# MATRIX GENERATORS FOR WEIL REPRESENTATIONS

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ABSTRACT. Let $r$ be an odd prime and $\mathbb{F}$ a field containing a primitive $r$th root of unity. Then for all $\ell \geq 1$, there is a faithful representation $f : \mathrm{Sp}_{2\ell}(r) \to \mathrm{GL}_{r^\ell}(\mathbb{F})$ called the *Weil representation*. We provide explicit matrices generating $\mathrm{Sp}_{2\ell}(r)$ in $\mathrm{GL}_{r^\ell}(\mathbb{F})$, which we have implemented in Magma. We also describe such generators for the *irreducible Weil representations* of $\mathrm{Sp}_{2\ell}(r)$, which are of degree $(r^\ell \pm 1)/2$ and arise as irreducible constituents of the Weil representations.

## 1. Introduction

Let $r$ be an odd prime, and let $\mathbb{F}$ be a field containing a primitive $r$th root of unity. Then for all $\ell \geq 1$, there is an embedding of $\mathrm{Sp}_{2\ell}(r)$ into $\mathrm{GL}_{r^\ell}(\mathbb{F})$, arising from the *Weil representation* of $\mathrm{Sp}_{2\ell}(r)$. The purpose of this note is to provide explicit matrices that generate this copy of $\mathrm{Sp}_{2\ell}(r)$ in $\mathrm{GL}_{r^\ell}(\mathbb{F})$. Originating from the work of Weil in [22], the Weil representation has been extensively studied in the literature, and there are several different proofs of its existence and construction [1], [21], [8], [9], [4], [5]. The construction described in this note gives yet another proof of the existence of the Weil representation. However, our main motivation here will be a computational implementation. We have implemented the generators in Magma [2], and the code is available at [14].

To describe the generators, we will first have to setup some basic facts about representations of extraspecial groups, we refer to [3, Chapter A, Section 20 and Chapter B, Section 9] for more details. Let $R$ be an extraspecial $r$-group of exponent $r$, so $|R| = r^{1+2\ell}$ for some $\ell \geq 1$. We note that $R$ is often denoted by $R = r_+^{1+2\ell}$ in the literature. Then every irreducible $\mathbb{F}[R]$-module is absolutely irreducible, and every irreducible $\mathbb{F}[R]$-module of dimension $> 1$ is faithful of dimension $r^\ell$ [3, Chapter B, Theorem 9.16]. Moreover faithful irreducible $\mathbb{F}[R]$-modules are quasi-equivalent [3, A, 20.8 (d) and B, 9.16(ii)], which implies that they define a unique conjugacy class of subgroups of $\mathrm{GL}_{r^\ell}(\mathbb{F})$.

Suppose then that $R < \mathrm{GL}(W)$, where $W$ is an absolutely irreducible $\mathbb{F}[R]$-module of dimension $r^\ell$. Let $\theta$ be a primitive $r$th root of unity in $\mathbb{F}$. Note that since the center $Z(R)$ is cyclic of order $r$, we have $Z(R) = \langle \theta I_W \rangle$. The basic observation needed to understand the structure of $N_{\mathrm{GL}(W)}(R)$ is that the quotient $R/Z(R)$ can be considered as a vector space of dimension $2\ell$ over the finite field $\mathbb{F}_r$ of order $r$. The group commutator then induces a non-degenerate alternating bilinear form $b : R/Z(R) \times R/Z(R) \to \mathbb{F}_r$ on $R/Z(R)$, with

$$[x, y] = \theta^{b(xZ(R), yZ(R))}$$

for all $x, y \in R$.

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It is clear that elements of $\mathrm{GL}(W)$ normalizing $R$ will leave invariant the bilinear form $b$ defined on $R/Z(R)$, so we have a homomorphism

$$\pi : N_{\mathrm{GL}(W)}(R) \to \mathrm{Sp}_{2\ell}(r), \tag{1.1}$$

where $\pi(g)$ is defined as the action of $g$ on $R/Z(R)$, for all $g \in N_{\mathrm{GL}(W)}(R)$.

It is then well known that the following hold, see for example [13, Lemma 3.1.1, Theorem 3.2.2 (ii)].

$$\operatorname{Ker}\pi = RZ, \text{ where } Z \cong \mathbb{F}^\times \text{ is the subgroup of scalar matrices;} \tag{1.2}$$

$$N_{\mathrm{GL}(W)}(R)/\operatorname{Ker}\pi \cong \mathrm{Sp}_{2\ell}(r). \tag{1.3}$$

It turns out that $N_{\mathrm{GL}(W)}(R)$ is a split extension of $\operatorname{Ker}\pi$, meaning that the normalizer contains a subgroup $G \cong \mathrm{Sp}_{2\ell}(r)$ with $G \cap \operatorname{Ker}\pi = 1$. This subgroup $G$ then defines the Weil representation of $\mathrm{Sp}_{2\ell}(r)$.

To describe the generators for $G \cong \mathrm{Sp}_{2\ell}(r)$, we first have to describe the embedding $R < \mathrm{GL}(W)$. We will also need a well-known set of generators for $N_{\mathrm{GL}(W)}(R)$, which go back to the work of Jordan (see Remark 1.2). Let $V$ be an $r$-dimensional $\mathbb{F}$-vector space with basis $v_0, v_1, \ldots, v_{r-1}$. We define linear maps $A, B, C, E : V \to V$ by

$$\begin{aligned} Av_\xi &= \theta^\xi v_\xi \\ Bv_\xi &= v_{\xi+1} \\ Cv_\xi &= \sum_{0 \le i < r} \theta^{i\xi} v_i \\ Ev_\xi &= \theta^{\frac{\xi(\xi-1)}{2}} v_\xi \end{aligned} \tag{1.4}$$

for all $0 \le \xi < r$, where the indices are interpreted modulo $r$. It is clear that $A, B, E$ are invertible, and $C$ is also invertible by the Vandermonde determinant.

Then $R_0 = \langle A, B \rangle \cong r_+^{1+2}$ is absolutely irreducible, and is normalized by $C$ and $E$. Let $W = V \otimes \cdots \otimes V$, where $V$ appears $\ell$ times in the tensor product. Since $R_0$ is absolutely irreducible, the tensor product $R = R_0 \otimes \cdots \otimes R_0$ ($\ell$ times) is an absolutely irreducible subgroup of $\mathrm{GL}(W)$, see for example [12, Lemma 4.4.3 (vi)]. Here $R$ is a central product of $\ell$ copies of $r_+^{1+2}$, which implies that $R \cong r_+^{1+2\ell}$.

As a basis for $W$, we take the vectors $v_{\xi_1,\ldots,\xi_\ell} := v_{\xi_1} \otimes \cdots \otimes v_{\xi_\ell}$ for $0 \le \xi_1, \ldots, \xi_\ell < r$. We define linear maps $A_t, B_t, C_t, D_{st}, E_t : W \to W$ for all $1 \le t \le \ell$ and $1 \le s < t \le \ell$ by

$$\begin{aligned} A_t v_{\xi_1,\ldots,\xi_\ell} &= \theta^{\xi_t} v_{\xi_1,\ldots,\xi_\ell} \\ B_t v_{\xi_1,\ldots,\xi_\ell} &= v_{\xi_1,\ldots,\xi_{t-1},\xi_t+1,\xi_{t+1},\ldots,\xi_\ell} \\ C_t v_{\xi_1,\ldots,\xi_\ell} &= \sum_{0 \le i < r} \theta^{i\xi_t} v_{\xi_1,\ldots,\xi_{t-1},i,\xi_{t+1},\ldots,\xi_\ell} \\ D_{st} v_{\xi_1,\ldots,\xi_\ell} &= \theta^{\xi_s \xi_t} v_{\xi_1,\ldots,\xi_\ell} \\ E_t v_{\xi_1,\ldots,\xi_\ell} &= \theta^{\frac{\xi_t(\xi_t-1)}{2}} v_{\xi_1,\ldots,\xi_\ell} \end{aligned} \tag{1.5}$$

for all $0 \le \xi_1, \ldots, \xi_\ell < r$, where as before the indices are interpreted modulo $r$. Note that $A_t = I_{r^{t-1}} \otimes A \otimes I_{r^{\ell-t}}$, $B_t = I_{r^{t-1}} \otimes B \otimes I_{r^{\ell-t}}$, $C_t = I_{r^{t-1}} \otimes C \otimes I_{r^{\ell-t}}$, $E_t = I_{r^{t-1}} \otimes E \otimes I_{r^{\ell-t}}$ for all $1 \le t \le \ell$.

The following theorem is to a large extent due to Jordan, see for example [13, Theorem 3.2.2] for a proof.

**Theorem 1.1.** *Let $Z$ be the group of scalar matrices in $\mathrm{GL}(W)$, and let $R$ be the subgroup of $\mathrm{GL}(W)$ generated by the linear maps $A_t$ and $B_t$. Then the following statements hold:*

(i) *$R < \mathrm{GL}(W)$ is an absolutely irreducible extraspecial group of exponent $r$, and order $r^{1+2\ell}$.*
(ii) *$N_{\mathrm{GL}(W)}(R)$ is generated by $Z$ together with the linear maps $A_t$, $B_t$, $C_t$, $D_{st}$, $E_t$;*
(iii) *$N_{\mathrm{GL}(W)}(R)/RZ \cong \mathrm{Sp}_{2\ell}(r)$.*

*Remark* 1.2. The generators $A_t$, $B_t$, $C_t$, $D_{st}$, $E_t$ defined in (1.5) are originally due to Jordan, see [10, (15), p. 437, and (25)–(27), p. 441] and [11, §X]. Later similar generators have appeared in various references, for example in [7, Section 9], [12, Section 4.6], and [20, Theorem 21.4].

We will now describe generators for $G \cong \mathrm{Sp}_{2\ell}(r)$. First on the representation $V$ of $R_0 = r_+^{1+2}$ we started with, we define $U : V \to V$ by $U = A^{(r+1)/2}E$, so

$$Uv_\xi = \theta^{\frac{\xi(\xi+r)}{2}} v_\xi$$

for all $0 \leq \xi < r$. Then for $1 \leq t \leq \ell$ we define similarly $U_t : W \to W$ by $U_t = A_t^{(r+1)/2}E_i$, so

$$U_t v_{\xi_1,\ldots,\xi_\ell} = \theta^{\frac{\xi_t(\xi_t+r)}{2}} v_{\xi_1,\ldots,\xi_\ell} \tag{1.6}$$

for all $0 \leq \xi_1, \ldots, \xi_\ell < r$. Note that then $U_t = I_{r^{t-1}} \otimes U \otimes I_{r^{\ell-t}}$ for all $1 \leq t \leq \ell$.

Next we define a scalar multiplier

$$\lambda = (-r)^{(r-1)/2} \det(C)^{-1} \tag{1.7}$$

that we will use to change each $C_t$ to a linear map of determinant one. Our main result is the following.

**Theorem 1.3.** *Let $G$ be the subgroup of $\mathrm{GL}(W)$ generated by the linear maps $\lambda C_t$, $D_{st}$, and $U_t$. Then $G \cong \mathrm{Sp}_{2\ell}(r)$, with an isomorphism given by restriction of the map $\pi$ to $G$.*

Thus by Theorem 1.3, with (1.5) and (1.6) we can write down generating matrices for $G$. Moreover, for every $g \in \mathrm{Sp}_{2\ell}(r)$, there is a fairly straightforward algorithm for writing down $g$ as a word in the generators $\pi(C_t)$, $\pi(D_{st})$, $\pi(U_t)$ of $\mathrm{Sp}_{2\ell}(r)$. Evaluating the corresponding word in the generators of $G$, we then get the action of $g$ in the Weil representation — see Remark 1.5.

One can also describe the submodules of $G$ on $W$ explicitly, thus giving the action of the generators of $G$ on the *irreducible Weil representations*, which are irreducible representations of $\mathrm{Sp}_{2\ell}(r)$ of degree $(r^\ell \pm 1)/2$. Specifically, if $\mathrm{char}\,\mathbb{F} \neq 2$, then $W$ is a direct sum of two irreducible $\mathbb{F}[G]$-submodules of degrees $(r^\ell + 1)/2$ and $(r^\ell - 1)/2$. And if $\mathrm{char}\,\mathbb{F} = 2$, then $W$ is uniserial with composition factors of degrees $(r^\ell - 1)/2$, 1, and $(r^\ell - 1)/2$. We discuss the $\mathbb{F}[G]$-submodules of $W$ in Section 5.

About the proof of Theorem 1.3, we note that since $U_t = A_t^{(r+1)/2}E_i$, it is clear from Theorem 1.1 that the image of $G$ under the map $\pi$ is equal to $\mathrm{Sp}_{2\ell}(r)$. Thus to prove Theorem 1.3, it suffices to check that $G \cong \mathrm{Sp}_{2\ell}(r)$. This will be done in Section 4, where our strategy for the proof of Theorem 1.1 is as follows. For $(r, \ell) = (3, 1)$ we will verify that the generators of $G$ satisfy relations defining $\mathrm{Sp}_2(3) = \mathrm{SL}_2(3)$. In the case $(r, \ell) \neq (3, 1)$, by calculations with the generators we will check that $G$ is a perfect central extension of $\mathrm{Sp}_{2\ell}(r)$. Since $\mathrm{Sp}_{2\ell}(r)$ has trivial Schur multiplier [19, Theorem 1.1], we can then conclude that $G \cong \mathrm{Sp}_{2\ell}(r)$.

*Remark* 1.4. Here the determinant involved in the definition (1.7) of $\lambda$ is a Vandermonde determinant, so

$$\det(C) = \prod_{0 \leq i < j < r} (\theta^j - \theta^i).$$

Over $\mathbb{F} = \mathbb{C}$, the precise evaluation of this determinant goes back to Schur, who used it to determine the value of the quadratic Gauss sum. Schur proved in [18, p. 149] that for $\theta = \exp(2\pi i/r)$ we have $\det(C) = r^{r/2} i^{r(r-1)/2}$, see for example [15, pp. 211–212]. (There is a version of this formula that works when $r$ is replaced by an arbitrary positive integer, see for example [17, Section 6.3, p. 109])

Over an arbitrary field $\mathbb{F}$ containing a primitive $r$th root of unity, we have $\det(C)^2 = r^r(-1)^{r(r-1)/2}$, so $\det(C)$ is still determined up to a sign as the square root of $r^r(-1)^{r(r-1)/2}$. We also have the formula

$$\det(C) = \left(\frac{2}{r}\right) r^{(r-1)/2} \sum_{0 \leq i < r} \theta^{i^2},$$

where $\left(\frac{a}{r}\right)$ is the Legendre symbol — see Remark 4.2.

*Remark* 1.5. Let $e_i$ and $f_i$ be the images in $R/Z(R)$ for $A_i$ and $B_i$, respectively. We have $[A_i, B_j] = \theta^{\delta_{i,j}}$ and $[A_i, A_j] = [B_i, B_j] = 1$ for all $1 \leq i, j \leq \ell$, so with respect to the alternating bilinear form defined on $R/Z(R)$ the vectors $e_1$, $f_1$, $\ldots$, $e_\ell$, $f_\ell$ form a basis of hyperbolic pairs. With respect to this basis, the action of $C_t$, $D_{st}$, and $E_t$ is given as follows:

$$\pi(C_t) : \begin{cases} e_t \mapsto -f_t \\ f_t \mapsto e_t \end{cases} \qquad \pi(D_{st}) : \begin{cases} f_t \mapsto f_t + e_s \\ f_s \mapsto f_s + e_t \end{cases}$$

$$\pi(E_t) : f_t \mapsto f_t + e_t$$

with all the other basis vectors fixed (see for example [13, (3.1), p. 74]). Note that $\pi(E_t) = \pi(U_t)$ for all $1 \leq t \leq \ell$.

By Theorem 1.1, we know that the elements $\pi(C_t)$, $\pi(D_{st})$, $\pi(E_t)$ generate $\mathrm{Sp}_{2\ell}(r)$. (This was first proven by Jordan in [10, Théorème 221, p. 174].) Moreover, there is a fairly straightforward algorithm for writing down a given element of $\mathrm{Sp}_{2\ell}(r)$ as a product of the generators $\pi(C_t)$, $\pi(D_{st})$, and $\pi(E_t)$. We omit the details, but such an algorithm can be extracted for example from the proof of [13, Theorem 3.2.2].

It follows then that for a given element $g \in \mathrm{Sp}_{2\ell}(r)$ written as a word in the generators $\pi(C_t)$, $\pi(D_{st})$, $\pi(E_t) = \pi(U_t)$, evaluating the corresponding word in the generators $\lambda C_t$, $D_{st}$, $U_t$ gives a matrix $g' \in G$ which is the action of $g$ in the Weil representation of $\mathrm{Sp}_{2\ell}(r)$. Furthermore, using the description of the $\mathbb{F}[G]$-submodules of $W$ discussed in Section 5, one can similarly calculate the action of $g$ on the irreducible Weil representations of degrees $(r^\ell \pm 1)/2$.

*Remark* 1.6. Let $s \geq 1$ be an integer. More generally, one defines Weil representations of $\mathrm{Sp}_{2\ell}(r^s)$ as follows. We can consider $\mathrm{Sp}_{2\ell}(r^s)$ as a field extension subgroup $\mathrm{Sp}_{2\ell}(r^s) \leq \mathrm{Sp}_{2\ell s}(r)$, see for example [12, Section 4.3] for the definition and construction. Then the Weil representations of $\mathrm{Sp}_{2\ell}(r^s)$ can be defined as the restriction of the Weil representations of $\mathrm{Sp}_{2\ell s}(r)$, and it is known that $\mathrm{Sp}_{2\ell s}(r)$

and $\mathrm{Sp}_{2\ell}(r^s)$ have the same submodules on $W$. Moreover, generators for $\mathrm{Sp}_{2\ell}(r^s)$ as a subgroup of $\mathrm{Sp}_{2\ell s}(r)$ are well known, and have been implemented in Magma [7, Proposition 6.4]. Using this embedding $\mathrm{Sp}_{2\ell}(r^s) \leq \mathrm{Sp}_{2\ell s}(r)$, we have also included the construction of Weil representations of $\mathrm{Sp}_{2\ell}(r^s)$ in the Magma code available at [14].

Similarly, with the embeddings $\mathrm{GL}_\ell(r^s) \leq \mathrm{Sp}_{2\ell}(r^s)$ and $\mathrm{GU}_\ell(r^s) \leq \mathrm{Sp}_{2\ell}(r^s)$, we can define Weil representations of finite general linear groups and unitary groups. Explicit matrix generators for these subgroups of $\mathrm{Sp}_{2\ell}(r^s)$ are also known and implemented in Magma, see [7, Proposition 5.3, Proposition 6.5]. Thus one can similarly write down matrix generators for the Weil representations of $\mathrm{GL}_\ell(r^s)$ and $\mathrm{GU}_\ell(r^s)$ over finite fields.

## 2. Some preliminary calculations

In this section, we provide some basic properties of the generators of the group $G$ from Theorem 1.3. All except the last lemma of this section are about the case $\ell = 1$, in which case $G = \langle \lambda C, U\rangle$.

**Lemma 2.1.** *For all $0 \leq \xi < r$, we have $C^2 v_\xi = r v_{-\xi}$.*

*Proof.* For $0 \leq i, j < r$, the $(i,j)$ entry of the matrix of $C^2$ (with respect to the basis $v_0, v_1, \ldots, v_{r-1}$) is equal to

$$\sum_{0\leq\xi<r} \theta^{i\xi+\xi j} = \sum_{0\leq\xi<r} \theta^{\xi(i+j)}. \tag{2.1}$$

Now if $i + j \not\equiv 0 \mod r$, then (2.1) is the sum of all $r$th roots of unity, and thus equals zero. If $i + j \equiv 0 \mod r$, then (2.1) evaluates to $\sum_{0\leq\xi<r} 1 = r$. From this the claim of the lemma follows. $\square$

**Lemma 2.2.** *We have $\det(C^2) = (-1)^{(r-1)/2} r^r$, and $(\lambda C)^2 v_\xi = (-1)^{(r-1)/2} v_{-\xi}$ for all $0 \leq \xi < r$.*

*Proof.* By Lemma 2.1, we have $C^2 = rP$, where $P$ is a permutation matrix that fixes the basis vector $v_0$ and swaps $v_i$ with $v_{r-i}$ for $0 < i < r$. The permutation corresponding to $P$ is a product of $(r-1)/2$ transpositions, so we conclude that $\det(C^2) = (-1)^{(r-1)/2} r^r$.

For the second claim, note that $\lambda^2 = r^r \det(C)^{-2} = r^{-1}(-1)^{(r-1)/2}$ by the first claim. Thus $(\lambda C)^2 v_\xi = (-1)^{(r-1)/2} v_{-\xi}$ follows from Lemma 2.1. $\square$

**Lemma 2.3.** *We have $\det(\lambda C) = 1$. Moreover $\det(U) = 1$ if $r > 3$, and $\det(U) = \theta$ if $r = 3$.*

*Proof.* First note that

$$\det(\lambda C) = \lambda^r \det(C) = (-r)^{r(r-1)/2} \det(C)^{-r} \det(C) = \left(-r^r \det(C)^{-2}\right)^{(r-1)/2}.$$

Since $\det(C)^2 = (-1)^{(r-1)/2} r^r$ (Lemma 2.2), we get that $\det(\lambda C) = 1$.

For the determinant of $U$, we have

$$\det(U) = \prod_{0\leq\xi<r} \theta^{\frac{\xi(\xi+r)}{2}} = \theta^{\sum_{0\leq\xi<r} \xi(\xi+r)/2}.$$

Now $\sum_{0\leq\xi<r} \xi^2 = (r-1)r(2r-1)/6$ is divisible by $r$ if $r > 3$, so it follows that $\det(U) = 1$. For $r = 3$ we have $U$ equal to a diagonal matrix $\mathrm{diag}(1, \theta^2, \theta^2)$, so $\det(U) = \theta$. $\square$

**Lemma 2.4.** *We have* $\mathrm{Tr}(U)^2 = (-1)^{(r-1)/2}r$.

*Proof.* First note that $\mathrm{Tr}(U)^2$ equals

$$\left(\sum_{0\leq i<r} \theta^{\frac{i(i+r)}{2}}\right)^2 = \sum_{0\leq i,j<r} \theta^{\frac{i(i+r)}{2}+\frac{j(j+r)}{2}} = \sum_{0\leq i,j<r} \theta^{\frac{i^2+j^2+(i+j)r}{2}} \tag{2.2}$$

Here the value of each summand is determined by the value of $i^2+j^2$ modulo $r$. We now make use of classical formulae for the number of solutions to $i^2+j^2 \equiv a \mod r$, which go back at least to the work of Jordan [10, §199 − 200, pp. 159 − 161], see for example [16, Lemma 6.24].

For the equation $i^2+j^2 \equiv 0 \mod r$, the number of solutions $(i,j)$ with $0 \leq i,j < r$ is $\alpha = 2r-1$ if $r \equiv 1 \mod 4$, and $\alpha = 1$ if $r \equiv 3 \mod 4$. Similarly for fixed $a \not\equiv 0 \mod r$, there are $\beta = r + (-1)^{(r+1)/2}$ solutions to $i^2+j^2 \equiv a \mod r$. Thus (2.2) evaluates to

$$\alpha + \beta(\theta + \theta^2 + \cdots + \theta^{r-1}) = \alpha - \beta.$$

It is readily checked that $\alpha - \beta = (-1)^{(r-1)/2}r$, from which the lemma follows. □

**Lemma 2.5.** *Suppose that* $\ell \geq 2$. *Then* $\det(D_{st}) = 1$ *for all* $1 \leq s < t \leq \ell$.

*Proof.* From the definition of $D_{st}$ in (1.5), one computes directly that

$$\det(D_{st}) = \theta^{r^{\ell-2}\sum_{0\leq \xi_s,\xi_t<r}\xi_s\xi_t} = \theta^{r^{\ell-2}(r(r-1)/2)^2} = 1,$$

as required. □

## 3. Basic properties of $G$

Let $G$ be the subgroup generated by $\lambda C_t$, $U_t$ (for $1 \leq t \leq \ell$) and $D_{st}$ for $1 \leq s < t \leq \ell$, as in Theorem 1.3.

**Lemma 3.1.** *Let* $\sigma : W \to W$ *be the involution defined by* $v_{\xi_1,\ldots,\xi_\ell} \mapsto v_{-\xi_1,\ldots,-\xi_\ell}$. *Then the following statements hold:*

(i) $\sigma$ *normalizes* $R$, *and acts on* $R/Z(R)$ *by inversion.*
(ii) $C_R(\sigma) = Z(R)$.
(iii) $\sigma = (-1)^{\ell(r-1)/2}(\lambda C_1)^2\cdots(\lambda C_\ell)^2$.
(iv) $G$ *centralizes* $\sigma$.

*Proof.* A direct calculation using the definition (1.5) shows that $\sigma A_i \sigma^{-1} = A_i^{-1}$ and $\sigma B_i \sigma^{-1} = B_i^{-1}$ for all $1 \leq i \leq \ell$, from which claim (i) follows.

For (ii), first note that $Z(R) \leq C_R(\sigma)$ since $Z(R)$ consists of scalar matrices. For the other direction, suppose that $x \in C_R(\sigma)$. Then $x = \sigma x \sigma^{-1} = x^{-1}y$ for some $y \in Z(R)$ by (i), so $x^2 \in Z(R)$. It follows then from $r > 2$ that $x \in Z(R)$. Thus we conclude that $C_R(\sigma) = Z(R)$.

Claim (iii) follows from Lemma 2.2. For (iv), we check that all the generators of $G$ are centralized by $\sigma$. First since the $C_t$'s commute with each other, it follows from (iii) that $\sigma$ centralizes $\lambda C_t$ for all $1 \leq t \leq \ell$. A direct calculation from the definition of $U_t$ shows that it is centralized by $\sigma$, using the fact that $\xi(\xi+r)/2 \equiv (-\xi)(-\xi+r)/2 \mod r$ for all $0 \leq \xi < r$. Similarly $D_{st}$ is centralized by $\sigma$, since $\xi\xi' = (-\xi)(-\xi')$ for all $0 \leq \xi, \xi' < r$. □

**Lemma 3.2.** *Let* $Z \leq \mathrm{GL}(W)$ *be the group of scalar matrices. The following properties hold for* $G$.

(i) $G \cap RZ = G \cap Z$.
(ii) $G/(G \cap Z) \cong \mathrm{Sp}_{2\ell}(r)$.
(iii) $\det(g)^r = 1$ *for all* $g \in G$.
(iv) $G/[G,G]$ *is an* $r$*-group.*

*Proof.* Claim (i) follows from Lemma 3.1 (ii) and (iv). Then claim (ii) is a consequence of (i) and the fact that $G$ maps surjectively to $\mathrm{Sp}_{2\ell}(r)$.

For claim (iii), it suffices to check the claim for the generators of $G$. We have $\det(\lambda C_t) = 1$ and $\det(U_t)^r = 1$ by Lemma 2.3, and $\det(D_{st}) = 1$ by Lemma 2.5. Thus $\det(g)^r = 1$ for all $g \in G$.

It remains to consider (iv). First note that by claim (iii), the subgroup $G \cap Z$ of scalar matrices is an $r$-group. Thus it suffices to check that the abelianization of $G/(G \cap Z)$ is an $r$-group. This follows from (ii), since $\mathrm{Sp}_{2\ell}(r)$ is perfect for $(r,\ell) \neq (3,1)$, and $\mathrm{Sp}_2(3)$ has abelianization of order $r = 3$. $\square$

## 4. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. At this point we already know (Lemma 3.2) that $G$ as in Theorem 1.3 is a central extension of $\mathrm{Sp}_{2\ell}(r)$. For $(r,\ell) \neq (3,1)$ our strategy will be to prove that $G$ is perfect, so in fact $G \cong \mathrm{Sp}_{2\ell}(r)$ since the Schur multiplier of $\mathrm{Sp}_{2\ell}(r)$ is trivial [19, Theorem 1.1]. In the special case $(r,\ell) = (3,1)$, we will verify $G \cong \mathrm{SL}_2(3)$ by checking that the generators satisfy suitable relations.

We will first prove two lemmas that verify some relations among the generators.

**Lemma 4.1.** *The following hold for generators of* $G$*:*
(i) $(\lambda C_t)^4 = 1$ *for all* $1 \leq t \leq \ell$.
(ii) $(\lambda C_t U_t)^3 = 1$ *for all* $1 \leq t \leq \ell$.

*Proof.* Claim (i) follows from Lemma 2.2. For claim (ii), from the definition of $C_t$ and $U_t$ as tensor products, it suffices to prove the lemma for $\ell = 1$. Suppose then that $\ell = 1$, in which case we aim to prove that $(\lambda CU)^3 = 1$. Under the homomorphism $\pi : G \to \mathrm{Sp}_2(r)$ from (1.1), we have

$$\pi(C) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \pi(U) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus $\pi((CU)^3)$ is trivial, so $(CU)^3$ is a scalar matrix by Lemma 3.2 (i) and (1.2). To check what the scalar multiplier is, we compute the $(0,0)$ entry of $(CU)^3$ with respect to the basis $v_0, v_1, \ldots, v_{r-1}$. To this end, we have $((CU)^3)_{00}$ equal to

$$\begin{aligned}\sum_{0 \leq i,j < r} (CU)_{0i}(CU)_{ij}(CU)_{j0} &= \sum_{0 \leq i,j < r} \theta^{\frac{i(i+r)}{2} + ij + \frac{j(j+r)}{2}} \\ &= \sum_{0 \leq i,j < r} \theta^{\frac{(i+j)(i+j+r)}{2}} \\ &= r \sum_{0 \leq i < r} \theta^{\frac{i(i+r)}{2}} = r\,\mathrm{Tr}(U),\end{aligned}$$

so $(CU)^3 = r\,\mathrm{Tr}(U) I_r$.

Now since $(\lambda CU)^3$ is a scalar matrix, by Lemma 3.2 (iii) it has $r$-power order. So since $r > 2$, it suffices to check that $(\lambda CU)^6 = 1$. For this, we first get

$$(\lambda CU)^6 = \lambda^6 (CU)^6 = \lambda^6 r^2\,\mathrm{Tr}(U)^2 I_r = \lambda^6 r^3 (-1)^{(r-1)/2} I_r \tag{4.1}$$

with Lemma 2.4. On the other hand, we have

$$\lambda^6 = \left(r^{r-1}\det(C)^{-2}\right)^3 = \left(r^{r-1}r^{-r}(-1)^{(r-1)/2}\right)^3 = r^{-3}(-1)^{(r-1)/2}$$

by Lemma 2.2. Combining this with (4.1), we get $(\lambda CU)^6 = 1$. □

*Remark* 4.2. As seen in the proof of Lemma 4.1, we have $(CU)^3 = r\,\mathrm{Tr}(U)I_r$. On the other hand $(\lambda CU)^3 = 1$ by Lemma 4.1, so $\lambda^3 r\,\mathrm{Tr}(U) = 1$. Since $\lambda^2 = (-1)^{(r-1)/2}r^{-1}$ by Lemma 2.2, it follows that $\lambda(-1)^{(r-1)/2}\,\mathrm{Tr}(U) = 1$. Then plugging in the definition (1.7) of $\lambda$ gives the formula

$$\det(C) = r^{(r-1)/2}\sum_{0\leq i<r}\theta^{\frac{i(i+r)}{2}} = \left(\frac{2}{r}\right)r^{(r-1)/2}\sum_{0\leq i<r}\theta^{i^2},$$

where the second equality follows from the fact that $\sum_{0\leq i<r}\theta^i = 0$.

**Lemma 4.3.** *Suppose that $\ell \geq 2$. Then the following hold the for generators of $G$:*

(i) $((\lambda C_t)^2 D_{st})^2 = 1$ *for all* $1 \leq s < t \leq \ell$.
(ii) *Denote* $X = C_t D_{st} C_t^{-1}$,*where* $1 \leq s < t \leq \ell$. *Then* $XU_tX^{-1}U_t^{-1} = U_sD_{st}$.

*Proof.* From the definition of $C_t$ and $D_{st}$ acting on a tensor product, it is clear that it suffices to prove the lemma in the case $\ell = 2$. Then $(s,t) = (1,2)$, and it follows from Lemma 2.2 that the action of $(\lambda C_1)^2 D_{12}$ on $W$ is given by

$$v_{\xi_1,\xi_2} \mapsto (-1)^{(r-1)/2}\theta^{\xi_1\xi_2}v_{-\xi_1,\xi_2}$$

for all $0 \leq \xi_1, \xi_2 < r$. This clearly defines an involution, so (i) holds.

For claim (ii), we first note that a direct calculation from the definitions (1.5) shows that $X$ acts on $W$ by

$$v_{\xi_1,\xi_2} \mapsto v_{\xi_1-\xi_2,\xi_2}$$

for all $0 \leq \xi_1, \xi_2 < r$. Note that then $X^{-1}$ acts on $W$ by $v_{\xi_1,\xi_2} \mapsto v_{\xi_1+\xi_2,\xi_2}$. Then a straightforward calculation shows that $XU_2X^{-1}U_2^{-1}$ acts on $W$ by

$$v_{\xi_1,\xi_2} \mapsto \theta^{\frac{\xi_1(\xi_1+r)}{2}+\xi_1\xi_2}v_{\xi_1,\xi_2}$$

for all $0 \leq \xi_1, \xi_2 < r$. In other words, we have $XU_1X^{-1}U_1^{-1} = U_1D_{12}$, as required. □

With this we can prove the main result.

*Proof of Theorem 1.3.* We already know by Theorem 1.1 that the restriction of $\pi$ to $G$ defines a surjective map $G \to \mathrm{Sp}_{2\ell}(r)$. Thus to prove the theorem, it suffices to prove that $G \cong \mathrm{Sp}_{2\ell}(r)$.

We first consider the case where $r = 3$ and $\ell = 1$, so $G = \langle \lambda C_1, U_1\rangle$. We will make use of the fact that $\mathrm{Sp}_2(3) = \mathrm{SL}_2(3)$ has a presentation

$$\mathrm{Sp}_2(3) \cong \langle x, y \mid x^4 = y^3 = (xy)^3 = [x^2, y] = 1\rangle. \tag{4.2}$$

Now note that $(\lambda C_1)^4 = (\lambda C_1 U_1)^3 = 1$ by Lemma 4.1, and $U_1^3 = 1$ is clear from the action of $U_1$. It follows from Lemma 3.1 that $(\lambda C_1)^2$ is central in $G$, so by (4.2) there is a surjective homomorphism $\mathrm{Sp}_2(3) \to G$. By Lemma 3.2 (iii) this is an isomorphism, which proves the theorem in this case.

Suppose then that $(r,\ell) \neq (3,1)$, in which case $\mathrm{Sp}_{2\ell}(r)$ is perfect. Moreover, we know from Lemma 3.2 (ii) that $G$ is a central extension of $\mathrm{Sp}_{2\ell}(r)$. Thus it will suffice to verify that $G$ is perfect, since the Schur multiplier of $\mathrm{Sp}_{2\ell}(r)$ is trivial [19, Theorem 1.1], and thus $\mathrm{Sp}_{2\ell}(r)$ has no non-trivial perfect central extension.

For the proof that $G$ is perfect, suppose first that $\ell > 1$. Now $G/[G,G]$ is an $r$-group by Lemma 3.2 (iv), and on the other hand $\lambda C_t$ has order coprime to $r$ by Lemma 4.1, so $\lambda C_t \in [G,G]$ for all $1 \leq t \leq \ell$. Similarly $(\lambda C_t)^2 D_{st}$ has order coprime to $r$ by Lemma 4.3 (i), so $(\lambda C_t)^2 D_{st} \in [G,G]$ and thus $D_{st} \in [G,G]$ for all $1 \leq s < t \leq \ell$. Finally by Lemma 4.3 (ii) we have $U_t D_{st} \in [G,G]$, so we conclude that $U_t \in [G,G]$ for all $1 \leq t \leq \ell$. Since $[G,G]$ contains all the generators of $G$, we have $G = [G,G]$.

It remains to consider the case $\ell = 1$, so here $r > 3$ by assumption. We have $G = \langle \lambda C_1, U_1 \rangle$ and it follows from Lemma 4.1 that $\lambda C_1$ and $\lambda C_1 U_1$ both have order coprime to $r$. Since $G/[G,G]$ is an $r$-group (Lemma 3.2 (iv)), we conclude as in the previous paragraph that $G = [G,G]$. This completes the proof of the theorem. □

## 5. Irreducible submodules of degrees $(r^\ell \pm 1)/2$

Let $G \cong \mathrm{Sp}_{2\ell}(r)$ be as in Theorem 1.3. Then the $\mathbb{F}[G]$-submodule structure of $W$ is known to be as follows, see for example [21, Theorem 2.5] and [6, Lemma 5.2].

- If $\mathrm{char}\,\mathbb{F} \neq 2$, then $W$ is a direct sum of two irreducible $\mathbb{F}[G]$-submodules of dimensions $(r^\ell - 1)/2$ and $(r^\ell + 1)/2$.
- If $\mathrm{char}\,\mathbb{F} = 2$, then $W$ is uniserial, with composition factors of dimensions $(r^\ell - 1)/2$, 1, $(r^\ell - 1)/2$. Specifically
$$0 \subsetneq A \subsetneq B \subsetneq W$$
with $A \cong W/B$ irreducible of dimension $(r^\ell - 1)/2$, and $\dim B/A = 1$.

The irreducible modules of dimensions $(r^\ell \pm 1)/2$ are known as *irreducible Weil representations* of $\mathrm{Sp}_{2\ell}(r)$. The submodules of $W$ can be described as follows.

Let $\sigma$ be the involution $v_{\xi_1,\ldots,\xi_\ell} \mapsto v_{-\xi_1,\ldots,-\xi_\ell}$ defined in Lemma 3.1. Since $G$ centralizes $\sigma$ (Lemma 3.1 (iv)), it follows that $G$ acts on the eigenspaces of $\sigma$. We will denote the $\pm 1$-eigenspace of $\sigma$ by $W^\pm$.

Suppose that $\mathrm{char}\,\mathbb{F} \neq 2$. Then it is straightforward to see that

$$\begin{aligned} W^+ &= \langle v_{\xi_1,\ldots,\xi_\ell} + v_{-\xi_1,\ldots,-\xi_\ell} : 0 \leq \xi_1, \ldots, \xi_\ell < r \rangle, \\ W^- &= \langle v_{\xi_1,\ldots,\xi_\ell} - v_{-\xi_1,\ldots,-\xi_\ell} : 0 \leq \xi_1, \ldots, \xi_\ell < r \rangle. \end{aligned}$$

Furthermore, we have $W = W^+ \oplus W^-$. Here $W^\pm$ is an irreducible $\mathbb{F}[G]$-submodule of dimension $(r^\ell \pm 1)/2$.

Suppose then that $\mathrm{char}\,\mathbb{F} = 2$. In this case the non-zero proper submodules of $W$ are

$$\begin{aligned} A &= \langle v_{\xi_1,\ldots,\xi_\ell} + v_{-\xi_1,\ldots,-\xi_\ell} : 0 \leq \xi_1, \ldots, \xi_\ell < r \rangle, \\ B &= W^+ = \langle A, v_{0,\ldots,0} \rangle. \end{aligned}$$

Here $A$ is an irreducible $\mathbb{F}[G]$-module of dimension $(r^\ell - 1)/2$. Furthermore $B/A$ is 1-dimensional, and $W/B \cong A$ is irreducible of dimension $(r^\ell - 1)/2$.

On each submodule, we can easily write down the action of the generators of $G$ on a basis, thus giving generating matrices for the irreducible Weil representation. For example if $\mathrm{char}\,\mathbb{F} \neq 2$, then $W^+$ has a basis consisting of the vectors

$$y_{\xi_1,\ldots,\xi_\ell} := v_{\xi_1,\ldots,\xi_\ell} + v_{-\xi_1,\ldots,-\xi_\ell},$$

where $0 \leq \xi_1, \ldots, \xi_\ell < r$. It is then easy to see from (1.5) and (1.6) that similarly to the action of $G$ on the basis vectors of $W$, we have

$$\begin{aligned} C_t y_{\xi_1,\ldots,\xi_\ell} &= \sum_{0 \leq i < r} \theta^{i\xi_t} y_{\xi_1,\ldots,\xi_{t-1},i,\xi_{t+1},\ldots,\xi_\ell} \\ D_{st} y_{\xi_1,\ldots,\xi_\ell} &= \theta^{\xi_s \xi_t} y_{\xi_1,\ldots,\xi_\ell} \\ U_t v_{\xi_1,\ldots,\xi_\ell} &= \theta^{\frac{\xi_t(\xi_t+r)}{2}} y_{\xi_1,\ldots,\xi_\ell} \end{aligned}$$

for all $0 \leq \xi_1, \ldots, \xi_\ell < r$.

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