

# A New Method of Constructing Hadamard Matrices, Circulant Hadamard Matrices, CZCS, GCS, CCC, and CZCSS

Piyush Priyanshu, Sudhan Majhi, and Subhabrata Paul

## Abstract

A Hadamard matrix  $H$  is a square matrix of order  $n$  with entries  $\pm 1$ , such that  $HH^\top = nI_n$ , where  $I_n$  is an identity matrix of order  $n$ . A circulant Hadamard matrix  $H$  is a Hadamard matrix that has rows of entries in cyclic order. There exist only 8 circulant Hadamard matrices of order 4, and here, we provide a novel construction of all such 8 circulant Hadamard matrices using a linear operator and generalized Boolean function (GBF). The constructed circulant Hadamard matrices are used recursively to construct a binary cross Z-complementary set (CZCS) of all lengths with an even phase, a binary Golay complementary set (GCS) of all lengths, and Hadamard matrices of order  $2^{n+2}$ , where  $n \geq 1$ . The construction of a binary CZCS covering all lengths was not available before. We also propose an alternative, lower-complexity construction of binary GCSs of all lengths and Hadamard matrices of order  $2^{a+1}10^b26^c$  using circulant matrices, where  $a, b, c \geq 0$ . The proposed binary GCS covers all lengths with a flexible flock size. The constructions of GCS are further extended to form binary complete complementary code (CCC) of the parameter  $(2N, 2N, 2N) - CCC$  where  $N = 2^a10^b26^c$ ,  $a, b, c \geq 0$ . The constructed binary CCC provides a flexible flock size. The construction of CZCS is further extended to form a binary optimal cross-Z complementary sequence set (CZCSS) of the parameter  $(2^{n+2}, 2^{n+2}, 2^{n+2}, 2^{n+1}) - CZCSS$ , where  $n \geq 1$ . Finally, we provide a relation between Hadamard matrices and GCS, which enables the study of the Hadamard conjecture in a new direction. We also provided a few properties of circulant matrices over aperiodic cross-correlation (ACCF) and aperiodic auto-correlation (AACF), which are used to prove the theorems. All proposed constructions are novel, and their parameters are compared with the existing state-of-the-art.

## Index Terms

Complete complementary code (CCC), circulant Hadamard matrix, cross Z-complementary sequence set (CZCSS), cross Z-complementary set (CZCS), generalized Boolean function (GBF), Golay complementary set (GCS), Hadamard matrix.

## I. INTRODUCTION

**T**he Hadamard conjecture states that a Hadamard matrix exists of order  $4k$ , where  $k \in \mathbb{N}$ , which was first proposed by Jacques Hadamard [1]. Hadamard studied square matrices with entries of  $+1$  or  $-1$ , with the property that all their rows or columns are pairwise orthogonal, such that  $\mathbf{H}\mathbf{H}^\top = n\mathbf{I}_n$ , where  $\mathbf{H}$  is a square matrix of order  $n$  and  $\mathbf{I}_n$  is the identity matrix of the same order. Hadamard posed the more general question of finding the maximal determinant of matrices whose entries lie on the unit disc.

Before Hadamard's work, in 1857, Sylvester had found Hadamard matrices of orders that are powers of two [2]. Sylvester observed that if  $\mathbf{H}$  is a Hadamard matrix of order  $n$ , then the matrix  $\begin{bmatrix} \mathbf{H} & \mathbf{H} \\ \mathbf{H} & -\mathbf{H} \end{bmatrix}$  is also a Hadamard matrix of order  $2n$ , also known as the Sylvester construction. Sylvester's work laid the foundation for the study of Hadamard matrices, which have since found numerous applications in coding theory, signal processing, and quantum computing [3]. However, Hadamard's contribution was to show the general existence conditions for Hadamard matrices.

The study of the construction of Hadamard matrices has attracted many researchers. In 1933, Paley provided two major theorems on the existence of Hadamard matrices, stating that if  $p$  is a prime number such that  $p \equiv 3 \pmod{4}$  and  $p \equiv 1 \pmod{4}$ , then there exists a Hadamard matrix of order  $(p+1)$  and  $2(p+1)$ , respectively [4]. In 1944, Williamson introduced matrices that later became known as Williamson-type matrices [5]. In 1965, Baumert *et al.* stated that a Hadamard matrix of order  $12t$  exists for every Williamson-type matrix of order  $4t$  [6], where  $t$  is a positive integer. In 1967, Goethals-Seidel proposed a strong relationship between orthogonal matrices with zero diagonal and Hadamard matrices [7]. In 1970, Cooper *et al.* constructed Hadamard matrices of order  $4t$ , where  $t \in \{1, 3, 5, 7, \dots, 19\}$  [8].

In the same year, Turyn proposed that if there is a complex Hadamard matrix of order  $2n$  and a Hadamard matrix of order  $4h$ , then there exists a Hadamard matrix of order  $8nh$  [9], where  $n, h \in \mathbb{I}^+$ . In 1972, Cooper *et al.* provided the construction of Hadamard matrices of order  $2^{t+2}q$  from T-matrices of order  $2^tq$ , where  $q$  and  $t$  are positive integers [8]. In 1973, Wallis constructed Hadamard matrices of order  $28m$ ,  $36m$ , and  $44m$  using T-matrices of order  $m$  [10], where  $m \in \mathbb{I}^+$ . In 1976, W. D. Wallis established a critical connection between the existence of Hadamard matrices and Williamson-type matrices. A quadruple of symmetric circulant matrices  $A$ ,  $B$ ,  $C$ , and  $D$  of order  $n$ , with entries  $-1$  or  $1$ , is of Williamson type if it satisfies:  $AA^\top + BB^\top + CC^\top + DD^\top = 4nI_n$ , where  $I_n$  is the identity matrix [11]. A circulant matrix is a Toeplitz matrix where each row is a cyclic shift of the row above it [12]. In 1985, Agayan-Sarukbanyan stated that if there are two Hadamard matrices of orders  $4h$  and  $4k$ , then there exists a Hadamard matrix of order  $8hk$  [13], where  $h, k \in \mathbb{I}^+$ .

In 1989, Koukouvinous *et al.* used T-matrices and a Golay complementary pair (GCP) to construct Hadamard matrices of order  $2^t q$ , where  $q$  is the sum of the lengths of two GCPs [14]. In 1991, Miyamoto established the existence of Hadamard matrices of order  $4q$  if there is a Hadamard matrix of order  $q - 1$ , where  $q$  is a prime power [15]. In 1992, Craigen *et al.* showed that if there are Hadamard matrices of orders  $4a, 4b, 4c, 4d$ , then there is a Hadamard matrix of order  $16abcd$  [16]. Many Hadamard matrices of different orders have since been found, including orders 428 [17], 1004 and 2524 [18], 268, 412, 436, and 604 [19], and 764 [20]. This motivated us to find new constructions of Hadamard matrices and study their structure.

A circulant matrix that satisfies the Hadamard condition is known as a circulant Hadamard matrix. Ryser conjectured that there is no circulant Hadamard matrix unless the order  $n$  is 1 or 4 [21]. Although this conjecture was partially solved in [22]–[24], it remained open until 2023, when Morris used congruence conditions to show that circulant Hadamard matrices exist only for  $n \leq 4$  [25]. In 2024, Gallardo confirmed Ryser's conjecture using stochastic matrix methods, affirming the nonexistence for orders greater than 4 [26]. There are a total of 10 circulant Hadamard matrices: 2 of order 1, and 8 of order 4 [26]. Interested readers can explore additional papers related to circulant Hadamard matrices [27], [28]. According to the literature, no relation has been established between circulant Hadamard matrices and generalized Boolean functions (GBF). The construction of circulant Hadamard matrices using GBFs has not yet been addressed.

A cross Z-complementary set (CZCS) is a set of sequences whose sum of the aperiodic auto-correlation function (AACF) of all sequences and the sum of the aperiodic cross-correlation function (ACCF) of adjacent sequences is zero in a specific zone called the zero correlation zone (ZCZ) [29]. The ZCZ is defined as the ratio of the ZCZ's width to the sequence's length. CZCS is used in spatial modulation (SM) systems over frequency-selective channels [29]. The first construction of a CZCS of set size 4 and various lengths was provided by Huang *et al.* [30], where they constructed CZCS using concatenation. In 2022, Huang *et al.* proposed CZCS of set size 4, length  $2^m$ , and a ZCZ ratio of 1, where  $m \geq 1$  [31]. In 2023, Das *et al.* used an indirect method to construct quaternary CZCS of lengths  $3L, 7L$ , and  $14L$ , with a ZCZ ratio less than  $1/2$ , where  $L$  is the length of the seed sequences [32]. In 2023, Kumar *et al.* constructed CZCS of length  $2^{m-1} + 2^\delta$  with a maximum ZCZ ratio of  $2/3$ , where  $m \geq 4$  and  $0 \leq \delta \leq m - 1$  [33]. In 2025, Huang *et al.* presented a flexible construction of  $(2^{k+1}, 2^{m-k}(2^k - 1) + 2^v, 2^{m-1})$ -CZCS and  $(2^{k_1+2}, 2^{m-1} + \sum_{\beta=1}^{k_1-1} a_\beta 2^{\pi(m-k_1+\beta)-1} + 2^{v_1}, 2^{m-1} + \sum_{\beta=1}^{k_1-1} a_\beta 2^{\pi(m-k_1+\beta)-1} + 2^{v_1})$ -CZCS, where  $m \geq 2$ ,  $1 \leq k, k_1 - 1 \leq m - 1$ ,  $0 \leq v \leq m - k$ ,  $\beta \in \mathbb{N}$ , and  $0 \leq v_1 \leq m - k_1$  [34]. To date, the construction of CZCS for all lengths remains an open problem.

The Golay complementary set (GCS) was extended from the Golay complementary pair (GCP) in 1972 [35]. A GCS is a set of sequences whose sum of the AACF is zero at every non-zero time shift. GCS has numerous applications, including channel estimation [36], synchronization [37], and peak-to-mean envelope power ratio (PMEPR) reduction in orthogonal frequency division multiplexing (OFDM) systems [38], [39]. Due to these applications, the study of GCS plays a vital role in wireless communication. Paterson proposed a method to construct GCS using GBFs [40]. Initially, constructions were limited to lengths that are powers of two, until Chen proposed GCS constructions with flexible lengths using GBFs in 2016 [41]. Several constructions using GBFs, Reed-Muller codes (RMC), generalized Reed-Muller (GRM) codes, and extended Boolean functions have been proposed in [42]–[47]. In [48], the authors introduced para-unitary (PU) matrices, where each element has unit magnitude, as a new method to construct GCS. A Hadamard matrix is a special case of a PU matrix. Wang *et al.* made significant progress in constructing GCS using PU matrices [49], [50]. However, no construction covering all lengths existed until Roy *et al.* proposed a binary GCS construction covering all lengths, though restricted to flock sizes that are powers of 2 [51]. All known binary GCS constructions to date have flock sizes limited to powers of 2. This limitation motivated the development of binary GCS constructions of all lengths with a flock size equal to  $2N$ , where  $N = 2^a 10^b 26^c$ ,  $a, b, c \geq 0$ , which has not been achieved yet.

In 1988, GCS was extended to a code set called the complete complementary code (CCC) by Suehiro and Hatori [52]. CCC is denoted as  $(N, N, L)$ -CCC, where  $N$  represents the code size and  $L$  the sequence length. CCCs have wide applications in coding, signal processing, and wireless communication [53]–[56]. Numerous constructions exist based on unitary matrices, Hadamard matrices, Boolean functions, generalized Boolean functions (GBFs), permutation polynomials, and para-unitary matrices [52], [57]–[62]. However, for the binary case, there are no construction which provides a binary CCC having a flock size and lengths that are the non-power of 2.

In 2024, Kumar *et al.* proposed an extension of CZCS and a generalization called symmetrical Z-complementary code sets (SZCCS) [63]. The authors proposed a direct construction of  $(2^{n+1}, 2^{n+1}, 2^{m-1} + 2, 2^{\pi(m-3)})$ -CZCCS, where  $n \geq 0$  and  $m \geq 4$ . In the same year, Huang *et al.* proposed three major constructions of CZCCS and named them Enhanced Cross Z-Complementary Set (E-CZCS). These constructions include  $(M, N, 2L, Z)$ -CZCCS using a  $(M, N, L, Z + 1)$ -ZCCS,  $(M, N, 2L, L)$ -CZCCS, and  $(2^k, 2^v, 2^m, 2^{\pi_1(1)-1})$ -CZCCS. The construction is optimal when  $\pi_1(1) = m - k + v$ , where  $m, k, v \in \mathbb{I}^+$  and  $v \leq k$  [64]. To date, no optimal construction of CZCCS is available using the circulant Hadamard matrices.

In this paper, we propose a new construction of Hadamard matrices, circulant Hadamard matrices, CZCS, GCS, CCC, and CZCCS by using linear operators and circulant matrices as follows:

- Properties of circulant matrices are provided over AACF/ACCF, which are used to prove the theorems.
- For the first time, we propose a direct construction of all 8 circulant Hadamard matrices of order 4 using a linear operator and GBFs.

- A new construction of  $(2^{n+2}, 2^{n+2} - k, 2^{n+1} - (k - 2^n) \lfloor \frac{k}{2^n} \rfloor)$ -CZCS is provided using the proposed circulant Hadamard matrices, where  $n \geq 0$  and  $0 \leq k \leq 2^{n+1} - 1$ . This is the first construction covering all CZCS lengths for even phases.
- A new construction of binary GCS for all lengths and Hadamard matrices of order  $2^{n+2}$ , where  $n \geq 1$ , is provided using circulant Hadamard matrices.
- Another novel construction of GCS for all lengths, with a flexible flock size  $2N$ , where  $N = 2^a 10^b 26^c$ ,  $a, b, c \geq 0$ , is proposed using circulant matrices, each generated using a GCP and its complementary pair as seed sequences.
- The  $(2N, 2N)$ -GCS construction yields Hadamard matrices of order  $2N$ , where  $N = 2^a 10^b 26^c$ ,  $a, b, c \geq 0$ .
- The constructed GCS is extended to form a  $(2N, 2N, 2N)$ -CCC, where  $N = 2^a 10^b 26^c$ ,  $a, b, c \geq 0$ .
- The constructed CZCS is extended to form an optimal  $(2^{n+2}, 2^{n+2}, 2^{n+2}, 2^{n+1})$ -CZCSS, where  $n \geq 1$ .
- Finally, we provide a theoretical relationship between GCS and Hadamard matrices. Also classified as GCS based on their property.

The rest of the paper is organized as follows. Section II provides basic notations, AACF, ACCF, linear operators, circulant matrices, and the definition of Hadamard matrices. Section III presents theorems related to circulant matrices and concatenated sequences. Section IV describes all proposed novel constructions of Hadamard matrices, circulant Hadamard matrices, CZCS, GCS, CCC, and CZCSS. Section V compares the proposed constructions with existing methods. Finally, Section VI concludes the paper and outlines open problems.

## II. PRELIMINARIES

This section introduces essential definitions, notations, and theorems used in the constructions presented later in the paper.

### A. Definitions

*Definition 1:* Let  $\mathbf{a} = (a_1, a_2, \dots, a_L)$  and  $\mathbf{b} = (b_1, b_2, \dots, b_L)$  be sequences of length  $L$ . The aperiodic cross-correlation function (ACCF) is defined as

$$\mathcal{C}(\mathbf{a}, \mathbf{b})(\lambda) = \begin{cases} \sum_{i=1}^{L-\lambda} a_i b_{i+\lambda}^*, & 0 \leq \lambda \leq L-1, \\ \sum_{i=1-\lambda}^L a_{i+\lambda} b_i^*, & -L+1 \leq \lambda \leq -1, \\ 0, & |\lambda| \geq L, \end{cases} \quad (1)$$

where  $b^*$  represents complex conjugate of  $b$ . When  $\mathbf{a} = \mathbf{b}$ , the ACCF becomes the aperiodic autocorrelation function (AACF), denoted  $\mathcal{A}(\mathbf{a})(\lambda)$ .

*Definition 2:* Let  $S = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{M-1}\}$  be a set of sequences of length  $L$ . Then  $S$  is called a Golay complementary set (GCS) if

$$\sum_{j=0}^{M-1} \mathcal{A}(\mathbf{a}_j)(\lambda) = 0, \quad \forall \lambda \neq 0. \quad (2)$$

When  $M = 2$ , the set  $S$  is referred to as a Golay complementary pair (GCP).

*Definition 3:* Let  $(\mathbf{a}, \mathbf{b})$  and  $(\mathbf{c}, \mathbf{d})$  be two GCPs of length  $L$ . The pair  $(\mathbf{c}, \mathbf{d})$  is a complementary mate of  $(\mathbf{a}, \mathbf{b})$  if

$$\mathcal{C}(\mathbf{a}, \mathbf{c})(\lambda) + \mathcal{C}(\mathbf{b}, \mathbf{d})(\lambda) = 0, \quad \forall \lambda \neq 0. \quad (3)$$

*Definition 4:* Let  $S = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{M-1}\}$  be a set of  $M$  sequences of length  $L$ . The set  $S$  is called a  $(M, L, Z)$ -cross Z-complementary set (CZCS) if it satisfies the following conditions:

$$\begin{aligned} \sum_{j=0}^{M-1} \mathcal{A}(\mathbf{a}_j)(\lambda) &= 0, \quad \forall |\lambda| \in \mathcal{T}_1 \cup \mathcal{T}_2, \\ \sum_{j=0}^{M-1} \mathcal{C}(\mathbf{a}_j, \mathbf{a}_{(j+1) \bmod M})(\lambda) &= 0, \quad \forall |\lambda| \in \mathcal{T}_2, \end{aligned} \quad (4)$$

where  $\mathcal{T}_1 = \{1, 2, \dots, Z\}$  and  $\mathcal{T}_2 = \{L-Z, L-Z+1, \dots, L-1\}$ . When  $M = 2$ , the set reduces to a cross Z-complementary pair (CZCP). The ratio  $Z/L$  is referred to as the ZCZ ratio.

*Definition 5:* A cyclic shift operator  $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is defined for a vector  $\mathbf{v} = (v_0, v_1, \dots, v_{n-1})$ . The  $k$ -th cyclic shift is given by:

$$T^k(\mathbf{v}) = (v_{k-1}, v_{k-2}, \dots, v_0, v_{n-1}, \dots, v_k), \quad (5)$$

for  $1 \leq k \leq n$ .

*Definition 6:* Define an another  $k$ -th cyclic shift operator  $T_1 : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is defined by:

$$T_1^k(\mathbf{v}) = (v_{n-1}, v_0, v_1, \dots, v_{n-2}), \quad (6)$$

where  $0 \leq k \leq n-1$  and  $v = (v_0, v_1, \dots, v_{n-1}) \in \mathbb{C}^n$ , i.e.,  $T_1^0(\mathbf{v}) = \mathbf{v}$ ,  $T_1^1(\mathbf{v}) = (v_{n-1}, v_0, v_1, \dots, v_{n-2})$ ,  $T_1^2(\mathbf{v}) = T_1^1(T_1^1(\mathbf{v})) = T_1^1(v_{n-1}, v_0, v_1, \dots, v_{n-2}) = (v_{n-2}, v_{n-1}, v_0, \dots, v_{n-3})$  and so on.

*Definition 7:* A circulant matrix of size  $n \times n$ , denoted by  $\text{Cir}(\mathbf{a})$ , is constructed using either the shift operator  $T$  or  $T_1$  as:

$$\text{Cir}(\mathbf{a}) = \begin{bmatrix} T(\mathbf{a}) \\ T^2(\mathbf{a}) \\ \vdots \\ T^n(\mathbf{a}) \end{bmatrix} = [T_1^0(\mathbf{a})^\top, T_1^1(\mathbf{a})^\top, \dots, T_1^{n-1}(\mathbf{a})^\top]. \quad (7)$$

*Example 1:* Let  $\mathbf{a} = (1, -1, -1, 1)$ . The circulant matrix  $\text{Cir}(\mathbf{a})$  is given by:

$$\text{Cir}(\mathbf{a}) = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

*Definition 8:* A matrix  $H$  of size  $n \times n$ , whose entries are either 1 or  $-1$ , is called a Hadamard matrix if it satisfies:

$$HH^\top = nI_n, \quad (8)$$

where  $I_n$  is the identity matrix of order  $n$ . Hadamard matrices exist for orders 1, 2, and  $4t$ , where  $t$  is a positive integer [1].

*Example 2:* Consider the matrix:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Then:

$$HH^\top = 4I_4,$$

verifying that  $H$  is a Hadamard matrix of order 4.

*Definition 9:* Let  $S = \{S^0, S^1, \dots, S^{N-1}\}$ , where each  $S^p = \{\mathbf{a}_0^p, \mathbf{a}_1^p, \dots, \mathbf{a}_{M-1}^p\}$  is a set of  $M$  sequences of length  $L$ . The collection  $S$  is called a mutually orthogonal Golay complementary set (MOGCS), denoted  $(N, M, L)$ -MOGCS, if it satisfies:

$$\mathcal{C}(S^p, S^{p'})(\lambda) = \sum_{i=0}^{M-1} \mathcal{C}(\mathbf{a}_i^p, \mathbf{a}_i^{p'}) (\lambda) = \begin{cases} ML, & \lambda = 0, p = p'; \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

When  $N = M$ , the MOGCS becomes a complete complementary code (CCC), denoted  $(N, N, L)$ -CCC.

*Definition 10:* Let  $S = \{S^0, S^1, \dots, S^{N-1}\}$  be the set of the  $N$  sequence set, where each  $S^p$  contains  $M$  sequences of length  $L$ , i.e.,  $S^p = \{\mathbf{a}_0^p, \mathbf{a}_1^p, \dots, \mathbf{a}_{M-1}^p\}$ , where  $0 \leq p \leq N-1$ . The set  $S$  is called a  $(K, M, L, Z)$  cross Z-complementary sequence set (CZCSS) if each  $S^p$  satisfies the following four conditions

$$\sum_{j=0}^{M-1} \mathcal{A}(\mathbf{a}_j^p)(\lambda) = 0, \quad \forall |\lambda| \in (\mathcal{T}_1 \cup \mathcal{T}_2) \cap \mathcal{T}, \quad (10)$$

$$\sum_{j=0}^{M-1} \mathcal{C}(\mathbf{a}_j^p, \mathbf{a}_{(j+1) \bmod M}^p)(\lambda) = 0, \quad \forall |\lambda| \in \mathcal{T}_2, \quad (11)$$

$$\sum_{j=0}^{M-1} \mathcal{C}(\mathbf{a}_i^p, \mathbf{a}_i^{p'}) (\lambda) = 0, \quad \forall |\lambda| \in \{0\} \cup \mathcal{T}_1 \cup \mathcal{T}_2, \quad (12)$$

and

$$\sum_{j=0}^{M-1} \mathcal{C}(\mathbf{a}_j^p, \mathbf{a}_{(j+1) \bmod M}^{p'}) (\lambda) = 0, \quad \forall |\lambda| \in \mathcal{T}_2, \quad (13)$$

where  $\mathcal{T}_1 = \{1, 2, \dots, Z\}$ ,  $\mathcal{T}_2 = \{L-Z, L-Z+1, \dots, L-1\}$ , and  $\mathcal{T} = \{1, 2, \dots, L-1\}$ . The optimality attends when the  $Z = NL/2M$  for binary sequences [64].

### B. Generalized Boolean Function

A Generalized Boolean Function (GBF) is a mapping  $f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$ , where  $m \geq 1$ ,  $q \in 2\mathbb{Z}^+$ , and  $x_i \in \{0, 1\}$  for all  $1 \leq i \leq m$ . The domain consists of all binary vectors of length  $m$ , and the codomain is the ring of integers modulo  $q$ .

We define  $2^m$  monomials of degree  $r$  as all possible products of up to  $r$  distinct variables from  $\{x_1, x_2, \dots, x_m\}$ . Examples include:

- Degree 0: 1
- Degree 1:  $x_1, x_2, \dots, x_m$
- Degree 2:  $x_1x_2, x_2x_3, \dots, x_{m-1}x_m$
- Degree  $m$ :  $x_1x_2 \dots x_m$

It is established in [65] that any GBF  $f$  can be uniquely represented as a linear combination of these monomials. The sequence  $\mathbf{f} = (f_0, f_1, \dots, f_{2^m-1})$  is generated by evaluating  $f$  at all binary inputs, where each  $f_i = f(i_1, i_2, \dots, i_m)$  and  $(i_1, i_2, \dots, i_m)$  is the binary representation of the integer  $I \in [0, 2^m - 1]$ , computed by:

$$I = \sum_{k=1}^m i_k 2^{k-1}. \quad (14)$$

The associated complex-valued sequence is  $\psi(f) = (\xi^{f_0}, \xi^{f_1}, \dots, \xi^{f_{2^m-1}})$ , where  $\xi = e^{2\pi i/q}$ .

*Example 3:* Let  $m = 3$  and  $q = 2$ . The Boolean function  $f : \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2$  yields the sequence:

$$\mathbf{f} = (f(0, 0, 0), f(1, 0, 0), f(0, 1, 0), f(1, 1, 0), \\ f(0, 0, 1), f(1, 0, 1), f(0, 1, 1), f(1, 1, 1)).$$

Its complex representation is:

$$\psi(f) = ((-1)^{f_0}, (-1)^{f_1}, \dots, (-1)^{f_7}),$$

where  $\xi = e^{2\pi i/2} = -1$ .

### C. Truncation

Let  $\mathbf{A} = [a_{ij}]_{m \times n}$  be a matrix. We define its truncation  $\mathbf{A}^k$  by removing the last  $k$  columns:

$$\mathbf{A}^k = [a_{ij}]_{m \times (n-k)}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n - k.$$

For a sequence  $\mathbf{a}$  of length  $L$ , the truncated sequence is denoted  $\mathbf{a}^{L-k}$ , indicating the last  $k$  elements are removed [66].

### D. Lemmas

*Lemma 1 ([67]):* Let  $\pi$  be a permutation of  $\{1, 2, \dots, m\}$ , where  $m \geq 1$ . Define a GBF:

$$f(x_1, x_2, \dots, x_m) = 2^{h-1} \sum_{i=1}^{m-1} x_{\pi(i)} x_{\pi(i+1)} + \sum_{k=1}^m c_k x_k, \quad (15)$$

where  $h \geq 1$ , and each  $c_k \in \mathbb{Z}_2$ . Then the sequences  $\mathbf{a} = \psi(f + \theta)$  and  $\mathbf{b} = \psi(f + 2^{h-1} x_{\pi(1)} + \theta')$ , for arbitrary  $\theta, \theta' \in \mathbb{Z}_2$ , form a GCP of length  $2^m$ .

*Lemma 2 ([68]):* Let  $(\mathbf{a}, \mathbf{b})$  and  $(\mathbf{c}, \mathbf{d})$  be GCPs of lengths  $m$  and  $n$ , respectively. Then the pair:

$$\mathbf{e} = \mathbf{a} \otimes \left( \frac{\mathbf{c} + \mathbf{d}}{2} \right) - \mathbf{b}^* \otimes \left( \frac{\mathbf{c} - \mathbf{d}}{2} \right),$$

and

$$\mathbf{f} = \mathbf{b} \otimes \left( \frac{\mathbf{c} + \mathbf{d}}{2} \right) + \mathbf{a}^* \otimes \left( \frac{\mathbf{c} - \mathbf{d}}{2} \right)$$

form a GCP of length  $mn$ . Here,  $\otimes$  denotes the Kronecker product,  $\mathbf{a}^*$  the complex conjugate, and  $\overleftarrow{\mathbf{a}}$  the reversed sequence.

*Lemma 3 ([69]):* Let  $(\mathbf{a}, \mathbf{b})$  be a GCP of length  $L$ . Then  $(\mathbf{c}, \mathbf{d}) = (\overleftarrow{\mathbf{b}^*}, -\overleftarrow{\mathbf{a}^*})$  is a complementary mate of  $(\mathbf{a}, \mathbf{b})$ .

### III. LIST OF PROPOSED LEMMAS

In this section, we propose lemmas, which play a crucial role in proving the subsequent theorems.

*Lemma 4:* For a fixed time shift  $\lambda$ , the solution of equation  $i - \lambda \equiv x_i \pmod{n}$  satisfies  $\cap_{i=1}^n x_i = \{\phi\}$ , where  $x_i \in \mathbf{I}^+$ ,  $0 \leq \lambda, i - 1 \leq n - 1$  and  $n \in \mathbb{I}^+$ .

*Proof:* See Appendix A. ■

*Lemma 5:* The value of  $T_1^i(\mathbf{a}) \cdot T_1^j(\mathbf{b})$  for  $j \leq i$  is given as

$$T_1^i(\mathbf{a}) \cdot T_1^j(\mathbf{b}) = \mathcal{C}(\mathbf{b}, \mathbf{a})(L - k') + \mathcal{C}(\mathbf{a}, \mathbf{b})(k'), \quad (16)$$

where  $\cdot$  denotes the dot product,  $k' = i - j$ ,  $L$  is the length of sequences and  $0 \leq i, j \leq L - 1$ .

*Proof:* See Appendix B. ■

*Lemma 6:* The value of  $T_1^i(\mathbf{a}) \cdot T_1^j(\mathbf{b})$  for  $i \leq j$  is given as

$$T_1^i(\mathbf{a}) \cdot T_1^j(\mathbf{b}) = \mathcal{C}(\mathbf{b}, \mathbf{a})(k') + \mathcal{C}(\mathbf{a}, \mathbf{b})(L - k'), \quad (17)$$

where  $k' = j - i$ ,  $L$  is the length of the sequences and  $0 \leq i, j \leq L - 1$ .

*Proof:* The proof can be done similarly to the *Lemma 5*. ■

*Lemma 7:* The value of  $T^i(\mathbf{a}) \cdot T^j(\mathbf{a})$  for  $i < j$  is given as

$$T^i(\mathbf{a}) \cdot T^j(\mathbf{a}) = \mathcal{A}(\mathbf{a})(k') + \mathcal{A}(\mathbf{a})(L - k'), \quad (18)$$

where  $k' = j - i$ ,  $L$  is the length of the sequences  $1 \leq i, j \leq L$  and  $i \neq j$ . When  $i = j$  then  $T^i(\mathbf{a})^2 = \mathcal{A}(\mathbf{a})(0) + \mathcal{A}(\mathbf{a})(L - 0) = L$ .

*Proof:* The proof can be done similarly to the *Lemma 5*. ■

*Lemma 8:* Let  $\mathbf{A} = \text{Cir}(\mathbf{a})$  be a circulant matrix of order  $n$  corresponding to a sequence  $\mathbf{a}$ . Then the sum of AACF of each row of the truncated matrix  $\mathbf{A}^k$  is given as

$$\sum_{i=1}^n \mathcal{A}(T^i(\mathbf{a}))(\lambda) = (n - \lambda - k)(\mathcal{A}(\mathbf{a})(\lambda) + \mathcal{A}(\mathbf{a})(n - \lambda)), \quad 0 \leq \lambda \leq n - 1 - k, \quad (19)$$

where  $0 \leq k \leq n - 1$ .

*Proof:* See Appendix C. ■

*Lemma 9:* Let  $\mathbf{a}$  and  $\mathbf{b}$  be sequences of length  $n$ , consider a matrix  $\mathbf{Z} = [\text{Cir}(\mathbf{a}), \text{Cir}(\mathbf{b})]$ . Then the sum of AACF of each row denoted by  $R_i$  of the truncated matrix  $\mathbf{Z}^k$ , where  $1 \leq i \leq n$  for  $0 \leq k \leq n - 1$  and  $n \leq k \leq 2n - 2$  are given as

$$\sum_{i=0}^n \mathcal{A}(R_i)(\lambda) = \begin{cases} (n - \lambda)(\mathcal{A}(\mathbf{a})(\lambda) + \mathcal{A}(\mathbf{a})(n - \lambda)) \\ + (n - \lambda - k)(\mathcal{A}(\mathbf{b})(\lambda) + \mathcal{A}(\mathbf{b})(n - \lambda)) \\ + \lambda(\mathcal{C}(\mathbf{a}, \mathbf{b})(\lambda) + \mathcal{C}(\mathbf{a}, \mathbf{b})(n - \lambda)), \\ \lambda \leq n - 1, \\ (\lambda - k)(\mathcal{A}(\mathbf{a})(\lambda \pmod{n}) \\ + \mathcal{A}(\mathbf{a})(n - \lambda \pmod{n})), \lambda \geq n, \end{cases} \quad (20)$$

and

$$\sum_{i=0}^n \mathcal{A}(R_i)(\lambda) = \begin{cases} (\lambda - k \pmod{n})(\mathcal{A}(\mathbf{a})(\lambda) \\ + \mathcal{A}(\mathbf{a})(n - \lambda)), \forall \lambda \end{cases} \quad (21)$$

respectively.

*Proof:* See Appendix D. ■

*Lemma 10:* Let  $\mathbf{R}_1 = (a_0, a_1, \dots, a_{L-1})$  and  $\mathbf{R}_2 = (b_0, b_1, \dots, b_{L-1})$  be complex-valued sequences. Let  $\mathbf{A} = (\mathbf{R}_1, -\mathbf{R}_1^{L-k})$  and  $\mathbf{B} = (\mathbf{R}_2, -\mathbf{R}_2^{L-k})$ , then

$$\mathcal{C}(\mathbf{A}, \mathbf{B})(\lambda) = \begin{cases} \mathcal{C}(\mathbf{R}_1, \mathbf{R}_2)(\lambda) + \mathcal{C}(\mathbf{R}_1^{L-k}, \mathbf{R}_2^{L-k})(\lambda) \\ - \mathcal{C}(\mathbf{R}_2, \mathbf{R}_1)(L - \lambda), 0 \leq \lambda \leq L - k - 1, \\ - \mathcal{C}(\mathbf{R}_2^{L-k}, \mathbf{R}_1^{L-k})(L - \lambda) \\ + \mathcal{C}(\mathbf{R}_1, \mathbf{R}_2)(\lambda), L - k \leq \lambda \leq L - 1, \\ - \mathcal{C}(\mathbf{R}_1^{2L-k-\lambda}, \mathbf{R}_2^{2L-k-\lambda})(\lambda \pmod{L}), \\ L \leq \lambda \leq 2L - 1 - k. \end{cases} \quad (22)$$

The  $\mathcal{A}(\mathbf{A})(\lambda)$  is calculated by replacing  $\mathbf{R}_2$  by  $\mathbf{R}_1$  in the above equation.

*Proof:* See Appendix E. ■

*Lemma 11:* Let  $D = \begin{bmatrix} Cir(\mathbf{a}) \\ Cir(\mathbf{c}) \end{bmatrix}$  be a  $2L \times 2L$  matrix. Then

$$T^i(\mathbf{a}) \cdot T^j(\mathbf{c}) = C(\mathbf{a}, \mathbf{c})(k') + C(\mathbf{c}, \mathbf{a})(L - k'), i < j, \quad (23)$$

where  $k' = (j - i) \bmod L$ ,  $L$  is the length of the sequences,  $1 \leq i \leq L$ ,  $L \leq j \leq 2L$  and  $i \neq j$ .

*Proof:* The proof can be done similarly to the *Lemma 5*. ■

#### IV. PROPOSED CONSTRUCTIONS

##### A. Proposed Construction of CZCS

In this sub-section, we provide the construction of circulant Hadamard matrices, CZCS, GCS and Hadamard matrices.

*Theorem 1:* Let us define a GBF  $f : \{0, 1\}^2 \rightarrow \mathbb{Z}_q$  such that  $f(x_1, x_2) = \frac{q}{2}(x_1x_2 + \theta_1x_1 + \theta_2x_2) + \theta_3$ , where  $\theta_1, \theta_2, \theta_3 \in \{0, 1, \dots, q-1\}$  and  $q$  is a positive even integer. Then the matrix

$$\mathbf{E}_4 = \begin{bmatrix} T(\mathbf{f}) \\ T^2(\mathbf{f}) \\ T^3(\mathbf{f}) \\ T^4(\mathbf{f}) \end{bmatrix}, \quad (24)$$

forms a circulant Hadamard matrix and complex circulant Hadamard matrix for  $q = 2$  and  $q \geq 4$ , respectively.

*Proof:* See Appendix F. ■

*Example 4:* Let us take  $\frac{q}{2} = \theta_1 = \theta_2 = \theta_3 = 1$ . Then the matrix  $\mathbf{E}_4$  corresponding to GBF  $f(x_1, x_2) = x_1x_2 + x_1 + x_2 + 1$  is given as

$$\mathbf{E}_4 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \quad (25)$$

forms a circulant Hadamard matrix, i.e.,  $\mathbf{E}_4\mathbf{E}_4' = 4I_4$ .

*Theorem 2:* Consider  $\mathbf{E}_4$  to be a circulant Hadamard matrix of order 4, generated from the *Theorem 1*. Let us define matrices

$$\mathbf{F}_{2^{n+2}} = \begin{bmatrix} \mathbf{E}_{2^{n+1}} & \mathbf{E}_{2^{n+1}} \\ \mathbf{E}_{2^{n+1}} & -\mathbf{E}_{2^{n+1}} \end{bmatrix}, \quad (26)$$

$$\mathbf{G}_{2^{n+2}} = \begin{bmatrix} \mathbf{E}_{2^{n+1}} & \mathbf{E}_{2^{n+1}} \\ -\mathbf{E}_{2^{n+1}} & \mathbf{E}_{2^{n+1}} \end{bmatrix}, \quad (27)$$

$$\mathbf{H}_{2^{n+2}} = \begin{bmatrix} -\mathbf{E}_{2^{n+1}} & \mathbf{E}_{2^{n+1}} \\ \mathbf{E}_{2^{n+1}} & \mathbf{E}_{2^{n+1}} \end{bmatrix}, \quad (28)$$

and

$$\mathbf{I}_{2^{n+2}} = \begin{bmatrix} \mathbf{E}_{2^{n+1}} & -\mathbf{E}_{2^{n+1}} \\ \mathbf{E}_{2^{n+1}} & \mathbf{E}_{2^{n+1}} \end{bmatrix}. \quad (29)$$

Then the truncated matrices  $\{\mathbf{F}_{2^{n+2}}^k, \mathbf{G}_{2^{n+2}}^k, \mathbf{H}_{2^{n+2}}^k, \mathbf{I}_{2^{n+2}}^k\}$  forms  $(2^{n+2}, 2^{n+2} - k, 2^{n+1} - (k - 2^n) \lfloor \frac{k}{2^n} \rfloor)$ -CZCS, where  $n \geq 1$  and  $0 \leq k \leq 2^{n+1} - 1$ .

*Proof:* See Appendix G. ■

*Example 5:* Let us take matrix  $\mathbf{E}_4$  of (25), such that

$$\mathbf{F}_8 = \begin{bmatrix} -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}.$$

Then the truncated matrix  $\mathbf{F}_8^k$  forms a  $(8, 8 - k, 4 - (k - 2) \lfloor \frac{k}{2} \rfloor)$ -CZCS, where  $0 \leq k \leq 3$  and matrix  $\mathbf{F}_8$  forms Hadamard matrix of order 8.

*Corollary 1:* Each of the truncated matrices of the set  $\{\mathbf{F}_{2^{n+2}}^k, \mathbf{G}_{2^{n+2}}^k, \mathbf{H}_{2^{n+2}}^k, \mathbf{I}_{2^{n+2}}^k\}$  also form  $(2^{n+2}, 2^{n+2} - k)$ -GCS, where  $n \geq 1$  and  $0 \leq k \leq 2^{n+1} - 2$ .

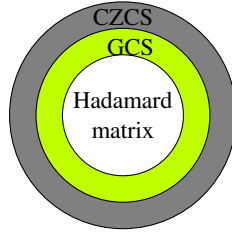


Fig. 1: This represents the relation among CZCS, GCS and Hadamard matrix generated from the proposed constructions.

*Remark 1:* Each matrix in the set  $\{\mathbf{F}_{2^{n+2}}, \mathbf{G}_{2^{n+2}}, \mathbf{H}_{2^{n+2}}, \mathbf{I}_{2^{n+2}}\}$  from *Theorem 2*, also forms a Hadamard matrix of order  $2^{n+2}$ , where  $n \geq 1$ .

*Remark 2:* From *Theorem 2*, *Corollary 1*, and *Remark 1*, we find that there are  $(2^{n+2}, 2^{n+2}, 2^{n+1})$ -CZCS and  $(2^{n+2}, 2^{n+2})$ -GCS which are Hadamard matrices of order  $2^{n+2}$ , where  $n \geq 1$ . The Venn diagram is given in Fig. 1.

TABLE I: Representation of each matrix  $R_i \odot \mathbf{G}$ , for all  $1 \leq i \leq 8$  generated from *Example 8*.

| $R_1 \odot \mathbf{G}$                                                                                                                                                                                                                                                                                                           | $R_2 \odot \mathbf{G}$                                                                                                                                                                                                                                                                                                           | $R_3 \odot \mathbf{G}$                                                                                                                                                                                                                                                                                                           | $R_4 \odot \mathbf{G}$                                                                                                                                                                                                                                                                                                           |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \end{bmatrix}$ | $\begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \end{bmatrix}$  | $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$  | $\begin{bmatrix} -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}$  |
| $R_5 \odot \mathbf{G}$                                                                                                                                                                                                                                                                                                           | $R_6 \odot \mathbf{G}$                                                                                                                                                                                                                                                                                                           | $R_7 \odot \mathbf{G}$                                                                                                                                                                                                                                                                                                           | $R_8 \odot \mathbf{G}$                                                                                                                                                                                                                                                                                                           |
| $\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$  | $\begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}$ | $\begin{bmatrix} -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ |

### B. Proposed Construction of GCS

In this sub-section, we present a novel construction of the GCS of all lengths using circulant matrices. We use GCP and its complementary mate as seed sequences and generate them into a circulant matrix.

*Theorem 3:* Let  $(\mathbf{a}, \mathbf{b})$  be a GCP of length  $N$  generated from the *Lemma 2* and  $(\mathbf{c}, \mathbf{d})$  be its complementary mate. Let  $\mathbf{A} = \text{Cir}(\mathbf{a})$ ,  $\mathbf{B} = \text{Cir}(\mathbf{b})$ ,  $\mathbf{C} = \text{Cir}(\mathbf{c})$ , and  $\mathbf{D} = \text{Cir}(\mathbf{d})$  be a circulant matrix of size  $N \times N$ . Then the matrix

$$\mathbf{G}^k = \begin{bmatrix} T(\mathbf{a}) & T(\mathbf{b}) \\ T^2(\mathbf{a}) & T^2(\mathbf{b}) \\ \vdots & \vdots \\ T^N(\mathbf{a}) & T^N(\mathbf{b}) \\ T(\mathbf{c}) & T(\mathbf{d}) \\ T^2(\mathbf{c}) & T^2(\mathbf{d}) \\ \vdots & \vdots \\ T^N(\mathbf{c}) & T^N(\mathbf{d}) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \quad (30)$$

forms a  $(2N, 2N - k)$ -GCS, where  $N = 2^a 10^b 26^c$ ,  $a, b, c \geq 0$ , and  $0 \leq k \leq 2N - 2$ .

*Proof:* See Appendix H. ■

*Remark 3:* This approach is designed to achieve reduced computational complexity while maintaining a significantly larger flock size than previous constructions. The structural properties of this construction also distinguish it from the previous construction of GCS, as it avoids the generation of CZCS.

*Corollary 2:* The matrix  $\mathbf{G} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$  forms a Hadamard matrix of order  $2N$ .

*Proof:* See Appendix I ■

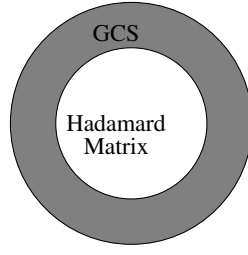


Fig. 2: This represents the relation between GCS and Hadamard matrix generated from the proposed constructions.

*Example 6:* Let us take  $\mathbf{a} = (1, 1, 1, -1)$  and  $\mathbf{b} = (1, 1, -1, 1)$  be a GCP of length 4 and the pair  $\mathbf{c} = (1, -1, 1, 1)$  and  $\mathbf{d} = (1, -1, -1, -1)$  be a complementary mate of the pair  $\mathbf{a}$  and  $\mathbf{b}$ . Then the matrix  $\mathbf{G}^k$  gives  $(8, 8 - k) - GCS$ , where  $0 \leq k \leq 6$  and

$$\mathbf{G}^k = \begin{bmatrix} 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix}.$$

The above matrix is also a Hadamard matrix of order 8, i.e.,  $\mathbf{G}\mathbf{G}^\top = 8\mathbf{I}_8$ .

*Remark 4:* It should be noted from *Theorem 3* that we get  $(2N, 2N) - GCS$  for  $k = 0$ , which is the Hadamard matrix of order  $2N$  or vice versa, where  $N = 2^a 10^b 26^c$ , and  $a, b, c \geq 0$ . The Venn diagram is shown in *Fig. 2*.

### C. Proposed Construction of CCC

In this sub-section, we propose the construction of CCC by extending the result of GCS.

*Theorem 4:* Let  $R_i$  be each row of matrix  $\mathbf{G} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ , where  $1 \leq i \leq 2N$ . Then  $\bigcup_{i=1}^{2N} R_i \odot \mathbf{G}$  forms a  $(2N, 2N, 2N) - CCC$ , where  $\mathbf{G}$  be a matrix from (30) and  $\odot$  denotes element wise product.

*Proof:* See Appendix J ■

*Example 7:* Consider  $\mathbf{a} = (1, 1, -1, 1, -1, 1, -1, -1, 1, 1)$  and  $\mathbf{b} = (1, 1, -1, 1, 1, 1, 1, 1, -1, -1)$  to be a GCP of length 10, and  $\mathbf{c} = (-1, -1, 1, 1, 1, 1, 1, -1, 1, 1)$  and  $\mathbf{d} = (-1, -1, 1, 1, -1, 1, -1, 1, -1, -1)$  its complementary mate, where

$$\mathbf{G} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}.$$

Then  $\bigcup_{i=1}^{20} R_i \odot \mathbf{G}$  forms a  $(20, 20, 20) - CCC$ , where  $R_i$  denotes the  $i$ th row of  $\mathbf{G}$ , and  $1 \leq i \leq 20$ . The sum of AACF of each  $R_i \odot \mathbf{G}$  is given in *Figure 3*. The sum of ACCF between  $R_i \odot \mathbf{G}$  and  $R_j \odot \mathbf{G}$  is given in *Figure 4*.

### D. Proposed Construction of CZCSS

In this sub-section, we propose the construction of optimal CZCSS by extending the result of CZCS.

*Theorem 5:* Let  $\mathbf{G}$  be a matrix from the set  $\{\mathbf{F}_{2^{n+1}}, \mathbf{G}_{2^{n+1}}, \mathbf{H}_{2^{n+1}}, \mathbf{I}_{2^{n+1}}\}$  generated from *Theorem 2*. Let  $R_i$  be each row of matrix  $\mathbf{G}$ , where  $1 \leq i \leq 2^{n+2}$ . Then  $\bigcup_{i=1}^{2^{n+2}} R_i \odot \mathbf{G}$  forms  $(2^{n+2}, 2^{n+2}, 2^{n+2}, 2^{n+1}) - CZCSS$  and  $(2^{n+2}, 2^{n+2}, 2^{n+2}) - CCC$  for  $n \geq 1$ .

*Proof:* See Appendix K. ■

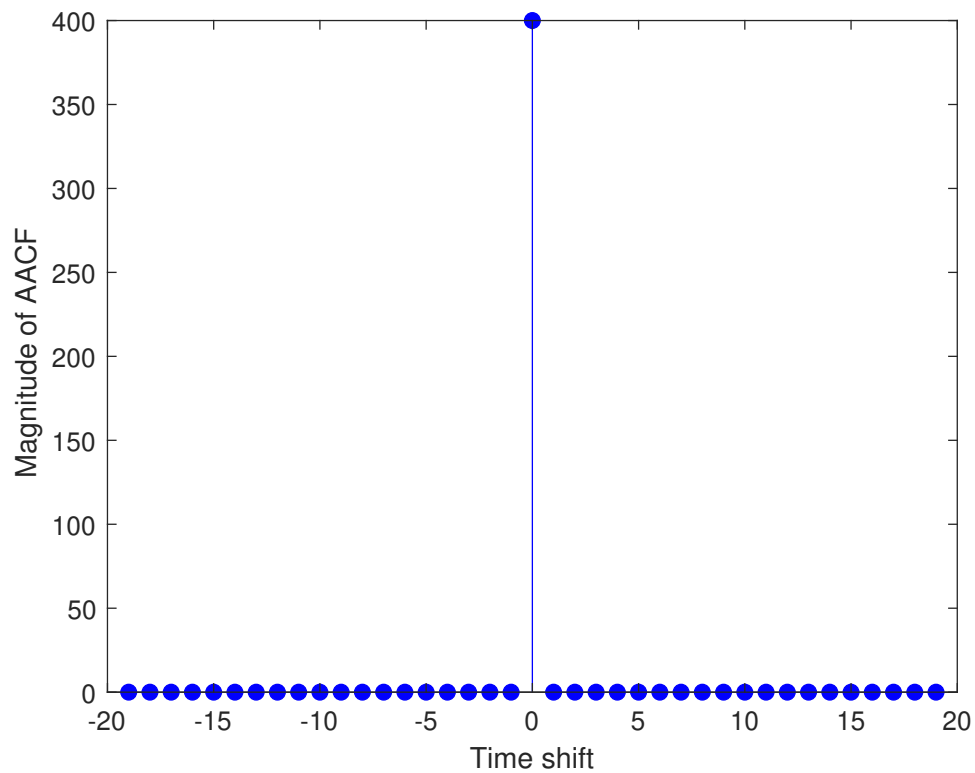
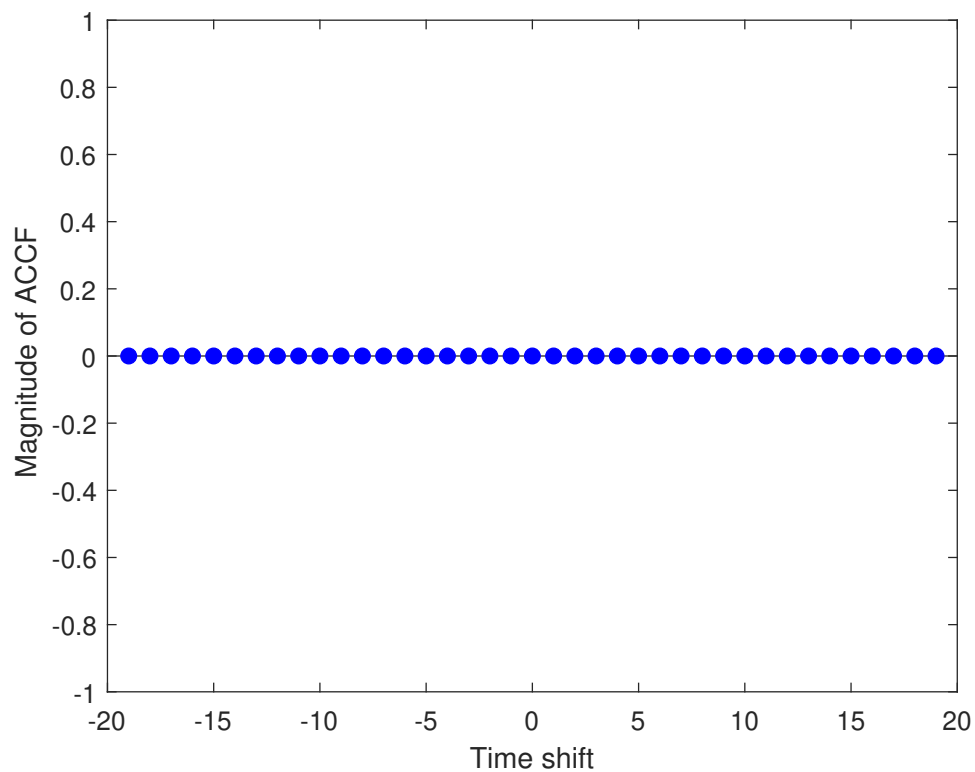
*Example 8:* Let us take a circulant Hadamard matrix  $\mathbf{E}_4$  from *Theorem 1* such that

$$\mathbf{E}_4 = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{bmatrix}, \text{ and } \mathbf{F}_8 = \begin{bmatrix} \mathbf{E}_4 & \mathbf{E}_4 \\ \mathbf{E}_4 & -\mathbf{E}_4 \end{bmatrix} = \mathbf{G}.$$

Then  $\bigcup_{i=1}^8 R_i \odot \mathbf{G}$  forms  $(8, 8, 8, 4) - CZCSS$ , where  $R_i$  denotes the  $i$ th row of  $\mathbf{G}$ , and  $1 \leq i \leq 8$ . Each code is presented in *Table 1*.

*Remark 5:* The above-proposed construction of  $(2^{n+2}, 2^{n+2}, 2^{n+2}, 2^{n+1}) - CZCSS$  is optimal [64].

*Remark 6:* All the proposed constructions also provide  $q$ -phase CZCS, GCS, CCC, and CZCSS, where  $q$  is an even positive integer.

Fig. 3: Sum of AACF of each row of  $R_i \odot \mathbf{G}$ Fig. 4: Sum of ACCF between each rows of  $R_i \odot \mathbf{G}$  and  $R_j \odot \mathbf{G}$

### E. Classification of GCS and its relation with Hadamard matrices

In this section, we divide the GCS into two parts according to their property, and also show the relationship between the GCS and Hadamard matrices.

- A matrix  $\mathbf{G}$  of order  $n \times n$  is called type-1 GCS if  $\mathbf{G}$  is Hadamard matrix, i.e.,  $\mathbf{G}$  forms  $(n, n) - GCS$  and  $\mathbf{G}\mathbf{G}^\top = n\mathbf{I}_n$ , where  $n$  is 1, 2 and multiple of 4.

*Example 9:* Let us take

$$\mathbf{G} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \quad (31)$$

$\mathbf{G}$  forms  $(4, 4) - GCS$  and  $\mathbf{G}\mathbf{G}^\top = 4\mathbf{I}_4$ , so  $\mathbf{G}$  is type-1 GCS.

- A matrix  $\mathbf{G}$  of order  $n \times n$  is called type-2 GCS if  $\mathbf{G}$  is not Hadamard matrix, i.e.,  $\mathbf{G}$  forms  $(n, n) - GCS$  and  $\mathbf{G}\mathbf{G}^\top \neq n\mathbf{I}_n$ , where  $n$  is 1, 2 and multiple of 4.

*Theorem 6:* All the Hadamard matrices of order  $n$  form a  $(n, n) - GCS$ , where  $n$  is 1, 2 and multiple of 4.

*Proof:* See Appendix L. ■

*Remark 7:* The converse of *Theorem 6* is not true.

*Example 10:* Let us have  $(8, 8) - GCS$  generated from [46], i.e.,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}. \quad (32)$$

The above matrix forms a  $(8, 8) - GCS$  but is not a Hadamard matrix of order 8. This is an example of type-2 GCS.

*Theorem 7:* From the collection of all  $(n, n) - GCS$ , at least one of them forms a Hadamard matrix of order  $n$ , where  $n = 1, 2$  or is a multiple of 4.

*Proof:* See Appendix M. ■

TABLE II: Comparison of the construction of Hadamard matrices with the proposed constructions

| Ref.     | Method                            | Order                  | Constraint                                                               |
|----------|-----------------------------------|------------------------|--------------------------------------------------------------------------|
| [2]      | Adjoining the matrices            | $2n$                   | $n$ is order of Hadamard matrix                                          |
| [4]      | Quadratic residue in finite field | $p+1$ and $2(p+1)$     | $p \equiv 3 \pmod{4}$ and $p \equiv 1 \pmod{4}$                          |
| [6]      | Williamson-type matrices          | $12t$                  | $t$ is a positive integer                                                |
| [8]      | Recursive approach                | $4t$                   | $t \in \{1, 3, 5, 7, 19\}$                                               |
| [9]      | Used complex Hadamard matrices    | $8nh$                  | $4h$ and $2h$ is order of real and complex Hadamard matrix, respectively |
| [10]     | T-matrices                        | $28m, 36m$ , and $44m$ | $m$ is order of T-matrices                                               |
| [14]     | T-matrices                        | $2^t q$                | $q$ is the length of two GCPs and $t \in \mathbb{I}^+$                   |
| Remark 1 | GBF                               | $2^{n+2}$              | $n \geq 1$                                                               |
| Remark 2 | Circulant matrices                | $2N$                   | $N = 2^a 10^b 26^c$ , where $a, b, c \geq 0$                             |

TABLE III: Comparison of existing CZCS with the proposed constructions

| Ref.      | Method                    | Set size    | Length                                                                      | Zone                                                                        | Constraint                                                                      |
|-----------|---------------------------|-------------|-----------------------------------------------------------------------------|-----------------------------------------------------------------------------|---------------------------------------------------------------------------------|
| [29]      | Indirect                  | $M$         | $L$                                                                         | $Z$                                                                         | Existing $(L, Z) - CZCP$                                                        |
| [30]      | Indirect                  | 4           | $N$                                                                         | $N$                                                                         | $N$ is exiting GCP                                                              |
|           |                           |             | $N_1 + N_2$                                                                 | $\min(N_2, N_1 + Z_2)$                                                      | $N_1$ is existing GCP and $(N_2, Z_2)$ is existing CZCP                         |
|           |                           |             | $2L$                                                                        | $L$                                                                         | $(4, L) - GCS$                                                                  |
|           |                           |             | $2N$                                                                        | $2N$                                                                        | $N$ is existing GCP                                                             |
|           |                           |             | $4L$                                                                        | $2L$                                                                        | $(4, L) - GCS$                                                                  |
| [31]      | Indirect                  | 4           | $N$                                                                         | $N$                                                                         | $N$ is exiting GCP                                                              |
| [32]      | Indirect                  | 4           | $3L$                                                                        | $L$                                                                         | $L$ is length of GCP                                                            |
|           |                           |             | $7L$                                                                        | $2L$                                                                        |                                                                                 |
|           |                           |             | $14L$                                                                       | $6L$                                                                        |                                                                                 |
| [33]      | GBF                       | $2^{k+1}$   | $2^{m-1} + 2^b$                                                             | $2^{n_k(1)-1} + 2^b$                                                        | $m \geq 4, 0 \leq b \leq m-1$ and $1 \leq k \leq m-1$                           |
| [34]      | GBF                       | $2^{k+1}$   | $2^{m-k} (2^k - 1) + 2^b$                                                   | $2^{m-1}$                                                                   | $m \geq 2, 2 \leq k \leq m$ and $0 \leq b \leq m-k$                             |
|           |                           | $2^{k_1+2}$ | $2^{m-1} + \sum_{\beta=1}^{k_1-1} a_\beta 2^{\pi(m-k_1+\beta)-1} + 2^{v_1}$ | $2^{m-1} + \sum_{\beta=1}^{k_1-1} a_\beta 2^{\pi(m-k_1+\beta)-1} + 2^{v_1}$ | $m \geq 2, 1 \leq k_1 \leq m, \beta \in \mathbb{N}$ and $0 \leq v_1 \leq m-k_1$ |
| Theorem 2 | Circulant Hadamard matrix | $2^{n+2}$   | $2^{n+2} - k$                                                               | $2^{n+1} - (k - 2^n) \lfloor \frac{k}{2^n} \rfloor$                         | $n \geq 1$ and $0 \leq k \leq 2^{n+1} - 1$                                      |

## V. COMPARISON

This section compares the proposed constructions with the current state of the work.

- The complete comparison of Hadamard matrices is provided in *Table II*. The previous work of Hadamard is based on T-matrices [10], quadratic residue [4], and Williamson-type matrices [2], [6]–[9]. In 1989, Koukouvinous *et al.* used T-matrices and GCP to provide the construction of Hadamard matrices of order  $2^t q$ , where  $q$  is the sum of the lengths of two GCP [14]. The circulant Hadamard matrices covered the order of,  $2^{n+2}$  and circulant matrices covered the order of  $2^{a+1}10^b26^c$  of Hadamard matrices, where  $n \geq 1$ , and  $a, b, c \geq 0$ . The proposed constructions show a relationship between GCS and Hadamard matrices. The proposed constructions provide ease in getting the Hadamard matrices, because the GCP of length  $N$  can be easily generated, where  $N = 2^a10^b26^c, a, b, c \geq 0$ . The constructions of Hadamard, which are based on T-matrices, and Williamson-type matrices, make it difficult to get the Hadamard matrices because of the unavailability of T-matrices and Williamson-type matrices.
- The complete comparison of CZCS is provided in *Table III*. The first known construction used concatenation with a set size of 4 and varying lengths [30]. CZCS with lengths  $2^m$  and ZCZ ratio 1, for  $m \geq 1$ , were later introduced [31]. Constructions with lengths  $3L$ ,  $7L$ , and  $14L$  and ZCZ ratios less than  $1/2$  were proposed using an indirect method [32]. Construction of CZCS of length  $2^{m-1} + 2^\delta$  achieving maximum ZCZ ratio  $2/3$  is presented in [33], where  $m \geq 4$  and  $0 \leq \delta \leq m-1$ . The proposed construction of CZCS is based on the circulant Hadamard matrix. The proposed construction covers all lengths and has flexible phases and set sizes, which were unavailable before. The maximum ZCZ ratio is  $2/3$ . It covers all the existing work and has a larger ZCZ ratio for some special cases.

TABLE IV: Comparison of existing GCS with the proposed constructions

| Ref.               | Method                    | Length                                                                 | Set Size  | Phase | Constraint                                                                           |
|--------------------|---------------------------|------------------------------------------------------------------------|-----------|-------|--------------------------------------------------------------------------------------|
| [40]               | GBF                       | $2^m$                                                                  | $2^m$     | $q$   | $m \geq 1, 2 q$                                                                      |
| [70]               | GBF                       | $2^m$                                                                  | $2^m$     | $q$   | $m \geq 1, 2 q$                                                                      |
| [41]               | GBF                       | $2^{m-1} + 2^v$                                                        | 4         | $q$   | $1 \leq v \leq m-1, m \geq 2, 2 q$                                                   |
|                    |                           | $2^{m-1} + 1$                                                          | $2^{k+1}$ |       |                                                                                      |
| [42]               | PU matrices               | $2^{m-1} + 2^v$                                                        | $2^{k+1}$ | $q$   | $1 \leq v \leq m-1, m \geq 2, 2 q$                                                   |
| [43]               | GBF                       | $2^{m-1} + 2^v$                                                        | $2^{k+1}$ | $q$   | $1 \leq v \leq m-1, m \geq 2, 2 q$                                                   |
| [44]               | GBF                       | $2^{m-1} + \sum_{\alpha=1}^{k-1} a_\alpha 2^{\pi(m-k+\alpha)-1} + 2^v$ | $2^{k+1}$ | 2     | $k, m \geq 2, v \geq 0$                                                              |
| [45]               | GBF                       | $N+1, N+2$                                                             | 4         | $q$   | $N = 2^a10^b26^c, a, b, c \geq 0, 2 q$                                               |
|                    |                           | $2N+3$                                                                 | 8         |       |                                                                                      |
| [46]               | Concatenation             | $N+M$                                                                  | 4         | $q$   | $N = 2^a10^b26^c, a, b, c \geq 0, 2 q$<br>$q$ $P$ is the length of GCS of set size 4 |
|                    |                           | $N+P$                                                                  | 8         |       |                                                                                      |
| [71]               | Indirect                  | $LN$                                                                   | 4         | $q$   | $L$ is the length of ESCP, $2 q$                                                     |
| [47]               | GBF                       | $2^{m-1} + 2^t$                                                        | $2^{k+1}$ | $q$   | $k \leq m-1, m \geq 2, 2 q$                                                          |
| [51]               | EBF                       | $L$                                                                    | $p^k$     | $q$   | $p, L \in \mathbb{N}$ and $p q$                                                      |
| <i>Theorem 3</i>   | Circulant matrix          | $2N - k$                                                               | $2N$      | $q$   | $N = 2^\alpha 10^\beta 26^\gamma, 2 q, 0 \leq k \leq 2N - 2$                         |
| <i>Corollary 1</i> | Circulant Hadamard matrix | $2^{n+2} - k$                                                          | $2^{n+2}$ | $q$   | $n \geq 1$ and $0 \leq k \leq 2^{n+1} - 2, 2 q$                                      |

- The complete comparison of GCS is provided in *Table IV*. Constructions of GCS with lengths that are powers of two were presented in [40], [70], while non-power-of-two lengths were addressed in [41]. GBF-based constructions included GCSs of set size 4 and length  $2^{m-1} + 2^v$  [42], and set size  $2^k + 1$  with the same length form [43]. More generalized lengths of the form  $2^{m-1} + \sum_{\alpha=1}^{k-1} a_\alpha 2^{\pi(m-k+\alpha)-1} + 2^v$  with set size  $2^k + 1$  were introduced for  $k, m \geq 2$  and  $v \geq 0$  [44]. GCSs of lengths  $N+1$ ,  $N+2$ , and  $2N+3$  with set sizes 4 and 8 were constructed by extending GCPs [72]. Further constructions with set sizes 4 and 8 of lengths  $N+M$  and  $N+P$  were given, where  $N, M$  are GCP lengths and  $P$  is the length of a GCS of set size 4 [46]. A construction using ESCPs yielded lengths  $LN$  with set size 4 [71]. GBF-based GCSs of length  $2^{m-1} + 2^t$  and set size  $2^{k+1}$  were developed for  $k \leq m-1$  [47]. A recent construction based on EBF was proposed to cover all lengths, though with a limited flock size [51]. All previous constructions were based on GBF, EBF, and the concatenation of GCPs. In this paper, two new constructions of GCS are proposed. The first construction is based on circulant Hadamard matrices, and the second is based on circulant matrices. The proposed constructions cover all lengths and offer flexible flock sizes and phases. A binary (20, 17)-GCS cannot be obtained from existing works but is obtained using the proposed construction. The proposed constructions cover all existing work and give flexibility in the choice of flock size and phase.
- The complete comparison of CCC is provided in *Table V*. Numerous methods exist to construct CCC, including unitary matrices, Hadamard matrices, Boolean functions, para-unitary matrices, and permutation polynomials [57], [58], [60], [61]. Previous CCC constructions based on GBF, EBF, and permutation polynomial have a flock size in powers of 2. Construction based on PU matrices has a flock size in non-power of 2, but the generation of PU matrices is highly complex [62], [73]. Few constructions are based on Hadamard matrices, where many Hadamard matrices are still unknown. The proposed construction offers flexible flock sizes using circulant matrices and circulant Hadamard matrices, providing

$(2N, 2N, 2N) - CCC$  and  $(2^{n+2}, 2^{n+2}, 2^{n+2}) - CCC$ , where  $N = 2^a 10^b 26^c$ ,  $a, b, c \geq 0$ , and  $n \geq 1$ . Since the seed sequences are GCP, the phase depends on the user's choice. Each proposed code forms a Hadamard matrix.

TABLE V: Comparison of the proposed method CCC with the previous constructions

| Ref.      | Method/Tool               | Phase                       | Parameter                                    | Constraint                                                                       |
|-----------|---------------------------|-----------------------------|----------------------------------------------|----------------------------------------------------------------------------------|
| [52]      | Unitary matrices          | $q$                         | $(M, M, M^N)$                                | $q, N \geq 2$                                                                    |
| [57]      | Hadamard matrices         | 2                           | $(2^{N-r}, 2^{N-r}, 2^N)$                    | $r = 1, 2, \dots, N-1$                                                           |
| [58]      | Boolean function          | $q$                         | $(2^{k+1}, 2^{k+1}, 2^m)$                    | $q \geq 2, m, k \geq 1$ and $k = m-1$                                            |
| [59]      | GBF                       | $\prod_{i=1}^k p_i^{n_i+1}$ | $\prod_{i=1}^k p_i^{m_i}$                    | $q = \prod_{i=1}^k p_i, p_i, m_i \geq 2, n_i \geq 0$ and $p_i$ is a prime number |
| [61]      | PU matrices               | $q$                         | $(M, M, M^N)$                                | $N \geq 0, q \geq 2$                                                             |
| [62]      | PU matrices               | Q                           | $(M, M, M^j M' \prod_{j=0}^J l_j^{N_j} l^N)$ | $1 \leq M' \leq M, l_j   M, l   M', \text{lcm}\{q, q_p\}_{p=0}^{P-1} = Q$        |
| [60]      | Permutation polynomial    | $q$                         | $(2^m, 2^m, 2^m)$                            | $m \geq 1$ and $q$ is an even integer                                            |
| Theorem 4 | Circulant matrix          | $q$                         | $(2N, 2N, 2N)$                               | $N = 2^a 10^b 26^c, a, b, c \geq 0, 2 q$                                         |
| Theorem 5 | Circulant Hadamard matrix | $q$                         | $(2^{n+2}, 2^{n+2}, 2^{n+2})$                | $n \geq 1, 2 q$                                                                  |

TABLE VI: Comparison of existing CZCSS with the proposed method

| Ref.      | Method                      | Code size | Set size  | Length        | Zone             | Optimality | Constraint                                                                    |
|-----------|-----------------------------|-----------|-----------|---------------|------------------|------------|-------------------------------------------------------------------------------|
| [63]      | GBF                         | $2^{n+1}$ | $2^{n+1}$ | $2^{m-1} + 2$ | $2^{\pi(m-3)}$   | No         | $n \in \mathbb{I}^+, m \geq 4$                                                |
| [64]      | Indirect                    | $M$       | $N$       | $2L$          | $Z$              | No         | $(M, N, L, Z+1) - ZCCS$                                                       |
|           | GBF                         | $2^k$     | $2^v$     | $2^m$         | $2^{\pi_1(1)-1}$ | No         | $(M, N, L) - MOGCS$                                                           |
| Theorem 5 | Circulant Hadamard matrices | $2^{n+2}$ | $2^{n+2}$ | $2^{n+2}$     | $2^{n+1}$        | Yes        | $k, m, v \in \mathbb{I}^+, v \leq k$ and $\pi_1$ is permutation<br>$n \geq 1$ |

- The complete comparison of CZCSS is provided in Table VI. The constructions of CZCSS are available by using GBF and concatenation. The proposed construction of an optimal  $(2^{n+2}, 2^{n+2}, 2^{n+2}, 2^{n+1}) - CZCSS$  is based on circulant Hadamard matrices and shows the relation between Hadamard matrices and CCC as it also forms  $(2^{n+2}, 2^{n+2}, 2^{n+2}) - CCC$ , where  $n \geq 1$ .

## VI. CONCLUSION

A GBF generates the all-circulant Hadamard matrix of order 4. Using this, a new recursive construction of CZCS of all lengths, GCS of all lengths, and Hadamard matrices of order  $2^{n+2}$ , where  $n \geq 1$ . This construction also shows a relation between the CZCS, GCS, and Hadamard matrices. The constructed CZCS achieves the maximum  $ZCZ$  ratio is  $2/3$ . The paper provides another low-complex construction of GCS using circulant matrices. This paper also shows how to construct GCS of all lengths and have an even phase from a circulant matrix. The constructed GCS forms a Hadamard matrix. The proposed constructions of GCS and CZCS are further extended to form CCC and CZCCS, respectively. All the proposed methods are new and can provide new directions for constructing sequences.

Further research can include the study of complete relations among CZCS, GCS, and Hadamard matrices.

## APPENDIX A PROOF OF LEMMA 4

For each  $i$ , we have the following equations:

$$1 - \lambda \equiv x_1 \pmod{n}, \quad (33)$$

$$2 - \lambda \equiv x_2 \pmod{n}, \quad (34)$$

$$\vdots$$

$$n - \lambda \equiv x_n \pmod{n}. \quad (35)$$

$$\text{When } \lambda = 0, (x_1, x_2, \dots, x_n) = (1, 2, 3, \dots, n). \quad (36)$$

$$\text{When } \lambda = 1, (x_1, x_2, \dots, x_n) = (n, 1, 2, \dots, n-1). \quad (37)$$

$$\text{When } \lambda = 2, (x_1, x_2, \dots, x_n) = (n-1, n, 1, \dots, n-2). \quad (38)$$

$$\vdots$$

$$\text{When } \lambda = n-1, (x_1, x_2, \dots, x_n) = (2, 3, 4, \dots, n, 1). \quad (40)$$

Hence, we get  $\cap_{i=1}^n x_i = \{\phi\}$ .

APPENDIX B  
PROOF OF LEMMA 5

We have,

$$\begin{aligned} T_1^i(\mathbf{a}) &= (a_{n-i}, a_{n-i+1}, \dots, a_{n-1}, a_0, a_1, \dots, a_{n-1-i}), \\ T_1^j(\mathbf{b}) &= (b_{n-j}, b_{n-j+1}, \dots, b_{n-1}, b_0, b_1, \dots, b_{n-1-j}). \end{aligned} \quad (41)$$

Then the value of  $T_1^i(\mathbf{a}) \cdot T_1^j(\mathbf{b})$  is given as

$$\begin{aligned} T_1^i(\mathbf{a}) \cdot T_1^j(\mathbf{b}) &= a_{n-i}b_{n-j} + a_{n-i+1}b_{n-j+1} + \dots \\ &\quad + a_{n-i+j+1}b_{n-1} + a_{n-i+j}b_0 + \dots \\ &\quad + a_{n-1}b_{i-j-1} + a_0b_{i-j} + a_1b_{i-j+1} \\ &\quad + \dots + a_{n-i-1}b_{n-j-1}, \\ &= a_{n-i+j}b_0 + a_{n-i+j+1}b_1 + \dots \\ &\quad + a_{n-1}b_{i-j-1} + a_0b_{i-j} + a_1b_{i-j+1} + \dots \\ &\quad + a_{n-i-1}b_{n-j-1} + a_{n-i}b_{n-j} \\ &\quad + a_{n-i+1}b_{n-j+1} + \dots + a_{n-i+j+1}b_{n-1}. \end{aligned} \quad (42)$$

Let us take  $i - j = k'$ , then  $0 \leq k' \leq n - 1$ . The above (42) becomes,

$$\begin{aligned} T_1^i(\mathbf{a}) \cdot T_1^j(\mathbf{b}) &= a_{n-k'}b_0 + a_{n-k'+1}b_1 + \dots + a_{n-1}b_{k'-1}, \\ &\quad + a_0b_{k'} + a_1b_{k'+1} + \dots + a_{n-i-1}b_{n-j-1} \\ &\quad + a_{n-i}b_{n-j} + a_{n-i+1}b_{n-j+1} + \dots \\ &\quad + a_{n-k'+1}b_{n-1}, \\ &= \mathcal{C}(\mathbf{b}, \mathbf{a})(n - k') + \mathcal{C}(\mathbf{a}, \mathbf{b})(k'), \\ &= \mathcal{C}(\mathbf{b}, \mathbf{a})(n - \lambda) + \mathcal{C}(\mathbf{a}, \mathbf{b})(\lambda). \end{aligned} \quad (43)$$

Hence, the lemma is proved.

APPENDIX C  
PROOF OF LEMMA 8

We prove this using the induction method.

*Base case:* When  $k = 0$ ,

we have,

$$Cir(\mathbf{a}) = \begin{bmatrix} T(\mathbf{a}) \\ T^2(\mathbf{a}) \\ \vdots \\ T^n(\mathbf{a}) \end{bmatrix}. \quad (44)$$

The sum of AACF of all rows of  $Cir(\mathbf{a})$  is given as

$$\sum_{i=1}^n \mathcal{A}(T^i(\mathbf{a}))(\lambda) = \left( \sum_{i=1}^{n-\lambda} T^i(\mathbf{a}) \cdot T^{x_i}(\mathbf{a}) \right). \quad (45)$$

where  $1 \leq i \leq n$ ,  $x_i \geq 1$ , and  $i - \lambda \equiv x_i \pmod{n}$ . Using *Lemma 4* and *Lemma 7* then

$$\begin{aligned} \sum_{i=1}^{n-0} T^i(\mathbf{a}) \cdot T^{x_i}(\mathbf{a}) &= T^1(\mathbf{a}) \cdot T^1(\mathbf{a}) + \dots \\ &\quad + T^{n-1}(\mathbf{a}) \cdot T^n(\mathbf{a}), \\ &= \mathcal{A}(\mathbf{a})(0) + \dots + \mathcal{A}(\mathbf{a})(0) \\ &= (n)\mathcal{A}(\mathbf{a})(0). \end{aligned} \quad (46)$$

$$\begin{aligned} \sum_{i=1}^{n-1} T^i(\mathbf{a}) \cdot T^{x_i}(\mathbf{a}) &= T^1(\mathbf{a}) \cdot T^n(\mathbf{a}) + \dots \\ &\quad + T^{n-1}(\mathbf{a}) \cdot T^{n-2}(\mathbf{a}), \\ &= \mathcal{A}(\mathbf{a})(1) + \mathcal{A}(\mathbf{a})(n-1) + \dots \\ &\quad + \mathcal{A}(\mathbf{a})(1) + \mathcal{A}(\mathbf{a})(n-1), \\ &= (n-1)(\mathcal{A}(\mathbf{a})(1) + \mathcal{A}(\mathbf{a})(n-1)). \end{aligned} \quad (47)$$

Similarly,

$$\begin{aligned} \sum_{i=1}^{n-(n-1)} T^i(\mathbf{a}) \cdot T^{x_i}(\mathbf{a}) &= T^1(\mathbf{a}) \cdot T^2(\mathbf{a}), \\ &= \mathcal{A}(\mathbf{a})(n-1) + \mathcal{A}(\mathbf{a})(1). \end{aligned} \quad (48)$$

By following the above pattern, the general term we get

$$\begin{aligned} \sum_{i=1}^n \mathcal{A}(T^i(\mathbf{a}))(\lambda) &= (n-\lambda)(\mathcal{A}(\mathbf{a})(\lambda) \\ &\quad + \mathcal{A}(\mathbf{a})(n-\lambda)). \end{aligned} \quad (49)$$

Hence, the result is true for  $k = 0$ .

*Induction hypothesis:* Let's assume that the result is true for  $k=m$ , i.e., the sum of AACF of all rows of the truncated matrix  $\mathbf{A}^m$  is given by

$$\begin{aligned} \sum_{i=1}^n \mathcal{A}(T^i(\mathbf{a}))(\lambda) &= (n-\lambda-m)(\mathcal{A}(\mathbf{a})(\lambda) \\ &\quad + \mathcal{A}(\mathbf{a})(n-\lambda)), \quad 0 \leq \lambda \leq n-1-m. \end{aligned} \quad (50)$$

Now, take  $k = m+1$  then the  $\mathbf{A}^m = \mathbf{A}^{m+1}|\mathbf{B}$ , where

$$\mathbf{B} = \begin{bmatrix} a_{m+1} \\ a_{m+2} \\ \vdots \\ a_0 \\ \vdots \\ a_m \end{bmatrix}, \quad (51)$$

i.e.,

$$\mathbf{A}^m = \begin{bmatrix} R_1 & a_{m+1} \\ R_2 & a_{m+2} \\ \vdots & \vdots \\ R_{n-m} & a_0 \\ \vdots & \vdots \\ R_n & a_m \end{bmatrix}, \quad (52)$$

where  $R_i$ 's is the row of the truncated matrix  $\mathbf{A}^{m+1}$ . Then

$$\begin{aligned} \sum_{i=1}^n \mathcal{A}(T^i(\mathbf{a}))(\lambda) &= \sum_{i=1}^n \mathcal{A}(R_i)(\lambda) \\ &\quad + T_1^{n-m}(\mathbf{a})^\top \cdot T_1^{n-m-\lambda}(\mathbf{a})^\top, \\ &= \sum_{i=1}^n \mathcal{A}(R_i)(\lambda) \\ &\quad + \mathcal{A}(\mathbf{a})(\lambda) + \mathcal{A}(\mathbf{a})(n-\lambda). \end{aligned} \quad (53)$$

From (53) and (50), we get

$$\begin{aligned} \sum_{i=1}^n \mathcal{A}(R_i)(\lambda) &= (n-\lambda-m-1)(\mathcal{A}(\mathbf{a})(\lambda) \\ &\quad + \mathcal{A}(\mathbf{a})(n-\lambda)), \quad 0 \leq \lambda \leq n-1-m-1. \end{aligned} \quad (54)$$

Hence, the lemma is proved.

APPENDIX D  
PROOF OF LEMMA 9

Let matrix  $\mathbf{Z}^k = Cir(\mathbf{a}) || Cir(\mathbf{b})$ , it is denoted as

$$\mathbf{Z}^k = [T_1^0(\mathbf{a})^\top, \dots, T_1^{n-1}(\mathbf{a})^\top, T_1^0(\mathbf{b})^\top, \dots, T_1^{n-1-k}(\mathbf{b})^\top]. \quad (55)$$

Let  $R_i$  denotes the rows of  $\mathbf{Z}^k$ , where  $1 \leq i \leq n$ . Then for  $0 \leq k \leq n-1$ ,

$$\sum_{i=1}^n \mathcal{A}(R_i)(\lambda) = \begin{cases} \sum_{i=0}^{n-\lambda-1} T_1^i(\mathbf{a})^\top \cdot T_1^{i+\lambda}(\mathbf{a})^\top \\ + \sum_{i=0}^{n-\lambda-1-k} T_1^i(\mathbf{b})^\top \cdot T_1^{i+\lambda}(\mathbf{b})^\top \\ + \sum_{i=0}^{\lambda-1} T_1^{i+n-\lambda}(\mathbf{a})^\top \cdot T_1^i(\mathbf{b})^\top, \\ 0 \leq \lambda \leq n-1, \\ \sum_{i=0}^{\lambda-k-1} T_1^i(\mathbf{a})^\top \cdot T_1^{n-\lambda} \text{ mod } n+i(\mathbf{a})^\top, \\ \lambda \geq n, \end{cases} \quad (56)$$

when  $n \leq k \leq 2n-2$ , then  $\sum_{i=1}^n \mathcal{A}(R_i)(\lambda)$

$$= \left\{ \sum_{i=0}^{\lambda-k} \text{ mod } n-1 T_1^i(\mathbf{a})^\top \cdot T_1^{i+\lambda-k} \text{ mod } n(\mathbf{a})^\top, \forall \lambda. \right. \quad (57)$$

Using the property of linear operator  $T_1^i(\mathbf{a})^\top \cdot T_1^j(\mathbf{b})^\top$  and  $T_1^i(\mathbf{a})^\top \cdot T_1^j(\mathbf{a})^\top$ , we get for  $0 \leq k \leq n-1$  and  $n \leq k \leq 2n-2$  are given as

$$\sum_{i=0}^n \mathcal{A}(R_i)(\lambda) = \begin{cases} (n-\lambda)(\mathcal{A}(\mathbf{a})(\lambda) + \mathcal{A}(\mathbf{a})(n-\lambda)) \\ + (n-\lambda-k)(\mathcal{A}(\mathbf{b})(\lambda) + \mathcal{A}(\mathbf{b})(n-\lambda)) \\ + \lambda(\mathcal{C}(\mathbf{a}, \mathbf{b})(\lambda) + \mathcal{C}(\mathbf{a}, \mathbf{b})(n-\lambda)), \\ \lambda \leq n-1, \\ (\lambda-k)(\mathcal{A}(\mathbf{a})(\lambda \text{ mod } n) \\ + \mathcal{A}(\mathbf{a})(n-\lambda \text{ mod } n)), \lambda \geq n, \end{cases} \quad (58)$$

and

$$\sum_{i=0}^n \mathcal{A}(R_i)(\lambda) = \begin{cases} (\lambda-k \text{ mod } n)(\mathcal{A}(\mathbf{a})(\lambda) \\ + \mathcal{A}(\mathbf{a})(n-\lambda)), \forall \lambda \end{cases} \quad (59)$$

respectively.

Hence, the lemma is proved.

APPENDIX E  
PROOF OF LEMMA 10

We know:

$$\sum_{i=0}^{2L-\lambda-1-k} \mathcal{C}(\mathbf{A}, \mathbf{B})(\lambda) = \sum_{i=0}^{2L-\lambda-1-k} A_i B_{i+\lambda}, \quad (60)$$

$$0 \leq \lambda \leq 2L-1-k.$$

Let us divide the  $\lambda$  in three cases:

- When  $0 \leq \lambda \leq L - k - 1$ , then

$$\begin{aligned}
\sum_{i=0}^{2L-\lambda-1-k} A_i B_{i+\lambda} &= \sum_{i=0}^{L-\lambda-1} A_i B_{i+\lambda} + \sum_{i=L-\lambda}^{L-1} A_i B_{i+\lambda} \\
&\quad + \sum_{i=L}^{2L-\lambda-1-k} A_i B_{i+\lambda}, \\
&= \mathcal{C}(\mathbf{R}_1, \mathbf{R}_2)(\lambda) + \sum_{i=L-\lambda}^{L-1} A_i B_{i+\lambda} \\
&\quad + \sum_{j=0}^{L-\lambda-1-k} A_j B_{j+\lambda}, \\
&= \mathcal{C}(\mathbf{R}_1, \mathbf{R}_2)(\lambda) - (B_0 A_{L-\lambda} \\
&\quad + B_1 A_{L-\lambda+1} + \dots \\
&\quad + B_{L-1+\lambda} A_{L-1}) \\
&\quad + \mathcal{C}(\mathbf{R}_1^{L-k}, \mathbf{R}_2^{L-k})(\lambda), \\
&= \mathcal{C}(\mathbf{R}_1, \mathbf{R}_2)(\lambda) - \mathcal{C}(\mathbf{R}_2, \mathbf{R}_1)(L - \lambda) \\
&\quad + \mathcal{C}(\mathbf{R}_1^{L-k}, \mathbf{R}_2^{L-k})(\lambda).
\end{aligned} \tag{61}$$

- When  $L - k \leq \lambda \leq L - 1$ , then

$$\begin{aligned}
\sum_{i=0}^{2L-\lambda-1-k} A_i B_{i+\lambda} &= \sum_{i=0}^{L-\lambda-1} A_i B_{i+\lambda} + \sum_{i=L-\lambda}^{L-1} A_i B_{i+\lambda}, \\
&= \mathcal{C}(\mathbf{R}_1, \mathbf{R}_2)(\lambda) - (B_0 A_{L-\lambda} \\
&\quad + B_1 A_{L-\lambda+1} + \dots \\
&\quad + B_{L-1-k-\lambda} A_{L-1-k}), \\
&= \mathcal{C}(\mathbf{R}_1, \mathbf{R}_2)(\lambda) \\
&\quad - \mathcal{C}(\mathbf{R}_2^{L-k}, \mathbf{R}_1^{L-k})(L - \lambda).
\end{aligned} \tag{62}$$

- When  $L \leq \lambda \leq 2L - 1 - k$ , let  $\lambda - L = \lambda \bmod L = k'$  then

$$\begin{aligned}
\sum_{i=0}^{2L-\lambda-1-k} A_i B_{i+\lambda} &= \sum_{i=0}^{L-1-k-k'} A_i B_{L+i+k'}, \\
&= -(A_0 B_{k'} + A_1 B_{k'+1} + \dots \\
&\quad + A_{L-k-k'-1} B_{L-1-k}), \\
&= -\mathcal{C}(\mathbf{R}_1^{L-k-k'}, \mathbf{R}_2^{L-k-k'})(k'),
\end{aligned} \tag{63}$$

Hence, the lemma is proved.

#### APPENDIX F PROOF OF THEOREM 1

We have

$$E = \begin{bmatrix} f_0 & f_3 & f_2 & f_1 \\ f_1 & f_0 & f_3 & f_2 \\ f_2 & f_1 & f_0 & f_3 \\ f_3 & f_2 & f_1 & f_0 \end{bmatrix}, \tag{64}$$

where  $f_0 = \xi^{\theta_3}$ ,  $f_1 = f_0 \xi^{\theta_1 \frac{q}{2}}$ ,  $f_2 = f_0 \xi^{\theta_2 \frac{q}{2}}$ , and  $f_3 = -f_0 \xi^{(\theta_1 + \theta_2) \frac{q}{2}}$ . Now let  $\mathbf{E}\mathbf{E}^\top = \mathbf{H} = (h_{i,j})$ . Then we have,

$$\begin{aligned} h_{1,2} &= h_{2,1} = f_0 f_1 + f_3 f_0 + f_2 f_3 + f_1 f_2 = 0, \\ h_{1,3} &= h_{3,1} = f_0 f_2 + f_3 f_1 + f_2 f_0 + f_1 f_3 = 0, \\ h_{1,4} &= h_{4,1} = f_0 f_3 + f_3 f_2 + f_2 f_1 + f_1 f_0 = 0, \\ h_{2,3} &= h_{3,2} = f_1 f_2 + f_0 f_1 + f_3 f_0 + f_2 f_3 = 0, \\ h_{2,4} &= h_{4,2} = f_1 f_3 + f_0 f_2 + f_3 f_1 + f_2 f_0 = 0, \\ h_{3,4} &= h_{4,3} = f_2 f_3 + f_1 f_2 + f_0 f_1 + f_3 f_0 = 0, \\ h_{1,1} &= h_{2,2} = h_{3,3} = h_{4,4} = 4, \end{aligned} \tag{65}$$

i.e.,

$$h_{i,j} = \begin{cases} 0, & i \neq j, \\ 4, & i = j. \end{cases} \tag{66}$$

Hence, we get  $\mathbf{E}\mathbf{E}^\top = 4\mathbf{I}_4$ .

#### APPENDIX G PROOF OF THEOREM 2

We prove this theorem by using the induction hypothesis

- When  $n = 0$ , then we have

$$\mathbf{E}_4 = \begin{bmatrix} f_0 & f_3 & f_2 & f_1 \\ f_1 & f_0 & f_3 & f_2 \\ f_2 & f_1 & f_0 & f_3 \\ f_3 & f_2 & f_1 & f_0 \end{bmatrix}. \tag{67}$$

The sum of AACF of each row is given as

$$\begin{aligned} \sum_{i=1}^4 \mathcal{A}(T^i(\mathbf{f}))(\lambda) &= (4 - \lambda - k)(\mathcal{A}(\mathbf{f})(\lambda) \\ &\quad + \mathcal{A}(\mathbf{f})(4 - \lambda)), \quad 1 \leq \lambda \leq 3 - k. \end{aligned} \tag{68}$$

The value of  $\mathcal{A}(\mathbf{f})(\lambda) + \mathcal{A}(\mathbf{f})(4 - \lambda)$  for every  $\lambda$  is calculated below

$$\begin{aligned} \mathcal{A}(\mathbf{f})(1) + \mathcal{A}(\mathbf{f})(3) &= f_0 f_1 + f_1 f_2 + f_2 f_3 + f_0 f_3 = 0, \\ \mathcal{A}(\mathbf{f})(2) + \mathcal{A}(\mathbf{f})(2) &= f_0 f_2 + f_1 f_3 + f_0 f_2 + f_1 f_3 = 0. \end{aligned} \tag{69}$$

We get,  $\sum_{i=1}^4 \mathcal{A}(T^i(\mathbf{f}))(\lambda) = 0 \quad \forall 1 \leq \lambda \leq 3 - k$ . Now, the value of the sum of ACCF of each row is calculated directly for  $2 + k \leq \lambda \leq 3 - k$  as

$$\sum_{i=1}^4 \mathcal{C}(T^i(\mathbf{f}), T^{(i+1) \bmod 4}(\mathbf{f}))(\lambda) = 0. \tag{70}$$

Hence,  $\mathbf{E}_4$  forms  $(4, 4 - k, 2 - k) - CZCS$ , where  $0 \leq k \leq 1$ .

- Let us consider the result is true for  $n=m$ , i.e.,  $\mathbf{E}_m$  forms  $(2^{m+2}, 2^{m+2} - k) - GCS$  and  $(2^{m+2}, 2^{m+2} - k', 2^{m+1} - (k' - 2^m) \lfloor \frac{k'}{2^m} \rfloor) - CZCS$ , where  $0 \leq k' \leq 2^m - 1$ .
- Take  $n=m+1$ , i.e.,

$$\mathbf{E}_{m+1} = \begin{bmatrix} \mathbf{E}_m & \mathbf{E}_m \\ \mathbf{E}_m & -\mathbf{E}_m \end{bmatrix}. \tag{71}$$

There are a total of  $2^{m+3}$  and  $2^{m+2}$  rows in  $\mathbf{E}_{m+1}$  and  $\mathbf{E}_m$ , respectively. Let us consider  $\mathbf{R}\mathbf{R}_i$  and  $\mathbf{R}_j$  denote the rows of  $\mathbf{E}_{m+1}$  and  $\mathbf{E}_m$ , respectively, where  $1 \leq i \leq 2^{m+3}$  and  $1 \leq j \leq 2^{m+2}$ . Using *Theorem 8*, we have,

$$\begin{aligned} \sum_{i=1}^{2^{m+3}} \mathcal{A}(\mathbf{R}\mathbf{R}_i)(\lambda) &= \begin{cases} 2 \sum_{j=1}^{2^{m+2}} (\mathcal{A}(\mathbf{R}_j)(\lambda) + \mathcal{A}(\mathbf{R}_j^{2^N-k})(\lambda)), \\ 1 \leq \lambda \leq 2^{m+2} - k, \\ 2 \sum_{j=1}^{2^{m+2}} \mathcal{A}(\mathbf{R}_j)(\lambda), 2^{m+2} - k + 1 \leq \lambda \leq 2^{m+2}, \\ 0, 2^{m+2} + 1 \leq \lambda \leq 2^{m+3} - 1 - k. \end{cases} \end{aligned} \tag{72}$$

Using the induction hypothesis that  $\mathbf{E}_m$  forms  $(2^{m+2}, 2^{m+2} - k) - GCS$ , where  $0 \leq k \leq 2^{m+1} - 1$ .

We get  $\sum_{i=1}^{2^{m+3}} \mathcal{A}(\mathbf{R}\mathbf{R}_i)(\lambda) = 0 \forall \lambda \neq 0$ . Now calculate the ACCF of  $\mathbf{E}_{m+1}$ , i.e.,

$$\begin{aligned} \sum_{i=1}^{2^{m+3}} \mathcal{C}(\mathbf{R}\mathbf{R}_i, \mathbf{R}\mathbf{R}_{i \bmod 2^{m+3}})(\lambda) \\ = \sum_{j=0}^{L-1} \mathcal{C}(\mathbf{R}_j || \mathbf{R}_j^{L-k}, \mathbf{R}_{j+1} || \mathbf{R}_{j+1}^{L-k})(\lambda) \\ + \sum_{j=0}^{L-1} \mathcal{C}(\mathbf{R}_j || -\mathbf{R}_j^{L-k}, \mathbf{R}_{j+1} || -\mathbf{R}_{j+1}^{L-k})(\lambda) \\ + \mathcal{C}(\mathbf{R}_L || \mathbf{R}_L^{L-k}, \mathbf{R}_1 || -\mathbf{R}_1^{L-k}) + \mathcal{C}(\mathbf{R}_L || -\mathbf{R}_L^{L-k}, \\ \mathbf{R}_1 || \mathbf{R}_1^{L-k}). \end{aligned} \quad (73)$$

Let us divide the value of  $L - k + (k - \frac{L}{2}) \lfloor \frac{2k}{L} \rfloor \leq \lambda \leq 2L - 1 - k$  into two cases, i.e., for  $k = 0$  and  $k \geq 1$ . Then the (73) becomes 0 and  $2 \sum_{j=1}^L \mathcal{C}(\mathbf{R}_j, \mathbf{R}_{j \bmod L})(\lambda)$  using the Lemma 10. Using the induction hypothesis as  $\mathbf{E}_m$  forms  $(2^{m+2}, 2^{m+2} - k', 2^{m+1} - (k' - 2^m) \lfloor \frac{k'}{2^m} \rfloor) - CZCS$ , where  $0 \leq k' \leq 2^m - 1$ , the above (73) becomes 0  $\forall L - k + (k - \frac{L}{2}) \lfloor \frac{2k}{L} \rfloor \leq \lambda \leq 2L - 1 - k$ , where  $0 \leq k \leq L - 1$  and  $L = 2^{m+2}$ . Thus, the result is true for  $n = m + 1$ . Hence,  $\mathbf{E}_n$  forms  $(2^{n+2}, 2^{n+2} - k, 2^{n+1} - (k - 2^n) \lfloor \frac{k}{2^n} \rfloor) - CZCS$ , where  $n \geq 0$  and  $0 \leq k \leq 2^{n+1} - 1$ .

#### APPENDIX H PROOF OF THEOREM 3

Let us consider

$$\mathbf{Z}_1^k = [Cir(\mathbf{a}), Cir(\mathbf{b})], \quad (74)$$

and

$$\mathbf{Z}_2^k = [Cir(\mathbf{c}), Cir(\mathbf{d})], \quad (75)$$

let us consider  $R_i(\mathbf{Z}_1^k)$  and  $R_i(\mathbf{Z}_2^k)$  denote the  $i$ th row of  $\mathbf{Z}_1$  and  $\mathbf{Z}_2$ , respectively, where  $1 \leq i \leq n = 2N$  and  $N = 2^a 10^b 26^c$ ,  $a, b, c \geq 0$ . Since  $(\mathbf{c}, \mathbf{d})$  is GCP mate of  $(\mathbf{a}, \mathbf{b})$ , which also implies that  $(\mathbf{a}, \mathbf{c})$  and  $(\mathbf{b}, \mathbf{d})$  forms a GCP. Then, using Lemma 9, we get,

$$\sum_{i=0}^N \mathcal{A}(R_i(\mathbf{Z}_1^k))(\lambda) + \sum_{i=0}^N \mathcal{A}(R_i(\mathbf{Z}_2^k))(\lambda) = 0, \forall \lambda \neq 0. \quad (76)$$

Hence,  $\mathbf{G} = \begin{bmatrix} \mathbf{Z}_1^k \\ \mathbf{Z}_2^k \end{bmatrix}$  forms  $(2N, 2N - k) - GCS$ , where  $0 \leq k \leq 2N - 2$ .

#### APPENDIX I PROOF OF COROLLARY 2

We have  $\mathbf{G} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ , then,  $\mathbf{G}^\top = \begin{bmatrix} \mathbf{A}^\top & \mathbf{C}^\top \\ \mathbf{B}^\top & \mathbf{D}^\top \end{bmatrix}$ .

$$\begin{aligned} \mathbf{G}\mathbf{G}^\top &= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{A}^\top & \mathbf{C}^\top \\ \mathbf{B}^\top & \mathbf{D}^\top \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top & \mathbf{A}\mathbf{C}^\top + \mathbf{B}\mathbf{D}^\top \\ \mathbf{C}\mathbf{A}^\top + \mathbf{D}\mathbf{B}^\top & \mathbf{C}\mathbf{C}^\top + \mathbf{D}\mathbf{D}^\top \end{bmatrix}. \end{aligned} \quad (77)$$

Let us calculate  $\mathbf{A}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top$ , using Lemma 5 and Lemma 6, we get

$$\mathbf{A}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top = \mathbf{C}\mathbf{C}^\top + \mathbf{D}\mathbf{D}^\top = 2NI_N, \quad (78)$$

similarly,

$$\mathbf{A}\mathbf{C}^\top + \mathbf{B}\mathbf{D}^\top = \mathbf{C}\mathbf{A}^\top + \mathbf{D}\mathbf{B}^\top = 0_N, \quad (79)$$

where  $0_N$  is the zero matrix of order  $N$ . From (78) and (79), we get

$$\mathbf{G}\mathbf{G}^\top = \begin{bmatrix} 2NI_N & 0_N \\ 0_N & 2NI_N \end{bmatrix} = 2NI_{2N}. \quad (80)$$

Hence,  $\mathbf{G}$  forms a Hadamard matrix of order  $2N$ , where  $N = 2^a 10^b 26^c$ ,  $a, b, c \geq 0$ .

APPENDIX J  
PROOF OF THEOREM 4

It is easy to show that each  $R_i \odot \mathbf{G}$  forms a  $(2N, 2N) - GCS$ , where  $1 \leq i \leq 2N$  and  $N = 2^a 10^b 26^c, a, b, c \geq 0$ . Now, to show the second condition of CCC, let's consider  $R_i$  denotes the  $i$ th row of matrix  $\mathbf{G}$ , take two matrices  $R_i \odot \mathbf{G}$  and  $R_j \odot \mathbf{G}$ . Then,

$$\begin{aligned} \sum_{k=1}^{2N} \mathcal{C}(R_i \odot R_k, R_j \odot R_k)(\lambda) &= \sum_{k=1}^{2N} \mathcal{C}(R_i, R_j) \mathcal{A}(R_k)(\lambda), \\ &= \mathcal{C}(R_i, R_j) \sum_{k=1}^{2N} \mathcal{A}(R_k)(\lambda). \end{aligned} \quad (81)$$

We know that  $\mathbf{G}$  forms  $(2N, 2N) - GCS$ , then,

$$\sum_{k=1}^{2N} \mathcal{A}(R_k)(\lambda) = 0 \quad \forall \lambda \neq 0. \quad (82)$$

The value of  $\mathcal{C}(R_i, R_j)(0)$  are  $T^i(\mathbf{a}) \cdot T^j(\mathbf{a})$  and  $T^i(\mathbf{a}) \cdot T^j(\mathbf{c})$  when  $1 \leq i, j \leq L$  and  $1 \leq i \leq L, L \leq j \leq 2L$ , respectively. Using *Theorem 5*, *Corollary 6*, *Corollary 7*, and (82), we get

$$\sum_{k=1}^{2N} \mathcal{C}(R_i \odot R_k, R_j \odot R_k)(\lambda) = 0, \quad \forall \lambda. \quad (83)$$

Hence  $\bigcup_{i=1}^{2N} R_i \odot \mathbf{G}$  forms a  $(2N, 2N, 2N) - CCC$ , where  $m \geq 1$ .

APPENDIX K  
PROOF OF THEOREM 5

Each of  $R_i \odot \mathbf{G}$  forms a  $(2^{n+2}, 2^{n+2}, 2^{n+1}) - CZCS$ . Let's take two matrices  $R_i \odot \mathbf{G}$  and  $R_j \odot \mathbf{G}$  for the (12). Then,

$$\begin{aligned} \sum_{k=1}^{2^{n+2}} \mathcal{C}(R_i \odot R_k, R_j \odot R_k)(\lambda) &= \sum_{k=1}^{2^{n+2}} \mathcal{C}(R_i, R_j) \mathcal{A}(R_k)(\lambda), \\ &= \mathcal{C}(R_i, R_j)(\lambda) \sum_{k=1}^{2^{n+2}} \mathcal{A}(R_k)(\lambda). \end{aligned} \quad (84)$$

We know that  $\mathbf{G}$  forms  $(2^{n+2}, 2^{n+2}) - GCS$ . Then,

$$\sum_{k=1}^{2^{n+2}} \mathcal{A}(R_k)(\lambda) = 0 \quad \forall \lambda \neq 0. \quad (85)$$

The value of  $\mathcal{C}(R_i, R_j)(0)$  are  $T^i(\mathbf{a}) \cdot T^j(\mathbf{a})$  and  $T^i(\mathbf{a}) \cdot T^j(\mathbf{c})$  when  $1 \leq i, j \leq L$  and  $1 \leq i \leq L, L \leq j \leq 2L$ , respectively. Using *Theorem 7*, *Theorem 11*, and (85), we get

$$\sum_{k=1}^{2^{n+2}} \mathcal{C}(R_i \odot R_k, R_j \odot R_k)(\lambda) = 0, \quad \forall \lambda. \quad (86)$$

Now, let us take two codes  $R_i \odot \mathbf{G}$  and  $R_j \odot \mathbf{G}$  for the (13). We have the following relation:

$$\begin{aligned} \sum_{k=1}^{2^{n+2}} \mathcal{C}(R_i \odot R_k, R_i \odot R_{k+1 \bmod 2^{n+2}})(\lambda) \\ = \mathcal{C}(R_i, R_j)(\lambda) \sum_{k=1}^{2^{n+2}} \mathcal{C}(R_k, R_{k+1 \bmod 2^{n+2}})(\lambda). \end{aligned} \quad (87)$$

We know that  $\mathcal{C}(R_k, R_{k+1 \bmod 2^{n+2}})(\lambda) = 0 \quad \forall \lambda \in \{2^{n+1}, 2^{n+1} + 1, \dots, 2^{n+2} - 1\}$ .

Hence,  $\bigcup_{i=1}^{2^{n+2}} R_i \odot \mathbf{G}$  forms  $(2^{n+2}, 2^{n+2}, 2^{n+2}, 2^{n+1}) - CZCSS$  and  $(2^{n+2}, 2^{n+2}, 2^{n+2}) - CCC$ , where  $n \geq 1$ .

## APPENDIX L

### PROOF OF THEOREM 6

Let us take  $A = [a_1, a_2, \dots, a_n]$  a Hadamard matrix of order  $n$ , where each  $a_i$  is a column vector of length  $n$ . We have  $a_i \cdot a_j = 0 \ \forall i \neq j$ . The sum of the AACF of every row sequence at every time shift  $\lambda$  is

$$\sum_{i=1}^{n-\lambda} a_i^\top \cdot a_{\lambda+i}^\top. \quad (88)$$

We know that in every Hadamard matrix, the columns are pairwise orthogonal, we get (88) zero for all  $\lambda \neq 0$ . Hence, the matrix  $A$  forms  $(n, n) - GCS$ .

## APPENDIX M

### PROOF OF THEOREM 7

Let's  $G$  denote the collection of all matrices of order  $n$  that form a  $(n, n) - GCS$ , where  $n = 2$  is a multiple of 4. Then, from the property of GCS, we have

$$\sum_{i=1}^{n-\lambda} a_i^\top \cdot a_{\lambda+i}^\top = 0 \ \forall \ \lambda \neq 0. \quad (89)$$

Now, break the above equation corresponding to  $\lambda$ ,

$$\begin{aligned} \text{When } \lambda = 1, \quad & \sum_{i=1}^{n-1} a_i^\top \cdot a_{1+i}^\top = 0. \\ \text{When } \lambda = 2, \quad & \sum_{i=1}^{n-2} a_i^\top \cdot a_{2+i}^\top = 0. \\ \text{When } \lambda = 3, \quad & \sum_{i=1}^{n-3} a_i^\top \cdot a_{3+i}^\top = 0. \\ & \vdots \\ \text{When } \lambda = n-1, \quad & a_1^\top \cdot a_n^\top = 0. \end{aligned} \quad (90)$$

We have  $n-1$  equations having  $(n-1)!$  unknowns; that is  $Ax = 0$ , where  $A$  is a matrix of  $(n-1) \times (n-1)!$  and  $x = (a_1^\top \cdot a_2^\top, \dots, a_1^\top \cdot a_n^\top, a_2^\top \cdot a_3^\top, \dots, a_2^\top \cdot a_n^\top, \dots, a_{n-1}^\top \cdot a_n^\top)$ . Since we know that every homogeneous system of equations has a trivial solution. Hence we get  $a_i \cdot a_j = 0 \ \forall i \neq j$ . This implies that from the set of matrices  $G$ , there is a Hadamard matrix.

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