On Diophantine equations involving intersection of Thabit and Williams numbers base b and some ternary recurrent sequences

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Abstract

Let $\mathcal{P}_n$ be the n-th Padovan number, E_n be the n-th Perrin number and N_n be the n-th Narayana's cows number. Let b be a positive integer such that $b \geq 2$. In this paper, we study the Diophantine equations

$$\mathcal{P}_n = (b \pm 1) \cdot b^l \pm 1,$$

$$E_n = (b \pm 1) \cdot b^l \pm 1,$$

and

$$N_n = (b \pm 1) \cdot b^l \pm 1,$$

in non-negative integers n, b and positive integer l. As a result, we determine the Padovan, Perrin and Narayana's cows numbers that are Thabit and Williams numbers base b. Moreover, we determine all solutions of the above equations within the range $2 \leq b \leq 10$.

Keywords: Thabit numbers, Williams numbers, Padovan numbers, Perrin numbers, Narayana's cows numbers, Linear forms in logarithms, Reduction method.
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1 Introduction

The Padovan sequence $\{P_n\}_{n\geq 0}$ and the Perrin sequence $\{E_n\}_{n\geq 0}$ are recursively defined by the following ternary recurrences

$$P_0 = P_1 = P_2 = 1, \quad P_{n+3} = P_{n+1} + P_n, \quad \text{for all} \quad n \geq 0,$$

and

$$E_0 = 3, E_1 = 0, E_2 = 2, \quad E_{n+3} = E_{n+1} + E_n, \quad \text{for all} \quad n \geq 0$$

respectively. The Padovan and Perrin sequences are the sequences A000931 and A001608 respectively, in the online encyclopedia of integer sequences (OEIS). The first few terms of the Padovan sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \cdots,$$

while the Perrin sequence begins with

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, \cdots$$

Another well-known ternary sequence is the Narayana's cows sequence $\{N_m\}_{m\geq 0}$, defined by

$$N_{m+3} = N_{m+2} + N_m \ \text{ for } m \geq 0,$$

with initials $N_0 = N_1 = N_2 = 1$. This sequence, listed as A000930 in the OEIS, begins with

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, 872, \ldots$$

Together with these recurrence sequences, a classical family of numbers introduced by Thâbit ibn Qurra has attracted significant attention.

A Thabit number (also called a *321 number*) is defined by $3 \cdot 2^n - 1$ for $n \geq 0$ and is of $n + 2$ digits long. This sequence corresponds to A055010 in the OEIS, with the first few terms are

$$2, 5, 11, 23, 47, 95, 191, 383, 767, 1535, 3071, 6143, \ldots$$

Motivated by their structure, generalized forms of Thabit numbers have been studied extensively. For an integer $b \geq 2$, a Thabit number of the first kind base b is a number of the form

$$(b+1)b^l - 1 \quad \text{for} \quad l \geq 1,$$

while a Thabit number of the second kind base b is a number of the form

$$(b+1)b^l + 1 \quad \text{for} \quad l \geq 1.$$

A closely related family is that of Williams numbers, which form another generalization defined for $b \geq 2$ by

$$(b-1)b^l \pm 1 \quad \text{for} \quad l \geq 1.$$

For example, Williams numbers with base $b = 2$ reduce precisely to Mersenne numbers.

In recent years, several papers have addressed Diophantine equations involving Thabit and Williams numbers in base b. For instance, Adédji et al. [2] studied equations where such numbers appear as sums or differences of two Lucas sequence terms. In a subsequent work [1], they investigated whether Thabit and Williams numbers in base b can be expressed as sums or differences of Fibonacci and Multau numbers, and vice versa. More recently, Adédji et al. [3] examined representations of Thabit and Williams numbers in base b as sums or differences of two g-repdigits. Motivated by these studies, we focus on the interaction between Thabit and Williams numbers in base b and the Padovan, Perrin, and Narayana's cows sequences. Specifically, we establish the following results.

Theorem 1.1. *Let $b \geq 2$ be an integer. Then, the Diophantine equation*

$$\mathcal{P}_n = (b \pm 1) \cdot b^l \pm 1, \tag{1.1}$$

has only finitely many solutions in integers (n, b, l). In particular, we have

$$n < 1.96 \times 10^{13} \log b \log(184b^3)(29.91 + \log(\log b) + \log(\log(184b^3))).$$

Theorem 1.2. *Let $b \geq 2$ be an integer. Then, the Diophantine equation*

$$E_n = (b \pm 1) \cdot b^l \pm 1, \tag{1.2}$$

has only finitely many solutions in integers (n, b, l). In particular, we have

$$n < 1.96 \times 10^{13} \log b(\log(8b^3))(29.91 + \log(\log b) + \log(\log(8b^3))).$$

Theorem 1.3. *Let $b \geq 2$ be an integer. Then, the Diophantine equation*

$$N_n = (b \pm 1) \cdot b^l \pm 1, \tag{1.3}$$

has only finitely many solutions in integers (n, b, l). In particular, we have

$$n < 1.95 \times 10^{13} \log b \log(248b^3)(29.91 + \log(\log b) + \log(\log 248b^3)).$$

The proofs of our theorems rely primarily on linear forms in logarithms of algebraic numbers, as developed by Matveev [8], together with the reduction algorithm of Dujella and Pethő [7]. We begin by presenting several preliminary results, which are discussed in detail in the following section.

2 Preliminary results

This section is devoted to gathering several definitions, notations, properties, and results that will be used in the rest of this study.

2.1 Linear forms in logarithms

Let γ be an algebraic number of degree d with a minimal primitive polynomial

$$f(Y) := b_0 Y^d + b_1 Y^{d-1} + \cdots + b_d = b_0 \prod_{j=1}^{d} (Y - \gamma^{(j)}) \in \mathbb{Z}[Y],$$

where the b_j's are relatively prime integers, $b_0 > 0$, and the $\gamma^{(j)}$'s are conjugates of γ. Then the *logarithmic height* of γ is given by

$$h(\gamma) = \frac{1}{d} \left(\log b_0 + \sum_{j=1}^{d} \log \left(\max\{|\gamma^{(j)}|, 1\} \right) \right).$$

With the above notation, Matveev (see [8] or [6, Theorem 9.4]) proved the following result.

Theorem 2.1. *Let $\eta_1, \ldots, \eta_s$ be positive real algebraic numbers in a real algebraic number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$. Let $a_1, \ldots, a_s$ be non-zero integers such that*

$$\Lambda := \eta_1^{a_1} \cdots \eta_s^{a_s} - 1 \neq 0.$$

Then

$$-\log |\Lambda| \leq 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) \cdot B_1 \cdots B_s,$$

where

$$D \geq \max\{|a_1|, \ldots, |a_s|\},$$

and

$$B_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \text{ for all } j = 1, \ldots, s.$$

2.2 The reduction method

Our next tool is a version of the reduction method of Baker and Davenport (see [4]). Here, we use a slight variant of the version given by Dujella and Pethö (see [7]). For a real number x, we write $||x||$ for the distance from x to the nearest integer.

Lemma 2.2. *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let*

$$\epsilon := ||\mu q|| - M \cdot ||\tau q||.$$

If $\epsilon > 0$, then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

Note that Lemma 2.2 cannot be applied when $\mu = 0$ (since then $\epsilon < 0$) or when μ is a multiple of τ. For this case, we use the following well-known technical result from Diophantine approximation, known as Legendre's criterion.

Lemma 2.3. *Let τ be a real number and r, s integers such that*

$$\left|\tau - \frac{r}{s}\right| < \frac{1}{2s^2}.$$

Then $r/s = p_k/q_k$ is a convergent of the continued fraction expansion $[a_0, a_1, \ldots]$ of τ (with some $k = 0, 1, \ldots$). Further, let M and N be nonnegative integers such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality

$$\left| \tau - \frac{r}{s} \right| > \frac{1}{(a(M) + 2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

2.3 Properties of Padovan and Perrin sequences

In this subsection, we begin by recalling some properties of these ternary recurrence sequences that will be used later. The characteristic equation of these sequences are $x^3 - x - 1 = 0$ which has one real root α and two complex conjugate roots β and $\gamma = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

with $r_1 = \sqrt[3]{108 + 12\sqrt{69}}$ and $r_2 = \sqrt[3]{108 - 12\sqrt{69}}$. The Binet's formula of $\mathcal{P}_n$ is

$$\mathcal{P}_n = p\alpha^n + q\beta^n + r\gamma^n, \text{ for all } n \geq 0, \tag{2.4}$$

and that for E_n is

$$E_n = \alpha^n + \beta^n + \gamma^n, \text{ for all } n \geq 0, \tag{2.5}$$

where

$$\begin{aligned} p &= \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1+\alpha}{-\alpha^2+3\alpha+1}, \\ q &= \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} = \frac{1+\beta}{-\beta^2+3\beta+1}, \\ r &= \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1+\gamma}{-\gamma^2+3\gamma+1} = \overline{q}. \end{aligned} \tag{2.6}$$

The minimal polynomial of p over $\mathbb{Z}$ is $23x^3 - 23x^2 + 6x - 1$ with all its zeros lie strictly inside the unit circle. Numerically, we have the following estimates:

$$\begin{aligned} 1.32 <&\alpha < 1.33, \\ 0.86 < |\beta| =&|\gamma| = \alpha^{-1/2} < 0.87, \\ 0.72 <&p < 0.73, \\ 0.24 < |q| =& |r| < 0.25. \end{aligned} \tag{2.7}$$

In particular, setting

$$\pi(n) := \mathcal{P}_n - p\alpha^n = q\beta^n + r\gamma^n, \quad \text{we have} \quad |\pi(n)| < \frac{1}{\alpha^{n/2}} \tag{2.8}$$

and

$$\psi(n) := E_n - \alpha^n = \beta^n + \gamma^n, \quad \text{we have} \quad |\psi(n)| < \frac{2}{\alpha^{n/2}} \tag{2.9}$$

holds for all $n \geq 1$. Further, using induction, one can prove that

$$\alpha^{n-3} \leq \mathcal{P}_n \leq \alpha^{n-1}, \quad \text{for all} \quad n \geq 1 \tag{2.10}$$

and

$$\alpha^{n-2} \leq E_n \leq \alpha^{n+1}, \quad \text{for all} \quad n \geq 2. \tag{2.11}$$

2.4 Properties of Narayana's cows sequence

The characteristic polynomial of Narayana's cows sequence is $f(x) = x^3 - x^2 - 1$. This polynomial is irreducible in $\mathbb{Q}[x]$ and has only real root φ which has absolute value greater than 1, while the other two conjugate complex roots are λ and $\delta = \bar{\lambda}$ with $|\lambda| = |\delta| < 1$. The Binet formula for N_n is given by

$$N_n = C_\varphi \varphi^n + C_\lambda \lambda^n + C_\delta \delta^n, \ \text{ for all } n \geq 0, \tag{2.12}$$

where

$$C_\varphi = \frac{\varphi}{(\varphi-\lambda)(\varphi-\delta)}, \qquad C_\lambda = \frac{\lambda}{(\lambda-\varphi)(\lambda-\delta)}, \qquad C_\delta = \frac{\delta}{(\delta-\varphi)(\delta-\lambda)}. \tag{2.13}$$

The coefficient C_φ has the minimal polynomial $31x^3 - 3x - 1$ over $\mathbb{Z}$ and all the zeros of this polynomial lie strictly inside the unit circle. Numerically, we have the following:

$$\begin{aligned} 1.46 <&\varphi < 1.47, \\ 0.82 < |\lambda| =& |\delta| < 0.83, \\ 0.41 <&C_\varphi < 0.42, \\ 0.27 < |C_\lambda| =& |C_\delta| < 0.28. \end{aligned} \tag{2.14}$$

Considering the estimates for $\varphi, \lambda, \delta, C_\varphi, C_\lambda, C_\delta$ and putting

$$e(n) := N_n - C_\varphi \varphi^n = C_\lambda \lambda^n + C_\delta \delta^n, \quad \text{we have that} \quad |e(n)| < \frac{1}{\varphi^{n/2}}, \tag{2.15}$$

for all $n \geq 1$. Using induction, it can be seen that

$$\varphi^{n-2} \leq N_n \leq \varphi^{n-1} \quad \text{hold for all} \quad n \geq 1. \tag{2.16}$$

2.5 Other useful lemmas

We conclude this section by recalling two lemmas that we will need in this work.

Lemma 2.4. ([5], Lemma 8) *For any non-zero real number x, we have*
(a) $0 < x < |e^x - 1|$.
(b) If $x < 0$ *and* $|e^x - 1| < 1/2$, *then* $|x| < 2|e^x - 1|$.

Lemma 2.5. ([9], Lemma 7) *If* $m \geq 1$, $S \geq (4m^2)^m$ *and* $\frac{x}{(\log x)^m} < S$, *then* $x < 2^m S (\log S)^m$.

3 Thabit and Williams numbers base b which are Padovan numbers

In this section, we will prove Theorem 1.1 and then give an illustration of this result for $2 \leq b \leq 10$.

3.1 Proof of Theorem 1.1

We start by assuming $n > 300$. Combining estimates (2.10) together with the equation (1.1) yields

$$(b-1)b^l - 1 \leq (b \pm 1)b^l \pm 1 = P_n \leq \alpha^{n-1},$$

which implies that

$$b^l < (b-1)b^l \leq \alpha^{n-1} + 1 < \alpha^{n+1}. \tag{3.17}$$

Taking logarithms on both sides of inequalities (3.17) , we obtain

$$l < n \quad \text{holds for all} \quad n > 300 \quad \text{and} \quad b \geq 2. \tag{3.18}$$

Using Binet's formula (2.8), we write the equation (1.1) as

$$(b \pm 1)b^l \pm 1 = p\alpha^n + \pi(n)$$

which further implies

$$(b \pm 1)b^l - p\alpha^n = \pi(n) \pm 1.$$

Taking absolute values on both sides, we obtain

$$\left|(b \pm 1)b^l - p\alpha^n\right| < |\pi(n)| + 1.$$

Dividing both sides of the above inequality by $p\alpha^n$, we get

$$|\Lambda_1| < \frac{2.77}{\alpha^n}, \tag{3.19}$$

where

$$\Lambda_1 = (b \pm 1)b^l p^{-1}\alpha^{-n} - 1. \tag{3.20}$$

To apply Theorem 2.1, we need to show that $\Lambda_1 \neq 0$. Indeed, $\Lambda_1 = 0$ implies

$$(b \pm 1)b^l = p\alpha^n.$$

If we conjugate the above relation with an automorphism σ of the Galois extension of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ given by $\sigma(\alpha) = \beta$ and then taking absolute values, we obtain

$$(b \pm 1)b^l = |q||\beta|^n$$

which leads us to a contradiction as left-hand side exceeds 1 for all $b \geq 2$, while its right-hand side is less than than 1. Hence, $\Lambda_1 \neq 0$. Thus, we apply Theorem 2.1 to Λ_1 given by (3.20) with the parameters:

$$\eta_1 := (b \pm 1)p^{-1}, \quad \eta_2 := b, \quad \eta_3 := \alpha,$$

and

$$b_1 := 1, \quad b_2 := l, \quad b_3 := -n.$$

Note that the three numbers η_1, η_2, η_3 are positive real numbers and belong to the field $\mathbb{L} := \mathbb{Q}(\alpha)$, so we can take $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 3$. Since $h(\eta_2) = \log b$ and $h(\eta_3) = \frac{\log \alpha}{3}$, it follows that

$$\max\{3h(\eta_2), |\log \eta_2|, 0.16\} = 3\log b := B_2$$

and

$$\max\{3h(\eta_3), |\log \eta_3|, 0.16\} = \log \alpha := B_3.$$

On the other hand, by using the properties of logarithmic height, it follows that

$$h(\eta_1) \leq h\left((b \pm 1)p^{-1}\right) < h\,(b \pm 1) + h(p) < \log b + \log 2 + \frac{\log 23}{3}.$$

Thus, we obtain

$$\max\{3h(\eta_1), |\log \eta_1|, 0.16\} = 3\log b + 3\log 2 + \log 23 := B_1.$$

We can take $D := \max\{|1|, |l|, |-n|\} = n$ by (3.18). Applying Matveev's theorem we get

$$-\log|\Lambda_1| < 1.4 \times 30^6 \times 3^{4.5} \times (3)^2(1+\log 3)(1+\log n)(3\log b + 3\log 2 + \log 23)(3\log b)(\log \alpha). \tag{3.21}$$

From the comparison of lower bound (3.21) and upper bound (3.19) of $|\Lambda_1|$ gives us

$$n\log\alpha - \log 2.77 < 2.74 \times 10^{12} \log n \log b \log(184b^3) \tag{3.22}$$

where we have used the fact $1 + \log n < 1.2 \log n$ for all $n > 300$. The last inequality (3.22), becomes

$$\frac{n}{\log n} < 9.78 \times 10^{12} \log b \log(184b^3).$$

Now, we apply Lemma 2.5 with $S := 9.78 \times 10^{12} \log b \log(184b^3)$, $x = n$ and $m = 1$. So, we have

$$n < 1.96 \times 10^{13} \log b \log(184b^3)(29.91 + \log(\log b) + \log(\log(184b^3))).$$

This completes the proof of Theorem 1.1.

3.2 Application for $2 \leq b \leq 10$

In this subsection, we will determine all solutions to the Diophantine equation (1.1) for $l \geq 1$ and $2 \leq b \leq 10$. So in this range we have the following result.

Theorem 3.1. *Let b be a positive integer such that $2 \leq b \leq 10$.*

1. *Then, the solutions (n, b, l) to the Diophantine equation $\mathcal{P}_n = (b+1)b^l - 1$ is $\{(7, 2, 1)\}$. More precisely, the Thabit numbers base b of first kind which are Padovan numbers is 5.*

2. *Then, the solutions (n, b, l) to the Diophantine equation $\mathcal{P}_n = (b+1)b^l + 1$ are in*

$$\{(8, 2, 1), (12, 4, 1), (14, 3, 2), (15, 2, 4), (19, 5, 2)\}.$$

   More precisely, the Thabit numbers base b of second kind which are Padovan numbers are 7, 21, 37, 49 and 151.

3. *Then, the solutions (n, b, l) to the Diophantine equation $\mathcal{P}_n = (b-1)b^l - 1$ are in*

$$\{(0, 2, 1), (1, 2, 1), (2, 2, 1), (5, 2, 2), (7, 3, 1), (8, 2, 3)\}.$$

   More precisely, the Williams numbers base b of first kind which are Padovan numbers are 1, 3, 5, 7.

4. *Then, the solutions (n, b, l) to the Diophantine equation $\mathcal{P}_n = (b-1)b^l + 1$ are in*

$$\{(5, 2, 1), (7, 2, 2), (8, 3, 1), (9, 2, 3), (12, 5, 1), (15, 4, 2), (16, 2, 6), (26, 6, 3)\}.$$

   More precisely, the Williams numbers base b of second kind which are Padovan numbers are 3, 5, 7, 9, 21, 49, 65, and 1081.

Proof. Since $2 \leq b \leq 10$, from Theorem 1.1 we obtain $n < 1.82 \times 10^{16}$. To reduce these bounds we need to apply Lemma 2.2. Let us define

$$\Gamma_1 := l \log b - n \log \alpha + \log(b \pm 1)p^{-1}. \tag{3.23}$$

Then $e^{\Gamma_1} - 1 := \Lambda_1$, where Λ_1 is defined by (3.20). Therefore, (3.19) implies that

$$|e^{\Gamma_1} - 1| < \frac{2.77}{\alpha^n}.$$

Note that $\Gamma_1 \neq 0$. Thus, we distinguish the following cases. If $\Gamma_1 > 0$, then we can apply Lemma 2.4 (a) to obtain

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 < \frac{2.77}{\alpha^n}.$$

If $\Gamma_1 < 0$, then from (3.19) we have that $|e^{\Gamma_1} - 1| < 1/2$ and therefore $e^{|\Gamma_1|} < 2$. Thus, by Lemma 2.4 (b), we have that

$$0 < |\Gamma_1| \leq e^{|\Gamma_1|} - 1 = e^{|\Gamma_1|}|e^{\Gamma_1} - 1| < \frac{5.54}{\alpha^n}.$$

So in both cases, we have

$$0 < |\Gamma_1| < 5.54 \cdot \alpha^{-n}. \tag{3.24}$$

Inserting (3.23) in (3.24) and dividing both sides by $\log \alpha$, it results that

$$0 < \left| l\left(\frac{\log b}{\log \alpha}\right) - n + \frac{\log(b \pm 1)p^{-1}}{\log \alpha} \right| < 19.7 \cdot \alpha^{-n}. \tag{3.25}$$

In order to apply Lemma 2.2, we set

$$\tau := \frac{\log b}{\log \alpha}, \quad \mu := \frac{\log(b \pm 1)p^{-1}}{\log \alpha}, \quad A := 19.7 \quad \text{and} \quad B := \alpha.$$

Note that $\tau = \frac{\log b}{\log \alpha} \notin \mathbb{Q}$ because if $\frac{\log b}{\log \alpha} = \frac{s}{t}$ for some coprime positive integers s and t, we would have $b^t = \alpha^s \in \mathbb{Z}$. Using the automorphism $\sigma \in Gal(\mathbb{L}/\mathbb{Q})$, the Galois group of the extension $\mathbb{L}/\mathbb{Q}$, such that $\sigma(\alpha) = \beta$. Applying this to the above relation and taking absolute values we get $1 < b^t = |\beta|^s < 1$, which is a contradiction. Now, we apply Lemma 2.2 with $M := 1.82 \times 10^{16}$ which is the upper bound of n. For every b such that $2 \leq b \leq 10$, we found that q_{44} the denominator of the 44-th convergent of τ exceeds $6M$ and $\epsilon(b) := ||\mu q_{44}|| - M||\tau q_{44}|| > 0$. Then, with the help of *Mathematica*, we can say that if the inequality (3.25) has a solution, then

$$n < \frac{\log(Aq_{44}/\epsilon)}{\log B} < 212,$$

contradicting the fact that $n > 300$. Now, we search for the solutions to the Diophantine equation (1.1) with

$$1 \leq l < n, \quad 0 \leq n \leq 300 \quad \text{and} \quad 2 \leq b \leq 10.$$

Using *Mathematica*, we checked that all the solutions of the Diophantine equation (1.1) are those listed in the statement of Theorem 3.1. This completes the proof. □

4 Thabit and Williams numbers base b which are Perrin numbers

In this section, we will prove Theorem 1.2 and then give an illustration of this result for $2 \leq b \leq 10$.

4.1 Proof of Theorem 1.2

We begin by considering $n > 350$. Using the inequality (2.11) and equation (1.2), we obtain

$$(b-1)b^l - 1 \leq (b \pm 1)b^l \pm 1 = E_n < \alpha^{n+1}$$

leads to

$$b^l < (b-1)b^l \leq \alpha^{n+1} + 1 < \alpha^{n+2}. \tag{4.26}$$

Taking logarithms on both sides of inequalities (4.26) , we obtain

$$l < n \quad \text{for all} \quad n > 350, \quad b \geq 2. \tag{4.27}$$

By employing Binet's formula (2.9), equation (1.2) can be expressed as

$$(b \pm 1)b^l \pm 1 = \alpha^n + \psi(n)$$

which we rewrite as

$$(b \pm 1)b^l - \alpha^n = \psi(n) \pm 1.$$

Taking absolute values on both sides, we obtain

$$\left|\alpha^n - (b \pm 1)b^l\right| < |\psi(n)| + 1.$$

Dividing both sides of the above inequality by α^n, we get

$$\left|(b \pm 1)b^l\alpha^{-n} - 1\right| < \frac{3}{\alpha^n}.$$

We deduce that

$$|\Lambda_2| < \frac{3}{\alpha^n}, \quad \text{where} \quad \Lambda_2 = (b \pm 1)b^l\alpha^{-n} - 1. \tag{4.28}$$

We can prove that $\Lambda_2 \neq 0$ by a similar method used to show that $\Lambda_1 \neq 0$. Now, let us apply Theorem 2.1 to Λ_2 given by (4.28) with the following parameters

$$s := 3, \qquad (\eta_1, a_1) := (b \pm 1, 1), \qquad (\eta_2, a_2) := (b, l), \quad \text{and} \quad (\eta_3, a_3) := (\alpha, -n).$$

The number field containing η_1, η_2, η_3 is $\mathbb{L} := \mathbb{Q}(\alpha)$, which has degree $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 3$. As calculated before, we take $B_2 = 3\log b$ and $B_3 = \log\alpha$. On the other hand, by using the properties of logarithmic height, it follows that

$$h(\eta_1) \leq h(b \pm 1) < \log b + \log 2.$$

Thus, we obtain

$$\max\{3h(\eta_1), |\log\eta_1|, 0.16\} = 3\log b + 3\log 2 := B_1.$$

In addition by (4.27), we can take $D := \max\{|1|, |l|, |-n|\} = n$. Therefore, according to Theorem 2.1, we have

$$-\log|\Lambda_2| < 1.4 \times 30^6 \times 3^{4.5} \times 3^2(1 + \log 3)(1 + \log n)(3\log b + 3\log 2)(3\log b)(\log\alpha). \tag{4.29}$$

Comparing this lower bound (4.29) with the upper bound of $|\Lambda_2|$ provided by (4.28), we arrive at

$$n\log\alpha - \log 3 < 2.74 \times 10^{12}\log n\log b\log(8b^3) \tag{4.30}$$

where we used the fact that inequality $1 + \log n < 1.2\log n$ holds for $n > 350$. Consequently, using (4.30) yields

$$\frac{n}{\log n} < 9.78 \times 10^{12}\log b\log(8b^3).$$

Applying the inequality from Lemma 2.5 with $S := 9.78 \times 10^{12}\log b\log(8b^3)$, $x = n$ and $m = 1$, we obtain

$$n < 1.96 \times 10^{13}\log b(\log(8b^3))(29.91 + \log(\log b) + \log(\log(8b^3))).$$

This establishes and finishes the proof of Theorem 1.2.

4.2 Application for $2 \leq b \leq 10$

In this part, we will search for all solutions to the Diophantine equation (1.2) for $l \geq 1$ and $2 \leq b \leq 10$. Thus, we get the following result.

Theorem 4.1. *Let b be a positive integer such that $2 \leq b \leq 10$.*

1. *Then, the solutions* (n, b, l) *to the Diophantine equation* $E_n = (b+1)b^l - 1$ *are in*

$$\{(5, 2, 1), (6, 2, 1), (12, 5, 1)\}.$$

More precisely, the Thabit numbers base b *of first kind which are Perrin numbers are* 5 *and* 29*.*

2. *Then, the solutions* (n, b, l) *to the Diophantine equation* $E_n = (b+1)b^l + 1$ *is* $\{(7, 2, 1)\}$*. More precisely, the Thabit numbers base* b *of second kind which are Perrin is* 7*.*

3. *Then, the solutions* (n, b, l) *to the Diophantine equation* $E_n = (b-1)b^l - 1$ *are in*

$$\{(0, 2, 2), (3, 2, 2), (5, 3, 1), (6, 3, 1), (7, 2, 3), (10, 3, 2), (12, 6, 1)\}.$$

More precisely, the Williams numbers base b *of first kind which are Perrin numbers are* 3*,* 5*,* 7*,* 17 *and* 29*.*

4. *Then, the solutions* (n, b, l) *to the Diophantine equation* $E_n = (b-1)b^l + 1$ *are in*

$$\{(0, 2, 1), (3, 2, 1), (5, 2, 2), (6, 2, 2), (7, 3, 1), (10, 2, 4)\}.$$

More precisely, the Williams numbers base b *of second kind which are Perrin numbers are* 3*,* 5*,* 7 *and* 17*.*

Proof. Since $2 \leq b \leq 10$, from Theorem 1.2 we obtain $n < 1.34 \times 10^{16}$. To reduce these bounds we need to apply Lemma 2.2. Let us define

$$\Gamma_2 := l \log b - n \log \alpha + \log(b \pm 1). \tag{4.31}$$

Therefore, inequality (4.28) can be rewritten as $|e^{\Gamma_2} - 1| < 3/\alpha^n$. In this case, $\Gamma_2 \neq 0$. If $\Gamma_2 > 0$, then we can apply Lemma 2.4 (a) to obtain $|\Gamma_2| < 3/\alpha^n$. Now, if $\Gamma_2 < 0$, then $|e^{\Gamma_2} - 1| < 1/2$. Thus, by Lemma 2.4 (b), we have that $|\Gamma_2| < 2|e^{\Gamma_2} - 1| < 6/\alpha^n$. So, in both cases we have

$$0 < |\Gamma_2| < 6 \cdot \alpha^{-n}. \tag{4.32}$$

Replacing (4.32) by (4.31) and dividing through by $\log \alpha$, we obtain

$$0 < |l\tau - n + \mu| < A \cdot B^{-n}. \tag{4.33}$$

where

$$\tau := \frac{\log b}{\log \alpha}, \quad \mu := \frac{\log(b \pm 1)}{\log \alpha}, \quad A := 21.34 \quad \text{and} \quad B := \alpha.$$

Now, we apply Lemma 2.2 to (4.33) by taking $M := 1.34 \times 10^{16}$ which is an upper bound on n. For every b such that $2 \leq b \leq 10$ we found that q_{43} the denominator of the 43-th convergent of τ exceeds $6M$. A quick computation with *Mathematica* gives us that the inequality $\epsilon(b) := ||\mu q_{43}|| - M||\tau q_{43}|| > 0$ except for the case $b = 2$. Then, with the help of Mathematica, we can say that if the inequality (4.33) has a solution for $3 \leq b \leq 10$, then

$$n < \frac{\log(Aq_{43}/\epsilon)}{\log B} < 219.$$

On the other hand, in the case $b = 2$; from (4.33), we have

$$0 < \left| l\left(\frac{\log b}{\log \alpha}\right) - n \right| < 21.34 \cdot \alpha^{-n}.$$

If we divide this inequality by l, we get

$$0 < \left| \frac{\log b}{\log \alpha} - \frac{n}{l} \right| < \frac{21.34}{l \cdot \alpha^n}.$$

For $n > 350$, it can be seen that

$$\frac{\alpha^n}{2(21.34)} > 1.7 \times 10^{41} > 1.34 \times 10^{16} > l,$$

and so we have

$$\left| \frac{\log b}{\log \alpha} - \frac{n}{l} \right| < \frac{21.34}{l \cdot \alpha^n} < \frac{1}{2l^2}.$$

Therefore, from Lemma 2.3, we conclude that the rational number $\frac{n}{l}$ is a convergent to $\tau = \frac{\log b}{\log \alpha}$ with $b = 2$. Let

$$[a_0, a_1, a_2, \dots] := [2, 2, 6, 1, 1, 1, 2, 1, 13, 3, 1, 1, 1, 1, 1, 8, 1, 3, 2, 2, 7, 1, 2,\ 5, 1, 2 \dots]$$

be the continued fraction expansion of τ. Since $l < 1.34 \times 10^{16}$, we can apply Lemma 2.3 with $M := 1.34 \times 10^{16}$. A quick search using *Mathematica* reveals that

$$q_{42} < 1.34 \times 10^{16} < q_{43}.$$

Furthermore, $a(M) := \max\{a_i : i = 0, 1, \dots, 43\} = 80$. So by Lemma 2.3, we have

$$\frac{1}{(a(M)+2) \cdot l^2} \leq \left| \frac{\log b}{\log \alpha} - \frac{n}{l} \right| < \frac{21.34}{l \cdot \alpha^n}.$$

This yields to

$$n < \frac{\log(21.34 \cdot (a(M)+2) \cdot 1.34 \times 10^{16})}{\log \alpha} < 159.$$

Therefore, in all cases, we have $n < 219$. This is a contradiction to the fact that $n > 350$. Now, we search for the solutions to the Diophantine equation (1.2) with

$$1 \leq l < n, \quad 0 \leq n \leq 350 \quad \text{and} \quad 2 \leq b \leq 10.$$

Using *Mathematica*, we checked that all the solutions of the Diophantine equation (1.2) are those listed in the statement of Theorem 4.1. This completes the proof. □

5 Thabit and Williams numbers base b which are Narayana's cows numbers

In this section, we will prove Theorem 1.3 and then give an illustration of this result for $2 \leq b \leq 10$.

5.1 Proof of Theorem 1.3

We start by assuming $n > 400$. By inequality (2.16), the equation (1.3) implies that

$$(b-1)b^l - 1 \leq (b \pm 1)b^l \pm 1 = N_n < \varphi^{n-1}$$

which gives

$$b^l < (b-1)b^l \leq \varphi^{n-1} + 1 < \varphi^{n+1}. \tag{5.34}$$

Taking logarithms on both sides of inequalities (5.34), we obtain

$$l < n \quad \text{holds for all} \quad n > 400 \quad \text{and} \quad b \geq 2. \tag{5.35}$$

Using Binet's formula (2.15), equation (1.3) can be rearranged as

$$(b \pm 1)b^l \pm 1 = C_\varphi \varphi^n + e(n)$$

and consequently we have

$$(b \pm 1)b^l - C_\varphi \varphi^n = e(n) \pm 1.$$

Dividing by $C_\varphi \varphi^n$ and taking absolute values, we obtain

$$|\Lambda_3| < \frac{3.4}{\varphi^n} \quad \text{where} \quad \Lambda_3 = (b \pm 1)b^l C_\varphi^{-1} \varphi^{-n} - 1. \tag{5.36}$$

We have $\Lambda_3 \neq 0$, otherwise we would get

$$(b \pm 1)b^l = C_\varphi \varphi^n$$

If we conjugate the above relation with an automorphism σ of the Galois extension of $\mathbb{Q}(\varphi)$ over $\mathbb{Q}$ given by $\sigma(\varphi) = \lambda$ and then taking absolute values, we obtain

$$(b \pm 1)b^l = |C_\lambda||\lambda|^n$$

which leads us to a contradiction as left-hand side exceeds 1 for all $b \geq 2$, while its right-hand side is less than than 1. Thus, $\Lambda_3 \neq 0$. Thus, we apply Theorem 2.1 to Λ_3 given by (5.36) with the parameters:

$$s := 3, \qquad (\eta_1, a_1) := \left((b \pm 1)C_\varphi^{-1}, 1\right), \qquad (\eta_2, a_2) := (b, l), \quad \text{and} \quad (\eta_3, a_3) := (\varphi, -n).$$

Then, we obtain

$$h(\eta_1) \leq h\left((b \pm 1)C_\varphi^{-1}\right) < \log b + \log 2 + \frac{\log 31}{3}, \quad h(\eta_2) = \log b \quad \text{and} \quad h(\eta_3) = \frac{\log \varphi}{3}.$$

It is clear that $\mathbb{L} := \mathbb{Q}(\varphi)$ contains η_1, η_2, η_3 and has degree $d_\mathbb{L} = [\mathbb{L} : \mathbb{Q}] = 3$. So, we take

$$B_1 := 3\log b + 3\log 2 + \log 31, \quad B_2 := 3\log b, \quad B_3 := \log \varphi \quad \text{and} \quad D := n.$$

Applying Theorem 2.1 and comparing the resulting inequality with (5.36), we obtain

$$n \log \varphi - \log 3.4 < 3.72 \times 10^{12} \log n \log b \log(248b^3) \tag{5.37}$$

where we used the fact that inequality $1 + \log n < 1.2 \log n$ holds for $n > 400$. So, inequality (5.37) becomes

$$\frac{n}{\log n} < 9.76 \times 10^{12} \log b \log(248b^3).$$

Thus, putting $S := 9.76 \times 10^{12} \log b \log(248b^3)$, $x = n$ and $m = 1$ in Lemma 2.5, we obtain

$$n < 1.95 \times 10^{13} \log b \log(248b^3)(29.91 + \log(\log b) + \log(\log 248b^3)).$$

This establishes and completes the proof of Theorem 1.3.

5.2 Application for $2 \leq b \leq 10$

In this subsection, we give all the solutions of the Diophantine equation (1.3) for $l \geq 1$ and $2 \leq b \leq 10$. Here is our result in this range.

Theorem 5.1. *Let b be a positive integer such that $2 \leq b \leq 10$.*

1. *Then, the solutions (n, b, l) to the Diophantine equation $N_n = (b+1)b^l - 1$ are in*

$$\{(9, 4, 1), (11, 6, 1)\}.$$

More precisely, the Thabit numbers base b of first kind which are Narayana's cows numbers are 19 and 41.

2. *Then, the solutions (n, b, l) to the Diophantine equation $N_n = (b+1)b^l + 1$ are in*

$$\{(8, 2, 2), (8, 3, 1), (22, 7, 3)\}.$$

More precisely, the Thabit numbers base b of second kind which are Narayana's cows numbers are 13 and 2745.

3. *Then, the solutions (n, b, l) to the Diophantine equation $N_n = (b-1)b^l - 1$ are in*

$$\{(0, 2, 1), (1, 2, 1), (2, 2, 1), (4, 2, 2), (9, 5, 1), (11, 7, 1)\}.$$

More precisely, the Williams numbers base b of second kind which are Narayana's cows numbers are 1, 3, 19 and 41.

4. *Then, the solutions (n, b, l) to the Diophantine equation $N_n = (b-1)b^l + 1$ are*

$$\{(4, 2, 1), (7, 2, 3), (8, 4, 1), (9, 3, 2), (14, 2, 7)\}.$$

More precisely, the Williams numbers base b of second kind which are Narayana's cows numbers are 3, 9, 13, 19 and 129.

Proof. Since $2 \leq b \leq 10$, from Theorem 1.3 we obtain $n < 1.85 \times 10^{16}$. To reduce these bounds we need to apply Lemma 2.2. Let us define

$$\Gamma_3 := l \log b - n \log \varphi + \log(b \pm 1) C_\varphi^{-1}. \tag{5.38}$$

From the inequality (5.36), one can see that

$$0 < |\Gamma_3| < 6.8 \cdot \varphi^{-n}. \tag{5.39}$$

Inserting (5.38) in (5.39) and dividing both sides by $\log \varphi$, we get

$$0 < \left| l \left(\frac{\log b}{\log \varphi} \right) - n + \frac{\log(b \pm 1) C_\varphi^{-1}}{\log \varphi} \right| < 17.79 \cdot \varphi^{-n}. \tag{5.40}$$

In order to apply Lemma 2.2, we set

$$\tau := \frac{\log b}{\log \varphi}, \quad \mu := \frac{\log(b \pm 1) C_\varphi^{-1}}{\log \varphi}, \quad A := 17.79 \quad \text{and} \quad B := \varphi.$$

Note that $\tau = \frac{\log b}{\log \varphi} \notin \mathbb{Q}$. Now, we apply Lemma 2.2 with $M := 1.85 \times 10^{16}$ which is the upper bound of n. For $2 \leq b \leq 10$, we found that q_{44} the denominator of the 44-th convergent of τ exceeds $6M$ and $\epsilon(b) := ||\mu q_{44}|| - M||\tau q_{44}|| > 0$. Then, with the help of *Mathematica*, we can say that if the inequality (5.40) has a solution, then

$$n < \frac{\log(Aq_{44}/\epsilon)}{\log B} < 169,$$

contradicting the fact that $n > 400$. Now, we search for the solutions to the Diophantine equation (1.3) with

$$1 \leq l < n, \quad 0 \leq n \leq 400 \quad \text{and} \quad 2 \leq b \leq 10.$$

Using *Mathematica*, we checked that all the solutions of the Diophantine equation (1.3) are those listed in the statement of Theorem 5.1. This completes the proof. □

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