

# A note on the number of non-cycle components in a pseudo 2-factor of graphs

Masaki Kashima\*

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## Abstract

A pseudo 2-factor of a graph is a spanning subgraph such that each component is  $K_1$ ,  $K_2$ , or a cycle. This notion was introduced by Bekkai and Kouider in 2009, where they showed that every graph  $G$  has a pseudo 2-factor with at most  $\alpha(G) - \delta(G) + 1$  components that are not cycles. For a graph  $G$  and a set of vertices  $S$ , let  $\delta_G(S)$  denote the minimum degree of vertices in  $S$ . In this note, we show that every graph  $G$  has a pseudo 2-factor with at most  $f(G)$  components that are not cycles, where  $f(G)$  is the maximum value of  $|I| - \delta_G(I) + 1$  among all independent sets  $I$  of  $G$ . This result is a common generalization of a result by Bekkai and Kouider and a previous result by the author on the existence of a 2-factor.

**Keywords:** 2-factor, pseudo 2-factor, minimum degree, independent set

## 1 Introduction

Throughout the paper, we only consider simple, finite, and undirected graphs. For a graph  $G$ , let  $\delta(G)$  and  $\alpha(G)$  denote the minimum degree and the independence number, respectively. For a graph  $G$  and a set  $S \subseteq V(G)$ , let  $N_G(S)$  denote the set of vertices in  $V(G) \setminus S$  that have neighbors in  $S$ . In particular, for a subgraph  $H$  of  $G$ , we abbreviate  $N_G(V(H))$  to  $N_G(H)$ . For a positive integer  $n$ , let  $K_n$  denote the complete graph of order  $n$ .

A *2-factor* of a graph  $G$  is a 2-regular spanning subgraph of  $G$ . A *Hamilton cycle* of a graph  $G$ , which is a cycle that passes through all vertices of  $G$ , is exactly a connected 2-factor. Thus, sufficient conditions for a graph to have a 2-factor have been actively studied in connection with Hamilton cycles.

As a relaxation of a 2-factor, Bekkai and Kouider [1] introduced a notion of pseudo 2-factor. The term “pseudo 2-factor” was coined by them, though the concept had already been studied by Enomoto and Li [5] in 2004. A *pseudo 2-factor* of a graph  $G$  is a spanning subgraph of  $G$  in which each component is isomorphic to  $K_1$ ,  $K_2$ , or a cycle. By allowing  $K_1$  and  $K_2$  as components, it is clear that every graph has a pseudo 2-factor. Thus sufficient and/or necessary conditions for a graph to have a “special” pseudo 2-factor have been studied in the literature.

Well before the term pseudo 2-factor established, Tutte [8] gave a sufficient and necessary condition for a graph to have a pseudo 2-factor without isolated vertices. Later, by Cornu  jols and Hartvigsen [3], the result was extended to a sufficient and necessary condition for a graph to have a pseudo 2-factor without isolated vertices and small odd cycles. In 2018, Egawa and Furuya [4] gave sufficient conditions, which are more easily checkable, for a graph to have a pseudo 2-factor with no isolated vertices and small odd cycles. From the other aspect, motivated by a result on 2-factor with prescribed number of components, Enomoto and Li [5] investigated the sufficient degree

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\*Keio University, Yokohama, Japan. email: masaki.kashima10@gmail.com

sum conditions for a graph to have a pseudo 2-factor with exactly  $k$  components. Recently, Chiba and Yoshida [2] considered an analogue of the result for bipartite graphs.

In this note, we focus on the number of components that are isomorphic to  $K_1$  or  $K_2$  in a pseudo 2-factor. A component of a pseudo 2-factor is called a *non-cycle component* if it is isomorphic to  $K_1$  or  $K_2$ . Since a pseudo 2-factor without non-cycle components is a 2-factor of a graph, we are interested in upper bounds of the number of non-cycle components in a pseudo 2-factor of a given graph. Bekkai and Kouider [1] gave the following upper bound.

**Theorem 1** ([1]). For any graph  $G$  with  $\alpha(G) \geq \delta(G)$ ,  $G$  has a pseudo 2-factor with at most  $\alpha(G) - \delta(G) + 1$  non-cycle components.

The bound in Theorem 1 is best possible. Indeed, for an arbitrary graph  $H$  and a positive integer  $p \geq |V(H)| + 1$ , let us consider the graph  $G_1$  obtained from  $H$  by joining  $p$  disjoint copies of  $K_2$ . Then it follows that  $\delta(G_1) = |V(H)| + 1$  and  $\alpha(G_1) = p \geq |V(H)| + 1$ , both of which are satisfied by vertices in copies of  $K_2$ . On the other hand, it is easy to see that every pseudo 2-factor of  $G_1$  has at least  $p - |V(H)| = \alpha(G_1) - \delta(G_1) + 1$  non-cycle components since  $G_1 - V(H)$  consists of  $p$  disjoint copies of  $K_2$ .

Their result with the case  $\alpha(G) = \delta(G)$  implies the following theorem by Niessen [7].

**Theorem 2** ([7]). Every graph  $G$  with  $\delta(G) \geq \alpha(G) + 1$  has a 2-factor.

Recently, the author showed the following result, which extends Theorem 2 in a different way. For a vertex set  $S$  of a graph  $G$ , let  $\delta_G(S)$  denote the minimum degree of the vertices in  $S$ .

**Theorem 3** ([6]). If every independent set  $I$  of  $G$  satisfies  $\delta_G(I) \geq |I| + 1$ , then  $G$  has a 2-factor.

If a graph  $G$  satisfies  $\delta(G) \geq \alpha(G) + 1$ , then every independence set  $I$  of  $G$  satisfies

$$\delta_G(I) \geq \delta(G) \geq \alpha(G) + 1 \geq |I| + 1,$$

and hence Theorem 2 holds from Theorem 3. In this note, we show the following result, which generalizes both Theorems 1 and 3 (and obviously Theorem 2). For a graph  $G$ , let

$$f(G) := \max\{|I| - \delta_G(I) + 1 \mid I \text{ is an independent set of } G\}.$$

Then the following holds.

**Theorem 4.** Every graph  $G$  has a pseudo 2-factor with at most  $\max\{0, f(G)\}$  non-cycle components.

We will give a proof of Theorem 4 in the next section.

When every independent set  $I$  of  $G$  satisfies  $\delta_G(I) \geq |I| + 1$ , then obviously  $f(G) \leq 0$  and Theorem 4 implies that  $G$  has a 2-factor. Also, for any graph  $G$  and any independent set  $I$  of  $G$ , we have  $|I| - \delta_G(I) + 1 \leq \alpha(G) - \delta(G) + 1$ , and hence  $f(G) \leq \alpha(G) - \delta(G) + 1$ . Thus, Theorem 4 implies both Theorems 1 and 3. By the tightness of the bound in Theorem 1, the bound in Theorem 4 is best possible as well.

We remark that the gap between  $f(G)$  and  $\alpha(G) - \delta(G) + 1$  can be arbitrarily large. For a positive integer  $k$ , we set two vertices  $v_1, v_2$ , two disjoint independent sets  $A_1, A_2$  of order  $k$ , and a complete graph  $B$  of order at least  $2k$ . Let  $G_2$  be a graph obtained by  $v_1, v_2, A_1, A_2$ , and  $B$  by joining  $A_i$  to  $B \cup \{v_i\}$  for each  $i \in \{1, 2\}$  (Figure 1). Then it follows that  $\delta(G_2) = d_G(v_1) = k$  and  $\alpha(G_2) = |A_1 \cup A_2| = 2k$ , implying that  $\alpha(G_2) - \delta(G_2) + 1 = 2k - k + 1 = k + 1$ . On the other hand, since all the maximal independent sets of  $G_2$  are  $I_1 = \{v_1, v_2, b\}$  with  $b \in B$ ,  $I_2 = v_1 \cup A_2$ ,  $I_3 = v_2 \cup A_1$ , and  $I_4 = A_1 \cup A_2$ , we have

$$f(G_2) = |I_2| - \delta_G(I_2) + 1 = (k + 1) - k + 1 = 2.$$

Thus, the bound in Theorem 4 is strictly smaller than that in Theorem 1.

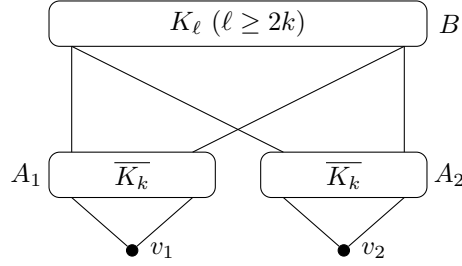


Figure 1: A graph  $G_2$  which has a gap between  $f(G_2)$  and  $\alpha(G_2) - \delta(G_2) + 1$ .

## 2 Proofs

We first show some statements used in our proof of Theorem 4. The following observation can be easily verified.

**Observation 5.** For every tree  $T$  and every leaf  $u$  of  $T$ ,  $T$  has a maximum independent set that contains  $u$ .

By using well-known König's theorem on matchings and vertex covers of bipartite graphs, we can show the following.

**Proposition 6.** Every tree  $T$  has a pseudo 2-factor with exactly  $\alpha(T)$  components.

*Proof.* The statement is trivial when  $|V(T)| = 1$ . Suppose that  $n := |V(T)| \geq 2$ . Let  $\beta(T)$  be the minimum cardinality of a vertex cover of  $T$ .

For every independent set  $I$  of  $T$ ,  $V(T) \setminus I$  is a vertex cover of  $T$  since every vertex in  $I$  has at least one neighbor in  $T$  that must be in  $V(T) \setminus I$ . Thus, we have  $\beta(T) = n - \alpha(T)$ . Since  $T$  is a bipartite graph, by König's theorem,  $T$  has a matching  $M$  with  $\beta(T) = n - \alpha(T)$  edges. Combining all the edges in  $M$  and the vertices in  $V(T) - V(M)$ , we obtain a pseudo 2-factor of  $T$  with

$$|M| + (n - 2|M|) = n - \alpha(T) + (n - 2(n - \alpha(T))) = \alpha(T)$$

components, as desired.  $\square$

By Observation 5, every tree  $T$  satisfies  $f(T) \leq \alpha(T) - 1 + 1 = \alpha(T)$ , and hence Proposition 6 states that Theorem 4 holds for trees. Furthermore, this directly implies the following.

**Proposition 7.** Every forest  $G$  has a pseudo 2-factor with exactly  $\alpha(G)$  components. In particular, every forest  $G$  has a pseudo 2-factor with at most  $f(G)$  components.

Now we prove Theorem 4.

### 2.1 Proof of Theorem 4

The case  $G$  is a forest is done by Proposition 7. Suppose that  $G$  has at least one cycle. Let  $F$  be a union of pairwise vertex-disjoint cycles of  $G$  such that

- (a)  $|V(F)|$  is as large as possible, and
- (b) subject to (a), the number of isolated vertices in  $G - V(F)$  is as small as possible.

If  $V(F) = V(G)$ , then obviously  $F$  is a 2-factor of  $G$  and we are done. Thus, we assume that  $V(G) \setminus V(F) \neq \emptyset$ . By the maximality of  $F$ , it follows that  $H := G - V(F)$  is a forest. Set  $\alpha := \alpha(H)$ . By Proposition 7,  $H$  has a pseudo 2-factor  $F_H$  of exactly  $\alpha$  components. Then  $F \cup F_H$  is a pseudo 2-factor of  $G$  with exactly  $\alpha$  non-cycle components. Thus, it suffices to show that  $\alpha \leq f(G)$ .

Assume to the contrary that  $\alpha > f(G)$ . We set one orientation of each cycle  $C$  of  $F$ . For each vertex  $v \in V(F)$ , let  $v^+$  denote the successor of  $v$  and let  $v^-$  denote the predecessor of  $v$  along the orientation of the cycle of  $F$  containing  $v$ .

**Claim 1.** For a vertex  $x$  of  $H$ , if  $x$  has two neighbors  $y_1$  and  $y_2$  in  $V(F)$ , then  $y_1^+ y_2^+ \notin E(G)$ .

*Proof.* Assume to the contrary that  $y_1^+ y_2^+ \in E(G)$  for some  $y_1, y_2 \in N_G(x) \cap V(F)$ . Then,  $F' = F \cup \{y_1 x, y_2 x, y_1^+ y_2^+\} - \{y_1 y_1^+, y_2 y_2^+\}$  is a 2-regular subgraph of  $G$  such that  $V(F') = V(F) \cup \{x\}$ , a contradiction by the maximality of  $F$ .  $\square$

**Claim 2.** For every isolated vertex  $x$  of  $H$ , there are two vertices  $y, y' \in N_G(x) \cap V(F)$  such that  $N_G(y^+) \cap V(H) \neq \emptyset$  and  $N_G(y'^+) \cap V(H) \neq \emptyset$ .

*Proof.* For an isolated vertex  $x$  of  $H$ , we set  $Y^+ = \{y^+ \mid y \in N_G(x) \cap V(F)\}$ . Suppose, for the sake of contradiction, that  $|Y^+ \cap N_G(H)| \leq 1$ . Let  $I$  be a maximum independent set of  $H$ . Note that  $x \in I$  since  $x$  is an isolated vertex of  $H$ . By Claim 1,  $Y^+$  is an independent set of  $G$ , and hence  $I' := I \cup (Y^+ \setminus N_G(H))$  is an independent set of  $G$ . Since  $|Y^+ \setminus N_G(H)| \geq |Y^+| - 1 = d_G(x) - 1$ ,

$$f(G) \geq |I'| - \delta_G(I') + 1 \geq (|I| + |Y^+ \setminus N_G(H)|) - d_G(x) + 1 \geq \alpha + d_G(x) - 1 - d_G(x) + 1 = \alpha,$$

a contradiction.  $\square$

**Claim 3.** For every vertex  $x$  of  $H$  with  $d_H(x) = 1$ , there is a vertex  $y \in N_G(x) \cap V(F)$  such that  $N_G(y^+) \cap V(H) \neq \emptyset$ .

*Proof.* For a vertex  $x$  of  $H$  with  $d_H(x) = 1$ , set  $Y^+ = \{y^+ \mid y \in N_G(x) \cap V(F)\}$ . Note that  $|Y^+| = |N_G(x) \cap V(F)| = d_G(x) - 1$ . Assume, for the sake of contradiction, that  $Y^+ \cap N_G(H) = \emptyset$ . By Observation 5,  $H$  has a maximum independent set  $I$  that contains  $x$ . By Claim 1,  $Y^+$  is an independent set of  $G$ . This, together with the assumption that  $Y^+ \cap N_G(H) = \emptyset$  implies that  $I' := I \cup Y^+$  is an independent set of  $G$  that contains  $x$ , and hence

$$f(G) \geq |I'| - \delta_G(I') + 1 \geq (|I| + |Y^+|) - d_G(x) + 1 = \alpha + d_G(x) - 1 - d_G(x) + 1 = \alpha,$$

a contradiction.  $\square$

In the rest of the proof, using Claims 2 and 3, we shall construct a 2-regular subgraph of  $G$  which contradicts the choice of  $F$ .

Let  $D_0$  be a component of  $H$  and choose  $x_0 \in V(D_0)$  with  $d_{D_0}(x_0) \leq 1$  arbitrarily. By Claims 2 and 3, there is a vertex  $y_0 \in N_G(x_0) \cap V(F)$  such that  $N_G(y_0^+) \cap V(H) \neq \emptyset$ . Let  $z_0$  be a vertex in  $N_G(y_0^+) \cap V(H)$ . For  $i = 1, 2, \dots$ , we sequentially define  $(D_i, x_i, y_i, z_i)$  in the following procedure until  $z_i \in \bigcup_{j=0}^i V(D_j)$ .

Suppose that  $(D_j, x_j, y_j, z_j)$  is defined for each  $0 \leq j \leq i - 1$ .

1. Let  $D_i$  be the component that contains  $z_{i-1}$
2. We define  $x_i \in D_i$  and  $y_i \in N_G(x_i) \cap V(F)$  as follows.
  - (a) If  $D_i$  is isomorphic to  $K_1$ , then let  $x_i = z_{i-1}$ . By Claim 2, there is a vertex  $y_i \in N_G(x_i) \cap (V(F) \setminus \{y_{i-1}^+\})$  such that  $N_G(y_i^+) \cap V(H) \neq \emptyset$ .

(b) If  $D_i$  is not isomorphic to  $K_1$ , then let  $x_i$  be a leaf of  $D_i$  distinct from  $z_{i-1}$ . By Claim 3, there is a vertex  $y_i \in N_G(x_i) \cap V(F)$  such that  $N_G(y_i^+) \cap V(H) \neq \emptyset$ . Note that it is possible that  $y_i = y_{i-1}^+$  in this case.

3. If there is a component  $D_j \in \{D_0, \dots, D_i\}$  such that  $N_G(y_i^+) \cap V(D_j) \neq \emptyset$ , then let  $z_i \in N_G(y_i^+) \cap V(D_j)$  so that the index  $j$  is as large as possible. Otherwise, let  $z_i$  be an arbitrary vertex in  $N_G(y_i^+) \cap V(H)$ .

Since the number of components of  $H$  is finite, this procedure must end. Without loss of generality, we may assume that the procedure stops at  $(D_r, x_r, y_r, z_r)$  and  $z_r \in V(D_0)$ . Furthermore, we may assume that  $y_r^+$  is not adjacent to any components in  $\{D_1, \dots, D_r\}$ , and hence  $y_r \notin \{y_1, \dots, y_{r-1}\}$ .

By the choice of  $r$ , we know that  $D_0, \dots, D_r$  are distinct components of  $H$ . Also, for every  $i$  and  $j$  with  $0 \leq i < j \leq r-1$ , if  $y_j \in \{y_i^-, y_i\}$ , then we can choose  $D_i$  or  $D_{i+1}$  as  $D_{j+1}$ , which contradicts the choice of  $r$ . We consider the following two cases.

*Case 1.*  $y_r = y_0^-$ .

For each  $i \in \{1, \dots, r\}$ , let  $W_i$  be the  $y_{i-1}^+ y_i$ -walk  $y_{i-1}^+ z_{i-1} P_i x_i y_i$ , where  $P_i$  is the unique  $z_{i-1} x_i$ -path in  $D_i$ . If  $D_i$  is isomorphic to  $K_1$ , then by 2(a), we chose  $y_i$  so that  $y_i \neq y_{i-1}^+$ . Thus, if  $y_i = y_{i-1}^+$ , then we know that  $x_i \neq z_{i-1}$ , and hence  $W_i$  is a cycle of  $G$ . Otherwise,  $W_i$  is a  $y_{i-1}^+ y_i$ -path of  $G$ .

Since  $D_1, \dots, D_r$  are pairwise distinct components of  $H$ ,  $V(W_i) \setminus \{y_{i-1}^+, y_i\}$  and  $V(W_j) \setminus \{y_{j-1}^+, y_j\}$  are disjoint for different  $i$  and  $j$ , and in particular,  $W_1, \dots, W_r$  are pairwise edge-disjoint. Let  $F_1$  be a graph obtained from  $F$  by removing the edges  $\{y_i y_i^+ \mid 0 \leq i \leq r\}$ , adding the walks  $W_1, \dots, W_r$ , and deleting  $y_0$ . We can check that  $F_1$  is a 2-regular subgraph of  $G$  as follows. It is easy to see that every vertex in  $V(F_1) \setminus \bigcup_{i=0}^r \{y_i, y_i^+\}$  has degree 2 in  $F_1$ . For each  $i \in \{1, \dots, r\}$ ,  $y_i$  originally has degree two in  $F$ , loses degree one by deleting  $y_i y_i^+$ , and gains degree one by adding  $W_i$ , resulting in  $d_{F_1}(y_i) = 2 + 1 - 1 = 2$ . Similarly, for each  $i \in \{0, \dots, r-1\}$ ,  $y_i^+$  originally has degree two in  $F$ , loses degree one by deleting  $y_i y_i^+$ , and gains degree one by adding  $W_{i+1}$ , and hence  $d_{F_1}(y_i^+) = 2 + 1 - 1 = 2$ . Note that when  $y_{i-1}^+ = y_i$ ,  $y_{i-1}^+$  loses degree two by deleting  $\{y_{i-1} y_{i-1}^+, y_i y_i^+\}$  and gains degree two by adding  $W_i$ . Combining these, we conclude that  $F_1$  is 2-regular.

If  $r \geq 2$ , then we have

$$|V(F_1)| = |V(F) \setminus \{y_0\}| + \sum_{i=1}^r |V(W_i) \setminus \{y_{i-1}^+, y_i\}| \geq |V(F)| - 1 + r > |V(F)|,$$

a contradiction by the maximality of  $|V(F)|$ . Similarly, if  $r = 1$  and  $D_1$  is not isomorphic to  $K_1$ , the choice of  $x_1$  implies that  $|V(W_1)| \geq 4$ , and hence

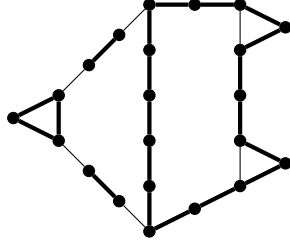
$$|V(F_1)| \geq |V(F)| - 1 + |V(W_1)| - 2 > |V(F)|,$$

a contradiction again. Thus, we conclude that  $r = 1$  and  $D_1$  is isomorphic to  $K_1$ , which implies that  $V(F_1) = (V(F) \setminus \{y_0\}) \cup \{z_0\}$ . Then, since  $y_0$  is adjacent to a component  $D_0$  of  $H$  and  $z_0$  is an isolated vertex of  $H$ , the number of isolated vertices of  $G - V(F_1)$  is strictly less than that of  $H = G - V(F)$ , a contradiction.

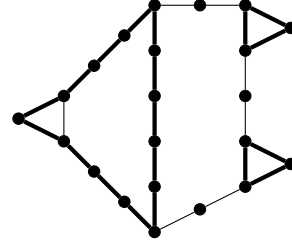
*Case 2.*  $y_r \neq y_0^-$ .

We define  $W_1, \dots, W_r$  similarly to Case 1. Also, since  $y_r^+ \neq y_0$ , let  $W_0$  be a  $y_r^+ y_0$ -walk  $y_r^+ z_r P_0 x_0 y_0$  of  $G$ , where  $P_0$  is the unique  $z_r x_0$ -path in  $D_0$ .

Using an argument similar to that in the previous case, we infer that  $V(W_i) \setminus \{y_{i-1}^+, y_i\}$  and  $V(W_j) \setminus \{y_{j-1}^+, y_j\}$  are disjoint for any  $0 \leq i < j \leq r$ , and that  $W_0, \dots, W_r$  are pairwise edge-disjoint. Let  $F_2$  be a subgraph of  $G$  obtained from  $F$  by removing the edges  $\{y_i y_i^+ \mid 0 \leq i \leq r\}$  and adding the walks  $W_0, \dots, W_r$ .



(a) A pseudo 2-factor with minimum number of non-cycle components.



(b) A maximum 2-regular subgraph.

Figure 2: An example of a graph in which every pseudo 2-factor with minimum number of non-cycle components does not contain maximum 2-regular subgraphs.

Then, we can check that  $F_2$  is a 2-regular subgraph of  $G$  with

$$|V(F_2)| = |V(F)| + \sum_{i=1}^r |V(W_i) \setminus \{y_{i-1}^+, y_i\}| \geq |V(F)| + r > |V(F)|,$$

a contradiction by the maximality of  $|V(F)|$ . This completes the proof of Theorem 4.

### 3 Remarks on algorithmic aspect of pseudo 2-factor

Our proof gives an algorithm to find a pseudo 2-factor with at most  $f(G)$  non-cycle components, but not a pseudo 2-factor with minimum number of non-cycle components. It is known that there is a polynomial-time algorithm to give a maximum 2-regular subgraph of a given graph. We remark that, for a given graph  $G$ , a pseudo 2-factor with minimum number of non-cycle components does not always contain a maximum 2-regular subgraph. For instance, a graph in Figure 2 of 22 vertices has a maximum 2-regular subgraph of order 19, but every pseudo 2-factor with minimum number of non-cycle components contains a 2-factor with 18 vertices. Thus, the following question remains open.

**Question 8.** Is there a polynomial-time algorithm to find a pseudo 2-factor with minimum number of non-cycle components?

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