that $\mathcal{T}$ has Rouquier dimension 0 if there exists a linear generator $G \cong G[1]$. $\square$

Corollary 2.18. *Let $\mathcal{T}$ be a triangulated category satisfying [Setup 2.1](#), and equipped with a linear generator G . Let $X \in \mathcal{T}$ be an indecomposable object, then there exists a triangle*

$$P \rightarrow X \rightarrow Q \rightarrow P[1],$$

such that $P, Q \in \langle G \rangle_1$, with P indecomposable, and Q either indecomposable or trivial.

Proof. Suppose $X \in \langle G \rangle_1$, then we may take $X \cong P$ and $Q = 0$ to obtain the identity triangle for X . Now suppose that $X \notin \langle G \rangle_1$, then $X \cong X_{Q[-1],P}$ by [Proposition 2.10](#) and [Theorem 2.16](#). $\square$

2.4. Cones

Throughout this section, let $\mathcal{T}$ be a triangulated category satisfying [Setup 2.1](#), and equipped with a linear generator G . The properties of linear generators are strong enough to afford control over cones of morphisms between indecomposable objects in $\mathcal{T}$. In this section, we present such results.

Lemma 2.19. *Let $Q \leq R < P$ be indecomposables in $\langle G \rangle_1$, and let $g : R \rightarrow X_{P,Q}$ be a non-zero morphism. Then, $\text{cone } g \cong P[1] \oplus X_{R,Q}$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} R & \longrightarrow & Q & \longrightarrow & X_{R,Q} & \longrightarrow & R[1] \\ \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \\ R & \xrightarrow{g} & X_{P,Q} & \longrightarrow & \text{cone } g & \longrightarrow & R[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P[1] & \xrightarrow{\sim} & P[1] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow u & & \downarrow \\ R[1] & \longrightarrow & Q[1] & \longrightarrow & X_{R,Q}[1] & \longrightarrow & R[2]. \end{array}$$

where the top leftmost square exists and commutes by [Lemma 2.8](#). Thus, the diagram commutes and has exact rows and columns by the octahedral axiom. Moreover, the morphism u is trivial, as $\mathcal{T}(P[1], X_{R,Q}[1]) = 0$ by [Lemma 2.6](#), hence $\text{cone } g \cong P[1] \oplus X_{R,Q}$. $\square$

Lemma 2.20. *Let $Q[1] < R \leq P[1]$, and let $g : X_{P,Q} \rightarrow R$ be a non-zero morphism. Then $\text{cone } g \cong Q[1] \oplus X_{P[1],R}$.*

Proof. By [Lemma 2.9](#), the top leftmost square in the diagram

$$\begin{array}{ccccccc} X_{P,Q} & \longrightarrow & P[1] & \longrightarrow & Q[1] & \longrightarrow & X_{P,Q}[1] \\ \downarrow \wr & & \downarrow h & & \downarrow & & \downarrow \\ X_{P,Q} & \xrightarrow{g} & R & \longrightarrow & \text{cone } g & \longrightarrow & X_{P,Q}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_{P[1],R} & \xrightarrow{\sim} & X_{P[1],R} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_{P,Q}[1] & \longrightarrow & P[2] & \longrightarrow & Q[2] & \longrightarrow & X_{P,Q}[2]. \end{array}$$

exists and commutes. Thus, the diagram exists, commutes and has exact rows and columns by the octahedral axiom.

Since $\mathcal{T}(X_{P[1],R}, Q[2]) = 0$ by [Lemma 2.5](#), then $\text{cone } g \cong Q[1] \oplus X_{P[1],R}$. $\square$

Lemma 2.21. *Let $P, Q \in \langle G \rangle_1$ and $X_{R,S} \in \mathcal{T}$ be indecomposable objects. Then, a nonzero morphism $f: P \rightarrow Q$ does not factor through $X_{R,S}$.*

Applying the functor $\mathcal{T}(-, Q)$ to the triangle

induced by [Lemma 2.19](#) yields the exact sequence

By [Definition 2.2](#), and [Lemma 2.5](#), and [Lemma 2.13](#) this simplifies to the following exact sequence of vector spaces.

This is only possible if $\mathcal{T}(g, Q) = 0$. That is, for any pair of maps $g: P \rightarrow X_{R,S}$ and $h: X_{R,S} \rightarrow Q$, then $h \cdot g = 0 \neq f$, and we may conclude that f does not factor through W . $\square$

Proof. Suppose first that $Q \leq Q' \leq R < P \leq P'$. Then, by [Lemma 2.6](#), there is a nonzero morphism $g: R \rightarrow X_{P', Q'}$. Moreover, by [Lemma 2.19](#), there is a triangle

By applying the functor $\mathcal{T}(-, X_{P,O})$ to this triangle, we obtain the long exact sequence

We claim that $\mathcal{T}(g, X_{P,Q})$ is an isomorphism, so that we may conclude that f factors through g . Well, by [Lemmas 2.6](#) and [2.13](#), the Hom spaces $\mathcal{T}(P'[1] \oplus X_{R,Q'}, X_{P,Q})$ and $\mathcal{T}(P' \oplus X_{R,Q'}[-1], X_{P,Q})$ vanish, and so we may conclude that $\mathcal{T}(g, X_{P,Q})$ is an isomorphism.

To prove the converse, suppose that $Q \leq Q' \leq R < P \leq P'$ does not hold. Observe that since f is nonzero, it must be that $Q \leq R < P$ by [Lemma 2.6](#). Moreover, if $\mathcal{T}(R, X_{P', Q'}) = 0$, then f cannot factor through $X_{P', Q'}$. Hence, it suffices to consider cases where $\mathcal{T}(R, X_{P', Q'}) \neq 0$. By [Lemma 2.6](#), this means that we may assume $Q' \leq R < P$.

Similarly, it suffices to consider cases where $\mathcal{T}(X_{P',Q'}, X_{P,Q}) \neq 0$. By [Lemma 2.13](#), we may assume that either (1) $Q \leq Q' < P \leq P'$ or (2) $Q'[1] < Q \leq P'[1] < P$. Combining (1) with the inequality $Q' \leq R < P$ implies that $Q \leq Q' \leq R < P \leq P'$, which is a contradiction since we assumed that this does not hold. Hence, (2) must hold. Combining (2) and the fact that $Q \leq R < P'$ implies $Q'[1] < Q \leq R < P' \leq P'[1] < P$. However, in this case, [Lemmas 2.6](#) and [2.13](#) imply that the exact sequence (2.C), seen as a sequence of vector spaces, becomes

$$k \xrightarrow{\mathcal{T}(g, X_P, Q)} k \rightarrow k^2 \rightarrow k.$$

This is only possible if $\mathcal{T}(g, X_{P,Q}) = 0$. Since this is true for any $g : R \rightarrow X_{P',Q'}$, given any pair of morphisms $g \in \mathcal{T}(R, X_{P',Q'})$ and $h \in \mathcal{T}(X_{P',Q'}, X_{P,Q})$, then $h \cdot g = 0 \neq f$. Thus, f does not factor through $X_{P',Q'}$, as required. $\square$

Lemma 2.23. *Let $R, X_{P,Q} \in \mathcal{T}$ be indecomposable objects, such that there exists a non-zero morphism $f : X_{P,Q} \rightarrow R$. Then f factors through $X_{P',Q'}$ if and only if $Q'[1] \leq Q[1] < R \leq P'[1] \leq P[1]$.*

Proof. The proof is analogous to that of Lemma 2.22, by applying the functor $\mathcal{T}(X_{P,Q}, -)$ to the triangle

$$X_{P',Q'} \rightarrow R \rightarrow Q'[1] \oplus X_{P',R} \rightarrow X_{P',Q'}[1].$$

$\square$

Lemma 2.24. *Let $X_{P,Q}, X_{P',Q'} \in \mathcal{T}$ and $R \in \langle G \rangle_1$ be indecomposable objects. Let $f : X_{P,Q} \rightarrow X_{P',Q'}$ be a non-zero morphism. Then f factors through R if and only if $Q[1] < Q' \leq R \leq P[1] < P'$.*

Proof. Suppose that $Q[1] < Q' \leq R \leq P[1] < P'$. Then, since $Q' \leq R < P'$, it follows from Lemma 2.6 that there is a nonzero morphism $g : R \rightarrow X_{P',Q'}$. Therefore, applying $\mathcal{T}(X_{P,Q}, -)$ to the triangle

$$R \xrightarrow{g} X_{P',Q'} \rightarrow P'[1] \oplus X_{R,Q'} \rightarrow R[1]$$

given by Lemma 2.19, induces the long exact sequence

$$\cdots \rightarrow \mathcal{T}(X_{P,Q}, P' \oplus X_{R,Q'}[-1]) \rightarrow \mathcal{T}(X_{P,Q}, R) \xrightarrow{\mathcal{T}(X_{P,Q}, g)} \mathcal{T}(X_{P,Q}, X_{P',Q'}) \rightarrow \mathcal{T}(X_{P,Q}, P'[1] \oplus X_{R,Q'}) \rightarrow \cdots$$

Therefore, it suffices to prove that $\mathcal{T}(X_{P,Q}, g)$ is an isomorphism. Well, it follows from Lemma 2.6 and Lemma 2.13 that under the assumptions $Q[1] < Q' \leq R \leq P[1] < P'$, then the Hom spaces $\mathcal{T}(X_{P,Q}, P' \oplus X_{R,Q'}[-1])$ and $\mathcal{T}(X_{P,Q}, P'[1] \oplus X_{R,Q'})$ vanish, so that we may conclude that $\mathcal{T}(X_{P,Q}, g)$ is an isomorphism.

To prove that converse, assume that $Q[1] < Q' \leq R \leq P[1] < P'$ does not hold. Observe that we may assume $\mathcal{T}(R, X_{P',Q'}) \neq 0$, $\mathcal{T}(X_{P,Q}, R) \neq 0$ and $\mathcal{T}(X_{P,Q}, X_{P',Q'}) \neq 0$, since otherwise, the statement is clear. Thus, we may assume that $Q' \leq R < P'$, $Q[1] < R \leq P[1]$ and that either (1) $Q' \leq Q < P' \leq P$ or (2) $Q[1] < Q' \leq P[1] < P'$. Combining (2) with $Q' \leq R \leq P[1]$ implies that $Q[1] < Q' \leq R \leq P[1] < P'$, which is a contradiction since we assumed this does not hold. Hence, (1) must hold. Combining (1) and $Q[1] \leq R < P'$ implies that $Q' \leq Q \leq Q[1] < R < P' \leq P$ holds. Similarly to the proof of Lemma 2.22, this implies that there is an exact sequence of vector spaces

$$k \xrightarrow{\mathcal{T}(X_{P,Q}, g)} k \rightarrow k^2 \rightarrow k$$

and so $\mathcal{T}(X_{P,Q}, g) = 0$. That is, the morphism f cannot factor through g . $\square$

We may now complete our computations for the cone of a morphism between indecomposable objects.

Lemma 2.25. *Let $X_{P,Q}, X_{P',Q'} \in \mathcal{T}$ be indecomposable objects and let $f : X_{P,Q} \rightarrow X_{P',Q'}$ be a non-zero morphism. Then,*

$$\text{cone } f \cong \begin{cases} X_{P,P'}[1] \oplus X_{Q,Q'} & \text{if } Q' \leq Q < P' \leq P, \\ X_{P',Q[1]} \oplus X_{P[1],Q'} & \text{if } Q[1] < Q' \leq P[1] < P'. \end{cases}$$

Equivalently, $\text{cone } f \cong X_{P,P'}[1] \oplus X_{Q,Q'}$ if f is a forwards morphism. Else, f is a backwards morphism and $\text{cone } g \cong X_{P',Q[1]} \oplus X_{P[1],Q'}$.

Proof. Let $Q' \leq Q < P' \leq P$, then by Lemma 2.22, the top left square of the diagram

$$\begin{array}{ccccccc}
Q & \longrightarrow & X_{P,Q} & \longrightarrow & P[1] & \longrightarrow & Q[1] \\
\downarrow \wr & & \downarrow f & & \downarrow u=(u_1, u_2) & & \downarrow \\
Q & \longrightarrow & X_{P',Q'} & \xrightarrow{h} & P'[1] \oplus X_{Q,Q'} & \longrightarrow & Q[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{cone } f & \xrightarrow{\sim} & \text{cone } f & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q[1] & \longrightarrow & X_{P,Q}[1] & \longrightarrow & P[2] & \longrightarrow & Q[2].
\end{array}$$

commutes. Hence, the full diagram with exact rows and columns exists and commutes by the octahedral axiom. By Lemma 2.6, $\mathcal{T}(P[1], X_{Q,Q'}) = 0$. Hence, if $u \neq 0$, then $\text{cone } f \cong \text{cone } u_1 \oplus X_{Q,Q'} \cong X_{P,P'}[1] \oplus X_{Q,Q'}$ by the definition of $X_{P,P'}$. To see that $u \neq 0$, observe that applying $\mathcal{T}(X_{P,Q}, -)$ to the second row of the diagram yields the exact sequence

$$\mathcal{T}(X_{P,Q}, Q) \rightarrow \mathcal{T}(X_{P,Q}, X_{P',Q'}) \xrightarrow{\mathcal{T}(X_{P,Q}, h)} \mathcal{T}(X_{P,Q}, P'[1] \oplus X_{Q,Q'}) \rightarrow \mathcal{T}(X_{P,Q}, Q[1])$$

It therefore follows from Lemma 2.5 that $\mathcal{T}(X_{P,Q}, h)$ is an isomorphism. Hence, $0 \neq h \cdot f = u \cdot g$ and we may conclude that $u \neq 0$.

Next, suppose that $Q[1] < Q' \leq P[1] < P'$, then by Lemma 2.24, the top left square of the following diagram commutes.

$$\begin{array}{ccccccc}
X_{P,Q} & \longrightarrow & Q' & \longrightarrow & Q[1] \oplus X_{P[1],Q'} & \longrightarrow & X_{P,Q}[1] \\
\downarrow \wr & & \downarrow & & \downarrow & & \downarrow \\
X_{P,Q} & \xrightarrow{f} & X_{P',Q'} & \longrightarrow & \text{cone } f & \longrightarrow & X_{P,Q}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P'[1] & \longrightarrow & P'[1] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow w=(w_1, w_2) & & \downarrow \\
X_{P,Q}[1] & \longrightarrow & Q'[1] & \longrightarrow & Q[2] \oplus X_{P[1],Q'}[1] & \longrightarrow & X_{P,Q}[2].
\end{array}$$

Therefore, this diagram commutes and has exact columns and rows by the octahedral axiom. By assumption, $P[1] < P'$ and so Lemma 2.6 implies that $\mathcal{T}(P', X_{P[1],Q'}) = 0$. Hence, $w_2 = 0$. It follows that $\text{cone } f \cong \text{cone } w_1[-1] \oplus X_{P[1],Q'}$. That is, $\text{cone } f \cong X_{P',Q[1]} \oplus X_{P[1],Q'}$, as required.

The final statement follows directly from Lemma 2.13. $\square$

2.5. Factorisation Properties, Composition and Triangles

In this section, we consolidate the results established thus far into a handful of propositions that are easier to track. Our first result summarises the factorisation properties of morphisms between indecomposable objects.

Proposition 2.26. *Let $Y, Z \in \mathcal{T}$ be non-isomorphic indecomposable objects such that $\mathcal{T}(Y, Z) \cong k$. Then, any nonzero morphism $f : Y \rightarrow Z$ factors through an indecomposable object W if and only if there exist non-zero morphisms $g : Y \rightarrow W$ and $h : W \rightarrow Z$, such that*

- (1) *if f is a forward morphism, then g and h are both forward morphisms,*

- (2) if f is a backwards morphism, then one of g or h is a backwards morphism and the other is a forwards morphism.

Proof. There are eight cases for us to consider, where each of Y, Z, W are either of the form $R \in \langle G \rangle_1$ or of the form $X_{P,Q}$. However, seven of these cases have been considered.

First, when $Y, Z, W \in \langle G \rangle_1$, then the statement follows from [Definition 2.2](#) and [Lemma 2.12 \(1\)](#). Further, for $Y, W \in \langle G \rangle_1$ and $Z = X_{P,Q}$, then the statement is a consequence of [Lemmas 2.8](#) and [2.12 \(1\)](#). The third case, where $Y = X_{P,Q}$ and $Z, W \in \langle G \rangle_1$, is implied by [Lemmas 2.9](#) and [2.12 \(2\)](#). Next, when $Y \in \langle G \rangle_1$, $Z = X_{P,Q}$ and $W = X_{P',Q'}$, then the statement is entailed by [Lemmas 2.12, 2.13](#) and [2.22](#). The fifth case, where $Y = X_{P,Q}$, $Z \in \langle G \rangle_1$ and $W = X_{P',Q'}$, can be deduced by applying [Lemmas 2.12, 2.13](#) and [2.23](#). Additionally, the situation where $Y = X_{P,Q}$, $Z = X_{P',Q'}$ and $W \in \langle G \rangle_1$ results from applying [Lemmas 2.12, 2.13](#) and [2.24](#). The seventh case, where $Y, Z \in \langle G \rangle_1$ and $W \cong X_{P,Q}$ follows from [Lemmas 2.12](#) and [2.21](#).

Consider the final case, where $Y \cong X_{P,Q}$, $Z \cong X_{P',Q'}$ and $W \cong X_{R,S}$. If there does not exist a nonzero morphism $g: Y \rightarrow W$, then the statement is clear. Hence, we may assume that such a morphism exists. The functor $\mathcal{T}(-, Z)$ induces the exact sequence

$$\mathcal{T}(Y[1], Z) \rightarrow \mathcal{T}(\text{cone } g, Z) \rightarrow \mathcal{T}(W, Z) \xrightarrow{\mathcal{T}(g, Z)} \mathcal{T}(Y, Z) \rightarrow \mathcal{T}(\text{cone } g[-1], Z) \rightarrow \mathcal{T}(W[-1], Z) \quad (2.D)$$

We begin by showing that if either (1) or (2) hold, then f factors through W . Suppose first that (1) holds. That is, $f: Y \rightarrow Z$ is a forwards morphism and there exist forwards morphisms $g: Y \rightarrow W$ and $h: W \rightarrow Z$. We will show that f factors through W by proving that $\mathcal{T}(g, Z)$ is an isomorphism. Due to [Lemma 2.13](#), it must be that $Q' \leq S \leq Q < P' \leq R \leq P$. Moreover, by [Lemma 2.25](#), $\text{cone } g \cong X_{P,R} \oplus X_{Q,S}$. It thus follows from [Lemma 2.13](#) that $\mathcal{T}(\text{cone } g, Z) = 0$, and so (2.D) implies that there is an exact sequence

$$0 \rightarrow k \xrightarrow{\mathcal{T}(g, Z)} k$$

Hence, we may conclude that $\mathcal{T}(g, Z)$ is an isomorphism, as required.

Next, assume that $f: Y \rightarrow Z$ is a backwards morphism and that there exist a forwards morphism $g: Y \rightarrow W$ and a backwards morphism $h: W \rightarrow Z$. Again, we will show that f factors through W by showing that $\mathcal{T}(g, Z)$ is an isomorphism. Our assumptions imply that $S \leq Q[1] < Q' \leq R[1] \leq P[1] < P'$ by [Lemma 2.13](#) and that $\text{cone } g \cong X_{P,R} \oplus X_{Q,S}$ by [Lemma 2.25](#). As before, this implies that $\mathcal{T}(\text{cone } g, Z) = 0$ and we may deduce that $\mathcal{T}(g, Z)$ is an isomorphism.

Finally, assume that $f: Y \rightarrow Z$ is a backwards morphism and that there exist a backwards morphism $g: Y \rightarrow W$ and a forwards morphism $h: W \rightarrow Z$. Then, $Q[1] < Q' \leq S \leq P[1] < P' \leq R$ by [Lemma 2.13](#) and $\text{cone } g \cong X_{R,Q} \oplus X_{P,S}$ by [Lemma 2.25](#). Therefore, $\mathcal{T}(\text{cone } g, Z) = 0$ and $\mathcal{T}(g, Z)$ is an isomorphism. Consequently, if either (1) or (2) hold, then f factors through W .

To prove the converse, suppose that $f: Y \rightarrow Z$ is such that $f = h \circ g$ where $g: Y \rightarrow W$ and $h: W \rightarrow Z$. Assume first that f is a forwards morphism. We will show that (1) holds by showing that all other cases lead to a contradiction. If f is forwards and both g and h are backwards, then [Lemma 2.13](#) implies that $S < Q' \leq Q \leq Q[1] < S$, which is a contradiction. Whence, g and h cannot both be backwards.

On the other hand, if f is forwards, g is backwards and h is forwards, then $Q' \leq Q[1] < S < P' \leq P[1] < R$ by [Lemma 2.13](#). By the same lemma, (2.D) induces the exact sequence

$$\mathcal{T}(Y[1], Z) \rightarrow k^2 \rightarrow k \xrightarrow{\mathcal{T}(g, Z)} k.$$

Since $\mathcal{T}(Y[1], Z)$ is at most one dimensional, then $\mathcal{T}(g, Z)$ cannot be an isomorphism, whence it is zero. We conclude that $f = h \circ g = 0$, which is a contradiction.

By the same argument, it cannot be that f is a forwards morphism, g is a forwards morphism and h is a backwards morphism. We thus conclude that if f is a forwards morphism, then both g and h are forwards morphisms. That is, (1) holds.

Next, suppose that f is a backwards morphism. We will show that (2) holds by showing that the two other possible cases lead to contradictions. If both g and h are backwards, then by Lemma 2.13, (2.D) induces again the exact sequence (2.5). Therefore, it must be that $\mathcal{T}(g, Z) = 0$ and $f = h \circ g = 0$, which is a contradiction. Furthermore, if g and h are both forwards, then Lemma 2.13 implies that $S \leq Q \leq Q[1] < Q' \leq S$, which is a contradiction. Consequently, if f is a backwards morphism, then (2) holds. $\square$

With Proposition 2.26 in hand, we are able to prove that for all indecomposable objects $X, Y \in \mathcal{T}$, there is a choice of generators $\alpha^{X,Y} \in \mathcal{T}(X, Y)$ which respects composition in a suitable sense. This is a subtle result, but it is the key technology we use to write down functors between triangulated categories with linear generators. It is also helpful for describing certain triangles in $\mathcal{T}$.

Proposition 2.27. *Let $\text{ind } \mathcal{T}$ denote the set of indecomposable objects in $\mathcal{T}$. Then, there is a choice of morphisms $\{\alpha^{Y,Z} : Y \rightarrow Z\}_{Y, Z \in \text{ind } \mathcal{T}}$ such that for any pair of indecomposable objects $Y, Z \in \mathcal{T}$*

$$\alpha^{Y,Z} = \alpha^{W,Z} \alpha^{Y,W},$$

whenever $\alpha^{Y,Z} : Y \rightarrow Z$ factors through an indecomposable object W .

Proof. The proof of this result is quite long and subtle, so we postpone it to Appendix A. $\square$

Finally, we may completely determine the triangles containing at least two indecomposable terms up to isomorphism of triangles.

Corollary 2.28. *Let $\{\alpha^{Y,Z}\}$ be a choice of generators as in Proposition 2.27. Then,*

(1) *Any triangle*

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

in $\mathcal{T}$ with three indecomposable objects is isomorphic to the triangle

$$X \xrightarrow{\alpha^{X,Y}} Y \xrightarrow{\alpha^{Y,Z}} Z \xrightarrow{\delta \alpha^{Z,X[1]}} X[1]$$

where $\delta \in k^\times$ is a scalar that does not depend (up to a sign) on the objects X, Y or Z .

(2) *Any non-split triangle in $\mathcal{T}$ with exactly two indecomposable summands is isomorphic to the triangle*

$$X \xrightarrow{\begin{pmatrix} \alpha^{X,Z} \\ \alpha^{X,Z'} \end{pmatrix}} Z \oplus Z' \xrightarrow{(\alpha^{Z,Y}, -\alpha^{Z',Y})} Y[1] \xrightarrow{\delta \alpha^{Y,X[1]}} X[1],$$

for indecomposable objects Z, Z' and $\delta \in k^\times$ is the same scalar as in (1).

Proof. Let us begin by proving (1). Since morphism spaces between indecomposable objects in $\mathcal{T}$ are at most one-dimensional, it follows that any triangle in $\mathcal{T}$ with three indecomposable objects is of the form

$$X \xrightarrow{\lambda_1 \alpha^{X,Y}} Y \xrightarrow{\lambda_2 \alpha^{Y,Z}} Z \xrightarrow{\lambda_3 \alpha^{Z,X[1]}} X[1]$$

with $\lambda_1, \lambda_2, \lambda_3 \in k^\times$. We may thus construct the following isomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{\lambda_1 \alpha^{X,Y}} & Y & \xrightarrow{\lambda_2 \alpha^{Y,Z}} & Z & \xrightarrow{\lambda_3 \alpha^{Z,X[1]}} & X[1] \\ \parallel & & \downarrow \lambda_1^{-1} \text{id}_Y & \wr & \downarrow \lambda_2^{-1} \lambda_1^{-1} \text{id}_Z & \wr & \parallel \\ X & \xrightarrow{\alpha^{X,Y}} & Y & \xrightarrow{\alpha^{Y,Z}} & Z & \xrightarrow{\lambda_1 \lambda_2 \lambda_3 \alpha^{Z,X[1]}} & X[1] \end{array}$$

as required.

It remains to show that δ does not depend on X, Y or Z . Due to the above and [Section 2.4](#), there we only need to consider four cases:

$$\begin{array}{ccccccc}
 P & \xrightarrow{\alpha^{P,Q}} & Q & \xrightarrow{\alpha^{Q,X_{P,Q}}} & X_{P,Q} & \xrightarrow{\delta \alpha^{X_{P,Q},P[1]}} & P[1] \\
 X_{R,Q} & \xrightarrow{\alpha^{X_{R,Q},X_{P,R}[1]}} & X_{P,R}[1] & \xrightarrow{\alpha^{X_{P,R},X_{P,Q}}} & X_{P,Q}[1] & \xrightarrow{\delta \alpha^{X_{P,Q},X_{R,Q}}} & X_{R,Q}[1] \\
 X_{P,S} & \xrightarrow{\alpha^{X_{P,S},X_{Q,S}}} & X_{Q,S} & \xrightarrow{\alpha^{X_{Q,S},X_{P,Q}}} & X_{P,Q}[1] & \xrightarrow{\delta \alpha^{X_{P,Q},X_{P,S}}} & X_{P,S}[1] \\
 X_{R,P}[1] & \xrightarrow{\alpha^{X_{R,P}[1],X_{R,Q}}} & X_{R,Q} & \xrightarrow{\alpha^{X_{R,Q},X_{P,Q}}} & X_{P,Q}[1] & \xrightarrow{\delta \alpha^{X_{P,Q},X_{R,P}[1]}} & X_{R,P}[1][1]
 \end{array}$$

for objects $P, Q, R, S \in \langle G \rangle_1$.

As in the proof of [Proposition 2.27](#), there are objects $\bar{P}, \bar{Q} \in \langle G \rangle_1$ and $\bar{X} = X_{\bar{P},\bar{Q}}$ such that for any $X_{P,Q}$, there are forwards morphisms $X_{\bar{P},Q} \rightarrow X_{P,Q}$ and $X_{\bar{P},Q} \rightarrow \bar{X}$. Hence, by the definition of forwards morphisms, the following commutative diagram exists

$$\begin{array}{ccccccc}
 P & \xrightarrow{\alpha^{P,Q}} & Q & \xrightarrow{\alpha^{Q,X_{P,Q}}} & X_{P,Q} & \xrightarrow{\delta \alpha^{X_{P,Q},P[1]}} & P[1] \\
 \alpha^{\bar{P},P} \uparrow & & \alpha^{Q,Q} \uparrow & & \alpha^{X_{\bar{P},Q},X_{P,Q}} \uparrow & & \alpha^{\bar{P},P[1]} \uparrow \\
 \bar{P} & \xrightarrow{\alpha^{\bar{P},Q}} & Q & \xrightarrow{\alpha^{Q,X_{\bar{P},Q}}} & X_{\bar{P},Q} & \xrightarrow{\delta' \alpha^{X_{\bar{P},Q},\bar{P}[1]}} & \bar{P}[1] \\
 \alpha^{\bar{P},\bar{P}} \downarrow & & \alpha^{Q,\bar{Q}} \downarrow & & \alpha^{X_{\bar{P},Q},\bar{X}} \downarrow & & \alpha^{\bar{P},\bar{P}[1]} \downarrow \\
 \bar{P} & \xrightarrow{\alpha^{\bar{P},\bar{Q}}} & \bar{Q} & \xrightarrow{\alpha^{\bar{Q},\bar{X}}} & X_{\bar{P},\bar{Q}} & \xrightarrow{\delta'' \alpha^{X_{\bar{P},\bar{Q}},\bar{P}[1]}} & \bar{P}[1]
 \end{array}$$

from which we may conclude that $\delta = \delta' = \delta''$. Thus, the scalar δ is the same for all triangles of the form

$$P \xrightarrow{\alpha^{P,Q}} Q \xrightarrow{\alpha^{Q,X_{P,Q}}} X_{P,Q} \xrightarrow{\delta \alpha^{X_{P,Q},P[1]}} P[1].$$

Now, consider an arbitrary triangle with three indecomposable objects. Then, a case by case analysis shows that there are objects P, Q such that (up to rotation) it is always possible to construct the left hand diagram below or the right hand diagram below.

$$\begin{array}{ccccccc}
 X & \xrightarrow{\alpha^{X,Y}} & Y & \xrightarrow{\alpha^{Y,Z}} & Z & \xrightarrow{\delta' \alpha^{Z,X[1]}} & X[1] \\
 \downarrow \alpha^{X,Q} & & \downarrow \alpha^{Y,X_{P,Q}} & & \downarrow h = \alpha^{Z,P[1]} & & \downarrow \alpha^{X,Q[1]} \\
 Q & \xrightarrow{\alpha^{Q,X_{P,Q}}} & X_{P,Q} & \xrightarrow{\alpha^{X_{P,Q},P[1]}} & P[1] & \xrightarrow{\delta \alpha^{P[1],Q[1]}} & Q[1]
 \end{array}
 \quad
 \begin{array}{ccccccc}
 X & \xrightarrow{\alpha^{X,Y}} & Y & \xrightarrow{\alpha^{Y,Z}} & Z & \xrightarrow{\delta' \alpha^{X[1],Z}} & X[1] \\
 \uparrow \alpha^{Q,X} & & \uparrow \alpha^{X_{P,Q},Y} & & \uparrow h' = \alpha^{P[1],Z} & & \uparrow \alpha^{Q,X[1]} \\
 Q & \xrightarrow{\alpha^{Q,X_{P,Q}}} & X_{P,Q} & \xrightarrow{\alpha^{X_{P,Q},P[1]}} & P[1] & \xrightarrow{\delta \alpha^{P[1],Q[1]}} & Q[1]
 \end{array}$$

In either diagram, the left most square commutes, and so the axioms of triangulated categories implies that there must exist an $h: Z \rightarrow P[1]$ or $h': P[1] \rightarrow Z$ making either diagram into a morphism of triangles. By commutativity of the central square, this forces $h = \alpha^{Z,P[1]}$ or $h' = \alpha^{P[1],Z}$. Whence, because the right most square commutes, we may conclude that $\delta' = \delta$. That is, δ is independent of the objects X, Y and Z .

Next, let us prove (2). Consider a non-split triangle with exactly 2 indecomposable summands. Then, we may assume all morphisms in the triangle are nonzero. Moreover, as in (1), we may use [Proposition 2.27](#) and the fact that Hom-sets

between indecomposable objects in $\mathcal{T}$ have dimension at most one to write

$$X \xrightarrow{\begin{pmatrix} \lambda_1 \alpha^{X,Z} \\ \lambda_2 \alpha^{X,Z'} \end{pmatrix}} Z \oplus Z' \xrightarrow{\begin{pmatrix} \lambda_3 \alpha^{Z,Y} & \lambda_4 \alpha^{Z',Y} \end{pmatrix}} Y \xrightarrow{\lambda_5 \alpha^{Y,X[1]}} X[1],$$

for $\lambda_i \in k^\times$ with $1 \leq i \leq 5$.

There are four cases to consider, depending on whether or not $X, Y \in \langle G \rangle_1$. By [Lemmas 2.19, 2.20](#) and [2.25](#), we have one of the four following triangles;

$$R \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} Q[1] \oplus X_{P[1],R} \xrightarrow{(f_2, g_2)} X_{P,Q}[1] \xrightarrow{h} R[1], \quad (2.E)$$

$$X_{P,Q} \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} P[1] \oplus X_{R,Q} \xrightarrow{(f_2, g_2)} R[1] \xrightarrow{h} X_{P,Q}[1], \quad (2.F)$$

$$X_{R,S} \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} X_{P,R}[1] \oplus X_{Q,S} \xrightarrow{(f_2, g_2)} X_{P,Q}[1] \xrightarrow{h} X_{R,S}[1], \quad (2.G)$$

$$X_{R,S} \xrightarrow{\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}} X_{R,Q}[1] \oplus X_{P[1],S} \xrightarrow{(f_2, g_2)} X_{P,Q}[1] \xrightarrow{h} X_{R,S}[1]. \quad (2.H)$$

It is straightforward to check, by using [Lemmas 2.5, 2.6](#) and [2.13](#) and the assumption that $Z \oplus Z'$ is not indecomposable, that in each case there are nonzero morphisms $X \rightarrow Y$. Moreover, at least one of f_1 or f_2 and at least one g_1 or g_2 are forwards morphisms. Hence, the compositions $f_2 \circ f_1$ and $g_2 \circ g_1$ are nonzero by [Proposition 2.26](#). Therefore, $f_2 \circ f_1 + g_2 \circ g_1 = (\lambda_1 \lambda_3 + \lambda_2 \lambda_4) \alpha^{X,Y}$ by [Proposition 2.27](#). However, $f_2 \circ f_1 + g_2 \circ g_1 = 0$, and so we may write $\lambda_4 = -\lambda_1 \lambda_3 \lambda_2^{-1}$. We can, thus, construct the isomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{\begin{pmatrix} \lambda_1 \alpha^{X,Z} \\ \lambda_2 \alpha^{X,Z'} \end{pmatrix}} & Z \oplus Z' & \xrightarrow{\begin{pmatrix} \lambda_3 \alpha^{Z,Y} & -\lambda_1 \lambda_3 \lambda_2^{-1} \alpha^{Z',Y} \end{pmatrix}} & Y & \xrightarrow{\lambda_5 \alpha^{Y,X[1]}} & X[1] \\ \parallel & & \downarrow \begin{pmatrix} \lambda_1^{-1} \alpha^{Z,Z} & 0 \\ 0 & \lambda_2^{-1} \alpha^{Z',Z'} \end{pmatrix} & & \downarrow (\lambda_1 \lambda_3)^{-1} \alpha^{Y,Y} & & \parallel \\ X & \xrightarrow{\begin{pmatrix} \alpha^{X,Z} \\ \alpha^{X,Z'} \end{pmatrix}} & Z \oplus Z' & \xrightarrow{\begin{pmatrix} \alpha^{Z,Y} & -\alpha^{Z',Y} \end{pmatrix}} & Y & \xrightarrow{\lambda_1 \lambda_3 \lambda_5 \alpha^{Y,X[1]}} & X[1] \end{array}$$

as required.

It remains to show that δ is independent of X, Y, Z , and Z' . Consider a triangle with exactly two indecomposable summands and assume that it is written as in one of the four cases (2.E)-(2.H). Then, a straightforward case by case analysis shows that there is an indecomposable Y' such that there is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\begin{pmatrix} \alpha^{X,Z} \\ \alpha^{X,Z'} \end{pmatrix}} & Z \oplus Z' & \xrightarrow{\begin{pmatrix} \alpha^{Z,Y} & -\alpha^{Z',Y} \end{pmatrix}} & Y & \xrightarrow{\delta' \alpha^{Y,X[1]}} & X[1] \\ \parallel & & \downarrow (\alpha^{Z,Z}, 0) & & \downarrow \alpha^{Y,Y'} & & \parallel \\ X & \xrightarrow{\alpha^{X,Z}} & Z & \xrightarrow{\alpha^{Z,Y'}} & Y' & \xrightarrow{\delta \alpha^{Y',X[1]}} & X[1] \end{array}$$

where both rows are triangles, and all maps in the diagram are nonzero. Therefore, $\delta' = \delta$. $\square$

2.6. Functors Preserving Linear Generators

We aim to understand functors between triangulated categories satisfying [Setup 2.1](#) with linear generators, and what properties these functors may preserve under mild assumptions.

Recall that a functor $\mathcal{F} : \mathcal{T} \rightarrow \mathcal{S}$ between triangulated categories commutes with the respective suspension functors if there exists a natural isomorphism $\phi : \mathcal{F} \circ [1]_{\mathcal{T}} \xrightarrow{\sim} [1]_{\mathcal{S}} \circ \mathcal{F}$. Moreover, the functor $\mathcal{F}$ preserves a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]_{\mathcal{T}},$$

in $\mathcal{T}$ if

$$\mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Z) \xrightarrow{\mathcal{F}(h)} \mathcal{F}(X[1]_{\mathcal{T}}),$$

is a triangle in $\mathcal{S}$.

Theorem 2.29. *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories satisfying [Setup 2.1](#) and admitting linear generators G and G' , respectively. If there is a fully faithful additive functor $F : \langle G \rangle_1 \rightarrow \langle G' \rangle_1$ that commutes with the respective suspension functors, then there exists a fully faithful additive functor $\mathcal{F} : \mathcal{T} \rightarrow \mathcal{S}$ which commutes with suspension and preserves triangles with at least two indecomposable terms.*

Proof. Let $\{\alpha^{Y,Z}\}_{Y,Z \in \text{ind } \mathcal{T}}$ be a choice of generators in as in [Proposition 2.27](#). Then, by [Corollary A.2](#), the generators $\{F(\alpha^{Y,Z})\}_{Y,Z \in \langle G' \rangle_1}$ can be extended to a choice of generators $\{\beta^{Y,Z}\}_{Y,Z \in \mathcal{S}}$ which is compatible with composition.

Let $P, Q \in \langle G \rangle_1$ be non-isomorphic indecomposable objects. Then, by [Corollary 2.28](#), any triangle with vertices $P \neq Q$ in $\mathcal{T}$ and any triangle with vertices $F(P), F(Q) \in \mathcal{S}$ is isomorphic to

$$P \xrightarrow{\alpha^{P,Q}} Q \xrightarrow{\delta_{\mathcal{T}} \alpha^{Q,X_{P,Q}}} X_{P,Q} \xrightarrow{\alpha^{X_{P,Q},P[1]}} P[1] \quad (2.I)$$

$$F(P) \xrightarrow{\beta^{F(P),F(Q)}} F(Q) \xrightarrow{\delta_{\mathcal{S}} \beta^{F(Q),X_{F(P),F(Q)}}} X_{F(P),F(Q)} \xrightarrow{\beta^{X_{F(P),F(Q)},F(P)[1]}} F(P)[1] \quad (2.J)$$

respectively. Here, $\delta_{\mathcal{T}}, \delta_{\mathcal{S}} \in k^\times$ are scalars that only depend on the generators for Hom spaces that we have chosen in $\mathcal{T}$ and $\mathcal{S}$ and not on the objects P, Q .

We may "change basis" $\{\beta^{X,Y}\}$ in $\mathcal{S}$ in the following way. We first let $\gamma^{P',Q'} = \beta^{P',Q'}$ for all indecomposable objects $P', Q' \in \langle G' \rangle_1$. Then, for indecomposable objects $F(Q) \in \langle G' \rangle_1$ and $Z = X_{F(P),F(Q)} \notin \langle G' \rangle_1$, let $\gamma^{F(Q),Z} = \delta_{\mathcal{S}} \delta_{\mathcal{T}}^{-1} \beta^{F(Q),Z}$. Moreover, for indecomposable objects $Q' \in \langle G' \rangle_1$ and $Z' = X_{P',Q'} \notin \langle G' \rangle_1$ which are not in the essential image of F , let $\gamma^{Q',Z'} = \beta^{Q',Z'}$. Then, we use [Proposition 2.27](#) to complete this choice of generators to $\{\gamma^{Y,Z}\}_{Y,Z \in \text{ind } \mathcal{S}}$. By the proof of [Proposition 2.27](#), it is straightforward to check that $\gamma^{Z,F(P)[1]} = \beta^{Z,F(P)[1]}$. Under this choice of basis, we may rewrite the triangles (2.I) to

$$P \xrightarrow{\alpha^{P,Q}} Q \xrightarrow{\delta_{\mathcal{T}} \alpha^{Q,X_{P,Q}}} X_{P,Q} \xrightarrow{\alpha^{X_{P,Q},P[1]}} P[1] \quad (2.K)$$

$$F(P) \xrightarrow{\gamma^{F(P),F(Q)}} F(Q) \xrightarrow{\delta_{\mathcal{T}} \gamma^{F(Q),X_{F(P),F(Q)}}} X_{F(P),F(Q)} \xrightarrow{\gamma^{X_{F(P),F(Q)},F(P)[1]}} F(P)[1]. \quad (2.L)$$

Let us now define the functor $\mathcal{F} : \mathcal{T} \rightarrow \mathcal{S}$. Note that, since $\mathcal{S}$ and $\mathcal{T}$ are Krull-Schmidt, it suffices to define $\mathcal{F}$ on indecomposable objects. For $Y \in \mathcal{T}$ indecomposable, if $Y \in \langle G \rangle_1$ then we let $\mathcal{F}(Y) = F(Y)$. Otherwise, if $Y \notin \langle G \rangle_1$, then $Y = X_{R,S}$ for some $R, S \in \langle G \rangle_1$ and so we let $\mathcal{F}(Y) := X_{\mathcal{F}(R),\mathcal{F}(S)}$. For a morphism $\alpha^{Y,Z} \in \mathcal{T}$, we let $\mathcal{F}(\alpha^{Y,Z}) = \gamma^{Y,Z}$ and extend linearly over k .

Given indecomposable objects $Y, Z \in \mathcal{T}$, the linear map $\mathcal{F} : \mathcal{T}(Y, Z) \rightarrow \mathcal{S}(\mathcal{F}(Y), \mathcal{F}(Z))$ is bijective since it maps basis to basis. Since our categories are Krull-Schmidt, this is enough to conclude that $\mathcal{F}$ is fully faithful.

Moreover, $\mathcal{F}$ commutes with the suspension functor. To see this, let $P, Q \in \langle G \rangle_1$. Then, there is a natural isomorphism

$$\mathcal{F}(P[1]) = F(P[1]) \cong F(P)[1] = \mathcal{F}(P)[1]$$

by assumption on F . On the other hand, for $Y = X_{P',Q'} \in \mathcal{S}$ and $Z = X_{P,Q} \in \mathcal{T}$, recall that by [Remark 2.14](#), there are canonical isomorphisms $Y[1] \rightarrow X_{P'[1],Q'[1]}$ and $Z[1] \rightarrow X_{P[1],Q[1]}$, so that we may identify $Y[1] = X_{P'[1],Q'[1]}$ and $Z[1] = X_{P[1],Q[1]}$. Therefore,

$$\mathcal{F}(Z[1]) = \mathcal{F}(X_{P[1],Q[1]}) = X_{F(P[1]),F(Q[1])} \cong X_{F(P)[1],F(Q)[1]} = X_{F(P),F(Q)}[1] = \mathcal{F}(Z)[1].$$

where the isomorphisms involved are natural by assumption on F . Again, since our categories are Krull-Schmidt, this is enough to conclude that $\mathcal{F}$ commutes with suspension.

Finally, given the definition of $\mathcal{F}$, then [Corollary 2.28](#) implies that $\mathcal{F}$ preserves all triangles with three indecomposable objects. Hence, it remains to show that $\mathcal{F}$ preserves triangles with exactly two indecomposable terms. Well, the image of a triangle in $\mathcal{T}$ with two indecomposable summands under $\mathcal{F}$ is

$$\mathcal{F}(X) \xrightarrow{\begin{pmatrix} \gamma^{\mathcal{F}(X), \mathcal{F}(Z)} \\ \gamma^{\mathcal{F}(X), \mathcal{F}(Z')} \end{pmatrix}} \mathcal{F}(Z) \oplus \mathcal{F}(Z') \xrightarrow{(\gamma^{\mathcal{F}(Z), \mathcal{F}(Y)}, \gamma^{\mathcal{F}(Z'), \mathcal{F}(Y)})} \mathcal{F}(Y) \xrightarrow{\delta_S \gamma^{\mathcal{F}(Y), \mathcal{F}(X)}[1]} \mathcal{F}(X)[1]$$

which, by [Corollary 2.28](#) and our choice of $\{\gamma^{Y, Z}\}$, is a triangle in $\mathcal{S}$. $\square$

We find an immediate corollary to [Theorem 2.29](#).

Corollary 2.30. *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories satisfying [Setup 2.1](#) and admitting linear generators G and G' , respectively. Let $F: \langle G \rangle_1 \rightarrow \langle G' \rangle_1$ be a fully faithful additive functor which commutes with suspension. Then, the functor $\mathcal{F}$ constructed in [Theorem 2.29](#) is an additive equivalence if and only if F is an equivalence.*

Corollary 2.31. *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories satisfying [Setup 2.1](#). Suppose that $\mathcal{T}$ admits a linear generator G and that $\mathcal{S}$ admits a generator G' . Let $F: \langle G \rangle_1 \rightarrow \langle G' \rangle_1$ be an additive equivalence which commutes with suspension and is such that $F(G) \cong G'$. Then, G' is a linear generator of $\mathcal{S}$ and the functor $\mathcal{F}$ constructed in [Theorem 2.29](#) is an additive equivalence.*

Proof. In view of [Corollary 2.30](#), it suffices to prove that G' is a linear generator of $\mathcal{S}$. Let $G = \bigoplus_{i=1}^n G_i$, where G_i are indecomposable. Then, since $G' = F(G)$ and F is an additive equivalence, $G' = \bigoplus_{i=1}^n F(G_i)$, with $F(G_i)$ indecomposable. Since F is an equivalence which commutes with suspension, it is thus straightforward to check that G' satisfies the conditions of [Definition 2.2](#). $\square$

Remark 2.32. It is possible that the functor $\mathcal{F}$ constructed in [Theorem 2.29](#) is in fact a triangulated functor which induces a triangulated equivalence under the conditions of [Corollary 2.30](#). In fact, it is even possible to show that for any triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]_{\mathcal{T}},$$

in $\mathcal{T}$, then there exists a triangle of the form

$$\mathcal{F}(X) \xrightarrow{u} \mathcal{F}(Y) \xrightarrow{g} \mathcal{F}(Z) \xrightarrow{w} \mathcal{F}(X)[1]_{\mathcal{S}},$$

in $\mathcal{S}$. However, it is a delicate question whether this triangle in $\mathcal{S}$ is isomorphic to the image of the above triangle in $\mathcal{T}$ under the functor $\mathcal{F}$.

3. Graded Endomorphism Algebras of Linear Generators

3.1. The Algebra Λ^G

In this subsection, we are interested in computing the graded endomorphism algebra of a *minimal* linear generator G of $\mathcal{T}$. Recall from [\[Mur25a\]](#) that a generator is minimal if there exists no proper direct summand H of G , such that H is a generator of $\mathcal{T}$.

Definition 3.1. Let $\mathcal{T}$ be a triangulated category. Recall that the *graded endomorphism ring* of an object $Y \in \mathcal{T}$ is the graded ring $\text{End}_{\mathcal{T}}^*(Y)$ which, at degree i , is given by

$$\text{End}_{\mathcal{T}}^i(Y) = \text{Ext}_{\mathcal{T}}^i(Y, Y) = \mathcal{T}(Y, Y[i]).$$

For $f \in \text{End}_{\mathcal{T}}^i(Y)$ and $g \in \text{End}_{\mathcal{T}}^j(Y)$, multiplication is given by the rule

$$g \cdot f := g[i] \circ f \in \text{End}_{\mathcal{T}}^{i+j}(M).$$

Notice that if $\mathcal{T}$ is Krull-Schmidt and $Y = \bigoplus_{i=1}^m Y_i$, then $\text{End}_{\mathcal{T}}^*(Y)$ can be written as a graded generalised $m \times m$ -matrix ring

$$\text{End}_{\mathcal{T}}^*(Y) = \begin{pmatrix} \text{End}_{\mathcal{T}}^*(Y_1) & \text{Ext}_{\mathcal{T}}^*(Y_2, Y_1) & \cdots & \text{Ext}_{\mathcal{T}}^*(Y_m, Y_1) \\ \text{Ext}_{\mathcal{T}}^*(Y_1, Y_2) & \text{End}_{\mathcal{T}}^*(Y_2) & \cdots & \text{Ext}_{\mathcal{T}}^*(Y_m, Y_2) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Ext}_{\mathcal{T}}^*(Y_1, Y_m) & \text{Ext}_{\mathcal{T}}^*(Y_2, Y_m) & \cdots & \text{End}_{\mathcal{T}}^*(Y_m) \end{pmatrix}$$

where $\text{End}_{\mathcal{T}}^*(Y_i)$ are seen as graded rings and $\text{Ext}_{\mathcal{T}}^*(Y_i, Y_j)$ are seen as graded $\text{End}_{\mathcal{T}}^*(Y_j)$ - $\text{End}_{\mathcal{T}}^*(Y_i)$ -bimodules.

Proposition 3.2. *Let $\mathcal{T}$ be a triangulated category satisfying [Setup 2.1](#) and admitting a minimal linear generator $G = \bigoplus_{i=1}^m G_i$ with m indecomposable summands. Then, $\text{End}_{\mathcal{T}}^*(G)$ is isomorphic to the upper triangular $m \times m$ -matrix algebra χ^G , with*

$$\chi_{i,j}^G := \begin{cases} k[x] & \text{if } i = j \text{ and } G_i < G_i[1] \\ 0 & \text{if } j < i \\ k[x^{\pm}] & \text{otherwise.} \end{cases}$$

where x is concentrated in degree -1 .

Proof. Let $\{\alpha^{Y,Z}\}_{Y,Z \in \text{ind } \mathcal{T}}$ be a choice of generators of Hom spaces as in [Proposition 2.27](#). Moreover, by reordering if necessary, we assume that $G_j < G_i$ for all $j < i$. By the definition of a linear generator, these assumptions imply that $G_j[t] < G_i$ for all $t \in \mathbb{Z}$. To simplify notation, let $\Lambda := \text{End}_{\mathcal{T}}^*(G)$. Throughout the proof, $k[x]$ and $k[x^{\pm}]$ are graded rings with grading defined by considering x as a homogeneous element of degree -1 .

We begin by constructing isomorphisms $\Lambda_{i,j} \cong \chi_{i,j}^G$ that respect grading. Observe first that the definition of a linear generator implies that

$$\Lambda_{i,j}^t = \text{Hom}_{\mathcal{T}}(G_j, G_i[t]) = k \cdot \alpha^{G_j, G_i[t]}$$

for all $i, j, t \in \mathbb{Z}$, and where $\alpha^{G_j, G_i[t]}$ might possibly be zero.

If $j < i$, then $G_j < G_i[t]$ for all $t \in \mathbb{Z}$ and so $\alpha^{G_j, G_i[t]} = 0$ by the definition of a linear generator. Hence, $\Lambda_{i,j} = 0 = \chi_{i,j}^G$.

Consider next the case $j = i$ and $G_i < G_i[1]$. If $0 < t$, then $G_i < G_i[t]$ so that $\Lambda_{i,i}^t = 0$. If, however, $t \leq 0$, then $\Lambda_{i,i}^t \neq 0$. In fact, For $t, s \leq 0$, observe that

$$\alpha^{G_i, G_i[t]} \cdot \alpha^{G_i, G_i[s]} = \alpha^{G_i, G_i[t]}[s] \circ \alpha^{G_i, G_i[s]} = \alpha^{G_i, G_i[t+s]},$$

and we may construct a graded algebra isomorphism

$$\Lambda_{i,i} \rightarrow k[x]: \alpha^{G_i, G_i[t]} \mapsto x^t$$

for all $t \leq 0$.

On the other hand, if $j = i$ and $G_i \cong G_i[1]$, then $\Lambda_{i,i}^t \neq 0$ for all t and so there is an algebra isomorphism

$$\Lambda_{i,i} \rightarrow k[x^{\pm}]: \alpha^{G_i, G_i[t]} \mapsto x^t$$

for all $t \in \mathbb{Z}$.

Next, suppose $i < j$, so that $G_i < G_j$ and $G_i[t] < G_j$ for all $t \in \mathbb{Z}$. Let $s, \ell \in \mathbb{Z}$. If $s \leq 0$ or $G_i = G_i[1]$, and if $\ell \leq 0$ or $G_j = G_j[1]$, then

$$\begin{aligned} \alpha^{G_i, G_i[s]} \cdot \alpha^{G_j, G_i[t]} &= \alpha^{G_i[t], G_i[s+t]} \circ \alpha^{G_j, G_i[t]} = \alpha^{G_j, G_i[s+t]} \\ \alpha^{G_j, G_i[t]} \cdot \alpha^{G_j, G_j[\ell]} &= \alpha^{G_j[\ell], G_i[\ell+t]} \circ \alpha^{G_j, G_j[\ell]} = \alpha^{G_j, G_i[\ell+t]}. \end{aligned}$$

Hence, under the identification $\Lambda_{i,i} \cong k[x]$ or $k[x^{\pm}]$ and $\Lambda_{j,j} \cong k[x]$ or $k[x^{\pm}]$, then there is a graded $\Lambda_{i,i}$ - $\Lambda_{j,j}$ -bimodule isomorphism

$$\Lambda_{i,j} \rightarrow k[x^{\pm}]: \alpha^{G_j, G_i[t]} \mapsto x^t$$

for all $t \in \mathbb{Z}$.

Finally, we let $\phi: \Lambda \rightarrow \chi^G$ be the graded ring morphism specified by

$$[\phi([\alpha^{G_i, G_j[t]}]_{i,j})]_{i,j} := \begin{cases} x^t & \text{if } i = j, G_i < G_i[1] \text{ and } t \leq 0 \\ 0 & \text{if } i = j, G_i < G_i[1] \text{ and } 0 < t \\ 0 & \text{if } j < i \\ x^t & \text{otherwise.} \end{cases}$$

It is straightforward to check that this is indeed a graded ring isomorphism. $\square$

Let $G \in \mathcal{T}$ be a linear generator of a triangulated category satisfying [Setup 2.1](#). Then we define Λ^G to be the differential graded algebra with underlying graded algebra χ^G and with a trivial differential, $d = 0$. Our goal for the remainder of this section is to show that Λ^G is a linear generator of the triangulated category $\text{perf}(\Lambda^G)$, so that we may construct an equivalence $\mathcal{T} \xrightarrow{\sim} \text{perf}(\Lambda^G)$. We first recall a well-known lemma that will be useful to us.

Lemma 3.3 (See e.g. [\[Kra15, Prop. 2.3\]](#) or [\[JY22, A.1 \(iv\)\]](#)). *Let $\mathcal{A}$ be an additive idempotent complete category and consider objects $X, Y, Z \in \mathcal{A}$. If $Y \in \text{add}_{\mathcal{A}} X$, then functor $\mathcal{A}X-$ induces an isomorphism*

$$\mathcal{A}(Y, Z) \rightarrow \text{Hom}_{\text{End}_{\mathcal{A}}(X)}(\mathcal{A}(X, Y), \mathcal{A}(X, Z)).$$

which is natural in Y and Z .

Proposition 3.4. *Let $G \in \mathcal{T}$ be a minimal linear generator of a triangulated category satisfying [Setup 2.1](#). Then, the functor*

$$\text{Ext}_{\mathcal{T}}^*(G, -): \langle G \rangle_1 \rightarrow \langle \Lambda^G \rangle_1$$

is an equivalence.

Proof. Let $G = \bigoplus_{i=1}^m G_i$. We begin by showing that the functor $\text{Ext}_{\mathcal{T}}^*(G, -)$ is fully faithful. This amounts to showing that the assignment

$$\mathcal{T}(G_i, G_j[t]) \rightarrow \text{Hom}_{\Lambda^G}(\text{Ext}_{\mathcal{T}}^*(G, G_i), \text{Ext}_{\mathcal{T}}^*(G, G_j[t]))$$

sending $f: G_i \rightarrow G_j[t]$ to $f \circ (-)$ is bijective. This is essentially a consequence [Lemma 3.3](#), as we will explain below.

Observe first that morphism $f \in \text{Hom}_{\Lambda^G}(\text{Ext}_{\mathcal{T}}^*(G, G_i), \text{Ext}_{\mathcal{T}}^*(G, G_j[t]))$ is equivalent to the data

$$(f^s: \mathcal{T}(G, G_i[s]) \rightarrow \mathcal{T}(G, G_j[s+t]))_{s \in \mathbb{Z}},$$

where for all $\alpha \in \mathcal{T}(G, G_i[s])$ and $\lambda \in (\Lambda^G)^\ell$, $f^s(\alpha) \cdot \lambda = f^{s+\ell}(\alpha \cdot \lambda)$.

We claim that f is uniquely determined by f^0 . To see this, note that, under the graded algebra isomorphism of [Proposition 3.2](#), then $\alpha \in \mathcal{T}(G, G_i[s])$ can be written as $\alpha = (a_\ell x^s)_{\ell=1}^m$ with $a_\ell \in k$ for all $1 \leq \ell \leq m$. Let $I_{-s} \in \Lambda^G$ denote the $m \times m$ diagonal matrix with entries x^{-s} along the diagonal. Then,

$$f^s(\alpha) \cdot I_{-s} = f^0((\alpha_\ell x^s)_{\ell=1}^m I_{-s}) = f^0((\alpha_\ell)_{\ell=1}^m),$$

and so

$$f^s(\alpha) = f^0((\alpha_\ell)_{\ell=1}^m) I_s.$$

Hence, the morphisms f^s are all uniquely determined by f^0 . That is, f is determined by its component f^0 .

Furthermore, morphisms f^0 are precisely the morphisms in $\text{Hom}_{(\Lambda^G)^0}(\mathcal{T}(G, G_i), \mathcal{T}(G, G_j[t]))$. By definition, $(\Lambda^G)^0 = \mathcal{T}(G, G)$ and $G_i \in \text{add}_{\mathcal{T}} G$, so that by [Lemma 3.3](#), there are isomorphisms

$$\mathcal{T}(G_i, G_j[t]) \xrightarrow{\sim} \text{Hom}_{(\Lambda^G)^0}(\mathcal{T}(G, G_i), \mathcal{T}(G, G_j[t]))$$

sending $h: G_i \rightarrow G_j[t]$ to $h \circ (-)$.

Since morphisms $f \in \text{Hom}_{\Lambda^G}(\text{Ext}_{\mathcal{T}}^*(G, G_i), \text{Ext}_{\mathcal{T}}^*(G, G_j[t]))$ are determined by the component f^0 , this is enough to conclude that the assignment

$$\mathcal{T}(G_i, G_j[t]) \rightarrow \text{Hom}_{\Lambda^G}(\text{Ext}_{\mathcal{T}}^*(G, G_i), \text{Ext}_{\mathcal{T}}^*(G, G_j[t]))$$

sending $f: G_i \rightarrow G_j[t]$ to $f \circ (-)$ is bijective.

To conclude that $\text{Ext}_{\mathcal{T}}^*(G, -): \langle G \rangle_1 \rightarrow \langle \Lambda^G \rangle_1$ is an equivalence, it remains to show that it is essentially surjective. This is essentially the same argument as in the proof of [Kra15, p. 2.3]. An object $Y \in \langle \Lambda^G \rangle_1$ is a summand of $(\Lambda^G)^s[t]$ for some integers s, t . Hence, there is an idempotent morphism $\bar{e}: (\Lambda^G)^s \rightarrow (\Lambda^G)^s$ and morphisms $\bar{p}: G^s \rightarrow Y[-t]$, $\bar{i}: Y[-t] \rightarrow G^s$ such that $\bar{i} \circ \bar{p} = \bar{e}$ and $\bar{p} \circ \bar{i} = \text{id}_{Y[-t]}$.

Since the functor $\text{Ext}_{\mathcal{T}}^*(G, -)$ is fully faithful and additive, there is an idempotent $e: G^s \rightarrow G^s \in \mathcal{T}$ such that $\bar{e} = e \circ (-)$. Because $\mathcal{T}$ is idempotent complete (since it is Krull-Schmidt), e splits so that there are morphisms $p: G^s \rightarrow X$ and $i: X \rightarrow G^s$ with $i \circ p = e$ and $p \circ i = \text{id}_X$.

It is straightforward to check that the morphisms

$$\begin{aligned} (p \circ (-)) \circ \bar{i}: Y[-t] &\rightarrow \text{Ext}_{\mathcal{T}}^*(G, X), \\ (i \circ (-)) \circ \bar{p}: \text{Ext}_{\mathcal{T}}^*(G, X) &\rightarrow Y[-t] \end{aligned}$$

are inverses. Hence, since $\text{Ext}_{\mathcal{T}}^*(G, -)$ commutes with suspension, $\text{Ext}_{\mathcal{T}}^*(G, X[t]) \cong Y$, as required. $\square$

Theorem 3.5. *Let $G \in \mathcal{T}$ be a minimal linear generator of a triangulated category satisfying Setup 2.1. Then, there is an additive equivalence $\mathcal{F}: \mathcal{T} \rightarrow \text{perf}(\Lambda^G)$ which commutes with the respective suspension functors and preserves triangles with at least two indecomposable terms.*

Proof. This is a direct consequence of Corollary 2.31 and Proposition 3.4. $\square$

Recall that an *algebraic triangulated category* is a triangulated category $\mathcal{T}$ such that there exists a Frobenius exact category $\mathcal{E}$ and a triangulated equivalence $\underline{\mathcal{E}} \cong \mathcal{T}$. Here, $\underline{\mathcal{E}}$ denotes the stable category of $\mathcal{E}$.

Corollary 3.6. *Let $G \in \mathcal{T}$ be a linear generator of a triangulated category satisfying Setup 2.1. Suppose that $\mathcal{T}$ is algebraic. If G is indecomposable, then there is a triangulated equivalence $\mathcal{T} \xrightarrow{\sim} \text{perf}(\Lambda^G)$.*

Proof. If G is indecomposable, then $\Lambda^G = k[x^{\pm}]$ or $\Lambda^G = k[x]$, depending on whether or not $G = G[1]$. If $G = G[1]$, then $\Lambda^G = k[x^{\pm}]$ and the morphisms in $\mathcal{T}$ are isomorphisms. Hence, the only triangles in $\mathcal{T}$ are triangles isomorphic to direct sums of shifts of the triangle $0 \rightarrow G \xrightarrow{\text{id}_G} G \rightarrow 0$. Thus, it is straightforward to see that the equivalence $\mathcal{F}$ of Theorem 3.5 preserves all triangles.

On the other hand, if $G < G[1]$, then $\Lambda^G = k[x]$, with x in degree -1 . It follows from [Aug+23, Lemma 3.1] that Λ^G is intrinsically formal, which, by definition, implies that Λ^G is quasi-isomorphic to $\text{End}_{\mathcal{T}}^{\text{DG}}(G)$. Hence, since $\mathcal{T}$ is algebraic, [Kel94, §4.3] implies that there is a triangulated equivalence $\mathcal{T} \xrightarrow{\sim} \text{perf}(\Lambda^G)$. $\square$

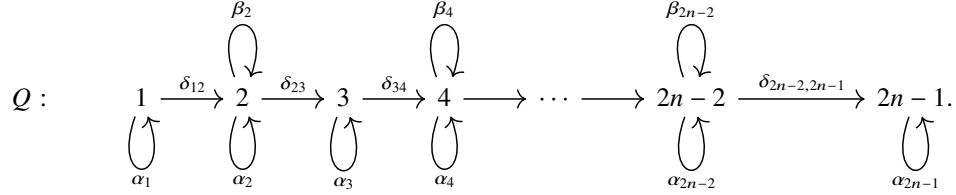
3.2. The Algebra Λ_n

With applications to Discrete Cluster Categories of type A_{∞} in mind, we consider a triangulated category $\mathcal{T}$ satisfying Setup 2.1 and equipped with a minimal linear generator $G \cong \bigoplus_{i=1}^{2n-1} P_i$ such that $P_i \cong P_i[1]$ if and only if $i \equiv 0 \pmod{2}$. For such a generator G , we know by Proposition 3.2, that χ^G is the upper triangular $(2n-1) \times (2n-1)$ -matrix algebra

$$\text{Ext}_{\mathcal{T}}^*(P_i, P_j) = \begin{cases} k[x] & \text{if } i = j \text{ and } i \text{ odd,} \\ 0 & \text{if } j < i, \\ k[x^{\pm}] & \text{otherwise.} \end{cases}$$

From now on, we shall denote χ^G by Λ_n . Let $e_i \in \Lambda_n$ correspond to the idempotent matrix whose entries are zero everywhere except at index (i, i) , in which case the entry is 1.

The algebra Λ_n admits a particularly nice presentation as the path algebra of a graded quiver. More precisely, let Q be the quiver



Let Q_0 denote the set of vertices of Q and for each $a \in Q_0$ write ι_a for the trivial idempotent path in Q at vertex a . Moreover, let $B \subset Q_0$ be the set of vertices such that there is no loop β_b for $b \in B$. Consider the ideal I is given by

$$I := \langle \alpha_a \beta_a - \iota_a, \beta_a \alpha_a - \iota_a, \alpha_b \delta_{b,b+1} - \delta_{b,b+1} \alpha_{b+1}, \beta_a \delta_{a,a+1} \delta_{a+1,a+2} - \delta_{a,a+1} \delta_{a+1,a+2} \beta_{a+2} \rangle,$$

for $1 \leq a, b \leq 2n-1$, and $a \notin B$. The goal of this section is to prove the following result.

Theorem 3.7. *Let G be a minimal linear generator of $\mathcal{T}$, with $G \cong \bigoplus_{i=1}^{2n-1} P_i$, such that $P_i \cong P_i[1]$ if and only if $i \equiv 0 \pmod{2}$. Then there is an isomorphism of graded k -algebras,*

$$\Lambda_n \xrightarrow{\sim} kQ/I.$$

To prove this theorem, we begin with some lemmas concerning submodules of kQ/I .

Lemma 3.8. *Let a, b be different vertices in Q_0 , and let $m \in \mathbb{Z}$. Then $(\iota_a(kQ/I)\iota_b)^m \cong (\iota_a(kQ/I)\iota_b)^0$ as k -vector spaces. Moreover, the dimension of $(\iota_a(kQ/I)\iota_b)^m$ as a k -vector space is at most 1. In fact, if there is a nonzero path $\sigma \in (\iota_a(kQ/I)\iota_b)^m$, then it is equivalent to*

$$\sigma \equiv \begin{cases} \delta \cdots \delta \delta \alpha^{-m} & \text{if } m \leq 0, \\ \delta \cdots \delta \delta \beta^m & \text{if } m \geq 0 \text{ and } b \notin B, \\ \delta \cdots \delta \beta^m \delta & \text{if } m \geq 0 \text{ and } b \in B. \end{cases}$$

Proof. In order to avoid cluttered notation, given arrows $\delta_{a,b}$, α_a and β_a , we may suppress the indices a and b when there is no ambiguity.

Suppose $\sigma = \rho \delta \rho \delta \cdots \rho \delta \rho$ is a path of degree $m \in \mathbb{Z}$, starting at a and ending at b where each ρ is some path consisting combinations of cycles α_c , β_c , and ι_c for a $c \in Q_0$.

By the relations in I , we know that $\alpha_a \delta_{ab} = \delta_{ab} \alpha_b$ and $\alpha_a \beta_a = \beta_a \alpha_a$, and so we may "commute" α with β and δ . Hence

$$\sigma \equiv \sigma' = \beta^{i_1} \delta \beta^{i_2} \delta \cdots \beta^{i_{p-1}} \delta \beta^{i_p} \alpha^i,$$

where $m = \sum_{l=1}^p (i_l) - i$.

Similarly, according to the relations in I , for a vertex $c \notin B$, $\beta_c \delta_{c,c+1} \delta_{c+1,c+2} = \delta_{c,c+1} \delta_{c+1,c+2} \beta_{c+2}$, and so we may "commute" β with paths starting and ending at vertices not in B . Hence, we have either

$$\sigma \equiv \sigma'_1 = \delta \cdots \delta \delta \alpha^i \beta^j,$$

or

$$\sigma \equiv \sigma'_2 = \delta \cdots \delta \beta^j \delta \alpha^i,$$

where, in both σ'_1 and σ'_2 , we must have $m = j - i$, as σ is in degree m and all paths δ are in degree 0.

Thus, one of the following is true;

$$\sigma \equiv \begin{cases} \delta \cdots \delta \delta \alpha^{-m} & \text{if } m \leq 0, \\ \delta \cdots \delta \delta \beta^m & \text{if } m \geq 0 \text{ and } b \notin B, \\ \delta \cdots \delta \beta^m \delta & \text{if } m \geq 0 \text{ and } b \in B. \end{cases}$$

That is, any non-zero path from a to b of degree m , is equivalent to one of the above paths in kQ/I , determined by m and b . Indeed, such a path exists if and only if a path of degree 0 between a and b exists, as any path starting and ending at different vertices must pass through a vertex not in B . Hence there is a k -vector space isomorphism $(\iota_a(kQ/I)\iota_b)^m \cong (\iota_a(kQ/I)\iota_b)^0$, and it follows that the dimension as a k -vector space is at most 1 as Q is a tree. $\square$

Lemma 3.9. *There is an isomorphism of graded k -algebras,*

$$\iota_a(kQ/I)\iota_a \cong e_a \Lambda_n e_a \cong \text{End}_{\mathcal{T}}^*(P_a),$$

for any $a \in Q_0$.

Proof. By Lemma 3.8, there exists at most one non-zero equivalence class of paths of degree i starting and ending at vertex a : If $i < 0$, then all nonzero paths are equivalent to α^i , and so $(\iota_a(kQ/I)\iota_a)^i = k \cdot \alpha^i$. On the other hand, if $i = 0$, then all nonzero paths can be represented by ι_a and so $(\iota_a(kQ/I)\iota_a)^i = k \cdot \iota_a$. If $i > 0$ and $a \in B$, then there is no loop β_a at a and so there are no nonzero paths of degree i . However, if $i > 0$ and $a \notin B$, then all nonzero paths are equivalent to β^i , so that $(\iota_a(kQ/I)\iota_a)^i = k \cdot \beta^i$.

Suppose first that $a \in B$, then a is odd and $P_a < P_a[1]$ by definition of the linear generator G . Hence, there is an isomorphism $\text{End}_{\mathcal{T}}^*(P_a) \cong k[x]$, with $|x| = -1$ by Proposition 3.2. Moreover, since $a \in B$, then, by the above, there are no nonzero paths starting and ending at a of degree $i > 0$. We may thus construct a graded k -linear map $\varphi_{a,a} : \iota_a(kQ/I)\iota_a \rightarrow k[x]$ sending $\alpha_a^{-i} \mapsto x^{-i}$ for $i < 0$ and sending $\iota_a \mapsto 1$. This is clearly a graded algebra isomorphism.

Next, consider the case $a \notin B$. Then, a is even, $P_a \cong P_a[1]$ and $\text{End}_{\mathcal{T}}^*(P_a) \cong k[x^{\pm}]$, with $|x| = -1$ by Proposition 3.2. Thus, similarly to before, we may construct the k -linear map $\varphi_{a,a} : \iota_a(kQ/I)\iota_a \rightarrow k[x^{\pm}]$, taking $\alpha_a^{-i} \mapsto x^{-i}$ for $i < 0$, $\iota_a \mapsto 1$, and $\beta_a^i \mapsto x^{-i}$ for $i > 0$. Again, it is clear that this is an isomorphism of graded k -algebras given the relations $\alpha_a \beta_a - \iota_a$ and $\beta_a \alpha_a - \iota_a$. $\square$

Lemma 3.10. *Let a, b be distinct vertices in Q_0 . Then there is an isomorphism of graded $\text{End}^*(P_a)$ - $\text{End}^*(P_b)$ -bimodules*

$$\iota_a(kQ/I)\iota_b \cong e_a \Lambda_n e_b \cong \text{Ext}_{\mathcal{T}}^*(P_a, P_b).$$

Proof. If $b < a$, then there are no paths from a to b , and so $\iota_a(kQ/I)\iota_b = 0$. Similarly, by definition of the linear generator G , $b < a$ implies that $\text{Ext}_{\mathcal{T}}^*(P_a, P_b) = 0$. Whence, we may assume $a < b$.

There are two cases to consider: the case where $a \in B$ and $b \notin B$ or vice-versa. Let us first assume that $a \in B$ and $b \notin B$. By Proposition 3.2 and the of G , we know that there are algebra isomorphisms $\text{End}^*(P_a) \cong k[x]$, $\text{End}^*(P_b) \cong k[x^{\pm}]$. Moreover, $\text{Ext}_{\mathcal{T}}^*(P_a, P_b) \cong k[x^{\pm}]$ as a $k[x]$ - $k[x^{\pm}]$ -bimodule. Hence, it suffices to show that $\iota_a(kQ/I)\iota_b \cong k[x^{\pm}]$ as a $k[x]$ - $k[x^{\pm}]$ -bimodule.

There is only one degree zero path from a to b given by $\sigma = \delta_{a,a+1} \cdots \delta_{b-1,b}$, so that $(\iota_a(kQ/I)\iota_b)^0 = k \cdot \sigma$. Moreover, by Lemma 3.8, any path $a \rightarrow b$ of degree $m < 0$ is equivalent to $\sigma \alpha^{-m}$, and any path $a \rightarrow b$ of degree $j > 0$ is equivalent to $\sigma \beta^j$. Hence, $(\iota_a(kQ/I)\iota_b)^m = k \cdot \sigma \alpha^m$ and $(\iota_a(kQ/I)\iota_b)^j = k \cdot \sigma \beta^j$. We may thus construct the graded bimodule isomorphism $\varphi_{a,b} : \iota_a(kQ/I)\iota_b \rightarrow k[x^{\pm}]$ via the assignment $\sigma \alpha^{-m} \mapsto x^{-m}$, $\sigma \mapsto 1$, and $\sigma \beta^j \mapsto x^{-j}$ for $m < 0$ and $j > 0$.

The case where $a \notin B$ and $b \in B$ is similar, and so we omit it. $\square$

We are now in a position to prove Theorem 3.7.

Proof of Theorem 3.7. We combine Lemmas 3.9 and 3.10, and Proposition 3.2 to construct a k -linear isomorphism $h : kQ/I \rightarrow \Lambda_n$ sending a path σ between vertices $a \rightarrow b$ to $\varphi_{a,b}(\sigma)$. Here, we mildly abuse notation and denote the composition of $\varphi_{a,b} : \iota_a kQ/I \iota_b \rightarrow e_a \Lambda_n e_b$ with the natural inclusion $i_{a,b} : e_a \Lambda_n e_b \rightarrow \Lambda_n$ as $\varphi_{a,b} : \iota_a kQ/I \iota_b \rightarrow \Lambda_n$.

All that is left to show is that h is an isomorphism of graded k -algebras. Let $\sigma \in kQ/I$ be a path $a \rightarrow b$ of degree i and let $\tau \in kQ/I$ be a path $b \rightarrow c$ of degree j . We would like to show that $h(\sigma\tau) = h(\sigma)h(\tau)$.

A case by case analysis of [Lemma 3.8](#) together with the definition of $\varphi_{a,b}$ and $\phi_{a,c}$ implies that

$$\begin{aligned} h(\sigma) &= \varphi_{a,b}(\sigma) = i_{a,b}(x^i) \\ h(\tau) &= \varphi_{b,c}(\tau) = i_{b,c}(x^j). \end{aligned}$$

Thus, $h(\sigma)h(\tau) = i_{a,c}(x^{i+j})$.

Moreover, since paths α "commute" with paths δ due to the relations I , then, by [Lemma 3.8](#), there is a degree zero path γ_0 such that

$$\sigma\tau = \begin{cases} \gamma_0\alpha^{-i-j} & \text{if } i+j < 0, \\ \gamma_0\beta^{i+j} & \text{if } i+j > 0 \text{ and } c \notin B, \\ \gamma_0\beta^{i+j}\delta & \text{if } i+j > 0 \text{ and } c \in B, \\ \gamma_0 & \text{if } i+j = 0. \end{cases}$$

In any of the cases above,

$$h(\sigma\tau) = \phi_{a,c}(\sigma\tau) = i_{a,c}(x^{i+j}) = h(\sigma)h(\tau),$$

as required. $\square$

4. Discrete Cluster Categories

Discrete cluster categories of Dynkin type A_∞ , labelled $\mathcal{C}_n$ for $n \in \mathbb{Z}^+$, were introduced by Igusa–Todorov [[IT15](#)], as an example cluster categories arising from cyclic posets. These categories may be thought of as an infinite rank version of the cluster categories of Dynkin type A_n , first studied independently in both [[Bua+06](#); [CCS05](#)]. Also, discrete cluster categories of Dynkin type A_∞ can be seen as generalisation of a category studied by Holm–Jørgensen [[HJ10](#)], namely the perfect derived category of $k[x]$ viewed as a differential graded algebra, with x in homogeneous degree -1 and equipped with a trivial differential. In the case $n = 1$, there is a triangulated equivalence between $\mathcal{C}_1$ and $\text{perf } k[x]$.

The categories $\mathcal{C}_n$ are Hom-finite, k -linear, Krull-Schmidt, 2-Calabi-Yau, triangulated categories. Because these categories are well-behaved examples of cluster categories of infinite rank, they have been studied by many authors since their introduction [[Fis14](#); [GHJ19](#); [GZ23](#); [LP17](#); [Mur25b](#)]. The category $\mathcal{C}_n$ admits a combinatorial model, with a discrete set of infinitely many marked points on the boundary of a closed orientated disc, and n so-called accumulation points. The indecomposable objects of $\mathcal{C}_n$ are in bijection to isotopy classes of non-contractible curves between marked points, such that the curves are non-isotopic to the boundary.

Paquette–Yıldırım introduced a completion of $\mathcal{C}_n$ in [[PY21](#)] by considering a Verdier localisation of $\mathcal{C}_{2n}$ by a thick subcategory, labelled $\overline{\mathcal{C}}_n$. The category $\overline{\mathcal{C}}_n$ also admits the same combinatorial model as $\mathcal{C}_n$, however we now also consider isotopy classes of curves with endpoints in the set of accumulation points and marked points.

The categories $\overline{\mathcal{C}}_n$ have since been studied in their own right [[Aug+23](#); [Fra25b](#); [Mur25a](#); [Mur25b](#)], and are our primary example of triangulated categories admitting a linear generator.

4.1. Paquette–Yıldırım Completions of Discrete Cluster Categories of Dynkin Type A_∞

By adapting a construction found in [[GHJ19](#)], we construct the category $\overline{\mathcal{C}}_n$ via a combinatorial model, namely the closed disk D^2 with marked points on the boundary. We will consider D^2 to have an anti-clockwise orientation on the boundary $\partial D^2 = S^1$.

Definition 4.1 ([[GHJ19](#)]). A subset $\mathcal{M}$ of $\partial D^2 = S^1$ is called *admissible* if it satisfies the following conditions:

- (1) $\mathcal{M}$ has infinitely many elements,
- (2) $\mathcal{M} \subset S^1$ is a discrete subset,
- (3) $\mathcal{M}$ satisfies the *two-sided limit condition*, i.e. each $x \in S^1$ which is the limit of a sequence in $\mathcal{M}$ is a limit of both an increasing and decreasing sequence from $\mathcal{M}$ with respect to the cyclic order.

The points at which the two-sided limits in $\mathcal{M}$ converge are called the *accumulation points*. Note that these accumulation points are not in $\mathcal{M}$, as they do not have defined successors and predecessors, meaning $\mathcal{M}$ would not be discrete if they were to be included. As in [GHJ19], we may think of $\mathcal{M}$ as the vertices of the ∞ -gon. For an admissible subset $\mathcal{M} \in S^1$, we label the set of accumulation points $L(\mathcal{M})$, and give them a cyclic ordering induced by the orientation of S^1 .

Let $\mathfrak{a} \in L(\mathcal{M})$ be an accumulation point, we define $\mathfrak{a}_+ \in L(\mathcal{M})$ (resp. $\mathfrak{a}_- \in L(\mathcal{M})$) to be the accumulation point such that $\mathfrak{a} \leq \mathfrak{b} \leq \mathfrak{a}_+$ with $\mathfrak{b} \in L(\mathcal{M})$ (resp. $\mathfrak{a}_- \leq \mathfrak{c} \leq \mathfrak{a}$ with $\mathfrak{c} \in L(\mathcal{M})$) implies $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{b} = \mathfrak{a}_+$ (resp. $\mathfrak{c} = \mathfrak{a}_-$ or $\mathfrak{c} = \mathfrak{a}$). Moreover, we say that $\mathfrak{a}$ is its own *successor* and *predecessor*, that is $\mathfrak{a} = \mathfrak{a}^+ = \mathfrak{a}^-$. Let $\overline{\mathcal{M}} := L(\mathcal{M}) \cup \mathcal{M}$.

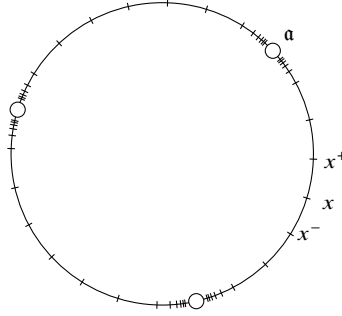


FIGURE 1. An admissible subset $\mathcal{M}$ of S^1 . The marked points in $\mathcal{M}$ converge to the accumulation points represented as small circles, and each marked point x has both a predecessor and a successor, labelled x^- and x^+ respectively.

Definition 4.2 (See [GHJ19]). A *arc* of $\overline{\mathcal{M}}$ is a subset $\ell_X = \{x_0, x_1\} \subset \overline{\mathcal{M}}$ with $x_1 \notin \{x_0^-, x_0, x_0^+\}$, where x^+ and x^- are the successor and predecessor respectively to $x \in \overline{\mathcal{M}}$. If $\ell_Y = \{y_0, y_1\}$ is another arc, then ℓ_X and ℓ_Y *cross* if $x_0 < y_0 < x_1 < y_1 < x_0$ or $x_0 < y_1 < x_1 < y_0 < x_0$. We will call the points x_0 and x_1 the *endpoints* of ℓ_X .

We may think of an arc $\ell_X = \{x_0, x_1\}$ as being a representative of the isotopy class of non-self-intersecting curves in D^2 between the marked points x_0 and x_1 . Two arcs cross if any of their representative curves cross in the interior of D^2 .

There are four types of arcs for us to consider; *short arcs*, *long arcs*, *limit arcs*, and *double limit arcs*. An arc $\ell = \{x, y\}$ with endpoints in $\mathcal{M}$ is a short arc if $\mathfrak{a} < x < y < \mathfrak{a}_+$, and is a long arc if $\mathfrak{a}_- < x < \mathfrak{a} < y < \mathfrak{a}_+$, for some $\mathfrak{a} \in L(\mathcal{M})$. An arc $\ell = \{x, y\}$ is a limit arc if $x \in L(\mathcal{M})$ and $y \in \mathcal{M}$, or vice versa, and ℓ is a double limit arc if $x, y \in L(\mathcal{M})$.

Note that there are no long arcs, nor double limit arcs when $n = 1$. We write *(double) limit arcs* to mean the set of all limit arcs and double limit arcs.

In [PY21], the authors construct a Hom-finite, Krull-Schmidt, k -linear, triangulated category, denoted $\mathcal{C}(D^2, \overline{\mathcal{M}})$ here, such that the indecomposable objects are in bijection to the arcs of $\overline{\mathcal{M}}$, for $|L(\mathcal{M})| = n$. The suspension functor of $\mathcal{C}(D^2, \overline{\mathcal{M}})$ acts on indecomposable objects by taking endpoints in $\mathcal{M}$ to their predecessor, and taking an endpoint in $\overline{\mathcal{M}}$ to itself. The authors of [PY21] show that $\mathcal{C}(D^2, \overline{\mathcal{M}})(X, Y) \cong k$ if and only if ℓ_X and $\ell_{Y[-1]}$ cross for X, Y indecomposable objects in $\mathcal{C}(D^2, \overline{\mathcal{M}})$, and $\mathcal{C}(D^2, \overline{\mathcal{M}})(X, Y) = 0$ otherwise.

These categories are known as the *Paquette-Yildirim completions of discrete cluster categories of Dynkin type A_∞* , however we shall hereafter refer to them as *Paquette-Yildirim categories*.

Let $\mathcal{M}$ and $\mathcal{N}$ be admissible subsets of S^1 such that $|L(\mathcal{M})| = |L(\mathcal{N})| = n$, then there is an equivalence of categories between $\mathcal{C}(D^2, \overline{\mathcal{M}})$ and $\mathcal{C}(D^2, \overline{\mathcal{N}})$. Therefore for all $n \in \mathbb{Z}_{\geq 1}$, we will consider an admissible subset of S^1 with n accumulation points, $\mathcal{M}_n$, and so we consider the category $\overline{\mathcal{C}}_n := \mathcal{C}(S^1, \overline{\mathcal{M}}_n)$ as the representative of the equivalence

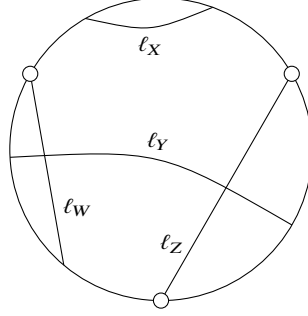


FIGURE 2. Four arcs of $\overline{\mathcal{M}}$, where ℓ_X is a short arc, ℓ_Y is a long arc, ℓ_W is a limit arc, and ℓ_Z is a double limit arc.

class of discrete cluster categories of Dynkin type A_∞ with n accumulation points. From now on we shall make no distinction between indecomposable objects in $\overline{\mathcal{C}}_n$ and arcs of $\overline{\mathcal{M}}$.

Let $X = \{x_0, x_1\}$ and $Y = \{y_0, y_1\}$ be indecomposable objects in $\overline{\mathcal{C}}_n$ such that $y_0 < x_0 < y_1 < x_1$, then there exists two triangles in $\overline{\mathcal{C}}_n$;

$$\begin{aligned} X &\rightarrow A \oplus B \rightarrow Y \rightarrow X[1], \\ Y &\rightarrow C \oplus D \rightarrow X \rightarrow Y[1]. \end{aligned}$$

where $A = \{y_1, x_1\}$, $B = \{y_0, x_0\}$, $C = \{y_0, x_1\}$, and $D = \{x_0, y_1\}$. Moreover, for indecomposable objects $U = \{a, u\}$ and $V = \{a, v\}$ in $\overline{\mathcal{C}}_n$ with $a < u < v < a$, then there also exists a triangle in $\overline{\mathcal{C}}_n$ of the form;

$$X \rightarrow Z \rightarrow Y \rightarrow X[1],$$

where $Z = \{u, v\}$.

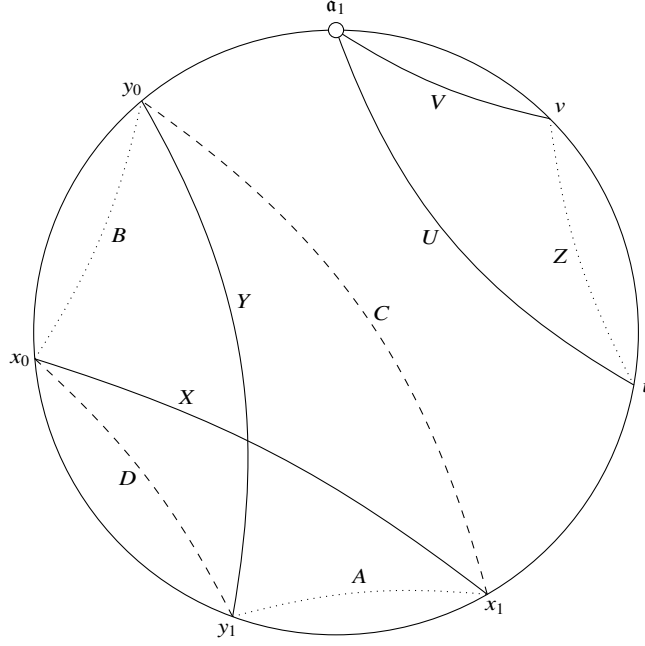


FIGURE 3. The arcs corresponding to triangles coming from non-split extensions between indecomposable objects.

Definition 4.3. Let X be an indecomposable object in $\bar{\mathcal{C}}_n$. Let $\bar{\mathcal{M}}_X \subseteq \bar{\mathcal{M}}$ be the set of marked points that are the endpoints of the arcs corresponding to suspensions and desuspensions of X . If $A \cong \bigoplus_{i=1}^l X_i$, with all X_i indecomposable, then $\bar{\mathcal{M}}_A = \bigcup_{i=1}^l \bar{\mathcal{M}}_{X_i}$. We call $\bar{\mathcal{M}}_A$ the *orbit* of A in $\bar{\mathcal{M}}$. If we have $\bar{\mathcal{M}}_A = \bar{\mathcal{M}}$, then we say A has a *complete orbit* in $\bar{\mathcal{M}}$.

The following lemma shows how the Ext^1 -spaces are given in $\bar{\mathcal{C}}_n$.

Proposition 4.4 ([PY21], Prop. 3.14). *Let $X, Y \in \bar{\mathcal{C}}_n$ be indecomposable objects. Then $\bar{\mathcal{C}}_n(X, Y[1])$ is at most one dimensional. It is one dimensional if and only if one of the following conditions are met for the arcs ℓ_X and ℓ_Y :*

- X, Y cross,
- $X \neq Y$ share exactly one accumulation point, and we can go from X to Y by rotating X about their common endpoint following the orientation of S^1 ,
- $X = Y$ are double limit arcs.

Lemma 4.5. *Let $Y, Z \in \bar{\mathcal{C}}_n$ be non-isomorphic indecomposable objects such that $\bar{\mathcal{C}}_n(Y, Z) \cong k$, and write $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$. Then any morphism $f : Y \rightarrow Z$ factors through an indecomposable object $W = \{w_1, w_2\}$ if and only if $z_1 \geq w_1 \geq y_1$ and $z_2 \geq w_2 \geq y_2$ with respect to $(S^1, \bar{\mathcal{M}})$.*

Proof. This is a direct consequence of [IT15, Lemma 2.4.2], and [PY21, Lemmas 3.8, 3.9]. $\square$

4.2. Linear Generators of $\bar{\mathcal{C}}_n$

The classical generators of $\bar{\mathcal{C}}_n$ were classified in [Mur25a], using a notion of an object being homologically connected. An object X in a triangulated category $\mathcal{T}$ is *homologically connected*, if for any two indecomposable objects $Y, Z \in \langle X \rangle$,

there exists a sequence of non-zero, irreducible morphisms between indecomposables in $\langle X \rangle$, starting and ending at Y and Z . Note that there is no requirement on the direction or composition of the morphisms.

Theorem 4.6 ([Mur25a]). *Let G be an object in $\overline{\mathcal{C}}_n$. Then G is a classical generator if and only if G is homologically connected and G has a complete orbit in $\mathcal{M}$.*

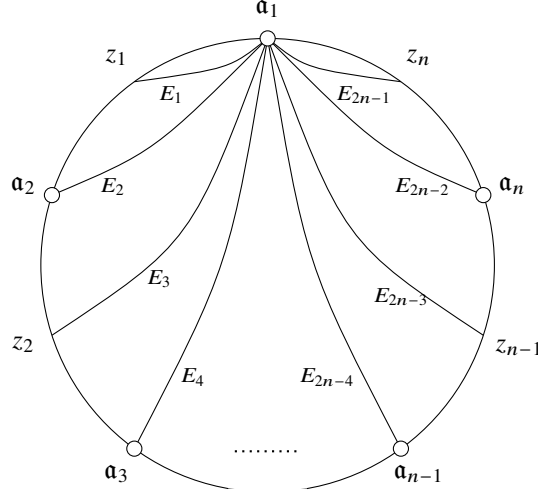


FIGURE 4. A fan generator of $\overline{\mathcal{C}}_n$.

Let $E \in \overline{\mathcal{C}}_n$ be the object in Figure 4, then E is a generator by [Mur23, Lemma 4.3.2], and we call E a *fan generator* of $\overline{\mathcal{C}}_n$.

Lemma 4.7. *Let $E \in \overline{\mathcal{C}}_n$ be a fan generator. Then E is a linear generator.*

Proof. By Proposition 4.4, we have the Hom spaces $\overline{\mathcal{C}}_n(E_i, E_j[l]) \cong k$ if and only if $i < j$, or $i = j$ and $l \leq 0$. Hence we may impose a linear ordering on the indecomposable objects in $\langle E \rangle_1$, such that $E_j[l] < E_i$ if and only if $\overline{\mathcal{C}}_n(E_i, E_j[l]) \cong k$. It also follows from Proposition 4.4 that $P \leq P[1]$ for all indecomposable objects $P \in \langle E \rangle_1$.

Suppose that $P \leq Q \leq P[1]$ for some $Q \in \langle E \rangle_1$, and suppose that $P = \{\alpha_1, x\}$ for some $x \in \overline{\mathcal{M}}$. Then $P[1] = \{\alpha_1, x^-\}$, and let $Q = \{\alpha_1, y\}$. By Proposition 4.4, and the construction of the linear ordering above, $P \leq Q$ if and only if $\alpha_1 < y \leq x < \alpha_1$, similarly $Q \leq P[1]$ if and only if $\alpha_1 < x^- \leq y < \alpha_1$. Hence either $y = x$ or $y = x^-$, and so either $Q \cong P$ or $Q \cong P[1]$.

Finally, if $R \leq Q \leq P$ in $\langle E \rangle_1$, then a morphism from P to R factors non-trivially through Q by Lemma 4.5. $\square$

Lemma 4.7 means that we may apply the results of Sections 2 and 3 to the category $\overline{\mathcal{C}}_n$, as it is a Hom-finite, Krull-Schmidt, triangulated category with a linear generator. Importantly, this means that by Theorem 3.5, there is an additive equivalence $\overline{\mathcal{C}}_n \xrightarrow{\sim} \text{perf}(\Lambda^E)$.

In fact, it follows from subsection Section 3.2 (see also [Mur23, Theorem 4.3.10]) that the graded endomorphism ring of E is isomorphic to Λ_n . We thus have the following result.

Theorem 4.8. *Let $\overline{\mathcal{C}}_n$ be a Paquette-Yildirim category with n accumulation points. Then there is a additive equivalence*

$$\mathcal{F} : \overline{\mathcal{C}}_n \xrightarrow{\sim} \text{perf} \Lambda_n,$$

such that $\mathcal{F}$ commutes with the respective suspension functors, and preserves triangles with two indecomposable terms.

Moreover, let $\mathcal{S}$ be a triangulated category satisfying [Setup 2.1](#), admitting a linear generator H . Then $\mathcal{S}$ is additively equivalent to a thick subcategory of $\bar{\mathcal{C}}_n$ for some $n \in \mathbb{Z}$.

Proof. It is clear that the fan generator E satisfies the assumptions of [Section 3.2](#). Hence, the existence of the additive equivalence $\mathcal{F}: \bar{\mathcal{C}}_n \xrightarrow{\sim} \text{perf} \Lambda_n$ follows from [Theorem 3.5](#).

For the second statement, let $\mathcal{S}$ be a thick subcategory of $\bar{\mathcal{C}}_n$ for some $n \in \mathbb{Z}$, and suppose that H is a linear generator for $\mathcal{S}$. To prove our claim, it is enough to show that there exists a fully faithful additive functor $F': \langle H \rangle_1 \rightarrow \langle E \rangle_1 \subset \bar{\mathcal{C}}_n$ for some $n \in \mathbb{Z}^+$ by [Theorem 4.8](#) and that the image $\mathcal{F}'(\langle H \rangle)$ of $\langle H \rangle$ under the induced functor $\mathcal{F}'$ is a thick subcategory of $\bar{\mathcal{C}}_n$.

Suppose that $H \cong \bigoplus_{i=1}^m H_i$, and let $E \in \bar{\mathcal{C}}_{m+1}$. Then we define the functor $F': \langle H \rangle_1 \rightarrow \langle E \rangle_1$ such that;

$$F'(H_i) \cong \begin{cases} E_{2i-1} & \text{if } H_i < H_i[1], \\ E_{2i} & \text{if } H_i \cong H_i[1]. \end{cases}$$

Then it is clear that F' is a fully faithful additive functor by [Definition 2.2](#), and the construction of $E \in \bar{\mathcal{C}}_{m+1}$.

Finally, to see that $\mathcal{F}'(\langle H \rangle)$ is a thick subcategory of $\bar{\mathcal{C}}_n$, consider that we need to prove that $\mathcal{F}'(\langle H \rangle) = \langle E' \rangle$ for the direct summand $E' = F'(H)$ of E . Since $\mathcal{F}'$ is an equivalence, and $\langle E' \rangle$ is a full subcategory, it suffices to show that the objects in $\langle E' \rangle$ are the same as the objects in $\mathcal{F}'(\langle H \rangle)$. Given that E' is itself a linear generator of $\langle E' \rangle$, then every indecomposable object of $\langle E' \rangle$ fits into a triangle with at least two indecomposable terms in $\langle E' \rangle_1$ by [Corollary 2.18](#). However, the functor $\mathcal{F}'$ preserves triangles with at least two indecomposable terms by [Theorem 4.8](#), and $\mathcal{F}'(\langle H \rangle_1) = \langle E' \rangle_1$, an indecomposable object is in the thick closure of E' if and only if it is in the image of $\mathcal{S} = \langle H \rangle$ under $\mathcal{F}'$. Hence $\mathcal{S}$ is additively equivalent to a thick subcategory of $\bar{\mathcal{C}}_{m+1}$. $\square$

Remark 4.9. We note that the characterisation of $\mathcal{S}$ as additively equivalent to a thick subcategory of $\bar{\mathcal{C}}_{m+1}$ is not unique. For instance, we may consider a linear generator $H \cong \bigoplus_{i=1}^m H_i$ such that $H_i < H_i[1]$ for all i . Then one may check that $\mathcal{S}$ is also additively equivalent to a thick subcategory of $\bar{\mathcal{C}}_m$, under that assignment $F'(H_i) = E_{2i-1}$.

Appendix A. Choice of Generators for Hom-sets in Categories with Linear Generators

The goal of this section is to present a proof and explore some corollaries of the following result.

Proposition A.1. *Let $\text{ind } \mathcal{T}$ denote the set of indecomposable objects in $\mathcal{T}$. Then, there is a choice of morphisms $\{\alpha^{Y,Z}: Y \rightarrow Z\}_{Y,Z \in \text{ind } \mathcal{T}}$ such that for any pair of indecomposable objects $Y, Z \in \mathcal{T}$*

$$\alpha^{Y,Z} = \alpha^{W,Z} \alpha^{Y,W},$$

whenever $\alpha^{Y,Z}: Y \rightarrow Z$ factors through an indecomposable object W .

Proof. Since $\mathcal{T}$ is Krull-Schmidt, then G is a finite direct sum of indecomposable objects $\{P_i\}_{i=1}^n$. Throughout this proof, let $\bar{P} = P_j$ and $\bar{Q} = P_k$ where $\bar{Q} = P_k \leq P_i \leq P_j = \bar{P}$ for all $1 \leq i \leq n$.

(1) We first choose $\{\alpha^{P,Q}\}_{P,Q \in \text{ind } \langle G \rangle_1}$.

Let $\alpha^{\bar{P},\bar{P}}$ be the identity, and fix nonzero morphisms $\alpha^{\bar{P},Q}$ for $Q < \bar{P}$. For $0 < j \in \mathbb{Z}$, choose nonzero morphisms $\alpha^{\bar{P}[j],\bar{P}}$, and let $\alpha^{\bar{P}[j],Q} := \alpha^{\bar{P}[j],\bar{P}} \circ \alpha^{\bar{P},Q}$. Extend this choice to include all shifts of the morphisms defined.

Let $P, Q \in \langle G \rangle_1$ such that there is a nonzero morphism $P \rightarrow Q$. Equivalently, $Q \leq P$. Then, there are integers i, j and s, t such that $P \cong P_i[s]$ and $Q \cong P_j[t]$. Let $m > \max(s, t)$. Then, by [Proposition 2.26](#) there is

a morphism h making the diagram

$$\begin{array}{ccc}
 & \bar{P}[m] & \\
 \alpha^{\bar{P}[m-s], P_i[s]} \swarrow & & \searrow \alpha^{\bar{P}[m-t], P_j[t]} \\
 P \cong P_i[s] & \xrightarrow{\quad h \quad} & Q \cong P_j[t]
 \end{array}$$

commute. We let $\alpha^{P,Q} := h$. In fact, h is independent of choice of m . To see this, let $m_1, m_2 > \max(s, t)$ and let $h_1: P \rightarrow Q$ and $h_2: P \rightarrow Q$ be the morphisms induced by m_1 and m_2 , respectively. Without loss of generality, suppose $m_2 < m_1$ so that there is a morphism $\alpha^{\bar{P}[m_1], \bar{P}[m_2]}$. Then, we have the following diagram.

$$\begin{array}{ccccc}
 & & \bar{P}[m_1] & & \\
 & \swarrow \alpha^{\bar{P}[m_1], P} & \downarrow \alpha^{\bar{P}[m_1], \bar{P}[m_2]} & \searrow \alpha^{\bar{P}[m_1], Q} & \\
 & & \bar{P}[m_2] & & \\
 \swarrow \alpha^{\bar{P}[m_2], P} & & & & \searrow \alpha^{\bar{P}[m_2], Q} \\
 P & \xrightarrow{\quad h_1 \quad} & & \xrightarrow{\quad h_2 \quad} & Q
 \end{array}$$

By construction of the maps $\alpha^{\bar{P}[k], R}$, the top two triangles commute. Hence,

$$h_1 \circ \alpha^{\bar{P}[m_1], P} = \alpha^{\bar{P}[m_1], Q} = \alpha^{\bar{P}[m_2], Q} \circ \alpha^{\bar{P}[m_1], \bar{P}[m_2]} = h_2 \circ \alpha^{\bar{P}[m_2], P} \circ \alpha^{\bar{P}[m_1], \bar{P}[m_2]} = h_2 \circ \alpha^{\bar{P}[m_1], P}$$

Since the Hom spaces are one-dimensional, $\mathcal{T}(\alpha^{\bar{P}[m_1], P}, Q)$ is either zero or an isomorphism. Since $\alpha^{\bar{P}[m_1], Q}$ is nonzero, then it must be an isomorphism. Hence, $h_1 = h_2$, as required.

- (2) The choice of $\alpha^{P,Q}$ is compatible with composition.

To see this, note first that $\alpha^{P,Q}$ cannot factor through $W \notin \langle G \rangle_1$ by [Proposition 2.26](#). So, suppose that $\alpha^{P,Q}$ factors through $R \in \langle G \rangle_1$. Then, $Q \leq R \leq P$ and we may choose a large enough m such that diagram

$$\begin{array}{ccccc}
 & & \bar{P}[m] & & \\
 & \swarrow \alpha^{\bar{P}[m], P} & \downarrow \alpha^{\bar{P}[m], R} & \searrow \alpha^{\bar{P}[m], Q} & \\
 & & R & & \\
 \swarrow \alpha^{P,R} & & & & \searrow \alpha^{R,Q} \\
 P & \xrightarrow{\quad \alpha^{P,Q} \quad} & & \xrightarrow{\quad \alpha^{P,Q} \quad} & Q
 \end{array}$$

exists. The top two triangles and the outer triangle commute by construction of $\alpha^{P,R}$, $\alpha^{R,Q}$ and $\alpha^{P,Q}$. Therefore,

$$\alpha^{R,Q} \circ \alpha^{P,R} \circ \alpha^{\bar{P}[m], P} = \alpha^{\bar{P}[m], Q} = \alpha^{P,Q} \circ \alpha^{\bar{P}[m], P}.$$

As before, this is enough to conclude that $\alpha^{R,Q} \circ \alpha^{P,R} = \alpha^{P,Q}$, as required.

- (3) We next choose morphisms $\{\alpha^{P,Z}\}_{P \in \text{ind} \langle G \rangle_1, Z \in \text{ind} \mathcal{T} \setminus \text{ind} \langle G \rangle_1}$.

For each indecomposable object $Z \notin \langle G \rangle_1$, [Theorem 2.16](#) implies that $Z \cong X_{P,Q}$ for $P, Q \in \langle G \rangle_1$. For each such Z , choose a nonzero morphism $\alpha^{Q,Z} \in \mathcal{T}(Q, Z)$. Moreover, for $R \in \langle G \rangle_1$ such that there is a nonzero morphism $R \rightarrow Z$, then there is a morphism $R \rightarrow Q$ by [Proposition 2.26](#). Thus, define $\alpha^{R,Z} := \alpha^{Q,Z} \circ \alpha^{R,Q}$.

- (4) Let us choose forwards morphisms $\{\alpha^{Y,Z}\}_{Y,Z \in \text{ind } \mathcal{T} \setminus \text{ind } \langle G \rangle_1}$.

Since $Y, Z \in \text{ind } \mathcal{T} \setminus \text{ind } \langle G \rangle_1$, then $Y \cong X_{P,Q}$ and $Z \cong X_{P',Q'}$. Therefore, by [Proposition 2.26](#), there exists for any $Q \leq R < P$ a morphism h making the diagram

$$\begin{array}{ccc} & R & \\ \alpha^{R,Y} \swarrow & & \searrow \alpha^{R,Z} \\ X_{P,Q} & \xrightarrow{\quad h \quad} & X_{P',Q'} \end{array}$$

commute. We let $\alpha^{Y,Z} := h$. As in the proof of (1), h is actually independent of the choice of R by construction of the morphisms $\alpha^{R,Y}$ and $\alpha^{R,Z}$.

We will postpone showing compatibility of this choice with composition until we have defined our choices for backwards morphisms.

- (5) We now choose morphisms $\{\alpha^{Y,Q}\}_{Q \in \text{ind } \langle G \rangle_1, Y \in \text{ind } \mathcal{T} \setminus \text{ind } \langle G \rangle_1}$.

To ease notation, let $\bar{X}^t := X_{\bar{P}[t], \bar{Q}}$. Choose a nonzero morphism $\alpha^{\bar{X}^0, \bar{P}[1]}: \bar{X}^0 \rightarrow \bar{P}[1]$. By [Proposition 2.26](#), for $S \in \langle G \rangle_1$ and $t \geq 0$, there exist morphisms $\alpha^{\bar{X}^t, \bar{P}[t+1]}$ and $\alpha^{X_{\bar{P}[t], S}, \bar{P}[t+1]}$ making the following diagram commute.

$$\begin{array}{ccccc} X_{\bar{P}[t], S} & \xrightarrow{\alpha^{X_{\bar{P}[t], S}, \bar{X}^t}} & X_{\bar{P}[t], \bar{Q}} & \xrightarrow{\alpha^{\bar{X}^t, \bar{X}^0}} & X_{\bar{P}, \bar{Q}} \\ \alpha^{X_{\bar{P}[t], S}, \bar{P}[t+1]} \downarrow & & \alpha^{\bar{X}^t, \bar{P}[t+1]} \downarrow & & \alpha^{\bar{X}, \bar{P}[1]} \downarrow \\ \bar{P}[t+1] & \xrightarrow{\alpha^{\bar{P}, \bar{P}[t+1]}} & \bar{P}[t+1] & \xrightarrow{\alpha^{\bar{P}[t+1], \bar{P}[1]}} & \bar{P}[1] \end{array}$$

Now, let $Y \notin \langle G \rangle_1$, so that we may assume $Y = X_{R,S}$, where $R = P_i[t]$ for integers i and t . By choosing $k_{\max} > \max(t, 0)$, we have that $R \leq \bar{P}[k_{\max}]$ so that, by [Lemma 2.13](#), there is a forwards morphism $X_{\bar{P}[k_{\max}], S} \rightarrow X_{R,S}$. Thus, by [Proposition 2.26](#), there is a nonzero morphism $\alpha^{Y, R[1]}$ making the following diagram commute.

$$\begin{array}{ccc} X_{\bar{P}[k_{\max}], S} & \xrightarrow{\alpha^{X_{\bar{P}[k_{\max}], S}, Y}} & X_{R,S} \\ \downarrow \alpha^{X_{\bar{P}[k_{\max}], S}, \bar{P}[k_{\max}+1]} & & \downarrow \alpha^{Y, R[1]} \\ \bar{P}[k_{\max}+1] & \xrightarrow{\alpha^{\bar{P}[k_{\max}+1], R[1]}} & R[1] \end{array}$$

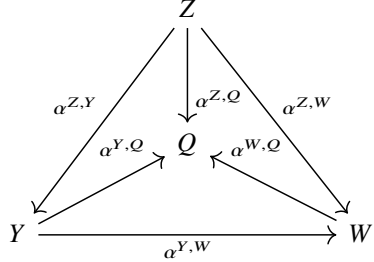
As before, our choice of $\alpha^{Y, R[1]}$ does not depend on which $k_{\max}$ we choose.

Next, suppose that there is a nonzero morphism $Y \rightarrow Q$. Then, $S \leq Q < R$ by [Lemma 2.5](#), and so we let $\alpha^{Y, Q} := \alpha^{R[1], Q} \circ \alpha^{Y, R[1]}$.

- (6) It remains to check that our choice of $\{\alpha^{Y, Q}\}_{Q \in \text{ind } \langle G \rangle_1, Y \in \text{ind } \mathcal{T} \setminus \text{ind } \langle G \rangle_1}$ is compatible with composition.

To see this, suppose that a morphism $Y \rightarrow Q$ factors through an indecomposable object W . If $W \in \langle G \rangle_1$, then it is easy to check that, by construction, $\alpha^{Y, Q} = \alpha^{W, Q} \circ \alpha^{Y, W}$. So, suppose that $W \notin \langle G \rangle_1$. Then, $W \cong X_{P', Q'}$ and, by [Proposition 2.26](#), morphisms $Y \rightarrow W$ are forwards morphisms. Moreover, by taking low enough $k_{\min}$ and high enough $k_{\max}$, there are forwards morphisms $X_{\bar{P}, \bar{Q}[\Delta]}[k_{\max}] \rightarrow Y$ and $X_{\bar{P}, \bar{Q}[\Delta]}[k_{\max}] \rightarrow X$. For

brevity, let $Z := X_{\bar{P}, \bar{Q}[\Delta]}[k_{\max}]$. We thus have the diagram



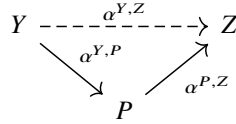
where the top two triangles and the outer triangle commute by construction of the morphisms in the diagram. It is straightforward to check that

$$\alpha^{Y,Q} \circ \alpha^{Z,Y} = \alpha^{Z,Q} = \alpha^{W,Q} \circ \alpha^{Z,W} = \alpha^{W,Q} \circ \alpha^{Y,W} \circ \alpha^{Z,Y}$$

which, as in the proof of (1), is enough to conclude that $\alpha^{Y,Q} = \alpha^{W,Q} \circ \alpha^{Y,W}$, as required.

- (7) We complete our choice by choosing backwards morphisms $\{\alpha^{Y,Z}\}_{Y,Z \in \text{ind } \mathcal{T} \setminus \text{ind } \langle G \rangle_1}$.

Write $Y = X_{R,S}$ and $Z = X_{P',Q'}$, and suppose that $Y \rightarrow Z$ is a backwards morphism. Then, for any $Q' \leq P \leq R[1]$, [Proposition 2.26](#) implies the existence of the diagram



where we define $\alpha^{Y,Z}$ to be the morphism making the diagram commute. As before, this choice is independent of P .

- (8) Finally, we show that our choices are compatible with composition.

Consider $\alpha^{Y,Z}: Y \rightarrow Z$, and suppose it factors through an indecomposable object W . If both $Y, Z \in \langle P \rangle_1$, then $W \in \langle G \rangle_1$ by [Proposition 2.26](#), and we have shown in (2) that compatibility of composition holds.

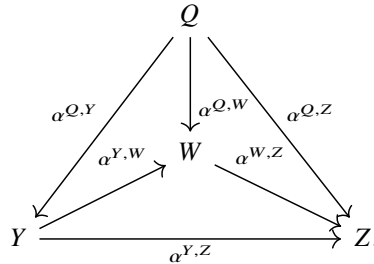
Suppose that $Y \in \langle G \rangle_1$ and $Z = X_{R,S}$. If and $W \in \langle G \rangle_1$, then by (1) and the definition of $\alpha^{Y,Z}$ and $\alpha^{W,Z}$,

$$\alpha^{W,Z} \circ \alpha^{Y,W} = \alpha^{S,Z} \circ \alpha^{W,S} \circ \alpha^{Y,W} = \alpha^{S,Z} \circ \alpha^{Y,S} = \alpha^{Y,Z}$$

as required. If, on the other hand, $W = X_{P,Q}$, then morphisms $W \rightarrow Z$ are forwards by [Proposition 2.26](#). Hence, $\alpha^{W,Z} \circ \alpha^{Y,W} = \alpha^{Y,Z}$ by definition of forwards morphisms $\alpha^{W,Z}$ in (4).

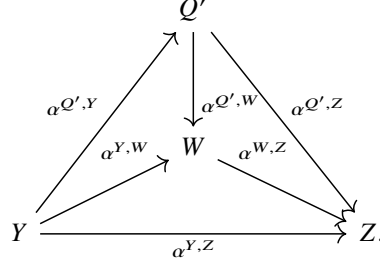
The case where $Y = X_{P,Q}$ and $Z \in \langle G \rangle_1$ was shown in (6). Hence, it remains to consider the setting of $Y = X_{P,Q}$ and $Z = X_{R,S}$.

Under the assumption $Y = X_{P,Q}$ and $Z = X_{R,S}$, suppose that $\alpha^{Y,Z}$ is a forwards morphism. Then by [Proposition 2.26](#), $W = X_{P',Q'}$ and the morphisms $\alpha^{Y,W}$ and $\alpha^{W,Z}$ are forwards. It follows from [Lemmas 2.6](#) and [2.13](#), as well as the choice of forwards morphisms in (4) that there is that there is a diagram



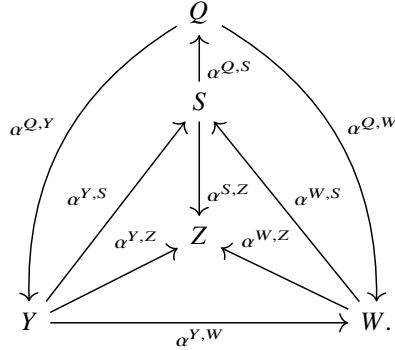
where the top two triangles and the outer triangle commute. It is easily checked that this implies that the entire diagram commutes (see e.g. (2)), which proves compatibility with composition.

Finally, suppose that $Y = X_{P,Q}$, $Z = X_{R,S}$, and $\alpha^{Y,Z}$ is a backwards morphism. If $W \in \langle G \rangle_1$, then compatibility follows from the definition of $\alpha^{Y,Z}$ in (7). If $W = X_{P',Q'}$ and $\alpha^{Y,W}$ is a backwards morphism, then $\alpha^{W,Z}$ is forwards and the following diagram exists by Lemmas 2.6 and 2.13 and Proposition 2.26.



Here, the top left triangle and the outer triangle commute by the choice of backwards morphisms in (7), and the top right triangle commutes by the choice of forwards morphism in (4). As before, it is easily checked that this implies that the entire diagram commutes, as required.

On the other hand, if, under the assumption that $W = X_{P',Q'}$, $\alpha^{Y,W}$ is a forwards morphism, then $\alpha^{W,Z}$ is backwards and the following diagram exists by Lemmas 2.6 and 2.13 and Proposition 2.26.



The triangle with vertices Y, Q, W commutes by construction of forwards morphisms in (4), and the triangle with vertices Y, S, W commutes by (6). Moreover, the triangles with vertices Y, S, Z and Z, S, W commute by choice of backwards morphisms in (7). Hence,

$$\alpha^{W,Z} \circ \alpha^{Y,W} \circ \alpha^{Q,Y} = \alpha^{W,Z} \circ \alpha^{Q,W} = \alpha^{S,Z} \circ \alpha^{W,S} \circ \alpha^{Q,W} = \alpha^{S,Z} \circ \alpha^{Y,S} \circ \alpha^{Q,Y} = \alpha^{Y,Z} \circ \alpha^{Q,Y}$$

which implies that $\alpha^{W,Z} \circ \alpha^{Y,W} = \alpha^{Y,Z}$, as required.

□

Corollary A.2. Let $\mathcal{T}$ be a triangulated category satisfying Setup 2.1 and admitting a linear generator G . Let G' be a summand of G , let $\mathcal{S} = \langle G' \rangle \subseteq \mathcal{T}$ and suppose that $\{\alpha^{X,Y}\}_{X,Y \in \text{ind } \mathcal{S}}$ is a choice of generators for $\mathcal{S}$ as in Proposition 2.27. Then, there is a choice of generators $\{\beta^{X,Y}\}_{X,Y \in \text{ind } \mathcal{T}}$ such that $\beta^{X,Y} = \alpha^{X,Y}$ whenever $X, Y \in \mathcal{S}$.

Proof. As in the proof of Proposition 2.27, $\bar{P}$ and $\bar{P}'$ be the biggest (with respect to the order on $\langle G \rangle_1$) indecomposable summands of G and G' , respectively. Observe that $\bar{P}' \leq \bar{P}$.

- (1) We first construct a collection $\{\beta^{P,Q}\}_{P,Q \in \text{ind } \langle G \rangle_1}$ compatible with composition such that $\beta^{P,Q} = \alpha^{P,Q}$ whenever $P, Q \in \langle G' \rangle_1$.

We are given morphisms $\alpha^{\bar{P}', Q}$ for all $Q \leq \bar{P}'$ and $Q \in \langle G' \rangle_1$. Extend this collection by making choices of morphisms $\alpha^{\bar{P}', Q}$ for all $Q \leq \bar{P}'$ and $Q \in \langle G \rangle_1 \setminus \langle G' \rangle_1$. Moreover, we are also given morphisms $\alpha^{\bar{P}'[j], \bar{P}'}$ for $0 < j \in \mathbb{Z}$.

Next, fix a morphism $\beta^{\bar{P}, \bar{P}'}$. Note that for $0 < j \in \mathbb{Z}$, $\bar{P}' \leq \bar{P} \leq \bar{P}[j]$. Hence, by the definition of linear generators, there is an h such that

$$\alpha^{\bar{P}'[j], \bar{P}'} \circ \beta^{\bar{P}, \bar{P}'}[j] = \beta^{\bar{P}, \bar{P}'} \circ h.$$

We let $\beta^{\bar{P}[j], \bar{P}} := h$.

Furthermore, for all $Q \leq \bar{P}' \in \langle G \rangle_1$, let $\beta^{\bar{P}, Q} = \alpha^{\bar{P}', Q} \circ \beta^{\bar{P}, \bar{P}'}$. On the other hand, for $Q \in \langle G \rangle_1$ with $\bar{P}' < Q < \bar{P}$, just fix a choice of morphisms $\beta^{\bar{P}, Q}$. Finally, let $\beta^{\bar{P}, \bar{P}}$ be the identity. With this initial choice of morphisms, apply [Proposition 2.27\(1\)](#) to get a collection $\{\beta^{P, Q}\}_{P, Q \in \text{ind } \langle G \rangle_1}$.

It remains to prove that $\beta^{P, Q} = \alpha^{P, Q}$ whenever $P, Q \in \langle G' \rangle_1$. Well, by the proof of [Proposition 2.27](#) and our choices for $\alpha^{P, Q}$ and $\beta^{P, Q}$, there is an integer m such that there is a diagram

$$\begin{array}{ccccc}
 & \bar{P}[m] & & & \\
 & \swarrow \beta^{\bar{P}[m], P} & \downarrow \beta^{\bar{P}, \bar{P}'}[m] & \searrow \beta^{\bar{P}[m], Q} & \\
 P & \swarrow \alpha^{\bar{P}'[m], P} & \bar{P}'[m] & \searrow \alpha^{\bar{P}'[m], Q} & Q \\
 & \xleftarrow{\alpha^{P, Q}} & & \xrightarrow{\alpha^{P, Q}} & \\
 & \xleftarrow{\beta^{P, Q}} & & \xrightarrow{\beta^{P, Q}} &
 \end{array}$$

where the triangle with vertices $P, Q, \bar{P}[m]$ and edge $\beta^{P, Q}$, as well as the triangle with vertices $P, Q, \bar{P}'[m]$ and edge $\alpha^{P, Q}$ commute by construction of $\beta^{P, Q}$ and $\alpha^{P, Q}$. Moreover, the top left triangle commutes since

$$\begin{aligned}
 \alpha^{\bar{P}'[m], P} \circ \beta^{\bar{P}, \bar{P}'}[m] &= \alpha^{\bar{P}', P} \circ \alpha^{\bar{P}'[m], \bar{P}'} \circ \beta^{\bar{P}, \bar{P}'}[m] && \text{(Since } \{\alpha^{P, Q}\} \text{ is compatible with composition)} \\
 &= \alpha^{\bar{P}', P} \circ \beta^{\bar{P}, \bar{P}'} \circ \beta^{\bar{P}[m], \bar{P}} && \text{(By definition of } \beta^{\bar{P}[m], \bar{P}}) \\
 &= \beta^{\bar{P}[m], P}. && \text{(By definition of } \beta^{\bar{P}[m], P})
 \end{aligned}$$

Similarly, the top right triangle also commutes. Hence,

$$\alpha^{P, Q} \circ \beta^{\bar{P}[m], P} = \alpha^{P, Q} \circ \alpha^{\bar{P}'[m], P} \circ \beta^{\bar{P}, \bar{P}'}[m] = \alpha^{\bar{P}'[m], Q} \circ \beta^{\bar{P}, \bar{P}'}[m] = \beta^{\bar{P}[m], Q} = \beta^{P, Q} \circ \beta^{\bar{P}[m], P}$$

which is enough to conclude that $\alpha^{P, Q} = \beta^{P, Q}$, as required.

- (2) For the morphisms $\{\beta^{P, Z}\}_{P \in \text{ind } \langle G \rangle_1, Z \in \text{ind } \mathcal{T} \setminus \langle G \rangle_1}$, we make the obvious choice: If $Z = X_{R, S} \in \mathcal{S}$, then we let $\beta^{S, Z} = \alpha^{S, Z}$, otherwise if $Z = X_{R, S} \notin \mathcal{S}$, then we fix some morphism $\beta^{S, Z}$. For any $R \in \langle G \rangle_1$ such that there is a nonzero morphism $R \rightarrow S$, we let $\beta^{R, Z} = \beta^{S, Z} \circ \beta^{R, S}$. Thus, it is clear that $\beta^{P, Z} = \alpha^{P, Z}$ whenever $P, Z \in \mathcal{S}$.
- (3) The construction forwards morphisms $\{\beta^{Y, Z}\}_{Y, Z \in \text{ind } \mathcal{T} \setminus \langle G \rangle_1}$ follows exactly as in [Proposition 2.27\(4\)](#). It is clear from (1) that $\beta^{Y, Z} = \alpha^{Y, Z}$ for $Y, Z \in \mathcal{S}$.
- (4) To choose morphisms $\{\beta^{Y, P}\}_{P \in \text{ind } \langle G \rangle_1, Y \in \text{ind } \mathcal{T} \setminus \langle G \rangle_1}$, consider the object $\bar{X}' := X_{\bar{P}', \bar{Q}'}$ and note that we are given a morphism $\alpha^{\bar{X}', \bar{P}'[1]}: \bar{X}' \rightarrow \bar{P}'[1]$. Since there are forwards morphisms $X_{\bar{P}, \bar{Q}'} \rightarrow X_{\bar{P}, \bar{Q}}$ and $X_{\bar{P}, \bar{Q}} \rightarrow \bar{X}'$,

Proposition 2.26 implies that there are morphisms g and h making the diagram

$$\begin{array}{ccccc} X_{\bar{P}, \bar{Q}} & \xleftarrow{\beta^{X_{\bar{P}, \bar{Q}}, X_{\bar{P}, \bar{Q}}}} & X_{\bar{P}, \bar{Q}'} & \xrightarrow{\beta^{X_{\bar{P}, \bar{Q}'}, \bar{X}'}} & \bar{X}' \\ g \downarrow & & h \downarrow & & \downarrow \alpha^{\bar{X}', \bar{P}'[1]} \\ \bar{P}[1] & \xleftarrow{\beta^{\bar{P}, \bar{P}[1]}} & \bar{P}[1] & \xrightarrow{\beta^{\bar{P}, \bar{P}'[1]}} & \bar{P}'[1] \end{array}$$

commute. Let $\bar{X} := X_{\bar{P}, \bar{Q}}$ and $\beta^{\bar{X}, \bar{P}[1]} := g$. Then, construct the morphisms $\{\beta^{Y, P}\}_{P \in \text{ind } \langle G \rangle_1, Y \in \text{ind } \mathcal{T} \setminus \langle G \rangle_1}$ as in Proposition 2.27(5).

It remains to show that this choice of morphisms $\beta^{Y, P}$ restricts to $\alpha^{Y, P}$. To see this, let $S \in \langle G' \rangle_1$. We may construct the diagram

$$\begin{array}{ccccccc} X_{\bar{P}, \bar{Q}} & \xleftarrow{\beta^{X_{\bar{P}, \bar{Q}}, X_{\bar{P}, \bar{Q}}}} & X_{\bar{P}, \bar{Q}'} & \xleftarrow{\beta^{X_{\bar{P}, \bar{Q}'}, X_{\bar{P}, \bar{Q}'}}} & X_{\bar{P}, S} & \xrightarrow{\beta^{X_{\bar{P}, S}, X_{\bar{P}', S}}} & X_{\bar{P}', S} & \xrightarrow{\beta^{X_{\bar{P}', S}, X_{\bar{P}', \bar{Q}'}}} & \bar{X}' \\ \downarrow \beta^{X_{\bar{P}, \bar{Q}}, \bar{P}[1]} & & \downarrow \beta^{X_{\bar{P}, \bar{Q}'}, \bar{P}[1]} & & \downarrow \beta^{X_{\bar{P}, S}, \bar{P}[1]} & & \downarrow \alpha^{X_{\bar{P}', S}, \bar{P}[1]} & & \downarrow \alpha^{X', \bar{P}[1]} \\ \bar{P}[1] & \xrightarrow{\beta^{\bar{P}, \bar{P}[1]}} & \bar{P}[1] & \xrightarrow{\beta^{\bar{P}, \bar{P}[1]}} & \bar{P}[1] & \xrightarrow{\beta^{\bar{P}, \bar{P}'[1]}} & \bar{P}'[1] & \xrightarrow{\beta^{\bar{P}', \bar{P}'[1]}} & \bar{P}'[1] \end{array}$$

where all the morphisms in the first row are forwards. The two leftmost squares and the rightmost square commute by construction of the vertical morphisms. The outermost square commutes by (4). It is then straightforward to check that the third inner square commutes.

Next, let $X_{R, S} \in \mathcal{S}$, then, as in the proof of Proposition 2.27(5), there is an integer t such that there are forwards morphisms $X_{\bar{P}'[t], S} \rightarrow X_{R, S}$ and $X_{\bar{P}[t], S} \rightarrow X_{R, S}$, and so following diagram exists

$$\begin{array}{ccccccc} & & \xleftarrow{\beta^{X_{\bar{P}'[t], S}, X_{\bar{P}'[t], S}}} & & X_{\bar{P}[t], S} & & \\ & & & & \parallel & & \\ & & X_{\bar{P}'[t], S} & \xrightarrow{\beta^{X_{\bar{P}'[t], S}, X_{R, S}}} & X_{R, S} & \xleftarrow{\beta^{X_{\bar{P}[t], S}, X_{R, S}}} & X_{\bar{P}[t], S} \\ \alpha^{X_{\bar{P}'[t], S}, \bar{P}'[t+1]} \downarrow & & & & \downarrow \alpha^{X_{R, S}, R[1]} & & \downarrow \beta^{X_{R, S}, R[1]} \\ \bar{P}'[t+1] & \xrightarrow{\beta^{\bar{P}'[t+1], R[1]}} & R[1] & \xleftarrow{\beta^{\bar{P}[t+1], R[1]}} & \bar{P}[t+1] & & \\ \parallel & & & & \parallel & & \\ \bar{P}'[t+1] & \xleftarrow{\beta^{\bar{P}[t+1], \bar{P}'[t+1]}} & \bar{P}[t+1] & & & & \end{array}$$

Here, the outer square commutes by (4). Moreover, the two middle squares commute by construction of the morphisms $\alpha^{X_{R, S}, R[1]}$ and $\beta^{X_{R, S}, R[1]}$. It is, thus, straightforward to check that this implies $\alpha^{X_{R, S}, R[1]} = \beta^{X_{R, S}, R[1]}$, as required.

- (5) Finally, we construct backwards morphisms $\{\beta_{Y, Z}^{Y, Z}\}_{Y, Z \in \text{ind } \mathcal{T} \setminus \langle G \rangle_1}$ exactly as in Proposition 2.27(7). It is clear from the construction that if $Y, Z \in \mathcal{S}$ then $\beta_{Y, Z}^{Y, Z} = \alpha^{Y, Z}$.

□

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