Qualification-free convex analysis via the joint supporting subspace

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Abstract

In convex analysis, qualification conditions (also termed constraint qualifications) help avoid pathological behavior at domain boundaries. In this work we remove the need for such conditions by localizing to an affine subspace—the *joint supporting subspace*—that contains the feasible region and ensures qualification conditions hold after localization.

Our theory generalizes Borwein and Wolkowicz's facial reduction beyond conic programs to convex programs of the form $f(x) + g(Ax)$. Intuitively, the joint supporting subspace corresponds to a bilateral facial reduction between any two convex sets. It enables simple qualification-free generalizations for a host of central results of convex analysis. These include: an exact Fenchel-Rockafellar dual; Karush-Kuhn-Tucker (KKT) optimality conditions; attained infimal convolution for convex conjugates; subdifferential sum and chain rules; and a characterization of the normal cones of the intersection of two convex sets. All generalizations reduce seamlessly to their original formulations when qualification conditions hold.

We offer a number of characterizations for the joint supporting subspace, one of which is constructive. Our proofs are self-contained, and introduce a novel theoretical framework consisting of *nested normals* and *supporting subspaces*, which simultaneously describe both the boundary of convex sets and the lattice of faces.

1 Introduction

In convex analysis, many results depend on *qualification conditions*, of which one prominent example found in the seminal work [32] by Rockafellar, is that for two convex functions $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$,

$$\operatorname{relint} \operatorname{dom}(f) \cap \operatorname{relint} \operatorname{dom}(g) \neq \emptyset, \tag{1}$$

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where relint denotes the relative interior. This condition ensures that the feasible region $\mathrm{dom}(f+g)$ is not contained in the relative boundary of either $\mathrm{dom}(f)$ or $\mathrm{dom}(g)$. When Equation (1) does not hold, the geometry of the boundary of the convex sets $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ becomes important, and requires special treatment. Such cases are typically excluded from consideration with qualification conditions because of the added complexity, and also because such cases are often avoidable in practice.

The sum rule of subdifferentials,

$$\partial(f+g)(x) = \partial f(x) + \partial g(x),$$

is guaranteed by [32, Theorem 23.8], subject to Equation (1) (amongst other possible sufficient conditions). Other results of convex analysis with similar conditions include, e.g., the subdifferential chain rule, attainment of the infimal convolution in convex conjugates, strong duality of the Fenchel-Rockafellar dual, and Karush-Kuhn-Tucker (KKT) optimality conditions. For results concerned with optimization problems, this type of condition is usually called *constraint qualifications.*

Qualification conditions are not merely theoretical artifacts; they are justified by counterexamples, and therefore reflect more or less accurately the true scope of applicability of the associated rules. We introduce one such counterexample for the sum rule of subdifferentials, to later show how our theory resolves this case. Take the function $f(x) = -\sqrt{x}$ for $x \geq 0$ and $f(x) = \infty$ otherwise. Consider the indicator function $g(x) = \iota_{\mathbb{R}_-}(x)$, which is 0 on $\mathbb{R}_-$ and ∞ elsewhere. Naive application of the sum rule yields

$$\partial(f+g)(0) = \partial f(0) + \partial g(0) = \emptyset + \mathbb{R}_+ = \emptyset.$$

In fact, $\partial(f+g)(0) = \partial\iota_0(0) = \mathbb{R}$ so the sum rule does not hold. There is no conflict with [32, Theorem 23.8] because qualification conditions are violated; the relative interiors of the domains are $(-\infty, 0)$ and $(0, \infty)$, and do not intersect. The standard sum rule fails in this case, though we will show that it is recovered after localization to the joint supporting subspace.

We tackle the problem of qualification conditions directly by developing geometric results describing the boundary of convex sets, allowing us to generalize a collection of results from convex analysis in one fell swoop in Section 5, with the classical formulations trivially recovered when qualifications hold (see Proposition 5.1).

The key geometric insight is as follows. For any two intersecting convex sets $C, D \subseteq \mathbb{R}^n$ in $\mathbb{R}^n$, there exists a special affine subspace T_a—the joint supporting subspace—that contains $C \cap D$ and that is uniquely determined by C and D, i.e., $T_a := T_a(C, D)$. We provide multiple characterizations of the joint supporting subspace in Section 4. Supporting subspaces (see Definition 6.1) are a generalization of supporting hyperplanes, first defined, to our knowledge, in [7, Exercise 5.4]. As intersections with *supporting hyperplanes* reveal exposed faces of convex sets, so do intersections with *supporting subspaces* reveal faces—which need not be exposed. As the name suggests, T_a is a supporting subspace of both

C and D, which means that two faces of C and D are simultaneously revealed by intersection with T_a. Each of the convex sets presents a face to the other, and they meet within T_a. The utility of T_a in convex analysis stems from two simple facts:

1. $C \cap D \subseteq T_a$.
2. $\mathrm{relint}(C \cap T_a) \cap \mathrm{relint}(D \cap T_a) \neq \emptyset$.

We prove these facts in Corollary 4.1 and Corollary 4.5 respectively. These two properties can be understood as follows: when reducing the ambient space to T_a, the the reduced (localized, intersected) sets $C' := C \cap T_a$ and $D' := D \cap T_a$ preserve the original intersection ($C' \cap D' = C \cap D$) while satisfying the qualification condition $\mathrm{relint}(C') \cap \mathrm{relint}(D') \neq \emptyset$. Similarly, for convex functions $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$, with $T_a := T_a(\mathrm{dom}(f), \mathrm{dom}(g))$, the localized functions $f' := f + \iota_{T_a}, g' := g + \iota_{T_a}$ preserve the original sum ($f' + g' = f + g$) while satisfying Equation (1). Standard convex analysis results can then be applied to the localized functions f' and g', while still accurately describing the original sum $f + g$. For instance, in Corollary 5.1 we state that

$$\partial(f + g)(x) = \partial f'(x) + \partial g'(x), \tag{2}$$

without any condition on f and g other than they be proper convex functions. A host of other generalizations are obtained via minor variations on this procedure in Section 5.

This modification to the sum rule resolves the previously encountered counterexample. In that problem, where $f(x) = -\sqrt{x}$ and $g(x) = \iota_{(-\infty,0]}$, the joint supporting subspace from Definition 4.1 is $T_a = \{0\}$, and so

$$\partial(f + \iota_{T_a})(x) + \partial(g + \iota_{T_a})(x) = \partial\iota_{\{0\}} + \partial\iota_{\{0\}} = \mathbb{R},$$

which indeed matches $\partial(f + g)(x)$.

1.1 Facial reduction and computational outlook

As we show in Section 5.3.1, our theory is a generalization of facial reduction, a preprocessing technique with strong theoretical guarantees in conic programming. Facial reduction reduces the problem to the smallest face of the conic constraint that contains the feasible set [23, 6, 4]. After reducing to this face, Slater's condition is provably satisfied. As mentioned in [9], facial reduction leads to both a reduction in the size of the problem and improved numerical stability, and has consequently been implemented in solvers for semi-definite programs (e.g., in the MOSEK solver [27, 10]). The impact of facial reduction in computation inspires hope that our theory could lead to improvements in optimization algorithms in non-conic settings where constraint qualifications may fail.

Where facial reduction reduces to the minimal face of a cone, we reduce jointly to the smallest faces of two convex sets, both selected by intersection with the single joint supporting subspace. For simplicity, we frame our results as a reduction of the relevant objects (convex sets or convex functions), while keeping the ambient space $\mathbb{R}^n$ fixed. Our results are however naturally suited to a reduction of the ambient space to the joint supporting subspace, i.e., the joint supporting subspace would then become the new ambient space (see, e.g., Theorem 5.1). This alternative mathematical formulation would better model a reduction of the problem in a preprocessing step. Reduction of the ambient space to the joint supporting subspace in a preprocessing step may seem counterintuitive, as results in convex analysis are commonly specified only for real vector spaces, not affine spaces. This discrepancy can be mitigated by translating the affine subspace jointly with all the objects it contains until it reaches origin. But even this minor adjustment seems redundant, because, we believe, affine spaces seem to be the natural home for many results in convex analysis.

In full abstraction, an affine space is a pair (L, U) consisting of a set L and a real vector space U (the tangent space), equipped with an addition operation $+ : L \times U \to L$ with the property that $\forall l_1, l_2 \in L$ there exists a unique $u \in U$ such that $l_1 + u = l_2$. For L an affine subspace of $\mathbb{R}^n$, the tangent space is $U = L - L = \{l_1 - l_2 \mid l_1, l_2 \in L\}$. Results in convex analysis can then be generalized by distinguishing tangent vectors from points in the affine space. For example, that affine subspaces are closed under convex combinations follows from reformulating $\lambda x + (1 - \lambda)y$ for $x, y \in L$ as $y + \lambda(x - y)$, by noticing that $y \in L$ and that $\lambda(x - y)$ is in the tangent space $L - L$. The tangent space appears in our definition for the separation of convex sets (Definition 4.3), and justifies our repeated use of self-differences of sets $S - S$ (see, e.g., Definition 6.2, Corollary 4.1, or Definition 4.2).

1.2 Theoretical contribution

An important contribution of the present work is the theoretical framework that we build in Section 6. The central objects are *nested normals* (Definition 6.3) and *supporting subspaces* (Definition 6.2), which describe local subsets of the lattice of faces of a convex set. Their combination forms a mathematical object that is an exact dual for convex cones (see Theorem 6.2). As such, the nested normals can also be understood as a local description of the boundary of convex sets, an idea that we discuss in Section 6. With these objects, we derive results describing the intersection of convex sets, as well as the facial structure of differences of convex sets. Notable technical results include Theorem 6.3, Theorem 6.4, and Theorem 6.6. We therefore contribute a geometric framework describing the boundaries, intersections, and differences of convex sets.

1.3 Paper outline

Section 2 surveys prior works, notably facial reduction in conic programming, limiting approaches to exact subdifferential calculus, and the lexicographic characterization of the faces of convex sets. In Section 3 we introduce some notation. In Section 4 we state our results describing the joint supporting subspace, with proofs deferred to Section 7. There, we define the joint supporting subspace in Definition 4.1, characterize it with a distinct analytical formula in Theorem 4.1, and with a constructive iterative procedure in Corollary 5.7 which terminates in a number of steps no more than the ambient dimension. In Theorem 4.2 we state the main property of the joint supporting subspace: that qualification conditions hold when localizing to it. In Section 5, we state and prove generalizations of central results in convex analysis. In Section 5.3.1, we show that results of facial reduction in conic programming can be recovered from our results. In Section 6, we provide a narrative, motivated introduction to our mathematical framework, introducing key concepts and proving associated results along the way. In Section 7, we prove the results of Section 4 with the tools from Section 6.

2 Prior works

Constraint qualifications (CQs) in finite dimensions evolved from Slater's condition [35], later weakened to the algebraic Mangasarian-Fromovitz CQ [20] and to the geometric Abadie CQ [1]. The weak Slater condition, introduced by Rockafellar [32], uses the notion of relative interiors. Subsequent relaxed CQs include the constant rank constraint qualification condition [13], further relaxed in [29, 2], and a CQ by Robinson [31], which takes the form of a stability condition now known as *linear metric subregularity*. We note that in his seminal work [32], Rockafellar used the same qualification conditions for optimization problems and subdifferential calculus, which enabled our unified treatment of optimization and subdifferential calculus.

There has been two major research directions generalizing results beyond qualification conditions. The geometric stream, to which our work belongs, focuses on the geometry of constraints and feasible sets. The sequential stream considers limits of increasingly accurate approximations, with exact results in the limit, and is related to variational analysis [24, 33]. The geometric stream focuses on constraint qualifications for optimization problems, while the sequential stream emphasizes subdifferential calculus, with extensions to sequential optimality conditions. Our work extends the geometric stream to subdifferential calculus. We survey qualification-free results in finite dimensions that are comparable to our results in Section 5.

2.1 Geometric stream

In the setting of inequality-constrained convex programs, the Fritz-John condition [16] is an optimality condition that holds without qualification conditions. Unlike the KKT condition, it is not a sufficient condition for optimality, and is satisfied either when KKT conditions hold, or when some degeneracy in the constraints is detected, akin to applying the KKT conditions with a suitable constraint qualification.

A qualification-free necessary and sufficient condition was formulated by

Ben-Tal, Ben-Israel, and Zlobec [3], and extended to the nonsmooth case by Wolkowicz [39]. These works add a term to the KKT condition to deal with pathological cases; see [8, Chapter 6] for a treatment. Our qualification-free KKT condition in Corollary 5.6 is formulated for programs of the form $f(x) + g(Ax)$, instead of inequality-constrained convex programs, and uses faces of convex sets instead of an additional term.

Facial Reduction (FR) generalized preprocessing ideas from linear programming to the setting of conic programming [23, 6, 4], with subsequent developments providing efficient computational methods for the reduction. An early method iteratively reduces the dimension of the problem [4, 19, 38] until the minimal face is reached. Ideas of facial reduction can alternatively be framed with *exact dual problems* [5], with important applications to semidefinite programming [30, 25]. For in-depth literature reviews of facial reduction and its numerous applications, we refer the reader to [37, 9, 28].

Many results in our work have analogues in facial reduction. For instance, the notion of a *minimal face* appears in different forms in Definition 4.1 and Theorem 4.1. The iterative reduction procedure of Borwein and Wolkowicz [4] finds its analogue in Theorem 7.1. We derive an exact Fenchel-Rockafellar dual in Corollary 5.5, echoing exact duals in conic programming. Finally, our qualification-free optimality condition in Corollary 5.6 generalizes similar results in the conic setting.

2.2 Sequential stream

The sequential approach to relaxing qualification conditions began with subdifferential calculus. One approach is that of ε-subdifferentials, which defines subgradients satisfying the subgradient inequality up to an ε-error, recovering the exact subdifferential in the limit of vanishing ε [11, 26, 24]. A second, similar approach is that of fuzzy sum rules, which approximate the subdifferential by perturbing the point x at which the subdifferential is taken, deriving exact additivity rules in the limit of vanishing perturbations [17, 12].

These ideas were then applied to optimality conditions, where the condition of optimality takes the form of the existence of some sequence converging to the optimal point. One approach is to consider sequences of points converging to the point of interest [36], similarly to fuzzy sum rules, while another is analogous to ε-subdifferentials [15, 14] relating them to the epigraphs of conjugate functions.

2.3 Theory of convex faces

A geometric insight that was critical to the development of our theory was the *lexicographical characterization of faces* [22, 21]. Martínez-Legaz and Singer [22] derive a separation theorem from which the characterization follows. It turned out that the lexicographical framework was not flexible enough for our needs, prompting us to develop an analogous yet independent and self-contained theory. In this paper, we recover the lexicographical characterization of convex faces in Theorem 6.4, though in different terms. Our proof is direct, and does not make use of a separation theorem, though some separation ideas are implicit in the definition of supporting subspaces (Definition 6.1).

3 Preliminaries

For two sets $A, B \subseteq \mathbb{R}^n$, $A + B := \{a + b \mid a \in A, b \in B\}$ is the Minkowski sum, with $A - B := \{a - b \mid a \in A, b \in B\}$, and for $z \in \mathbb{R}^n$, let $A + z := A + \{z\}$. For a set $S \subseteq \mathbb{R}^n$, we denote by $\iota_S(x)$ the indicator function which is 0 if $x \in S$ and ∞ otherwise.

For a set $S \subseteq \mathbb{R}^n$, denote by $\text{conv}(S)$ its convex hull (smallest containing convex set). For $y_1, y_2 \in \mathbb{R}^n$, let $[y_1, y_2] := \text{conv}\{y_1, y_2\}$, and $]y_1, y_2[:= \text{conv}\{y_1, y_2\} \setminus \{y_1, y_2\}$. We also denote $\text{cone}(S)$ for the conical hull of a set; the smallest cone containing S.

Intuitively the notion of a *face* of a convex set C is a subset of C for which there is no line segment in C that "goes through" F. The following definitions can be found inline in [34, p. 16] and [34, p.75] respectively.

Definition 3.1 (Face of a convex set)**.** Let C be convex. A subset $F \subseteq C$ is defined to be a *face* of C when

$$\forall y, z \in C, (\exists x \in]y, z[\cap F \implies y, z \in F).$$

Definition 3.2 (Exposed face of a convex set)**.** Let $C \subseteq \mathbb{R}^n$ be convex. A subset $F \subseteq C$ is defined to be an *exposed face* of C when

$$\exists u \in \mathbb{R}^n \setminus \{0\} : F = \arg\max_{c \in C} \langle c, u \rangle.$$

For any set $S \subseteq C$, we denote by $F_C(S)$ the face of C *generated* by S; it is the smallest face of C that contains S (it is also the intersection of all faces containing S, which, it can be shown, is itself a face).

For $x \in \mathbb{R}^n$ and a convex set $C \subseteq \mathbb{R}^n$, the normal cone of C at x is

$$N_C(x) = \{y \mid \forall c \in C, \langle y, c - x \rangle \leq 0\}$$

if $x \in C$, and $N_C(x) = \emptyset$ otherwise.

We also recall some objects from convex analysis. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For a convex function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, the subdifferential is

$$\partial f(x) := \{v \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n, f(x) + \langle v, y - x \rangle \leq f(y)\},$$

with $\partial f(x) = \emptyset$ when $x \notin \text{dom}(f)$. The convex conjugate is

$$f^*(y) := \sup_{x \in \mathbb{R}^n} \langle y, x \rangle - f(x).$$

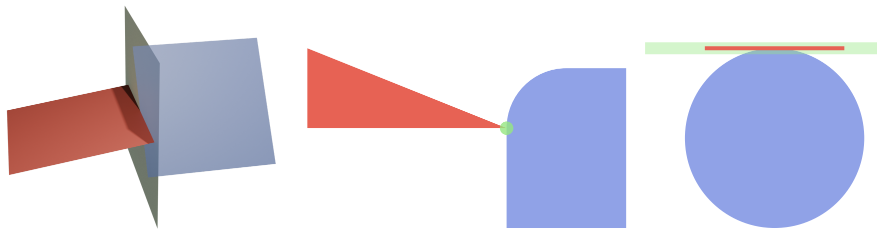

Figure 1: In each figure, the green subspace is the joint supporting subspace for the red and blue convex sets.

4 The joint supporting subspace

In this section we investigate the following object.

Definition 4.1 (The joint supporting subspace)**.** The *joint supporting subspace* of two convex sets $C, D \subseteq \mathbb{R}^n$ is

$$T(C, D) := \operatorname{span} F_{C-D}(0).$$

We also define its affine counterpart: $T_a(C, D) := T(C, D) + C \cap D$.

The Minkowski sum $(+C \cap D)$ in the definition of T_a is in fact an affine translation. This follows from the next corollary.

Corollary 4.1 (Containment of the intersection)**.** *Let $C, D \subseteq \mathbb{R}^n$ be convex. Then*

$$C \cap D - C \cap D \subseteq T(C, D).$$

All proofs for results presented in this section are deferred to Section 7, because they require the machinery introduced in Section 6. Corollary 4.1 follows from Theorem 6.6; for details, the proof can be found at the end of Section 6.2.

From Corollary 4.1, we see that picking any $x \in C \cap D$, we have $T_a = T + x$, independently of the choice of x. Corollary 4.1 also implies that $C \cap D \subseteq T_a$, showing that T_a is the unique translation of T that contains $C \cap D$.

In this section we provide two main characterizations of the joint supporting subspace. The next result builds towards the first characterization, describing the relation of the joint supporting subspace with the individual convex sets.

Lemma 4.1 (The joint supporting subspace reveals faces)**.** *For convex sets $C, D \subseteq \mathbb{R}^n$, and $T_a = T_a(C, D)$,*

$$T_a \cap C = F_C(C \cap D).$$

Next we introduce the *generated supporting subspace*, which is central to our theory. This is made clear in Section 6, where we re-introduce this object in motivated fashion in Definition 6.5.

Definition 4.2 (Preliminary definition of the generated supporting subspace)**.** Given a convex set $C \subseteq \mathbb{R}^n$ with $S \subseteq C$ non-empty, define the *supporting subspace* of C *generated* by $S \subseteq C$ to be

$$H_C(S) := \operatorname{span}(F_C(S) - F_C(S)).$$

The next theorem is our first characterization of the joint supporting subspace.

Theorem 4.1 (Characterization through generated faces)**.** *The joint supporting subspace of $C, D \subseteq \mathbb{R}^n$ is*

$$T(C, D) = H_C(C \cap D) + H_D(C \cap D).$$

We provide two other characterizations that follow from Theorem 4.1.

Corollary 4.2 (Joint supporting subspace as hull of faces)**.** *For convex sets $C, D \subseteq \mathbb{R}^n$,*

$$T_a(C, D) = \operatorname{aff}(F_C(C \cap D) \cup F_D(C \cap D)).$$

Corollary 4.3 (Joint supporting subspace as difference of faces)**.** *For convex sets $C, D \subseteq \mathbb{R}^n$,*

$$T(C, D) = \operatorname{span}(F_C(C \cap D) - F_D(C \cap D)).$$

So far, we have characterized the joint supporting subspace with various analytical formulas. Our next characterization instead specifies how to obtain the joint supporting subspace constructively, from an iterative process with a number of steps bounded by the ambient dimension. This result, Theorem 7.1, is best stated in terms of *nested normals*, a concept that we introduce in Definition 6.3, and motivate throughout Section 6. For this reason, we defer it to Section 7.3. For the moment, we state a direct corollary which does not require this object.

Corollary 4.4 (Iterative bilateral facial reduction)**.** *Let $C, D \subseteq \mathbb{R}^n$ be convex sets and fix any $x \in C \cap D$. Beginning with $T_0 = \mathbb{R}^n$, let*

$$T_{i+1} = T_i \cap (N_C(x) \cap [-N_D(x)] \cap T_i)^\perp.$$

Let ℓ be the smallest number such that $T_{\ell+1} = T_\ell$. Then:

1. *$T_\ell = T(C, D)$.*
2. *$\ell \leq n$.*

Now that the joint supporting subspace has been thoroughly identified, we move on to describing its core property. Recall that for two convex functions $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$, a common qualification condition is that for convex sets $C := \operatorname{dom}(f), D := \operatorname{dom}(g)$,

$$\operatorname{relint}(C) \cap \operatorname{relint}(D) \neq \emptyset. \tag{3}$$

It was shown in [32, Theorem 11.3] that this notion is equivalent to a *lack of proper separation* of the domains. We define the notions of *separation*, and *proper separation* below. We define these notions in affine subspaces, an intuitive generalization of separation in Euclidean spaces.

Definition 4.3 (Separation of sets)**.** Given an affine subspace $V \subseteq \mathbb{R}^n$, two convex sets $C, D \subseteq V$ are said to be *separated* in V when there exists some $u \in (V - V) \setminus \{0\}$ such that

$$\forall c \in C, d \in D, \langle u, c\rangle \leq \langle u, d\rangle. \tag{4}$$

The concept of *proper separation* is slightly stronger.

Definition 4.4 (Proper separation of sets)**.** For an affine subspace $V \subseteq \mathbb{R}^n$, convex sets $C, D \subseteq \mathbb{R}^n$ are *properly separated* in V when there exists some $u \in (V - V) \setminus \{0\}$ such that Equation (4) holds, and additionally such that there exists some pair $(c, d) \in C \times D$ for which the inequality in Equation (4) is strict.

Proper separation excludes the case where the two sets are merely separated by virtue of being both contained in a single hyperplane. Since separation follows from proper separation, and lack of proper separation is equivalent to intersection of relative interiors (by [32, Theorem 11.3]), the contrapositive statement is that

$$\text{lack of separation} \implies \text{intersection of relative interiors}. \tag{5}$$

In the joint supporting subspace there is always a "lack of separation" of the convex sets.

Theorem 4.2 (Qualification conditions always hold in the joint supporting subspace)**.** *For two convex sets $C, D \subseteq \mathbb{R}^n$ with $T_a := T_a(C, D)$, $C \cap T_a$ and $D \cap T_a$ are not separated in T_a.*

The proof of Theorem 4.2 is the subject of Section 7.1.

Corollary 4.5 (rint qualification holds in joint supporting subspace)**.** *For any convex sets $C, D \subseteq \mathbb{R}^n$ such that $C \cap D \neq \emptyset$,*

$$\operatorname{relint}(C \cap T_a) \cap \operatorname{relint}(D \cap T_a) \neq \emptyset.$$

Proof of Corollary 4.5. The result follows from Theorem 4.2 by Equation (5), with the caveat that we defined separation in affine subspaces, which are not necessarily vector spaces, while in [32, theorem 11.3] this notion is defined in $\mathbb{R}^n$. Yet the argument still holds because the notions of separation, proper separation, and relative interiors are invariant under translation of T_a to T. For any point $x_0 \in T_a$, $C \cap T_a$ is (properly) separated from $C \cap T_a$ iff $C \cap T_a - x_0$ is (properly) separated from $D \cap T_a - x_0$ in T. A similar equivalence holds for intersection of relative interiors. ☐

5 Main results

With Corollary 4.5 we extend standard results of convex analysis by first localizing to T_a. We thus generalize the following beyond qualification conditions: subdifferential additivity; normal cones of the intersection of convex sets; attained infimal convolution for the conjugate of a sum $(f+g)^*$; the subdifferential chain rule; Fenchel-Rockafellar duality; and KKT conditions.

The next result implies that when standard qualification conditions hold, the classical formulation of our generalizations is trivially recovered.

Proposition 5.1 (Recovery of the classical results)**.** *If the relative interiors of two convex sets $C, D \subseteq \mathbb{R}^n$ intersect, then*

$$C, D \subseteq T_a(C, D).$$

5.1 Subdifferential calculus

Corollary 5.1 (Subdifferential sum rule)**.** *For $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$ proper convex, $x \in \mathbb{R}^n$, and $T_a = T_a(\mathrm{dom}(f), \mathrm{dom}(g))$,*

$$\partial(f+g)(x) = \partial(f + \iota_{T_a})(x) + \partial(g + \iota_{T_a})(x).$$

Proof of Corollary 5.1. By Corollary 4.5 applied to $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$, the result follows by application of [32, Theorem 23.8] to the functions $f + \iota_{T_a}$ and $g + \iota_{T_a}$. □

Corollary 5.2 (Normal cone of the intersection of convex sets)**.** *For $C, D \subseteq \mathbb{R}^n$ convex, and $x \in C \cap D$, for $T_a = T_a(C, D)$,*

$$N_{C \cap D}(x) = N_{C \cap T_a}(x) + N_{D \cap T_a}(x).$$

Proof of Corollary 5.2. With Corollary 4.5 we can use [32, Corollary 23.8.1], from which we find that

$$N_{C \cap D}(x) = N_{(C \cap T_a) \cap (D \cap T_a)}(x) = N_{C \cap T_a}(x) + N_{D \cap T_a}(x).$$

□

In results that follow, we abuse notation by associating a matrix $A \in \mathbb{R}^{m \times n}$ with the corresponding linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. For a set $S \subseteq \mathbb{R}^n$, we denote by AS the set image $A(S)$, and by $A^{-1}(\cdot)$ the preimage of the linear map. We make no assumption on the invertibility of the matrix A.

Corollary 5.3 (Subdifferential chain rule)**.** *Let $g \in \mathbb{R}^m \to \bar{\mathbb{R}}$ be a proper convex function, and $A \in \mathbb{R}^{m \times n}$ a matrix. Then for $T := T(\mathrm{dom}(g), \mathrm{range}(A))$,*

$$\partial(g \circ A)(x) = A^T \partial(g + \iota_T)(Ax).$$

Proof of Corollary 5.3. Denote $T_a = T_a(\mathrm{dom}(g), \mathrm{range}(A))$. Note that $\forall x \in \mathbb{R}^n$,

$$(g \circ A)(x) = ([g + \iota_{T_a}] \circ A)(x),$$

because $T_a \supseteq \mathrm{dom}(g) \cap \mathrm{range}(A)$. So it suffices to calculate the subdifferential of the r.h.s. of the above equation.

If $\mathrm{dom}(g) \cap \mathrm{range}(A) = \emptyset$, then the result holds trivially because $g(Ax) \equiv \infty$, and $(g + \iota_{T_a})(Ax) \equiv \infty$ as well. Otherwise, by Corollary 4.5, $\mathrm{relint}\,\mathrm{dom}(g + \iota_{T_a}) \cap \mathrm{range}(A) \neq \emptyset$, which lets us apply [32, Theorem 23.9] to the composition $[g + \iota_{T_a}] \circ A$. Additionally, note that the only face of $\mathrm{range}(A)$ is itself, whence by Lemma 4.1 $\mathrm{range}(A) \subseteq T_a$. Since $0 \in \mathrm{range}(A)$, this in fact implies that $T_a = T$, and the result follows. □

5.2 Infimal convolution

Recall that for two convex functions $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$, the infimal convolution is

$$(f \,\square\, g)(z) := \inf_{x+y=z} f(x) + g(y).$$

In the following results, we denote by f^* the convex conjugate of f.

Corollary 5.4 (Attained infimal convolution)**.** *For two proper convex functions $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$, and $T_a := T_a(\mathrm{dom}(f), \mathrm{dom}(g))$,*

$$(f + g)^* = (f + \iota_{T_a})^* \square (g + \iota_{T_a})^*,$$

where the infimum in the infimal convolution is attained.

Proof of Corollary 5.4. Whenever $\mathrm{dom}(f) \cap \mathrm{dom}(g) = \emptyset$, we have that $T_a = \emptyset$, in which case the equality holds trivially, with the infimum attained at any $y \in \mathbb{R}^n$. Otherwise, by Corollary 4.5 the result follows from [32, Theorem 20.1]. □

5.3 Convex optimization

We next examine the following general convex program.

Problem 5.1 (Convex program)**.** Given lower semicontinuous proper convex functions $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, $g : \mathbb{R}^m \to \bar{\mathbb{R}}$, and a matrix $A \in \mathbb{R}^{m \times n}$,

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\, f(x) + g(Ax).$$

A standard regularity condition for this problem is that

$$A(\mathrm{relint}\,\mathrm{dom}(f)) \cap \mathrm{relint}\,\mathrm{dom}(g) \neq \emptyset,$$

which is a weakening of Slater's condition. A program with this property is called *strongly consistent* [32, p. 300].

The next result shows the sense in which Problem 5.1 can be localized to the joint facial subspace T_a, guaranteeing strong consistency whenever the program is feasible, while retaining all the crucial information of the original program.

Theorem 5.1 (Strong consistency of the localized program)**.** *For $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $g : \mathbb{R}^m \to \bar{\mathbb{R}}$ lower semicontinuous proper convex functions, a matrix $A \in \mathbb{R}^{n\times m}$, $T_a := T_a(A \operatorname{dom}(f), \operatorname{dom}(g))$, and any $x \in \mathbb{R}^n$,*

$$f(x) + g(Ax) = (f(x) + \iota_{A^{-1}(T_a)}(x)) + (g(x) + \iota_{T_a}(x)). \tag{6}$$

And whenever $\operatorname{dom}(f + g \circ A) \neq \emptyset$,

$$A \operatorname{relint} \operatorname{dom}(f + \iota_{A^{-1}(T_a)}) \cap \operatorname{relint} \operatorname{dom}(g + \iota_{T_a}) \neq \emptyset.$$

Proof of Theorem 5.1. By Corollary 4.1, $(A \operatorname{dom}(f)) \cap \operatorname{dom}(g) \subseteq T_a$, and therefore $\iota_{T_a}(Ax) = 0$ for all $x \in \operatorname{dom}(f) \cap A^{-1} \operatorname{dom}(g)$. Additionally,

$$\operatorname{dom}(f) \cap A^{-1} \operatorname{dom}(g) \subseteq A^{-1}(A \operatorname{dom}(f) \cap \operatorname{dom}(g)) \subseteq A^{-1} T_a,$$

which means that $\iota_{A^{-1}T_a}(x)$ is 0 on $\operatorname{dom}(f + g \circ A)$. And for any $x \notin \operatorname{dom}(f + g \circ A)$, it is easily checked that both sides of Equation (6) evaluate to ∞, so Equation (6) holds for all $x \in \mathbb{R}^n$.

For the second part of the result,

$$\begin{aligned}
&A \operatorname{relint} \operatorname{dom}(f + \iota_{A^{-1}(T_a)}) \cap \operatorname{relint}(\operatorname{dom}(g + \iota_{T_a})) && (7)\\
&= \operatorname{relint}(A \operatorname{dom}(f + \iota_{A^{-1}(T_a)})) \cap \operatorname{relint}(\operatorname{dom}(g + \iota_{T_a})) && (8)\\
&= \operatorname{relint}(A(\operatorname{dom}(f) \cap A^{-1}(T_a))) \cap \operatorname{relint}(\operatorname{dom}(g) \cap T_a) && (9)\\
&= \operatorname{relint}((A \operatorname{dom}(f)) \cap T_a) \cap \operatorname{relint}(\operatorname{dom}(g) \cap T_a) \neq \emptyset. && (10)
\end{aligned}$$

where Equation (8) holds by [32, Theorem 6.6]; Equation (10) uses the identity that $A(S_1 \cap A^{-1}(S_2)) = AS_1 \cap S_2$ for any sets $S_1 \subseteq \mathbb{R}^n$, $S_2 \subseteq \mathbb{R}^m$; and the last inequality is guaranteed by Corollary 4.5. □

We present an exact Fenchel–Rockafellar dual for Problem 5.1: a dual problem for which strong duality always holds and the dual attained.

Corollary 5.5 (Exact Fenchel-Rockafellar dual)**.** *For $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $g : \mathbb{R}^m \to \bar{\mathbb{R}}$ l.s.c. proper convex functions, and a matrix $A \in \mathbb{R}^{n\times m}$, let $T_a := T_a(A \operatorname{dom}(f), \operatorname{dom}(g))$. Then*

$$\inf_{x\in\mathbb{R}^n} [f(x) + g(Ax)] = -\min_{y\in\mathbb{R}^m} [(f + \iota_{A^{-1}(T_a)})^*(A^T y) + (g + \iota_{T_a})^*(-y)].$$

Proof of Corollary 5.5. The result follows from Theorem 5.1 with the use of [32, Corollary 31.2.1]. □

Definition 5.1 (Minimizer)**.** A *minimizer* of a given convex program is any point $x \in \mathbb{R}^n$ that attains the infimum of the program, provided that the infimum is finite.

Next, we provide a characterization of the minimizers of Problem 5.1, i.e., KKT optimality conditions. Our KKT condition holds without any further assumptions on the program; we do not require strong consistency.

Corollary 5.6 (KKT conditions)**.** *Consider Problem 5.1, and let* $T_a := T_a(A\,\mathrm{dom}(f), \mathrm{dom}(g))$*. A point* $x \in \mathbb{R}^n$ *is a minimizer of Problem 5.1 if, and only if, the following conditions hold:*

1. $Ax \in \partial(g + \iota_{T_a})^*(y)$.
2. $A^T y \in \partial(f + \iota_{A^{-1}(T_a)})(x)$.

Proof of Corollary 5.6. From Theorem 5.1, the result follows from application of [32, Corollary 31.3.1] to the functions $f + \iota_{A^{-1}(T_a)}$ and $g + \iota_{T_a}$. □

Infeasible case as convention In the above results, the domains of the function under consideration may not intersect, in which case $f + g \circ A \equiv \infty$. Intersection is required for the existence of a meaningful joint supporting subspace, so it might be reasonable to make this non-trivial intersection a requirement. This kind of requirement was dubbed a "universal qualification condition" in the context of conic programming [37]. We observe that such a condition is not a true limitation of the theory, because the behaviour outside this condition is known and simple. Therefore, we instead use a convention: whenever $C \cap D = \emptyset$, we let $T_a(C, D) := \emptyset$ as well. As a result, $f + \iota_{A^{-1}(T_a)} \equiv \infty$, and the equality in Corollary 5.5 trivially holds because both sides evaluate to ∞, with the value of ∞ attained in the dual.

5.3.1 Facial reduction

We derive two known facial reduction results, thereby showing that the joint supporting subspace is indeed a generalization of facial reduction.

Problem 5.2 (Conic program)**.** For $K \subseteq \mathbb{R}^n$ a closed convex cone, $A \in \mathbb{R}^{m\times n}$ a matrix, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$,

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \langle x, c\rangle \quad \text{s.t.} \quad Ax = b, \quad x \in K.$$

We apply Corollary 5.5 and Corollary 5.6 to this problem, yielding analogues of [37, Theorem 4.3] and [37, Corollary 4.4]. The derived results differ from those in [37] only in that our primal problem is their dual.

Corollary 5.7 (Exact conic dual)**.** *For* $A \in \mathbb{R}^{m\times n}$*,* $b \in \mathbb{R}^m$*,* $c \in \mathbb{R}^n$*, and a closed convex cone* $K \subseteq \mathbb{R}^n$*,*

$$\begin{aligned} &\inf\{\langle x, c\rangle \mid x \in \mathbb{R}^n, Ax = b, \ \textit{and } x \in K\} \\ &= \max\{\langle b, y\rangle \mid y \in \mathbb{R}^m \ \textit{and } A^T y - c \in F^\circ\}, \end{aligned}$$

where $F = K \cap A^{-1}(H_{AK}(\{b\}))$*.*

Proof of Corollary 5.7. We use Corollary 5.5 with $f(x) := \langle c, x\rangle + \iota_K(x)$, and $g := \iota_{\{b\}}$. We find

$$T := T(A\,\mathrm{dom}(f), \mathrm{dom}(g)) = T(AK, \{b\}) = H_{AK}(\{b\}),$$

where the last equality follows from Theorem 4.1. Since AK is a cone, any face includes the origin, and therefore $0 \in H_{AK}(\{b\}) = T_a$, which means that $T_a = T$. The functions in the dual program are

$$\begin{aligned} f(x) + \iota_{A^{-1}(T)}(x) &= \langle c, x\rangle + \iota_K(x) + \iota_{A^{-1}(H_{AK}(\{b\}))}(x) \\ &= \langle c, x\rangle + \iota_{K\cap A^{-1}(H_{AK}(\{b\}))}(x) \end{aligned}$$

and

$$g(Ax) + \iota_T(Ax) = \iota_{\{b\}}(Ax) + \iota_{H_{AK}(\{b\})}(Ax) = \iota_{\{b\}}(Ax).$$

Let $F = K \cap A^{-1}(H_{AK}(\{b\}))$, the subset of K selected by the joint supporting subspace. The set F is a cone, because $H_{AK}(\{b\}) \cap AK$ is a face of the cone AK by Definition 6.5 and Corollary 6.1, and F is therefore itself a cone.

Then with [32, Theorem 12.3, Theorem 14.1] we compute that

$$\begin{aligned} (f + \iota_{A^{-1}(T)})^*(A^T y) + (g + \iota_T)^*(-y) &= (\langle c, \cdot\rangle + \iota_F)^*(A^T y) + (\iota_{\{b\}})^*(-y) \\ &= \iota_{F^\circ}(A^T y - c) - \langle b, y\rangle. \end{aligned}$$

We thereby find the dual problem, for which, by Corollary 5.5, strong duality holds, and the dual is attained. □

Corollary 5.8 (Conic KKT conditions)**.** *A point $x \in \mathbb{R}^n$ is a minimizer of Problem 5.2 if, and only if, there exists some $y \in \mathbb{R}^m$ such that all of the following hold.*

1. $Ax = b$.
2. $A^T y - c \in F^\circ$.
3. $\langle x, A^T y - c\rangle = 0$.

Proof of Corollary 5.8. We apply Corollary 5.6 to the case where $f(x) := \langle c, x\rangle + \iota_K(x)$ and $g := \iota_{\{b\}}$. As in the proof of Corollary 5.7, we find that $f(x) + \iota_{A^{-1}(T_a)} = \langle c, x\rangle + \iota_F(x)$ and $g(\bar{y}) + \iota_{T_a}(\bar{y}) = \iota_{\{b\}}(\bar{y})$.

Then $(g+\iota_{T_a})^*(y) = \langle b, y\rangle$, and the first KKT condition, that $\partial(g+\iota_{T_a})^*(y) = \{b\}$ reduces to $Ax = b$. The second KKT condition is

$$\begin{aligned} A^T y &\in \partial(f + \iota_{A^{-1}(T_a)})(x) \\ &= \partial(\langle c, \cdot\rangle + \iota_F)(x) = c + N_F(x). \end{aligned}$$

But since $N_F(x) = F^\circ \cap x^\perp$, this second condition holds if, and only if, both $x \perp A^T y - c$ and $A^T y - c \in F^\circ$. □

6 A framework for local face lattices

Inspired by the lexicographic characterization of convex faces of [21], we develop a framework describing the lattice of convex faces at a point $x \in C$, which we will then use to show the results of Section 4. We begin with a characterization of faces of convex sets. In the next few results, we denote X for a real vector space.

Theorem 6.1 (Characterization of convex faces by supporting subspaces)**.** *Let $C \subseteq X$ be a convex set. A subset $F \subseteq C$ is a face of C if, and only if the two following conditions hold.*

1. $\operatorname{aff}(F) \cap C = F$,
2. $C \setminus F$ *is convex.*

Proof of Theorem 6.1. Let F be a face. We argue by contradiction that $\operatorname{aff}(F) \cap [C \setminus F] = \emptyset$; let us assume that there is some $z \in \operatorname{aff}(F) \cap [C \setminus F]$.

If F is a singleton, then $\operatorname{aff}(F) = F$, and $\operatorname{aff}(F) \cap [C \setminus F] = \emptyset$. Otherwise, we may assume that $0 \in F \setminus \{z\}$ without loss of generality, because the faces of a convex set are invariant under translation. With this assumption, $\operatorname{aff}(F) = \operatorname{span}(F) = \operatorname{cone}(F - F)$. Then $z \in \operatorname{cone}(F - F)$, so there are $y_1, y_2 \in F$ such that $z = \alpha(y_1 - y_2)$ for some $\alpha \geq 0$. We can rule out the possibility that $z \in [y_1, y_2]$, because by convexity of F it would follow that $z \in F$. The remaining possibility is that $y_1 \in [y_2, z[$. Further, note that $y_1 \neq y_2$, because otherwise $z = 0$, which disagrees with our assumption that $z \in F \setminus \{0\}$. Therefore, $y_1 \in]y_2, z[$. The line segment $[y_2, z]$ has a point $y_1 \in]y_2, z[\cap F$. Since $y_2, z \in C$, the fact that F is a face implies that z must then be in F, in contradiction with our definition of z. We have therefore shown the claim.

We next argue that $C \setminus F$ is convex. $\forall y, z \in C \setminus F$, any $x \in]y, z[$ is in C by convexity of C. Also, x is not in F because F is a face. So any $x \in]y, z[$ is also in $C \setminus F$, which means that $C \setminus F$ is convex.

We now show the converse: let $F \subseteq C$ be such that $C \setminus F$ is convex and $\operatorname{aff}(F) \cap C = F$, we show that F is a face of C.

Take any $y, z \in C$, $x \in F \cap]y, z[$. If $y \in \operatorname{aff}(F)$ then $z \in \operatorname{aff}(F)$ as well, because $z = \alpha(x - y) + y$ for some $\alpha > 0$. So either z, y are both in $\operatorname{aff}(F)$, or neither are. But if neither are, then $z, y \in C \setminus \operatorname{aff}(F)$, and by convexity of $C \setminus \operatorname{aff}(F)$, $x \in C \setminus \operatorname{aff}(F)$ also, which is a contradiction with $x \in F$. Therefore, z, y are both in F, which shows that F is a face. $\square$

Theorem 6.1 motivates the next mathematical object.

Definition 6.1 (Supporting subspace)**.** Let $C \subseteq X$ be convex. A supporting subspace $L \subseteq X$ of C is an affine subspace such that $C \setminus L$ is convex.

Though supporting subspaces are affine subspaces, we omit the adjective "affine" for brevity, mirroring the term "supporting hyperplane". The notion of supporting subspaces seems to be surprisingly sparse in the literature; the only previous appearance of this notion the author could find is in [7, Exercise 5.4]. This idea is of central importance to this paper; it allows us to shift our focus away from the faces themselves to the comparatively simpler affine subspaces which select them through intersections with the convex set. The next result describes the relationship between supporting subspaces and faces.

Corollary 6.1 (Relation of supporting subspaces to faces)**.** *Let $C \subseteq X$ be convex and $S \subseteq C$ non-empty.*

1. *For any affine subspace $L \subseteq X$,*

$$L \text{ is a supporting subspace} \iff L \cap C \text{ is a face.}$$

2. *For any subset $F \subseteq C$,*

$$F \text{ is a face of } C \iff \exists L \text{ a supporting subspace s.t.} F = L \cap C.$$

Proof of Corollary 6.1. 1), ($\Longrightarrow$) we show that if L is a supporting subspace, then $F := C \cap L$ satisfies both conditions of Theorem 6.1:

1. That $\operatorname{aff}(F) \cap C = F$: note that $\operatorname{aff}(F) \subseteq L$ because $F \subseteq L$, and so $\operatorname{aff}(F) \cap C \subseteq L \cap C =: F$. On the other hand, $L \cap C = (L \cap C) \cap C \subseteq \operatorname{aff}(L \cap C) \cap C$, and so $F \subseteq \operatorname{aff}(F) \cap C$.

2. That $C \setminus F$ is convex: note that

$$C \setminus F = C \setminus [C \cap L] = C \setminus L,$$

which is convex by definition of L as a supporting subspace.

1), ($\Longleftarrow$) Since $L \cap C$ is a face, $C \setminus (L \cap C)$ is convex by Theorem 6.1, and therefore $C \setminus L$ is convex, since it is the same set.

2), ($\Longrightarrow$) If F is a face, then we argue that $\operatorname{aff}(F)$ is a suitable supporting subspace. Indeed, by Theorem 6.1 we know that $F = C \cap \operatorname{aff}(F)$ and that $C \setminus F$ is convex. Note that $C \setminus F = C \setminus (C \cap \operatorname{aff}(F)) = C \setminus \operatorname{aff}(F)$, and therefore $C \setminus \operatorname{aff}(F)$ is convex, while $F = C \setminus \operatorname{aff}(F)$, which shows that $\operatorname{aff}(F)$ is a supporting subspace that selects F by intersection.

2), ($\Longleftarrow$) We are given a supporting subspace L such that $F = L \cap C$. Since $\operatorname{aff}(F) \subseteq L$, $\operatorname{aff}(F) \cap C \subseteq L \cap C$. On the other hand, $L \cap C = F \subseteq \operatorname{aff}(F) \cap C$ because $F \subseteq C$. Therefore, $F = \operatorname{aff}(F) \cap C$, the first of the two properties in Theorem 6.1. For the second property, $C \setminus F = C \setminus (C \cap L) = C \setminus L$, and therefore $C \setminus F$ is convex. □

In this work, we focus on "local" supporting subspaces; the subset of supporting subspaces that contain a point $x \in C$. These subsets have some additional nice algebraic properties. We find it useful to generalize this idea of "locality" to supporting subspaces that are localized at a set $S \subseteq C$ instead of only at points, i.e., we consider all supporting subspaces of C that also contain S. To emphasize this connection, we say that a supporting subspace is "at S" to describe its containment of S. The reader should keep in mind the important special case of S being a singleton $S = \{x\}$, for which the results are localized at a point.

Any supporting subspace L at S has its affine offset determined by S. Recall that $L - L$ is the tangent space of L, i.e., the translation of L that becomes a (non-affine) subspace. With this, it becomes clear that $L = (L - L) + S$. We can always deduce the correct affine offset from S alone. Therefore, we find it convenient to describe the set of supporting subspaces that contain S through their tangent spaces. We henceforth call the tangent spaces of supporting subspaces "supporting subspaces" as well, and whether we mean the affine subspace or its tangent space will be clear from context.

Definition 6.2 (Supporting subspace at S)**.** Let $C \subseteq X$ be a convex set and $S \subseteq X$ a non-empty subset. We call a subspace $U \subseteq X$ a *supporting subspace* of C at S if:

1. $S - S \subseteq U$
2. $U + S$ is an (affine) supporting subspace of C as defined in Definition 6.1.

We define $\mathcal{F}(C; S)$ to be the set of all subspaces U satisfying both conditions above for fixed C and S.

When S is the singleton $\{x\}$, the requirement that $S - S \subseteq U$ is trivially satisfied. When S is not a singleton, this assumption guarantees that $\forall s_1, s_2 \in S, U + s_1 = U + s_2$. This implies that $\forall s \in S, U + s = U + S$, and so the Minkowski sum $U + S$ is a simple affine offset of U, for which $S \subseteq U + S$. In other words, the supporting subspace $U + S$ contains S.

Next, we introduce a generalization of the notion of a normal cone. A normal cone of a convex set $C \subseteq \mathbb{R}^n$ is defined as

$$N_C(x) := \{z \in \mathbb{R}^n \mid \forall y \in C \langle y - x, z\rangle \leq 0\}.$$

When C is a cone and $x = 0$, the normal cone $N_C(0) =: C^\circ$ is called the *polar* of C. The polar is a dual operation, in the sense that

$$C^{\circ\circ} = \mathrm{cl}(C).$$

We can see that a normal cone is a dual object which "describes" the original cone only up to its closure.

We require a dual object which has sufficient descriptive power to go beyond the closure, by encoding information about the boundary. To this end, we introduce the notion of a *nested normal cone*. Let $C \subseteq \mathbb{R}^n$ be convex and $U \subseteq \mathbb{R}^n$ be a subspace, then $\forall x \in \mathbb{R}^n$,

$$N_C^U(x) := N_{C \cap U}(x) \cap U.$$

We would also like for our normal cones to located *at a set* $S \subseteq C$, similarly to supporting subspaces.

$$N_C(S) := \{v \in \mathbb{R}^n \mid \forall s \in S, c \in C, \langle c - s, v\rangle \leq 0\}.$$

The hyperplanes corresponding to the vectors in this normal cone are supporting hyperplanes of C that contain S. Note also that $N_C(S) = \bigcap_{s\in S} N_C(s)$. The idea of normal cones that are located at sets appears in the earlier work [18], where normal cones are at faces of polytopes, instead at points. The normal cone at a set has an elegant connection to the notion of relative interior of that set: the normal cone at any point in the relative interior of a convex set $S \subseteq C$ will match the normal cone of S.

Combining the two generalizations, we get the following.

Definition 6.3 (Nested normals)**.** Let $C \subseteq \mathbb{R}^n$ be a convex set, and S a non-empty subset of C. Then for a subspace $U \subseteq \mathbb{R}^n$ such that $S - S \subseteq U$, and with $C' := C \cap (U + S)$,

$$N_C^U(S) := \{v \in U \mid \forall c \in C', s \in S, \langle c - s, v\rangle \leq 0\}$$

is a *nested normal cone* of C at S, and we label any $v \in N_C^U(S)$ to be a *nested normal* of C at S in U.

The idea of normal cone defined at a set has an implicit relation with the notion of relative interior.

Lemma 6.1 (Normal cones in the relative interior are representative)**.** *Let $C \subseteq \mathbb{R}^n$ be convex and $S \subseteq C$ a non-empty convex set. Let $U \subseteq \mathbb{R}^n$ be a subspace, and $x \in \operatorname{relint}(S)$. Then*

$$N_C^U(S) = N_C^U(x).$$

Proof of Lemma 6.1. That $N_C^U(S) \subseteq N_C^U(x)$ follows from simple inclusion $x \in S$.

For the containment ($\supseteq$), WLOG let $U = \mathbb{R}^n$. We can make this assumption because the normal cones are invariant under the translation $U + S \to U$, and so we have an equivalent normal cone inside the subspace U. Since U is a finite dimensional vector space, it is isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Let $v \in N_C(x)$. Then $\forall s \in S, \langle v, s\rangle \leq \langle v, x\rangle$ because $S \subseteq C$. We show that this inequality is in fact always an equality. Arguing by contradiction, assume that there is an $s \in S$ such that $\langle v, s\rangle < \langle v, x\rangle$. Then since $x \in \operatorname{relint}(S)$, there is a $\varepsilon > 0$ such that $(\varepsilon B + x) \cap \operatorname{aff}(S) \subseteq S$ for B the unit ball in $\mathbb{R}^n$. There is a $\lambda > 0$ such that $s' := x + \lambda(s - x) \in \varepsilon B + x$. Then note that $s' \in (\varepsilon B + x) \cap \operatorname{aff}(S) \subseteq S$, and

$$\langle v, s'\rangle = \langle v, x\rangle - \lambda\langle v, s - x\rangle.$$

The second term is strictly positive, therefore $\langle v, s'\rangle > \langle v, x\rangle$. But this is a contradiction with $v \in N_C(x)$. Therefore, we find that $\forall s \in S$, $\langle v, s\rangle = \langle v, x\rangle$. Then $\forall c \in C, s \in S$ $\langle c - s, v\rangle = \langle c - x, v\rangle \leq 0$, and therefore $v \in N_C(S)$. $\square$

The nested normals *at a set* are orthogonal with that set.

Lemma 6.2 (Normal cone everywhere is the orthogonal subspace)**.** *Let $C \subseteq \mathbb{R}^n$ be a convex set, and $U \subseteq \mathbb{R}^n$ a subspace. Then for any non-empty $S \subseteq C$ such that $S - S \subseteq U$,*

$$N_C^U(S) \subseteq (S - S)^\perp \cap U.$$

Further, when $S = C \cap (U + S)$, the sets are equal.

Proof of Lemma 6.2. Let $y \in N_C^U(S)$. By Definition 6.3, $\forall z, x \in S$, since $x \in S$ and $z \in C$, $\langle x - z, y\rangle \leq 0$, and since $x \in C$ and $z \in S$ $\langle x - z, y\rangle \geq 0$. Then $y \perp S - S$, which shows the inclusion.

Next we show the containment ($\supseteq$), in the case that $S = C \cap (U + S)$. Let $y \in (S - S)^\perp \cap U$. For any $x \in S, z \in C \cap (U + S)$ we need to show that $\langle z - x, y\rangle \leq 0$. Note that $z \in S$, so $x - z \in S - S$, and by definition of y, $\langle z - x, y\rangle = 0$, which proves the statement. $\square$

As previously discussed, one motivating property of the nested normals is that they can encode information about the boundary of convex sets. We make this idea concrete.

The tangent cone of a convex set C at $x \in C$ is the smallest closed cone T such that $C \subseteq x + T$. We can think of this cone as a local description of C around x. The normal cone is the polar of the tangent cone, and therefore "describes" the convex set C around x to the same extent as the tangent cone. In contrast, nested normals in supporting subspaces of C at x describe the *cone of feasible directions*.

Definition 6.4 (Cone of feasible directions)**.** For a convex set $C \in \mathbb{R}^n$ with $x \in C$, we define the *cone of feasible directions* of C at x to be

$$D_C(x) := \text{cone}(C - x).$$

The closure of the cone of feasible directions is the tangent cone. The difference is that the cone of feasible directions captures additional information about the boundary of C around x. The following result shows that any convex cone (including the cone of feasible directions at any point) can be recovered from the nested normals in supporting subspaces.

Theorem 6.2 (Local polar without closure)**.** *Given any convex cone T with $0 \in T$,*

$$T = \{x \in \mathbb{R}^n \mid \forall U \in \mathcal{F}(T; 0), v \in N_T^U(0), x \in U \implies \langle x, v \rangle \leq 0\}.$$

Proof of Theorem 6.2. Let $T' := \{x \in \mathbb{R}^n \mid \forall U \in \mathcal{F}(T; 0), v \in N_T^U(0), x \in U \implies \langle x, v \rangle \leq 0\}$. By the definition of nested normals, $T \subseteq T'$. For the containment ($\supseteq$), we argue by contradiction: we assume that there is some $x \in T' \setminus T$. If $x \in \text{int}(T^c)$, then x is strictly separated from T by a supporting hyperplane of T (e.g., see [32, Theorem 11.3]). Any such hyperplane induces a nested normal of the form $v \in N_T^{\mathbb{R}^n}(0)$, for which $v^\perp$ is the hyperplane, and $\langle v, x \rangle > 0$. But then $x \notin T'$, and so $x \notin \text{int}(T^c)$. Since we assumed that $x \notin T$, the only possibility is that $x \in \partial T$. Then x is contained in a supporting hyperplane, so there is some nested normal $v \in N_T^{\mathbb{R}^n}(0)$ such that $\langle v, x \rangle = 0$. Let $T_1 = T \cap v^\perp$, and narrow the ambient space to the subspace T_1. Inside T_1, we repeat the same argument, and keep narrowing the ambient space in this manner. This process must eventually reduce the ambient space to span(x), because $\mathbb{R}^n$ is finite-dimensional. In this one-dimensional space, with the narrowed cone $T_\ell = T \cap \text{span}(x)$, note that T_ℓ is a closed cone. Therefore, either $x \in T_\ell$ or $x \in \text{int}(T_\ell^c)$, both of which yields a contradiction, from which the result follows. $\square$

Another important property of nested normals is that they describe the containment structure of supporting subspaces, which is closely related to the lattice structure of the faces. We make this precise in the next two results.

Theorem 6.3 (Nested normals induce supporting subspaces)**.** *Let $C \subseteq \mathbb{R}^n$ be convex, $S \subseteq C$ non-empty, $U \subseteq \mathbb{R}^n$ a subspace, and $v \in N_C^U(S)$. Let $V \in \mathcal{F}(C;S)$ be such that $V \subseteq U$. Then*

$$V \cap v^\perp \in \mathcal{F}(C;S).$$

Proof of Theorem 6.3. First, note that if $v = 0$, then $V \cap v^\perp = V$, and there result holds. We now consider the case where $v \neq 0$.

By Lemma 6.2, $v \perp S - S$, and therefore $S - S \subseteq V \cap v^\perp$.

We make two observations. First, since $v \in N_C^U(S)$ and $V \subseteq U$, $\forall y \in C \cap (V+S)$, $\forall s \in S$, we have that $\langle y, v\rangle \leq \langle s, v\rangle$. Second, this inequality is tight precisely for those values of y that are also contained in $v^\perp + S$, i.e.,

$$\begin{aligned}&\{y \in C \cap (V+S) \mid \forall s \in S, \langle y, v\rangle = \langle s, v\rangle\}\\=&((V+S)\cap C) \cap (v^\perp + S) = (V \cap v^\perp + S) \cap C.\end{aligned}$$

These two facts imply that

$$(V \cap v^\perp + S) \cap C = \operatorname*{arg\,max}_{y \in C \cap (V+S)} \langle y, v\rangle.$$

The r.h.s. is an exposed face of the face $C \cap (V+S)$, so it is itself a face, which by Corollary 6.1 implies that $V \cap v^\perp \in \mathcal{F}(C;S)$. $\square$

We next show that all supporting subspaces are "reached" by nested normals in supporting subspaces.

Theorem 6.4 (Completeness of the composition of nested normals)**.** *Let $C \subseteq \mathbb{R}^n$ be convex, $S \subseteq C$ non-empty, and $V, U \in \mathcal{F}(C;S)$ such that $U \subseteq V$. Then there is a sequence $V_0, \ldots, V_k \in \mathcal{F}(C;S)$ with $k = \dim(V) - \dim(U)$ such that $V_0 = V$, $V_k = U$, and for each $i \in \{1, \ldots, k\}$ there is a $v_i \in N_C^{V_{i-1}}(S) \setminus \{0\}$ such that $V_i = V_{i-1} \cap v_i^\perp$.*

This result is similar to the lexicographic characterization of the faces of convex sets in [21, Proposition 6], which states that every face can be obtained by a sequence of faces where each face is an exposed face of the preceding face. Let us state a technical lemma that we will need for the proof.

Lemma 6.3 (Affine slices preserve supporting subspaces)**.** *Let $C \subseteq \mathbb{R}^n$ be convex and $S \subseteq C$ non-empty. Let $U \in \mathcal{F}(C;S)$, and V any subspace such that $S - S \subseteq V$. Then $U \cap V \in \mathcal{F}(C \cap (V+S); S)$.*

Proof of Lemma 6.3. Since $C \setminus (U+S)$ is convex, $(V+S) \cap (C \setminus (U+S))$ is also convex. We can write this set as $[(V+S)\cap C] \setminus [(V+S) \cap (U+S)]$, which is the same as $((V+S)\cap C) \setminus ((V \cap U) + S)$, because $S - S \in V \cap U$. Since this set is convex, and $S - S \subseteq V \cap U$, we find that $V \cap U \in \mathcal{F}((V+S) \cap C; S)$. $\square$

Proof of Theorem 6.4. It suffices to show the following statement: given any $U, V \in \mathcal{F}(C;S)$ such that $V \supsetneq U$, there exists a $v \in N_C^V(S) \setminus \{0\}$ such that $v \perp U$.

If this statement is shown to hold, applying it recursively generates the sequence of supporting subspaces $V_1, \dots, V_k$ in a finite number of steps. Note that the dimension of the supporting subspaces decreases by one at each iteration while always containing U, which implies that $U \subseteq V_k$ with $\dim(U) = \dim(V_k)$, and therefore $U = V_k$.

We now show the above statement. Let $C' = C \cap (V + S)$, $\bar{C} = \Pi_{U^\perp} C'$, and $\bar{U} = \Pi_{U^\perp}(U + S)$. Note that $\bar{C}$ is a convex set and $\bar{U}$ a singleton $\{x\} \subseteq \bar{C}$. We argue that $\bar{U}$ is a face of $\bar{C}$. Arguing by contradiction, we hypothesize the $\bar{U}$ is not a face of $\bar{C}$, in which case there are $\bar{y}, \bar{z} \in \bar{C} \backslash \bar{U}$ such that $x = \lambda \bar{y} + (1-\lambda)\bar{z}$ for $\lambda \in (0, 1)$. This implies that there are elements $y, z \in C'$ such that $\bar{y} = \Pi_{U^\perp} y$ and $\bar{z} = \Pi_{U^\perp} z$, with $w = \lambda y + (1 - \lambda) z$. Note that the line segment $[y, z]$ has endpoints $y, z \in C' \setminus (U + S)$ with a point $w \in]y, z[$ that is also contained in $C' \cap (U + S)$. This implies that $C' \cap (U + S)$ is not a face of C', which, by Corollary 6.1, means that $U \notin \mathcal{F}(C'; S)$. This is a contradiction because in fact, $U \in \mathcal{F}(C'; S)$, as we now show. We know that $U \in \mathcal{F}(C; S)$, so by Lemma 6.3, $U = U \cap V \in \mathcal{F}(C \cap (V + S); S)$, i.e., $U \in \mathcal{F}(C'; S)$. Therefore, we indeed have a contradiction, from which we find that $\bar{U}$ is a face of $\bar{C}$.

Note that $\Pi_{U^\perp}(V + S) \subseteq V + S$. This becomes apparent from the following equality: $V + S = (V + S) + U = \Pi_{U^\perp}(V + S) + U$, where the first equality holds because $U \subseteq V$ implies that $V = V + U$.

Then since $U + S, C \cap (V + S) \subseteq V + S$, we find that $\bar{U}, \bar{C} \subseteq (V + S) \cap U^\perp$. Since $\bar{U}$ is a face of $\bar{C}$, there exists a supporting hyperplane H of $\bar{C}$, where the hyperplane is relative to the affine subspace $(V + S) \cap U^\perp$, such that $\bar{U} \subseteq H$. Indeed, if there is no such hyperplane H, $\bar{U}$ would then be a singleton in the interior of $\bar{C}$ (where the interior is relative to $(V+S) \cap U^\perp$, i.e., with the induced topology in $(V + S) \cap U^\perp$).

Take $\bar{v} \in V \cap U^\perp$ to be the unit normal to H that is also in $N_{\bar{C}}^{V \cap U^\perp}(x)$. Then note that $\bar{v} \in N_C^V(S) \cap U^\perp$, which is the vector we had to find. □

Together, Theorem 6.3 and Theorem 6.4 enables a powerful tool that is central to our proofs: we can perform structural induction on the tree of supporting subspaces, using only nested normals. Indeed, the set of supporting subspaces forms a tree, where the nodes are the supporting subspaces, the root is $\mathbb{R}^n$ (note that it is always a supporting subspace, and that it contains any other supporting subspace), and every edge of the tree is associated with a nested normal. Crucially, Theorem 6.4 shows that starting from the root, and generating the tree by using the nested normals, we attain all supporting subspaces. Induction on this tree then takes the following form.

Corollary 6.2 (Induction by nested normal). *Let C be a convex set, $S \subseteq C$ a non-empty subset, and a statement $A : \mathcal{F}(C; S) \to \{true, false\}$. If the following statements hold:*

1. $A(\mathbb{R}^n)$,
2. $\forall U \in \mathcal{F}(C; S)$, $v \in N_C^U(S)$, $A(U) \implies A(U \cap v^\perp)$, *then it follows that*

$$\forall V \in \mathcal{F}(C; S),\ A(V).$$

Proof of Corollary 6.2. Consider any $U \in \mathcal{F}(C; S)$, and let the statement A satisfy both stated conditions. Then by Theorem 6.4, we have supporting subspaces $V_0, \ldots, V_k \in \mathcal{F}(C; S)$ and nested normals $v_1, \ldots, v_k$ as in Theorem 6.4. $A(V_0) =$ true because $V_0 = \mathbb{R}^n$. Also, $\forall i \in [k]$, $A(V_{i-1}) \implies A(V_i)$ because of the existence of the nested normal $v_i \in N_C^{V_{i-1}}(S) \setminus \{0\}$ with the property that $V_i = V_{i-1} \cap v_i^\perp$. Then $A(U)$ holds by classical induction, because $U = V_k$. □

6.1 Minimal supporting subspace

Recall that the smallest face of a convex set C containing some $x \in C$ is $F_C(x)$, the face of C *generated* by x. It is the only face with x in its relative interior. In this section, we develop an analogous notion for supporting subspaces, and characterize this supporting subspace by using nested normals.

Definition 6.5 (The generated supporting subspace)**.** Given a convex set $C \subseteq \mathbb{R}^n$ with $S \subseteq C$ non-empty, we define the *generated supporting subspace* $H_C(S)$ of C, generated by S, to be the supporting subspace that is contained in every other supporting subspace in $\mathcal{F}(C; S)$.

We show existence and uniqueness for the above definition, starting with a lemma.

Lemma 6.4 (Supporting subspaces are closed under intersections)**.** *For $C \subseteq \mathbb{R}^n$ convex and $S \subseteq C$ non-empty, let $U, V \in \mathcal{F}(C; S)$. Then $U \cap V \in \mathcal{F}(C; S)$.*

Proof of Lemma 6.4. The subspace $U \cap V$ is such that

$$(U \cap V + S) \cap C = ((U + S) \cap C) \cap ((V + S) \cap C).$$

The r.h.s. is the intersection of two convex faces of C. Their intersection is then also a face (this is immediate from Definition 3.1). By Theorem 6.1, $C \setminus [(U \cap V + S) \cap C]$ is convex. The convexity of this set, which we can write as $C \setminus ((U \cap V) + S)$, implies that $U \cap V \in \mathcal{F}(C; S)$. □

Proposition 6.1. *The generated supporting subspace of Definition 6.5 exists and is unique.*

Proof of Proposition 6.1. For existence, starting at any supporting subspace $U \in \mathcal{F}(C; S)$, either U contains no other supporting subspace in $\mathcal{F}(C; S)$, or we have a $U' \in \mathcal{F}(C; S)$ such that $U' \subsetneq U$. Repeating this argument, we eventually find a supporting subspace that contains no other, because the dimension is reduced at every step, and the ambient dimension is finite.

For uniqueness, let $V_1, V_2 \in \mathcal{F}(C; S)$ be such that they properly contain no other supporting subspace in $\mathcal{F}(C; S)$. Then by Lemma 6.4, $V_1 \cap V_2 \in \mathcal{F}(C; S)$ is properly contained in both V_1 and V_2, which is a contradiction. □

As might be expected, the generated supporting subspace is closely related to the generated face.

Lemma 6.5 (Relation between the generated face and the generated supporting subspace)**.** *With $C \subseteq \mathbb{R}^n$ convex and $S \subseteq C$ non-empty,*

1. $F_C(S) = C \cap (H_C(S) + S)$.
2. $H_C(S) = \operatorname{span}(F_C(S) - F_C(S))$.

Proof of Lemma 6.5. By Theorem 6.1, $H_C(S)$ being the smallest supporting subspace implies that $C \cap (H_C(S)+S)$ is the smallest face containing S, because any properly contained, smaller face would have its own supporting subspace (by Corollary 6.1), which would have to be properly contained in $H_C(S)$.

For the second statement, $\operatorname{span}(F_C(S) - F_C(S))$ is a supporting subspace because $\operatorname{span}(F_C(S)-F_C(S))+S = \operatorname{aff}(F_C(S))$, which is a supporting subspace, as implied by Theorem 6.1. Further, $\operatorname{span}(F_C(S) - F_C(S))$ is contained in any other supporting subspace in $\mathcal{F}(C;S)$, because given $U \in \mathcal{F}(C;S)$ such that $U \subsetneq \operatorname{span}(F_C(S) - F_C(S))$, we would then have $C \cap (U + S)$ as a face of C containing S that is properly contained in $F_C(S)$, in contradiction with the definition of $F_C(S)$. $\square$

We can now restate Definition 4.1 with specialized terms.

Definition 6.6 (Reformulated definition of the joint supporting subspace)**.** The *joint supporting subspace* of two convex sets $C, D \subseteq \mathbb{R}^n$, as defined in Definition 4.1, is

$$T(C, D) := H_{C-D}(0).$$

The next result characterizes generated supporting subspaces in terms of nested normals.

Theorem 6.5 (The generated supporting subspace is the only supporting subspace with no nested normals)**.** *Given a convex set $C \subseteq \mathbb{R}^n$ with $S \subseteq C$ non-empty, the* generated supporting subspace $H_C(S)$ *is the only supporting subspace of C at S such that $N_C^{H_C(S)}(S) = \{0\}$.*

Proof of Theorem 6.5. The generated supporting subspace does not contain any non-trivial nested normals, because any non-trivial nested normal $v \in N_C^{H(C)}(S) \setminus \{0\}$ induces the supporting subspace $H_C(S) \cap v^{\perp} \subsetneq H_C(S)$ by Theorem 6.3.

For any supporting subspace $V \in \mathcal{F}(C;S)$ such that $N_C^V(S) = \{0\}$, we argue that $V = H_C(S)$. Note that $H_C(S) \subseteq V$ by Definition 6.5. The inclusion cannot be proper, because otherwise, by Theorem 6.4, there would be a nontrivial nested normal $v \in N_C^V(S) \setminus \{0\}$. $\square$

6.2 Nested normals of a difference of sets

In this section, we show the following important result.

Theorem 6.6 (Characterization of the nested normals of a difference of sets)**.** *Let $C, D \subseteq \mathbb{R}^n$ be convex with $S \subseteq C \cap D$ non-empty, and $U \in \mathcal{F}(C - D; 0)$. Then $U \in \mathcal{F}(C;S) \cap \mathcal{F}(D;S)$, and*

$$N_{C-D}^U(0) = N_C^U(S) \cap (-N_D^U(S)).$$

To show Theorem 6.6, we require the following lemma.

Lemma 6.6 (Normal cone of the difference of convex sets)**.** *Given convex sets $C, D \subseteq \mathbb{R}^n$ with $S \subseteq C \cap D$ non-empty,*

$$N_{C-D}(0) = N_C(S) \cap (-N_D(S)).$$

Proof of Lemma 6.6. We start with the inclusion ($\subseteq$). Any $v \in N_{C-D}(0)$ has the property that $\forall c \in C, d \in D$, $\langle v, (c-d) - 0\rangle \leq 0$. Then $\forall c \in C, s \in S$, $\langle v, c - s\rangle \leq 0$ because $s \in D$, and therefore $v \in N_C(S)$. By a similar argument, $v \in -N_D(S)$, which shows the inclusion.

For the containment ($\supseteq$), fix some $s \in S$. let $v \in N_C(S) \cap (-N_D(S))$. Then $\forall c \in C, d \in D$

$$\langle v, c - d\rangle = \langle v, c - s\rangle + \langle v, s - d\rangle \leq 0,$$

and so $v \in N_{C-D}(0)$. $\square$

Proof of Theorem 6.6. We argue by structured induction over the supporting subspaces of $C - D$, as described in Corollary 6.2. Fixing some $S \subseteq C \cap D$, the statement $A : \mathcal{F}(C - D; 0) \to \{\text{true}, \text{false}\}$ that we show is the following: $A(U)$ is true if, and only if, all of the following hold.

1. $U \in \mathcal{F}(C; S) \cap \mathcal{F}(D; S)$.
2. $N^U_{C-D}(0) = N^U_C(S) \cap (-N^U_D(S))$.
3. $C \cap (U + S) - D \cap (U + S) = (C - D) \cap U$. The additional point 3) has a supportive role in this proof.

The base case, where $U = \mathbb{R}^n$, holds because $\mathbb{R}^n$ is always a supporting subspace. Point 2) is given by Lemma 6.6, and point 3) holds trivially.

For the inductive step, let $V \in \mathcal{F}(C - D; 0)$ be such that $A(V)$ is true. Let $v \in N^V_{C-D}(0) \setminus \{0\}$, and $U = V \cap v^\perp$. In the rest of the proof, we show that $A(U)$ is true. Note that by point 2) of $A(V)$, $v \in N^V_C(S) \cap (-N^V_D(S))$, which by Theorem 6.3 implies that $U \in \mathcal{F}(C; S) \cap \mathcal{F}(D; S)$, and this shows point 1) of $A(U)$.

We next show that for $C' = C \cap (V + S)$, $D' = D \cap (V + S)$,

$$(C' - D') \cap U = (C' \cap (U + S)) - (D' \cap (U + S)). \tag{11}$$

We start with the inclusion ($\subseteq$): Take $z \in C'$, $y \in D'$ with $z - y \in U$. Then since $U \subseteq v^\perp$, $\langle z - y, v\rangle = 0$, and therefore $\forall s \in S$, $\langle z - s, v\rangle - \langle y - s, v\rangle = 0$. That $\langle z - s, v\rangle \leq 0$ follows from $v \in N^V_C(S)$ since $z \in C' \subseteq C$ and $z - s \in V$, and similarly, $\langle y - s, v\rangle \geq 0$ because $v \in N^V_D(S)$. But then the only way that $\langle z - s, v\rangle - \langle y - s, v\rangle = 0$ is if $\langle z - s, v\rangle = 0$ and $\langle y - s, v\rangle = 0$ individually. It follows that $z, y \in U + S$, and so the inclusion holds. The containment ($\supseteq$) is trivial, so we have shown Equation (11).

By point 3) of $A(V)$, $C' - D' = (C - D) \cap V$. Also, since $U \subseteq V$, $C' \cap (U + S)$ $= C \cap (U + S)$, and similarly for D. Applying these identities on both sides of Equation (11), we find that

$$(C - D) \cap U = C \cap (U + S) - D \cap (U + S).$$

This shows that point 3) of $A(U)$ is true.

To show point 2) of $A(U)$, we compute

$$\begin{aligned} N_C^U(S) \cap (-N_D^U(S)) &= [N_{C\cap(U+S)}(S) \cap (-N_{D\cap(U+S)}(S))] \cap U && (12)\\ &= N_{C\cap(U+S)-D\cap(U+S)}(0) \cap U && (13)\\ &= N_{(C-D)\cap U}(0) \cap U && (14)\\ &= N_{C-D}^U(0), && (15) \end{aligned}$$

where Equation (12) holds by definition of nested normals, Equation (13) follows from Lemma 6.6 applied to the sets $C \cap (U + S)$ and $D \cap (U + S)$, and Equation (14) follows from point 3) of $A(U)$. This shows that point 2) of $A(U)$ holds for U, and the proof is complete. □

We now have the tools to prove Corollary 4.1.

Proof of Corollary 4.1. By Definition 6.6, $T(C, D) = H_{C-D}(0)$. So $T \in \mathcal{F}_{C-D}(0)$, which by Theorem 6.6 implies that, taking $S = C \cap D$, $T \in \mathcal{F}_C(C \cap D) \cap \mathcal{F}_D(C \cap D)$. By the definition of supporting subspaces (Definition 6.2), $C \cap D - C \cap D \subseteq T$. □

7 Proofs

We begin with the proof of Proposition 5.1, after which each subsection is dedicated to one result from Section 4.

Proof of Proposition 5.1. First we show that for $C \subseteq \mathbb{R}^n$ convex, and $x \in \operatorname{relint}(C)$, $F_C(x) = C$. Note that

$$N_C^{\operatorname{span}(C-C)}(x) = N_C^{\operatorname{span}(C-C)}(C) = (C - C)^\perp \cap \operatorname{span}(C - C) = \{0\},$$

where the first equality follows by Lemma 6.1 and the second by Lemma 6.2. Then by Theorem 6.5, $H_C(x) = \operatorname{span}(C - C)$, and by Lemma 6.5, $F_C(x) = (\operatorname{span}(C - C) + x) \cap C = C$.

From Lemma 4.1, for $x \in \operatorname{relint}(\operatorname{dom}(f)) \cap \operatorname{relint}(\operatorname{dom}(g))$ and $T_a = T_a(\operatorname{dom}(f), \operatorname{dom}(g))$,

$$T_a \cap \operatorname{dom}(f) = F_{\operatorname{dom}(f)}(\operatorname{dom}(f) \cap \operatorname{dom}(g)) \supseteq F_{\operatorname{dom}(f)}(x) = \operatorname{dom}(f).$$

From this the results follows. □

7.1 Regularity in the joint supporting subspace

We now show the main property of the joint supporting subspace, Theorem 4.2. We first state and prove a lemma which connects the notion of separation of convex set to a statement reminiscent of the qualification condition by Mordukhovich in variational analysis: [24, Theorem 2.19].

Lemma 7.1 (Separation from local normal cones)**.** *For convex sets $C, D \subseteq \mathbb{R}^n$ and $S \subseteq C \cap D$ nonempty, the following equivalence holds:*

$$N_C(S) \cap (-N_D(S)) \neq \{0\} \iff C \text{ is separated from } D. \tag{16}$$

Proof of Lemma 7.1. We start with ($\implies$). There exists a $v \in [N_C(S) \cap (-N_D(S))] \setminus \{0\}$, and from Lemma 6.2 $v \perp S - S$, whence the hyperplane $v^\perp + S$ separates C from D.

For the reverse direction ($\impliedby$), suppose C and D are separated by a hyperplane H. Then $S \subseteq H$, and there exist $v \in \mathbb{R}^n \setminus \{0\}$ normal to H and $\beta \in \mathbb{R}$ such that

1. $\langle v, s\rangle = \beta$ for all $s \in S$;
2. $\langle v, c\rangle \geq \beta$ for all $c \in C$;
3. $\langle v, d\rangle \leq \beta$ for all $d \in D$.

Thus, for all $y \in C$ and $s \in S$, $\langle v, y - s\rangle \geq 0$, so $v \in N_C(S)$. Similarly, $-v \in N_D(S)$, whence $v \in N_C(S) \cap (-N_D(S)) \neq \{0\}$.\}$. ☐

Proof of Theorem 4.2. Recall that from Definition 6.6, $T = H_{C-D}(0)$. Then by Theorem 6.5, $N^T_{C-D}(C \cap D) = \{0\}$. Applying Theorem 6.6, we find that $N^T_C(C \cap D) \cap (-N^T_D(C \cap D)) = \{0\}$.

By applying Lemma 7.1 in the space T_a, the result follows. Note that Lemma 7.1 is stated in $\mathbb{R}^n$, yet we can apply it in T_a because T is a finite-dimensional subspace, thereby isomorphic to $\mathbb{R}^n$, and Lemma 7.1 is invariant when translating T to T_a. ☐

7.2 Analytical characterizations

In this section we prove Lemma 4.1 and Theorem 4.1.

Lemma 7.2 (Nested normals of constrained set)**.** *Let $C \subseteq \mathbb{R}^n$ be a convex set and $S \subseteq C$ a nonempty set. Let $U, V \subseteq \mathbb{R}^n$ be subspaces such that $S - S \subseteq U \subseteq V$ and $C \cap (V + S) \subseteq U + S$. Then*

$$N^U_C(S) + (U^\perp \cap V) = N^V_{C \cap (U+S)}(S).$$

Proof of Lemma 7.2. We begin with the containment ($\supseteq$). Let $x \in N^V_C(S)$. Then $\forall s \in S, y \in C \cap (V + S)$, note that $y - s \in U$, and so

$$\langle \Pi_U\, x, y - s\rangle = \langle x, y - s\rangle \leq 0.$$

This shows that $\Pi_U x \in N_C^U(S)$. Now $x = \Pi_U x + \Pi_{U^\perp} x$, and $x, \Pi_U x \in V \implies \Pi_{U^\perp} x \in V$, therefore $x \in N_C^U(S) + (U^\perp \cap V)$.

For the inclusion ($\subseteq$), we consider any $q \in N_C^U(S)$ and $z \in U^\perp \cap V$. Let $y \in C \cap (U + S), s \in S$, and note that $\langle z, y - s\rangle = 0$ because $y - s \in U$. Then

$$\langle q + z, y - s\rangle = \langle q, y - s\rangle \leq 0.$$

Therefore $q + z \in N^V_{C\cap(U+S)}(S)$, which shows the inclusion and completes the proof. $\square$

Lemma 7.3 (Conical projections yield supporting subspaces)**.** *Let $C \subseteq \mathbb{R}^n$ be convex, $S \subseteq C$ non-empty, and $U \in \mathcal{F}(C; S)$. Then for any $K \subseteq N_C^U(S)$,*

$$U \cap K^\perp \in \mathcal{F}(C; S).$$

Proof of Lemma 7.3. Since K is finite dimensional, there is a finite number of vectors $v_1, \ldots, v_m \in K$ such that $\operatorname{span}(K) = \operatorname{span}(v_1, \ldots, v_m)$. Then since $v_1, \ldots, v_m \in N_C^U(S)$, by repeated application of Theorem 6.3, $U \cap v_1^\perp \ldots \cap v_m^\perp$ is a supporting subspace, which is the same set as $U \cap K^\perp$. $\square$

Lemma 7.4 (Generated supporting subspace of one of the two convex sets)**.** *For convex sets $C, D \subseteq \mathbb{R}^n$, $T = T(C, D)$, $T_a = T + C \cap D$, we have*

$$H_C(C \cap D) = T \cap N_C^T(C \cap T_a)^\perp.$$

Proof of Lemma 7.4. Let $U = T \cap N_C^T(C \cap T_a)^\perp$. we first show that $U \in \mathcal{F}(C \cap D; C)$. Note that $T \in \mathcal{F}(C \cap D; C)$ by Theorem 6.6. Then we can use Lemma 7.3 because $N_C^T(C \cap T_a) \subseteq N_C^T(C \cap D)$, which shows that $U \in \mathcal{F}(C \cap D; C)$ as well.

In the remainder of the proof we show that the assumption of Theorem 6.5 holds for U, namely that $N_C^U(C \cap D) = \{0\}$.

We first show that $N_C^U(C \cap T_a) = \{0\}$. By Lemma 6.2, $N_C^T(C \cap T_a) = (C \cap T_a - C \cap T_a)^\perp \cap T$. Also, note that by the identity in Lemma 7.2, we find that $N_C^U(C \cap T_a) = N_C^T(C \cap T_a) \cap U$. Combining these two facts,

$$N_C^U(C \cap T_a) = N_C^T(C \cap T_a) \cap U = (C \cap T_a - C \cap T_a)^\perp \cap U.$$

But note that $C \cap T_a - C \cap T_a \subseteq U$, therefore it fo $N_C^U(C \cap T_a) = \{0\}$. That $C \cap T_a - C \cap T_a \subseteq U$ holds because $C \cap T_a - C \cap T_a \subseteq T$ and $(C \cap T_a - C \cap T_a) \perp N_C^T(C \cap T_a)$ by Lemma 6.2.

It only remains to show that $N_C^U(C \cap D) = N_C^U(C \cap T_a)$. By Corollary 4.5, we may pick some $x \in \operatorname{relint}(C \cap T_a) \cap \operatorname{relint}(D \cap T_a)$. Then by Lemma 6.1,

$$N_C^U(C \cap T_a) = N_C^U(x) \supseteq N_C^U(C \cap D).$$

Then $N_C^U(C \cap D) = \{0\}$, and so the result follows from Theorem 6.5. $\square$

We are now in a position to prove the analytical characterizations in Section 4.

Proof of Lemma 4.1. From Lemma 7.4, $H_C(C \cap D) = T \cap N_C^T(C \cap T_a)^\perp$. Since $(C \cap T_a - C \cap T_a) \perp N_C^T(C \cap T_a)$ by Lemma 6.2, $(C \cap T_a - C \cap T_a) \subseteq H_C(C \cap D)$, i.e., $H_C(C \cap D) + C \cap T_a \supseteq C \cap T_a$. But $H_C(C \cap D) + C \cap T_a = H_C(C \cap D) + C \cap D$, therefore

$$H_C(C \cap D) + C \cap D \supseteq C \cap T_a. \tag{17}$$

Additionally, since we know that $T \in \mathcal{F}(C \cap D; C)$ from Theorem 6.6, by definition of the generated supporting subspace we know that $H_C(C \cap D) \subseteq T$. Therefore,

$$H_C(C \cap D) + C \cap D \subseteq T_a. \tag{18}$$

From Equation (17) and Equation (18) it follows that $(H_C(C \cap D) + C \cap D) \cap C = C \cap T_a$. Then the statement follows from Lemma 6.5. $\square$

Proof of Theorem 4.1. From Lemma 4.1, we have that $T \in \mathcal{F}(C; C \cap D)$ is a supporting subspace of C that contains $C \cap D$, and so by Definition 6.5, $H_C(C \cap D) \subseteq T$. Therefore, for $T' := H_C(C \cap D) + H_D(C \cap D)$, $T' \subseteq T$.

For the other containment, we argue by contradiction: let $T' \subsetneq T$. Then there is a unit vector $v \in T$ with $v \perp T'$. Then $v \perp H_C(C \cap D)$, which by Lemma 7.4, implies that $v \in N_C^T(C \cap T)$. By Lemma 6.2, $-v \in N_C^T(C \cap T)$ as well. And since $N_C^T(C \cap T) \subseteq N_C^T(C \cap D)$, we find that $v, -v \in N_C^T(C \cap D)$. By a similar argument, $v, -v \in N_D^T(C \cap D)$. But then $v \in N_C^T(C \cap D) \cap (-N_D^T(C \cap D))$, which by Theorem 6.6 means that $v \in N_{C-D}^T(0)$. But by Theorem 6.5, T cannot contain such a nested normal because by Definition 6.6, $T = H_{C-D}(0)$. Therefore, by contradiction $T = T'$. $\square$

Next, we present the proofs for the corollaries of Theorem 4.1, starting with a simple lemma.

Lemma 7.5 (Formulation of the tangent space)**.** *For a convex set $C \subseteq \mathbb{R}^n$ and $S \subseteq C$ non-empty,*

$$\operatorname{span}(C - C) = \operatorname{span}(C - S).$$

Proof of Lemma 7.5. That $\operatorname{span}(C - C) \supseteq \operatorname{span}(C - S)$ is trivial, so we only show $\operatorname{span}(C - C) \subseteq \operatorname{span}(C - S)$. It suffices to show that $C - C \subseteq \operatorname{span}(C - S)$; let $u, v \in C$, then for any fixed $x \in S$,

$$u - v = (u - x) - (v - x) \in \operatorname{span}(C - S).$$

$\square$

Proof of Corollary 4.3. In this proof, we denote $F_C := F_C(C \cap D)$ and $F_D :=$

$F_D(C \cap D)$.

$$
\begin{aligned}
\operatorname{span}(F_C - F_D) &= \operatorname{span}(F_C - F_D) + (C \cap D - C \cap D) && (19)\\
&= \operatorname{span}(F_C - F_D + (C \cap D - C \cap D)) && (20)\\
&= \operatorname{span}((F_C - C \cap D) - (F_D - C \cap D)) && (21)\\
&= \operatorname{span}(F_C - C \cap D) + \operatorname{span}(F_D - C \cap D) && (22)\\
&= \operatorname{span}(F_C - F_C) + \operatorname{span}(F_D - F_D), && (23)\\
&= H_C(C \cap D) + H_D(C \cap D). && (24)\\
& && (25)
\end{aligned}
$$

where Equation (19) holds because

$$(C \cap D - C \cap D) \subseteq F_C - F_D \subseteq \operatorname{span}(F_C - F_D),$$

and Equation (23) follows from Lemma 7.5. The result then follows from Theorem 4.1. □

Proof of Corollary 4.2. Throughout this proof, we omit the argument $C \cap D$, i.e., $H_C := H_C(C \cap D)$, and similarly for H_D, F_C, F_D. We also define $\bar{F}_C = F_C - C \cap D$ and $\bar{F}_D = F_D - C \cap D$. By Lemma 7.5,

$$H_C = \operatorname{span}(F_C - F_C) = \operatorname{span}(\bar{F}_C).$$

With this, starting with T_a as in Theorem 4.1,

$$
\begin{aligned}
T_a(C, D) &= H_C + H_D + C \cap D\\
&= \operatorname{span}(\bar{F}_C) + \operatorname{span}(\bar{F}_D) + C \cap D\\
&= \operatorname{span}(\operatorname{span}(\bar{F}_C) \cup \operatorname{span}(\bar{F}_D)) + C \cap D\\
&= \operatorname{span}(\bar{F}_C \cup \bar{F}_D) + C \cap D\\
&= \operatorname{aff}(\bar{F}_C \cup \bar{F}_D) + C \cap D\\
&= \operatorname{aff}(\bar{F}_C \cup \bar{F}_D + C \cap D)\\
&= \operatorname{aff}(F_C \cup F_D).
\end{aligned}
$$

□

7.3 Second characterization

We can now state the second characterization in full.

Theorem 7.1 (Iterative bilateral facial reduction)**.** *Let $C, D \subseteq \mathbb{R}^n$ be convex sets and $S \subseteq C \cap D$ non-empty. Beginning with $T_0 = \mathbb{R}^n$, let*

$$T_{i+1} = T_i \cap (N_C^{T_i}(S) \cap [-N_D^{T_i}(S)])^{\perp}.$$

Let ℓ be the first index for which $T_{\ell+1} = T_\ell$. Then:

1. $T_\ell = T(C, D)$.

2. $\ell \leq n$.

Proof of Theorem 7.1. First, $(N_C^{T_i}(S) \cap [-N_D^{T_i}(S)]) = N_{C-D}^{T_i}(S)$ by Theorem 6.6. Applying Lemma 7.3 at each step, we find that $\forall i \in \{0, \ldots, \ell\}$, $T_i \in \mathcal{F}(C-D; S)$. Finally, note that whenever $N_C^{T_i}(S) \cap [-N_D^{T_i}(S)] \neq \{0\}$, $\dim(T_{i+1}) \leq \dim(T_i) - 1$. Therefore, the dimension may be reduced at most n times, which implies that $\ell \leq n$. On the other hand, the fact that $T_\ell = T_{\ell+1}$ implies that $N_{C-D}^{T_\ell}(S) = \{0\}$. Then from Theorem 6.5 it follows that $T_\ell = H_{C-D}(S)$, which is $T(C, D)$ from Definition 6.6. □

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