Exact bounds for efficient consistent matrices obtained from a reciprocal matrix

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Abstract

For a given reciprocal matrix A, we give a union of matrix intervals in which any consistent matrix obtained from an efficient vector for A lies, and, conversely, any consistent matrix in this union comes from an efficient vector for A. The maximal sets of entries in the lower and upper bound matrices of each interval that are attainable by some consistent matrix in the interval are described. This allows us to understand which subsets of the alternatives lie above which other subsets in all efficient orders for each interval. As a result, the partial order on the alternatives dictated by the efficient vectors follows. Then, we use the tools developed to also show that, when the n-by-n reciprocal matrices A, B are simple perturbed consistent

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matrices, or $n = 4$, the sets of efficient vectors for A and B coincide only if $A = B$.

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1 Introduction

An n-by-n entry-wise positive matrix $A = [a_{ij}]$ is called *reciprocal* if $a_{ji} = \frac{1}{a_{ij}}$, for each pair $1 \leq i, j \leq n$. In particular, all diagonal entries are 1. The entries represent pair-wise ratio comparisons among n alternatives, so that such an A is also called a *pair-wise comparison matrix*. We denote the set of all such matrices by $\mathcal{PC}_n$. In a variety of multi-criterion decision models, a cardinal ranking vector $w \in \mathbb{R}^n_+$ (the set of positive n-vectors) is desired, for an $A \in \mathcal{PC}_n$, to represent relative weights for the n alternatives.

If $a_{ij}a_{jk} = a_{ik}$ for all triples $1 \leq i, j, k \leq n$, $A \in \mathcal{PC}_n$ is called *consistent*. In this case, $A = ww^{(-T)}$, in which $w \in \mathbb{R}^n_+$ and $w^{(-T)}$ is the entry-wise inverse of the transpose of w. Vector w, which is projectively unique (that is, unique up to a factor of scale), is the natural ranking vector. However, in practice, consistent matrices seldom arise. So, we may wish to approximate $A \in \mathcal{PC}_n$ by a consistent matrix, in order to choose w. In an early model [13], Saaty recommended the right Perron eigenvector of $A \in \mathcal{PC}_n$ as the cardinal ranking vector. When A is consistent, this is the correct vector, but, otherwise, the Perron eigenvector may not be the best choice [2, 3, 8].

A ranking vector w obtained from $A \in \mathcal{PC}_n$ should, at least, be one upon which no global improvement is possible (Pareto optimality). This means that $\left|A - vv^{(-T)}\right| \leq \left|A - ww^{(-T)}\right|$, entry-wise in absolute value, implies that $v \in \mathbb{R}^n_+$ is proportional to w. In this event, w is called *efficient* for $A \in \mathcal{PC}_n$ [2, 5] and the ordering of the alternatives indicated by the magnitudes of the entries of w is called an *efficient order*. We denote the set of all such vectors by $\mathcal{E}(A)$. This set is (projectively) infinite, unless A is consistent, in which case it consists of the positive multiples of any column of A. A characterization of efficiency of a vector $w \in \mathbb{R}^n_+$ in terms of a directed graph appeared in [2] (see also [6]).

There are some vectors that may be calculated from $A \in \mathcal{PC}_n$, via a fixed formula, and are universally efficient. In [2] it was shown that the geometric mean of the columns of A is efficient for A. Later [6], this result was extended to any weighted geometric mean of the columns. More recently [11], the geometric mean of the right and entry-wise inverse left Perron vectors of A was shown to be efficient for A.

An inductive description of $\mathcal{E}(A)$ was presented in [10]. A characterization of $\mathcal{E}(A)$, for inconsistent $A \in \mathcal{PC}_n$, as a union of at most $\frac{(n-1)!}{2}$ convex sets was given in [7]. Each of these sets is associated with a Hamiltonian cycle product < 1 in the matrix.

In [12] we presented a matrix interval, depending on the entries of $A \in \mathcal{PC}_n$, for the matrix $W = ww^{(-T)}$ obtained from a $w \in \mathcal{E}(A)$. As a single matrix interval, this is the best possible, since any entry of the lower and upper bound matrices is attainable for some $w \in \mathcal{E}(A)$. This interval allowed us to characterize the matrices $A \in \mathcal{PC}_n$ for which all efficient orders are the same. This is especially important in applications in which an ordinal ranking is the goal. However, the obtained matrix interval may include consistent matrices that do not result from an efficient vector. We also have noticed that, for $A, B \in \mathcal{PC}_n$, if $\mathcal{E}(A) = \mathcal{E}(B)$ then the given matrix intervals coincide for A and B. If $n = 3$, this implies that $A = B$.

Here we give a union of at most $\frac{(n-1)!}{2}$ matrix intervals in which any matrix $W = ww^{(-T)}$ obtained from a $w \in \mathcal{E}(A)$ lies and such that any consistent matrix in it comes from an efficient vector. This improves the bounds in [12] and gives another way of describing $\mathcal{E}(A)$. This new result can give more precise information than the one in [12] concerning the possible efficient orders for A, or about how some alternatives are positioned relative to the others. We may see when some alternatives lie above the remaining ones for all efficient vectors from a particular interval, thus giving a complete description of the partial order based on the pair-wise comparisons. It may happen that one alternative, or a set of alternatives, is higher than the others in every efficient order. We also show that $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$ when $A, B \in \mathcal{PC}_n$ are simple perturbed consistent matrices, that is, are obtained from a consistent matrix by modifying one pair of reciprocal entries (this includes the matrices in $\mathcal{PC}_3$), and when $A, B \in \mathcal{PC}_4$.

We start in Section 2 with some useful background. In Section 3 we describe the so called path matrices associated with a Hamiltonian cycle in $A \in \mathcal{PC}_n$ and give some of their properties. These matrices are used in Section 4 to give a union of a finite number of matrix intervals such that a consistent matrix lies in it if and only if it is obtained from an efficient vector. In Section 5 we describe the maximal sets of entries of the lower (upper) bound matrix of each interval attained by a consistent matrix in that interval. Some consequences are given. In Section 6 we study the possible efficient orders corresponding to the consistent matrices in each interval of our union of intervals. In particular, we characterize when one set of alternatives is above another set in all efficient orders. In the remaining sections we focus on the problem of when $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$. In Section 7 we give some general comments and identify some circumstances in which this implication holds. Then, in Section 8, we show this result when both A and B are simple perturbed consistent matrices. In Section 9 we give necessary conditions for $\mathcal{E}(A) = \mathcal{E}(B)$ that allow us to obtain a condition under which $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$. The given necessary conditions are also used in Section 10 to show that the claimed implication holds when $n = 4$. We conclude with some observations in Section 11.

2 Preliminaries

We start with some additional notation that we will use. Let $n > 1$ be an integer. By M_n we denote the set of n-by-n real matrices. A product of a permutation and a positive diagonal matrix is called a *monomial matrix*. We denote $N = \{1, \ldots, n\}$ and $\mathcal{N} = \{(x, y) \in N \times N : x \neq y\}$. Given a set $S \subseteq N \times N$ and $A \in M_n$, we denote by $A[S]$ the entries of A indexed by S. We write $A[S] = B[S]$ if, in each position indexed by S, the entries in A and $B \in M_n$ coincide. We denote by $A^{(-T)}$ the entry-wise inverse of the transpose of A. Note that $A \in \mathcal{PC}_n$ if and only if A is positive and $A = A^{(-T)}$. For $A \in M_n$, a_{ij} denotes the i, j entry of A. Given a vector $w \in \mathbb{R}^n_+$, we denote its ith entry by w_i. By W we denote the consistent matrix $ww^{(-T)} = [\frac{w_i}{w_j}]$.

Any matrix in $\mathcal{PC}_2$ is consistent. We denote by $\mathcal{PC}^0_n$, $n \geq 3$, the set of matrices in $\mathcal{PC}_n$ that are not consistent. Each of $\mathcal{PC}_n$, $\mathcal{PC}^0_n$ and the set of consistent matrices is closed under monomial similarity. Such similarities interface well with $\mathcal{E}(A)$.

Lemma 1 *[5] Suppose that $A \in \mathcal{PC}_n$. If $S \in M_n$ is a monomial matrix, then $\mathcal{E}(SAS^{-1}) = S\mathcal{E}(A)$.*

We call a sequence $\tau : \tau_1\tau_2 \cdots \tau_n\tau_1$, in which $\tau_1\tau_2 \cdots \tau_n$ is a permutation of N, a *Hamiltonian cycle* (H-cycle) in N. Note that an H-cycle can be listed starting with any index. For $A \in \mathcal{PC}_n$, we denote by $\tau(A)$ the product of the entries of A along τ. We denote by $\Gamma(A)$ the set of H-cycles τ in N such that $\tau(A) < 1$. If A is inconsistent, there is at least one H-cycle in $\Gamma(A)$. Also, there are at most $\frac{(n-1)!}{2}$ such cycles.

If $A \in \mathcal{PC}_n$ and τ is an H-cycle in N, we denote by $P_{A,\tau}(i,j)$, $i \neq j$, the product of the entries in A along the path from i to j in τ. We have $P_{A,\tau}(\tau_1, \tau_{i+1}) = P_{A,\tau}(\tau_1, \tau_i)P_{A,\tau}(\tau_i, \tau_{i+1}) = P_{A,\tau}(\tau_1, \tau_i)a_{\tau_i,\tau_{i+1}}$, $i = \{2, \ldots, n-1\}$, and $\tau(A) = P_{A,\tau}(\tau_1, \tau_n)P_{A,\tau}(\tau_n, \tau_1)$.

Example 2 *If* $A = [a_{ij}] \in \mathcal{PC}_4$ *and* $\tau = 12341$, *then* $\tau(A) = a_{12}a_{23}a_{34}a_{41}$. *Also,* $P_{A,\tau}(1,3) = a_{12}a_{23}$, $P_{A,\tau}(3,2) = a_{34}a_{41}a_{12}$ *and* $P_{A,\tau}(1,4) = P_{A,\tau}(1,3)P_{A,\tau}(3,4) = a_{12}a_{23}a_{34}$.

From now on, we focus on matrices in $A \in \mathcal{PC}_n^0$.

Next we describe $\mathcal{E}(A)$ as a finite union of convex sets [7]. For $\tau \in \Gamma(A)$, let

$$\mathcal{E}_\tau(A) = \left\{w \in \mathbb{R}^n_+ : w_{\tau_1} \geq P_{A,\tau}(\tau_1, \tau_2)w_{\tau_2} \geq \cdots \geq P_{A,\tau}(\tau_1, \tau_n)w_{\tau_n} \geq \tau(A)w_{\tau_1}\right\}.$$

The set $\mathcal{E}_\tau(A)$ is convex and is the cone generated by the (projectively distinct) n vectors satisfying

$$w_{\tau_1} \geq P_{A,\tau}(\tau_1, \tau_2)w_{\tau_2} \geq \cdots \geq P_{A,\tau}(\tau_1, \tau_n)w_{\tau_n} \geq \tau(A)w_{\tau_1}, \qquad (1)$$

with $n-1$ inequalities replaced by equalities. The n rays corresponding to these vectors are the extremes of the cone $\mathcal{E}_\tau(A)$. Since $\tau(A) \neq 1$, there is no vector w satisfying (1) with all inequalities replaced by equalities.

Theorem 3 *[7] Let* $A \in \mathcal{PC}_n^0$. *Then*

$$\mathcal{E}(A) = \bigcup_{\tau \in \Gamma(A)} \mathcal{E}_\tau(A).$$

Since $\mathcal{E}(A)$ is known to be connected [2, 10], for any $\tau \in \Gamma(A)$, there is another $\mu \in \Gamma(A)$ such that $\mathcal{E}_\tau(A)$ and $\mathcal{E}_\mu(A)$ intersect. However, $\mathcal{E}(A)$ is not generally convex, but it is, for example, when $\#\Gamma(A) = 1$ (in which case $n \leq 4$), or if A is obtained from a consistent matrix by modifying one pair of reciprocal entries [1, 4] (See Section 8).

An important fact noticed in [12] is the following.

Theorem 4 *Let $A \in \mathcal{PC}_n^0$. If $\tau \in \Gamma(A)$ and $w \in \mathcal{E}_\tau(A)$, then $\frac{w_i}{w_j} \geq P_{A,\tau}(i,j)$ for any $i,j \in N$, $i \neq j$. Moreover, there is a $w \in \mathcal{E}_\tau(A)$ such that $\frac{w_i}{w_j} = P_{A,\tau}(i,j)$.*

In [12], the best single matrix interval for the consistent matrices $ww^{(-T)} = \left[\frac{w_i}{w_j}\right]$ obtained from an efficient vector w for $A \in \mathcal{PC}_n^0$ has been given. Let $L_A = [l_{ij}] \in M_n$ be defined by $l_{ii} = 1$ and

$$l_{ij} = \min_{\tau \in \Gamma(A)} P_{A,\tau}(i,j),$$

$i,j \in N$, $i \neq j$. Let $U_A = (L_A)^{(-T)}$ be the entry-wise inverse of L_A^T.

Theorem 5 *[12] Let $A \in \mathcal{PC}_n^0$. Let $w \in \mathcal{E}(A)$ and $W = ww^{(-T)}$. Then*

$$L_A \leq W \leq U_A.$$

In [9] the existence of a unique efficient order for $A \in \mathcal{PC}_n^0$ was first studied. Then, in [12], this was characterized using the matrices L_A and U_A.

Theorem 6 *[12] Let $A \in \mathcal{PC}_n^0$ and $L_A = [l_{ij}]$. The following are equivalent:*

1. *there is only one efficient order for A;*
2. *there is a permutation $i_1 i_2 \cdots i_n$ of N such that $l_{i_t,i_{t+1}} \geq 1$ for $t = 1, , \ldots, n-1$; and*
3. *L_A has exactly $\frac{n^2-n}{2}$ off-diagonal entries ≥ 1.*

3 Path matrices

Given $A \in \mathcal{PC}_n^0$ and τ an H-cycle in N, we call the matrix $P_{A,\tau} = [p_{ij}]$ defined by $p_{ij} = P_{A,\tau}(i,j)$ and $p_{ii} = 1$, $i,j \in N$, $i \neq j$, the τ *path matrix* for A. The matrix $P_{A,\tau}$ is determined by the H-cycle τ and the entries in A along it. In the next section we will use the path matrices for A and its H-cycles in $\Gamma(A)$ to give an exact collection of intervals in which the consistent matrices constructed from efficient vectors for A lie.

Recall that

$$p_{ij}p_{ji} = \tau(A), \tag{2}$$

for $i \neq j$, and observe that

$$P_{A,\tau} = (P_{A,\mathrm{rev}\,\tau})^{(-T)},$$

in which $\mathrm{rev}\,\tau$ denotes the H-cycle that is reverse to τ.

Remark 7 *Let $A, B \in \mathcal{PC}_n^0$. Let τ and ν be H-cycles in N. If there are $i, j \in N$, $i \neq j$, such that $P_{B,\nu}(i,j) \leq P_{A,\tau}(i,j)$ and $P_{B,\nu}(j,i) \leq P_{A,\tau}(j,i)$, then $\nu(B) \leq \tau(A)$.*

Example 8 *Let*

$$A = \begin{bmatrix} 1 & 1 & a_{13} & a_{14} \\ 1 & 1 & 1 & a_{24} \\ \frac{1}{a_{13}} & 1 & 1 & 1 \\ \frac{1}{a_{14}} & \frac{1}{a_{24}} & 1 & 1 \end{bmatrix} \in \mathcal{PC}_4^0.$$

There are 6 *H-cycles in* $\{1,2,3,4\}$:

$$\alpha = 12341; \quad \beta = 12431; \quad \gamma = 13241; \tag{3}$$

and their reverses

$$\alpha' = 14321; \quad \beta' = 13421; \quad \gamma' = 14231. \tag{4}$$

We have

$$\begin{aligned} \alpha(A) &= \frac{1}{a_{14}}, \quad \beta(A) = \frac{a_{24}}{a_{13}}, \quad \gamma(A) = \frac{a_{13}a_{24}}{a_{14}}, \\ \alpha'(A) &= a_{14}, \quad \beta'(A) = \frac{a_{13}}{a_{24}}, \quad \gamma'(A) = \frac{a_{14}}{a_{13}a_{24}}. \end{aligned}$$

Also,

$$P_{A,\alpha} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{a_{14}} & 1 & 1 & 1 \\ \frac{1}{a_{14}} & \frac{1}{a_{14}} & 1 & 1 \\ \frac{1}{a_{14}} & \frac{1}{a_{14}} & \frac{1}{a_{14}} & 1 \end{bmatrix}, \quad P_{A,\alpha'} = \begin{bmatrix} 1 & a_{14} & a_{14} & a_{14} \\ 1 & 1 & a_{14} & a_{14} \\ 1 & 1 & 1 & a_{14} \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$P_{A,\beta} = \begin{bmatrix} 1 & 1 & a_{24} & a_{24} \\ \beta(A) & 1 & a_{24} & a_{24} \\ \frac{1}{a_{13}} & \frac{1}{a_{13}} & 1 & \beta(A) \\ \frac{1}{a_{13}} & \frac{1}{a_{13}} & 1 & 1 \end{bmatrix}, \quad P_{A,\beta'} = \begin{bmatrix} 1 & \beta'(A) & a_{13} & a_{13} \\ 1 & 1 & a_{13} & a_{13} \\ \frac{1}{a_{24}} & \frac{1}{a_{24}} & 1 & 1 \\ \frac{1}{a_{24}} & \frac{1}{a_{24}} & \beta'(A) & 1 \end{bmatrix},$$

$$P_{A,\gamma} = \begin{bmatrix} 1 & a_{13} & a_{13} & a_{13}a_{24} \\ \frac{a_{24}}{a_{14}} & 1 & \gamma(A) & a_{24} \\ \frac{a_{24}}{a_{14}} & 1 & 1 & a_{24} \\ \frac{1}{a_{14}} & \frac{a_{13}}{a_{14}} & \frac{a_{13}}{a_{14}} & 1 \end{bmatrix}, \quad P_{A,\gamma'} = \begin{bmatrix} 1 & \frac{a_{14}}{a_{24}} & \frac{a_{14}}{a_{24}} & a_{14} \\ \frac{1}{a_{13}} & 1 & 1 & \frac{a_{14}}{a_{13}} \\ \frac{1}{a_{13}} & \gamma'(A) & 1 & \frac{a_{14}}{a_{13}} \\ \frac{1}{a_{13}a_{24}} & \frac{1}{a_{24}} & \frac{1}{a_{24}} & 1 \end{bmatrix}.$$

We next see when two path matrices associated with H-cycles with cycle products < 1 (possibly, for two different matrices) are equal. We first give an important lemma that shows how $P_{A,\tau}$ changes under a monomial similarity on A. It allows us to consider the matrix A in a special form.

Lemma 9 *Let $A \in \mathcal{PC}_n$, let $D \in M_n$ be a positive diagonal matrix and $Q = [q_{ij}] \in M_n$ be a permutation matrix. Let τ be an H-cycle in N. Then, $D^{-1}P_{A,\tau}D = P_{D^{-1}AD,\tau}$, and $Q^T P_{A,\tau} Q = P_{Q^TAQ,\rho}$, in which ρ is the H-cycle satisfying $q_{\tau_i,\rho_i} = 1$.*

Proof. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$ and $i, j \in N$, $i \neq j$. Let $\tau = \tau_1 \ldots \tau_n \tau_1$ with $\tau_1 = i$ and $\tau_k = j$. We have

$$\begin{aligned} P_{D^{-1}AD,\tau}(i,j) &= \frac{d_{\tau_2}}{d_{\tau_1}} a_{\tau_1\tau_2} \frac{d_{\tau_3}}{d_{\tau_2}} a_{\tau_2\tau_3} \cdots \frac{d_{\tau_k}}{d_{\tau_{k-1}}} a_{\tau_{k-1}\tau_k} \\ &= \frac{d_{\tau_k}}{d_{\tau_1}} a_{\tau_1\tau_2} a_{\tau_2\tau_3} \cdots a_{\tau_{k-1}\tau_k} = \frac{d_j}{d_i} P_{A,\tau}(i,j). \end{aligned}$$

By definition of ρ, for $B \in M_n$, the ρ_i, ρ_j entry of Q^TBQ is the τ_i, τ_j entry of B (that is, the sequence of entries in B and in Q^TBQ along τ and ρ, respectively, coincide). Thus, the ρ_i, ρ_j entry of $Q^TP_{A,\tau}Q$ is $P_{A,\tau}(\tau_i, \tau_j)$. If $i < j$, the ρ_i, ρ_j entry of $P_{Q^TAQ,\rho}$ is $(Q^TAQ)_{\rho_i,\rho_{i+1}} \cdots (Q^TAQ)_{\rho_{j-1},\rho_j} = a_{\tau_i,\tau_{i+1}} \cdots a_{\tau_{j-1},\tau_j} = P_{A,\tau}(\tau_i, \tau_j)$. The proof is similar if $i > j$. ■

As in the proof of Lemma 9, it follows that, if $D \in M_n$ is a positive diagonal matrix and τ is an H-cycle in N, then $\tau(A) = \tau(D^{-1}AD)$.

Theorem 10 *Let $A, B \in \mathcal{PC}_n^0$. If $\tau \in \Gamma(A)$, $\nu \in \Gamma(B)$ and $P_{A,\tau} = P_{B,\nu}$, then $\tau = \nu$. Moreover, the sequence of entries in A and in B along τ $(= \nu)$ coincide. In particular, $\tau(A) = \nu(A)$.*

Proof. Taking into account Lemma 9, by a simultaneous monomial similarity on A and B, we assume that $\tau = 123\cdots n1$ and $a_{i,i+1} = 1$ for $i = 1,\ldots,n-1$. Then $a_{n1} = \tau(A)$. Let $\nu = \nu_1\nu_2\cdots\nu_n\nu_1$, with $\nu_1 = 1$. Suppose that $P_{A,\tau} = P_{B,\nu}$. In order to get a contradiction, suppose that $\nu \neq \tau$, that is, there is $i \in \{2,\ldots,n-1\}$ such that $\nu_i > \nu_{i+1}$. For such an i, we have $P_{A,\tau}(1,\nu_i) = P_{A,\tau}(1,\nu_{i+1}) = 1$. Also, $P_{A,\tau}(\nu_i,\nu_{i+1}) = \tau(A)$. Thus, if $P_{A,\tau} = P_{B,\nu}$, we have

$$\begin{aligned}1 &= P_{A,\tau}(1,\nu_{i+1}) = P_{B,\nu}(1,\nu_{i+1}) = P_{B,\nu}(1,\nu_i)P_{B,\nu}(\nu_i,\nu_{i+1}) \\ &= P_{A,\tau}(1,\nu_i)P_{A,\tau}(\nu_i,\nu_{i+1}) = \tau(A),\end{aligned}$$

a contradiction, since $\tau(A) < 1$. Thus, $\tau = \nu$. As, for $i \in \{1,\ldots,n-1\}$, we have $a_{i,i+1} = P_{A,\tau}(i,i+1) = P_{B,\tau}(i,i+1) = b_{i,i+1}$ and $a_{n,1} = P_{A,\tau}(n,1) = P_{B,\tau}(n,1) = b_{n,1}$, the second claim also follows. ■

Taking Theorem 4 into account, if $\mathcal{E}_\tau(A) = \mathcal{E}_\nu(B)$, then $P_{A,\tau}(i,j) = P_{B,\nu}(i,j)$, for any $i,j \in N$, that is, $P_{A,\tau} = P_{B,\nu}$. Thus, from Theorem 10, we get the following.

Corollary 11 *Let $A, B \in \mathcal{PC}_n^0$. Let $\tau \in \Gamma(A)$ and $\nu \in \Gamma(B)$. Then, $\mathcal{E}_\tau(A) = \mathcal{E}_\nu(B)$ if and only if $\tau = \nu$ and the sequence of entries in A and in B along τ $(= \nu)$ coincide.*

4 Exact Bounds for W

Now, we may give our finite union of matrix intervals in which all consistent matrices are exactly those that come from vectors in $\mathcal{E}(A)$.

Theorem 12 *Let $A \in \mathcal{PC}_n^0$. Let $\tau \in \Gamma(A)$, $w \in \mathbb{R}_+^n$ and $W = \left[\frac{w_i}{w_j}\right]$. Then, $w \in \mathcal{E}_\tau(A)$ if and only if*

$$P_{A,\tau} \leq W \leq P_{A,\tau}^{(-T)}. \tag{5}$$

Proof. Suppose that $w \in \mathcal{E}_\tau(A)$. By Theorem 4, $\frac{w_i}{w_j} \geq P_{A,\tau}(i,j)$ for all i,j, implying the lower bound. The upper bound follows from the lower bound taking into account that $W = W^{(-T)}$.

Now suppose that (5) holds. Then, $\frac{w_{\tau_i}}{w_{\tau_{i+1}}} \geq P_{A,\tau}(\tau_i,\tau_{i+1})$, $i = 1,\ldots,n-1$, and $\frac{w_{\tau_n}}{w_{\tau_1}} \geq P_{A,\tau}(\tau_n,\tau_1)$. Thus, $w \in \mathcal{E}_\tau(A)$. ■

Since $P_{A,\tau} \leq W$ is equivalent to $W \leq P_{A,\tau}^{(-T)}$, condition (5) is equivalent to $P_{A,\tau} \leq W$. Also, observe that, because of (2), if p_{ij}, $i \neq j$, is the i,j entry of $P_{A,\tau}$, then the i,j entry of $P_{A,\tau}^{(-T)}$ is $\frac{1}{\tau(A)} p_{ij}$.

Corollary 13 *Let $A \in \mathcal{PC}_n^0$ and $\tau, \nu \in \Gamma(A)$. If $P_{A,\tau} \leq P_{A,\nu}$ then $\mathcal{E}_\nu(A) \subseteq \mathcal{E}_\tau(A)$.*

Proof. Let $w \in \mathcal{E}_\nu(A)$. By Theorem 12, $P_{A,\nu} \leq W$. Then, by the hypothesis, $P_{A,\tau} \leq W$, implying $W \leq P_{A,\tau}^{(-T)}$. By Theorem 12, $w \in \mathcal{E}_\tau(A)$. ■

As an immediate consequence of Theorems 3 and 12, we give the main result of this section.

Theorem 14 *Let $A \in \mathcal{PC}_n^0$, $w \in \mathbb{R}_+^n$ and $W = ww^{(-T)}$. Then, $w \in \mathcal{E}(A)$ if and only if there is $\tau \in \Gamma(A)$ such that*

$$P_{A,\tau} \leq W \leq P_{A,\tau}^{(-T)}.$$

Notice that the matrix L_A in Theorem 5 is the entry-wise minimum of the matrices $P_{A,\tau}$, $\tau \in \Gamma(A)$.

Example 15 *Let*

$$A = \begin{bmatrix} 1 & 1 & 3 & 7 \\ 1 & 1 & 1 & 2 \\ \frac{1}{3} & 1 & 1 & 1 \\ \frac{1}{7} & \frac{1}{2} & 1 & 1 \end{bmatrix}.$$

We have $\Gamma(A) = \{\alpha, \beta, \gamma\}$, with α, β, γ as in (3). Moreover, $\alpha(A) = \frac{1}{7}$, $\beta(A) = \frac{2}{3}$ and $\gamma(A) = \frac{6}{7}$. We have (see Example 8)

$$P_{A,\alpha} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{7} & 1 & 1 & 1 \\ \frac{1}{7} & \frac{1}{7} & 1 & 1 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 1 \end{bmatrix}, \quad P_{A,\beta} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ \frac{2}{3} & 1 & 2 & 2 \\ \frac{1}{3} & \frac{1}{3} & 1 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 & 1 \end{bmatrix}, \quad P_{A,\gamma} = \begin{bmatrix} 1 & 3 & 3 & 6 \\ \frac{2}{7} & 1 & \frac{6}{7} & 2 \\ \frac{2}{7} & 1 & 1 & 2 \\ \frac{1}{7} & \frac{3}{7} & \frac{3}{7} & 1 \end{bmatrix}.$$

If $w \in \mathcal{E}(A)$, one of the following holds:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{7} & 1 & 1 & 1 \\ \frac{1}{7} & \frac{1}{7} & 1 & 1 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 1 \end{bmatrix} \leq W \leq \begin{bmatrix} 1 & 7 & 7 & 7 \\ 1 & 1 & 7 & 7 \\ 1 & 1 & 1 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix}; \tag{6}$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ \frac{2}{3} & 1 & 2 & 2 \\ \frac{1}{3} & \frac{1}{3} & 1 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 & 1 \end{bmatrix} \leq W \leq \begin{bmatrix} 1 & \frac{3}{2} & 3 & 3 \\ 1 & 1 & 3 & 3 \\ \frac{1}{2} & \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix}; \tag{7}$$

or

$$\begin{bmatrix} 1 & 3 & 3 & 6 \\ \frac{2}{7} & 1 & \frac{6}{7} & 2 \\ \frac{2}{7} & 1 & 1 & 2 \\ \frac{1}{7} & \frac{3}{7} & \frac{3}{7} & 1 \end{bmatrix} \leq W \leq \begin{bmatrix} 1 & \frac{7}{2} & \frac{7}{2} & 7 \\ \frac{1}{3} & 1 & 1 & \frac{7}{3} \\ \frac{1}{3} & \frac{7}{6} & 1 & \frac{7}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}. \tag{8}$$

5 Maximal sets of attainable bounds

For any $A \in \mathcal{PC}_n^0$, $\tau \in \Gamma(A)$ and $i, j \in N$, there is a $w \in \mathcal{E}_\tau(A)$ such that $\frac{w_i}{w_j} = P_{A,\tau}(i,j)$ (Theorem 4). Here we describe the maximal sets of positions in which $W = \left[\frac{w_i}{w_j}\right]$ and $P_{A,\tau}$ can coincide.

Definition 16 *Let $\tau = \tau_1\tau_2\cdots\tau_n\tau_1$, with $\tau_1 = k$, be an H-cycle in N. For $i = 1, \ldots, n-1$, let $S^{(\tau_i)}$ be the set of pairs (τ_i, τ_j), with $j = i+1, \ldots, n$. Let*

$$S_k^{(\tau)} = S^{(\tau_1)} \cup S^{(\tau_2)} \cup \cdots \cup S^{(\tau_{n-1})}.$$

Remark 17 *Let $\tau = \tau_1\tau_2\cdots\tau_n\tau_1$, with $\tau_1 = k$, be an H-cycle in N. We have that $(p,q) \in S_k^{(\tau)}$ if and only if there are $i, j \in N$, $i < j$, such that $p = \tau_i$ and $q = \tau_j$. In particular, $(p,q) \in S_k^{(\tau)}$ for all $q \in N$, $q \neq p$, if and only if $p = k$, and $(p,q) \in S_k^{(\tau)}$ for all $p \in N$, $p \neq q$, if and only if $q = \tau_n$. Moreover, if (p,q), $(q,r) \in S_k^{(\tau)}$ then $(p,r) \in S_k^{(\tau)}$. The set $S_k^{(\tau)}$ has $\frac{n^2-n}{2}$ pairs, none of which is the reversal of another. In fact, for any $p, q \in N$, $p \neq q$, exactly one of (p,q) or (q,p) is in $S_k^{(\tau)}$.*

Example 18 *Let $n = 4$, and α, β, γ be as (3). We have*

$$\begin{aligned} S_1^{(\alpha)} &= \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}, \\ S_2^{(\alpha)} &= \{(2,3),(2,4),(2,1),(3,4),(3,1),(4,1)\}, \\ S_3^{(\alpha)} &= \{(3,4),(3,1),(3,2),(4,1),(4,2),(1,2)\}, \\ S_4^{(\alpha)} &= \{(4,1),(4,2),(4,3),(1,2),(1,3),(2,3)\}. \end{aligned}$$

$$S_1^{(\beta)} = \{(1,2),(1,4),(1,3),(2,4),(2,3),(4,3)\},$$
$$S_2^{(\beta)} = \{(2,4),(2,3),(2,1),(4,3),(4,1),(3,1)\},$$
$$S_3^{(\beta)} = \{(3,1),(3,2),(3,4),(1,2),(1,4),(2,4)\},$$
$$S_4^{(\beta)} = \{(4,3),(4,1),(4,2),(3,1),(3,2),(1,2)\}.$$

$$S_1^{(\gamma)} = \{(1,3),(1,2),(1,4),(3,2),(3,4),(2,4)\},$$
$$S_2^{(\gamma)} = \{(2,4),(2,1),(2,3),(4,1),(4,3),(1,3)\},$$
$$S_3^{(\gamma)} = \{(3,2),(3,4),(3,1),(2,4),(2,1),(4,1)\},$$
$$S_4^{(\gamma)} = \{(4,1),(4,3),(4,2),(1,3),(1,2),(3,2)\}.$$

The sets $S_k^{(\mathrm{rev}\,\alpha)}$, $S_k^{(\mathrm{rev}\,\beta)}$, $S_k^{(\mathrm{rev}\,\gamma)}$, $k = 1,2,3,4$, *are obtained from the sets* $S_k^{(\alpha)}$, $S_k^{(\beta)}$, $S_k^{(\gamma)}$ *by reversing the pairs* (i,j) *with* $i \neq k$.

Each set $S_k^{(\tau)}$ uniquely determines τ and k.

Proposition 19 *Let* $k_1, k_2 \in N$ *and* τ, ν *be H-cycles in* N. *If* $k_1 \neq k_2$ *or* $\tau \neq \nu$, *then* $S_{k_1}^{(\tau)} \neq S_{k_2}^{(\nu)}$.

Proof. The fact that $k_1 \neq k_2$ implies $S_{k_1}^{(\tau)} \neq S_{k_2}^{(\nu)}$ follows from Remark 17. If $k \in N$, $\tau = \tau_1\tau_2\cdots\tau_n\tau_1$ and $\nu = \nu_1\nu_2\cdots\nu_n\nu_1$, with $\tau_1 = \nu_1 = k$, and $\tau \neq \nu$, then there are $i, j \in N$ such that i precedes j in $\tau_1\tau_2\cdots\tau_n$, and j precedes i in $\nu_1\nu_2\cdots\nu_n$. Then $(i,j) \in S_k^{(\tau)}$ but $(i,j) \notin S_k^{(\nu)}$ (in fact $(j,i) \in S_k^{(\nu)}$). ■

Each set $S_k^{(\tau)}$ determines a set of entries in the matrix $P_{A,\tau}$. It contains the off-diagonal entries in the kth row and in the τ_nth column of $P_{A,\tau}$ (assuming $\tau_1 = k$). Also note that the set of entries determined by $S_k^{(\tau)}$ is the transpose of the one determined by $S_{\tau_n}^{(\mathrm{rev}\,\tau)}$. We include examples when $n = 4$ that will be helpful later.

Example 20 *Let* A *be the matrix in Example 8 and* α, β, γ *and* α', β', γ' *be the H-cycles given in (3) and (4). The entries in* $P_{A,\alpha}$ *indexed by the sets*

$S_k^{(\alpha)}$ *(Example 18) are highlighted next:*

$$S_1^{(\alpha)}: \begin{bmatrix} 1 & \boxed{1} & \boxed{1} & \boxed{1} \\ \frac{1}{a_{14}} & 1 & \boxed{1} & \boxed{1} \\ \frac{1}{a_{14}} & \frac{1}{a_{14}} & 1 & \boxed{1} \\ \frac{1}{a_{14}} & \frac{1}{a_{14}} & \frac{1}{a_{14}} & 1 \end{bmatrix}, \quad S_2^{(\alpha)}: \begin{bmatrix} 1 & 1 & 1 & 1 \\ \boxed{\frac{1}{a_{14}}} & 1 & \boxed{1} & \boxed{1} \\ \boxed{\frac{1}{a_{14}}} & \frac{1}{a_{14}} & 1 & \boxed{1} \\ \boxed{\frac{1}{a_{14}}} & \frac{1}{a_{14}} & \frac{1}{a_{14}} & 1 \end{bmatrix}$$

$$S_3^{(\alpha)}: \begin{bmatrix} 1 & \boxed{1} & 1 & 1 \\ \frac{1}{a_{14}} & 1 & 1 & 1 \\ \boxed{\frac{1}{a_{14}}} & \boxed{\frac{1}{a_{14}}} & 1 & \boxed{1} \\ \boxed{\frac{1}{a_{14}}} & \boxed{\frac{1}{a_{14}}} & \frac{1}{a_{14}} & 1 \end{bmatrix}, \quad S_4^{(\alpha)}: \begin{bmatrix} 1 & \boxed{1} & \boxed{1} & 1 \\ \frac{1}{a_{14}} & 1 & \boxed{1} & 1 \\ \frac{1}{a_{14}} & \frac{1}{a_{14}} & 1 & 1 \\ \boxed{\frac{1}{a_{14}}} & \boxed{\frac{1}{a_{14}}} & \boxed{\frac{1}{a_{14}}} & 1 \end{bmatrix}.$$

The case of α' will not be used. The entries in $P_{A,\beta}$ indexed by the sets $S_k^{(\beta)}$ are highlighted next:

$$S_1^{(\beta)}: \begin{bmatrix} 1 & \boxed{1} & \boxed{a_{24}} & \boxed{a_{24}} \\ \beta(A) & 1 & \boxed{a_{24}} & \boxed{a_{24}} \\ \frac{1}{a_{13}} & \frac{1}{a_{13}} & 1 & \beta(A) \\ \frac{1}{a_{13}} & \frac{1}{a_{13}} & \boxed{1} & 1 \end{bmatrix}, \quad S_2^{(\beta)}: \begin{bmatrix} 1 & 1 & a_{24} & a_{24} \\ \boxed{\beta(A)} & 1 & \boxed{a_{24}} & \boxed{a_{24}} \\ \boxed{\frac{1}{a_{13}}} & \frac{1}{a_{13}} & 1 & \beta(A) \\ \boxed{\frac{1}{a_{13}}} & \frac{1}{a_{13}} & \boxed{1} & 1 \end{bmatrix}$$

$$S_3^{(\beta)}: \begin{bmatrix} 1 & \boxed{1} & a_{24} & \boxed{a_{24}} \\ \beta(A) & 1 & a_{24} & \boxed{a_{24}} \\ \boxed{\frac{1}{a_{13}}} & \boxed{\frac{1}{a_{13}}} & 1 & \boxed{\beta(A)} \\ \frac{1}{a_{13}} & \frac{1}{a_{13}} & 1 & 1 \end{bmatrix}, \quad S_4^{(\beta)}: \begin{bmatrix} 1 & \boxed{1} & a_{24} & a_{24} \\ \beta(A) & 1 & a_{24} & a_{24} \\ \boxed{\frac{1}{a_{13}}} & \boxed{\frac{1}{a_{13}}} & 1 & \beta(A) \\ \boxed{\frac{1}{a_{13}}} & \boxed{\frac{1}{a_{13}}} & \boxed{1} & 1 \end{bmatrix}.$$

The entries in $P_{A,\beta'}$ indexed by the sets $S_k^{(\beta')}$ are highlighted next:

$$S_1^{(\beta')}: \begin{bmatrix} 1 & \boxed{\beta'(A)} & \boxed{a_{13}} & \boxed{a_{13}} \\ 1 & 1 & a_{13} & a_{13} \\ \frac{1}{a_{24}} & \boxed{\frac{1}{a_{24}}} & 1 & \boxed{1} \\ \frac{1}{a_{24}} & \boxed{\frac{1}{a_{24}}} & \beta'(A) & 1 \end{bmatrix}, \quad S_2^{(\beta')}: \begin{bmatrix} 1 & \beta'(A) & \boxed{a_{13}} & \boxed{a_{13}} \\ \boxed{1} & 1 & \boxed{a_{13}} & \boxed{a_{13}} \\ \frac{1}{a_{24}} & \frac{1}{a_{24}} & 1 & \boxed{1} \\ \frac{1}{a_{24}} & \frac{1}{a_{24}} & \beta'(A) & 1 \end{bmatrix}$$

$$S_3^{(\beta')}: \begin{bmatrix} 1 & \beta'(A) & a_{13} & a_{13} \\ \boxed{1} & 1 & a_{13} & a_{13} \\ \boxed{\frac{1}{a_{24}}} & \boxed{\frac{1}{a_{24}}} & 1 & \boxed{1} \\ \boxed{\frac{1}{a_{24}}} & \boxed{\frac{1}{a_{24}}} & \beta'(A) & 1 \end{bmatrix}, \quad S_4^{(\beta')}: \begin{bmatrix} 1 & \beta'(A) & \boxed{a_{13}} & a_{13} \\ \boxed{1} & 1 & \boxed{a_{13}} & a_{13} \\ \frac{1}{a_{24}} & \frac{1}{a_{24}} & 1 & 1 \\ \boxed{\frac{1}{a_{24}}} & \boxed{\frac{1}{a_{24}}} & \boxed{\beta'(A)} & 1 \end{bmatrix}.$$

The entries in $P_{A,\gamma}$ indexed by the sets $S_k^{(\gamma)}$ are highlighted next:

$$S_1^{(\gamma)}: \begin{bmatrix} 1 & \boxed{a_{13}} & \boxed{a_{13}} & \boxed{a_{13}a_{24}} \\ \frac{a_{24}}{a_{14}} & 1 & \gamma(A) & \boxed{a_{24}} \\ \frac{a_{24}}{a_{14}} & \boxed{1} & 1 & \boxed{a_{24}} \\ \frac{1}{a_{14}} & \frac{a_{13}}{a_{14}} & \frac{a_{13}}{a_{14}} & 1 \end{bmatrix}, \quad S_2^{(\gamma)}: \begin{bmatrix} 1 & a_{13} & \boxed{a_{13}} & a_{13}a_{24} \\ \boxed{\frac{a_{24}}{a_{14}}} & 1 & \boxed{\gamma(A)} & \boxed{a_{24}} \\ \frac{a_{24}}{a_{14}} & 1 & 1 & a_{24} \\ \boxed{\frac{1}{a_{14}}} & \frac{a_{13}}{a_{14}} & \boxed{\frac{a_{13}}{a_{14}}} & 1 \end{bmatrix}$$

$$S_3^{(\gamma)}: \begin{bmatrix} 1 & a_{13} & a_{13} & a_{13}a_{24} \\ \boxed{\frac{a_{24}}{a_{14}}} & 1 & \gamma(A) & \boxed{a_{24}} \\ \boxed{\frac{a_{24}}{a_{14}}} & \boxed{1} & 1 & \boxed{a_{24}} \\ \boxed{\frac{1}{a_{14}}} & \frac{a_{13}}{a_{14}} & \frac{a_{13}}{a_{14}} & 1 \end{bmatrix}, \quad S_4^{(\gamma)}: \begin{bmatrix} 1 & \boxed{a_{13}} & \boxed{a_{13}} & a_{13}a_{24} \\ \frac{a_{24}}{a_{14}} & 1 & \gamma(A) & a_{24} \\ \frac{a_{24}}{a_{14}} & \boxed{1} & 1 & a_{24} \\ \boxed{\frac{1}{a_{14}}} & \boxed{\frac{a_{13}}{a_{14}}} & \boxed{\frac{a_{13}}{a_{14}}} & 1 \end{bmatrix}.$$

The entries in $P_{A,\gamma'}$ indexed by the sets $S_k^{(\gamma')}$ are highlighted next:

$$S_1^{(\gamma')}: \begin{bmatrix} 1 & \boxed{\frac{a_{14}}{a_{24}}} & \boxed{\frac{a_{14}}{a_{24}}} & \boxed{a_{14}} \\ \frac{1}{a_{13}} & 1 & \boxed{1} & \frac{a_{14}}{a_{13}} \\ \frac{1}{a_{13}} & \gamma'(A) & 1 & \frac{a_{14}}{a_{13}} \\ \frac{1}{a_{13}a_{24}} & \boxed{\frac{1}{a_{24}}} & \boxed{\frac{1}{a_{24}}} & 1 \end{bmatrix}, \quad S_2^{(\gamma')}: \begin{bmatrix} 1 & \frac{a_{14}}{a_{24}} & \frac{a_{14}}{a_{24}} & \boxed{a_{14}} \\ \boxed{\frac{1}{a_{13}}} & 1 & \boxed{1} & \boxed{\frac{a_{14}}{a_{13}}} \\ \boxed{\frac{1}{a_{13}}} & \gamma'(A) & 1 & \boxed{\frac{a_{14}}{a_{13}}} \\ \frac{1}{a_{13}a_{24}} & \frac{1}{a_{24}} & \frac{1}{a_{24}} & 1 \end{bmatrix}$$

$$S_3^{(\gamma')}: \begin{bmatrix} 1 & \boxed{\frac{a_{14}}{a_{24}}} & \frac{a_{14}}{a_{24}} & \boxed{a_{14}} \\ \frac{1}{a_{13}} & 1 & 1 & \frac{a_{14}}{a_{13}} \\ \boxed{\frac{1}{a_{13}}} & \boxed{\gamma'(A)} & 1 & \boxed{\frac{a_{14}}{a_{13}}} \\ \frac{1}{a_{13}a_{24}} & \boxed{\frac{1}{a_{24}}} & \frac{1}{a_{24}} & 1 \end{bmatrix}, \quad S_4^{(\gamma')}: \begin{bmatrix} 1 & \frac{a_{14}}{a_{24}} & \frac{a_{14}}{a_{24}} & a_{14} \\ \boxed{\frac{1}{a_{13}}} & 1 & \boxed{1} & \frac{a_{14}}{a_{13}} \\ \boxed{\frac{1}{a_{13}}} & \gamma'(A) & 1 & \frac{a_{14}}{a_{13}} \\ \boxed{\frac{1}{a_{13}a_{24}}} & \boxed{\frac{1}{a_{24}}} & \boxed{\frac{1}{a_{24}}} & 1 \end{bmatrix}.$$

Remark 21 *Let $A, B \in \mathcal{PC}_n^0$ and $\tau \in \Gamma(A)$. Let $k_1, k_2 \in N$, $k_1 \neq k_2$. Because of (2) and the fact that, for $i \neq j$, either (i,j) or (j,i) is in $S_{k_1}^{(\tau)}$, the matrix $P_{A,\tau}$ is uniquely determined given $\tau(A)$ and $P_{A,\tau}[S_{k_1}^{(\tau)}]$. Also, $P_{A,\tau}$ is uniquely determined by $P_{A,\tau}[S_{k_1}^{(\tau)} \cup S_{k_2}^{(\tau)}]$, as this set of entries of $P_{A,\tau}$ contains a pair of symmetrically located entries and, thus, determines $\tau(A)$.*

We now give the main result of this section. It shows that the sets $S_k^{(\tau)}$, $k = 1, \ldots, n$, of $\frac{n^2-n}{2}$ positions are the maximal sets in which $P_{A,\tau}$ and $W = \left[\frac{w_i}{w_j}\right]$ can coincide, for an efficient vector $w \in \mathcal{E}_\tau(A)$.

Theorem 22 *If $A \in \mathcal{PC}_n^0$ and $\tau \in \Gamma(A)$, then the following holds:*

1. *for each $k \in N$, there is a $w \in \mathcal{E}_\tau(A)$ such that $P_{A,\tau}[S_k^{(\tau)}] = W[S_k^{(\tau)}]$. The vector w is the solution of the inequalities in (1), with all of them replaced by equalities, except the one relating k and the index adjacently preceding it in τ (w is an extreme of $\mathcal{E}_\tau(A)$);*

2. *if $w \in \mathcal{E}_\tau(A)$ and $P_{A,\tau}[S] = W[S]$, for some $\emptyset \neq S \subseteq \mathcal{N}$, then there is a $k \in N$ such that $S \subseteq S_k^{(\tau)}$.*

Proof. Let $\tau = \tau_1\tau_2\cdots\tau_n\tau_1$, with $\tau_1 = k$. Let w be such that

$$w_{\tau_1} = P_{A,\tau}(\tau_1,\tau_2)w_{\tau_2} = P_{A,\tau}(\tau_1,\tau_3)w_{\tau_3} = \cdots = P_{A,\tau}(\tau_1,\tau_n)w_{\tau_n}.$$

Clearly, $w \in \mathcal{E}_\tau(A)$. Let $(p,q) \in S_k^{(\tau)}$. Then $q \neq k$. If $p = k$, then $\frac{w_p}{w_q} = \frac{w_k}{w_q} = P_{A,\tau}(k,q)$. If $p \neq k$, then $P_{A,\tau}(k,q) = P_{A,\tau}(k,p)P_{A,\tau}(p,q)$ and we have

$$\frac{w_p}{w_q} = \frac{\frac{w_k}{P_{A,\tau}(k,p)}}{\frac{w_k}{P_{A,\tau}(k,q)}} = \frac{P_{A,\tau}(k,q)}{P_{A,\tau}(k,p)} = P_{A,\tau}(p,q).$$

This proves the first claim.

Suppose that $w \in \mathcal{E}_\tau(A)$ and $P_{A,\tau}[S] = W[S]$. Vector w is a solution of the inequalities in (1), with at least one, and at most $n-1$, of them replaced by equalities. Any set $S_k^{(\tau)}$ corresponding to $n-1$ equalities in (1) that includes the aforementioned equalities verifies the second claim. ∎

For $S \subseteq \mathcal{N}$, we have $P_{A,\tau}[S] = W[S]$ if and only if $P_{A,\tau}^{(-T)}[S'] = W[S']$, in which the set S' is the transpose of S. Thus, the maximal sets of entries in which W coincides with $P_{A,\tau}^{(-T)}$ are the transposes of the sets $S_k^{(\tau)}$, $k \in N$, that is, the sets $S_k^{(\mathrm{rev}\,\tau)}$, $k \in N$. So, we focus on the sets of entries that can attain the lower bound in (5).

We finally show that, if $P_{A,\tau}$ and $P_{B,\nu}$ coincide in the positions indexed by $S_k^{(\tau)}$, then $\tau = \nu$ (though not necessarily does $\tau(A) = \nu(B)$).

Remark 23 *Suppose that the matrices $W = ww^{(-T)}$ and $P_{A,\tau}$, $\tau \in \Gamma(A)$, coincide in the positions indexed by $S_k^{(\tau)}$. If Q is a permutation matrix, then $(Q^Tw)(Q^Tw)^{(-T)} = Q^TWQ$ and $Q^TP_{A,\tau}Q$ coincide in the corresponding positions. By Theorem 22, this means that this set of positions is an*

$S_{k'}^{(\rho)}$, *where* ρ *is the H-cycle given in Lemma 9. If* D *is a diagonal matrix, then* $(D^{-1}w)(D^{-1}w)^{(-T)} = D^{-1}WD$ *and* $D^{-1}P_{A,\tau}D$ *coincide in the positions indexed by* $S_k^{(\tau)}$.

Theorem 24 *Let* $A, B \in \mathcal{PC}_n^0$, $\tau \in \Gamma(A)$, $\nu \in \Gamma(B)$ *and* $k \in N$. *If* $P_{A,\tau}[S_k^{(\tau)}] = P_{B,\nu}[S_k^{(\tau)}]$, *then* $\tau = \nu$.

Proof. Taking into account Remark 23, by a simultaneous monomial similarity on A and B, suppose that $\tau = 123\cdots n1$ and $a_{i,i+1} = 1$ for $i = 1, \ldots, n-1$. Then $a_{n1} = \tau(A)$. Let $\nu = \nu_1\nu_2\cdots\nu_n\nu_1$, with $\nu_1 = 1$. Suppose that $\nu \neq \tau$, in order to get a contradiction. In that case, there is an $i \in \{2, \ldots, n-1\}$ such that $\nu_i > \nu_{i+1}$. We have $P_{A,\tau}(1, \nu_i) = P_{A,\tau}(1, \nu_{i+1}) = 1$. Also, $P_{A,\tau}(\nu_i, \nu_{i+1}) = \tau(A)$.

Notice that, if $(p,q) \notin S_k^{(\tau)}$, $p \neq q$, then $(q,p) \in S_k^{(\tau)}$ and, by assumption, $P_{A,\tau}(q,p) = P_{B,\nu}(q,p)$. Since $P_{A,\tau}(p,q)P_{A,\tau}(q,p) = \tau(A)$ and $P_{B,\nu}(p,q)P_{B,\nu}(q,p) = \nu(B)$, we get $P_{A,\tau}(p,q) = sP_{B,\nu}(p,q)$, with $s = \frac{\tau(A)}{\nu(B)}$.

Since $\nu(B) < 1$, we will arrive to a contradiction in each of the possible situations that could occur.

Case 1: Suppose that $(1, \nu_i) \in S_k^{(\tau)}$. Then, $(1, \nu_{i+1}), (\nu_{i+1}, \nu_i) \in S_k^{(\tau)}$, implying that $(\nu_i, \nu_{i+1}) \notin S_k^{(\tau)}$. Thus,

$$\begin{aligned} 1 &= P_{A,\tau}(1, \nu_{i+1}) = P_{B,\nu}(1, \nu_{i+1}) = P_{B,\nu}(1, \nu_i)P_{B,\nu}(\nu_i, \nu_{i+1}) \\ &= \frac{1}{s}P_{A,\tau}(1, \nu_i)P_{A,\tau}(\nu_i, \nu_{i+1}) = \frac{\tau(A)}{s} = \nu(B). \end{aligned}$$

Case 2: Suppose that $(1, \nu_i) \notin S_k^{(\tau)}$. We consider two subcases.

Case 2.1: Suppose that $(1, \nu_{i+1}) \notin S_k^{(\tau)}$. As $(\nu_i, 1), (\nu_{i+1}, 1) \in S_k^{(\tau)}$, it follows that $(\nu_{i+1}, \nu_i) \in S_k^{(\tau)}$, implying that $(\nu_i, \nu_{i+1}) \notin S_k^{(\tau)}$. Thus,

$$\begin{aligned} 1 &= P_{A,\tau}(1, \nu_{i+1}) = sP_{B,\nu}(1, \nu_{i+1}) = sP_{B,\nu}(1, \nu_i)P_{B,\nu}(\nu_i, \nu_{i+1}) \\ &= \frac{1}{s}P_{A,\tau}(1, \nu_i)P_{A,\tau}(\nu_i, \nu_{i+1}) = \frac{\tau(A)}{s} = \nu(B). \end{aligned}$$

Case 2.2: Suppose that $(1, \nu_{i+1}) \in S_k^{(\tau)}$. Then $(\nu_{i+1}, \nu_i) \notin S_k^{(\tau)}$, as $(1, \nu_i) \notin S_k^{(\tau)}$, implying that $(\nu_i, \nu_{i+1}) \in S_k^{(\tau)}$. Thus,

$$\begin{aligned} 1 &= P_{A,\tau}(1, \nu_{i+1}) = P_{B,\nu}(1, \nu_{i+1}) = P_{B,\nu}(1, \nu_i)P_{B,\nu}(\nu_i, \nu_{i+1}) \\ &= \frac{1}{s}P_{A,\tau}(1, \nu_i)P_{A,\tau}(\nu_i, \nu_{i+1}) = \frac{\tau(A)}{s} = \nu(B). \end{aligned}$$

$\blacksquare$

We note that the first claim in Theorem 10 follows from Theorem 24.

6 Orders occurring among the efficient vectors

First we ask which orders occur among vectors in $\mathcal{E}_\tau(A)$.

Example 25 *In Example 15 we can see that there is a unique order for the efficient vectors w for which the matrix W satisfies each of the inequalities (6), (7) and (8). In case (6), all vectors are decreasing as $\frac{w_i}{w_{i+1}} \geq 1$ for $i = 1, 2, 3$. In case (7), all vectors have the order*

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix},$$

as $\frac{w_1}{w_2} \geq 1$, $\frac{w_2}{w_4} \geq 1$ and $\frac{w_4}{w_3} \geq 1$. Similarly, in case (8), all vectors have the order

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}.$$

Thus, among all the efficient vectors only the 3 mentioned orders occur. In particular, we can observe that, in all efficient vectors, alternative 1 is above the other alternatives. The latter fact can also be deduced from the matrix interval for W, with $w \in \mathcal{E}(A)$, given in Theorem 5:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{7} & 1 & \frac{6}{7} & 1 \\ \frac{1}{7} & \frac{1}{7} & 1 & \frac{2}{3} \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 1 \end{bmatrix} \leq W \leq \begin{bmatrix} 1 & 7 & 7 & 7 \\ 1 & 1 & 7 & 7 \\ 1 & \frac{7}{6} & 1 & 7 \\ 1 & 1 & \frac{3}{2} & 1 \end{bmatrix}. \tag{9}$$

However, these bounds give less information than (6), (7) and (8). For instance, from (6), (7) and (8), we have that, if $w \in \mathcal{E}(A)$ and $\frac{w_2}{w_3} = \frac{6}{7}$, then $\frac{w_2}{w_4} \geq 2$ and $\frac{w_1}{w_4} \geq 6$. This does not follow from (9).

It may happen that all efficient vectors have a unique order, even when $\#\Gamma(A) > 1$.

Example 26 *Let*

$$A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ \frac{1}{3} & \frac{1}{2} & 1 & 1 \\ \frac{1}{4} & \frac{1}{3} & 1 & 1 \end{bmatrix}.$$

We have $\Gamma(A) = \{\alpha, \gamma'\}$, *with* α, γ' *as in (3) and (4). Moreover,* $\alpha(A) = \frac{1}{2}$ *and* $\gamma'(A) = \frac{8}{9}$. *We have*

$$P_{A,\alpha} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ \frac{1}{2} & 1 & 2 & 2 \\ \frac{1}{4} & \frac{1}{4} & 1 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}, \quad P_{A,\gamma'} = \begin{bmatrix} 1 & \frac{4}{3} & \frac{8}{3} & 4 \\ \frac{2}{3} & 1 & 2 & \frac{8}{3} \\ \frac{1}{3} & \frac{4}{9} & 1 & \frac{4}{3} \\ \frac{2}{9} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}.$$

All efficient vectors have (weakly) decreasing order since $P_{A,\alpha}(i, i+1) \geq 1$ *and* $P_{A,\gamma'}(i, i+1) \geq 1$, $i = 1, 2, 3$. *This can also be deduced from* L_A *using Theorem 5.*

It may happen that a set of alternatives always lies above the remaining alternatives without any particular order within the set.

Example 27 *Let*

$$A = \begin{bmatrix} 1 & \frac{1}{2} & * & 4 \\ 2 & 1 & 6 & * \\ * & \frac{1}{6} & 1 & \frac{1}{2} \\ \frac{1}{4} & * & 2 & 1 \end{bmatrix}.$$

(A reciprocal pair of $*$*'s means an arbitrary reciprocal pair.) Then* $\alpha = 12341 \in \Gamma(A)$. *We have*

$$P_{A,\alpha} = \begin{bmatrix} 1 & \frac{1}{2} & 3 & \frac{3}{2} \\ \frac{3}{4} & 1 & 6 & 3 \\ \frac{1}{8} & \frac{1}{16} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} & \frac{3}{4} & 1 \end{bmatrix}.$$

We can see that alternatives 1 *and* 2 *are above alternatives* 3 *and* 4 *in all vectors in* $\mathcal{E}_\alpha(A)$. *On the other hand, since any entry* i, j *of* $P_{A,\alpha}$ *is attained* *by $\frac{w_i}{w_j}$ for some $w \in \mathcal{E}_\alpha(A)$, there are $u, v \in \mathcal{E}_\alpha(A)$ such that $u_1 > u_2$ and $v_2 > v_1$. Similarly, there are $u, v \in \mathcal{E}_\alpha(A)$ such that $u_3 > u_4$ and $v_4 > v_3$. In fact, the possible orders of the vectors in $\mathcal{E}_\alpha(A)$ are*

$$\begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

Each is attained by an extreme vector of $\mathcal{E}_\alpha(A)$. Vector $w \in \mathcal{E}_\alpha(A)$ such that $W[S_k^{(\alpha)}] = P_{A,\alpha}[S_k^{(\alpha)}]$ has the first, second and third orders above for $k = 1$, 4 and 2, respectively. Note that the existence of such vectors w is guaranteed by Theorem 22. The decreasing order cannot occur since in that case, for some $w \in \mathcal{E}_\alpha(A)$ we would have $\frac{w_1}{w_2} \geq 1$ and $\frac{w_3}{w_4} \geq 1$. On the other hand, by Theorem 12, $\frac{w_3}{w_2} \leq \frac{1}{6}$ and $\frac{w_1}{w_4} \leq 4$. This gives the contradiction $6 \leq 4$.

By Theorems 4 and 12, we have the following.

Theorem 28 *Let $A \in \mathcal{PC}_n^0$, $\tau \in \Gamma(A)$, and $i, j \in N$, $i \neq j$. We have $P_{A,\tau}(i,j) \geq 1$ if and only if $w_i \geq w_j$ for all $w \in \mathcal{E}_\tau(A)$.*

Let $i_1, \ldots, i_p, j_1, \ldots, j_q \in N$ be distinct and $p + q = n$, $p, q \geq 1$. We have $P_{A,\tau}(i,j) \geq 1$ for any $i \in \{i_1, \ldots, i_p\}$ and $j \in \{j_1, \ldots, j_q\}$ if and only if alternatives $i_1, \ldots, i_p$ are above alternatives $j_1, \ldots, j_q$ in all $w \in \mathcal{E}_\tau(A)$. If the same holds for the matrix obtained from $P_{A,\tau}$ by deleting rows and columns $i_1, \ldots, i_p$, then we get a new set of alternatives that is above the remaining ones. In that way, we may identify a partition $S_1, \ldots, S_t$ of N such that any alternative in S_i is (weakly) above any alternative in S_{i+1} for all $w \in \mathcal{E}_\tau(A)$, and no such partition is possible within any set S_i (we may have $t = 1$). This gives a partial order for the alternatives in $\mathcal{E}_\tau(A)$.

In particular, if all entries of the kth row (kth column) of $P_{A,\tau}$ are ≥ 1, then alternative k is above (below) all the remaining alternatives, for all $w \in \mathcal{E}_\tau(A)$.

The next result gives necessary and sufficient conditions for all efficient vectors in a set $\mathcal{E}_\tau(A)$ to have a unique order.

Theorem 29 *Let $A \in \mathcal{PC}_n^0$ and $\tau \in \Gamma(A)$. The following are equivalent:*

1. *all vectors in* $\mathcal{E}_\tau(A)$ *have the same order;*

2. *there is a permutation* $i_1 i_2 \cdots i_n$ *of* N *such that* $P_{A,\tau}(i_t, i_{t+1}) \geq 1$ *for* $t = 1, \ldots, n-1$*;*

3. $P_{A,\tau}$ *has exactly* $\frac{n^2-n}{2}$ *off-diagonal entries* ≥ 1*;*

4. *for* $i, j \in N$*,* $i > j$*,* $P_{A,\tau}(i,j) \geq 1$ *or* $P_{A,\tau}(j,i) \geq 1$*.*

Proof. The proof of the equivalence between 1., 2. and 3. is similar to the proof of Theorem 6, given in [12]. Since $P_{A,\tau}(i,j)P_{A,\tau}(j,i) = \tau(A) < 1$, conditions 3. and 4. are equivalent. ■

If $w_i \geq w_j$ for every w in every $\mathcal{E}_\tau(A)$, $\tau \in \Gamma(A)$, then, of course, $w_i \geq w_j$ for every $w \in \mathcal{E}(A)$. But then, since $l_{ij} = \min_{\tau\in\Gamma(A)} P_{A,\tau}(i,j) \geq 1$ (Theorem 28), the same conclusion could have been drawn using L_A and Theorem 5. Similarly, facts that result from observations about orders that are common to every $\mathcal{E}_\tau(A)$ may also be derived from L_A. For example, if the partial order on the alternatives in each set is the same, it also follows from L_A in a similar way. See Examples 25 and 26.

7 $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$: background

We conjecture that, for $A, B \in \mathcal{PC}_n$, if $\mathcal{E}(A) = \mathcal{E}(B)$, then $A = B$. Here we give some general background and facts relevant to this conjecture, before proving it in some particular situations in later sections. First we note a simple case in which the conjecture is true.

If $A, B \in \mathcal{PC}_n$, A is consistent and $\mathcal{E}(A) = \mathcal{E}(B)$, then $A = B$. This is so since all columns of A are proportional and $\mathcal{E}(A)$ consists of the positive multiples of the columns of A. Since the columns of B are in $\mathcal{E}(B)$ [5], $\mathcal{E}(A) = \mathcal{E}(B)$ implies that all columns of A and B are proportional. They are uniquely determined because the diagonal entries are 1. So, $A = B$. Thus, we focus on the conjecture when $A, B \in \mathcal{PC}_n^0$.

In [12] we have noticed the following.

Theorem 30 *Let* $A, B \in \mathcal{PC}_n^0$*. If* $\mathcal{E}(A) = \mathcal{E}(B)$*, then* $L_A = L_B$*.*

It may happen that $L_A = L_B$ does not imply $\mathcal{E}(A) = \mathcal{E}(B)$.

Example 31 *Let*

$$A=\begin{bmatrix} 1 & 1 & \frac{1}{10} & 9 & 500 \\ 1 & 1 & 1 & 5 & 11 \\ 10 & 1 & 1 & 1 & 30 \\ \frac{1}{9} & \frac{1}{5} & 1 & 1 & 1 \\ \frac{1}{500} & \frac{1}{11} & \frac{1}{30} & 1 & 1 \end{bmatrix} \text{ and } B=\begin{bmatrix} 1 & 1 & \frac{1}{10} & 9 & 500 \\ 1 & 1 & 1 & 5 & 11 \\ 10 & 1 & 1 & 1 & \mathbf{30.1} \\ \frac{1}{9} & \frac{1}{5} & 1 & 1 & 1 \\ \frac{1}{500} & \frac{1}{11} & \frac{\mathbf{1}}{\mathbf{30.1}} & 1 & 1 \end{bmatrix}.$$

We have

$$L_A=L_B=\begin{bmatrix} 1 & \frac{1}{110} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{500} & 1 & \frac{1}{1000} & \frac{11}{5000} & \frac{1}{10} \\ \frac{1}{500} & \frac{1}{500} & 1 & \frac{99}{500} & 1 \\ \frac{1}{500} & \frac{1}{5000} & \frac{1}{5000} & 1 & \frac{11}{90} \\ \frac{1}{500} & \frac{1}{25000} & \frac{1}{5000} & \frac{1}{5000} & 1 \end{bmatrix}.$$

However, $\mathcal{E}(A)\neq\mathcal{E}(B)$. For example,

$$w=\begin{bmatrix} 5 & 5 & \frac{3}{10} & 1 & \frac{1}{100} \end{bmatrix}^T\in\mathcal{E}(A),$$

but $w\notin\mathcal{E}(B)$. See [2, 6] for a test of efficiency.

We have the following consequence of Theorem 30, noticed in [12].

Corollary 32 *Let $A,B\in\mathcal{PC}_n^0$. If $A\neq B$ and A and B are positive diagonally similar then $\mathcal{E}(A)\neq\mathcal{E}(B)$.*

Proof. If $B=DAD^{-1}$, in which $D=\operatorname{diag}(d_1,\ldots,d_n)$ is a positive diagonal matrix, then $L_B=DL_AD^{-1}$. Since there are i,j such that $\frac{d_i}{d_j}\neq 1$, it follows that the i,j entries of L_A and L_B are distinct. Thus, the result follows from Theorem 30. ■

When $n=3$, our conjecture is known to be true [12]. In the next section we consider a class of matrices in $\mathcal{PC}_n$ that coincides with $\mathcal{PC}_3$ when $n=3$.

8 $\mathcal{E}(A)=\mathcal{E}(B)$ implies $A=B$: the simple perturbed case

A simple perturbed consistent (SPC) matrix $A\in\mathcal{PC}_n$ is a reciprocal matrix obtained from a consistent one by modifying one symmetrically placed pair of reciprocal entries [1]. Such a matrix (inconsistent) is diagonally similar to a matrix $S_{l,k}(x)$ with entries $x \neq 1$ and $\frac{1}{x}$ in positions l, k and k, l $(l \neq k)$, respectively, and all the remaining entries 1. With an additional permutation similarity, l and k can be made 1 and n, respectively, and x may be assumed > 1. (If A is consistent it is diagonally similar to J_n.) From Theorem 3 (see also [4]), $w \in \mathcal{E}(S_{1,n}(x))$, $x > 1$, if and only if

$$w_n \leq w_i \leq w_1 \leq x w_n,\ i = 2, \ldots, n-1.$$

If $A = QS_{1,n}(x)Q^{-1}$, in which Q is a monomial matrix, by Lemma 1, $v \in \mathcal{E}(A)$ if and only if $Q^{-1}v \in \mathcal{E}(S_{1,n}(x))$.

Theorem 33 *Let $A, B \in \mathcal{PC}_n^0$ be SPC matrices. Then, $L_A = L_B$ implies $A = B$. Therefore, $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$.*

Proof. The second claim follows from the first one by Theorem 30. To show the first claim, by a simultaneous monomial similarity, suppose, without loss of generality, that $A = S_{1,n}(x)$, $x > 1$, and $B = DS_{l,k}(y)D^{-1}$, in which $D = \operatorname{diag}(d_1, \ldots, d_n)$ is a positive diagonal matrix and $y > 1$. If $n = 3$, we may assume $l = 1$ and $k = 3$, or $l = 3$ and $k = 1$.

Note that there are $(n-2)!$ H-cycles in $\Gamma(A)$, which are precisely those for which 1 adjacently succeeds n. All have cycle products equal to $\frac{1}{x}$. Similarly, the H-cycles in $\Gamma(B)$ are precisely those for which l adjacently succeeds k. The matrix L_A has the entries in row 1 and the entries in column n equal to 1, and all the remaining off-diagonal entries equal to $\frac{1}{x}$. The matrix L_B has the l, j and i, k entries, with $j \neq l$ and $i \neq k$, equal to $\frac{d_l}{d_j}$ and $\frac{d_i}{d_k}$, and the remaining entries in positions i, j, with $i \neq j$, equal to $\frac{d_i}{d_j}\frac{1}{y}$. Suppose that $L_A = L_B$. We want to see that D is scalar, $l = 1$, $k = n$ and $x = y$.

Suppose that $k, l \notin \{1, n\}$. We have $1 = L_A(1, l) = L_B(1, l) = \frac{d_1}{d_l}\frac{1}{y}$ and $\frac{1}{x} = L_A(l, 1) = L_B(l, 1) = \frac{d_l}{d_1}$. Thus $x = y$. Also, $\frac{1}{x} = L_A(l, k) = L_B(l, k) = \frac{d_l}{d_k}$ and $\frac{1}{x} = L_A(k, l) = L_B(k, l) = \frac{d_k}{d_l}\frac{1}{x}$. Thus, $\frac{d_l}{d_k} = \frac{1}{x}$ and $\frac{d_l}{d_k} = 1$, a contradiction.

Suppose that $l = n$ and $k \neq 1$. We have $1 = L_A(1, k) = L_B(1, k) = \frac{d_1}{d_k}$ and $\frac{1}{x} = L_A(k, 1) = L_B(k, 1) = \frac{d_k}{d_1}\frac{1}{y}$, implying $y = x$. Also, for $i \neq 1, k, n$, $\frac{1}{x} = L_A(i, k) = L_B(i, k) = \frac{d_i}{d_k}$ and $\frac{1}{x} = L_A(k, i) = L_B(k, i) = \frac{d_k}{d_i}\frac{1}{y}$, implying $y = x^2$. Thus, $y = x = 1$, a contradiction.

Suppose that $k = 1$ and $l \neq n$. We have $1 = L_A(l, n) = L_B(l, n) = \frac{d_l}{d_n}$ and $\frac{1}{x} = L_A(n, l) = L_B(n, l) = \frac{d_n}{d_l}\frac{1}{y}$, implying $y = x$. Also, for $j \neq 1, l, n$,

$\frac{1}{x} = L_A(l,j) = L_B(l,j) = \frac{d_l}{d_j}$ and $\frac{1}{x} = L_A(j,l) = L_B(j,l) = \frac{d_j}{d_l}\frac{1}{y}$, implying $y = x^2$. Thus, $y = x = 1$, a contradiction.

Suppose that $k = 1$ and $l = n$. Equating the last rows, and the first columns, of L_A and L_B, it follows that $d_1 = \cdots = d_n$. Then, $1 = L_A(1,2) = L_B(1,2) = \frac{d_1}{d_2}\frac{1}{y} = \frac{1}{y}$, a contradiction.

Suppose that $l = 1$. Equating the first rows of L_A and L_B, it follows that $d_1 = \cdots = d_n$. If $k \neq n$, for any $i \neq 1, k$, we have $\frac{1}{x} = L_A(i,k) = L_B(i,k) = \frac{d_i}{d_k} = 1$, a contradiction. Thus, $k = n$ and $x = y$, as desired.

Suppose that $k = n$. Equating the last columns of L_A and L_B, it follows that $d_1 = \cdots = d_n$. If $l \neq 1$, for any $j \neq n, l$ we have $\frac{1}{x} = L_A(l,j) = L_B(l,j) = \frac{d_l}{d_j} = 1$, a contradiction. Thus, $l = 1$ and $x = y$, as desired. ■

Any matrix in $\mathcal{PC}_3$ is an SPC matrix. Thus, Theorem 33 implies that, when $A, B \in \mathcal{PC}_3$, $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$, as noticed in [12]. In Section 10, we show that this result is valid when $A, B \in \mathcal{PC}_4$. Next we give some important tools for this purpose.

9 Necessary conditions for $\mathcal{E}(A) = \mathcal{E}(B)$ for general n

Here we give other necessary conditions for $\mathcal{E}(A) = \mathcal{E}(B)$ that will be important in the next section.

Given matrices $A, B \in M_n$ and $S \subseteq N \times N$, we say that $A[S]$ *dominates* $B[S]$ if $A[S] \leq B[S]$ entry-wise. The dominance is strict if the inequality is strict for at least one entry. Note that, from Theorem 24, for $A, B \in \mathcal{PC}_n^0$, $\tau \in \Gamma(A)$ and $k \in N$, if there is a $\nu \in \Gamma(A)$, $\nu \neq \tau$, such that $P_{A,\nu}[S_k^{(\tau)}]$ dominates $P_{B,\tau}[S_k^{(\tau)}]$, then the dominance is strict.

If $A \in \mathcal{PC}_n^0$ and $\tau \in \Gamma(A)$, we say that $P_{A,\tau}[S]$ is (A,S)-*undominated* if there is no $\nu \in \Gamma(A)$, $\nu \neq \tau$, such that $P_{A,\nu}[S]$ dominates $P_{A,\tau}[S]$. Note that, if $P_{A,\tau}[S_k^{(\tau)}]$ is $(A, S_k^{(\tau)})$-undominated, then $w \in \mathbb{R}_+^n$ such that $W[S_k^{(\tau)}] = P_{A,\tau}[S_k^{(\tau)}]$ is an extreme of $\mathcal{E}_\tau(A)$ (by Theorem 22) and also of $\mathcal{E}(A)$, as there is no $\nu \in \Gamma(A)$, $\nu \neq \tau$, such that $w \in \mathcal{E}_\nu(A)$ (by Theorem 14).

From Theorems 4 and 5 it follows that $\mathcal{E}(A) \subseteq \mathcal{E}(B)$ implies $L_B \leq L_A$ entry-wise. Next we give an improvement on this observation.

Theorem 34 *Let $A, B \in \mathcal{PC}_n^0$ and suppose that $\mathcal{E}(A) \subseteq \mathcal{E}(B)$. Then, for each $k \in N$ and each $\tau \in \Gamma(A)$, there is $\nu \in \Gamma(B)$ such that $P_{B,\nu}[S_k^{(\tau)}] \leq P_{A,\tau}[S_k^{(\tau)}]$.*

Proof. Suppose that $\mathcal{E}(A) \subseteq \mathcal{E}(B)$. By Theorem 22, there is $w \in \mathcal{E}(A)$ such that $P_{A,\tau}[S_k^{(\tau)}] = W[S_k^{(\tau)}]$. Since $w \in \mathcal{E}(B)$, there is $\nu \in \Gamma(B)$ such that $w \in \mathcal{E}_\nu(B)$. Thus, by Theorem 12, $P_{B,\nu} \leq W$. Therefore, $P_{B,\nu}[S_k^{(\tau)}] \leq W[S_k^{(\tau)}] = P_{A,\tau}[S_k^{(\tau)}]$. ■

Next we state the main result of this section. It gives conditions under which, when $\mathcal{E}(A) = \mathcal{E}(B)$, certain H-cycles in $\Gamma(A)$ are also in $\Gamma(B)$. We will use the next two auxiliary results.

Lemma 35 *Let $A \in \mathcal{PC}_n^0$ and $\tau \in \Gamma(A)$. If $w \in \mathcal{E}_\tau(A)$ is such that $\frac{w_i}{w_j} > P_{A,\tau}(i,j)$, for some $i, j \in N$, $i \neq j$, then there is an arbitrarily small perturbation w' of w such that $w' \in \mathcal{E}_\tau(A)$ and $\frac{w'_i}{w'_j} < \frac{w_i}{w_j}$.*

Proof. Let $\tau = \tau_1 \cdots \tau_n \tau_1$, with $\tau_1 = i$. Let k be such that $\tau_k = j$. Since $\frac{w_i}{w_j} > P_{A,\tau}(i,j)$, there is $s \in \{2, \ldots, k\}$ such that

$$\begin{aligned} w_{\tau_1} &\geq P_{A,\tau}(\tau_1, \tau_2) w_{\tau_2} \geq \cdots \geq P_{A,\tau}(\tau_1, \tau_{s-1}) w_{\tau_{s-1}} \\ &> P_{A,\tau}(\tau_1, \tau_s) w_{\tau_s} \geq \cdots \geq P_{A,\tau}(\tau_1, \tau_k) w_{\tau_k} \geq \cdots \\ &\geq P_{A,\tau}(\tau_1, \tau_n) w_{\tau_n} \geq \tau(A) w_{\tau_1}. \end{aligned}$$

The vector w' obtained from w by a small decrement of $w_{\tau_{s-1}}$ so that $w'_{\tau_{s-1}} > P_{A,\tau}(\tau_{s-1}, \tau_s) w_{\tau_s}$ verifies the claim. ■

Lemma 36 *Let $A \in \mathcal{PC}_n^0$, $\tau \in \Gamma(A)$ and $w \in \mathcal{E}_\tau(A)$. Let $k \in N$ and $(i,j) \in S_k^{(\tau)}$. If $P_{A,\tau}[S_k^{(\tau)}] = W[S_k^{(\tau)}]$ and $P_{A,\tau}[S_k^{(\tau)}]$ is $(A, S_k^{(\tau)})$-undominated, then there is a neighborhood $\mathcal{V}$ of w such that, for any $w' \in \mathcal{V}$ with $\frac{w'_i}{w'_j} < \frac{w_i}{w_j}$, we have $w' \notin \mathcal{E}(A)$.*

Proof. If $\#\Gamma(A) = 1$, the result follows from Theorem 14. Suppose that $\#\Gamma(A) > 1$. Let

$$\delta = \min_{\rho \in \Gamma(A) \setminus \{\tau\}} \max_{(p,q) \in S_k^{(\tau)}} \left(P_{A,\rho}(p,q) - P_{A,\tau}(p,q) \right).$$

Note that $\delta > 0$, since $P_{A,\tau}[S_k^{(\tau)}]$ is $(A, S_k^{(\tau)})$-undominated. Any w' such that $\frac{w'_i}{w'_j} < P_{A,\tau}(i,j)$ and $\left|\frac{w'_p}{w'_q} - \frac{w_p}{w_q}\right| < \delta$ for all $(p,q) \in S_k^{(\tau)}$ verifies the claim, as there is no $\nu \in \Gamma(A)$ such that $w' \in \mathcal{E}_\nu(A)$. ■

Theorem 37 *Let $A, B \in \mathcal{PC}_n^0$ and suppose that $\mathcal{E}(A) = \mathcal{E}(B)$. Let $\tau \in \Gamma(A)$ and $k \in N$. If $P_{A,\tau}[S_k^{(\tau)}]$ is $(A, S_k^{(\tau)})$-undominated, then $\tau \in \Gamma(B)$ and $P_{B,\tau}[S_k^{(\tau)}] = P_{A,\tau}[S_k^{(\tau)}]$.*

Proof. By Theorem 22, there is $w \in \mathcal{E}_\tau(A)$ such that $P_{A,\tau}[S_k^{(\tau)}] = W[S_k^{(\tau)}]$. Since $\mathcal{E}(A) = \mathcal{E}(B)$, we have $w \in \mathcal{E}(B)$. By Theorem 3, there is $\nu \in \Gamma(B)$ such that $w \in \mathcal{E}_\nu(B)$. By Theorem 12, $P_{B,\nu}[S_k^{(\tau)}] \leq W[S_k^{(\tau)}] = P_{A,\tau}[S_k^{(\tau)}]$. Suppose that $P_{B,\nu}[S_k^{(\tau)}]$ strictly dominates $P_{A,\tau}[S_k^{(\tau)}]$. Then, there is $(i,j) \in S_k^{(\tau)}$ such that $P_{B,\nu}(i,j) < P_{A,\tau}(i,j) = \frac{w_i}{w_j}$. By Lemma 35, there is a vector $w' \in \mathcal{E}_\nu(B)$, arbitrarily close to w, such that $\frac{w'_i}{w'_j} < \frac{w_i}{w_j}$. Since $P_{A,\tau}[S_k^{(\tau)}]$ is $(A, S_k^{(\tau)})$-undominated, by Lemma 36, $w' \notin \mathcal{E}(A)$, a contradiction since $\mathcal{E}(A) = \mathcal{E}(B)$. Thus, $P_{B,\nu}[S_k^{(\tau)}] = P_{A,\tau}[S_k^{(\tau)}]$. By Theorem 24, $\nu = \tau$. Thus the claim follows. ■

Recall that $P_{B,\tau}[S_k^{(\tau)}] = P_{A,\tau}[S_k^{(\tau)}]$ does not imply $P_{B,\tau} = P_{A,\tau}$, unless $\tau(A) = \tau(B)$.

Corollary 38 *Let $A, B \in \mathcal{PC}_n^0$. Suppose that there are $\tau \in \Gamma(A)$ and $k_1, k_2 \in N$, $k_1 \neq k_2$, such that $P_{A,\tau}[S_{k_1}^{(\tau)}]$ is $(A, S_{k_1}^{(\tau)})$-undominated and $P_{A,\tau}[S_{k_2}^{(\tau)}]$ is $(A, S_{k_2}^{(\tau)})$-undominated. If $\mathcal{E}(A) = \mathcal{E}(B)$, then $\tau \in \Gamma(B)$ and $P_{A,\tau} = P_{B,\tau}$. Thus, the sequence of entries in A and in B along τ coincide, that is, $\mathcal{E}_\tau(A) = \mathcal{E}_\tau(B)$.*

Proof. By Theorem 37, if $\mathcal{E}(A) = \mathcal{E}(B)$, then $\tau \in \Gamma(B)$ and $P_{B,\tau}[S_{k_1}^{(\tau)}] = P_{A,\tau}[S_{k_1}^{(\tau)}]$ and $P_{B,\tau}[S_{k_2}^{(\tau)}] = P_{A,\tau}[S_{k_2}^{(\tau)}]$. By Remark 21, $P_{B,\tau} = P_{A,\tau}$. By Theorem 10, the entries in A and B along τ coincide. ■

Remark 39 *Theorem 37 says that, if $\mathcal{E}(A) = \mathcal{E}(B)$, and w is an extreme of $\mathcal{E}(A)$ lying in a subset $\mathcal{E}_\tau(A)$, with $\tau \in \Gamma(A)$, and satisfying the undominance condition, then $\tau \in \Gamma(B)$ and w, which is also an extreme of $\mathcal{E}(B)$, lies in $\mathcal{E}_\tau(B)$. By Corollary 38, if there are two such extremes of $\mathcal{E}(A)$ lying in a subset $\mathcal{E}_\tau(A)$, then $\tau \in \Gamma(B)$ and $\mathcal{E}_\tau(B) = \mathcal{E}_\tau(A)$.*

We now can give a condition under which $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$.

Corollary 40 *Let $A, B \in \mathcal{PC}_n^0$. Suppose that the entries of a matrix in $\mathcal{PC}_n$ along the H-cycles in $\Gamma(A)$ determine the matrix (this happens if $\#\Gamma(A) = \frac{(n-1)!}{2}$). Suppose that for each $\tau \in \Gamma(A)$ there are $k_1, k_2 \in N$, $k_1 \neq k_2$, such that $P_{A,\tau}[S_{k_1}^{(\tau)}]$ is $(A, S_{k_1}^{(\tau)})$-undominated and $P_{A,\tau}[S_{k_2}^{(\tau)}]$ is $(A, S_{k_2}^{(\tau)})$-undominated. If $\mathcal{E}(A) = \mathcal{E}(B)$, then $A = B$.*

Proof. If $\mathcal{E}(A) = \mathcal{E}(B)$, by Corollary 38, for each $\tau \in \Gamma(A)$, we have that $\tau \in \Gamma(B)$ and the sequence of entries in A and in B along τ coincide. Since, by hypothesis, the entries of A and B along the H-cycles in $\Gamma(A) \subseteq \Gamma(B)$ determine A and B, respectively, and those entries coincide in both matrices, it follows that $A = B$. ■

10 $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$: the case $n = 4$

Here, we show that $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$, when $A, B \in \mathcal{PC}_4^0$. In this case, there are at most 3 H-cycles in $\Gamma(A)$.

We first show that, if $A, B \in \mathcal{PC}_4^0$ and $\mathcal{E}(A) = \mathcal{E}(B)$, then the minimum cycle products in A and B are the same. Given $A \in \mathcal{PC}_4^0$, we denote by θ_A an H-cycle τ such that $\tau(A) = \min_{\rho \in \Gamma(A)} \rho(A)$. The H-cycle θ_A may not be uniquely determined, but the numerical value $\theta_A(A) = \tau(A)$ is uniquely determined (is the minimum cycle product in A).

Theorem 41 *Let $A, B \in \mathcal{PC}_4^0$. If $\mathcal{E}(A) = \mathcal{E}(B)$ then $\theta_A(A) = \theta_B(B)$.*

Proof. Without loss of generality, suppose that $\theta_A(A) \leq \theta_B(B)$. By Theorem 34, and because $\#\Gamma(B) < 4$, there are $k_1, k_2 \in \{1, 2, 3, 4\}$, $k_1 \neq k_2$, and $\rho \in \Gamma(B)$ such that $P_{B,\rho}[S_{k_1}^{(\theta_A)} \cup S_{k_2}^{(\theta_A)}] \leq P_{A,\theta_A}[S_{k_1}^{(\theta_A)} \cup S_{k_2}^{(\theta_A)}]$. By Remark 7, $\rho(B) \leq \theta_A(A)$. Since $\theta_B(B) \leq \rho(B)$ and, by assumption, $\theta_A(A) \leq \theta_B(B)$, we get $\theta_B(B) = \theta_A(A)$. ■

To prove that $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$, first we show that, if $\mathcal{E}(A) = \mathcal{E}(B)$, then there is a common H-cycle in A and B with the same entries along it in both A and B and with minimal cycle product (which is equal in

A and B by Theorem 41). Based on this, we prove the result assuming that A and B are in a special form attained by monomial similarity.

Proof of the existence of a common H-cycle with minimum product

We consider the case in which the minimum cycle product in A occurs just for one H-cycle in $\Gamma(A)$, and the case in which it occurs for more than one H-cycle in $\Gamma(A)$. The following lemma states that, in the former case, there are two extremes of the set $\mathcal{E}_\tau(A)$ associated with the minimum cycle product that are not in the other sets $\mathcal{E}_\nu(A)$ (so they are extremes of $\mathcal{E}(A)$).

Lemma 42 *Let $A \in \mathcal{PC}_4^0$ with $\#\Gamma(A) > 1$. Suppose that $\theta_A(A) < \tau(A)$ for any $\tau \in \Gamma(A)$, $\tau \neq \theta_A$. Then, there are $k_1, k_2 \in \{1,2,3,4\}$, $k_1 \neq k_2$, such that $P_{A,\theta_A}[S_{k_1}^{(\theta_A)}]$ is $(A, S_{k_1}^{(\theta_A)})$-undominated and $P_{A,\theta_A}[S_{k_2}^{(\theta_A)}]$ is $(A, S_{k_2}^{(\theta_A)})$-undominated.*

Proof. Suppose that there are at least 3 k's for which there is $\tau \in \Gamma(A)$, $\tau \neq \theta_A$, such that $P_{A,\tau}[S_k^{(\theta_A)}]$ dominates $P_{A,\theta_A}[S_k^{(\theta_A)}]$ (the dominance is strict by Theorem 24). Then, there are $k_1, k_2 \in \{1,2,3,4\}$, $k_1 \neq k_2$, and $\nu \in \Gamma(A)$, $\nu \neq \theta_A$, such that $P_{A,\nu}[S_{k_1}^{(\theta_A)} \cup S_{k_2}^{(\theta_A)}]$ dominates $P_{A,\theta_A}[S_{k_1}^{(\theta_A)} \cup S_{k_2}^{(\theta_A)}]$. By Remark 7, $\nu(A) \leq \theta_A(A)$. Then, $\nu(A) = \theta_A(A)$ and, because of the hypothesis, $\nu = \theta_A$, a contradiction. ■

When the minimum cycle product occurs for at least two H-cycles in $\Gamma(A)$, then the conclusion in Lemma 42 may not hold.

Example 43 *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \frac{1}{a} & 1 & 1 & 1 \end{bmatrix},$$

$a > 1$. *We have $\Gamma(A) = \{\alpha, \gamma\}$, with $\alpha = 12341$ and $\gamma = 13241$. Moreover, $\alpha(A) = \gamma(A) = \frac{1}{a}$. We have (see Example 8)*

$$P_{A,\alpha} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{a} & 1 & 1 & 1 \\ \frac{1}{a} & \frac{1}{a} & 1 & 1 \\ \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & 1 \end{bmatrix} \text{ and } P_{A,\gamma} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{a} & 1 & \frac{1}{a} & 1 \\ \frac{1}{a} & 1 & 1 & 1 \\ \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & 1 \end{bmatrix}.$$

The matrix $P_{A,\alpha}[S_{k_1}^{(\alpha)}]$ *is* $(A, S_{k_1}^{(\alpha)})$*-undominated if and only if* $k_1 = 3$*; the matrix* $P_{A,\gamma}[S_{k_2}^{(\gamma)}]$ *is* $(A, S_{k_2}^{(\gamma)})$*-undominated if and only if* $k_2 = 2$*.*

The extremes of $\mathcal{E}_\alpha(A)$ *are (projectively)* $(1,1,1,1)$, $(a,a,a,1)$, $(a,1,1,1)$, *and* $(a,a,1,1)$, *associated with* $S_k^{(\alpha)}$ *for* $k = 1$, 4, 2 *and* 3, *respectively. The extremes of* $\mathcal{E}_\gamma(A)$ *are* $(1,1,1,1)$, $(a,a,a,1)$, $(a,1,1,1)$ *and* $(a,1,a,1)$, *associated with* $S_k^{(\gamma)}$ *for* $k = 1$, 4, 3 *and* 2, *respectively.*

Lemma 44 *Let* $A \in \mathcal{PC}_4^0$*. Suppose that there are* $\tau, \nu \in \Gamma(A)$, $\tau \neq \nu$, *such that* $\tau(A) = \nu(A) = \theta_A(A)$*. If there are* 3 *distinct* k*'s in* $\{1,2,3,4\}$ *such that* $P_{A,\nu}[S_k^{(\tau)}]$ *dominates* $P_{A,\tau}[S_k^{(\tau)}]$*, then there is an H-cycle product in A equal to* 1*.*

Proof. Without loss of generality, we may assume that $\tau = 12341$ and $a_{i,i+1} = 1$, $i = 1,2,3$. Suppose that there are 3 distinct k's such that $P_{A,\nu}[S_k^{(\tau)}]$ dominates $P_{A,\tau}[S_k^{(\tau)}]$. Note that the union of all such sets $S_k^{(\tau)}$ contains all pairs of symmetrically placed positions, except (at most) one (it indexes 3 rows and 3 columns). Then, except for one pair of symmetrically placed entries ($P_{A,\nu} \neq P_{A,\tau}$, by Theorem 10), all entries coincide in $P_{A,\nu}$ and $P_{A,\tau}$. In fact, by the dominance, each entry in a pair of symmetrically placed entries in $P_{A,\nu}$, indexed by the union of the $S_k^{(\tau)}$'s, is less than or equal to the corresponding entry in $P_{A,\tau}$, and the product of the entries in each pair is $\nu(A) = \tau(A)$ in both $P_{A,\nu}$ and $P_{A,\tau}$.

Taking into account Example 8 (and using the notation there), we next consider the cases in which $P_{A,\nu}$ and $P_{A,\tau}$ coincide in all entries except in those placed in one pair of symmetric positions. Recall that τ is α as in (3).

- If $\nu = \beta$, then $P_{A,\nu}$ and $P_{A,\tau}$ differ in the entries in positions $(3,4)$ and $(4,3)$. Moreover, $a_{24} = 1$ and $a_{13} = a_{14}$, implying $\gamma(A) = \gamma'(A) = 1$.

- If $\nu = \beta'$, then $P_{A,\nu}$ and $P_{A,\tau}$ differ in the entries in positions $(1,2)$ and $(2,1)$. Moreover, $a_{24} = a_{14}$ and $a_{13} = 1$, implying $\gamma(A) = \gamma'(A) = 1$.

- If $\nu = \gamma$, then $P_{A,\nu}$ and $P_{A,\tau}$ differ in the entries in positions $(2,3)$ and $(3,2)$. Moreover, $a_{13} = a_{24} = 1$, implying $\beta(A) = \beta'(A) = 1$.

- If $\nu = \gamma'$, then $P_{A,\nu}$ and $P_{A,\tau}$ differ in the entries in positions $(1,4)$ and $(4,1)$. Moreover, $a_{13} = a_{14} = a_{24}$, implying $\beta(A) = \beta'(A) = 1$.

■

Lemma 45 *Let $A \in \mathcal{PC}_4^0$ and suppose that $\#\Gamma(A) = 3$. Moreover, suppose that there are at least two H-cycles in $\Gamma(A)$ whose cycle products in A are $\theta_A(A)$. If $\nu \in \Gamma(A)$ is such that $\nu(A) = \theta_A(A)$, then there is a $k \in \{1, 2, 3, 4\}$ such that $P_{A,\nu}[S_k^{(\nu)}]$ is $(A, S_k^{(\nu)})$-undominated.*

Proof. Let $\Gamma(A) = \{\nu, \tau, \rho\}$ with $\nu(A) = \tau(A) = \theta_A(A)$. Without loss of generality, we may assume that $\nu = 12341$ and $a_{i,i+1} = 1$, $i = 1, 2, 3$. Then, $a_{41} = \frac{1}{a_{14}} = \theta_A(A) < 1$.

Suppose that $\rho(A) > \theta_A(A)$ and that there is no k such that $P_{A,\nu}[S_k^{(\nu)}]$ is $(A, S_k^{(\nu)})$-undominated. Note that there is no $\pi \in \Gamma(A)\backslash\{\nu\}$ such that, for all $k \in \{1, 2, 3, 4\}$, $P_{A,\pi}[S_k^{(\nu)}]$ dominates $P_{A,\nu}[S_k^{(\nu)}]$, as otherwise $P_{A,\pi} \leq P_{A,\nu}$, implying $P_{A,\pi} = P_{A,\nu}$, as $\nu(A) = \theta_A(A)$. This is impossible by Theorem 10, since $\pi \neq \nu$. Thus, there is exactly one $k \in \{1, 2, 3, 4\}$ such that $P_{A,\rho}[S_k^{(\nu)}]$ strictly dominates $P_{A,\nu}[S_k^{(\nu)}]$, as, if there were at least two, we would have $\rho(A) \leq \theta_A(A)$, a contradiction. Therefore, there are 3 k's such that $P_{A,\tau}[S_k^{(\nu)}]$ strictly dominates $P_{A,\nu}[S_k^{(\nu)}]$. By Lemma 44, this contradicts the fact that $\#\Gamma(A) = 3$. Thus, there is k such that $P_{A,\nu}[S_k^{(\nu)}]$ is $(A, S_k^{(\nu)})$-undominated.

Now suppose that $\rho(A) = \theta_A(A)$, that is, all H-cycles in $\Gamma(A)$ have the same product in A. We consider all possible sets $\Gamma(A)$. We use the notation in Example 8 (see also Example 20). Recall that $\nu = \alpha$.

- If $\Gamma(A) = \{\alpha, \beta, \gamma\}$ then $a_{13}a_{24} = 1$ and $\frac{1}{a_{13}} = \frac{1}{\sqrt{a_{14}}} > \frac{1}{a_{14}}$. In this case, $P_{A,\nu}[S_3^{(\nu)}]$ is $(A, S_3^{(\nu)})$-undominated.
- If $\Gamma(A) = \{\alpha, \beta, \gamma'\}$ then $a_{13}a_{24} = a_{14}^2$ and $\frac{a_{24}}{a_{13}} = \frac{1}{a_{14}}$. This implies $a_{24} = \sqrt{a_{14}} > 1$. In this case, $P_{A,\nu}[S_1^{(\nu)}]$ is $(A, S_1^{(\nu)})$-undominated.
- If $\Gamma(A) = \{\alpha, \beta', \gamma\}$ then $a_{13}a_{24} = 1$ and $\frac{a_{24}}{a_{13}} = a_{14}$. This implies $\frac{1}{a_{24}} = \frac{1}{\sqrt{a_{14}}} > \frac{1}{a_{14}}$. In this case, $P_{A,\nu}[S_3^{(\nu)}]$ is $(A, S_3^{(\nu)})$-undominated.
- If $\Gamma(A) = \{\alpha, \beta', \gamma'\}$ then $a_{13}a_{24} = a_{14}^2$ and $\frac{a_{24}}{a_{13}} = a_{14}$. This implies $a_{13} = \sqrt{a_{14}} > 1$. In this case, $P_{A,\nu}[S_1^{(\nu)}]$ is $(A, S_1^{(\nu)})$-undominated.

■

Now we can state the claim that allows us to consider both A and B in a convenient form attained by monomial similarity.

Theorem 46 *Let $A, B \in \mathcal{PC}_4^0$. If $\mathcal{E}(A) = \mathcal{E}(B)$, then $\theta_A(A) = \theta_B(B)$ and there is a $\nu \in \Gamma(A) \cap \Gamma(B)$ such that $\nu(A) = \theta_A(A) = \theta_B(B) = \nu(B)$ and the sequence of entries in A and in B along ν coincide.*

Proof. By Theorem 41, $\theta_B(B) = \theta_A(A)$. By interchanging the roles of A and B, it is enough to consider the following cases.

Case 1: Suppose that $\#\Gamma(A) = 1$ or $\theta_A(A) < \tau(A)$ for any $\tau \in \Gamma(A)$, $\tau \neq \theta_A$. In the former case, the claim follows from Corollary 38, and in the latter case it follows from Lemma 42 and Corollary 38.

Case 2: Suppose that there are two H-cycles in A and in B whose cycle products are $\theta_A(A) = \theta_B(B)$.

Case 2.1: Suppose that $\#\Gamma(A) = 3$. Let $\Gamma(A) = \{\pi, \tau, \rho\}$ with $\pi(A) = \tau(A) = \theta_A(A)$. By Lemma 45, there are k_1, k_2 such that $P_{A,\pi}[S_{k_1}^{(\pi)}]$ is $(A, S_{k_1}^{(\pi)})$-undominated and $P_{A,\tau}[S_{k_2}^{(\tau)}]$ is $(A, S_{k_2}^{(\tau)})$-undominated. By Theorem 37, $\pi, \tau \in \Gamma(B)$, $P_{A,\pi}[S_{k_1}^{(\pi)}] = P_{B,\pi}[S_{k_1}^{(\pi)}]$ and $P_{A,\tau}[S_{k_2}^{(\tau)}] = P_{B,\tau}[S_{k_2}^{(\tau)}]$. Taking into account the hypothesis, either $\pi(B) = \theta_B(B)$ or $\tau(B) = \theta_B(B)$. In the first case $P_{A,\pi} = P_{B,\pi}$ and, taking into account Theorem 10, $\nu = \pi$ satisfies the claim; otherwise $P_{A,\tau} = P_{B,\tau}$ and $\nu = \tau$ satisfies the claim.

Case 2.2: Suppose that $\#\Gamma(A) = \#\Gamma(B) = 2$. Then $\Gamma(A) = \{\tau, \rho\}$ with $\tau(A) = \rho(A) = \theta_A(A)$. Then, for any $\pi \in \Gamma(A)$, there is k such that $P_{A,\pi}[S_k^{(\pi)}]$ is $(A, S_k^{(\pi)})$-undominated, as otherwise $P_{A,\pi'} \leq P_{A,\pi}$ for $\pi' \in \Gamma(A)$, $\pi \neq \pi'$. In that case, because $\pi(A) = \pi'(A)$, we would have $P_{A,\pi'} = P_{A,\pi}$, implying $\pi = \pi'$ by Theorem 10, a contradiction. Thus, by Theorem 37, if $\mathcal{E}(A) = \mathcal{E}(B)$ then $\Gamma(A) = \Gamma(B)$. Taking into account the hypothesis, for any $\pi \in \Gamma(A) = \Gamma(B)$ we have $\theta_A(A) = \theta_B(B) = \pi(B)$. By Theorem 37 and Remark 21, $P_{A,\pi} = P_{B,\pi}$. Thus, the claim follows from Theorem 10. (In fact, the same entries in A and B along two distinct H-cycles, none of which is the reverse of the other, implies $A = B$.) ■

Proof of the main result

Consider the matrices in $\mathcal{PC}_4^0$,

$$A = \begin{bmatrix} 1 & 1 & a_{13} & a_{14} \\ 1 & 1 & 1 & a_{24} \\ \frac{1}{a_{13}} & 1 & 1 & 1 \\ \frac{1}{a_{14}} & \frac{1}{a_{24}} & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & b_{13} & a_{14} \\ 1 & 1 & 1 & b_{24} \\ \frac{1}{b_{13}} & 1 & 1 & 1 \\ \frac{1}{a_{14}} & \frac{1}{b_{24}} & 1 & 1 \end{bmatrix}. \tag{10}$$

Then, for $\alpha = 12341$, we have $\alpha(A) = \alpha(B) = \frac{1}{a_{14}}$. Suppose that $a_{14} > 1$ and $\theta_A(A) = \theta_B(B) = \frac{1}{a_{14}}$. Each of $\Gamma(A)$ and $\Gamma(B)$ is contained in one of the sets

$$\{\alpha, \beta, \gamma\}, \quad \{\alpha, \beta, \gamma'\}, \quad \{\alpha, \beta', \gamma\}, \quad \text{or } \{\alpha, \beta', \gamma'\},$$

with $\beta, \beta', \gamma, \gamma'$ as in (3) and (4). The matrices $P_{B,\rho}$, $\rho \in \Gamma(B)$, can be obtained from the matrices $P_{A,\rho}$ in Example 8 by replacing a_{13} and a_{24} by b_{13} and b_{24}, respectively.

In the next lemmas, Examples 8 and 20 are helpful.

Lemma 47 *Let $A, B \in \mathcal{PC}_4^0$ be as in (10), with $\theta_A(A) = \theta_B(B) = \frac{1}{a_{14}}$. If $\mathcal{E}(A) = \mathcal{E}(B)$ and $\beta \in \Gamma(A)$, then $A = B$.*

Proof. Suppose that $\beta \in \Gamma(A)$. Then, $a_{24} < a_{13}$, as $\beta(A) < 1$. Since $\mathcal{E}(A) \subseteq \mathcal{E}(B)$, by Theorem 34, there is $\rho \in \Gamma(B)$ such that $P_{B,\rho}[S_3^{(\beta)}] \leq P_{A,\beta}[S_3^{(\beta)}]$. We will see that this implies $\rho = \beta$.

- Since $\beta(A) < 1$, it follows that $P_{B,\alpha}[S_3^{(\beta)}]$ does not dominate $P_{A,\beta}[S_3^{(\beta)}]$.
- Suppose that $\gamma \in \Gamma(B)$. Since $\gamma(B) \geq \alpha(B)$, we have $b_{13}b_{24} \geq 1$. Then, $P_{B,\gamma}[S_3^{(\beta)}]$ does not dominate $P_{A,\beta}[S_3^{(\beta)}]$, since $b_{13} \leq 1$ and $b_{24} \leq \beta(A) < 1$ do not occur simultaneously.
- Suppose that $\beta' \in \Gamma(B)$. Since $\beta(A) < 1$, it follows that $P_{B,\beta'}[S_3^{(\beta)}]$ does not dominate $P_{A,\beta}[S_3^{(\beta)}]$.
- Suppose that $\gamma' \in \Gamma(B)$. If $\frac{b_{24}}{a_{14}} \geq 1$, then $\frac{a_{14}}{b_{13}} \geq 1$, as, from $\gamma'(B) \geq \alpha(B)$, we get $\frac{a_{14}}{b_{13}} \geq \frac{b_{24}}{a_{14}}$. Thus, either $\frac{a_{14}}{b_{24}} > 1$ or $\frac{a_{14}}{b_{13}} \geq 1$. In any case, we have that $P_{B,\gamma'}[S_3^{(\beta)}]$ does not dominate $P_{A,\beta}[S_3^{(\beta)}]$.

Then, $\beta \in \Gamma(B)$ and $P_{B,\beta}[S_3^{(\beta)}]$ dominates $P_{A,\beta}[S_3^{(\beta)}]$. Similarly, since $\mathcal{E}(B) \subseteq \mathcal{E}(A)$ and $\beta \in \Gamma(B)$, we get that $P_{A,\beta}[S_3^{(\beta)}]$ dominates $P_{B,\beta}[S_3^{(\beta)}]$. Thus, $P_{A,\beta}[S_3^{(\beta)}] = P_{B,\beta}[S_3^{(\beta)}]$, implying $A = B$. ■

Lemma 48 *Let $A, B \in \mathcal{PC}_4^0$ be as in (10), with $\theta_A(A) = \theta_B(B) = \frac{1}{a_{14}}$. If $\mathcal{E}(A) = \mathcal{E}(B)$ and $\beta' \in \Gamma(A)$, then $A = B$.*

Proof. Suppose that $\beta' \in \Gamma(A)$. Then, $\beta \notin \Gamma(B)$, as otherwise, by Lemma 47, $A = B$, which contradicts the fact that $\beta \notin \Gamma(A)$. We have $a_{13} < a_{24}$, as $\beta'(A) < 1$. Since $\mathcal{E}(A) \subseteq \mathcal{E}(B)$, by Theorem 34, there is $\rho \in \Gamma(B)$ such that $P_{B,\rho}[S_1^{(\beta')}] \leq P_{A,\beta'}[S_1^{(\beta')}]$. We will see that this implies $\rho = \beta'$.

- Since $\beta'(A) < 1$, it follows that $P_{B,\alpha}[S_1^{(\beta')}]$ does not dominate $P_{A,\beta'}[S_1^{(\beta')}]$.

- Suppose that $\gamma \in \Gamma(B)$. Since $\gamma(B) \geq \alpha(B)$, we have $b_{13}b_{24} \geq 1$. Then, $P_{B,\gamma}[S_1^{(\beta')}]$ does not dominate $P_{A,\beta'}[S_1^{(\beta')}]$, since $b_{24} \leq 1$ and $b_{13} \leq \beta'(A) < 1$ do not occur simultaneously.

- Suppose that $\gamma' \in \Gamma(B)$. If $\frac{b_{24}}{a_{14}} > 1$, then $\frac{a_{14}}{b_{13}} > 1$, as, from $\gamma'(B) \geq \alpha(B)$, we have $\frac{a_{14}}{b_{13}} \geq \frac{b_{24}}{a_{14}}$. Thus, either $\frac{a_{14}}{b_{24}} \geq 1$ or $\frac{a_{14}}{b_{13}} > 1$. In any case, we have that $P_{B,\gamma'}[S_1^{(\beta')}]$ does not dominate $P_{A,\beta'}[S_1^{(\beta')}]$.

Then $\beta' \in \Gamma(B)$ and $P_{B,\beta'}[S_1^{(\beta')}]$ dominates $P_{A,\beta'}[S_1^{(\beta')}]$. Similarly, since $\mathcal{E}(B) \subseteq \mathcal{E}(A)$ and $\beta' \in \Gamma(B)$, we get that $P_{A,\beta'}[S_1^{(\beta')}]$ dominates $P_{B,\beta'}[S_1^{(\beta')}]$. Thus, $P_{A,\beta'}[S_1^{(\beta')}] = P_{B,\beta'}[S_1^{(\beta')}]$, implying $A = B$. ■

Lemma 49 *Let $A, B \in \mathcal{PC}_4^0$ be as in (10), with $\theta_A(A) = \theta_B(B) = \frac{1}{a_{14}}$. If $\mathcal{E}(A) = \mathcal{E}(B)$ and $\gamma \in \Gamma(A)$, then $A = B$.*

Proof. Suppose that $\gamma \in \Gamma(A)$. If $\beta \in \Gamma(A) \cup \Gamma(B)$ or $\beta' \in \Gamma(A) \cup \Gamma(B)$, by Lemmas 47 and 48, $A = B$. Suppose that $\beta, \beta' \notin \Gamma(A) \cup \Gamma(B)$. Since $\mathcal{E}(A) \subseteq \mathcal{E}(B)$, by Theorem 34, there is $\rho \in \Gamma(B)$ such that $P_{B,\rho}[S_2^{(\gamma)}] \leq P_{A,\gamma}[S_2^{(\gamma)}]$. We will see that this implies $\rho = \gamma$.

- Since $\gamma(A) < 1$, it follows that $P_{B,\alpha}[S_2^{(\gamma)}]$ does not dominate $P_{A,\gamma}[S_2^{(\gamma)}]$.

- If $\gamma' \in \Gamma(B)$, $P_{B,\gamma'}[S_2^{(\gamma)}]$ does not dominate $P_{A,\gamma}[S_2^{(\gamma)}]$.

Then, $\gamma \in \Gamma(B)$ and $P_{B,\gamma}[S_2^{(\gamma)}]$ dominates $P_{A,\gamma}[S_2^{(\gamma)}]$. Similarly, since $\mathcal{E}(B) \subseteq \mathcal{E}(A)$ and $\gamma \in \Gamma(B)$, we get that $P_{A,\gamma}[S_2^{(\gamma)}]$ dominates $P_{B,\gamma}[S_2^{(\gamma)}]$. Thus, $P_{A,\gamma}[S_2^{(\gamma)}] = P_{B,\gamma}[S_2^{(\gamma)}]$, implying $A = B$. ■

Lemma 50 *Let $A, B \in \mathcal{PC}_4^0$ be as in (10), with $\theta_A(A) = \theta_B(B) = \frac{1}{a_{14}}$. If $\mathcal{E}(A) = \mathcal{E}(B)$ and $\gamma' \in \Gamma(A)$, then $A = B$.*

Proof. Suppose that $\gamma' \in \Gamma(A)$. If $\beta \in \Gamma(A) \cup \Gamma(B)$ or $\beta' \in \Gamma(A) \cup \Gamma(B)$, by Lemmas 47 and 48, $A = B$. Suppose that $\beta, \beta' \notin \Gamma(A) \cup \Gamma(B)$. We have $\gamma \notin \Gamma(B)$, as otherwise, by Lemma 49, $A = B$, which contradicts the fact that $\gamma \notin \Gamma(A)$. Since $\mathcal{E}(A) \subseteq \mathcal{E}(B)$, by Theorem 34, there is $\rho \in \Gamma(B)$ such that $P_{B,\rho}[S_4^{(\gamma')}] \leq P_{A,\gamma'}[S_4^{(\gamma')}]$.

Since $\gamma'(A) < 1$, we have $\frac{1}{a_{13}a_{24}} < \frac{1}{a_{14}}$. Thus, $P_{B,\alpha}[S_4^{(\gamma')}]$ does not dominate $P_{A,\gamma'}[S_4^{(\gamma')}]$. Then, $\gamma' \in \Gamma(B)$, and $P_{B,\gamma'}[S_4^{(\gamma')}]$ dominates $P_{A,\gamma'}[S_4^{(\gamma')}]$. Similarly, since $\mathcal{E}(B) \subseteq \mathcal{E}(A)$ and $\gamma' \in \Gamma(B)$, we get that $P_{A,\gamma'}[S_4^{(\gamma')}]$ dominates $P_{B,\gamma'}[S_4^{(\gamma')}]$. Thus, $P_{A,\gamma'}[S_4^{(\gamma')}] = P_{B,\gamma'}[S_4^{(\gamma')}]$, implying $A = B$. ■

Theorem 51 *Let $A, B \in \mathcal{PC}_4$. If $\mathcal{E}(A) = \mathcal{E}(B)$ then $A = B$.*

Proof. If A or B is consistent, the result has already been mentioned. Suppose that $A, B \in \mathcal{PC}_4^0$. By Theorem 46, if $\mathcal{E}(A) = \mathcal{E}(B)$, there is a common cycle in A and B with the same entries along it in both matrices A and B and with minimal cycle product. So, by a monomial similarity, we may assume that both A and B are as in (10) with $\theta_A(A) = \theta_B(B) = \frac{1}{a_{14}}$. If $\#\Gamma(A) = \#\Gamma(B) = 1$, then $\beta(A) = \gamma(A) = \beta(B) = \gamma(B) = 1$ (for β and γ as in (3)), implying $A = B$. If $\#\Gamma(A) > 1$, the result follows from Lemmas 47 to 50. ■

11 Conclusions

For a given n-by-n inconsistent reciprocal matrix A, we have given a union of at most $\frac{(n-1)!}{2}$ matrix intervals, depending on the entries of A, such that a consistent matrix lies in it if and only if it is obtained from an efficient vector for A. Each interval is associated with a Hamiltonian cycle product < 1 in A. Based on this result, the partial order on the alternatives dictated by the efficient vectors follows. The existence of a unique order for the efficient vectors corresponding to each matrix interval is considered in detail. We then focused on the question of when $\mathcal{E}(A) = \mathcal{E}(B)$ implies $A = B$. We gave some general sufficient conditions for the implication to hold, and have shown that it is correct when both matrices A and B are simple perturbed consistent matrices (which includes the case $n = 3$) and when $n = 4$. It is not yet known if there are counterexamples in any dimension greater than 4.

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