Dynamically generated tilt of isocurvature fluctuations

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Light scalar fields acquire isocurvature fluctuations during inflation. While these fluctuations could lead to interesting observable signatures at small scales, they are strongly constrained on large scales by cosmic microwave background observations. When the mass of the scalar is much lighter than the inflationary Hubble scale, $m \ll H_I$, the spectrum of these fluctuations is flat. Meanwhile, if $m \gg H_I$, the fluctuations are suppressed. A blue-tilted isocurvature spectrum which exhibits enhanced structure on small scales but avoids observational constraints on large scales therefore requires a coincidence of scales $m \sim H_I$ for a free massive scalar. In this *Letter*, we show that if a scalar field possesses a nontrivial potential, its inflationary dynamics naturally cause this condition to be satisfied, and so a blue-tilted spectrum is generically expected for a large class of potentials. Specifically, if its potential V exhibits a region which satisfies the slow-roll condition $V'' < 3H_I^2$, the scalar condensate will spend most of inflation close to the boundary of this region, so that its effective mass is typically close to H_I. The resulting blue tilt is inversely proportional to the number of e-folds of inflation prior to horizon crossing. If the scalar is long-lived, this mechanism leads to an attractor prediction for its relic abundance, which is insensitive to initial conditions of the scalar. In particular, a scalar field with quartic self-interactions can achieve the correct abundance to constitute all of the dark matter for a wide range of masses. We compute the relationship between the mass and self-coupling of quartic dark matter predicted by this mechanism.

I. INTRODUCTION

Although the Standard Model (SM) of particle physics explains many observed phenomena, a number of open problems necessitate the existence of new physics beyond the SM. Perhaps the simplest possible modification to the SM is the addition of a new scalar field. Most notably, if such a new field is cosmologically stable, it could constitute the dark matter (DM) [1–8]. Scalar fields also appear in several theoretical models designed to address other open problems, including the strong CP problem [9–11], the hierarchy problem [12–14], the origin of the matter-antimatter asymmetry [15–17], and the cosmological constant problem [18–21].

Extensive cosmological observations indicate that the universe is nearly homogeneous, with correlated (adiabatic) primordial fluctuations $P_\delta^{\mathrm{ad}} \sim 10^{-9}$ in all components [22]. In order to explain this observation, a period of early expansion, known as inflation, is often proposed [23, 24]. One generic feature of scalar fields is that they can acquire independent (isocurvature) fluctuations during inflation. Specifically, a free scalar field with mass much smaller than the inflationary Hubble scale, $m \ll H_I$, acquires isocurvature fluctuations on all scales, while a heavy scalar with $m \gg H_I$ acquires no fluctuations [25, 26]. On small scales, isocurvature fluctuations can lead to interesting signatures, such as gravitational waves [27–29] or non-Gaussianities [30–32]. However, they are constrained to have $P_\delta^{\mathrm{iso}} \lesssim 10^{-10}$ on large scales $k \lesssim 10^{-2}\,\mathrm{Mpc}^{-1}$ [33]. In order for a new non-interacting scalar field to present interesting signatures from enhanced fluctuations, it must therefore have $m \sim H_I$, which results in a blue-tilted isocurvature spectrum [28]. Lighter scalars may gain a large effective mass m_{eff} during inflation via non-minimal couplings to gravity or interactions with other fields, in order to avoid isocurvature constraints [34–37]. By contrast, a non-interacting vector field naturally exhibits a blue-tilted spectrum [38].

The purpose of this *Letter* is two-fold. The first purpose is to demonstrate that self-interactions can generically result in a blue-tilted spectrum for a scalar spectator ϕ during inflation. Specifically, if ϕ possess a nontrivial potential $V(\phi)$ and inflation does not last too long, the resulting spectrum will exhibit a blue tilt, for a large class of potentials. Similar scenarios have been studied in the case where inflation lasts a long time, so that the condensate (or zero mode) ϕ_0 settles near the minimum of its potential [36, 39, 40]. In our scenario, the condensate will instead spend the majority of inflation undergoing slow-roll dynamics. In particular, if ϕ_0 begins outside the regime where the slow-roll approximation holds, but rolls toward it, then it will quickly enter the slow-roll regime and remain near its boundary for the duration of inflation. In other words, the slow-roll condition will be nearly saturated for the majority of inflation. As we will see, the slow-roll condition is precisely $m_{\mathrm{eff}}^2 = V''(\phi_0) \lesssim H_I^2$. Therefore, the dynamics of the condensate naturally produce the conditions necessary for a blue-tilted spectrum! We will see that the tilt of the spectrum is inversely proportional to the number of e-folds before horizon crossing, i.e. if inflation began $\mathcal{O}(10)$ e-folds before the CMB modes exited the horizon, a generic potential will result

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in an $\mathcal{O}(0.1)$ blue tilt at CMB scales. Particular choices of potential can result in larger tilts (see Fig. 1).

The second purpose is to demonstrate that if ϕ is long-lived, this mechanism can produce the correct relic abundance for ϕ to constitute the DM. Moreover, because the slow-roll solution is an attractor, this relic abundance is insensitive to initial conditions of the scalar field. For fixed inflationary parameters, this implies a predictive relationship between the parameters of the potential, similar to the misalignment mechanism for axion DM [5]. In Fig. 2, we show this relationship between the mass m and self-coupling λ of a quartic potential.

This work is organized as follows. In Sec. II, we studying the dynamics of ϕ during inflation, focusing first on its condensate and then on its perturbations. We derive the slow-roll condition for the condensate, and show that the slow-roll parameter controls the tilt of the perturbation spectrum. In Sec. III, we study the subsequent dynamics during radiation domination, in the case where ϕ makes up the DM. In this section, we choose to focus on a quartic potential [see Eq. (25)]. We first track the relic abundance, and then see how the perturbations derived in Sec. II evolve after inflation. In Sec. IV, we conclude. In Appendix A, we elaborate on the details of the relic density calculation, and in Appendix B, we study how the density perturbations of a scalar field evolve. We make all the code used in this work publicly available on Github [41].

II. INFLATION

We begin by studying the evolution of a spectator scalar field ϕ during inflation. We will consider the scenario where the spectator consists of small perturbations on top of a homogeneous condensate[1]

$$\phi(x,t) = \phi_0(t) + \delta\phi(x,t). \tag{1}$$

First, we will study the dynamics of the condensate during inflation in the presence of a nontrivial potential $V(\phi)$. We will derive the slow-roll condition

$$|\alpha| \equiv \frac{|V''(\phi_0)|}{3H^2} \ll 1 \tag{2}$$

for the condensate dynamics (where $H = \dot{a}/a$ is the Hubble scale), and see that for a large class of potentials, $\alpha \lesssim 1$ for most of inflation. Then we will study the perturbations of the spectator. In particular, we will show that the spectrum of the perturbations receive a blue tilt from a nonzero α. We therefore conclude that a small blue tilt can generically arise in the power spectrum of a spectator.

Throughout this section, we will ignore the dynamics of the inflaton, and we will assume an exact de Sitter spacetime for inflation, given by the scale factor

$$a = e^{H_I t} = e^N = -\frac{1}{H_I \tau}, \tag{3}$$

in terms of physical time t, e-folds N, or conformal time τ. This corresponds to a constant Hubble scale $H(t) = H_I$. Accounting for the evolution of $H(t)$ introduces subleading corrections to the tilt of the power spectrum (see footnote 3).

A. Condensate

The evolution of the spectator condensate during inflation is given by[2]

$$\ddot{\phi}_0 + 3H_I\dot{\phi}_0 + V'(\phi_0) = 0, \tag{4}$$

where dots represent derivatives with respect to t. Let us suppose that the inital value $\phi_{0,i}$ satisfies $\alpha_i \gg 1$. In this case, the final term in Eq. (4) dominates over the second term, so that ϕ_0 oscillates in its potential. The second term, however, damps its motion, leading to a redshifting of the energy density

$$\rho \propto a^{-3(1+w)} = e^{-3(1+w)N}, \tag{5}$$

where w is the equation of state of the spectator, which depends on its potential. For instance, in a monomial potential

$$V(\phi) = \Lambda^{4-p}\phi^p, \tag{6}$$

the equation of state is given by [see Eq. (B-9)]

$$w = \frac{p-2}{p+2}. \tag{7}$$

Noting that $\rho \propto V(p)$, this leads to a scaling

$$\alpha \propto \phi^{p-2} \propto e^{-\frac{6(p-2)}{p+2}N} \tag{8}$$

during the initial stages of inflation. In other words, if ϕ_0 begins in a region where α is large, it will quickly evolve so that α decreases exponentially (as a function of N).

[1] Note that even if the scalar field begins in an inhomogeneous configuration, inflation will effectively "zoom in" on a small region of the configuration, leading to a homogeneous scenario. More specifically, the fluctuations of the field over a Hubble patch are $\delta\phi \sim (aH_I)^{-1}\nabla\phi$. Even in the absence of dynamics of the field, this redshifts relative to ϕ_0, so that Eq. (1) will eventually become a valid perturbative expansion.

[2] The nontrivial potential $V(\phi)$ can allow for fragmentation of the condensate into particles [36]. The couplings considered in this work are weak enough that this fragmentation is negligible during inflation (see Fig. 2).

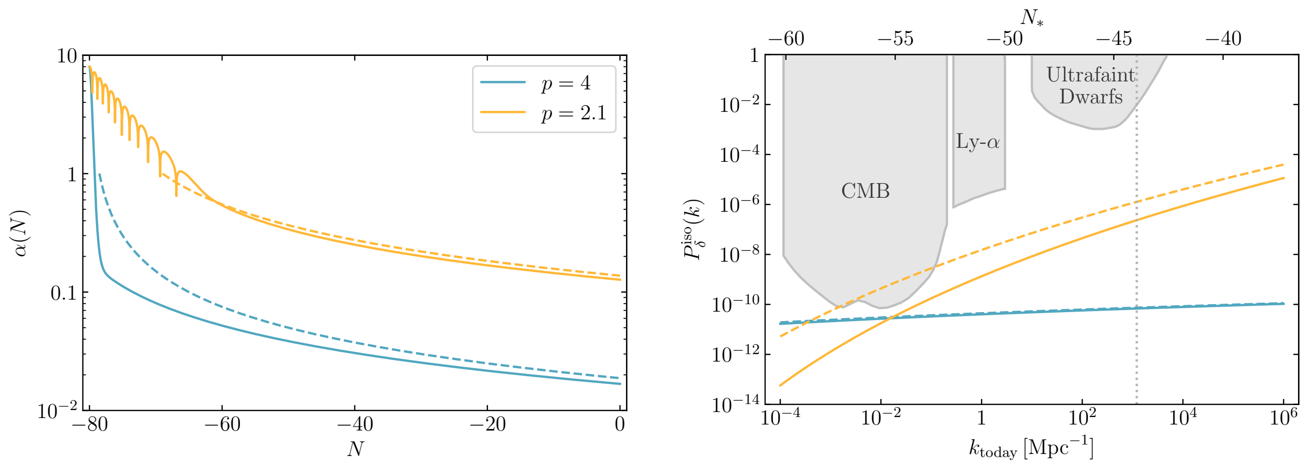

FIG. 1. *Left:* Evolution of α [see Eq. (2)] as a function of e-folds N during inflation. We take the inflationary Hubble scale to be $H_I = 10^{12}$ GeV in this plot. The spectator begins with $\alpha_i \gg 1$ at the start of inflation $N_i = -80$. Initially, $\alpha(N)$ oscillates and decreases in magnitude exponentially. Once $\alpha < 1$, it decreases as $1/N$. Dashed lines show the analytic approximation in Eq. (12). We show evolutions for two choices of potential $V(\phi) \sim \phi^p$. The blue curve has $\kappa = \mathcal{O}(1)$, while the orange has small κ. In accordance with Eq. (12), the latter results in much larger values of α. *Right:* Primordial isocurvature spectrum for scalar DM with same potentials as in left plot. Dashed lines show the analytic approximation in Eq. (21). In both numerical and analytic curves, we include the constant factor from late-time evolution [see Eq. (33)], which for quartic DM with mass $m = 10\,\text{eV}$ applies to the left of the grey dotted line [see Eq. (31)]. The tilt of the spectrum is related to α at horizon crossing N_* (shown on top axis), as in Eq. (22). For $\kappa = \mathcal{O}(1)$, this generically results in a small blue tilt. For smaller κ, as in the orange curve, the tilt can be much stronger. In grey, we show constraints on the primordial isocurvature spectrum from observations of the CMB [42], Lyman-α forest [43, 44], and ultrafaint dwarf galaxies [44]. In order to avoid these constraints, the normalizations of the potentials are fixed at $\lambda = 10^{-9}$ [as in Eq. (25)] for the blue curve and $\Lambda = 4.4 \times 10^{11}$ GeV [as in Eq. (6)] for the orange curve. Note that the total matter power spectrum will be the sum of the isocurvature and adiabatic fluctuations [see Eq. (33)].

Let us now understand when ϕ_0 begins to evolve slowly. When the first term in Eq. (4) can be neglected, the condensate follows the slow-roll solution

$$\dot{\phi}_0 = -\frac{V'(\phi_0)}{3H_I}. \tag{9}$$

The slow-roll solution holds when $|\ddot{\phi}_0| \ll H|\dot{\phi}_0|$, or equivalently when $|\alpha| \ll 1$.[3] The slow-roll solution in Eq. (9) can be re-expressed in terms of α, as

$$\frac{d\alpha}{dN} = -\kappa\alpha^2, \qquad \kappa \equiv \frac{V'''(\phi_0)V'(\phi_0)}{V''(\phi_0)^2}. \tag{10}$$

For a monomial potential, $\kappa = \frac{p-2}{p-1}$. This differential equation is solved by

$$\alpha(N) = \frac{1}{\kappa(N - N_{\rm sr}) + 1}, \tag{11}$$

where $\alpha(N_{\rm sr}) \equiv 1$.

We can now outline the scenario of interest. Any potential $V(\phi)$ will consist of fast-roll regimes with $|\alpha| \gg 1$ and slow-roll regimes with $|\alpha| \ll 1$. We are interested in the case where ϕ_0 begins inflation in the fast-roll regime and rolls toward the slow-roll regime. In such a scenario, α will decrease exponentially in N, so that ϕ_0 quickly reaches the slow-roll regime. From then on, it will decrease as $1/N$. If inflation begins at N_i, then we generically expect

$$\alpha(N) \approx \frac{1}{\kappa(N - N_i)}. \tag{12}$$

In the left plot of Fig. 1, we show sample evolutions of α for two monomial potentials, which demonstrate this behavior.

In summary, the necessary conditions on the potential and initial values for this scenario to occur are:

$$\begin{aligned}
V''(\phi_{0,i}) &> 3H_I^2 && \text{(begin in fast-roll regime)} \\
V''(\phi) &< 3H_I^2 \text{ for some } \phi && \text{(slow-roll regime exists)} \\
V'''(\phi)V'(\phi) &> 0 && \text{(roll toward slow-roll regime)} \\
V(\phi_{0,i}) &\ll 3H_I^2 M_{\rm pl}^2 && \text{(spectator subdominant)}
\end{aligned}$$

Note that the third condition implies that this mechanism does not work for certain potentials, e.g. a cosine

[3] If one allows for $H(t)$ to be time-dependent, one arrives at a second slow-roll condition from differentiating the denominator of Eq. (9). This condition is precisely the first slow-roll condition for inflation, $\epsilon \equiv -\dot{H}/H^2 \ll 1$. If ϕ were the inflaton (so that $V = 3H^2 M_{\rm pl}^2$), rather than a spectator, the condition $|\alpha| \ll 1$ would be the usual second slow-roll condition for the inflaton, $|\eta| \equiv M_{\rm pl}^2 |V''|/V \ll 1$.

potential $V(\phi) \sim \cos(\phi)$.

B. Perturbations

Next we consider the perturbations $\delta\phi$ and compute the isocurvature power spectrum imprinted on the field during inflation. In the absence of any potential, the resulting power spectrum is flat, while a nonzero $V''(\phi_0)$ imprints a blue tilt on the spectrum. See Refs. [26, 36, 45] for similar derivations. The perturbations of ϕ evolve as

$$\delta\ddot{\phi}_k + 3H_I\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} + V''(\phi_0)\right)\delta\phi_k = 0, \tag{13}$$

where $\delta\phi_k(t)$ are the Fourier modes of $\delta\phi(x,t)$. This can be rewritten in terms of $f_k \equiv a\delta\phi_k$ and conformal time τ as

$$\partial_\tau^2 f_k + \left(k^2 - \frac{\partial_\tau^2 a}{a} + a^2 V''(\phi_0)\right) f_k = 0. \tag{14}$$

Before the mode exits the horizon, $|k\tau| \gg 1$, Eq. (14) takes the form of a harmonic oscillator with frequency k. As initial conditions for this oscillator, we take the Bunch-Davies vacuum

$$f_k(\tau \to -\infty) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{15}$$

In the superhorizon limit, $|k\tau| \ll 1$, Eq. (14) becomes

$$\partial_\tau^2 f_k'' + \frac{3\alpha(\tau) - 2}{\tau^2} f_k = 0, \tag{16}$$

where $\alpha(\tau)$ encodes the evolution of the condensate $\phi_0(\tau)$. For constant (or slowly varying) $\alpha \ll 1$, this is solved by

$$f_k \sim \frac{(-k\tau)^{\alpha-1}}{\sqrt{k}} \implies \delta\phi_k \sim H_I^{1-\alpha} k^{\alpha-\frac{3}{2}} e^{-\alpha N}. \tag{17}$$

Note that $\delta\phi_k$ decreases exponentially (as a function of N) while outside the horizon. This means that longer wavelength modes, which exit the horizon earlier, will exhibit suppressed power compared to shorter wavelength modes. In other words, this implies a blue tilt of the power spectrum. Specifically, the power spectrum of the spectator perturbations (at the end of inflation) is approximately

$$P_{\delta\phi}(k)|_{\text{rh}} \equiv \frac{k^3}{2\pi^2}|\delta\phi_k|^2 \tag{18}$$

$$\approx \left(\frac{H_I}{2\pi}\right)^2 \cdot \exp\left(-2\int_{N_*}^{N_{\text{rh}}} \alpha(N)dN\right), \tag{19}$$

where N_* denotes horizon crossing ($k = a_* H_I$) and $N_{\text{rh}} = 0$ denotes the end of inflation. The first factor in Eq. (19) represents the well-known result for a free massless ($\alpha = 0$) spectator [25, 46], while the second factor represents the damping due to the nonzero potential.

Ultimately, we are interested in the statistics of the density perturbations[4]

$$\delta_k \equiv \frac{\delta\rho_k}{\rho} \approx \frac{V'(\phi_0)}{V(\phi_0)}\delta\phi_k, \tag{20}$$

where $\delta\rho_k$ are the Fourier modes of $\rho(x,t)$. From Eq. (19), we can compute the power spectrum of the density perturbations

$$P_\delta(k)|_{\text{rh}} \approx \frac{1}{12\pi^2\alpha_{\text{rh}}}\left[\frac{V'^2 V''}{V^2}\right]_{\text{rh}} \exp\left(-2\int_{N_*}^{N_{\text{rh}}} \alpha(N)dN\right). \tag{21}$$

Importantly, P_δ scales with k as

$$\frac{d\log P_\delta}{d\log k} = 2\alpha(N_*). \tag{22}$$

We see that the tilt of the spectrum is set by the value of α when the relevant mode exits the horizon. As the modes we measure in the CMB exited the horizon around $N_* \approx -60$, then the tilt of the spectrum on large scales will be approximately $2/\kappa(N_{\text{tot}} - 60)$, where N_{tot} is the total number of e-folds during inflation. For instance, for $\kappa = \mathcal{O}(1)$ and $N_{\text{tot}} - 60 = \mathcal{O}(10)$, we expect an $\mathcal{O}(0.1)$ blue tilt. However, the tilt may be larger for smaller κ or fewer e-folds. In the right plot of Fig. 1, we show the primordial isocurvature spectrum [with the constant factor from late-time evolution; see Sec. III B] for two monomial potentials. These are shown as a function of current momentum scale (see Appendix A)

$$k_{\text{today}} = \frac{a_{\text{rh}}}{a_{\text{today}}} k \approx 10^{-28}\sqrt{\frac{10^{12}\,\text{GeV}}{H_I}} \cdot k. \tag{23}$$

The potential with small κ yields large isocurvature fluctuations at small scales.

III. RADIATION DOMINATION

Now that we have understood how the condensate and perturbations of the spectator field are generated during inflation, we study how these further evolve during radiation domination. While in Sec. II, we primarily dealt with the field ϕ itself, during radiation domination, it will be more useful to deal with the energy density ρ. This is because ϕ_0 begins to oscillate, and so if $V(\phi)$ is nontrivial, the final term in Eq. (13) can lead to parametric resonance of $\delta\phi_k$. This implies that the perturbative

[4] Here we assume that the energy density is dominated by the potential energy and neglect the kinetic and gradient energy contributions (see Eq. (B-3) for full expression for ρ).

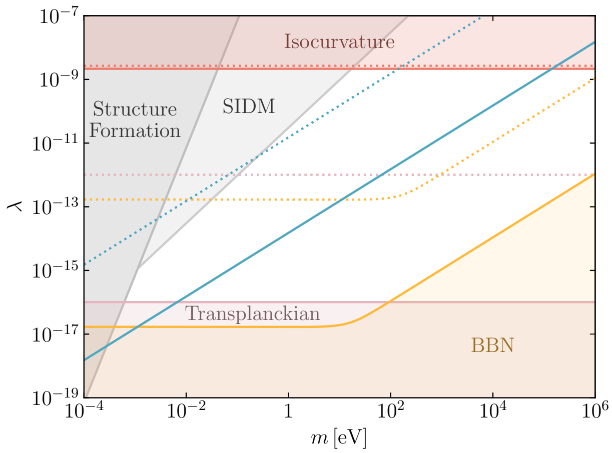

FIG. 2. Parameter space for inflationary production of scalar DM with a quartic potential [see Eq. (25)]. The blue line indicates parameters which produce the correct DM relic abundance for $H_I = 10^{10}$ GeV and $N_{\rm tot} = 80$ (assuming instantaneous reheating). Colored regions indicate constraints on this production scenario, including: overproduction of isocurvature fluctuations (see right plot of Fig. 1), violation of $N_{\rm eff}$ constraints at the time of BBN, and transplanckian initial conditions $\phi_{0,i} > M_{\rm pl}$. (Note that all parameters below the blue line are also constrained due to overclosure of the universe.) Dotted lines show these curves for $H_I = 10^{12}$ GeV. In grey, we show late-time constraints on self-interacting dark matter (SIDM) and disruption of structure formation [47].

decomposition of the field in Eq. (1) will break down. The energy density will, however, remain nearly homogeneous.

For simplicity, in this section, we will assume instantaneous reheating at $t_{\rm rh} = 0$ (or $\tau_{\rm rh} = -1/H_I$) into radiation domination. This is described by the scale factor

$$a = \sqrt{1 + 2H_I t} = 2 + H_I \tau \tag{24}$$

for $t > t_{\rm rh}$ (or $\tau > \tau_{\rm rh}$), which corresponds to a Hubble scale $H(a) = H_I a^{-2}$. In order to study the dynamics of the energy density, we will also need to fix a potential. We will focus on a quartic potential with a small mass

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{25}$$

We note that, for the parameter space of interest in this work, this potential is not technically natural, as the quartic term can lead to large radiative corrections for the mass term. Nevertheless, similarly unnatural models are often considered for the inflaton or for DM [6, 33, 48, 49].

We will first study how the average energy density of the scalar evolves through radiation domination. We will find viable parameter space where a scalar field with the potential in Eq. (25) can constitute the entirety of DM (see Fig. 2). Then we will track the density perturbations and see that the primordial power spectrum is related to the power spectrum at reheating by an $\mathcal{O}(1)$ factor. Therefore, the tilt derived in Eq. (22) remains valid at late times.

A. Relic density

First let us consider the evolution of the average energy density. We saw in Sec. II A that, at the beginning of inflation, the scalar quickly reaches a slow-roll solution, which acts as an attractor. Thus for a fixed length of inflation $N_{\rm tot}$, the energy density in the scalar at the end of inflation, and therefore also its energy density today, is fixed regardless of initial conditions. This implies a predictive relationship between the parameters of the potential in order to produce the correct DM relic abundance, analogous to the relationship between the mass and decay constant of axion DM implied by the misalignment mechanism [5].

As radiation domination progresses, the scalar will exhibit various different equations of state, which will affect how its energy density redshifts. The scalar will always begin radiation domination as it ended inflation: slow rolling. This corresponds to an equation of state $w = -1$, or equivalently a constant energy density $\rho \propto a^0$. If the scalar is to be the DM, it should end up redshifting like matter, i.e. $w = 0$ or $\rho \propto a^{-3}$. In the case of the potential in Eq. (25), there will also be an intermediate period where the quartic term dominates. In this case, Eq. (7) implies that $w = 1/3$ and so the scalar redshifts like radiation, $\rho \propto a^{-4}$, during this period.

More specifically, the scalar will end inflation with energy density

$$\rho_{\rm rh} \approx V(\phi_{\rm rh}) = \frac{3\alpha_{\rm rh}^2 H_I^4}{2\lambda}. \tag{26}$$

It will begin to oscillate and behave as radiation at a_r defined by

$$\alpha_r \approx \frac{V''(\phi_{\rm rh})}{3H(a_r)^2} \sim 1 \implies a_r \sim \alpha_{\rm rh}^{-1/4}. \tag{27}$$

When the scalar behaves as radiation, the amplitude of the field redshifts as $\phi \sim a^{-1}$. Finally, it will begin to behave as matter at a_m defined by

$$\frac{1}{2}m^2\phi_m^2 \sim \frac{\lambda}{4!}\phi_m^4 \implies a_m \sim \sqrt[4]{\frac{\alpha_{\rm rh}}{4}}\frac{H_I}{m}. \tag{28}$$

For $a > a_m$, the energy density in the scalar is therefore

$$\rho a^3 \sim \rho_{\rm rh}\left(\frac{a_{\rm rh}}{a_r}\right)^0\left(\frac{a_r}{a_m}\right)^4 a_m^3 \sim \frac{3\alpha_{\rm rh}^{3/4} m H_I^3}{\sqrt{2}\lambda}. \tag{29}$$

Setting this equal to the correct DM relic abundance fixes a relationship between m and λ (for fixed H_I and $N_{\rm tot}$).

In Fig. 2, we show a precise calculation of this relationship (see Appendix A for details). We also show other constraints on this production scenario. If λ is too large, the isocurvature fluctuations predicted in the right plot of Fig. 1 will contradict CMB observations. On the other hand, if λ is too small, the energy density in DM at the time of Big Bang Nucleosynthesis (BBN) will correspond to a large $\Delta N_{\rm eff}$, which is constrained to be $\Delta N_{\rm eff} < 0.2$ [50]. Additionally, if λ is too small, the first condition at the end of Sec. II A would require $\phi_{0,i} > M_{\rm pl}$. Finally, we also show late-time constraints on quartic DM. Astrophysical observations constrain self interactions of DM to be $\sigma/m \lesssim 1\,{\rm cm}^2/{\rm g}$ [51]. Additionally, large quartic interactions can inhibit structure formation during matter domination [47].

B. Density perturbations

Finally, we study how the density perturbations that we derived in Eq. (21) evolve through radiation domination. To do so, we will apply a "separate universes" argument [14, 52–54]. On large length scales, gradient terms in the equation of motion for the scalar can be ignored. In this case, each point in space evolves independently of surrounding points. In particular, each point will undergo the same dynamics that was outlined in Sec. III A for the average energy density, only they will each begin with different local initial conditions.

The separate universes approach holds for momentum modes k which are small enough that gradient terms have no effect on the dynamics of the density perturbations. In Appendix B, we study the evolution of the density perturbations for a scalar field. For a quartic potential, the effect of finite momentum k is to induce oscillations of the perturbations at frequency $k/\sqrt{3}a$. The separate universes approach therefore holds when the total accumulated phase

$$\int dt \frac{k}{\sqrt{3}a} = \int da \frac{k}{\sqrt{3}H_I} \tag{30}$$

is small. Oscillations will cease at $a = a_m$, when the potential is no longer dominantly quartic. We can therefore estimate that the separate universes approach holds for

$$k_{\rm today} \lesssim \frac{a_{\rm rh}}{a_{\rm today}} \frac{\sqrt{3}H_I}{a_m} \sim 1200\,{\rm Mpc}^{-1} \cdot \left(\frac{m}{10\,{\rm eV}}\right). \tag{31}$$

For scales satisfying this requirement, it is easy to see how the density perturbations evolve after reheating. Each separate universe undergoes the dynamics outlined in Sec. III A with a different initial energy density $\rho_{\rm rh}$, as well as a different local Hubble scale H_I. The latter is due to local variations in the metric and can be related to the local gravitational potential Φ by $H_I \propto 1 - \Phi$. As such, density fluctuations caused by variations in H_I are adiabatic perturbations, which correlate with the perturbations in all other components of the universe. On the other hand, variations due to different initial $\rho_{\rm rh}$ are isocurvature perturbations, which only affect DM. The dependence of the final energy density on these can easily be read off from Eqs. (26) and (29)[5]

$$\rho \propto \rho_{\rm rh}^{3/8} H_I^{3/2}. \tag{32}$$

One then finds that the final density perturbations (for $a > a_m$) are

$$\delta = \frac{3}{8}\delta_{\rm rh} - \frac{3}{2}\Phi. \tag{33}$$

The former term represents isocurvature perturbations, while the latter represents adiabatic perturbations. These are the total perturbations which constitute the primordial matter power spectrum [on scales which satisfy Eq. (31)].

IV. CONCLUSION

In this work, we have shown that the dynamics of a scalar with a nontrivial potential during inflation can naturally give rise to a blue-tilted spectrum for its perturbations. For a free scalar field, such a spectrum would typically require the mass to be comparable to the inflationary Hubble scale, $m \sim H_I$. However, if the scalar has a potential $V(\phi)$ which satisfies the conditions at the end of Sec. II A, its condensate will spend most of inflation near the boundary of the slow-roll regime, defined by $\alpha = V''(\phi_0)/3H_I^2 \sim 1$. As $V''(\phi_0)$ plays the role of an effective mass for the perturbations $\delta\phi$, then this is precisely the condition for a blue-tilted spectrum. Such a spectrum can lead to interesting signatures at small scales, while avoiding CMB constraints at large scales. A crucial condition for this mechanism is that inflation does not last too long, as the final blue tilt is inversely proportional to the number of e-folds before horizon crossing [see Eq. (12)]. This tilt can be enhanced for potentials with small κ, as in Eq. (10).

We also highlighted that when ϕ is long-lived, this mechanism leads to a final relic abundance, which is insensitive to the initial conditions of the scalar field. For fixed inflationary parameters, this implies a predictive relationship between the parameters of the potential in order to produce the correct relic abundance to be DM. In Fig. 2, we computed the relationship between m and λ for a quartic potential, and plotted it in relation to constraints on this mechanism and quartic DM in general.

[5] It is not difficult to see that the final energy density will always have this dependence on H_I [and therefore the same adiabatic perturbations in Eq. (33)], regardless of the potential. This is because the ratio a_r/a_m in Eq. (29) depends only on $\rho_{\rm rh}$ and parameters of the potential. In general, the amount of time spent progressing through different regions of the potential (with different equations of state) will not depend on H_I. The Hubble scale H_I will only affect the time at which the scalar begins to roll.

We demonstrated that there is viable parameter space where a scalar field with a quartic potential can constitute the entirety of DM. If the DM contains no interactions with the SM, enhanced small-scale isocurvature (and associated signatures, such as gravitational waves or non-Gaussianities) may be one of the most promising ways of searching for it.

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The code used for this research is made publicly available through Github [41] under CC-BY-NC-SA.

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Appendix A: Relic density calculation

In this appendix, we outline the details for the computation of the relationship in Fig. 2. A precise calculation of the relic density for the small masses appearing in Fig. 2 is too computationally difficult, as it requires evolving over a very long period of time. Instead, we compute the relic density for a much larger mass, and then extrapolate using the scalings derived in Eq. (29). This relic density must then be connected to the current DM density.

To begin, the condensate can be evolved during inflation as a function of t (or N), as in Eq. (4). During radiation domination, the equation of motion for the condensate becomes

$$\phi_0'' + 2a(\tau)H(\tau)\phi_0' + a(\tau)^2 V'(\phi_0) = 0, \tag{A-1}$$

as a function of conformal time τ. Once the condensate begins rolling, the second term will become small, and the condensate will oscillate with frequency given by the final term. Notice that when the quartic term in Eq. (25) dominates, this frequency is $\propto a^2\phi_0^2 \sim a^0$. In other words, when written in terms of τ, the oscillation frequency is constant during the quartic phase. For this reason, it is preferable to evolve as a function of τ (as opposed to t or N) during radiation domination.

In order to get a robust prediction for the relic density, the condensate should be evolved until the quadratic term in Eq. (25) dominates. (After this point, it will redshift as matter.) For the masses in Fig. 2, this would require evolving to very large a, and therefore also large τ. Instead, we evolve the spectator for a much larger mass $m_{\rm ref}$, and compute the quantity ρa^3 (which is constant at late times). From Eq. (29), we can see this quantity scales as $\propto m^1 H_I^3 \lambda^{-1}$, and so can readily be extrapolated to other values of interest.

To connect this quantity with the current DM abundance, we must compute $a_{\rm today}$. (Recall that our convention is $a_{\rm rh} = 1$.) This can be done by tracking the Hubble scale backwards in time to determine when it reaches H_I. In the ΛCDM model, at late times, the Hubble constant is given by

$$H(a) = H_0\sqrt{\Omega_\Lambda + \Omega_m \cdot \frac{a_{\rm today}^3}{a^3} + \Omega_m \cdot \frac{a_{\rm eq}a_{\rm today}^3}{a^4}}, \tag{A-2}$$

where the final term represents the radiation component, normalized so that the matter and radiation components are equal at $a_{\rm eq} = a_{\rm today}/3400$ [22].

At early times, the first two terms in Eq. (A-2) can be neglected. However, the radiation component evolves nontrivially when the number of SM degrees of freedom is changing. The entropy density evolves as $s \propto g_{*,s}T^3 \propto a^{-3}$, where $g_{*,s}$ is the entropy-weighted number of SM degrees of freedom. The energy density therefore evolves as

$$\rho \propto g_{*,\rho}T^4 \propto g_{*,\rho}g_{*,s}^{-4/3}a^{-4}, \tag{A-3}$$

where $g_{*,\rho}$ is the energy-weighted number of degrees of freedom. One can then see that, at early times, the Hubble scale is given by

$$H(a) = \frac{g_{*,\rho}(a)^{1/2}g_{*,s}(a_{\rm today})^{2/3}}{g_{*,\rho}(a_{\rm today})^{1/2}g_{*,s}(a)^{2/3}}H_0\sqrt{\frac{a_{\rm eq}a_{\rm today}^3\Omega_m}{a^4}}. \tag{A-4}$$

Setting $a = a_{\rm rh} = 1$ and $H(a_{\rm rh}) = H_I$, one can solve for $a_{\rm today}$. Using $\Omega_m = 0.32$ and $H_0 = 70\,{\rm km/s/Mpc}$ [22], as well as the SM values for the degrees of freedom $g_{*,\rho}(a_{\rm today}) = 3.4$, $g_{*,s}(a_{\rm today}) = 3.9$, and $g_{*,s}(a_{\rm rh}) = g_{*,\rho}(a_{\rm rh}) = 106.75$ [55], we find the result in Eq. (23).

The current DM abundance is given by

$$\rho_{\rm DM} = \frac{3\Omega_c H_0^2}{8\pi G}, \tag{A-5}$$

where $\Omega_c = 0.26$ [22] and G is the gravitational constant. The relationship between m and λ can then be determined by equating

$$\rho_{\rm DM}a_{\rm today}^3 = \left[\rho a^3\right]_{\rm ref}\left(\frac{m}{m_{\rm ref}}\right)\left(\frac{H_I}{H_{I,\rm ref}}\right)^3\left(\frac{\lambda_{\rm ref}}{\lambda}\right), \tag{A-6}$$

where we compute $\left[\rho a^3\right]_{\rm ref}$ with the reference values $m_{\rm rm} = 10^8\,{\rm GeV}$, $\lambda_{\rm ref} = 10^{-9}$, and $H_{I,\rm ref} = 10^{12}\,{\rm GeV}$. (Note that $a_{\rm today} \propto H_I^{1/2}$, so ultimately $\lambda/m \propto H_I^{3/2}$.)

Finally, let us elaborate on the calculation of the BBN bound in Fig. 2. This constraint is set by demanding that the DM abundance at the time of BBN has less energy density than $\Delta N_{\rm eff,max} = 0.2$ neutrino species. In order to determine the BBN abundance for parameter values of interest, we can again compute the abundance $\rho_{\rm ref}(a)$ over time for reference values and extrapolate. From Eq. (26), we see that the initial abundance scales as $\propto H_I^4\lambda^{-1}$. The abundance then scales as $\rho \propto a^{-4}$, until $a_m \propto H_I m^{-1}$ [from Eq. (28)]. We can therefore relate the physical abundance to the reference calculation as

$$\rho(a) = \rho_{\rm ref}\left(a \cdot \frac{m}{m_{\rm ref}} \cdot \frac{H_{I,\rm ref}}{H_I}\right)\left(\frac{m}{m_{\rm ref}}\right)^4\left(\frac{\lambda_{\rm ref}}{\lambda}\right). \tag{A-7}$$

This must be less than

$$\rho_{\rm BBN,max} = \frac{\pi^2}{30}\cdot\frac{7}{4}\cdot\left(\frac{4}{11}\right)^{4/3}\cdot\Delta N_{\rm eff,max}\cdot T_{\rm BBN}^4 \tag{A-8}$$

at

$$a_{\rm BBN} = a_{\rm today}\cdot\left(\frac{T_{\rm today}}{T_{\rm BBN}}\right)\cdot\left(\frac{g_{*,s}(a_{\rm today})}{g_{*,s}(a_{\rm BBN})}\right)^{1/3}, \tag{A-9}$$

where $T_{\rm BBN} = 1\,{\rm MeV}$ and $g_{*,s}(a_{\rm BBN}) = 10.6$ [55].

Appendix B: Evolution of density perturbations

In this appendix, we derive the evolution of the density perturbations of a scalar field with nontrivial potential.

The dynamics of the field itself are governed by the Klein-Gordon equation in an expanding spacetime

$$\ddot{\phi} + 3H\dot{\phi} - a^{-2}\nabla^2\phi + V(\phi) = 0. \tag{B-1}$$

If we multiply Eq. (B-1) by $\dot{\phi}$, the result can be written as

$$\dot{\rho} + 3H(\rho + P) + \nabla \cdot \vec{q} = 0, \tag{B-2}$$

where

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}a^{-2}\left(\nabla\phi\right)^2 + V(\phi), \tag{B-3}$$

$$P = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}a^{-2}\left(\nabla\phi\right)^2 - V(\phi), \tag{B-4}$$

$$\vec{q} = -a^{-2}\dot{\phi}\nabla\phi. \tag{B-5}$$

Eq. (B-2) is the continuity equation for a fluid with energy density ρ, pressure P, and momentum density $\vec{q}$ in an expanding spacetime. Similarly, if we multiply Eq. (B-1) by $-a^{-2}\nabla\phi$, the result can be written as

$$\dot{\vec{q}} + 5H\vec{q} + a^{-2}\nabla P + a^{-2}\nabla_j\Pi_{ij} = 0, \tag{B-6}$$

where

$$\Pi_{ij} = a^{-2}\left(\nabla_i\phi\nabla_j\phi - \frac{1}{3}(\nabla\phi)^2\delta_{ij}\right) \tag{B-7}$$

is the anisotropic stress of the scalar field. When written in terms of the velocity field, $\vec{v} = \vec{q}/(\rho + P)$, Eq. (B-6) can be understood as the Euler equation for a fluid.

Let us make some simplifying assumptions for Eqs. (B-2) and (B-6). First, if the production of ϕ is isotropic, we can expect that $\Pi_{ij} \approx 0$ on average. Second, we would like to relate the energy density and pressure via an equation of state $P = w\rho$. This can occur if all relevant modes are non-relativistic,[6] so that the second terms in Eqs. (B-3) and (B-4) are negligible, and moreover if we are interested in timescales longer than the oscillation frequency. In this case, the virial theorem implies

$$\langle\dot{\phi}^2\rangle = \langle V'(\phi)\phi\rangle, \tag{B-8}$$

where brackets represent an average over oscillations. Note that this holds at any point in space. Therefore, we can make the identification $\rho = wP$ locally with

$$w = \frac{\langle V'(\phi)\phi\rangle - 2\langle V(\phi)\rangle}{\langle V'(\phi)\phi\rangle + 2\langle V(\phi)\rangle}. \tag{B-9}$$

With these simpflications, Eqs. (B-2) and (B-6) can be combined to form

$$\ddot{\rho} + (8+3w)H\dot{\rho} + 3(1+w)\left(\dot{H} + 5H^2\right)\rho - wa^{-2}\nabla^2\rho = 0. \tag{B-10}$$

If we write Eq. (B-10) in terms of the density perturbations $\delta = \rho/\rho_0$, where $\rho_0 \propto a^{-3(1+w)}$ is the average energy density, then we find

$$\ddot{\delta} + (2 - 3w)H\dot{\delta} - wa^{-2}\nabla^2\delta = 0. \tag{B-11}$$

When a mode of interest is outside the horizon, $k \ll aH$, then the final term in Eq. (B-11) can be neglected. Regardless of the equation of state w, the dominant solution for δ will then be constant. In this case, the nontrivial potential has no effect on the evolution of the density perturbations. However, once the mode has entered the horizon, $k \gg aH$, then it begins to oscillates at frequency $\sqrt{w} \cdot k/a$. The dominant effect of a nontrivial potential is therefore to induce these oscillations once the mode enters the horizon. In the case of scalar DM, these oscillations will continue until the scalar reaches the part of its potential which is dominantly quadratic $V(\phi) \sim \phi^2$ (at which point $w = 0$).

[6] More specifically, we require $k^2/a^2 \ll \langle V''(\phi)\rangle$. This is satisfied at the end of inflation for all modes well outside the horizon. For $V(\phi) \sim \phi^p$ with $2 \le p \le 4$, the left-hand side decreases at least as fast as the right-hand side through radiation domination, and so this remains satisfied.