EXTENSIONS OF OPERATOR-VALUED KERNELS ON $\mathbb{F}_d^+$

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Abstract. We study the problem of extending a positive-definite operator-valued kernel, defined on words of a fixed finite length from a free semigroup, to a global kernel defined on all words. We show that if the initial kernel satisfies a natural one-step dominance inequality on its interior, a global extension that preserves this interior data and the dominance property is always possible. This extension is constructed explicitly via a Cuntz-Toeplitz model. For the problem of matching the kernel on the boundary, we introduce an intrinsic shift-consistency condition. We prove this condition is sufficient to guarantee the existence of a global extension that agrees with the original kernel on its entire domain.

Contents

1. Introduction

Positive definite kernels provide a unifying tool for extension, interpolation, and realization problems across analysis and operator theory [Aro50]. In one complex variable, this circle of ideas runs through Szegő-Toeplitz theory [Toe10, Sze39, SNFBK10], Pick-Nevanlinna interpolation [Pic15, Nev19, Sar67, AM02], and the classical moment problems [Hau23, Akh21, CF96, Sim05a, Sim05b, Las10]. In several noncommuting variables, the story is driven by dilation and model theory for row contractions and by the operator algebras of free semigroup actions [Fra82, Bun84, Pop89, DP98a, Pop99, DK11], together with developments in free/noncommutative RKHS and Schur-Agler theory [KVV14, PR16, BMV18, BB24].

The classical truncated moment problem asks whether a finite sequence of real numbers can be realized as the power moments of a positive measure on a given subset of $\mathbb{R}$. A well known case is the Hausdorff problem on $[0,1]$: a finite sequence $(s_0,\ldots,s_N)$ extends to the moments of a measure on $[0,1]$ if and only if it is

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completely monotone in the sense of finite differences [Hau23, Akh21, Sim05a]. This theory is elegant and complete, but fundamentally commutative and scalar-valued.

In recent decades, noncommutative analysis has revealed rich structures that generalize classical function theory to multivariable and genuinely free settings. A central object is the free semigroup $\mathbb{F}_d^+$ on d generators, which provides the natural index set for multi-indexed kernels and connects tightly to the theory of row contractions on Hilbert space [Pop89, Pop10, DP98a].

We take a deliberately minimal viewpoint. Fix $N \geq 1$ and write $\Lambda_N = \{\alpha \in \mathbb{F}_d^+ : |\alpha| \leq N\}$. Given a kernel $K : \Lambda_N \times \Lambda_N \to L(H)$, assume positivity on Λ_N and a single one-step dominance along the free shifts $K_\Sigma \leq K$, where

$$K_\Sigma(\alpha, \beta) := \sum_{i=1}^{d} K(\alpha i, \beta i), \quad \alpha, \beta \in \Lambda_{N-1}.$$

This inequality is the free analogue of a discrete energy decay; it can be read as a Schur-type positivity condition for the shift and is stable under compression and truncation. Under this hypothesis, our first main result (Theorem 3.1) produces a global positive definite extension $\widetilde{K} : \mathbb{F}_d^+ \times \mathbb{F}_d^+ \to L(H)$ that (i) agrees with K on the interior block $\Lambda_{N-1} \times \Lambda_{N-1}$ and (ii) satisfies $\widetilde{K}_\Sigma \leq \widetilde{K}$ on $\mathbb{F}_d^+ \times \mathbb{F}_d^+$. Moreover, $\widetilde{K}$ admits a concrete Cuntz-Toeplitz realization

$$\widetilde{K}(\alpha, \beta) = W^* S^\alpha P (S^\beta)^* W, \quad \alpha, \beta \in \mathbb{F}_d^+,$$

with $S = (S_1, \dots, S_d)$ a row isometry with orthogonal ranges, P an orthogonal projection, and $W : H \to \mathcal{K}$ bounded, placing the construction within the Frazho-Bunce-Popescu construction [Fra82, Bun84, Pop89].

A natural strengthening is *boundary agreement*: beyond preserving the interior block, can one choose $\widetilde{K}$ that also matches K on the full level $\Lambda_N \times \Lambda_N$? We identify a *shift-consistency* condition on the level-N boundary (Definition 2.5) that is sufficient for boundary agreement (Theorem 3.2). Shift-consistency gives a quick, verifiable test that separates the interior extension question from the sharper boundary-matching question, but it is not claimed to be necessary.

The operator-theoretic ideas behind Theorems 3.1 and 3.2 are simple to state. From the truncated data we construct the Kolmogorov space and the compressed shifts $B = (B_1, \dots, B_d)$ acting on it. The one-step dominance is equivalent to the contractivity of the interior column operator, and the positive operator

$$A_{N-1} := \sum_{i=1}^{d} B_i^* B_i$$

plays the role of an interior density. Dilating B to a row isometry with orthogonal ranges produces $\widetilde{K}$ with the desired global inequality; when the level-N boundary is shift-consistent, one can choose the dilation to reproduce the mixed and boundary-boundary pairings exactly, yielding boundary agreement.

Connections to the classical moment literature are straightforward. In the commutative ($d = 1$) setting, Hausdorff moment kernels $K(m, n) = \int_0^1 x^{m+n} d\mu(x)$ automatically satisfy the one-step dominance (see Lemma 3.4) so classical $[0, 1]$-moment data fit our hypotheses. Our extension results, however, do not assume Hankel or measure structure, and outside the commutative case the global $\widetilde{K}$ need not be induced by a measure and is typically nonunique—features familiar from truncated Toeplitz/Carathéodory-Fejér/Pick and truncated moment problems.

A closely related viewpoint appears in work on noncommutative disc algebras and moment problems on free (or product) semigroups [DP98b], where one prescribes "moments" on an admissible set and asks for a completely positive map on the associated operator algebra realizing those values; solvability is characterized by positivity of a Toeplitz-type kernel and proved via dilation to isometric representations. Our problem differs in both data and objective: we prescribe a kernel K on $\Lambda_N \times \Lambda_N$ and impose a shift dominance $K_\Sigma \le K$ on Λ_{N-1}, seeking a global p.d. kernel $\widetilde{K}$ with a Cuntz-Toeplitz realization that (i) preserves the interior and (ii) under a separate, sufficient shift-consistency condition, agrees on the full level N.

Organization. Section 2 fixes notation and standing assumptions, introduces the graded spaces $H_K^{(n)}$, the interior shift $B = (B_i)$, the one-step dominance $K_\Sigma \le K$, and the level-N matching conditions (Definition 2.5), and records basic consequences. Section 3 proves the two extension theorems: Theorem 3.1 (existence preserving the interior) and Theorem 3.2 (boundary agreement under shift-consistency). It includes the $d = 1$ link to the truncated Hausdorff moment problem (Remark 3.3). Section 4 presents finite-dimensional examples illustrating the scope and sharpness of the hypotheses (one with shift-consistency, one without).

2. Local extension on the free semigroup

Here we formulate the local-to-global extension problem for positive definite kernels on the free semigroup $\mathbb{F}_d^+$. Given a level N and a truncated kernel K on $\Lambda_N \times \Lambda_N$, we isolate the interior Λ_{N-1} and build its Kolmogorov space $H_K^{(N-1)}$ together with the compressed shifts $(B_1, \ldots, B_d)$. Under the natural one-step dominance $K_\Sigma \le K$ on $\Lambda_{N-1} \times \Lambda_{N-1}$, this yields a canonical row contraction model on $H_K^{(N-1)}$ and an associated interior density $A_{N-1} = \sum B_i^* B_i$ (Lemma 2.2).

Fix $d \in \mathbb{N}$ and the free semigroup $\mathbb{F}_d^+$ on $\{1, \ldots, d\}$ with neutral element $\emptyset$. For a word $\alpha = i_1 \cdots i_k \in \mathbb{F}_d^+$ write $|\alpha| = k$ and $\tilde{\alpha} = i_k \cdots i_1$. Concatenation is denoted by juxtaposition, so $\alpha i := i_1 \cdots i_k i$.

For $N \in \mathbb{N}$, set

$$\Lambda_N := \left\{ \alpha \in \mathbb{F}_d^+ : |\alpha| \le N \right\}, \quad \partial\Lambda_N := \left\{ \alpha \in \mathbb{F}_d^+ : |\alpha| = N \right\}.$$

We refer to Λ_{N-1} as the interior and $\partial\Lambda_N$ as the boundary at level N.

Let H be a Hilbert space and $L(H)$ the space of bounded linear operators on H. A map

$$K : E \times E \longrightarrow L(H)$$

defined on a finite index set $E \subset \mathbb{F}_d^+$ is called positive definite (p.d.) on E if for every finitely family $\{(\alpha_j, u_j)\}_{j=1}^m \subset E \times H$,

$$\sum_{j,k=1}^{m} \langle u_j, K(\alpha_j, \alpha_k) u_k \rangle_H \ge 0.$$

If $E = \Lambda_N$ we simply say "K is p.d. on Λ_N."

Whenever $\alpha i, \beta i \in E$ for all $i = 1, \ldots, d$, we define the (one-step) shifted kernel

$$K_\Sigma(\alpha, \beta) := \sum_{i=1}^{d} K(\alpha i, \beta i).$$

When both sides make sense on a set $F \subset E$, we write $K_\Sigma \leq K$ on F to mean that $K - K_\Sigma$ is p.d. on F.

Problem. Given a level N and a p.d. kernel

$$K : \Lambda_N \times \Lambda_N \to L(H),$$

decide when there exists a global p.d. kernel $\widetilde{K} : \mathbb{F}_d^+ \times \mathbb{F}_d^+ \to L(H)$ such that:

E1. (interior preservation)

$$\widetilde{K}(\alpha, \beta) = K(\alpha, \beta), \quad \alpha, \beta \in \Lambda_{N-1};$$

E2. (shift dominance)

$$\widetilde{K}_\Sigma \leq \widetilde{K} \quad \text{on } \mathbb{F}_d^+ \times \mathbb{F}_d^+;$$

E3. (boundary agreement)

$$\widetilde{K}(\alpha, \beta) = K(\alpha, \beta), \quad \alpha, \beta \in \Lambda_N.$$

We will separate the existence of an extension satisfying (E1) & (E2) from the sharper boundary agreement question (E3).

For later use, we single out two natural families of admissible completions.

Definition 2.1 (admissible extension classes)**.** Fix N and a p.d. K on Λ_N.

(1) $E_{int}(K)$ denotes the set of all global p.d. kernels $\widetilde{K}$ on $\mathbb{F}_d^+$ such that (E1) and (E2) hold.

(2) $E_{bd}(K)$ denotes the subset of $E_{int}(K)$ consisting of those $\widetilde{K}$ that also satisfy (E3) (full agreement on Λ_N).

To phrase the boundary question intrinsically on the given data, we use the Kolmogorov (RKHS) space of the truncation and the compressed shifts.

Lemma 2.2. *Let $K : \Lambda_N \times \Lambda_N \to L(H)$ be a p.d. kernel. Let $H_K^{(N)}$ be the associated Kolmogorov space, with canonical vectors $V_\alpha u$ ($\alpha \in \Lambda_N$, $u \in H$), so that*

$$\langle V_\alpha u, V_\beta v \rangle_{H_K^{(N)}} = \langle u, K(\alpha, \beta) v \rangle_H$$

for all $u, v \in H$, and all $\alpha, \beta \in \Lambda_N$. That is (see, e.g., [Ped58, Sza21, Tia25b]),

$$K(\alpha, \beta) = V_\alpha^* V_\beta, \quad \alpha, \beta \in \Lambda_N.$$

Set

$$H_K^{(N-1)} := \overline{span}\{V_\alpha u : \alpha \in \Lambda_{N-1},\ u \in H\} \subset H_K^{(N)},$$
$$H_K^{(N-2)} := \overline{span}\{V_\alpha u : \alpha \in \Lambda_{N-2},\ u \in H\} \subset H_K^{(N-1)},$$

and let $P_{N-2} : H_K^{(N-1)} \to H_K^{(N-2)}$ be the orthogonal projection.

Define B_i on $D_{N-1} := span\{V_\alpha u : \alpha \in \Lambda_{N-1}\}$ by

$$B_i(V_\alpha u) := \begin{cases} V_{\alpha i} u, & \text{if } \alpha \in \Lambda_{N-2} \\ 0 & \text{if } \alpha \in \partial\Lambda_{N-1} \end{cases}; \quad i = 1, \ldots, d$$

and extend linearly.

If $K_\Sigma \leq K$ on $\Lambda_{N-1} \times \Lambda_{N-1}$, the column map

$$Bx := (B_1 x, \ldots, B_d x), \quad D_{N-1} \to (H_K^{(N-1)})^{\oplus d}$$

is contractive, hence B extends by continuity to $H_K^{(N-1)}$, and

$$\sum\nolimits_i B_i^* B_i \leq I_{H_K^{(N-1)}}.$$

Proof. Take $x \in D_{N-1}$ and write $x = y + z$, where

$$y = P_{N-2}x, \quad z = x - P_{N-2}x.$$

By definition $B_i z = 0$ (since z is a linear combination of vectors with $|\alpha| = N-1$), hence

$$B_i x = B_i y, \quad i = 1, \dots, d,$$

and

$$\sum\nolimits_{i=1}^d \|B_i x\|^2_{H_K^{(N-1)}} = \sum\nolimits_{i=1}^d \|B_i y\|^2_{H_K^{(N-1)}}. \tag{2.1}$$

Write $y = \sum_{\alpha \in \Lambda_{N-2}} V_\alpha u'_\alpha$. Using the RKHS inner product,

$$\begin{aligned}
\sum\nolimits_{i=1}^d \|B_i y\|^2_{H_K^{(N)}} &= \sum\nolimits_{i=1}^d \left\langle \sum\nolimits_{\alpha \in \Lambda_{N-2}} V_{\alpha i} u'_\alpha, \sum\nolimits_{\beta \in \Lambda_{N-2}} V_{\beta i} u'_\beta \right\rangle_{H_K^{(N)}} \\
&= \sum\nolimits_{\alpha, \beta \in \Lambda_{N-2}} \sum\nolimits_{i=1}^d \left\langle u'_\alpha, K(\alpha i, \beta i)\, u'_\beta \right\rangle_H \\
&= \sum\nolimits_{\alpha, \beta \in \Lambda_{N-2}} \left\langle u'_\alpha, K_\Sigma(\alpha, \beta)\, u'_\beta \right\rangle_H \\
&\leq \sum\nolimits_{\alpha, \beta \in \Lambda_{N-2}} \left\langle u'_\alpha, K(\alpha, \beta)\, u'_\beta \right\rangle_H = \|y\|^2_{H_K^{(N)}},
\end{aligned}$$

where the inequality uses $K_\Sigma \leq K$ on $\Lambda_{N-1} \times \Lambda_{N-1}$. Combining this with (2.1) yields

$$\sum\nolimits_{i=1}^d \|B_i x\|^2_{H_K^{(N)}} \leq \|y\|^2_{H_K^{(N)}} = \|P_{N-2}x\|^2_{H_K^{(N)}} \leq \|x\|^2_{H_K^{(N)}}.$$

Thus B is contractive on D_{N-1}, extends to $H_K^{(N-1)}$, and $\sum_i B_i^* B_i \leq I_{H_K^{(N-1)}}$ follows. ∎

Corollary 2.3 (Interior density at level $N-1$)**.** *Assume K is p.d. on Λ_N and $K_\Sigma \leq K$ on $\Lambda_{N-1} \times \Lambda_{N-1}$. With $H_K^{(N-1)}$ and the operators $B_1, \dots, B_d$ as in Lemma 2.2, set*

$$A_{N-1} := \sum_{i=1}^d B_i^* B_i \in \mathcal{L}(H_K^{(N-1)}), \quad 0 \leq A_{N-1} \leq I_{H_K^{(N-1)}}.$$

Then for all $\alpha, \beta \in \Lambda_{N-2}$,

$$K_\Sigma(\alpha, \beta) \;=\; V_\alpha^* \, A_{N-1} \, V_\beta.$$

Proof. Let $y = \sum_{\alpha \in \Lambda_{N-2}} V_\alpha u_\alpha \in H_K^{(N-2)} \subset H_K^{(N-1)}$. Then

$$\sum_{i=1}^d \|B_i y\|^2_{H_K^{(N)}} = \sum_{\alpha, \beta \in \Lambda_{N-2}} \sum_{i=1}^d \langle u_\alpha, K(\alpha i, \beta i)\, u_\beta \rangle_H = \sum_{\alpha, \beta \in \Lambda_{N-2}} \langle u_\alpha, K_\Sigma(\alpha, \beta)\, u_\beta \rangle_H,$$

while also

$$\sum_{i=1}^d \|B_i y\|^2_{H_K^{(N)}} = \langle y, A_{N-1} y \rangle_{H_K^{(N)}} = \sum_{\alpha, \beta \in \Lambda_{N-2}} \langle u_\alpha, V_\alpha^* A_{N-1} V_\beta u_\beta \rangle_H.$$

Identifying the two quadratic forms on the dense span of $\{V_\alpha u : \alpha \in \Lambda_{N-2}\}$ gives $K_\Sigma(\alpha, \beta) = V_\alpha^* A_{N-1} V_\beta$. ∎

Remark 2.4. The operator A_{N-1} is the Radon-Nikodym derivative of the interior shift $K_\Sigma|_{\Lambda_{N-2}\times\Lambda_{N-2}}$ with respect to $K|_{\Lambda_{N-2}\times\Lambda_{N-2}}$ in the Kolmogorov space $H_K^{(N-1)}$. We record the concrete formula $A_{N-1} = \sum_i B_i^* B_i$ here because it is the only case needed in this paper. For the general Radon-Nikodym theorem for positive definite kernels, see e.g., [Tia25a, Section 3].

With this notation, and under the assumption that $K_\Sigma \leq K$ on $\Lambda_{N-1}\times\Lambda_{N-1}$, the interior is equipped with a row contraction model $B = (B_1, \dots, B_d)$ on $H_K^{(N-1)}$. The boundary agreement amounts to asking whether the prescribed level N cross-terms are generated by a row contraction $T = (T_1, \dots, T_d)$ on $H_K^{(N-1)}$ that extends the interior shift. Necessarily (by Corollary 2.3), the interior Radon-Nikodym density $A_{N-1} := \sum_i B_i^* B_i$ is realized as the compression of $\sum_i T_i^* T_i$ to $H_K^{(N-2)}$.

Definition 2.5. We say the boundary data of K at level N are *shift consistent* if there exist bounded operators $T_1, \dots, T_d$ on $H_K^{(N-1)}$ such that

$$\sum_i T_i^* T_i \leq I_{H_K^{(N-1)}}, \quad T_i V_\alpha u = V_{\alpha i} u \quad (\alpha \in \Lambda_{N-2})$$

and, for all $\alpha, \beta \in \Lambda_{N-1}$ and $u, v \in H$,

$$\langle V_\alpha u, T_i V_\beta v\rangle_{H_K^{(N)}} = \langle u, K(\alpha, \beta i)\, v\rangle_H, \tag{2.2}$$

and

$$\langle T_i V_\alpha u, T_j V_\beta v\rangle_{H_K^{(N)}} = \langle u, K(\alpha i, \beta j)\, v\rangle_H. \tag{2.3}$$

In this case one necessarily has

$$P_{H_{N-2}}\Big(\sum_{i=1}^d T_i^* T_i\Big)\big|_{H_K^{(N-2)}} = \sum_{i=1}^d B_i^* B_i = A_{N-1}, \tag{2.4}$$

so the interior density A_{N-1} is the compression of $\sum_i T_i^* T_i$.

Remark 2.6. At level N, the two matching conditions play complementary roles.

Condition (2.2) determines, for each i and $\beta \in \Lambda_{N-1}$, the vector

$$T_i V_\beta u \in H_K^{(N-1)}$$

uniquely via its pairings with $H_K^{(N-1)}$ (pairings taken in $\langle\cdot,\cdot\rangle_{H_K^{(N)}}$), thereby realizing the interior-boundary entries indexed by $(\alpha, \beta i)$.

With these vectors fixed, (2.3) requires that the mutual inner products

$$\langle T_i V_\alpha u, T_j V_\beta v\rangle_{H_K^{(N)}}$$

match the entries indexed by $(\alpha i, \beta j)$, which pins down the boundary-boundary block.

In particular, the compression identity (2.4) is necessary (e.g., from (2.2) or from (2.3) together with Corollary 2.3) but does not determine the boundary-boundary block on its own; (2.3) is needed for that. Together, these matching conditions ensure agreement with K on $\Lambda_N \times \Lambda_N$.

3. Existence and boundary agreement

The results below address: (i) existence of global p.d. extensions $\widetilde{K}$ that preserve the interior data and satisfy the shift inequality, with a Cuntz-Toeplitz realization (Theorem 3.1); and (ii) a criterion (shift consistency) for full agreement on the boundary level N (Theorem 3.2).

Theorem 3.1 (Existence with interior preservation)**.** *If $K : \Lambda_N \times \Lambda_N \to L(H)$ is p.d. and satisfies $K_\Sigma \le K$ on $\Lambda_{N-1} \times \Lambda_{N-1}$, then $E_{int}(K) \neq \emptyset$. In particular, there exists a global p.d. extension $\widetilde{K}$ with (E1)-(E2) and a Cuntz-Toeplitz realization*

$$\widetilde{K}(\alpha,\beta) = W^* S^\alpha P \left(S^\beta\right)^* W, \quad \alpha,\beta \in \mathbb{F}_d^+,$$

where $S = (S_1,\dots,S_d)$ is a row isometry with orthogonal ranges on a Hilbert space $\mathcal{K}$, P is an orthogonal projection on $\mathcal{K}$, and $W : H \to \mathcal{K}$ is bounded.

Proof. **Step 1.** Let $H_K^{(N)}$ be the Kolmogorov space of the truncated kernel on Λ_N with canonical vectors $V_\alpha u$:

$$\langle V_\alpha u, V_\beta v\rangle_{H_K^{(N)}} = \langle u, K(\alpha,\beta) v\rangle_H, \quad \alpha,\beta \in \Lambda_N,\ u,v \in H.$$

Set

$$\begin{aligned} H_K^{(N-1)} &:= \overline{span}\{V_\alpha u : \alpha \in \Lambda_{N-1},\ u \in H\} \subset H_K^{(N)},\\ H_K^{(N-2)} &:= \overline{span}\{V_\alpha u : \alpha \in \Lambda_{N-2},\ u \in H\} \subset H_K^{(N-1)}. \end{aligned}$$

On $D_{N-1} := span\{V_\alpha u : \alpha \in \Lambda_{N-1}\}$ which is dense in $H_K^{(N-1)}$, define

$$B_i(V_\alpha u) = \begin{cases} V_{\alpha i} u, & \alpha \in \Lambda_{N-2},\\ 0, & \alpha \in \partial\Lambda_{N-1}, \end{cases} \qquad (i = 1,\dots,d)$$

and extend linearly. By Lemma 2.2 and the hypothesis $K_\Sigma \le K$ on $\Lambda_{N-1} \times \Lambda_{N-1}$, the column map

$$B : H_{N-1} \to (H_K^{(N-1)})^{\oplus d}, \quad x \mapsto (B_1 x,\dots,B_d x),$$

is contractive; equivalently,

$$\sum_{i=1}^{d} B_i^* B_i \le I_{H_K^{(N-1)}}.$$

Step 2. By the multivariable dilation theorem for column contractions (see e.g., [Fra82, Bun84, Pop89]), there exist a Hilbert space $\mathcal{K}$, a row isometry $S = (S_1,\dots,S_d)$ on $\mathcal{K}$ with orthogonal ranges (i.e., $S_i^* S_j = \delta_{ij} I_{\mathcal{K}}$), and an isometry

$$J : H_K^{(N-1)} \to \mathcal{K}$$

such that

$$S_i^* J = J B_i, \quad i = 1,\dots,d. \tag{3.1}$$

In particular, $B_i = J^* S_i^* J$.

Consequently, for any word $\gamma = i_1 \cdots i_k$ with $|\gamma| \le N-1$,

$$J^* (S^\gamma)^* J = J^* S_{i_k}^* \cdots S_{i_1}^* J = B_{i_k} \cdots B_{i_1} =: B^{\tilde{\gamma}}, \tag{3.2}$$

where $\tilde{\gamma} = i_k \cdots i_1$ is the reversed word. Taking adjoints in (3.2) gives

$$J^* S^\gamma J = \left(J^* (S^\gamma)^* J\right)^* = \left(B^{\tilde{\gamma}}\right)^*. \tag{3.3}$$

Let $P := JJ^*$ be the orthogonal projection of $\mathcal{K}$ onto $JH_K^{(N-1)}$.

Step 3. Define $W : H \to \mathcal{K}$ by

$$Wu := JV_{\emptyset}u, \quad u \in H.$$

(We do not need W to be an isometry.) Define a kernel $\widetilde{K}$ on $\mathbb{F}_d^+ \times \mathbb{F}_d^+$ by

$$\begin{aligned}\widetilde{K}(\alpha,\beta) &:= W^* S^\alpha P \left(S^\beta\right)^* W \\ &= V_\emptyset^* J^* S^\alpha P \left(S^\beta\right)^* JV_\emptyset.\end{aligned} \tag{3.4}$$

For any finite family $\{(\alpha_k, u_k)\}$ in $\mathbb{F}_d^+ \times H$,

$$\sum\nolimits_{j,k} \left\langle u_j, \widetilde{K}(\alpha_j,\alpha_k) u_k \right\rangle_H = \left\| P \sum\nolimits_k (S^{\alpha_k})^* W u_k \right\|_{\mathcal{K}}^2 \geq 0,$$

so $\widetilde{K}$ is positive definite.

Step 4. Fix $\alpha, \beta \in \Lambda_{N-1}$. Using (3.3), (3.2) and $P = JJ^*$, we get

$$\begin{aligned}\widetilde{K}(\alpha,\beta) &= V_\emptyset^* J^* S^\alpha P \left(S^\beta\right)^* JV_\emptyset \\ &= V_\emptyset^* \left(J^* S^\alpha J\right) \left(J^* \left(S^\beta\right)^* J\right) V_\emptyset \\ &= V_\emptyset^* \left(B^{\tilde{\alpha}}\right)^* B^{\tilde{\beta}} V_\emptyset.\end{aligned}$$

We now justify the identification $B^{\tilde{\gamma}} V_\emptyset u = V_\gamma u$ for all $\gamma \in \Lambda_{N-1}$: if $\gamma = i_1 \cdots i_k$ with $k \leq N-1$, then recursively

$$B_{i_1} V_\emptyset u = V_{i_1} u, \quad B_{i_2} B_{i_1} V_\emptyset u = V_{i_1 i_2} u, \ \ldots, \ B_{i_k} \cdots B_{i_1} V_\emptyset u = V_{i_1 \cdots i_k} u,$$

and all applications are legal because at each step the current word length is $\leq N-2$ before applying $B_{i_{r+1}}$. Hence $B^{\tilde{\beta}} V_\emptyset = V_\beta$ and $B^{\tilde{\alpha}} V_\emptyset = V_\alpha$. Therefore,

$$\widetilde{K}(\alpha,\beta) = V_\emptyset^* \left(B^{\tilde{\alpha}}\right)^* B^{\tilde{\beta}} V_\emptyset = V_\alpha^* V_\beta = K(\alpha,\beta),$$

i.e.,$\widetilde{K}$ agrees with K on $\Lambda_{N-1} \times \Lambda_{N-1}$.

Step 5. Let

$$M := \overline{span}\left\{ (S^\gamma)^* WH : \gamma \in \mathbb{F}_d^+ \right\} \subset \mathcal{K}.$$

We claim

$$\sum_{i=1}^d S_i P S_i^* \leq P \quad \text{on } M. \tag{3.5}$$

Indeed, fix a finite sum $y = \sum_r (S^{\gamma_r})^* W u_r \in M$. Then

$$J^* y = \sum_r J^* (S^{\gamma_r})^* JV_\emptyset u_r = \sum_r B^{\tilde{\gamma}_r} V_\emptyset u_r \in H_K^{(N-1)},$$

and, for each i,

$$J^* S_i^* y = \sum_r J^* \left(S^{\gamma_r i}\right)^* JV_\emptyset u_r = \sum_r B_i B^{\tilde{\gamma}_r} V_\emptyset u_r = B_i (J^* y).$$

Hence

$$\sum_{i=1}^d \|J^* S_i^* y\|^2 = \sum_{i=1}^d \|B_i (J^* y)\|^2 \leq \|J^* y\|^2,$$

by $\sum_i B_i^* B_i \leq I_{H_{N-1}}$. Rewriting this in $\mathcal{K}$ yields

$$\sum_{i=1}^d \langle y, S_i P S_i^* y \rangle_{\mathcal{K}} \leq \langle y, P y \rangle_{\mathcal{K}},$$

i.e., (3.5). Now let $\{(\alpha_j, u_j)\}_{j=1}^m \subset \mathbb{F}_d^+ \times H$ and set

$$y := \sum_{j=1}^{m} (S^{\alpha_j})^* W u_j \in M.$$

Then, by (3.4) and (3.5),

$$\begin{aligned}\sum\nolimits_{j,k=1}^{m} \left\langle u_j, \widetilde{K}_\Sigma(\alpha_j, \alpha_k) u_k \right\rangle_H &= \left\langle y, \left(\sum\nolimits_{i=1}^{d} S_i P S_i^*\right) y \right\rangle_{\mathcal{K}} \le \langle y, Py \rangle_{\mathcal{K}} \\ &= \sum\nolimits_{j,k=1}^{m} \left\langle u_j, \widetilde{K}(\alpha_j, \alpha_k) u_k \right\rangle_H,\end{aligned}$$

which is exactly the quadratic form inequality $\widetilde{K}_\Sigma \le \widetilde{K}$ on $\mathbb{F}_d^+ \times \mathbb{F}_d^+$.

Combining Steps 3–5, $\widetilde{K}$ is a global p.d. kernel satisfying (E1)-(E2). The displayed formula is its Cuntz-Toeplitz realization. This shows $E_{int}(K) \neq \emptyset$ and completes the proof. □

Theorem 3.2 (Boundary extension criterion)**.** *Let K be p.d. on Λ_N and $K_\Sigma \le K$ on $\Lambda_{N-1} \times \Lambda_{N-1}$. Suppose the boundary data of K at level N are shift consistent (Definition 2.5). Then $E_{bd}(K) \neq \emptyset$. There exists a global p.d. kernel $\widetilde{K}$ on $\mathbb{F}_d^+ \times \mathbb{F}_d^+$ with $\widetilde{K}_\Sigma \le \widetilde{K}$ and*

$$\widetilde{K}(\alpha, \beta) = K(\alpha, \beta), \quad \alpha, \beta \in \Lambda_N.$$

Proof. Assume there exist bounded operators $T_1, \dots, T_d$ on $H_K^{(N-1)}$ with

$$\sum_{i=1}^{d} T_i^* T_i \le I_{H_K^{(N-1)}}, \qquad T_i V_\alpha u = V_{\alpha i} u, \quad \alpha \in \Lambda_{N-2},$$

and, for all $\alpha, \beta \in \Lambda_{N-1}$, $u, v \in H$,

$$\langle V_\alpha u, T_i V_\beta v \rangle_{H_K^{(N)}} = \langle u, K(\alpha, \beta i) v \rangle_H, \tag{3.6}$$

$$\langle T_i V_\alpha u, T_j V_\beta v \rangle_{H_K^{(N)}} = \langle u, K(\alpha i, \beta j) v \rangle_H. \tag{3.7}$$

Apply Frazho-Bunce-Popescu to the row contraction $T = (T_i)$: there exist a Hilbert space $\mathcal{K}$, a row isometry $S = (S_i)$ with orthogonal ranges, and an isometry $J : H_K^{(N-1)} \to \mathcal{K}$ such that $S_i^* J = J T_i$. Moreover,

$$J^*(S^\gamma)^* J = T^{\tilde{\gamma}}, \qquad J^* S^\gamma J = (T^{\tilde{\gamma}})^*. \tag{3.8}$$

Set $P := JJ^*$ and $W := JV_\emptyset$, and define the global kernel

$$\widetilde{K}(\alpha, \beta) := W^* S^\alpha P (S^\beta)^* W, \qquad \alpha, \beta \in \mathbb{F}_d^+.$$

This $\widetilde{K}$ is positive definite by construction. We also note that $T^{\tilde{\gamma}} V_\emptyset = V_\gamma$ for all $|\gamma| \le N-1$, by iterating $T_i V_\eta = V_{\eta i}$ on Λ_{N-2}.

Agreement on $\Lambda_N \times \Lambda_N$.

If $|\alpha|, |\beta| \leq N-1$:

$$\begin{aligned}\left\langle u, \widetilde{K}(\alpha,\beta) v\right\rangle_H &= \left\langle V_\emptyset u, (J^* S^\alpha J)\,(J^* \left(S^\beta\right)^* J) V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \left\langle V_\emptyset u, \left(T^{\tilde\alpha}\right)^* T^{\tilde\beta} V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \left\langle T^{\tilde\alpha} V_\emptyset u, T^{\tilde\beta} V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \langle V_\alpha u, V_\beta v\rangle_{H_K^{(N)}} \\ &= \langle u, K(\alpha,\beta) v\rangle_H .\end{aligned}$$

If $|\alpha| \leq N-1$ and $\beta = \beta' j$ with $|\beta'| \leq N-1$:

$$\begin{aligned}\left\langle u, \widetilde{K}(\alpha,\beta) v\right\rangle_H &= \left\langle u, W^* S^\alpha P \left(S^{\beta' j}\right)^* W v\right\rangle_{H_K^{(N)}} \\ &= \left\langle V_\emptyset u, J^* S^\alpha J J^* \left(S^{\beta' j}\right)^* J V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \left\langle V_\emptyset u, (J^* S^\alpha J)\left(J^* S_j^* J\right)\left(J^* \left(S^{\beta'}\right)^* J\right) V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \left\langle V_\emptyset u, \left(T^{\tilde\alpha}\right)^* T_j T^{\tilde\beta'} V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \left\langle T^{\tilde\alpha} V_\emptyset u, T_j T^{\tilde\beta'} V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \langle V_\alpha u, T_j V_{\beta'} v\rangle_{H_K^{(N)}} \\ &= \langle u, K(\alpha,\beta' j) v\rangle_H .\end{aligned}$$

The case $|\alpha| = N$, $|\beta| \leq N-1$ is symmetric.

If $\alpha = \alpha' i$ and $\beta = \beta' j$ with $|\alpha'|, |\beta'| \leq N-1$:

$$\begin{aligned}\left\langle u, \widetilde{K}(\alpha' i,\beta' j) v\right\rangle_H &= \left\langle V_\emptyset u, \left(J^* S^{\alpha' i} J\right)\left(J^* \left(S^{\beta' j}\right)^* J\right) V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \left\langle V_\emptyset u, \left((T^{\widetilde{\alpha' i}})^*\right)\left(T^{\widetilde{\beta' j}}\right) V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \left\langle V_\emptyset u, (T^{\tilde\alpha'})^* T_i^* T_j T^{\tilde\beta'} V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \left\langle T^{\tilde\alpha'} V_\emptyset u, T_i^* T_j T^{\tilde\beta'} V_\emptyset v\right\rangle_{H_K^{(N)}} \\ &= \langle V_{\alpha'} u, T_i^* T_j V_{\beta'} v\rangle_{H_K^{(N)}} \\ &= \langle T_i V_{\alpha'} u, T_j V_{\beta'} v\rangle_{H_K^{(N)}} \\ &= \langle u, K(\alpha i,\beta j) v\rangle_H .\end{aligned}$$

Thus $\widetilde{K} = K$ on $\Lambda_N \times \Lambda_N$.

Shift inequality. Let $M := \overline{\operatorname{span}}\{(S^\gamma)^* W H : \gamma \in \mathbb{F}_d^+\}$. For $y = \sum_r (S^{\gamma_r})^* W u_r \in M$, (3.8) gives $J^* y = \sum_r T^{\tilde\gamma_r} V_\emptyset u_r$ and $J^* S_i^* y = T_i(J^* y)$, hence $\sum_i \|J^* S_i^* y\|^2 = \sum_i \|T_i(J^* y)\|^2 \leq \|J^* y\|^2$. Equivalently, $\sum_i \langle y, S_i P S_i^* y\rangle \leq \langle y, P y\rangle$ on M, which is exactly $\widetilde{K}_\Sigma \leq \widetilde{K}$. □

Remark 3.3. We note that the classical truncated Hausdorff moment problem (THMP) on $[0,1]$ fits our setting. Here, let $d = 1$, so $\mathbb{F}_1^+ \simeq \mathbb{N}$. For a finite positive Borel measure μ on $[0,1]$, consider the kernel

$$K(m,n)=\int_0^1 x^{m+n}d\mu(x),\quad m,n\in\mathbb{N}. \tag{3.9}$$

In the classical THMP one is given finitely many moments $s_0,\ldots,s_M$ and asks for a measure μ on $[0,1]$ with $s_k=\int_0^1 x^k d\mu(x)$. This imposes (i) a Hankel constraint $K(m,n)=s_{m+n}$ and (ii) the Hausdorff feasibility (complete monotonicity) conditions on (s_k). Whenever these hold, the associated kernel (3.9) is positive definite. Moreover, with $K_\Sigma(m,n):=K(m+1,n+1)$, one has $K_\Sigma\le K$ (see Lemma 3.4). Hence, THMP feasibility $\Longrightarrow$ the hypotheses used in our extension theorems (for $d=1$).

The converse fails in general: our assumptions (p.d. on a truncation plus $K_\Sigma\le K$ on the interior) do not enforce Hankel structure or the Hausdorff inequalities, so a dilation-based global extension need not arise from a measure. When, in addition, one enforces the Hankel form $\widetilde{K}(m,n)=s_{m+n}$ and and the truncated Hausdorff (complete-monotonicity) conditions, the two formulations coincide: the model operator can be chosen as a self-adjoint contraction (multiplication by x on $L^2(\mu)$), and $\widetilde{K}(m,n)=\int_0^1 x^{m+n}d\mu(x)$.

Lemma 3.4. *Let K be as in (3.9). Then $K\ge 0$ and $K_\Sigma\le K$. Moreover, for every N, the truncations satisfy $K|_{\Lambda_N\times\Lambda_N}\ge 0$, and $K_\Sigma\le K$ on the interior block Λ_{N-1}. (Recall that $\Lambda_N=\{0,\ldots,N\}$ for every $N\in\mathbb{N}$.)*

Proof. For any finitely supported complex scalars (c_m),

$$\sum\nolimits_{m,n\ge 0}\overline{c_m}c_nK(m,n)=\int_0^1\left|\sum\nolimits_{m\ge 0}c_mx^m\right|^2 d\mu(x)\ge 0.$$

Also, $K_\Sigma(m,n)=\int_0^1 x^{m+n+2}d\mu(x)$, hence

$$\sum\nolimits_{m,n\ge 0}\overline{c_m}c_n\left(K(m,n)-K_\Sigma(m,n)\right)=\int_0^1\left|\sum\nolimits_{m\ge 0}c_mx^m\right|^2\left(1-x^2\right)d\mu(x)\ge 0,$$

so $K_\Sigma\le K$. Restricting sums to $0\le m,n\le N-1$ gives the gives the truncated interior inequality. □

4. Examples

Let $K:\Lambda_N\times\Lambda_N\to L(H)$ be a p.d. kernel. We consider the scalar case $H=\mathbb{C}$ with $d=2$. The index sets are $\Lambda_1=\{0,1,2\}$, $\Lambda_2=\{0,1,2,11,12,21,22\}$, $\partial\Lambda_2=\{11,12,21,22\}$, and $\Lambda_2=\Lambda_1\cup\partial\Lambda_2$.

Example 4.1. (Shift-consistency boundary $\Longrightarrow$ Theorem 3.2 applies) Work in $\mathbb{C}^3$ (the Kolmogorov space) with standard orthonormal basis $\{e_0,e_1,e_2\}$. Set

$$V_0=e_0,\quad V_1=\frac{1}{2}e_1,\quad V_2=\frac{1}{2}e_2$$

Then $K(\alpha,\beta)=\langle V_\alpha,V_\beta\rangle$ on Λ_1 gives

$$K|_{\Lambda_1\times\Lambda_1}=diag\left(1,1/4,1/4\right)\quad(\text{order } 0,1,2).$$

Define T_1, $T_2 : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$T_1 e_0 = \frac{1}{2} e_1, \quad T_1 e_1 = 0, \quad T_1 e_2 = 0;$$
$$T_2 e_0 = \frac{1}{2} e_2, \quad T_2 e_1 = 0, \quad T_2 e_2 = 0.$$

Set

$$A := T_1^* T_1 + T_2^* T_2 = diag\,(1/2, 0, 0) \leq I_{\mathbb{C}^3},$$

so (T_1, T_2) is a row contraction on the level-1 space.

Using $V_{\alpha i} := T_i V_\alpha$ for $|\alpha| \leq 1$, we get

$$V_{11} = T_1 V_1 = 0, \quad V_{12} = T_2 V_1 = 0,$$
$$V_{21} = T_1 V_2 = 0, \quad V_{22} = T_2 V_2 = 0,$$

and extend K to Λ_2.

For $\alpha, \beta \in \Lambda_1$,

$$K_\Sigma\,(\alpha, \beta) = \sum_{i=1}^{2} K\,(\alpha i, \beta i) = \sum_{i=1}^{2} \langle T_i V_\alpha, T_i V_\beta \rangle = \langle A V_\alpha, V_\beta \rangle\,.$$

Hence, for any scalars c_α supported on Λ_1,

$$\sum_{\alpha,\beta} \overline{c_\alpha} c_\beta\,(K\,(\alpha, \beta) - K_\Sigma\,(\alpha, \beta)) = \left\| (I - A)^{1/2} \sum c_\alpha V_\alpha \right\|^2 \geq 0,$$

so $K_\Sigma \leq K$ on Λ_1. Concretely,

$$K|_{\Lambda_1 \times \Lambda_1} = diag\,(1, 1/4, 1/4)\,, \quad K_\Sigma|_{\Lambda_1 \times \Lambda_1} = diag\,(1/2, 0, 0)\,,$$

and so the difference $diag\,(1/2, 1/4, 1/4)$ is p.d.

By construction,

$$K\,(\alpha i, \beta j) = \langle T_i V_\alpha, T_j V_\beta \rangle\,, \quad K\,(\alpha, \beta i) = \langle V_\alpha, T_i V_\beta \rangle\,,$$

for all $\alpha, \beta \in \Lambda_1$. This is the shift-consistency condition. Hence the boundary is shift consistent.

Conclusion. Since $K_\Sigma \leq K$ on Λ_1, Theorem 3.1 yields a global positive-definite extension preserving the interior; and since the boundary is shift-consistent, Theorem 3.2 gives a global extension that also agrees with K on $\Lambda_2 \times \Lambda_2$.

Remark 4.2. Let H_K denote the Kolmogorov space for the truncated kernel K (defined on $\Lambda_N \times \Lambda_N$), and set

$$H_K^{(m)} := span\{V_\gamma : |\gamma| \leq m\} \subset H_K, \quad m = 1, 2.$$

In Example 4.1 we have $H_K^{(1)} = H_K^{(2)} = H_K = \mathbb{C}^3$, because the length-2 vectors add no new directions. In this special situation, a global p.d. extension that preserves the interior already follows from the Λ_1 data and the inequality $K_\Sigma \leq K$ (Theorem 3.1); the level-2 block plays no role for mere existence. The additional shift-consistency requirement is only needed if one further demands boundary agreement on $\Lambda_2 \times \Lambda_2$ (Theorem 3.2).

By contrast, when $H_K^{(2)} \supsetneq H_K^{(1)}$ the boundary introduces genuinely new directions, and the shift-consistency condition becomes a substantive compatibility constraint for achieving boundary agreement.

Example 4.3 (Not shift-consistent $\implies$ Theorem 3.2 not applicable)**.** Keep the scalar case $H = \mathbb{C}$, $d = 2$, with index sets Λ_1, $\partial\Lambda_2$, $\Lambda_2 = \Lambda_1 \cup \partial\Lambda_2$ as above.

Work in $\mathbb{C}^4$ with orthonormal basis $\{e_0, e_1, e_2, e_3\}$. Set

$$V_0 = e_0, \quad V_1 = \tfrac{1}{2}e_1, \quad V_2 = \tfrac{1}{2}e_2,$$

so

$$K|_{\Lambda_1\times\Lambda_1} = \mathit{diag}\,(1, 1/4, 1/4) \quad (\text{order } (0,1,2)).$$

Define

$$V_{11} = 0, \quad V_{21} = 0, \quad V_{22} = 0, \quad V_{12} = \tfrac{1}{4}e_3.$$

Extend K to Λ_2 by the Gram rule $K(\gamma, \delta) = \langle V_\gamma, V_\delta\rangle$.

Then we have

$$\begin{aligned}
&K_\Sigma(0,0) = K(1,1) + K(2,2) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2},\\
&K_\Sigma(1,1) = K(11,11) + K(12,12) = 0 + \|V_{12}\|^2 = \tfrac{1}{16},\\
&K_\Sigma(2,2) = K(21,21) + K(22,22) = 0 + 0 = 0,\\
&K_\Sigma(1,2) = K(11,12) + K(12,22) = 0, \qquad K_\Sigma(0,1) = K(1,11) + K(2,12) = 0,\\
&K_\Sigma(0,2) = K(1,21) + K(2,22) = 0.
\end{aligned}$$

Hence (order $(0,1,2)$)

$$K_\Sigma|_{\Lambda_1\times\Lambda_1} = \mathit{diag}\,(1/2, 1/16, 0)\,, \quad K|_{\Lambda_1\times\Lambda_1} - K_\Sigma|_{\Lambda_1\times\Lambda_1} = \mathit{diag}\,(1/2, 3/16, 1/4) \geq 0.$$

Thus $K_\Sigma \leq K$ on Λ_1, and by Theorem 3.1 a global p.d. extension exists that preserves the interior.

Shift-consistency (Definition 2.5) requires operators T_1, T_2 on the *level-1 space*

$$H_K^{(1)} := \mathit{span}\{V_0, V_1, V_2\} = \mathit{span}\{e_0, e_1, e_2\}$$

such that $V_{\alpha i} = T_i V_\alpha \in H_K^{(1)}$ for all $|\alpha| \leq 1$. But here $V_{12} = \frac{1}{4}\,e_3 \notin H_1$, so no such $T_2 : H_K^{(1)} \to H_K^{(1)}$ can satisfy $T_2 V_1 = V_{12}$. Therefore the level-2 boundary is *not* shift-consistent, and Theorem 3.2 is not applicable (we make no claim about the existence/nonexistence of a boundary-preserving extension).

Note that, with these choices

$$H_K^{(1)} = \mathit{span}\{e_0, e_1, e_2\} \subsetneq H_K^{(2)} = \mathit{span}\{e_0, e_1, e_2, e_3\},$$

so the level-2 data introduce a new direction.

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