

Sensitivity dependence of the Navier-Stokes turbulence of a two-dimensional Rayleigh-Bénard convection on time-step

Shijie Qin^{1,2}, Kun Xu¹, and Shijun Liao^{2,3*}

¹ *Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong, China*

² *State Key Laboratory of Ocean Engineering, Shanghai 200240, China and*

³ *School of Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China*

A two-dimensional Rayleigh-Bénard convection governed by the Navier-Stokes (NS) equations is solved by traditional direct numerical simulation (DNS) using double precision and various time-steps. It is found that there are two kinds of final flow types, one is vortical flow, the other is zonal flow, and their statistics are completely different. Especially, the two flow types frequently alternate as the time-step decreases to a very small value, suggesting that the time-step corresponding to each type of turbulent flows should be densely distributed. Thus, stochastic numerical noise has huge influences on final flow type and statistics of numerical simulation of the NS turbulence (i.e. turbulence governed by NS equations), since time-step has a close relationship with numerical noise. However, the NS equations as turbulence model have such an assumption that all small stochastic disturbances for $t > 0$ are negligible. This leads to a logic paradox in theory. Obviously, more investigations are necessary to reveal the essential differences between the NS turbulence, its numerical simulation, and real turbulence.

In 1890 Poincaré [1] discovered that spatiotemporal trajectories of some dynamic systems (called chaos later) have sensitivity dependence on initial condition, i.e. the famous “butterfly-effect” named by Lorenz [2] in 1963 that leads to the collapse of determinism in scientific community. Many researchers [3–6] reported that spatiotemporal trajectories of the Navier-Stokes turbulence (i.e. turbulent flow governed by the Navier-Stokes equations) have sensitivity dependence on initial conditions, corresponding to “trajectory instability”. In other words, the Navier-Stokes (NS) turbulence is chaotic, i.e. any tiny (natural/artificial) stochastic disturbances increase exponentially to a macroscopic level of spatiotemporal trajectories.

In 2006 Lorenz [7] further discovered that numerical noises (i.e. truncation error and round-off error) also have huge influences on numerical simulations of chaotic systems, say, *not only* on their spatiotemporal trajectories *but also* even on their statistics. Especially, Lorenz [7] illustrated that the maximum Lyapunov exponent of a chaotic system might alternate its sign as time-step decreases to a very small value. It should be emphasized that initial condition has physical meanings, but numerical noise is “artificial” and thus “artificially” uncertain since it is strongly dependent of numerical algorithms that might have a large number of different choices for the same equations and the same initial conditions.

How large are the influences of numerical noise on the NS turbulence? Whether can numerical noise lead to essential and statistic differences of the NS turbulence or not? These are fundamental and essential questions for turbulence community.

To answer these questions, let us consider here a kind of two-dimensional (2D) turbulent Rayleigh-Bénard convection (RBC) [8–13], as shown in FIG. 1, which comprises a thin layer of fluid confined between two parallel

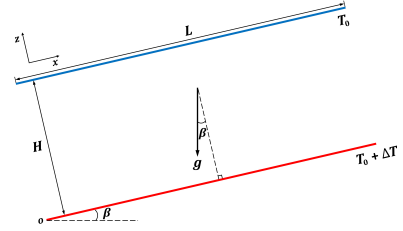


FIG. 1. Schematic of 2D turbulent RBC tilted at an angle β . The fluid layer, between two parallel plates separated by a distance H , obtains heat from the bottom (red) plate and is cooled by the top (blue) plate due to a constant temperature difference $\Delta T > 0$. Here L is the width of computational domain and the downward direction of gravity acceleration g is as indicated.

plates separated by a distance H and tilted at an angle β (positive when the plates rotate anti-clockwise) [14, 15]. The fluid gains heat from the bottom plate and is cooled by the top plate. The x -axis is parallel with the plates, the z -axis is perpendicular to the plate, T_0 and $T_0 + \Delta T$ (with $\Delta T > 0$) denote the temperatures of the top (blue) plate and bottom (red) plate, and g is the acceleration due to gravity, respectively.

Taking the length scale H , velocity scale $\sqrt{g\alpha H\Delta T}$ (where α is the thermal expansion coefficient) and temperature scale ΔT as characteristic scales, the dimensionless Navier-Stokes equations with Boussinesq approximation [16] and tilting angle β are as follows:

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, z)} - \frac{\partial \theta}{\partial x} \cos \beta + \left(\frac{\partial \theta}{\partial z} - 1 \right) \sin \beta \\ - \sqrt{\frac{Pr}{Ra}} \nabla^4 \psi = 0, \end{aligned} \quad (1)$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial(\psi, \theta)}{\partial(x, z)} - \frac{\partial \psi}{\partial x} - \frac{\nabla^2 \theta}{\sqrt{Pr Ra}} = 0, \quad (2)$$

subject to the boundary conditions

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = \theta = 0, \quad \text{at } z = 0 \text{ and } 1 \quad (3)$$

on the upper and lower plate, and the periodic condition

$$\psi(x, z, t) = \psi(x + \Gamma, z, t), \quad \theta(x, z, t) = \theta(x + \Gamma, z, t), \quad (4)$$

where ψ is the stream function with $u = -\partial\psi/\partial z$ and $w = \partial\psi/\partial x$, in which u and w are velocity components in the x - and z -directions, t denotes time, $x \in [0, \Gamma]$ and $z \in [0, 1]$ are position coordinates, $\Gamma = L/H$ is an aspect ratio, and θ denotes the temperature departure from a linear variation background (i.e., the temperature is expressed as $T = 1 - z + \theta$), ∇^2 denotes the Laplace operator with $\nabla^4 = \nabla^2 \nabla^2$, and $\partial(A, B)/\partial(x, z) = (\partial A/\partial x)(\partial B/\partial z) - (\partial B/\partial x)(\partial A/\partial z)$ is the Jacobi operator, respectively. The dimensionless Rayleigh number Ra and Prandtl number Pr are defined by

$$Ra = \frac{g\alpha H^3 \Delta T}{\nu \kappa}, \quad Pr = \frac{\nu}{\kappa}, \quad (5)$$

where ν is the kinematic viscosity and κ is the thermal diffusivity, respectively. The initial temperature and velocity fields are randomly generated as thermal fluctuations in the form of Gaussian white noise, with the temperature standard deviation $\sigma_T = 10^{-10}$ and velocity standard deviation $\sigma_u = 10^{-9}$. It should be emphasized that *all* numerical simulations reported below in this letter have the *same* initial condition. We consider here the case $\Gamma = 2\sqrt{2}$ (which is large enough for the approximation of heat flux at infinite aspect ratio [16, 17]), $Pr = 6.8$ (corresponding to water at room temperature 20°C) and $Ra = 6.8 \times 10^8$ (corresponding to a double-well system involving two completely different turbulent flow types).

The Navier-Stokes turbulence described above is solved by means of the traditional direct numerical simulation (DNS) [18–22] using the double-precision for all data, the fourth-order Runge-Kutta method with the time-step $\Delta t \in [0.0001, 0.002]$ for time integration, and the Fourier spectral method with the 3/2 rule for dealiasing on the same uniform mesh $N_x \times N_z = 1024 \times 1024$. The corresponding spatial resolution is sufficiently fine that the horizontal (maximum) grid spacing is less than the Kolmogorov scale, as previously verified by Pope [23]. Besides, we emphasize that the so-called Courant-Friedrichs-Lewy (CFL) condition, i.e. Courant number < 1 , is always satisfied for all simulations given by different time-step Δt and various values of β reported in this letter. Therefore, from the traditional viewpoint of DNS, all of our DNS results are reliable and can be regarded as benchmark solutions.

Without loss of generality, FIG. 2 shows the temperature departure fields θ at $t = 500$ (that is long enough for a stable flow) of the 2D turbulent RBC tilted at the angles $\beta = 10^\circ, 15^\circ, 18^\circ$, and 20° , given by DNS

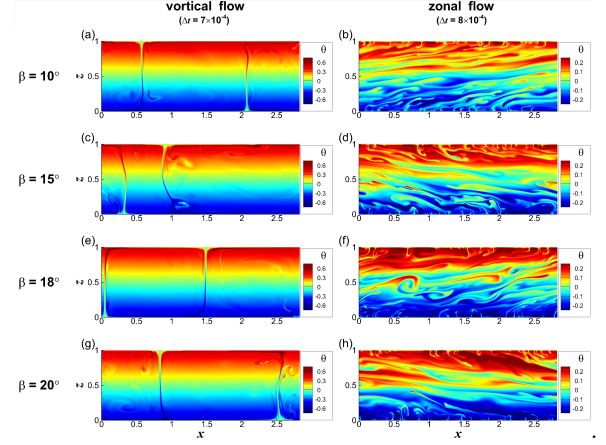


FIG. 2. θ (temperature departure from the linear variation background) fields at $t = 500$ of the turbulent RBC tilted at the angles (a)-(b) $\beta = 10^\circ$, (c)-(d) $\beta = 15^\circ$, (e)-(f) $\beta = 18^\circ$, and (g)-(h) $\beta = 20^\circ$, given by DNS with the time steps $\Delta t = 7 \times 10^{-4}$ (left, vortical flow) and $\Delta t = 8 \times 10^{-4}$ (right, zonal flow).

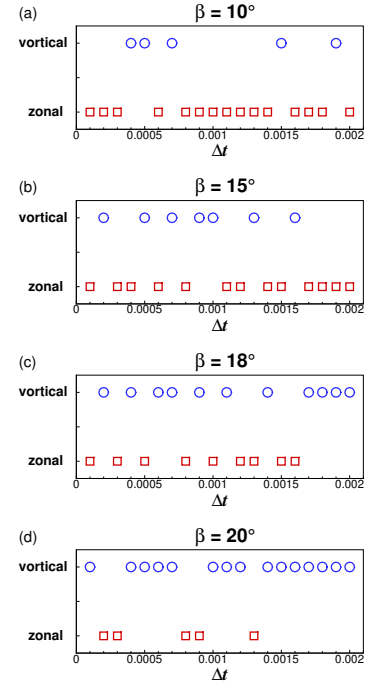


FIG. 3. Final flow type of the tilted turbulent RBC versus time step Δt of DNS: either vortical/roll-like flow (blue circle) or zonal flow (red square). Tilt angle: (a) $\beta = 10^\circ$, (b) $\beta = 15^\circ$, (c) $\beta = 18^\circ$, and (d) $\beta = 20^\circ$.

using the two different time-steps $\Delta t = 7 \times 10^{-4}$ and 8×10^{-4} , respectively. When $\beta = 10^\circ, 15^\circ, 18^\circ$, and 20° , two completely different types of turbulent flows, i.e. vortical/roll-like flow (left of FIG. 2) and zonal flow (right of FIG. 2), are observed.

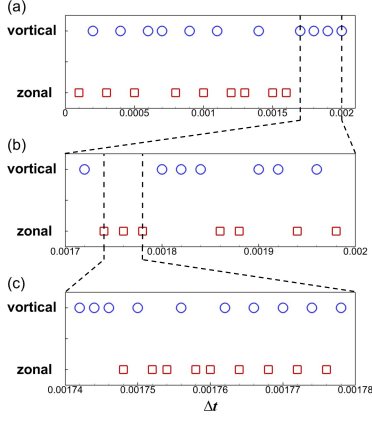


FIG. 4. Final flow type of the turbulent RBC tilted at the angle $\beta = 18^\circ$ versus time step Δt of DNS: either vortical/roll-like flow (blue circle) or zonal flow (red square). (a) $\Delta t \in [0.0001, 0.002]$, (b) $\Delta t \in [0.00172, 0.00198]$, and (c) $\Delta t \in [0.001742, 0.001778]$.

Furthermore, to investigate the sensitivity dependence of final flow type of the 2D turbulent RBC on the time-step Δt , we use various time-step $\Delta t \in [0.0001, 0.002]$ with the increment 0.0001 for the four cases of $\beta = 10^\circ$, 15° , 18° , and 20° . It is found that vortical and zonal flow states alternate randomly for Δt , as illustrated in FIG. 3.

Such kind of randomness can be shown clearly by means of smaller time-step increment. Without loss of generality, let us consider the case $\beta = 18^\circ$. As shown in FIG. 3(c) and FIG. 4(a), the DNS results given by $\Delta t = 0.0017, 0.0018, 0.0019$, and 0.002 with the time-step increment 10^{-4} correspond to the *same* type of turbulence, i.e. the vortical flow. However, using a smaller time-step increment 2×10^{-5} within $\Delta t \in [0.00172, 0.00198]$, both of the vortical and zonal flow types appear randomly again in our DNS results, as shown in FIG. 4(b). Similarly, only the zonal flow appears within $\Delta t \in [0.001742, 0.001778]$, as shown in FIG. 4(c). However, using an even smaller time-step increment 2×10^{-6} , both of the vortical and zonal flow types again appear randomly, as shown in FIG. 4(c). Clearly, FIG. 4 suggests that the time-step corresponding to each type of turbulent flows should be densely distributed. Thus, the final flow type of the 2D turbulent RBC is rather sensitive to the time-step Δt .

It should be emphasized that, except the time-step Δt , all other things such as the initial condition, the uniform mesh, the adapted numerical algorithms and so on, keep the same. Thus, the time-step Δt is the only reason that leads to such kind essential difference of final flow types. Therefore, given that the two different values of the time-step Δt correspond to two different numerical noises, the final type of the NS turbulence is extremely sensitive to numerical noise that can be regarded as tiny “artificial” stochastic disturbances for all the cases of $\beta = 10^\circ, 15^\circ,$

18° , and 20° . Note that the statistics of different types of the NS turbulence are certainly different. Thus, numerical noise can have huge influences even in statistics of the NS turbulence, i.e. the turbulent Rayleigh-Bénard convection considered in this letter.

It is widely accepted by turbulence community that DNS results are reliable as long as the following two conditions are satisfied:

1. the horizontal (maximum) grid spacing is less than the Kolmogorov scale;
2. the so-called Courant-Friedrichs-Lewy (CFL) condition, i.e. Courant number < 1 .

We check all of our DNS results very carefully and it is found that all of our DNS results reported in this letter satisfy the above-mentioned two conditions. Thus, from the traditional viewpoint of DNS, all of our DNS results should be reliable and can be used as benchmark solution. However, the final type of the NS turbulence is sensitive to the time-step Δt , which unfortunately is an “artificial” factor that is “artificially” determined by the “free will” of human being. Which DNS result should we believe? Obviously, turbulent flow is a kind of natural phenomena and certainly should *not* be determined by our “free will”. Thus, there should be something wrong here. We will discuss and reveal the reason below.

Note that the turbulent Rayleigh-Bénard convection considered in this letter is a kind of NS turbulence governed by the *deterministic* equations (1)-(4), which neglect all small physical stochastic disturbances for $t > 0$. It is a pity that such kind of nonlinear partial differential equations have no analytic solution so that one had to use numerical methods. But all numerical methods contain numerical noises caused by truncation error and round-off error. In other words, numerical noise is unavoidable for numerical methods. For a non-chaotic system, numerical noises do not enlarge and thus are indeed negligible. However, for a chaotic system, numerical noises might have huge influence even in statistics, as reported by Lorenz [7]. Unfortunately, the NS turbulence is chaotic, as pointed out by many researchers [3–6]. Thus, due to the famous butterfly-effect of chaos, the unavoidable numerical noises of DNS exponentially increase to a macro-level, which can change the type of the final flow (and the corresponding statistics). This indicates that the numerical noise as a small stochastic disturbance has huge influences on the NS turbulence. However, the NS turbulence as a *deterministic* model neglect *all* stochastic disturbances. This leads to a paradox in logic for the NS turbulence as a mathematical model.

It should be emphasized that the NS turbulence is just a mathematical model for turbulence. It is *not* the real turbulence in nature. In fact, the *true* solution of the NS turbulence is different from the DNS result of the NS turbulence: the latter is badly contaminated by numeri-

cal noise, but the former is not. Note that the following three things, i.e.

(A) the NS turbulence,

(B) numerical simulation of the NS turbulence,

(C) real turbulence,

might be completely different. Whether or not the NS turbulence can describe correctly the essential characteristics of the real turbulence need a lots of theoretical, numerical and experimental supports. Although many numerical and experimental results support the NS turbulence as a good model, our DNS results reported in this letter reveals a logic paradox of the NS turbulence in theory, and thus highly suggest that small stochastic disturbances should be considered in turbulence modeling, say, tiny physical disturbances should be considered in the Navier-Stokes equations for $t > 0$, just like Landau-Lifshitz-Navier-Stokes equations [24] that are stochastic partial differential equations.

Recall that, as reported by Lorenz [7], the maximum Lyapunov exponent of a chaotic system might alternate its sign as time-step decreases to a very small value. Thus Lorenz's results [7] highly support our conclusion, i.e. flow type and statistics of the NS turbulence might be sensitive to small stochastic disturbances. Certainly, more investigations are necessary in future so as to reveal the essential differences between the NS turbulence, its numerical simulation, and real turbulence.

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* email address: sjliao@sjtu.edu.cn

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