

2-Factors in Graphs^{*}

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Abstract

An account of 2-factors in graphs and their history is presented. We give a direct graph-theoretic proof of the 2-Factor Theorem and a new variant of it, and also a new complete characterisation of the maximal graphs without 2-factors. This is based on the important works of Tibor Gallai on 1-factors and of Hans-Boris Belck on k -factors, both published in 1950 and independently containing the theory of alternating chains. We also present an easy proof that a $(2k+1)$ -regular graph with at most $2k$ leaves has a 2-factor, and we describe all connected $(2k+1)$ -regular graphs with exactly $2k+1$ leaves without a 2-factor. This generalises Julius Petersen's famous theorem, that any 3-regular graph with at most two leaves has a 1-factor, and it generalises the extremal graphs Sylvester discovered for that theorem.

1 Introduction

We use standard graph-theoretic notation, unless otherwise indicated. Our graphs are undirected and finite, but we do allow multiple edges and loops.

Given a graph G and $X, Y \subseteq V(G)$, we use $E(X, Y)$ to denote the set of edges with one end in X and the other end in Y in G , and $e(X, Y)$ to denote their number. The subgraph of G induced by X is denoted $G[X]$.

An edge in a graph not contained in a cycle is a *bridge* of the graph. The removal of a bridge from a graph results in two new connected components replacing the connected component containing the bridge. A *leaf* of a graph is an induced bridgeless connected

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subgraph joined to the rest of the graph by one bridge. Two different leaves of a graph are disjoint. The number of leaves in a connected graph is at most the number of bridges, except that a connected graph with exactly one bridge has two leaves.

The degree of a vertex x is denoted $d(x)$, where a loop adds 2 to the degree of its vertex. A k -factor of a graph is a spanning subgraph of that graph in which all vertices have degree k .

Graph theory in the form we know it was created in the collaboration between James Joseph Sylvester in Oxford and Julius Petersen in Copenhagen. The collaboration started with Sylvester's visit to Copenhagen in September 1889, and was followed by an exchange of letters and a visit by Petersen to England in December 1889 and January 1890, resulting in Petersen's famous paper [16]. The main results of the paper are the following two theorems

Theorem 1.1 (Petersen, 1891 [16]).

Let G be a 3-regular graph with at most two leaves. Then G has a 1-factor.

Theorem 1.2 (Petersen, 1891 [16]).

Let G be a $2r$ -regular graph, for some positive integer r . Then G has a 2-factor.

Dénes Kőnig wrote the first extensive monograph of graph theory in 1936 [12] and gave Petersen's results a prominent place. Kőnig presented different ways to prove and relate factorisation theorems for general and for bipartite graphs. In particular, Theorem 1.2 above and Kőnig's own Theorem 1.3 on bipartite graphs below may be considered equivalent since each may be deduced from the other.

Theorem 1.3 (Kőnig, 1936 [12]).

Let G be a regular bipartite graph. Then G has a 1-factor.

In 1947, William T. Tutte obtained the general 1-Factor Theorem.

Theorem 1.4 (1-Factor Theorem; Tutte, 1947 [18]).

A graph G has a 1-factor if and only if for every $A \subseteq V(G)$ we have that the graph $G - A$ has at most $|A|$ connected components with an odd number of vertices.

Tutte's proof is based on a characterisation of the maximal graphs G without a 1-factor (called *hyperprime graphs* in Tutte's paper [18]). If G is maximal without a 1-factor, $|V(G)|$ is even, and B denotes the set of vertices joined to all other vertices, then the connected components of $G - B$ are all complete. Theorem 1.4 follows easily from this characterisation. Tutte's proof of this characterisation is algebraic, using skew-symmetric determinants. In 1950 Hans-Boris Belck [3] and Tibor Gallai [8] independently published graph theoretic proofs, while in 1952 Frederic George Maunsell [15] replaced the algebraic part of Tutte's proof by arguments using graph properties only. In 1975 László Lovász [13] gave a short graph-theoretic proof.

Gallai [8] obtained a proof of Theorem 1.4 based on his general theory of alternating chains. Gallai's method of proof gives a structural description of all graphs in terms of their maximum matchings, now known as the Gallai-Edmonds Decomposition Theorem (see [14]). Applying the theory of alternating chains, Gallai generalised in elegant fashion many of the known factorisation theorems and proved new ones. Among other results he generalised Baepler's theorem [2] that any $(2r + 1)$ -regular graph without bridges contains a 2-factor. Gallai's paper has an intriguing footnote, that he had obtained a similar structural theory for general graphs (as opposed to regular graphs) in terms of their 2-factors: "*With the present method I have succeeded in getting factorisation theorems for general graphs besides $\sigma = 1$ only for the case $\sigma = 2$. I shall discuss these results on another occasion.*"

Unfortunately, Gallai never published his theory of 2-factors. One may guess that the reason was that he discovered that Belck [3], also in 1950, published a general k -factor theorem, generalizing Theorem 1.4. As Belck's paper shows, and we explain in this paper, the theory of alternating chains is the key to a proof of the general k -factor theorem. For a graph without a k -factor, Gallai considered an induced subgraph with all degrees at most k with minimum deficiency from being k -regular, whereas Belck added new edges to get a maximal graph without a k -factor (called *hyper- k -prime graphs* in [3]), like Tutte did for 1-factors in 1947. Thus Belck and Gallai independently gave graph-theoretic proofs of Tutte's 1-Factor Theorem, and Belck was the first to obtain the general k -factor theorem [3].

Theorem 1.5 (Belck, 1950 [3]).

Let k be a positive integer. A graph G has a k -factor if and only if for every disjoint $A, B \subseteq V(G)$, setting $C = V(G) \setminus (A \cup B)$, we have that the graph $G[C]$ has at most $2e(A, A) - k|A| + k|B| + e(A, C)$ connected components C_i with $k|V(C_i)| + e(A, V(C_i))$ odd.

In particular for 2-factors we obtain the following.

Corollary 1.6.

A graph G has a 2-factor if and only if for every disjoint $A, B \subseteq V(G)$, setting $C = V(G) \setminus (A \cup B)$, we have that the graph $G[C]$ has at most $2e(A, A) - 2|A| + 2|B| + e(A, C)$ connected components joined to A by an odd number of edges.

As we shall see, in the condition of this corollary one may consider only independent sets A .

Theorem 1.7 (2-Factor Theorem).

A graph G has a 2-factor if and only if for every disjoint $A, B \subseteq V(G)$, A an independent set, and setting $C = V(G) \setminus (A \cup B)$, we have that the graph $G[C]$ has at most $-2|A| + 2|B| + e(A, C)$ connected components joined to A by an odd number of edges.

In 1952, Tutte [19], citing the works of Belck [3] and Gallai [8], subsequently obtained a general f -factor theorem, giving a necessary and sufficient condition for the existence of an f -factor in a general graph (where f is a mapping of $V(G)$ into the non-negative integers). A simple proof of the f -factor theorem, deducing it from the 1-Factor Theorem 1.4, is also due to Tutte [20]. This attractive proof may also be found in the book by Bollobás [5]. Already Claude Berge in 1958 in 'the second book' on graph theory [4] described Gallai's theory and used Tutte's method to prove the 2-Factor Theorem (formulated in [4] with misprints), calling 2-factors *semi-factors*. New comprehensive texts on factorisation, containing the general f -factor theorem, are due to Yu and Liu [21] and to Akiyama and Kano [1].

For maximum matchings in general graphs, Jack Edmonds [6] in 1965 obtained a polynomial algorithm, showing that the "good" characterisation of Tutte [18] is accompanied by a "good" algorithm. The output of the algorithm, with a general graph as input, gives, in a different language, the same structure as Gallai's method [8]. That structure is now usually called the *Gallai-Edmonds decomposition* of a graph.

1.1 Our work

Since an edge-multiplicity of 3 or more or two loops or more at the same vertex will not help to create a 2-factor, from now on we will only consider the class $\mathcal{M}_2$ of graphs having edge-multiplicity at most 2 and at most one loop at each vertex.

One aim of the present paper is to describe the complete structure of the maximal graphs without 2-factors. None of the above cited works contains such a complete description. Since cycles in graphs and 2-factors are basic structures in graph theory, it seems appropriate to fill this gap.

Theorem 1.8.

Let $\mathcal{M}_2$ be the class of all graphs with all multiplicities at most 2 and with each vertex having at most one loop. Then G is maximal without a 2-factor within the class $\mathcal{M}_2$ if and only if the following is satisfied.

Let the set A contain all vertices of G without loops, the set B contain all vertices in G with loops and joined to all other vertices in G by two edges, and set $C = V(G) \setminus (A \cup B)$. Suppose $G[C]$ has q connected components $\mathcal{C} = \{C_1, \dots, C_q\}$. Then we have the following.

- *The set A is independent.*
- *The components in $\mathcal{C}$ are all complete (with a loop at every vertex and two edges between any two vertices in the same component).*
- *Each component C_i is joined to A by an odd matching (of size $e(A, V(C_i))$).*
- *$2|A| + q = 2|B| + e(A, C) + 2$, which is equivalent to $|A| = |B| + 1 + \sum_{i=1}^q \frac{1}{2}(e(A, V(C_i)) - 1)$.*
- *For all $A' \subseteq A$, $A' \neq \emptyset$, and all $\mathcal{C}' \subseteq \mathcal{C}$ we have: $2|A'| + |\mathcal{C}'| \geq 2 + \sum_{C_i \in \mathcal{C}'} (e(A', V(C_i)))$.*

On the way to proving this theorem we shall give a simple direct and elegant proof of the 2-Factor Theorem, using the method of Belck and Gallai. Moreover, we deduce in a simple way from the 2-Factor Theorem the following result.

Theorem 1.9.

Let G be a $(2k + 1)$ -regular graph with at most $2k$ leaves. Then G has a 2-factor.

The case $k = 1$ is Petersen's classical Theorem 1.1 on 3-regular graphs (since a 1-factor is the complement of a 2-factor in a 3-regular graph). Being a special case of a general 2-factor theorem, Petersen's theorem should be regarded as a 2-factor theorem, and not just a 1-factor theorem. According to Roland Häggkvist [10], he announced the result in 1.9 at a meeting in Oberwolfach in 1977.

Theorem 1.9 with the stronger condition that there are no bridges is due to Baebler [2], and in generalised forms (but still for graphs without bridges) to Belck [3] and Gallai [8]. Hanson, Loten and Toft [9] proved a result similar to Theorem 1.9, allowing up to $2k$ bridges. In hindsight their proof looks unnecessarily complicated now. A better proof was obtained by Douglas West (private communication), and a generalisation was published in 2021 by Kostochka, Raspaud, Toft, West, and Zirlin [11]. That paper also contained a description of all $(2k + 1)$ -regular graphs with exactly $2k + 1$ bridges without a 2-factor; we also describe these graphs, but in a slightly different and generalised way. These extremal graphs generalise the Sylvester graphs that Sylvester communicated to Petersen in 1889 [16].

The final section of our paper contains some of the information we discovered about Hans-Boris Belck, whose work seems to be often overlooked by those working on factors in graphs. A reason might be that the paper was written in quite complicated German.

2 The proof of the 2-Factor Theorem

2.1 The ‘only if’ part of the 2-Factor Theorem

Recall that $\mathcal{M}_2$ denotes the class of graphs having edge-multiplicity at most 2 and at most one loop at each vertex.

Let G be a graph from $\mathcal{M}_2$ containing a 2-factor F . Let A and B be two disjoint subsets of $V(G)$. Set $C = V(G) \setminus (A \cup B)$ and let $C_1, C_2, \dots$ be the connected components of $G[C]$, the subgraph of G induced by C .

The factor F consists of disjoint cycles. Each such cycle may contain vertices from A , and between these vertices there are paths with vertices from $B \cup C$. We shall distinguish four different types of these paths, and use these to construct three disjoint subsets B_1, B_2, B_3 of B .

- Paths in which both end-vertices are from B (which includes the case of a path with one vertex (from B)). For those paths, add one of the end-vertices to B_1 .
- Paths in which both end-vertices are from the same component C_i of $G[C]$.
- Paths in which the end-vertices are from two different components C_i and C_j of $G[C]$. Then the path must contain a vertex from B ; add one such vertex from B to B_2 .
- Paths with one end-vertex from B and one from C ; add the end-vertex from B to B_3 .

Note that the number of edges from $E(A, B)$ in F is equal to $2|B_1| + |B_3|$. Hence the number of edges from $E(A, B)$ not in F is equal to $e(A, B) - 2|B_1| - |B_3|$.

Let q denote the number of components C_i in $G[C]$ with an odd number of edges to A , i.e. with $e(A, V(C_i))$ odd. At most $2|B_2| + |B_3|$ of these q components can have all the edges to A contained in F , since the $(A, V(C_i))$ -edges from the second type of paths above come in pairs. Hence at least $q - 2|B_2| - |B_3|$ of the (A, C) -edges are not in F .

From this we deduce

$$\begin{aligned}
 \sum_{x \in A} (d(x) - 2) &= 2|\{e \in E(A, A) \mid e \notin F\}| + |\{e \in E(A, B) \mid e \notin F\}| + |\{e \in E(A, C) \mid e \notin F\}| \\
 &\geq |\{e \in E(A, B) \mid e \notin F\}| + |\{e \in E(A, C) \mid e \notin F\}| \\
 &\geq e(A, B) - 2|B_1| - |B_3| + q - 2|B_2| - |B_3| \geq e(A, B) + q - 2|B|.
 \end{aligned}$$

Since we also have $\sum_{x \in A} (d(x) - 2) = 2e(A, A) + e(A, C) + e(A, B) - 2|A|$, it follows that $q \leq 2e(A, A) - 2|A| + 2|B| + e(A, C)$. This proves the easy part of Corollary 1.6 and thus also of Theorem 1.7, the ‘only if’ direction. $\square$

In case of equality, i.e. if $q = 2e(A, A) - 2|A| + 2|B| + e(A, C)$, we have equality in all the above inequalities. From this we obtain the following.

Theorem 2.1.

Let G be a graph with a 2-factor F . Let $A, B \subseteq V(G)$ be disjoint subsets. Set $C = V(G) \setminus (A \cup B)$, and suppose the graph $G[C]$ has $q = 2e(A, A) - 2|A| + 2|B| + e(A, C)$ connected components joined to A with an odd number of edges. Construct B_1, B_2, B_3 as above. Then all edges of $E(A, C)$ belong to F except exactly one edge from $q - 2|B_2| - |B_3|$ of the components joined to A with an odd number of edges. Moreover, we have $B = B_1 \cup B_2 \cup B_3$.

2.2 The ‘if’ part of the 2-Factor Theorem

To prove the ‘if’ part of the 2-Factor Theorem (Theorem 1.6) we shall use the theory of alternating chains of Gallai from his beautiful 1950 paper [8]. A main result of that paper was also obtained independently by Belck [3], submitted before Gallai’s paper had appeared. Moreover, we shall use a simple but useful lemma by Belck. In that sense, the proof of the 2-Factor Theorem we present is due to Belck.

Let G be a graph, possibly with multiple edges and loops, and let each edge of G be coloured either blue or red. An *alternating chain* is a walk without repeated edges and with the edges coloured alternating blue and red. (A single vertex is considered both as a red-blue and a blue-red alternating chain.) We fix some vertex p and consider alternating chains in G starting from p with a blue edge. The vertices of G can then be divided into four groups: 1) *BR-vertices* can be reached by an alternating chain as described ending in blue and also by an alternating chain ending in red; 2) *B-vertices* can be reached only by alternating chains ending in blue; 3) *R-vertices* can be reached only by alternating chains ending in red; and finally 4) the *unreachable vertices*.

The single vertex p is itself a blue-red alternating chain, so p is either an R-vertex or a BR-vertex. Belck and Gallai independently proved the following.

Theorem 2.2 (Belck [3], Gallai [8]).

Let C be a connected component in the graph induced by the BR-vertices. If p does not belong to C , then C has exactly one entering edge, either a blue edge from an R-vertex to a vertex in C or a red edge from a B-vertex to a vertex in C , such that any alternating chain from p starting with a blue edge to any vertex in C must enter C via the entering edge.

The proof can be found in Belck [3, § 4], and can be more easily, and with all details, be found in Gallai’s paper [8, § 3]. It was later presented by Berge in ‘the second book’ of graph theory [4]. The idea of the proof is that if there were a second entering edge, then using that walking into C and later the first entering edge walking out of C again, the end-vertex of the first entering edge outside C would be a BR-vertex and thus belong to C .

We next formulate a lemma by Belck [3, Zusatz I,1] that we use only in some special situations. (Belck’s version applies to k -factors in general.)

Lemma 2.3 (Belck [3]).

Let G be a maximal graph without a 2-factor, i.e. adding any new edge to G (which can be a parallel edge or a loop) will result in a graph with a 2-factor. Let e_1 and e_2 be two new edges. Let F_1 be a 2-factor in $G + e_1$ and F_2 a 2-factor in $G + e_2$. Colour the edges in $E(F_1) \setminus E(F_2)$ red and the edges in $E(F_2) \setminus E(F_1)$ blue. Then $H = G + e_1 + e_2$ contains a closed alternating chain containing e_1 and e_2 .

We give an outline of the proof. Let $e_1 = xy$ and start a chain in H from x with the red $E(F_1) \setminus E(F_2)$ edge xy . Then y is incident to a blue $E(F_2) \setminus E(F_1)$ edge yz . Then z is incident to a new red $E(F_1) \setminus E(F_2)$ edge, and so on. This process can only stop at x with a blue edge from $E(F_2) \setminus E(F_1)$. The obtained chain must contain e_2 , since otherwise changing its colours would result in a red 2-factor of G . This proves the lemma.

A consequence of Lemma 2.3, which again is a special case of a result for k -factors with the case $k = 1$ proved also in [13], is the following.

Lemma 2.4.

Let G be a maximal graph without a 2-factor. Let a, b, c, d be vertices in G , where a and b are joined by two edges, b and c are also joined by two edges, and b and d are joined by at most one edge. Then a and c are joined by two edges.

We just give an outline of the proof again. Assume otherwise, and let e_1 and e_2 be new edges between b and d , and between a and c , respectively. Follow Lemma 2.3 to colour some edges red or blue. Then by the lemma, there is a closed alternating chain containing the new red edge e_1 and the new blue edge e_2 . Hence there is an alternating subchain S from b to either a or c , starting with the red edge e_1 and ending with a red edge, say in a . One of the two edges between a and b is not red. Colour that edge red and change colours on S . The result is a red 2-factor in G , which is a contradiction.

We are now ready to prove the ‘if’ part of the 2-Factor Theorem 1.7. Let H be a graph without a 2-factor. Add edges to H to obtain a maximal graph G without a 2-factor. Remember that we operate within the class $\mathcal{M}_2$ of graphs with edge-multiplicities at most 2 and at most one loop at each vertex. We shall prove that G has an independent set $A \subseteq V(G)$, and a set $B \subseteq V(G)$ disjoint from A , such that if we set $C = V(G) \setminus (A \cup B)$, then the number q of components of $G[C]$ joined to A by an odd number of edges is strictly greater than $-2|A| + 2|B| + e(A, C)$. Then the same statement holds for H , proving the ‘if’ part of Corollary 1.6 and Theorem 1.7.

Let A be the set of vertices of G without loops, and let B be the set of vertices with a loop and joined to all other vertices of G by exactly two edges. Let C be the remaining vertices. The set A is non-empty, since otherwise each vertex of G would have a loop and all the loops together would form a 2-factor. Let $p \in A$ and add a loop e_1 at p to G . By maximality of G , the resulting graph has a 2-factor F_1 . Colour the edges in F_1 red and all other edges of G blue. Now consider alternating chains starting in p with blue edges. (We might alternatively say starting with the red loop at p .) Such an alternating chain cannot end at p with a blue edge, since changing red and blue on it would result in a red 2-factor of G . We are then in the situation described in the start of this section and we may use the theory and Theorem 2.2 of Belck and Gallai.

Since p is not reachable by an alternating chain ending with a blue edge, it must be an R-vertex. And in fact all vertices of A are R-vertices. Let namely $r \in A$ be a vertex different from p . Then adding a loop e_2 at r to G gives a 2-factor F_2 . Using Belck’s lemma 2.3 (with e_1 the loop added at p in the previous paragraph, and e_2 the new loop at r), we get two edge-disjoint alternating chains in G starting from p with new blue edges and ending in r with red edges. By Belck’s lemma these new blue edges are not red, hence they are also blue in the original colouring. So r is either an R-vertex or a BR-vertex. If r is a BR-vertex, then walk from r along the two alternating chains in the direction of p . On both chains we eventually move out of the BR-component containing r , but then that component has two entering edges, contradicting the Gallai-Belck theorem 2.2. Hence r must be an R-vertex, as claimed.

Since all vertices in A are R-vertices, the vertex p is therefore not joined to any $r \in A$ by a blue edge, and not by a red edge either (because of the red loop at p). Hence p has no neighbours in A . Because p was chosen as an arbitrary vertex of A , we obtain that A is an independent set.

Take $b \in B$. Then b is joined to p by two blue edges, one of which may be considered as an alternating chain ending with a blue edge. Assume there is an alternating chain from p

starting with a blue edge and ending at b with a red edge. If it uses both blue edges between p and b , then change blue and red on it to obtain a 2-factor of G , which is a contradiction. Otherwise add one of the blue edges to the chain and obtain an alternating chain to the R-vertex p which ends with a blue edge, again a contradiction. Thus the vertices of B are B-vertices.

The vertices of C have loops, but are not completely joined to everything else. Take $c \in C$ and let ct be an edge missing in G . As before, by Belck's lemma, there is an alternating chain from p to c starting with a blue edge and ending with a red edge. By definition, there is a loop at c . If that loop is red it must be the end-edge of the chain, hence by removing it we get an alternating chain from p to c starting and ending with a blue edge. If the loop is blue and has been used on the chain, then part of the chain ends in c with a blue edge. If the loop is blue, but not part of the chain, then add it and get an alternating chain from p to c starting with a blue edge and ending with a blue edge. The conclusion is that all vertices of C are BR-vertices.

The components of $G[C]$ are thus the BR-components. Since each vertex of C has red degree 2, the number of red edges from a component to $V(G - C)$ is even. By Theorem 2.2, each component has exactly one entering edge. There are now two possibilities. If the entering edge is red, then it enters from a B-vertex, all other edges from B to the component must be blue, and all edges from A to the component must be red and odd in number. If the entering edge is blue, then it enters from an R-vertex, i.e. from a vertex in A , all other edges from A to the component must be red and even in number, and all edges from the component to B must be blue since there is only one entering edge.

In conclusion we get that all components of $G[C]$ have a odd number of edges to A . Suppose that q_1 components have a red entering edge and q_2 a blue. Then the number of red edges from B to A equals $2|B| - q_1$. The number of red edges from C to A equals $e(A, C) - q_2$. Since the total number of red edges between A and $B \cup C$ is $2|A| - 2$, we find

$$2|A| - 2 = 2|B| - q_1 + e(A, C) - q_2 = 2|B| + e(A, C) - q,$$

and thus

$$q = -2|A| + 2|B| + e(A, C) + 2 > -2|A| + 2|B| + e(A, C).$$

This completes the proof of the 'if' part of Corollary 1.6 and Theorem 1.7. $\square$

For later use we note the following. In the maximal graph G without a 2-factor, the vertex $p \in A$ is joined to C only by blue edges, and thus only by entering edges. By Theorem 2.2, there is at most one such edge to each component of $G[C]$, hence p is joined by at most one edge to each component. Since $p \in A$ was chosen arbitrarily, it follows that each vertex of A is joined to each component of $G[C]$ by at most one edge.

Also, suppose that a vertex $c \in C$ is joined to p by an edge $e_1 = pc$ and by another edge e_2 also to A . Since p is joined to each component of C by at most one edge, the edge pc is single. If we add a new edge $e_3 = pc$ to G , rather than adding the loop at p , then we obtain a new 2-factor F^* in $G + e_3$. Theorem 2.1 now applies. (The number of odd components is now one less and the number of edges between A and C one more.) Thus all edges e_1, e_2, e_3 belong to F^* , which is impossible since they are all three incident with x . Again, $p \in A$ was chosen arbitrarily, so we can conclude that each vertex of C is joined in G to at most one vertex of A . The conclusion is that the edges joining vertices in A with the vertices of a component of $G[C]$ form a matching.

Finally, note that in the maximal graph G without a 2-factor the components of $G[C]$ are all complete in the sense that any two vertices are joined by a double edge and any vertex has a loop. This follows since adding anything missing still keeps too many components of $G[C]$ joined to A by an odd number of edges.

3 Regular graphs without 2-factors

Now let G be a $(2k + 1)$ -regular graph without a 2-factor. Then the 2-Factor Theorem 1.7 gives that there exist disjoint subsets $A, B \subseteq V(G)$, A an independent set, such that if we set $C = V(G) \setminus (A \cup B)$, then the number q of connected components of $G[C]$ joined to A by an odd number of edges satisfies

$$q > -2|A| + 2|B| + e(A, C) = (2k - 1)|A| - e(A, B) + 2|B|.$$

Since q and $e(A, C)$ have the same parity we deduce

$$q + e(A, B) \geq (2k - 1)|A| + 2|B| + 2. \quad (3.1)$$

Let q_1 denote the number of components of $G[C]$ that are joined to A with exactly one edge and not joined to B at all, and let q_2 denote the number of components of $G[C]$ that are joined to A with exactly one edge and to B with at least one edge. Finally, let q_3 denote the number of components of $G[C]$ that are joined to A with an odd number of at least 3 edges. Then of course $q = q_1 + q_2 + q_3$. Looking at edges incident with the vertices of A we get

$$(2k + 1)|A| \geq e(A, B) + q_1 + q_2 + 3q_3. \quad (3.2)$$

Similarly, looking at the edges incident with vertices from B we get

$$(2k + 1)|B| \geq e(A, B) + q_2. \quad (3.3)$$

Adding $\frac{1}{2}(2k + 1)$ times (3.1), $\frac{1}{2}(2k - 1)$ times (3.2), and 1 times (3.3) together, we get:

$$q_1 \geq (2k + 1) + (2k - 2)q_3. \quad (3.4)$$

This proves that G has at least $2k + 1$ leaves, which is exactly Theorem 1.9.

Sylvester, in his correspondence with Petersen in 1889 (see [12, 17]), gave examples of $(2k + 1)$ -regular graphs without 2-factors, so-called *primitive graphs*, having exactly $2k + 1$ leaves. Because of Theorem 1.2, the primitive graphs are exactly those without 2-factors. The above inequalities can be used to describe all the primitive $(2k + 1)$ -regular graphs having exactly $2k + 1$ leaves, because there is then equality in the inequalities. The case $|A| = 1$ in the following theorem gives the Sylvester graphs.

Theorem 3.1.

Let G be a $(2k + 1)$ -regular graph, with $k \geq 2$, without a 2-factor and with exactly $2k + 1$ leaves. Then there exists disjoint subsets $A, B \subseteq V(G)$, where A and B are both independent (i.e. $A \cup B$ induces a bipartite subgraph of G) with $|A| = |B| + 1$. Moreover, $2k + 1$ of the connected components of $G - (A \cup B)$ are joined to A with exactly one edge and not to B , while all other components are joined to A with exactly one edge and to B with exactly one edge.

We may construct these graph as follows. Start from a bipartite graph H with A and B the sets of vertices in its two sides, where $|A| = |B| + 1$ and with all vertices of B of degree $2k + 1$ and all vertices of A of degree at most $2k + 1$. Add $2k + 1$ disjoint graphs to H , each having all vertices of degree $2k + 1$ except one vertex of degree $2k$ and join each to A by one edge to get a $(2k + 1)$ -regular graph with $2k + 1$ leaves. (To get exactly $2k + 1$ leaves, we should use $2k + 1$ disjoint graphs obtained as follows: take a connected $(2k + 1)$ -regular graph with exactly two leaves and remove a bridge; a connected component in the remaining graph is suitable as one of the $2k + 1$ graphs.) Moreover for a number of edges xy of the bipartite graph H , replace the edge xy by a $(2k + 1)$ -regular bridgeless graph, removing an edge from it, and joining its two endvertices to x and to y to obtain a $(2k + 1)$ -regular graph.

In the case of 3-regular graphs, i.e. if $k = 1$, we may have $q_3 > 0$ and still have equality in (3.4), hence we have to add the following to the above description: Finally in addition, for some vertices $x \in B$ joined to three vertices of A , replace the vertex x by a 3-regular bridgeless graph, removing a vertex from it and joining its three neighbours to the three neighbours of x in A .

4 The maximal graphs without 2-factors

Remember that we operate within the class $\mathcal{M}_2$ of graphs with edges of multiplicity at most 2 and with at most one loop at each vertex. Let therefore G be a graph in $\mathcal{M}_2$ and let us assume that G does not have a 2-factor, but that G is maximal with respect to this property, i.e. if a new edge or a new loop is added and we stay within the class $\mathcal{M}_2$, then the resulting graph has a 2-factor.

From our proof of the ‘if’ part of the 2-Factor Theorem we immediately get the following partial characterisation, where the only missing part is a precise description of how the vertices of C are joined to the vertices of A .

Theorem 4.1.

Let $\mathcal{M}_2$ be the class of all graphs with all multiplicities at most 2 and with each vertex having at most one loop, and let G be maximal without a 2-factor within the class $\mathcal{M}_2$. Let the set A contain all vertices of G without loops, the set B contain all vertices in G with loops and joined to all other vertices in G by two edges, and set $C = V(G) \setminus (A \cup B)$. Suppose $G[C]$ has q connected components $\mathcal{C} = \{C_1, \dots, C_q\}$. Then we have the following.

- *The set A is independent.*
- *The components in $\mathcal{C}$ are all complete (with a loop at every vertex and two edges between any two vertices in the same component).*
- *Each component C_i is joined to A by an odd matching (of size $e(A, V(C_i))$).*
- $2|A| + q = 2|B| + e(A, C) + 2$, which is equivalent to $|A| = |B| + 1 + \sum_{i=1}^q \frac{1}{2}(e(A, V(C_i)) - 1)$.

The case $q = 0$ is possible; in this case $|A| = |B| + 1$. The resulting graph is maximal. (Remember that A is an independent set and each vertex of B has a loop and is joined to all other vertices by two edges.) For $q = 1$ and $q = 2$ the conditions described above are necessary and sufficient for G to be maximal without a 2-factor. For $q \geq 3$ we need to add extra conditions; for example for $q = 3$:

- *No three components of $G[C]$ are all joined to the same two different vertices $x, y \in A$.*

Assume namely that this was the case. Then adding a loop at x gives a graph G^* having a 2-factor F and satisfying Theorem 2.1. The three edges from the components to x are not in F , because the loop at x is. But then the three edges to y are all in F , a contradiction.

In what follows we shall generalise this property to a general necessary and sufficient condition for $G \in \mathcal{M}_2$ to be maximal without a 2-factor, as exhibited in the final condition of Theorem 1.8. The above is the case with $A' = \{x, y\}$ and $C' = \{C_1, C_2, C_3\}$.

4.1 The necessary and sufficient conditions

Let again G be a graph with all multiplicities at most 2 and with each vertex having at most one loop. Let us assume that G does not have a 2-factor, and define $A, B, C, q, \mathcal{C}$ as in Theorem 4.1. For $i = 1, \dots, q$, set $t_i = e(A, C_i)$. Also assume that the four conditions in the theorem are satisfied.

In this subsection we shall determine some extra properties that allow us establish necessary and sufficient conditions for G to be maximal within $\mathcal{M}_2$ without a 2-factor. There are four possible types of additions of a new edge e to G : 1) a loop at a vertex $x \in A$; 2) an edge between two different vertices $x, y \in A$; 3) an edge between a vertex $x \in A$ and a vertex y in a component of $\mathcal{C}$; and, finally, 4) an edge between two vertices x and y in two different components of $\mathcal{C}$.

In all four situations the graphs $G^* = G + e$, with the subsets A and B defined as above, satisfy the condition of Theorem 2.1 that $q = 2e(A, A) - 2|A| - 2|B| + e(A, C)$. We shall now in each case find a necessary and sufficient condition for G^* to contain a 2-factor.

The conditions will be given in terms of properties of a bipartite graph H . The vertices in the two sides of H are A and $\mathcal{C}$, respectively. Moreover $x \in A$ and $C_i \in \mathcal{C}$ are joined by an edge in H if and only if $x \in A$ is joined to C_i in G . The bipartite graph H is uniquely determined from G , and vice versa (up to isomorphism). There is a one-to-one correspondence between the edges of H and the edges between A and $G[C]$ in G , where $(x, C_i) \in E(H)$ corresponds to an edge $(x, y) \in E(G)$ with $y \in V(C_i)$. From Theorem 4.1 we have that the vertex C_i has odd degree t_i in H .

Case 1: G^* is the graph obtained from G by adding a loop to a vertex $x \in A$.

Suppose there is a 2-factor F^* in G^* containing the loop at x . The edges from F^* joining vertices from A to vertices of $\mathcal{C}$ correspond to a set of edges F of the bipartite graph H .

By Theorem 2.1, F satisfies the following.

- (1.1) The vertex $x \in A$ is not incident with an edge from F .
- (1.2) Each vertex $y \in A \setminus \{x\}$ is incident with at most two edges from F .
- (1.3) For each $C_i \in \mathcal{C}$, all t_i edges incident with C_i , except perhaps one, are in F .

If all edges in H incident with C_i are in F , then we remove one of the edges for each such C_i from F and obtain a subset F' of $E(H)$ satisfying the following.

- (1.1') The vertex $x \in A$ is not incident with an edge from F' .
- (1.2') Each vertex $y \in A \setminus \{x\}$ is incident with at most two edges from F' .
- (1.3') For each $C_i \in \mathcal{C}$, all t_i edges incident with C_i , except exactly one, are in F' .

We have shown above that if a loop added to the vertex $x \in A$ in G gives rise to a 2-factor in the resulting graph G^* , then $E(H)$ has a subset F' satisfying (1.1'), (1.2') and (1.3'). We shall now show that the converse is true as well.

So assume that $E(H)$ has a subset F' satisfying (1.1'), (1.2') and (1.3'). Then $|F'| = \sum_{i=1}^q (t_i - 1)$. For $i = 0, 1, 2$, let A_i denote the set of vertices in A incident with i edges from F' . Then $|F'| = |A_1| + 2|A_2| = \sum_{i=1}^q (t_i - 1)$. Using Theorem 4.1, we deduce that

$|B| = |A_0| + \frac{1}{2}|A_1| - 1$. A 2-factor F^* in the graph G^* can now be obtained as follows. F^* consists of the loop at x , the edges in G corresponding to the edges F' in H , for each vertex of $A_0 \setminus \{x\}$ a double-edge to a vertex in B , and for each vertex of A_1 one edge to a vertex in B . Because of the size of B , this can be done in such a way that each vertex of B becomes incident to exactly two edges of F^* . For each component $C_i \in \mathcal{C}$ there are now $t_i - 1$ matching edges from F^* coming in from A . We finally add the endvertices in C_i two and two by an edge of C_i to F^* , and add the loop at all other vertices of C_i to F^* . The result is a 2-factor F^* of G^* .

In conclusion, the existence of a subset F' of edges of the graph H satisfying (1.1'), (1.2'), (1.3') is equivalent to the graph G^* having a 2-factor, where G^* is obtained from G by adding a loop at the vertex $x \in A$.

Case 2: G^* is the graph obtained from G by adding an edge between two different vertices $x, y \in A$.

By arguments similar to those in Case 1, the graph G^* has a 2-factor if and only if the graph H has a subset F' of edges satisfying the following.

- (2.1') The vertices $x, y \in A$ are each incident with at most one edge from F' .
- (2.2') Each vertex $z \in A \setminus \{x, y\}$ is incident with at most two edges from F' .
- (2.3') For each $C_i \in \mathcal{C}$, all t_i edges incident with C_i , except exactly one, are in F' .

Case 3: G^* is the graph obtained from G by adding an edge between a vertex $x \in A$ and a vertex $y \in V(C_i)$ for some component $C_i \in \mathcal{C}$.

By arguments similar to those in Case 1, the graph G^* has a 2-factor if and only if the graph H has a subset F' of edges satisfying the following.

- (3.1') The vertex $x \in A$ is incident with at most one edge from F' .
- (3.2') Each vertex $z \in A \setminus \{x\}$ is incident with at most two edges from F' .
- (3.3') For each $C_j \in \mathcal{C}$ with $j \neq i$, all t_j edges incident with C_j , except exactly one, are in F' .
- (3.4') All t_i edges incident with C_i are in F' .

Case 4: G^* is the graph obtained from G by adding an edge between a vertex $x \in V(C_i)$ and a vertex $y \in V(C_j)$, for two different components $C_i, C_j \in \mathcal{C}$.

By arguments similar to that in Case 1, the graph G^* has a 2-factor if and only if the graph H has a subset F' of edges satisfying the following.

- (4.1') Each vertex $x \in A$ is incident with at most two edges from F' .
- (4.2') For each $C_k \in \mathcal{C}$ with $k \notin \{i, j\}$, all t_k edges incident with C_k , except exactly one, are in F' .
- (4.3') All t_i edges incident with C_i are in F' .
- (4.4') All t_j edges incident with C_j are in F' .

As we shall see in the next subsection, the four sets of conditions are, not surprisingly, closely related. In fact, the first set of conditions for all $x \in A$ imply the second, third and fourth sets of conditions.

4.2 Reformulation of the conditions; putting it all together

Let G be a maximal graph without a 2-factor within the class $\mathcal{M}_2$ satisfying Theorem 4.1 and the extra conditions of the previous subsection. Let again H be the bipartite graph with vertex sets A and $\mathcal{C}$, where $\mathcal{C}$ has q vertices $C_1, \dots, C_q$ of degrees $t_1, \dots, t_q$, all odd integers. We direct all edges from A to $\mathcal{C}$ and give them capacity 1, add a source and join it by a

directed edge of capacity 2 to each vertex of A , except to one vertex $x \in A$, and add a sink with a directed edge of capacity $t_i - 1$ from each $C_i \in \mathcal{C}$. In this way we obtain from the bipartite graph H a flow network N with integer capacities. The existence of a subset F' of the edges of H satisfying the conditions (1.1'), (1.2') and (1.3') is then equivalent to the existence of a flow of size $\sum_{i=1}^q (t_i - 1)$ in N . (If there is a flow of that size there is also an integer flow of that size, and the existence of such an integer flow may be translated directly into the existence of F' .) By the Max Flow Min Cut Theorem [7] the existence of a flow of this size is equivalent to any cut of N having capacity at least $\sum_{i=1}^q (t_i - 1)$.

A cut in N may be thought of as subsets $A' \subseteq A$ and $\mathcal{C}' \subseteq \mathcal{C}$, where A' and $\mathcal{C}'$ are the parts of A and $\mathcal{C}$ belonging to the same side of the cut as the sink. We may assume that x belongs to A' , since moving it there from $A \setminus A'$ can only give the same or smaller capacity. The capacity of the cut is then

$$2|A'| - 2 + e(A \setminus A', \mathcal{C}') + \sum_{C_i \in \mathcal{C} \setminus \mathcal{C}'} (t_i - 1).$$

So the condition for the existence of F' is

$$2|A'| - 2 + e(A \setminus A', \mathcal{C}') + \sum_{C_i \in \mathcal{C} \setminus \mathcal{C}'} (t_i - 1) \geq \sum_{C_i \in \mathcal{C}} (t_i - 1),$$

which is equivalent to

$$2|A'| \geq 2 + \sum_{C_i \in \mathcal{C}'} (t'_i - 1),$$

where t'_i is equal to the number of neighbours from A' the vertex $C_i \in \mathcal{C}$ has in H . Since the last inequality holds for all $x \in A'$ we can write the condition in the following way:

$$(4.1) \text{ for all } A' \subseteq A, A' \neq \emptyset, \text{ and all } \mathcal{C}' \subseteq \mathcal{C} \text{ we have: } 2|A'| \geq 2 + \sum_{C_i \in \mathcal{C}'} (t'_i - 1).$$

The condition in (4.1) is equivalent to the existence of F' in Case 1 above. In a similar way, conditions equivalent to the existence of F' in Cases 2, 3 and 4 can be obtained:

$$(4.2) \text{ for all } A' \subseteq A, |A'| \geq 2, \text{ and all } \mathcal{C}' \subseteq \mathcal{C} \text{ we have: } 2|A'| \geq 2 + \sum_{C_i \in \mathcal{C}'} (t'_i - 1);$$

$$(4.3) \text{ for all } A' \subseteq A, A' \neq \emptyset, \text{ and all } \mathcal{C}' \subseteq \mathcal{C}, \mathcal{C}' \neq \emptyset, \text{ we have: } 2|A'| \geq 2 + \sum_{C_i \in \mathcal{C}'} (t'_i - 1);$$

$$(4.4) \text{ for all } A' \subseteq A \text{ and all } \mathcal{C}' \subseteq \mathcal{C}, |\mathcal{C}'| \geq 2, \text{ we have: } 2|A'| \geq 2 + \sum_{C_i \in \mathcal{C}'} (t'_i - 1).$$

All these four conditions are implied by (4.1), which completes the proof of Theorem 1.8. $\square$

5 Conclusion

Cycles in graphs and 2-factors are important structures in graph theory, so even if partial characterisations of the maximal graphs without 2-factors have existed in the mathematical literature since 1950, it seems worthwhile to obtain a complete characterisation, as has been one of the goals of this paper. We have also pointed out the important role maximal graphs have played in the development of factorisation theory, and how the theorem of Belck and

Gallai (Theorem 2.2) may be used to prove the 2-Factor Theorem, and indeed the general k -factor theorem.

For general k -factors, $k \geq 3$, it is possible to carry out a similar analysis as carried out above, but the outcome will be much more complicated. A component C_i of $G[C]$ is said to be odd if the number $k|V(C_i)| + e(A, V(C_i))$ is odd. The connections between the set A and the components C_i are no longer restricted to be matchings. Moreover, the sets A and B are not always unique for maximal graphs (as they are in the case $k = 2$).

Any simple graph H that is maximal without a 2-factor within the class of simple graphs is equal to the simple restriction of a graph G as described in Theorem 1.8, i.e. H is equal to G with each multiple edge reduced to a single edge and all loops removed. But not all such reduced versions of graphs satisfying Theorem 1.8 are maximal without a 2-factor within the class of simple graphs. When $|V(G)| \leq 4$ there is exactly one exception: $G \cong K_{1,3}$, with $|A| = 1$, $|B| = 0$ and $q = 3$. The exceptions for $|V(G)| \geq 5$ are exactly the following three (overlapping) types of graphs G :

- there is a component $C_i \in \mathcal{C}$ such that $e(A, V(C_i)) = 1$ and $|V(C_i)| \in \{2, 3\}$;
- $|B| = 1$, and there is an $x \in A$ that is not joined to any component in $\mathcal{C}$;
- there is an $x \in A$ joined to at least two components $C_i, C_j \in \mathcal{C}$, $j \neq i$, with $|V(C_i)| = 1$.

Proving that this covers all exceptions requires some tedious but fairly straightforward case checking.

In some cases the 2-factor obtained in a maximal graph after adding a new edge may be chosen as a Hamiltonian cycle. It might be interesting to characterize those simple graphs, maximal without a 2-factor, that are also maximal without a Hamiltonian cycle.

6 Hans-Boris Alexander Belck, 1929–2007

At the conference *Combinatorics in Cambridge* in August 2003, Roland Häggkvist gave a lecture *Factors Galore* [10]. In his abstract he mentioned and asked for data of Hans-Boris Belck who in 1950 published his one and only mathematical paper, a 25-page opus in *Journal für die Reine und Angewandte Mathematik* (often called *Crelle's Journal*). In this paper Belck not only gave the first general k -factor criterium, including the first purely graph-theoretical proof of Tutte's 1-Factor Theorem, but also settled the following question completely: For what values of (l, r, k) is an l -edge-connected r -regular graph (of even order if k is odd) guaranteed to have a k -factor?

The second author took an interest in Häggkvist's question, and a few years later, with the help of the internet, he stumbled upon a Belck patent from Brazil. Searching then for Belck families in Brazil, he got in contact with Hans-Boris Belck's two sons, one living in Sao Paulo in Brazil, the other in Germany. They provided details from the life of their father and explained, at least partially, why the magnificent paper [3] was his only mathematical contribution.

We now know the following.

- Hans-Boris Alexander Belck was born January 19, 1929, in Apolda, Thüringen, Germany.
- In 1931 the family moved to Bromberg (now Bydgoszcz in Poland), where they stayed almost until the end of the war.

- He finished high school at age 16 and was admitted to the University of Frankfurt. Later he had a 1-year stay at the University of Bern.
- In December 1948 he obtained the degree of Diplom-Mathematiker from the University of Frankfurt; and in June 1949 (aged 20) the degree of Doktor der Naturwissenschaften from the same university (thesis title: *Reguläre Faktoren von Graphen*). Supervisors were Ruth Moufang and Wolfgang Franz.
- His paper [3] was submitted in November 1949, containing the theory of alternating chains and the k -factor theorem.
- Belck emigrated to the United States in 1950. He worked for the General Electric Company in Massachusetts.
- He became a member of AMS as Mr. Hans Boris Belck (since as a 21-year-old his doctorate was not recognized).
- In 1954 he obtained Ph.D. in physics from Rensselaer Polytechnic Institute (thesis title: *The Application of a Magnetic Tape Recorder in Analog Computing*). He submitted a US patent application in 1953, which was granted in 1960: *System for recording and reproducing signal waves*.
- Belck moved to Brazil in 1956. He married in 1957 and had three children (Andreas 1960, Monica 1962, and Alexander 1964).
- In 1964 he founded Amelco S A Industria Electronika. He also obtained several patents in Brazil.
- Belck was involved in the development of the Plano Cruzado, an economical plan to fight inflation, which was adopted by the Brazilian government in 1986.
- He visited Germany and the University of Frankfurt in 1987, where he had a hearty rendezvous with Wolfgang Franz.
- Belck passed away September 29, 2007, aged 78, in Sao Paulo, Brazil.

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At the meeting *Combinatorics in Cambridge* celebrating Béla Bollobás’s 60th birthday, August 4–7, 2003, Häggkvist presented a lecture *Factors Galore* with material from the report and asked for information about Hans-Boris Belck.
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¹ Although Petersen wrote the paper in German, which normally uses capitals for nouns, he consistently uses “graph(s)” without capital. He explains the reason for that early in the paper: “Englische Verfasser haben für ähnliche Figuren den Namen *graph* eingeführt; ich werde diesen Namen beibehalten [...]”.