

WITHOUT REAL VECTOR SPACES ALL REGULATORS ARE RATIONAL

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ABSTRACT. Every LCA group has a Haar measure unique up to rescaling by a positive scalar. Clausen has shown that the Haar measure describes the universal determinant functor of the category LCA in the sense of Deligne. We show that when only working with LCA groups without allowing real vector spaces, any conceivable determinant functor is unique up to rescaling by at worst *rational* values. As a result, no transcendental real nor p -adic regulators could ever show up in special L -value conjectures (as in Tamagawa number conjectures or Weil-étale cohomology) if anyone had the, admittedly outlandish and bizarre, idea to try to circumvent incorporating a real (Betti) realization of the motive.

1. OVERVIEW

The main result of this note will not shock anyone: It is hard to come by a transcendental real number without using real numbers. Let that sink in. However, the result we show is more precise and we mean different things: Real numbers as an object in the category of LCA groups, versus real numbers showing up as regulators. The standard conjectures on special L -values intertwine arithmetic cohomological values with transcendental regulator values. Dirichlet’s analytic class number formula or the B-SD conjecture are probably the most famous examples. Conjecturally, in the picture devised by Deligne and Beilinson, the regulators occur from determinant mismatches resulting from the comparison of various determinant lines $\Lambda^{\max} H^{\bullet}(-)$ defined through groups,

- of p -adic type, coming from p -adic realizations for all primes p ,
- of real type, coming from the Betti realization,
- and of integral or rational type, coming from motivic cohomology.

There are various formulations of conjectures for this picture, each a bit different: For example, the Bloch–Kato picture of motivic Tamagawa numbers [BK90, FPR94], or Lichtenbaum’s picture based on Weil-étale cohomology [Lic09, Lic24]. We are mostly inspired by recent work in the direction of Weil-étale cohomology theories with coefficients in LCA groups and output as LCA groups. This has recently been featured prominently in the work of Flach–Morin [FM18] and Geisser–Morin [GM24]. As well as in the work of Artusa [Art24, Art25], but also in older works like Kottwitz–Shelstad [KS99, Appendix E], Oesterlé [Oes83].

However, to show our claim, we will not even touch motives nor Weil-étale cohomology anywhere. We just show that no transcendental (real) values can occur from determinant functors when working on the category of LCA groups, *but disallowing real vector space summands*. This is possible because the determinant lines in the various forms of special L -value conjectures all come from determinant functors in the sense of Deligne. We can

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prove that *any* determinant functor on LCA groups – as long as no real vector spaces ever show up – must factor through *rational* numbers.

Write $\mathbf{LCA}$ for the quasi-abelian category of locally compact abelian (LCA) groups. Write $\mathbf{LCA}_{\text{vf}} \subset \mathbf{LCA}$ for the full subcategory of those groups not having a real line $\mathbb{R}$ as a direct summand (in group theory, such groups are called ‘*vector-free*’, whence the subscript originates). This is an exact category, see §2. Real quotients, like $\mathbb{R}^n/\Lambda$ for a full rank lattice Λ , remain allowed.

Theorem 1.1. *The restriction of the Haar functor*

$$Ha: \mathbf{LCA}^\times \rightarrow \mathbf{Tors}(\mathbb{R}_{>0}^\times)$$

to $\mathbf{LCA}_{\text{vf}}$ only attains rational values:

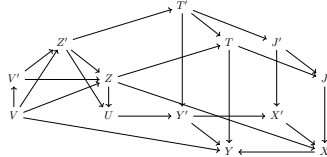
$$Ha^{\mathbb{Q}}: \mathbf{LCA}_{\text{vf}}^\times \rightarrow \mathbf{Tors}(\mathbb{Q}_{>0}^\times).$$

This functor $Ha^{\mathbb{Q}}$ is the universal determinant functor of $\mathbf{LCA}_{\text{vf}}$, i.e., for any determinant functor $\mathcal{D}: \mathbf{LCA}_{\text{vf}}^\times \rightarrow \mathbf{P}$ there exists a factorization

$$\begin{array}{ccc} \mathbf{LCA}_{\text{vf}}^\times & \xrightarrow{Ha^{\mathbb{Q}}} & (\mathbf{Tors}(\mathbb{Q}_{>0}^\times), \otimes) \\ & \searrow \mathcal{D} & \downarrow f \\ & & (\mathbf{P}, \boxtimes) \end{array}$$

with f a symmetric monoidal functor of Picard groupoids.

See Theorem 6.6 for a precise statement and the proof, but let us illustrate an easy consequence: Suppose you are given any connected diagram of objects in $\mathbf{LCA}$ and all arrows are isomorphisms. Then fixing a Haar measure on any object and pushing it forward along the arrows to any other object, we may get several *distinct* normalizations



depending on what path we follow. The first point is: If all objects are in $\mathbf{LCA}_{\text{vf}}$, then the normalization can only differ by rational numbers – irrespective of the shape of the diagram. Secondly (and this is the true content of the theorem): *Any* determinant functor in the sense of Deligne [Del87] (or see Def. 3.5 below) must have the same property. Of course, you may still decide to normalize the Haar measure on, say $\mathbb{Q}_p^n$, such that

$$\text{vol}(\mathbb{Z}_p^n) := \sqrt{2\pi}, \zeta(3) \text{ or } e \text{ etc.}$$

if you so please, so there would suddenly a transcendental number be involved – but then any pushforwards of this measure along various paths in any connected diagram can only differ by rational factors.

To rule out a possible misunderstanding: If we consider the diagram

$$(1.1) \quad \begin{array}{ccc} \mathbb{Q}_p & \xrightarrow{\cdot 1} & \mathbb{Q}_p \\ & \searrow \cdot \log_p(*) & \\ & & \mathbb{Q}_p \end{array}$$

with some transcendental logarithm value, then clearly the pushforwards of the (trivial) determinant line on the left side along either arrow will differ by $\log_p(*)$, which need not be rational. But the point is: The p -adic determinant line does not extend to a determinant functor on $\mathbf{LCA}$ or $\mathbf{LCA}_{\text{vf}}$. It is impossible to systematically extend it to respect exact sequences like

$$\mathbb{Z}_p \hookrightarrow \mathbb{Q}_p \rightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

where it is unclear¹ how to attach a line to $\mathbb{Q}_p/\mathbb{Z}_p$ such that all axioms of a determinant functor remain in place. The Haar measure, however, does indeed prolong to all of $\mathbf{LCA}$, but it only sees the valuation of p -adic numbers. So, writing $\log_p(*)$ as a power $p^r u$ with $u \in \mathbb{Z}_p^\times$ a p -adic unit, one can check that the ratio of volumes in Eq. 1.1 will be p^{-r} , which is rational.

Strategy of the proof: Our result is basically just a computation in K -theory. By an old idea of Deligne, every exact category has a universal determinant functor which all determinants must factor through. This stems from [Del87]. He also showed that this universal determinant is completely determined by the specifics of the K -theory spectrum of the category, truncated to degrees $[0, 1]$. This is a tiny bit more information than just knowing K_0 and K_1 , and also involves the glueing data of the Postnikov tower of these two layers (the stable k -invariant). It turns out that it is not too hard to compute these invariants. While this argument exhibits the group $\mathbb{Q}_{\geq 0}^\times$ abstractly, the tricky part is to link this up to the computation of the rescaling factor under the Haar measure.

Conventions: In this text, K denotes non-connective K -theory, K^{conn} denotes connective (Quillen) K -theory, $\mathcal{U}^{\text{loc}}$ denotes the localizing non-commutative motive, ${}_n A := \{a \in A \mid a^n = 1\}$ denotes the elements in an abelian group killed by n .

2. LCA GROUPS WITHOUT REAL LINE SUMMANDS

Let $\mathbf{LCA}$ (resp. $\mathbf{LCA}_{\text{vf}}$) be the category whose objects are locally compact Hausdorff topological abelian groups (resp. without a real line direct summand) and morphisms are continuous group homomorphisms. This category has all kernels and cokernels and is quasi-abelian. Its natural exact structure is such that

$$G' \hookrightarrow G \twoheadrightarrow G''$$

is exact if the first arrow is an injective closed map (these are the admissible monics) and the second arrow is a surjective closed map (these are the admissible epics) and the underlying sequence of abelian groups is exact.

Proposition 2.1 (Structure theorem for $\mathbf{LCA}$). *Every group $G \in \mathbf{LCA}$ is (non-canonically) isomorphic to $G \simeq G_0 \oplus \mathbb{R}^n$ for some $n < \infty$ and G_0 has a (non-unique) clopen compact subgroup C , i.e., there is an exact sequence*

$$C \oplus \mathbb{R}^n \hookrightarrow G \twoheadrightarrow D$$

with D discrete.

See for example [ADGB22, Theorem 14.2.18]. We immediately obtain that for $G \in \mathbf{LCA}_{\text{vf}}$ the same is true, but we additionally know that $n = 0$.

¹and as one can show: impossible

Caution 2.2 ([Büh10, §10.5]). Note that $\mathbf{LCA}_{\text{vf}}$ is not a fully exact subcategory of $\mathbf{LCA}$, as is witnessed by the exact sequence $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{T}$ ($\mathbb{T}$ the circle group), which shows that $\mathbf{LCA}_{\text{vf}}$ is not closed under extensions inside $\mathbf{LCA}$. Correspondingly, we cannot expect $D_{\infty}^b(\mathbf{LCA}_{\text{vf}}) \rightarrow D_{\infty}^b(\mathbf{LCA})$ to be fully faithful.

Example 2.3. The group $\mathbb{R}_d$ of real numbers, but with the discrete topology, is still allowed in $\mathbf{LCA}_{\text{vf}}$. However, all its Haar measures are rescalings of the counting measure and the measure is only finite on finite subsets. A comparison of volumes like $[-1, +1]$ vs. $[-\frac{1}{2}, +\frac{1}{2}]$ as in $\mathbb{R}$ is impossible in $\mathbb{R}_d$.

3. DETERMINANT FUNCTORS

Definition 3.1. A Picard groupoid $(\mathbf{P}, \otimes)$ is a (1-categorical) groupoid $\mathbf{P}$, equipped with a unital symmetric monoidal structure

$$\otimes: \mathbf{P} \times \mathbf{P} \longrightarrow \mathbf{P}$$

such that all objects are $\otimes$ -invertible. We write $1_{\mathbf{P}}$ for the neutral element of the $\otimes$ -structure.²

Write $\mathbf{Picard}$ for the 2-category of Picard groupoids with symmetric monoidal functors as 1-arrows and natural equivalences as 2-arrows.

We write $\pi_0(\mathbf{P}, \otimes)$ for the group of isomorphism classes in $\mathbf{P}$ with $\otimes$ as its multiplication. For every object $X \in \mathbf{P}$, the self-symmetry $s_{X,X}: X \otimes X \xrightarrow{\sim} X \otimes X$ must satisfy $s_{X,X} \circ s_{X,X} = \text{id}_{X \otimes X}$ by the symmetry axiom of a symmetric monoidal category. Multiplying from the right with the identity $(X \otimes X)^{-1} \xrightarrow{\sim} (X \otimes X)^{-1}$ induces an isomorphism

$$(3.1) \quad s_{X,X} \otimes X^{-1} \otimes X^{-1}: 1_{\mathbf{P}} \xrightarrow{\sim} 1_{\mathbf{P}},$$

known as the *signature* ε_X [SR72, §2.5.3]. One finds that

$$\varepsilon_{X \otimes Y} = \varepsilon_X \cdot \varepsilon_Y.$$

This induces a well-defined group homomorphism

$$\varepsilon: \pi_0(\mathbf{P}, \otimes) \longrightarrow {}_2\pi_1(\mathbf{P}, \otimes)$$

to the 2-torsion elements of $\pi_1(\mathbf{P}, \otimes) := \text{Aut}_{\mathbf{P}}(1_{\mathbf{P}})$.

Definition 3.2. The morphism ε is the stable k -invariant of $(\mathbf{P}, \otimes)$.

Let A be an abelian group.

Definition 3.3. Write $\text{Tors}(A)$ for the groupoid of A -torsors³:

- (1) Objects are left A -torsors, i.e., a set X with a free simply transitive left A -action

$$A \times X \longrightarrow X,$$

²Being $\otimes$ -invertible means that for every $X \in \mathbf{P}$, there exists some $X^{-1} \in \mathbf{P}$ such that $X^{-1} \otimes X \simeq 1_{\mathbf{P}}$ (with no requirements on naturality). There exist various variants of this definition, the strongest being the existence of a functor of inversion $(-)^{-1}: (\mathbf{P}, \otimes) \longrightarrow (\mathbf{P}, \otimes^{\text{op}})$. Essentially, one can show that merely assuming the existence of $\otimes$ -inverses, it is always possible to extend this to a functor of inversion. However, we shall not need to know any of this.

³if a torsor is a sheaf for you, regard it as a sheaf on an unnamed one-point set with the trivial topology.

- (2) and morphisms are bijections ϕ of A -sets, preserving the left A -action

$$\phi(a \cdot x) = a \cdot \phi(x).$$

There is a designated object, the trivial torsor:

$$1_{\mathbf{Tors}(A)} := A$$

with its natural left action given by multiplication in A .

The automorphisms of $1_{\mathbf{Tors}(A)}$ in this category are canonically isomorphic to A itself, $\text{Aut}(1_{\mathbf{Tors}(A)}) \cong A$. The groupoid $\mathbf{Tors}(A)$ can be promoted to become a Picard groupoid:

Definition 3.4. Write $(\mathbf{Tors}(A), \otimes)$ for the Picard groupoid of A -torsors:

- (1) The monoidal multiplication is

$$X \otimes Y := (X \times Y) / \sim,$$

where $(x, y) : \sim (ax, a^{-1}y)$ for all $a \in A$. Equip this set with the left A -action $a \cdot (x, y) := (ax, y) = (x, ay)$.

- (2) The inverse object X^{-1} is defined to be the same set X , but with the left action $a \cdot_{X^{-1}} x := a^{-1} \cdot x$. Then

$$X \otimes X^{-1} \xrightarrow{\sim} 1_{\mathbf{Tors}(A)}$$

sending $x' \otimes x$ to the unique element $a \in A$ such that $a \cdot x = x'$.

- (3) The associativity and symmetry constraint are trivial, e.g.,

$$\begin{aligned} s_{X,Y} : Y \otimes X &\longrightarrow X \otimes Y \\ (y, x) &\longmapsto (x, y). \end{aligned}$$

Since all objects in this category are pairwise isomorphic, we have $\pi_0(\mathbf{Tors}(A), \otimes) = 0$ and $\pi_1(\mathbf{Tors}(A), \otimes) \cong A$.

If $\psi : A \rightarrow B$ is a homomorphism of abelian groups, there is an induced (symmetric monoidal) basechange functor of Picard groupoids

$$(3.2) \quad \begin{aligned} \psi_* : (\mathbf{Tors}(A), \otimes) &\longrightarrow (\mathbf{Tors}(B), \otimes) \\ X &\longmapsto (B \times X) / \sim \quad (\text{some people write } B \otimes_A X) \end{aligned}$$

with $(x, y) : \sim (\psi(a)x, a^{-1}y)$ for all $a \in A$, $x \in B$ and $y \in X$. Equip this set with the left B -action $b \cdot (x, y) := (bx, y)$.

Let $\mathbf{C}$ be an exact category. Let $\mathbf{C}^\times$ denote its maximal inner groupoid, i.e., the category with the same objects, but we only keep isomorphisms as morphisms.

Definition 3.5 ([Del87, §4.3]). Let $\mathbf{C}$ be an exact category and let $(\mathbf{P}, \otimes)$ be a Picard groupoid. A determinant functor on $\mathbf{C}$ is a functor

$$\mathcal{D} : \mathbf{C}^\times \longrightarrow \mathbf{P}$$

along with the following extra structure and axioms:

- (1) For any exact sequence $\Sigma : G' \hookrightarrow G \twoheadrightarrow G''$ in $\mathbf{C}$, we are given an isomorphism

$$(3.3) \quad \mathcal{D}(\Sigma) : \mathcal{D}(G) \xrightarrow{\sim} \mathcal{D}(G') \otimes_{\mathbf{P}} \mathcal{D}(G'')$$

in $\mathbf{P}$. This isomorphism is required to be functorial in morphisms of exact sequences.

- (2) For every zero object Z of $\mathbf{C}$, we are provided with an isomorphism $z : \mathcal{D}(Z) \xrightarrow{\sim} 1_{\mathbf{P}}$ to the neutral object of the Picard groupoid.

(3) Suppose $f: G \rightarrow G'$ is an isomorphism in $\mathcal{C}$. We write

$$\Sigma_l: 0 \hookrightarrow G \twoheadrightarrow G' \quad \text{and} \quad \Sigma_r: G \hookrightarrow G' \twoheadrightarrow 0$$

for the depicted exact sequences. We demand that the composition

$$(3.4) \quad \mathcal{D}(G) \xrightarrow[\mathcal{D}(\Sigma_l)]{\sim} \mathcal{D}(0) \otimes_{\mathbf{P}} \mathcal{D}(G') \xrightarrow[z \otimes 1]{\sim} 1_{\mathbf{P}} \otimes_{\mathbf{P}} \mathcal{D}(G') \xleftarrow[g_{\mathcal{D}(G')}]{\sim} \mathcal{D}(G')$$

and the natural map $\mathcal{D}(f): \mathcal{D}(G) \xrightarrow{\sim} \mathcal{D}(G')$ agree. We further require that $\mathcal{D}(f^{-1})$ agrees with a variant of Equation 3.4 using Σ_r instead of Σ_l .

(4) If a two-step filtration $G_1 \hookrightarrow G_2 \hookrightarrow G_3$ is given, we demand that the diagram

$$(3.5) \quad \begin{array}{ccc} \mathcal{D}(G_3) & \xrightarrow{\sim} & \mathcal{D}(G_1) \otimes \mathcal{D}(G_3/G_1) \\ \sim \downarrow & & \downarrow \sim \\ \mathcal{D}(G_2) \otimes \mathcal{D}(G_3/G_2) & \xrightarrow{\sim} & \mathcal{D}(G_1) \otimes \mathcal{D}(G_2/G_1) \otimes \mathcal{D}(G_3/G_2) \end{array}$$

commutes.

(5) Given objects $G, G' \in \mathcal{C}$ consider the exact sequences

$$\Sigma_1: G \hookrightarrow G \oplus G' \twoheadrightarrow G' \quad \text{and} \quad \Sigma_2: G' \hookrightarrow G \oplus G' \twoheadrightarrow G$$

with the natural inclusion and projection morphisms. Then the diagram

$$\begin{array}{ccc} & \mathcal{D}(G \oplus G') & \\ \mathcal{D}(\Sigma_1) \swarrow & & \searrow \mathcal{D}(\Sigma_2) \\ \mathcal{D}(G) \otimes \mathcal{D}(G') & \xrightarrow[s_{G, G'}]{} & \mathcal{D}(G') \otimes \mathcal{D}(G) \end{array}$$

commutes, where $s_{G, G'}$ denotes the symmetry constraint of $\mathbf{P}$.

At the end of [Del87, §4.3], Deligne considers the category of determinant functors $\det(\mathcal{C}, \mathbf{P})$:

- (1) objects are determinant functors in the sense of the above definition, and
- (2) morphisms are natural transformations of determinant functors.

Details can be found spelled out in [Bre11, §2.3], especially a full description of a morphism of determinant functors is [Bre11, Definition 2.5]. We also took over his notation $\det(\mathcal{C}, \mathbf{P})$ for this category.

Definition 3.6. A determinant functor $\mathcal{D}: \mathcal{C}^\times \rightarrow \mathbf{P}$ is called universal if for every given Picard groupoid $\mathbf{P}'$ the functor

$$\mathrm{Hom}^\otimes(\mathbf{P}, \mathbf{P}') \rightarrow \det(\mathcal{C}, \mathbf{P}'), \quad \varphi \mapsto \varphi \circ \mathcal{D}$$

is an equivalence of categories.

This is in Deligne [Del87, §4.3], but perhaps a little more detailed in [Bre11, §4.1].

For an LCA group G , we write $C_c(G, \mathbb{R})$ to denote the (possibly non-unital) Banach algebra of continuous real-valued functions with compact support.

The following seems to be known among all specialists, but we are not aware of any detailed account in the literature:

Definition 3.7. Write

$$Ha: \mathrm{LCA}^\times \rightarrow \mathrm{Tors}(\mathbb{R}_{>0}^\times)$$

for the determinant functor sending an LCA group to its set of Haar measures.

- (1) This is a left $\mathbb{R}_{>0}^\times$ -torsor by multiplying the Haar measure with a positive real scalar.

(2) For any exact sequence $\Sigma: G' \hookrightarrow G \twoheadrightarrow G''$ in $\mathbf{LCA}$, the isomorphism

$$(3.6) \quad Ha(\Sigma): Ha(G) \xrightarrow{\sim} Ha(G') \otimes Ha(G'')$$

in $\mathbf{Tors}(\mathbb{R}_{>0}^\times)$ is the inverse of the following construction: Given Haar measures $\mu_{G'}$ and $\mu_{G''}$ on the closed subgroup G' , and quotient group G'' , there is a unique normalization of the Haar measure on G such that

$$(3.7) \quad \int_{G''} \int_{G'} f(x\xi) d\mu_{G'}(\xi) d\mu_{G''}(\bar{x}) = \int_G f(x) d\mu(x)$$

holds for all $f \in C_c(G, \mathbb{R})$.

(3) For the zero group $\{0\}$, $z: \mathcal{D}(\{0\}) \xrightarrow{\sim} 1_{\mathbf{P}}$ is the choice of multiples of the counting measure $\mu_{\{0\}}(\{0\}) = r \cdot 1$ for $r \in \mathbb{R}_{>0}^\times$.

We supply some details: (1) The existence and uniqueness up to positive scalars of Haar measures can for example be found in [Fol16, Theorem 2.10, Theorem 2.20] or [Loo53, Theorem 29C and D]. This defines Ha on objects. Given an arrow $G' \xrightarrow{F} G$ in $\mathbf{LCA}^\times$, i.e., an isomorphism (as we had switched to the maximal inner groupoid of the category), the pushforward measure

$$Ha(G') \longrightarrow Ha(G) \quad \mu \longmapsto F_*\mu$$

with $(F_*\mu)(X) := \mu(F^{-1}(X))$ is also a Haar measure: This is true because (a), since the Borel σ -algebra is generated by open sets and F is continuous, $F^{-1}(X)$ of a measurable set $X \subseteq G$ is also measurable in G' , (b) $F^{-1}(X + g) = F^{-1}(X) + F^{-1}(g)$, so the translation-invariance of μ implies that $F_*\mu$ is translation-invariant, (c) the properties to be a finite measure on compact sets, to be inner and outer regular, all just hinge on inclusion properties of open or measurable sets, and since F is a homeomorphism, these can all be transported back and forth along F and F^{-1} .

Since $F_*\mu$ is also a Haar measure, it pins down a unique element of $Ha(G)$. This defines Ha on morphisms.

(2) The integral in Eq. 3.7 requires justification: The function

$$x \mapsto \int_{G'} f(x\xi) d\mu_{G'}(\xi)$$

is constant on each coset of the closed subgroup G' in G . Hence, it defines a well-defined function on G'' by taking x in Eq. 3.7 to be any preimage of $\bar{x} \in G''$ in x . Now use [Fol16, Theorem 2.49], using that the modular character Δ ([Fol16, §2.4]) is trivial for abelian groups as there cannot be a difference between left and right Haar measures. A similar discussion can be found in [Loo53, §33]. Axiom (4) of Def. 3.5 follows from a triple integral version of Eq. 3.7,

$$\begin{aligned} \int_{G_3} f(x) d\mu(x) &= \int_{G_3/G_1} \int_{G_1} f(x\xi) d\mu_{G_1}(\xi) d\mu_{G_3/G_1}(\bar{x}) \text{ (by Eq. 3.7)} \\ &= \int_{G_3/G_2} \int_{G_2/G_1} \int_{G_1} f(x\xi\zeta) d\mu_{G_1}(\xi) d\mu_{G_2/G_1}(\bar{\zeta}) d\mu_{G_3/G_2}(\bar{x}) \end{aligned}$$

by another use of Eq. 3.7, where x is a lift of $\bar{x} \in G_3/G_2$ to G_3 , and ζ is a lift of $\bar{\zeta} \in G_2/G_1$ to G_2 , where by $G_2 \subseteq G_3$ it can be regarded an element in G_3 . Contracting the inner integrals along Eq. 3.7,

$$= \int_{G_3/G_2} \int_{G_2} f(x\zeta) d\mu_{G_2}(\zeta) d\mu_{G_3/G_2}(\bar{x}),$$

so that the left side corresponds to the upper left vertex in Diagram 3.5 and the right sides of the three lines of this computation correspond to the three remaining vertices.

Axiom (5) of Def. 3.5 is harmless since the symmetry constraint of $\text{Tors}(\mathbb{R}_{>0}^\times)$ is the identity.

Example 3.8. A distinctive feature of the Haar measure is its compatibility with exact sequences for *all* LCA groups, may they be discrete, compact, real or p -adic, across all primes. This makes it well-defined even on the derived category $D_\infty^b(\text{LCA})$ [Knu02], [MTW15, §1.3, Corollary 2.1.1]. For example, multiplication by 5 acts on

$$Ha(\mathbb{Q}_5^A \oplus \mathbb{R}^B \oplus \mathbb{Q}_3^C) \quad \text{for any} \quad A, B, C \in \mathbb{Z}_{\geq 0}$$

by multiplication with 5^{B-A} since it stretches volumes in the reals, shrinks them in the 5-adics, and is volume-preserving on $\mathbb{Q}_3$. Exactly whenever $A = B$, the map is volume-preserving as a whole. Writing the same objects as cones

$$\mathbb{Q}_5^A \cong \text{cone}[(\mathbb{Q}_5/\mathbb{Z}_5)^A \longrightarrow \Sigma \mathbb{Z}_5^A] \quad \mathbb{R}^B \cong \text{cone}[\mathbb{T}^B \longrightarrow \Sigma \mathbb{Z}^B],$$

multiplication by 5 has a kernel of order 5^A on the $\mathbb{Q}_5/\mathbb{Z}_5$ -summands, and a cokernel of order 5^A on the Σ -shift of the $\mathbb{Z}_5$ -summands (so that the total map has cone zero, as it is necessary for an automorphism), resp. a kernel of order 5^B on the $\mathbb{T}$ -summands (where the map is a degree 5^B covering space) and a cokernel of order 5^B on the Σ -shift of the $\mathbb{Z}$ -summands. And on the cone $[\mathbb{Q}_3/\mathbb{Z}_3 \longrightarrow \Sigma \mathbb{Z}_3]$ multiplication by 5 is an invertible map in both terms as $5 \in \mathbb{Z}_3^\times$ is a unit.

Example 3.9. The graded determinant lines

$$\begin{aligned} \det: \text{Vect}_{fd}(F) &\longrightarrow \text{Pic}_F^{\mathbb{Z}} & (\text{Vect}_{fd}(F) \text{ are finite-dimensional } F\text{-vector spaces}) \\ V &\longmapsto \left(\bigwedge^{\dim V} V, \dim V \right) \end{aligned}$$

for a field F can be regarded as being defined on full subcategories of LCA in the cases where F is a locally compact field⁴. These can be richer than the Haar measure, e.g., the two different pushforwards of any chosen trivialization of the p -adic determinant line along the two arrows in Eq. 1.1 would differ by $\log_p(*)$, a possibly transcendental p -adic value. However, this functor does not extend to all of LCA. For example, there is no sensible way to implement the exact sequence functoriality of Eq. 3.3 to $\mathbb{Q} \hookrightarrow \mathbb{A} \twoheadrightarrow \mathbb{Q}^\vee$, where $\mathbb{A}$ are the adèles of $\mathbb{Q}$. It would require intermingling real and p -adic determinant lines. The Haar measure accomplishes this compatibility across all types over objects in LCA, but at the price of being less precise. For example, it is oblivious to all the dimension gradings, and it *has* to be since, for example in $\mathbb{Q} \hookrightarrow \mathbb{A} \twoheadrightarrow \mathbb{Q}^\vee$, $\mathbb{A}$ has a 1-dimensional real number summand, but neither $\mathbb{Q}$ nor $\mathbb{Q}^\vee$ has. See [Bra19, Prop. 13.3] for details on what data is forgotten under switching to the Haar measure.

4. THE RATIONALIZED HAAR MEASURE

In this section we describe the rationalized Haar measure as it occurs in Theorem 1.1. The basic question is: How can we attach a Haar measure to a group such that it is well-defined up to a *rational* factor?

⁴Famously, this means that F must be a finite extension of $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{F}_p((t))$ or a discrete field. All finite-dimensional vector spaces over these fields are naturally objects in LCA.

Step 1: Given an object $X \in \mathbf{LCA}_{\text{vf}}$, pick a compact open subgroup $C \subseteq X$. This involves a choice, generally, but is always possible by Prop. 2.1. This choice induces an exact sequence

$$(4.1) \quad \Sigma: C \hookrightarrow X \twoheadrightarrow X/C$$

in $\mathbf{LCA}$ with X/C discrete (since C was *open* in X). On the compact group, we may pick the canonical normalized Haar measure μ_C such that $\mu_C(C) = 1$. This is possible because a Haar measure, by definition, assigns a *finite* volume to compact sets. On the discrete group, we may pick the canonical counting measure, i.e., $\mu_{X/C}(\{*\}) = 1$ for any singleton set. This is tautologically a translation-invariant measure. By Eq. 3.6 the exact sequence Σ induces a natural isomorphism

$$Ha(\Sigma): Ha(X) \xrightarrow{\sim} Ha(C) \otimes Ha(X/C)$$

and we define the *root measure* $\mu_{\text{root}}^C := Ha(\Sigma)^{-1}(\mu_C \otimes \mu_{X/C})$. Equivalently, this is the *unique* normalization of a Haar measure on X such that Eq. 3.7 holds, given that we use the measures μ_C and $\mu_{X/C}$ on the compact and discrete piece.

(This might sound more complicated than it is: If we start with an arbitrary Haar measure on X , we just need to rescale it to give C the volume +1. This yields exactly the measure just described).

Step 2: Now define

$$\begin{aligned} Ha^{\mathbb{Q}}: \mathbf{LCA}_{\text{vf}}^{\times} &\longrightarrow \mathbf{Tors}(\mathbb{Q}_{>0}^{\times}) \\ X &\longmapsto \mathbb{Q}_{>0}^{\times} \cdot \mu_{\text{root}}^C, \end{aligned}$$

i.e., $Ha^{\mathbb{Q}}$ sends X to the $\mathbb{Q}_{>0}^{\times}$ -torsor of all positive rational multiples of the root measure. We need to check that this is well-defined: The root measure only depended on the choice of C in Eq. 4.1. If we pick a further compact open C' such that $C' \subseteq C$, then $[C : C'] := \#(C/C')$ is finite (since C is compact and C' open, so C/C' must be both compact and discrete). We compute

$$\begin{aligned} \int_X f(x) d\mu_{\text{root}}^C(x) &= \int_{X/C} \int_C f(x\xi) d\mu_C(\xi) d\mu_{X/C}(\overline{x}) \\ &= \sum_{\overline{x} \in X/C} \sum_{c \in C/C'} \int_{C'} f(x\xi c) d\mu_C(\xi) \\ &= \frac{1}{[C : C']} \sum_{\overline{x} \in X/C} \sum_{c \in C/C'} \int_{C'} f(x\xi c) d\mu_{C'}(\xi) \\ &= \frac{1}{[C : C']} \sum_{\overline{x} \in X/C'} \int_{C'} f(x\xi) d\mu_{C'}(\xi) \\ &= \frac{1}{[C : C']} \int_{X/C'} \int_{C'} f(x\xi) d\mu_{C'}(\xi) d\mu_{X/C'}(\overline{x}) \\ &= \frac{1}{[C : C']} \int_X f(x) d\mu_{\text{root}}^{C'}(x) \end{aligned}$$

for any function $f \in C_c(X, \mathbb{R})$, where the equalities are, in succession, (1) Eq. 3.7 for μ_{root}^C , (2) $\mu_{X/C}$ is the counting measure, (3) $\mu_C = \frac{1}{[C : C']} \mu_{C'}$ by our normalization that $\mu_C(C) = 1$, and analogously for C' , (4) C decomposes into C' -tiles indexed over C/C' , (5) $\mu_{X/C'}$ is the counting measure and (6) Eq. 3.7 for $\mu_{\text{root}}^{C'}$.

We learn that the two choices of the root measure only differ by a rational factor, so both choices pin down the same $\mathbb{Q}_{>0}^\times$ -subtorsor $\mathbb{Q}_{>0}^\times \cdot \mu_{\text{root}}^C$ inside $Ha(X)$. An entirely analogous argument works for compact objects C' such that $C \subseteq C'$ is bigger than the previous choice. All in all, a zig-zag argument using that any two choices C, C' of compact opens in X have a joint compact sub-open, for example $C' \cap C$, proves that $Ha^\mathbb{Q}$ is well-defined on objects in LCA_{vf} .⁵

Example 4.1. The definition of $Ha^\mathbb{Q}$ cannot be extended to all of LCA : The group $\mathbb{R}$ does not possess a compact open subgroup, so already Step 1 fails.

Example 4.2. $Ha^\mathbb{Q}(\mathbb{F}_q((t)))$ is the subset of all Haar measures on the LCA group $\mathbb{F}_q((t))$ with the property that $\text{vol}(\mathbb{F}_q[[t]])$ is a positive rational number.

It remains to show that this definition, so far only on objects, really extends to a determinant functor satisfying all the axioms of Definition 3.5.

Below, write $i: \mathbb{Q}_{>0}^\times \rightarrow \mathbb{R}_{>0}^\times$ for the inclusion of abelian groups and i_* for the induced basechange of Picard groupoids (Eq. 3.2).

Theorem 4.3. *$Ha^\mathbb{Q}$ extends to a determinant functor on LCA_{vf} such that the composition of functors*

$$\text{LCA}_{\text{vf}}^\times \xrightarrow{Ha^\mathbb{Q}} \text{Tors}(\mathbb{Q}_{>0}^\times) \xrightarrow{i_*} \text{Tors}(\mathbb{R}_{>0}^\times)$$

agrees with the restriction of the Haar measure to the full subcategory $\text{LCA}_{\text{vf}} \subset \text{LCA}$. As a result $Ha^\mathbb{Q}$ can be described as follows: On every object X , $Ha^\mathbb{Q}(X)$ singles out a subset of Haar measures in $Ha(X)$, and under all arrows in LCA_{vf} , the natural maps of the Haar determinant functor Ha respect these subsets.

This is a self-contained description of $Ha^\mathbb{Q}$ which saves us the work to formulate the remaining axioms in Definition 3.5 for $Ha^\mathbb{Q}$: They all agree with the ones for the usual Haar measure, just restricted to a subset of values. We defer the proof to §6.

5. COMPUTATIONS

Lemma 5.1. *The inclusion of finite abelian groups Ab_{fin} into all abelian groups Ab induces a Verdier localization sequence*

$$D_\infty^b(\text{Ab}_{\text{fin}}) \longrightarrow D_\infty^b(\text{Ab}) \longrightarrow D_\infty^b(\text{Ab}/\text{Ab}_{\text{fin}}).$$

Proof. It suffices to note that Ab_{fin} is a Serre subcategory of the abelian category Ab . $\square$

Lemma 5.2. *The inclusion of compact abelian groups $\mathbb{C}$ into LCA_{vf} induces a Verdier localization sequence, up to equivalence of the shape,*

$$D_\infty^b(\mathbb{C}) \longrightarrow D_\infty^b(\text{LCA}_{\text{vf}}) \longrightarrow D_\infty^b(\text{Ab}/\text{Ab}_{\text{fin}}).$$

This equivalence is given by an exact equivalence of exact categories $\Xi: \text{Ab}/\text{Ab}_{\text{fin}} \xrightarrow{\sim} \text{LCA}_{\text{vf}}/\mathbb{C}$.

Proof. Again, we note that $\mathbb{C}$ is a Serre subcategory of LCA_{vf} . Since LCA_{vf} is not an abelian category, one needs to verify a few more properties to obtain the Verdier localization sequence

$$D_\infty^b(\mathbb{C}) \longrightarrow D_\infty^b(\text{LCA}_{\text{vf}}) \longrightarrow D_\infty^b(\text{LCA}_{\text{vf}}/\mathbb{C}),$$

⁵alternatively: instead of working with a common subobject, it is also possible to work with $C + C'$, a compact open containing both constituents

and one may follow [Hv19, Corollary 5.11]. Hence, our claim is proven once we exhibit an exact equivalence of exact categories

$$(5.1) \quad \Xi: \mathbf{Ab}/\mathbf{Ab}_{\text{fin}} \longrightarrow \mathbf{LCA}_{\text{vf}}/\mathbf{C}.$$

To this end, we need to describe the quotient categories on either side. In the setting of [Hv19], $\mathbf{C}$ is inflation-percolating in $\mathbf{LCA}_{\text{vf}}$, so by [Hv19, Prop. 4.4] the system $S_{\mathbf{C}}$ generated by (1) admissible epics with kernel in $\mathbf{C}$, (2) admissible monics with cokernel in $\mathbf{C}$, (3) any finite composition thereof, is *left* multiplicative⁶, so $\mathbf{LCA}_{\text{vf}}/\mathbf{C}$ can be modelled through right roofs [KS06, Remark 7.1.7]. We note that for Serre subcategories in abelian categories, $\mathbf{Ab}/\mathbf{Ab}_{\text{fin}} = \mathbf{Ab}[S_{\mathbf{Ab}_{\text{fin}}}^{-1}]$ is a localization by a both left and right multiplicative system. The functor Ξ sends an abelian group to itself, equipped with the discrete topology. (*Fullness and essential surjectivity*) Consider an arbitrary right roof in $\mathbf{LCA}_{\text{vf}}/\mathbf{C} = \mathbf{LCA}_{\text{vf}}[S_{\mathbf{C}}^{-1}]$ between discrete abelian groups X, Y . It has the shape of the solid arrows in

$$(5.2) \quad \begin{array}{ccccc} & & D & & \\ & \swarrow f' & \uparrow \text{---} & \nwarrow s' & \\ X & \text{---} & D & \text{---} & Y \\ & \searrow f & \downarrow q & \swarrow s \in S_{\mathbf{C}} & \\ & & Z & & \end{array}$$

(ignore the dashed arrows for the moment). Since $Z \in \mathbf{LCA}_{\text{vf}}$, there must exist a compact clopen $C \subseteq Z$ and we obtain an exact sequence

$$(5.3) \quad C \hookrightarrow Z \xrightarrow{q} D$$

with D discrete in $\mathbf{LCA}_{\text{vf}}$ (Prop. 2.1). Now we may add all the dashed arrows in Diagram 5.2: $f' := q \circ f$ and $s' := q \circ s$. Since $s \in S_{\mathbf{C}}$ and q is an admissible epic with kernel in $\mathbf{C}$, it follows that $s' \in S_{\mathbf{C}}$. As a result, Diagram 5.2 determines a valid equivalence of right roofs and shows that we may assume that Z was discrete to start with. It follows that $s \in S_{\mathbf{C}}$ is a morphism between two discrete groups. However, then we must have that $s \in S_{\mathbf{Ab}_{\text{fin}}}$ since finite groups are the only ones which are simultaneously discrete and compact. It follows that the right roof lies in the image under the functor Ξ of a right roof in $\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}$. Thus, Ξ is a full functor. For essential surjectivity, note that if $Z \in \mathbf{LCA}_{\text{vf}}$ is an arbitrary object, the same exact sequence as in Eq. 5.3 shows that $q: Z \simeq \Xi(D)$ is an isomorphism in the quotient category $\mathbf{LCA}_{\text{vf}}/\mathbf{C}$. (*Faithfulness*) Suppose a right roof in $\mathbf{Ab}[S_{\mathbf{Ab}_{\text{fin}}}^{-1}]$ that is equivalent to the zero map in $\mathbf{Ab}[S_{\mathbf{Ab}_{\text{fin}}}^{-1}]$. Then since $S_{\mathbf{Ab}_{\text{fin}}} \subseteq S_{\mathbf{C}}$, the same equivalence of roof also shows that the roof is equivalent to the zero map in $\mathbf{LCA}_{\text{vf}}/\mathbf{C}$. (*Exactness and reflection of exactness*) The functor is induced from the functor $\mathbf{Ab} \longrightarrow \mathbf{LCA}_{\text{vf}}$, which is obviously exact, to the quotient category $\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}$ by the universal property of sending $\mathbf{Ab}_{\text{fin}}$ to zero objects. Restricted onto the strict image, it is clear that Ξ reflects exactness since exactness on discrete groups in $\mathbf{LCA}_{\text{vf}}$ reduces to exactness of the underlying abelian groups. There is no topology to take into consideration. $\square$

⁶Unfortunately, some authors use left and right with opposite meaning in the context of a calculus of fractions. We follow the convention of Kashiwara–Shapira [KS06, Remark 7.1.8].

Lemma 5.3. *There is an equivalence of localizing non-commutative motives*

$$\mathcal{U}^{\text{loc}}(\text{LCA}_{\text{vf}}) \xrightarrow{\sim} \Sigma \mathcal{U}^{\text{loc}}(\text{Ab}_{\text{fin}}).$$

There is no urgent need for non-commutative motives, this is only one possible way to lead us to Corollary 5.5, which is all that we shall truly need.

Proof. From Lemma 5.1 and Lemma 5.2 we get the fiber sequences of localizing non-commutative motives

$$\begin{aligned} \mathcal{U}^{\text{loc}}(\text{Ab}_{\text{fin}}) &\longrightarrow \mathcal{U}^{\text{loc}}(\text{Ab}) \longrightarrow \mathcal{U}^{\text{loc}}(\text{Ab}/\text{Ab}_{\text{fin}}) \\ \mathcal{U}^{\text{loc}}(\text{C}) &\longrightarrow \mathcal{U}^{\text{loc}}(\text{LCA}_{\text{vf}}) \longrightarrow \mathcal{U}^{\text{loc}}(\text{Ab}/\text{Ab}_{\text{fin}}). \end{aligned}$$

By Tychonov's theorem arbitrary products of compact abelian groups are again compact, so C is a complete category. Hence $\mathcal{U}^{\text{loc}}(\text{C}) = 0$ by the Eilenberg swindle. Analogously, Ab is co-complete, so $\mathcal{U}^{\text{loc}}(\text{Ab}) = 0$. Combining these facts, the fiber sequences simplify to

$$\mathcal{U}^{\text{loc}}(\text{LCA}_{\text{vf}}) \xrightarrow{\sim} \mathcal{U}^{\text{loc}}(\text{Ab}/\text{Ab}_{\text{fin}}) \xrightarrow{\sim} \Sigma \mathcal{U}^{\text{loc}}(\text{Ab}_{\text{fin}}).$$

□

Remark 5.4. Actually, Ab and C are both complete and co-complete, so it makes no difference which property we use for the Eilenberg swindle. However, in the format as described in the proof, the inclusion functor $\text{Ab} \rightarrow \text{LCA}_{\text{vf}}$ preserves arbitrary colimits (resp. $\text{C} \rightarrow \text{LCA}_{\text{vf}}$ preserves arbitrary limits), so it is easier to visualize what is happening. The respectively opposite type of (co)limit is not preserved by these functors.

As a side result, we have computed the entire non-connective K -theory spectrum of LCA_{vf} :

Corollary 5.5. $K(\text{LCA}_{\text{vf}}) \cong \Sigma K(\text{Ab}_{\text{fin}})$.

This differs significantly from the counterpart where real vector spaces are allowed:

Theorem 5.6 (Clausen). $K(\text{LCA}) \cong \text{cofib}(K(\mathbb{Z}) \rightarrow K(\mathbb{R}))$.

Clausen's original proof is in [Cla17]. Another proof is in [Bra19].

6. PROOF OF THE MAIN THEOREM

Theorem 6.1. *We have*

$$K_1(\text{LCA}_{\text{vf}}) \cong \mathbb{Q}_{>0}^{\times} \quad \text{and} \quad K_1(\text{LCA}) \cong \mathbb{R}_{>0}^{\times}$$

and under the exact functor $\text{LCA}_{\text{vf}} \rightarrow \text{LCA}$, the induced map on K_1 is the inclusion of rational numbers,

$$\mathbb{Q}_{>0}^{\times} \subset \mathbb{R}_{>0}^{\times}.$$

Moreover,

$$K_0(\text{LCA}_{\text{vf}}) = K_0(\text{LCA}) = 0.$$

Proof. The computation $K_1(\text{LCA}) \cong \mathbb{R}_{>0}^{\times}$ goes back to Clausen [Cla17] (use Theorem 5.6, showing that $\dots \rightarrow \mathbb{Z}^{\times} \rightarrow \mathbb{R}^{\times} \rightarrow K_1(\text{LCA}) \xrightarrow{0} \mathbb{Z}$ is exact). A different proof is in [Bra19, Theorem 12.8]. The same techniques show that $K_0(\text{LCA}) = 0$. Hence, we focus on computing the vector-free variant $K_1(\text{LCA}_{\text{vf}})$: Every finite abelian group uniquely(!) splits into its p -primary torsion summands. This induces an equivalence of abelian categories,

$$\text{Ab}_{\text{fin}} \cong \bigoplus_p \text{Ab}_{\text{fin}}[p^{\infty}],$$

where $\mathbf{Ab}_{\text{fin}}[p^\infty]$ is the abelian category of finite p -power torsion abelian groups. Every such group has a finite filtration by quotients killed by p (or said differently: the simple objects of the category). Hence, these quotients are finite-dimensional $\mathbb{F}_p$ -vector spaces. By dévissage we deduce for connective (Quillen) K -theory that

$$K^{\text{conn}}(\mathbf{Ab}_{\text{fin}}[p^\infty]) \cong K^{\text{conn}}(\mathbb{F}_p)$$

for all primes p ([Wei13, Ch. V, Theorem 4.1]). Since both $\mathbf{Ab}_{\text{fin}}[p^\infty]$ and $\mathbf{Vect}_{fd}(\mathbb{F}_p)$ are Noetherian abelian categories, Schlichting's theorem [Sch06, §10.1, Theorem 7] implies that either category has connective non-connective K -theory (i.e., $\pi_i K(-) = 0$ for all $i < 0$). Since the categories are abelian, they are idempotent complete, so $K_0 = K_0^{\text{conn}}$ ([Sch06, §6.2, Remark 3]). It follows that for either category connective K -theory agrees with the non-connective K -theory [Sch06, §12.2], so

$$(6.1) \quad K(\mathbf{Ab}_{\text{fin}}) \cong \bigoplus_p K(\mathbf{Ab}_{\text{fin}}[p^\infty]) \cong \bigoplus_p K(\mathbb{F}_p).$$

Combining this with Corollary 5.5,

$$(6.2) \quad K_1(\mathbf{LCA}_{\text{vf}}) \underset{\text{Cor. 5.5}}{\cong} K_0(\mathbf{Ab}_{\text{fin}}) \underset{\text{Eq. 6.1}}{\cong} \bigoplus_p K_0(\mathbb{F}_p) \cong \bigoplus_p \mathbb{Z}.$$

The trickier part of the proof is now that while $\mathbf{LCA}_{\text{vf}} \rightarrow \mathbf{LCA}$ certainly induces a map

$$(6.3) \quad \begin{array}{ccc} K_1(\mathbf{LCA}_{\text{vf}}) & \longrightarrow & K_1(\mathbf{LCA}) \\ \downarrow & & \downarrow \\ \bigoplus_p \mathbb{Z} & \dashrightarrow & \mathbb{R}_{>0}^\times \end{array}$$

it is not so clear what the dashed arrow actually does (for all we know so far, it could be the zero map). In order to analyze this map, we shall construct a class $-\alpha_p \in K_1(\mathbf{LCA}_{\text{vf}})$ which is sent under the map in Eq. 6.2 to 1_p (i.e., 1 in the summand belonging to the prime p). To this end, we follow a technique due to Sherman and we rephrase the map of Corollary 5.5 in terms of a K -theory localization sequence, where it corresponds to the connecting map δ in

$$(6.4) \quad K^{\text{conn}}(\mathbf{Ab}_{\text{fin}}) \longrightarrow K^{\text{conn}}(\mathbf{Ab}) \underset{=0}{\longrightarrow} K^{\text{conn}}(\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}) \xrightarrow{\delta} \Sigma K^{\text{conn}}(\mathbf{Ab}_{\text{fin}})$$

in connective K -theory (this is only true because we work on K_1 , mapping to K_0 , and we had seen above that in this range connective K -theory agrees with its non-connective counterpart). We now claim that the automorphism of multiplication by p on the p -adics $\mathbb{Q}_p$ is a representative for the sought-for class α_p . To see this, we need to compute $\delta(\alpha_p)$ in the following commutative diagram:

$$\begin{array}{ccccc} \alpha_p \in & \pi_1 K(\mathbf{LCA}_{\text{vf}}) & & & \\ & \downarrow & \searrow \tilde{\delta} & & \\ & \pi_1 K^{\text{conn}}(\mathbf{LCA}_{\text{vf}}/C) & & & \\ & \Xi^{-1} \downarrow & & & \\ & \pi_1 K^{\text{conn}}(\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}) & \xrightarrow{\delta} & \pi_1 \Sigma K^{\text{conn}}(\mathbf{Ab}_{\text{fin}}) & \cong \bigoplus_p \mathbb{Z} \ni -1_p. \end{array}$$

The object $\mathbb{Q}_p \in \mathbf{LCA}_{\text{vf}}$ under Ξ^{-1} (Eq. 5.1) corresponds to the discrete group $\mathbb{Q}_p/\mathbb{Z}_p$ in $\mathbf{Ab}$ since

$$\mathbb{Z}_p \hookrightarrow \mathbb{Q}_p \xrightarrow{w_p} \mathbb{Q}_p/\mathbb{Z}_p$$

is an exact sequence in $\mathbf{LCA}_{\text{vf}}$ with $\mathbb{Z}_p$ compact, so that w_p becomes an isomorphism in the quotient category $\mathbf{LCA}_{\text{vf}}/\mathbf{C}$. As $\mathbb{Q}_p/\mathbb{Z}_p$ is already a discrete group, Ξ^{-1} just sends this group to itself, now living in the quotient category $\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}$. The horizontal map δ comes from the connecting homomorphism in the long exact sequence of homotopy groups attached to the fibration in Eq. 6.4. We can compute this explicitly by using a simplicial model of the underlying K -theory spaces⁷. We choose the Gillet–Grayson model $G_\bullet$ for this purpose and in §A we summarize all the facts we shall need to know in order to carry out the following computation. The map of multiplication by p on $\mathbb{Q}_p/\mathbb{Z}_p$ corresponds to the arrow

$$(6.5) \quad \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p$$

which is indeed an automorphism in $\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}$ (it is surjective, but has kernel $\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p$ in $\mathbf{Ab}$. Since this kernel is a finite abelian group, it is zero in the quotient category). This determines a class $\alpha_p \in \pi_1 K^{\text{conn}}(\mathbf{Ab}/\mathbf{Ab}_{\text{fin}})$, geometrically representable by a closed loop around the basepoint $(0, 0)$ in the Gillet–Grayson model. The boundary map δ on homotopy groups

$$\pi_1 |G_\bullet(\mathbf{Ab}/\mathbf{Ab}_{\text{fin}})| \longrightarrow \pi_0 |G_\bullet(\mathbf{Ab}_{\text{fin}})|$$

corresponds to lifting this loop to a path in $G_\bullet(\mathbf{Ab})$ and the output value of δ is the connected component in which the lifted path ends in $\pi_0 |G_\bullet(\mathbf{Ab}_{\text{fin}})|$. The homotopy lifting property of a fibration usually guarantees that such a lift exists for any loop. Unfortunately, even though Eq. 6.4 is a (homotopy) fibration sequence, the underlying map between the Gillet–Grayson simplicial sets need not be a simplicial (Kan) fibration. However, we may entirely bypass the path lifting property if we are able to manually exhibit a path lifting the loop.⁸ This is what we shall do: A 1-simplex in $G_\bullet(\mathbf{Ab})$ is given by a pair of exact sequences in $\mathbf{Ab}$ with the same cokernel (Eq. A.1), so define ξ by

$$\begin{array}{ccccc} \frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p & \hookrightarrow & \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{p} & \mathbb{Q}_p/\mathbb{Z}_p \\ & & & & \parallel \\ 0 & \hookrightarrow & \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{.1} & \mathbb{Q}_p/\mathbb{Z}_p, \end{array}$$

a path from $(\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p, 0)$ to $(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$. Now consider the path given solely by the *solid* arrows in

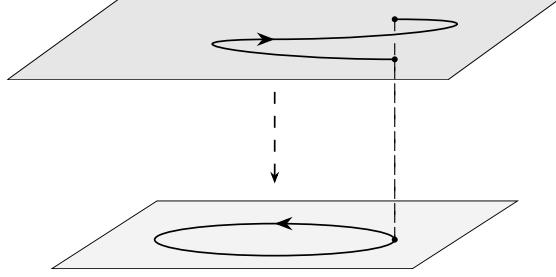
$$(6.6) \quad \begin{array}{ccc} (\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p, 0) & \xrightarrow{\xi} & (\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p) \\ & \nwarrow \rho \quad \nearrow \nu(\mathbb{Q}_p/\mathbb{Z}_p) & \\ & (0, 0) & \end{array}$$

and ν as in Eq. A.3. We shall show that under the map $|G_\bullet(\mathbf{Ab})| \longrightarrow |G_\bullet(\mathbf{Ab}/\mathbf{Ab}_{\text{fin}})|$ this path (essentially) maps to a closed loop: We note that both (distinct) objects 0 and $\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p$ from $\mathbf{Ab}$ become zero objects in the quotient category $\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}$, and thus become isomorphic

⁷note that since we use *connective* K -theory here, we may regard each K^{conn} as a space, equipped with the datum of an infinite loop space/grouplike E_∞ -space

⁸I thank Clayton Sherman and Alexander Nenashev, as I have learned this technique of working with explicit simplicial models of K -theory from their highly inspiring works [Nen96, She96, She98].

by a unique map. Therefore, up to replacing $\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}$ by an equivalent category, call it $\widetilde{\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}}$, we may identify these zero objects to become strictly the same object. Then $(\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p, 0)$ and $(0, 0)$ are the same 0-simplex in $G_\bullet(\widetilde{\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}})$. From the discussion in Example A.2 we now see that all arrows in Diagram 6.6 define a closed loop around the basepoint $(0, 0)$ of $|G_\bullet(\widetilde{\mathbf{Ab}/\mathbf{Ab}_{\text{fin}}})|$ representing the K_1 -class of the automorphism of Eq. 6.5. Hence, the solid arrows in Diagram 6.6 yield a lift of this path to $|G_\bullet(\mathbf{Ab})|$.



The endpoint of this path in $G_\bullet(\mathbf{Ab}_{\text{fin}})$ is the 0-simplex $(\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p, 0)$, which corresponds to the K_0 -class

$$[0] - \left[\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p \right]$$

by Example A.1. Under the dévissage of Eq. 6.1, this in turn identifies⁹ with (the K_0 -group negative of) a one-dimensional $\mathbb{F}_p$ -vector space $\mathbb{F}_p \simeq \frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p$, i.e., to $-1 \in \mathbb{Z}$ in the p -th direct summand of Eq. 6.2, all on the right. This finishes the proof that

$$\delta([\alpha_p]) = -1_p \in \bigoplus_p \mathbb{Z}.$$

We now need to check what α_p corresponds to under the map

$$K_1(\mathbf{LCA}_{\text{vf}}) \longrightarrow K_1(\mathbf{LCA})$$

induced from the exact functor $\mathbf{LCA}_{\text{vf}} \longrightarrow \mathbf{LCA}$. However, it was already computed in [Bra19, Example 2.3, Prop. 13.3] that multiplication by p on $\mathbb{Q}_p$ corresponds under the Haar torsor to multiplication with the p -adic valuation, i.e., $v_p(p) = \frac{1}{p} \in \mathbb{R}_{>0}^\times$. This means that the map

$$\begin{aligned} K_1(\mathbf{LCA}_{\text{vf}}) &\longrightarrow K_1(\mathbf{LCA}) \\ \bigoplus_p \mathbb{Z} &\longrightarrow \mathbb{R}_{>0}^\times \end{aligned}$$

agrees with $-1_p \mapsto \frac{1}{p}$. But then it is actually better to identify $\bigoplus_p \mathbb{Z} \simeq \mathbb{Q}_{>0}^\times$ with the positive rational numbers whose prime factor decomposition $+2^{a_2}3^{a_3}\dots$ corresponds to the vector $(a_2, a_3, \dots)$ in $\bigoplus_p \mathbb{Z}$. $\square$

Write $\mathbf{Sp}^{0,1}$ for the stable ∞ -category of spectra concentrated in degrees $[0, 1]$. Recall that we denote by **Picard** the 2-category of Picard groupoids.

⁹dévissage is trivial on this class as the underlying object is already simple

Proposition 6.2. *There is an equivalence of homotopy categories*

$$(6.7) \quad \Psi: \mathrm{Ho}(\mathrm{Picard}) \xrightarrow{\sim} \mathrm{Ho}(\mathrm{Sp}^{0,1}).$$

This correspondence preserves the notions of homotopy groups π_0, π_1 on either side and the stable k -invariant of the spectrum corresponds to the stable k -invariant for Picard groupoids of Definition 3.2 in Picard .

To elaborate on the stable k -invariant: Given a spectrum concentrated in degrees $[0, 1]$, let

$$\begin{array}{ccc} \Sigma H\pi_1(X) & \longrightarrow & X \longrightarrow H\pi_0(X) \\ \text{or } K(\pi_1 X, 1) & & \text{or } K(\pi_0 X, 0) \end{array},$$

be its fiber sequence of truncation in Sp , decomposing X into two (shifts) of Eilenberg–Mac Lane spectra (a tiny version of a stable Postnikov tower). Then the connecting homomorphism

$$H\pi_0(X) \longrightarrow \Sigma^2 H\pi_1(X)$$

determines a class in $[H\pi_0(X), \Sigma^2 H\pi_1(X)]$, which as an abelian group can be seen to correspond to the group of homomorphism $\pi_0(X) \otimes \mathbb{Z}/2 \rightarrow {}_2\pi_1(X)$, as the stable k -invariant of the attached Picard groupoid. Proofs are given in [Pat12, §5.1, Theorem 5.3] or [JO12, 1.5 Theorem], but already Grothendieck was aware of this correspondence.

Example 6.3. $\Psi(\mathrm{Tors}(A)) = \Sigma HA$ for all abelian groups. Said differently: The groupoid of A -torsors corresponds to the Eilenberg–Mac Lane spectrum of A , shifted to sit in degree one.

Example 6.4. The basechange of torsors from Eq. 3.2, $i_*: \mathrm{Tors}(A) \rightarrow \mathrm{Tors}(B)$, under Ψ gets sent to the Σ -shift of the natural map $HA \rightarrow HB$.

Proposition 6.5. *The virtual objects $V(\mathrm{LCA}_{\mathrm{vf}})$ are symmetric monoidally equivalent to the Picard groupoid $\mathrm{Tors}(\mathbb{Q}_{>0}^\times)$. Under this identification, the composition*

$$\mathrm{LCA}_{\mathrm{vf}}^\times \xrightarrow{u} \mathrm{Tors}(\mathbb{Q}_{>0}^\times) \xrightarrow{i_*} \mathrm{Tors}(\mathbb{R}_{>0}^\times),$$

where u is the universal determinant functor of $\mathrm{LCA}_{\mathrm{vf}}$, agrees with the Haar measure restricted to $\mathrm{LCA}_{\mathrm{vf}}$, i.e.,

$$(6.8) \quad i_* \circ u = Ha|_{\mathrm{LCA}_{\mathrm{vf}}}.$$

Proof. Following Deligne, the universal determinant functor of an exact category $\mathbf{C}$ can be modelled through the virtual objects $V(\mathbf{C})$ of [Del87]. This is the Picard groupoid belonging to truncated connective K -theory under the correspondence of homotopy categories of Prop. 6.2, i.e.,

$$\begin{aligned} \pi_0 V(\mathrm{LCA}_{\mathrm{vf}}) &\cong \pi_0 K^{\mathrm{conn}}(\mathrm{LCA}_{\mathrm{vf}}) = 0, \\ \pi_1 V(\mathrm{LCA}_{\mathrm{vf}}) &\cong \pi_1 K^{\mathrm{conn}}(\mathrm{LCA}_{\mathrm{vf}}) \cong \mathbb{Q}_{>0}^\times, \end{aligned}$$

both by Theorem 6.1. See also [MTW15] for more background on the link between K -theory and $V(\mathbf{C})$. A Picard groupoid (as well as a spectrum concentrated in degree $[0, 1]$) is uniquely determined by these values and the stable k -invariant

$$\pi_0 K^{\mathrm{conn}}(\mathrm{LCA}_{\mathrm{vf}}) \otimes \mathbb{Z}/2 \longrightarrow \pi_1 K^{\mathrm{conn}}(\mathrm{LCA}_{\mathrm{vf}}).$$

Since $\pi_0 K^{\mathrm{conn}}(\mathrm{LCA}_{\mathrm{vf}}) = 0$, this map is necessarily zero. Hence, it follows that $V(\mathrm{LCA}_{\mathrm{vf}})$ has trivial symmetry constraint. Just by comparison of invariants, we deduce that $V(\mathrm{LCA}_{\mathrm{vf}}) \cong \mathrm{Tors}(\mathbb{Q}_{>0}^\times)$, as this is (up to the homotopy classification in Picard) the unique connected

Picard groupoid with trivial stable k -invariant (Def. 3.2) and automorphism group $\mathbb{Q}_{>0}^\times$ of its tensor unit. Since Theorem 6.1 shows that the induced symmetric monoidal functor

$$(6.9) \quad V(\mathbf{LCA}_{\text{vf}}) \longrightarrow V(\mathbf{LCA})$$

on π_1 of the Picard groupoids is just the inclusion $\mathbb{Q}_{>0}^\times \subset \mathbb{R}_{>0}^\times$ and $V(\mathbf{LCA})$ corresponds to the usual Haar torsor, it follows that the universal determinant on $\mathbf{LCA}_{\text{vf}}$ can itself be interpreted as suitable choices of Haar measures, namely exactly those which only differ by positive rational multiples from any fixed initial choice. The symmetric monoidal functor of Eq. 6.9 (by Example 6.3) after applying Ψ turns into the map of spectra

$$(6.10) \quad i: \Sigma H\mathbb{Q}_{>0}^\times \longrightarrow \Sigma H\mathbb{R}_{>0}^\times$$

(or rather a homotopy class of maps of spectra). Hence, in order to prove that

$$\mathbf{LCA}_{\text{vf}}^\times \xrightarrow{u} \mathbf{Tors}(\mathbb{Q}_{>0}^\times) \xrightarrow{i_*} \mathbf{Tors}(\mathbb{R}_{>0}^\times)$$

agrees with $Ha|_{\mathbf{LCA}_{\text{vf}}}$, we just need to show that i_* also has the property that Ψ sends it to the map of Eq. 6.10. But this is just Example 6.4. $\square$

Now we are ready to prove a claim we have made much earlier.

Proof of Theorem 4.3. In §4 we introduced a functor $Ha^\mathbb{Q}: \mathbf{LCA}_{\text{vf}}^\times \rightarrow \mathbf{Tors}(\mathbb{Q}_{>0}^\times)$, but we did not supply the extra data needed to pin down a determinant functor as in Def. 3.5. The claim we have to prove here amounts to saying that it is possible to extend $Ha^\mathbb{Q}$ to a true determinant functor. We will prove this as follows: We will instead work with Deligne's universal determinant functor [Del87], which we denote by u ,

$$(6.11) \quad \mathbf{LCA}_{\text{vf}}^\times \xrightarrow{u} V(\mathbf{LCA}_{\text{vf}}) \cong \mathbf{Tors}(\mathbb{Q}_{>0}^\times)$$

(Prop. 6.5). It tautologically satisfies the demands of Def. 3.5 and we shall show, reversely, that on objects and arrows it can be identified with the description of $Ha^\mathbb{Q}$ in §4. To this end, let $\mathbf{LCA}_{\text{vf}}^{\text{dec}}$ (“decorated vector-free LCA groups”) be the category of pairs (X, C) with $X \in \mathbf{LCA}$ and C a compact open in X . Morphisms $(X', C') \rightarrow (X, C)$ are morphisms $X' \rightarrow X$ of the LCA groups and there is no interaction with the choices of C' or C . Evidently, the forgetful functor

$$\begin{aligned} T: \mathbf{LCA}_{\text{vf}}^{\text{dec}} &\longrightarrow \mathbf{LCA}_{\text{vf}} \\ (X, C) &\longmapsto X \end{aligned}$$

is an equivalence of categories. Equip $\mathbf{LCA}_{\text{vf}}^{\text{dec}}$ with the induced exact structure so that T becomes an exact equivalence of exact categories. A sequence is exact iff T sends it to an exact sequence. But this means that the universal determinant functor u^{dec} of $\mathbf{LCA}_{\text{vf}}^{\text{dec}}$ is just the same, or said differently: The diagram of solid arrows

$$(6.12) \quad \begin{array}{ccc} \mathbf{LCA}_{\text{vf}}^{\text{dec} \times} & \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{H} \end{array} & \mathbf{LCA}_{\text{vf}}^\times \\ & \begin{array}{c} \searrow u^{\text{dec}} \\ \swarrow h^{\text{dec}} \end{array} & \begin{array}{c} \swarrow u \\ \searrow Ha|_{\mathbf{LCA}_{\text{vf}}} \end{array} \\ & \mathbf{Tors}(\mathbb{Q}_{>0}^\times) & \\ & \downarrow i_* & \\ & \mathbf{Tors}(\mathbb{R}_{>0}^\times) & \end{array}$$

commutes by Eq. 6.8. Now recall the description of $Ha^{\mathbb{Q}}$ in §4: In Step 1, for any object $X \in \mathbf{LCA}_{\text{vf}}$ we pick a compact open $C \subseteq X$. Any such choice can be prolonged to a choice for any object, but that datum is just what we need to pick a concrete inverse equivalence H (the dashed arrow in Diagram 6.12). Now consider the Haar measure, restricted to $\mathbf{LCA}_{\text{vf}}$. It similarly admits a lift to the decorated category, denoted by h^{dec} above. We can now trivialize the torsors $Ha(X)$ for all objects in $\mathbf{LCA}_{\text{vf}}^{\text{dec}}$: In the torsor of Haar measures of X we can pick the unique element $\mu_{\text{root}} \in Ha(X)$ such that $\mu_{\text{root}}(C) = 1$. Then $h^{\text{dec}}(X, C)$ in Diagram 6.12 can be described as the multiples $\mathbb{R}_{>0}^{\times} \cdot \mu_{\text{root}}$ inside, and agreeing with all of, $Ha(X)$. Since this construction of μ_{root} matches the recipe in Step 2 of §4, we precisely get the characterization that

$$(6.13) \quad \begin{array}{ccc} Ha^{\mathbb{Q}}(X, C) & \subset & Ha(X, C) \\ \parallel & & \parallel \\ \mathbb{Q}_{>0}^{\times} \cdot \mu_{\text{root}} & \subset & \mathbb{R}_{>0}^{\times} \cdot \mu_{\text{root}} \end{array}$$

Any arrow $f: X' \rightarrow X$ in $\mathbf{LCA}_{\text{vf}}^{\times}$ lifts under H (of Diagram 6.12) to an arrow $(X', C') \rightarrow (X, C)$ in $\mathbf{LCA}_{\text{vf}}^{\text{dec} \times}$ and since the induced map in the Haar torsor is just basechanged from $\mathbb{Q}$ to $\mathbb{R}$ by

$$h^{\text{dec}} = i_* \circ u^{\text{dec}},$$

the distinguished subgroups of rational multiples, as in Eq. 6.13, are respected by h^{dec} . This finishes the proof. $\square$

Theorem 6.6. *Suppose $(\mathbf{P}, \boxtimes)$ is a Picard groupoid and*

$$\mathcal{D}: \mathbf{LCA}_{\text{vf}}^{\times} \longrightarrow \mathbf{P}$$

is any determinant functor. Then there exists a morphism of Picard groupoids f such that

$$(6.14) \quad \begin{array}{ccc} \mathbf{LCA}_{\text{vf}}^{\times} & \xrightarrow{Ha^{\mathbb{Q}}} & (\text{Tors}(\mathbb{Q}_{>0}^{\times}), \otimes) \\ & \searrow \mathcal{D} & \downarrow f \\ & & (\mathbf{P}, \boxtimes) \end{array}$$

commutes, where $Ha^{\mathbb{Q}}$ is the Haar measure determinant functor, restricted to only allowing rational multiples (§4). And more precisely, $Ha^{\mathbb{Q}}$ is the universal determinant functor of $\mathbf{LCA}_{\text{vf}}$ in the sense of Def. 3.6.

Proof. The functor $Ha^{\mathbb{Q}}$ of §4 extends by Theorem 4.3 to a¹⁰ universal determinant functor on $\mathbf{LCA}_{\text{vf}}$. The factorization in our claim, Diagram 6.14, therefore follows from the characterization of universality in Def. 3.6. $\square$

APPENDIX A. GILLET–GRAYSON MODEL

Let $\mathbf{C}$ be a pointed exact category, i.e., an exact category with a fixed choice of a zero object. This will be denoted by 0 . Following [GG87, GG03], define a simplicial set $G_{\bullet} \mathbf{C}$

¹⁰or: the

whose n -simplices are given by a pair of commutative diagrams

$$\begin{array}{ccc}
 & P_{n/(n-1)} & P_{n/(n-1)} \\
 & \uparrow \cdots \uparrow & \uparrow \cdots \uparrow \\
 & \cdots \hookrightarrow \cdots & \cdots \hookrightarrow \cdots \\
 & \uparrow \cdots \uparrow & \uparrow \cdots \uparrow \\
 & P_{2/1} \hookrightarrow \cdots \hookrightarrow P_{n/1} & P_{2/1} \hookrightarrow \cdots \hookrightarrow P_{n/1} \\
 & \uparrow \cdots \uparrow & \uparrow \cdots \uparrow \\
 & P_{1/0} \hookrightarrow P_{2/0} \hookrightarrow \cdots \hookrightarrow P_{n/0} & P_{1/0} \hookrightarrow P_{2/0} \hookrightarrow \cdots \hookrightarrow P_{n/0} \\
 & \uparrow \cdots \uparrow & \uparrow \cdots \uparrow \\
 P_0 \hookrightarrow P_1 \hookrightarrow P_2 \hookrightarrow \cdots \hookrightarrow P_n & & P'_0 \hookrightarrow P'_1 \hookrightarrow P'_2 \hookrightarrow \cdots \hookrightarrow P'_n
 \end{array}$$

such that (1) the diagrams agree strictly¹¹ above the bottom row, (2) each sequence $P_i \hookrightarrow P_j \twoheadrightarrow P_{j/i}$ is exact, (2') each sequence $P'_i \hookrightarrow P'_j \twoheadrightarrow P'_{j/i}$ is exact, (3) each sequence $P_{i/j} \hookrightarrow P_{m/j} \twoheadrightarrow P_{m/i}$ is exact. The face and degeneracy maps come from deleting the i -th row and column, resp. by duplicating them. For details we refer to the references. The 0-simplices are pairs (P, P') of objects. The 1-simplices are pairs of exact sequences

$$(A.1) \quad P_0 \hookrightarrow P_1 \twoheadrightarrow P_{1/0} \qquad P'_0 \hookrightarrow P'_1 \twoheadrightarrow P'_{1/0}$$

with the same cokernel¹². This pair corresponds to a 1-simplex from the point (P_0, P'_0) to the point (P_1, P'_1) . The main result of Gillet and Grayson is the equivalence

$$K^{\text{conn}}(\mathcal{C}) \cong |G_\bullet \mathcal{C}|,$$

or more specifically: They equip the space $|G_\bullet \mathcal{C}|$ with an infinite loop space structure and identifying it with a connective spectrum, it is a model for $K^{\text{conn}}(\mathcal{C})$.

Example A.1. The identification with the zero-th K -group is as follows: the 0-simplex (P, P') lies in the connected component $[P'] - [P] \in \pi_0 K^{\text{conn}}(\mathcal{C})$. Other authors use other sign conventions.¹³

Example A.2. The identification with the first K -group is more complicated. We only need to know that any automorphism $P \xrightarrow{\varphi} P$ of an object $P \in \mathcal{C}$ determines a unique class in $\pi_1 K^{\text{conn}}(\mathcal{C})$, corresponding to the loop

$$(A.2) \quad \begin{array}{ccc} (0, 0) & \xrightarrow{\xi} & (P, P) \\ & \nwarrow \nu(0) \quad \nearrow \nu(P) & \\ & (0, 0) & \end{array}$$

¹¹i.e., not just up to a natural isomorphism.

¹²i.e., not just up to a natural isomorphism.

¹³Weibel's K -book uses precisely the opposite signs.

around the basepoint $(0, 0)$ (where 0 is the designated zero object of the pointed category $\mathcal{C}$), where $\nu(P)$ and ξ come from the 1-simplices

$$(A.3) \quad \begin{array}{ccc} 0 \longrightarrow P \xrightarrow{1} P & & 0 \longrightarrow P \xrightarrow{\varphi} P \\ & \text{and} & \\ 0 \xrightarrow{\dots\dots\dots} P \xrightarrow{1} P & & 0 \xrightarrow{\dots\dots\dots} P \xrightarrow{1} P \end{array}$$

in $G_\bullet(\mathcal{C})$ respectively. See [Nen96] for details and proofs.

Remark A.3. It follows from Prop. 6.2 that the 1-truncation $\tau_{\leq 1} K^{\text{conn}}(\mathcal{C})$ can, up to stable homotopy type, be identified with a Picard groupoid. Deligne’s work [Del87] shows that the truncation map¹⁴ $K^{\text{conn}}(\mathcal{C}) \rightarrow \tau_{\leq 1} K^{\text{conn}}(\mathcal{C})$ essentially can be identified with the concept of a determinant functor. This entire text rests on making this idea explicit for LCA_{vf} .

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¹⁴the co-unit of the adjoint pair attached to the inclusion of 1-truncated spectra into all spectra

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