Einstein-Maxwell Equations on Mass-Centered GCM Hypersurfaces

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Abstract

The resolution of the nonlinear stability of black holes as solutions to the Einstein equations relies crucially on imposing the right geometric gauge conditions. In the vacuum case, the use of Generally Covariant Modulated (GCM) spheres and hypersurfaces has been successful in the proof of stability for slowly rotating Kerr spacetime [18]. For the charged setting, our companion paper [6] introduced an alternative mass-centered GCM framework, adapted to the additional difficulties of the Einstein–Maxwell system.

In this work, we solve the Einstein-Maxwell equations on such a mass-centered spacelike GCM hypersurface, which is equivalent to solving the constraint equations there. We control all geometric quantities of the solution in terms of some seed data, corresponding to the gauge-invariant fields describing coupled gravitational–electromagnetic radiation in perturbations of Reissner–Nordström or Kerr–Newman, first identified in [9] and expected to be governed by favorable hyperbolic equations. This provides the first step toward controlling gauge-dependent quantities in the nonlinear stability analysis of the Reissner–Nordström and Kerr–Newman families.

Contents

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1 Introduction

In this paper, we solve the Einstein–Maxwell equations on a mass-centered General Covariant Modulated (GCM) hypersurface. The mass-centered GCM framework, introduced in our companion work [6], is a refinement of the original GCM setup, which played a central role in the proof of the nonlinear stability of Kerr spacetimes [11, 18]. Since the hypersurface is spacelike, solving the restriction of the Einstein–Maxwell equations there amounts to solving the associated constraint equations. As seed data for these constraints, we prescribe the gauge-invariant quantities encoding the electromagnetic and gravitational radiation.

1.1 The GCM framework in the nonlinear stability of black holes

The nonlinear stability of black hole solutions to the Einstein equations has been a central topic of investigation in mathematical General Relativity, with significant progress achieved in recent years [5, 13, 17, 18]. As in most stability problems for hyperbolic PDEs, the goal is to establish global existence from initial data satisfying a smallness condition, typically via a continuity argument on a bootstrap region of finite size. Owing to the covariant nature of the Einstein equations, the analysis of the dynamical evolution of such data is deeply dependent on the choice of gauge. In other words, a suitable gauge fixing is essential in order to implement the continuity argument successfully.

In the recent proof of the nonlinear stability of slowly rotating Kerr black holes [11, 18, 20], the gauge choice underpinning the continuity argument is based on the construction of Generally Covariant Modulated (GCM) spheres [15, 16]. These spheres foliate a spacelike GCM hypersurface [20], on which several key geometric quantities are prescribed to vanish. More precisely, the bootstrap region (see Figure 1) has future boundary given by the union $\mathcal{A} \cup {}^{(top)}\Sigma \cup \Sigma_*$, where $\mathcal{A}$, ${}^{(top)}\Sigma$ and Σ_* are spacelike hypersurfaces, with Σ_* foliated by GCM spheres. The spacetime $\mathcal{M}$ also contains a timelike hypersurface $\mathcal{T}$ which divides $\mathcal{M}$ into an exterior region called ${}^{(ext)}\mathcal{M}$ and an interior region called ${}^{(int)}\mathcal{M}$. The hypersurface Σ_* is required to satisfy a dominance condition, ensuring it lies "far away" (in a precise quantitative sense) from the perturbed event horizon, as well as transversality conditions (see Section 2.1). Furthermore, parameters such as the final mass, angular momentum, and axis of rotation of the black hole are fixed on the last sphere S_* of Σ_*, where the bootstrap region terminates. As a consequence of the continuity argument, the GCM foliation of Σ_* induces a canonical foliation of null infinity, thereby resolving the supertranslation ambiguity [14].

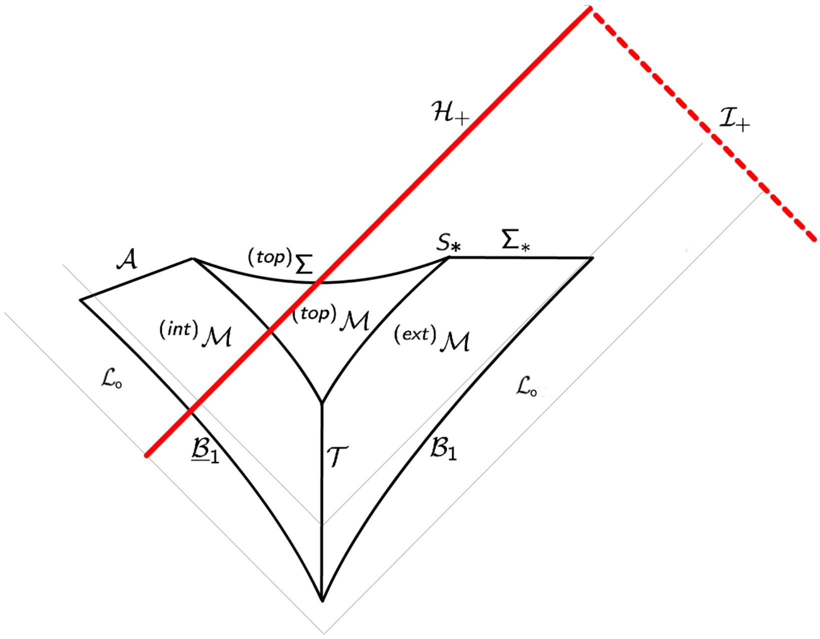

Figure 1: The GCM admissible spacetime of [18].

A critical condition imposed on the last sphere S_* is the vanishing of the spacetime's center of mass, encoded in the $\ell = 1$ mode[1] of the divergence of the curvature component β:

$$(\operatorname{div} \beta)_{\ell=1} = 0 \qquad \text{on } S_*,$$

see already Section 2 for the definition of the curvature components. Physically, this condition enforces that the last sphere of the bootstrap region shares the same center of mass as the perturbed spacetime, so that its limit as S_* approaches timelike infinity i^+ coincides with the final center of mass of the black hole [14].

In the vacuum case, this choice is possible due to an exceptional structure: the $\ell = 1$ mode of $\operatorname{div} \beta$ satisfies an outgoing null transport equation involving the spin $+2$ curvature component α, whose $\ell = 1$ projection remains regular even under weak decay assumptions [18]. Simultaneously, it satisfies a favorable transport equation along the vector field ν, tangent to Σ_* (see (2.2)), where the ingoing derivative dominates. This special structure is crucial to the GCM framework in vacuum, as it permits the controlled propagation of $(\operatorname{div} \beta)_{\ell=1}$ both along Σ_* and backwards along outgoing null directions into the exterior region.

1.2 Mass-centered GCM hypersurfaces in electrovacuum spacetimes

In the case of charged spacetimes satisfying the Einstein–Maxwell equations (see (1.2)), the exceptional structure present in vacuum breaks down: the $\ell = 1$ mode of the perturbations now carries genuine radiation, and the favorable transport properties of $(\operatorname{div} \beta)_{\ell=1}$ (or any of its renormalizations) are lost. More precisely, in the charged setting one can construct a renormalization of $(\operatorname{div} \beta)_{\ell=1}$ that admits either an acceptable ingoing transport equation or an improved outgoing one, but not both simultaneously. Consequently, the GCM conditions of [15, 16, 20] can no longer be applied directly in

[1] Defining $\ell = 1$ modes in a canonical way for general spacetimes is a subtle problem, addressed in [16], and used in the construction of GCM spheres and hypersurfaces [20].

the continuity argument for charged black holes, and a modification of the gauge conditions becomes necessary.

In [6], we introduced such a modification by imposing that the center of mass — defined as a suitable renormalization of $(\mathrm{div}\,\beta)_{\ell=1}$ with good outgoing transport equation — vanishes on every sphere foliating the GCM hypersurface Σ, and not only on its last sphere. This eliminates the need to propagate it along Σ_*, while still allowing backward propagation along outgoing null directions in the subsequent step of the proof. Concretely, the *center of mass function* is defined as

$$\boldsymbol{C} := \mathrm{div}\,\beta - \frac{Q}{r^2}\,\mathrm{div}\,{}^{(\mathbf{F})}\beta + \frac{2Q}{r^3}\,\widecheck{{}^{(\mathbf{F})}\rho},$$

where β and ${}^{(\mathbf{F})}\beta$ are the null components of the Weyl and electromagnetic fields, and $\widecheck{{}^{(\mathbf{F})}\rho}$ denotes the linearization of an electromagnetic component with respect to its Reissner–Nordström value. We then impose

$$\boldsymbol{C}_{\ell=1} = 0 \qquad \text{on } \Sigma_*. \tag{1.1}$$

Here the electric charge Q, like the other parameters, is defined on the last sphere S_* by

$$Q := \frac{1}{4\pi}\int_{S_*} {}^{(\mathbf{F})}\rho = \frac{1}{4\pi}\int_{S_*} {}^*\mathbf{F},$$

with the magnetic charge defined analogously (see (2.6)). A corresponding definition of the angular momentum function $\boldsymbol{J}$ is given in (2.9), and all parameter functions are recalled in Section 2.2.

This motivates the definition of *mass-centered* GCM spheres and hypersurfaces. The construction of such hypersurfaces is described in detail in our companion paper [6].

The choice of renormalization in the definition of $\boldsymbol{C}$ is guided by its improved transport along the outgoing null direction, namely (see (A.3))

$$\nabla_4(r^5\boldsymbol{C}) = -r^5\,\mathrm{div}\,\mathrm{div}\,\alpha + \text{nonlinear terms}.$$

Because of the weak decay of curvature components, however, $r^5\boldsymbol{C}$ fails to decay at infinity, which introduces significant subtleties when attempting to integrate this equation. In contrast to the vacuum case, where $(\mathrm{div}\,\beta)_{\ell=1}$ enjoys the exceptional structure discussed above, the transport of $\boldsymbol{C}_{\ell=1}$ along ν (the vector tangent to Σ_*, see Proposition 2.19) produces terms that decay too weakly to allow integration along the hypersurface. For this reason, rather than propagating $\boldsymbol{C}_{\ell=1}$ along Σ_*, we impose the condition (1.1) directly on every sphere of Σ_*. This turns the ν-transport equation into an algebraic relation among the perturbation quantities, which is then incorporated into the system of equations governing the hypersurface. This reformulation is precisely what leads to the notion of *mass-centered* GCM hypersurfaces, and it is on such hypersurfaces that we shall now solve the Einstein–Maxwell constraint equations.

1.3 Einstein-Maxwell equations on hypersurfaces satisfying bootstrap assumptions and the dominance condition

In this paper we solve the Einstein–Maxwell equations

$$\mathrm{Ric}[\mathbf{g}] = 2\mathbf{F}\cdot\mathbf{F} - \frac{1}{2}\mathbf{g}|\mathbf{F}|^2, \qquad \mathrm{div}\,\mathbf{F} = 0, \qquad d\mathbf{F} = 0, \tag{1.2}$$

where $\mathrm{Ric}[\mathbf{g}]$ is the Ricci curvature of the Lorentzian metric $\mathbf{g}$ and $\mathbf{F}$ is the electromagnetic 2-form, along a mass-centered GCM hypersurface Σ_* satisfying bootstrap assumptions and the dominance condition, as in the nonlinear stability proof of Kerr by Klainerman–Szeftel [18]. Since Σ_* is spacelike, restricting the Einstein–Maxwell equations to Σ_* is equivalent to solving the associated *constraint equations* on the hypersurface.

The hypersurface Σ_* is defined by the relation $u + r = c_*$, where r and u play the role of radial and retarded time functions, and c_* is a constant. Both the bootstrap assumptions and the dominance condition are formulated in terms of r and u:

(a) The *bootstrap assumptions*, compatible with the setup of the nonlinear stability of Kerr–Newman spacetimes, are decay conditions on the perturbation quantities, divided into two classes Γ_g and Γ_b, corresponding to "good" and "bad" r-decay:

$$\|\Gamma_b\|_{L^\infty(S)} \lesssim \frac{\epsilon}{ru^{1+\delta}}, \qquad \|\Gamma_g\|_{L^\infty(S)} \lesssim \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}}, \tag{1.3}$$

for some $\delta > 0$ and a small constant $\epsilon > 0$. Here $\|\cdot\|_{L^\infty(S)}$ denote the L^∞ norm on each sphere S of Σ_*.

(b) The *dominance condition* is a quantitative relation between r and u on S_*, which implies that along Σ_* the r-decay dominates as follows:

$$\frac{1}{r} \lesssim \frac{\epsilon_0}{u^{1+\delta}}, \tag{1.4}$$

with ϵ_0 another smallness parameter satisfying $\epsilon_0 \ll \epsilon$ and $\epsilon^2 \ll \epsilon_0$. This condition prescribes the position of S_* in the asymptotically flat region of the spacetime.

It is well known that the constraint equations form an underdetermined system, and the main difficulty lies in identifying suitable seed data parametrizing their solutions. Classical strategies include the conformal method [1, 19] and gluing constructions [2–4].

In our setting, we show that all geometric quantities on Σ_* describing perturbations of Reissner–Nordström or Kerr–Newman can be controlled in terms of seed data given by the *gauge-invariant radiation fields* for coupled electromagnetic–gravitational perturbations. These quantities,

$$\alpha, \qquad \underline{\alpha}, \qquad \mathfrak{f}, \qquad \underline{\mathfrak{f}}, \qquad \mathfrak{b}, \qquad \underline{\mathfrak{b}},$$

were identified in [9] and shown to satisfy a coupled system of Teukolsky equations. The first-order derived quantities of $\mathfrak{f}, \underline{\mathfrak{f}}, \mathfrak{b}, \underline{\mathfrak{b}}$, denoted by

$$\mathfrak{q}^{\mathbf{F}}, \qquad \underline{\mathfrak{q}}^{\mathbf{F}}, \qquad \mathfrak{p}, \qquad \underline{\mathfrak{p}},$$

satisfy generalized Regge–Wheeler equations. These equations are hyperbolic in nature, and thus the control of the radiation fields is expected to be obtainable independently of the rest of the system by energy and decay estimates. Results in this direction are already known: boundedness and decay estimates for these quantities have been established in the linearized setting for Reissner–Nordström in the full subextremal range [10], for slowly rotating and weakly charged Kerr–Newman [7], and for slowly rotating, strongly charged Kerr–Newman spacetimes in axial symmetry [12]. More recently, decay estimates have also been obtained in the fully nonlinear setting for the Regge–Wheeler system in Reissner–Nordström [21].

In this work, we go further by showing that, on a mass-centered GCM hypersurface Σ_* satisfying the bootstrap assumptions and dominance condition, *all geometric perturbation quantities* (i.e. the full sets Γ_g and Γ_b) can be expressed and controlled entirely in terms of the free radiation data above. This reduction of the Einstein–Maxwell constraint problem to a parametrization by gauge-invariant radiation fields is the content of our main theorem, stated in the next subsection.

1.4 The main theorem and sketch of the proof

The application we have in mind is the resolution of the constraint equations on Σ_* as part of the nonlinear stability problem for charged black holes. The gauge-invariant quantities introduced above will serve as the seed data for this underdetermined system. We now state a simplified version of our main result (see Theorem 4.3 for the precise formulation).

Theorem 1.1. *Let $\Sigma_* \subset \mathcal{M}$ be a mass-centered GCM hypersurface satisfying the dominance condition on a spacetime $\mathcal{M}$ solving the Einstein–Maxwell equations, and assume the bootstrap assumptions* (1.3) *hold for Γ_g and Γ_b. Then all geometric perturbation quantities along Σ_* (i.e. the sets Γ_g and Γ_b) can be expressed and controlled in terms of the gauge-invariant seed data*

$$seed := \{\alpha, \underline{\alpha}, \mathfrak{f}, \mathfrak{b}, \underline{\mathfrak{b}}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}\}.$$

In particular,

$$\|\Gamma_b\|_{L^\infty(S)} \lesssim \frac{\mathscr{F}[seed]}{ru^{1+\delta}}, \qquad \|\Gamma_g\|_{L^\infty(S)} \lesssim \mathscr{G}[seed] + \frac{\mathscr{F}[seed]}{r^2u^{1+\delta}}, \tag{1.5}$$

where $\mathscr{F}[seed]$ and $\mathscr{G}[seed]$ denote suitable flux energy or pointwise norms of the seed data.

In particular, if one can independently show that

$$\mathscr{F}[\text{seed}] \lesssim \epsilon_0, \qquad \mathscr{G}[\text{seed}] \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta}},$$

then (1.5) improves the bootstrap assumptions (1.3) on Σ_*.

The argument follows the proof of Theorem M3 in [18] (see Remark 1.2) and proceeds in three stages.

Step 1 (unconditional control of $\ell = 0$ and most $\ell = 1$ modes). Starting from the bootstrap assumptions (1.3) and the dominance condition (1.4), we first derive improved decay for the $\ell = 0$ and most of the $\ell = 1$ spherical modes of the perturbation, independently of the seed data. The $\ell = 0$ estimates are obtained from the transport equation for the Hawking mass (see Section 3.1), while the $\ell = 1$ estimates rely on the mass-centered condition imposed on all spheres of Σ_* (see Section 3.2). These arguments already improve the bootstrap bounds by ϵ_0, and also derive stronger decay in r and u.

Step 2 (control of Γ_b in terms of the seed data). We next show that the Γ_b quantities can be controlled directly from the prescribed seed data. This is achieved by establishing flux estimates along Σ_* (see Section 4.2), which yield L^2 bounds for Γ_b in terms of the seed data. The flux bounds also imply pointwise decay and, crucially, improved estimates for nonlinear interactions of the form $\Gamma_b \cdot \Gamma_g$ (see Corollary 4.7).

Step 3 (recovery of the remaining $\ell = 1$ modes and Γ_g). Finally, with Γ_b under control, we recover the remaining $\ell = 1$ modes and all Γ_g quantities. The nonlinear terms estimated in Step 2 appear on the right-hand side of the transport equation for the angular momentum function $\boldsymbol{J}$, which must be integrated along Σ_* (see Section 4.3). This provides control of $\boldsymbol{J}$ and hence of the missing $\ell = 1$ components in terms of the seed data. Elliptic estimates on the spheres foliating Σ_* are then used to propagate this control to all of Γ_g, completing the argument (see Section 4.4).

In summary, the constraint equations are resolved by a combination of transport, flux, and elliptic estimates, reducing the system entirely to the specification of the gauge-invariant seed data.

Remark 1.2. *Theorem 1.1 is formulated so as to be directly applicable to the setup of the nonlinear stability problem for Kerr–Newman. In the vacuum case of Kerr, the analogous result is Theorem M3 in [18], which builds on the prior control of gauge-invariant quantities established in Theorem M1 and*

M2 [11]. In contrast, in the present setting such control is not assumed: the gauge-invariant fields are treated as seed data for the induced Einstein-Maxwell equations. This separation ensures that the statement of Theorem 1.1 remains modular, so that it can later be combined with the independent hyperbolic estimates for the gauge-invariant quantities.

1.5 Organization of the Paper

The paper is organized as follows.

- Section 2 reviews the necessary preliminaries, including the mass-centered GCM hypersurfaces, the Einstein-Maxwell equations, the bootstrap assumptions and dominance condition. It also derives the equations for the renormalized quantities that will be important later.
- Section 3 collects the first consequences of the bootstrap assumptions and the dominance condition.
- Section 4 contains the proof of our main theorem.

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2 Preliminaries

We consider a 4-dimensional Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ with a 2-form $\mathbf{F}$, satisfying the Einstein-Maxwell equations (1.2).

We recall the following standard definition of the null connection coefficients and null Weyl curvature components relative to a null frame (e_3, e_4, e_1, e_2), where e_3, e_4 are respectively ingoing and outgoing null directions and e_1, e_2 are orthonormal frames orthogonal to them:

$$\begin{aligned}
\underline{\chi}_{ab} &= \mathbf{g}(\mathbf{D}_{e_a} e_3, e_b), & \chi_{ab} &= \mathbf{g}(\mathbf{D}_{e_a} e_4, e_b),\\
\underline{\xi}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_a), & \xi_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_a),\\
\underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3),\\
\underline{\eta}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_a), & \eta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_a),\\
\zeta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_a} e_4, e_3),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{ab} &= \mathbf{W}(e_a, e_4, e_b, e_4), & \underline{\alpha}_{ab} &= \mathbf{W}(e_a, e_3, e_b, e_3),\\
\beta_a &= \frac{1}{2}\mathbf{W}(e_a, e_4, e_3, e_4), & \underline{\beta}_a &= \frac{1}{2}\mathbf{W}(e_a, e_3, e_3, e_4),\\
\rho &= \frac{1}{4}\mathbf{W}(e_3, e_4, e_3, e_4), & {}^*\rho &= \frac{1}{4}{}^*\mathbf{W}(e_3, e_4, e_3, e_4),
\end{aligned}$$

where $\mathbf{D}$ is the spacetime covariant derivative, $\mathbf{W}$ is the Weyl curvature of the spacetime metric $\mathbf{g}$ and ${}^*\mathbf{W}$ is its Hodge dual. As usual, we denote

$$\operatorname{tr}\chi := \delta^{ab}\chi_{ab}, \qquad \operatorname{tr}\underline{\chi} := \delta^{ab}\underline{\chi}_{ab}.$$

We also define the following null electromagnetic components:

$$\begin{aligned} {}^{(\mathbf{F})}\beta_a &= \mathbf{F}(e_a, e_4), & {}^{(\mathbf{F})}\underline{\beta}_a &= \mathbf{F}(e_a, e_3), \\ {}^{(\mathbf{F})}\rho &= \frac{1}{2}\mathbf{F}(e_3, e_4), & {}^{*\,(\mathbf{F})}\rho &= \frac{1}{2}{}^*\mathbf{F}(e_3, e_4). \end{aligned}$$

2.1 Geometric setting of Σ_*

Here we introduce the main defining properties of the hypersurface Σ_* in $\mathcal{M}$. Such properties are the ones assumed to hold for the boundary of a GCM admissible spacetime in the context of nonlinear stability of Kerr or Kerr-Newman, as in [18] (see also [6]).

Let r and u be two functions on $\mathcal{M}$, such that r restricted to Σ_* induces the area radius function and moreover, there exists a constant c_* such that $u = c_* - r$ on Σ_*. We also impose the transversality conditions on Σ_*:

$$\xi = 0, \qquad \omega = 0, \qquad \underline{\eta} = -\zeta, \qquad e_4(r) = 1, \qquad e_4(u) = 0,$$

which allows us to make sense of all the Ricci coefficients of Σ_*, as well as of all first-order derivatives of r and u on Σ_*. We define

$$y := e_3(r), \qquad z := e_3(u).$$

Using the transversality condition for $e_4(r)$ and $e_4(u)$, and imposing that (e_1, e_2) is adapted to the r-foliation on Σ_*, we have

$$\nabla(r) = \nabla(u) = 0, \quad e_4(u) = 0, \quad e_4(r) = 1, \quad e_3(r) = y, \quad e_3(u) = z.$$

The following relations hold true:

$$\nabla y = -\underline{\xi} + \big(\zeta - \eta\big)y, \qquad \nabla z = (\zeta - \eta)z, \qquad b_* = -y - z. \tag{2.1}$$

We denote

$$\nu = e_3 + b_* e_4 \tag{2.2}$$

the vectorfield tangent to Σ_*, orthogonal to the foliation and normalized by the condition $\mathbf{g}(\nu, e_4) = -2$.

The hypersurface Σ_* terminates in a boundary sphere S_* on which the given function r is constant, i.e. S_* is a leaf of the r-foliation of Σ_*. Let r_*, u_* denote the values of r and u on S_*. On S_* there exist coordinates (θ, φ) such that the induced metric g on S_* takes the form

$$g = r^2 e^{2\phi}\Big((d\theta)^2 + \sin^2\theta (d\varphi)^2\Big), \tag{2.3}$$

and the functions

$$J^{(0)} := \cos\theta, \qquad J^{(-)} := \sin\theta\sin\varphi, \qquad J^{(+)} := \sin\theta\cos\varphi,$$

verify the balanced conditions

$$\int_{S_*} J^{(p)} = 0, \qquad p = 0, +, -.$$

We rely on a special orthonormal basis (e_1, e_2) of the tangent space of S_* given by

$$e_1 = \frac{1}{re^{\phi}}\partial_\theta, \qquad e_2 = \frac{1}{r\sin\theta e^{\phi}}\partial_\varphi, \quad \text{on} \quad S_*.$$

We define the 1-forms f_0, f_+ and f_- defined on S_* by:

$$\begin{aligned} (f_0)_1 &= 0, & (f_0)_2 &= \sin\theta, \\ (f_+)_1 &= \cos\theta\cos\varphi, & (f_+)_2 &= -\sin\varphi, \\ (f_-)_1 &= \cos\theta\sin\varphi, & (f_-)_2 &= \cos\varphi, \end{aligned}$$

and extend them to Σ_* by $\nabla_\nu f_p = 0$ for $p = 0, \pm$.

The coordinates (θ, φ) and the functions $J^{(p)}$ on S_* are extended to Σ_* by setting $\nu(\theta) = \nu(\varphi) = 0$ and $\nu(J^{(p)}) = 0$, for $p = 0, +, -$. We also impose the transversality conditions on Σ_* for (θ, φ) and $J^{(p)}$:

$$e_4(\theta) = 0, \qquad e_4(\varphi) = 0, \qquad e_4(J^{(p)}) = 0, \qquad p = 0, +, -.$$

For a scalar function λ on any sphere S of Σ_*, we define its projection to the $\ell = 1$ mode as the triplet functions

$$\lambda_{\ell=1} := \left\{ \int_S J^{(p)}\lambda, \qquad p \in \{-, 0, +\} \right\}.$$

We also denote by K the Gauss curvature of the spheres S of Σ_*.

2.2 Definitions of mass, charge and angular momentum at S_*

We now define the mass, charge and angular momentum parameters at the sphere S_*.

Definition 2.1. *We define the following constant parameters on Σ_*:*

1. *The* auxiliary mass m *is defined to be the Hawking mass of S_*, i.e.,*

$$\frac{2m}{r_*} := 1 + \frac{1}{16\pi}\int_{S_*} \mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi}. \tag{2.4}$$

2. *The* electric charge Q *is defined as the average of ${}^{(\mathbf{F})}\rho$ at S_*, i.e.*

$$Q := \frac{1}{4\pi}\int_{S_*} {}^{(\mathbf{F})}\rho = \frac{1}{4\pi}\int_{S_*} {}^*\mathbf{F}. \tag{2.5}$$

3. *The* magnetic charge e *is defined as the average of ${}^*\,{}^{(\mathbf{F})}\rho$ at S_*, i.e.*

$$e := \frac{1}{4\pi}\int_{S_*} {}^*\,{}^{(\mathbf{F})}\rho = \frac{1}{4\pi}\int_{S_*} \mathbf{F}. \tag{2.6}$$

4. *The* mass parameter M *is defined as*

$$M = m + \frac{Q^2}{2r_*}. \tag{2.7}$$

5. *The* angular momentum a *is defined as*

$$a := \frac{r_*^3}{8\pi M} \int_{S_*} J^{(0)} \boldsymbol{J}, \tag{2.8}$$

where $\boldsymbol{J}$ *is defined as*

$$\boldsymbol{J} := \operatorname{curl} \beta - \frac{Q}{r^2} \operatorname{curl} {}^{(\mathbf{F})}\beta + \frac{2Q}{r^3} {}^*{}^{(\mathbf{F})}\rho. \tag{2.9}$$

Remark 2.2. *Observe that by Stokes Theorem and Maxwell's equation $d\mathbf{F} = 0$, $d^*\mathbf{F} = 0$, the charges Q and e are independent of the choice of the sphere, i.e.*

$$Q = \frac{1}{4\pi} \int_S {}^{(\mathbf{F})}\rho, \qquad e = \frac{1}{4\pi} \int_S {}^*{}^{(\mathbf{F})}\rho,$$

for any sphere S. In particular, they are the same in the whole of Σ_*.

Remark 2.3. *Any rotation of the Maxwell field $\mathbf{F}_\lambda = \cos\lambda \mathbf{F} + \sin\lambda^* \mathbf{F}$ satisfies the Maxwell equations, therefore we can choose a parameter λ such that $e = 0$ on S_*, and therefore by Remark 2.2 $e = 0$ on Σ_*.*

Remark 2.4. *The definition of $\boldsymbol{J}$ is motivated by its favorable behavior under transport: it satisfies a suitable transport equation along the tangent vector ν to Σ_*,* as well as *an improved transport equation along the outgoing null direction e_4 (see already* (A.4)*). In contrast with the Kerr case, where $\boldsymbol{J} = \operatorname{curl} \beta$, the presence of electromagnetic contributions in the equations for β necessitates additional lower-order terms. These terms play a crucial role in canceling problematic contributions in the transport equation. This is in parallel with the definition of the center of mass $\boldsymbol{C}$ discussed below (see Remark 2.6).*

In addition to the constant parameters above, we also make use of the following definition of function masses on Σ_*.

Definition 2.5. *We define the following functions on Σ_*.*

1. *The* Hawking mass *of a sphere S in Σ_* is given by*

$$\frac{2m_H}{r} = 1 + \frac{1}{16\pi} \int_S \operatorname{tr}\chi \operatorname{tr}\underline{\chi}. \tag{2.10}$$

2. *The* mass aspect function μ *is defined as*

$$\mu := -\operatorname{div}\zeta - \rho + \frac{1}{2}\widehat{\chi} \cdot \widehat{\underline{\chi}}. \tag{2.11}$$

3. *The* center of mass function *is defined as*

$$\boldsymbol{C} := \operatorname{div} \beta - \frac{Q}{r^2} \operatorname{div} {}^{(\mathbf{F})}\beta + \frac{2Q}{r^3} \widetilde{{}^{(\mathbf{F})}\rho} \tag{2.12}$$

where $\widetilde{{}^{(\mathbf{F})}\rho} := {}^{(\mathbf{F})}\rho - \dfrac{Q}{r^2}$.

Remark 2.6. *The definition of the quantity $\boldsymbol{C}$ is motivated by the fact that it satisfies an improved transport estimate along the outgoing null direction e_4 (see already* (A.3)*), given by*

$$\nabla_4 \boldsymbol{C} = -\frac{5}{r}\boldsymbol{C} - \operatorname{div}\operatorname{div}\alpha + \textit{nonlinear terms}.$$

Contrary to the case of $\boldsymbol{J}$, as in Remark 2.4 above, it is not possible to find a renormalization of $(\operatorname{div}\beta)_{\ell=1}$ *which satisfies an improvement* both *in the transport equations along Σ_* and along e_4. For this reason, the GCM condition $\boldsymbol{C}_{\ell=1}=0$ is imposed on all the spheres of the hypersurface as opposed to only at S_* as in [18].*

Remark 2.7. *Observe that, using the operator $\not{d}_1 = (\operatorname{div}, \operatorname{curl})$, the quantities $\boldsymbol{C}$ and $\boldsymbol{J}$ can be written concisely as*

$$(\boldsymbol{C}, \boldsymbol{J}) := \not{d}_1\beta - \frac{Q}{r^2}\not{d}_1\,{}^{(\mathbf{F})}\beta + \frac{2Q}{r^3}(\widetilde{{}^{(\mathbf{F})}\rho}, {}^*\,{}^{(\mathbf{F})}\rho). \tag{2.13}$$

The mass and charge parameters given in Definition 2.1 are used to define the linearized quantities with respect to their expected value in Reissner-Nordström.

Definition 2.8. *We define the following linearized quantities:*

$$\widetilde{\operatorname{tr}\chi} := \operatorname{tr}\chi - \frac{2}{r}, \qquad \widetilde{\operatorname{tr}\underline{\chi}} := \operatorname{tr}\underline{\chi} + \frac{2\Upsilon}{r}, \qquad \underline{\check{\omega}} := \underline{\omega} - \frac{M}{r^2} + \frac{Q^2}{r^3},$$
$$\check{y} := y + \Upsilon, \qquad \check{z} := z - 2, \qquad \check{b}_* := b_* + 2 - \Upsilon, \qquad \check{K} := K - \frac{1}{r^2},$$
$$\check{\rho} := \rho + \frac{2M}{r^3} - \frac{2Q^2}{r^4}, \qquad \check{\mu} := \mu - \frac{2M}{r^3} + \frac{2Q^2}{r^4}, \qquad \widetilde{{}^{(\mathbf{F})}\rho} := {}^{(\mathbf{F})}\rho - \frac{Q}{r^2},$$

where M and Q are the mass and electric charge defined in Definition 2.1 and $\Upsilon = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.

In addition to the above, we also introduce the following renormalized quantities:

$$\widetilde{\nabla J^{(0)}} := \nabla J^{(0)} + \frac{1}{r}{}^*f_0, \qquad \widetilde{\nabla J^{(+)}} := \nabla J^{(+)} - \frac{1}{r}f_+, \qquad \widetilde{\nabla J^{(-)}} := \nabla J^{(-)} - \frac{1}{r}f_-,$$
$$\widetilde{\operatorname{curl}(f_0)} := \operatorname{curl}(f_0) - \frac{2}{r}\cos\theta, \qquad \widetilde{\operatorname{div}(f_\pm)} := \operatorname{div}(f_\pm) + \frac{2}{r}J^{(\pm)}.$$

2.3 The electrovacuum mass-centered GCM conditions on Σ_*

Here we recall the mass-centered GCM conditions imposed on Σ_* in [6]. The conditions partly coincide with the ones in Kerr in vacuum, with the most important difference being that each sphere along Σ_* has vanishing center of mass.

Definition 2.9. *We say that Σ_* is an* electrovacuum mass-centered GCM hypersurface *if the following conditions are satisfied.*

1. *On the sphere S_* we have:*

$$\widetilde{\operatorname{tr}\chi} = 0, \qquad \widetilde{\operatorname{tr}\underline{\chi}} = 0, \qquad \boldsymbol{C}_{\ell=1} = 0, \tag{2.14}$$

and

$$\int_{S_*} J^{(+)}\boldsymbol{J} = 0, \qquad \int_{S_*} J^{(-)}\boldsymbol{J} = 0. \tag{2.15}$$

2. *On any sphere of the r-foliation of Σ_*, we have*

$$
\begin{aligned}
\widetilde{\operatorname{tr}\chi} &= 0, \\
\widetilde{\operatorname{tr}\underline{\chi}} &= \underline{C}_0 + \sum_{p=0,+,-} \underline{C}_p J^{(p)}, \\
\check{\mu} &= M_0 + \sum_{p=0,+,-} M_p J^{(p)},
\end{aligned}
\tag{2.16}
$$

where $\underline{C}_0$, $\underline{C}_p$, M_0, M_p are scalar functions on Σ_* and are constant on the leaves of the foliation. Also,

$$
\boldsymbol{C}_{\ell=1} = 0, \qquad \overline{b_*} = -1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2.17}
$$

where $\overline{b_*}$ denotes the average of b_* on the spheres foliating Σ_*, and

$$
(\operatorname{div}\underline{\xi})_{\ell=1} = 0. \tag{2.18}
$$

2.4 Bootstrap assumptions and dominance condition

We collect here the two additional assumptions on the mass-centered GCM hypersurface Σ_*: the bootstrap assumptions and the dominance condition.

Recall the linearized quantities of Definition 2.8. We divide all the linearized geometrical quantities (connection coefficients, curvature and electromagnetic components) into two sets, denoted Γ_g, Γ_b, depending on their expected decay:

$$
\Gamma_g := \Big\{ \widetilde{\operatorname{tr}\chi},\ \widehat{\chi},\ \zeta,\ \widetilde{\operatorname{tr}\underline{\chi}},\ r\alpha,\ r\beta,\ {}^{(\mathbf{F})}\beta,\ r\check{\rho},\ r\,{}^*\rho,\ {}^{*\,(\mathbf{F})}\rho,\ \widetilde{{}^{(\mathbf{F})}\rho},\ r\check{\mu},\ r\check{K} \Big\}, \tag{2.19}
$$

$$
\Gamma_b = \Big\{ \eta,\ \widehat{\underline{\chi}},\ \check{\underline{\omega}},\ \underline{\xi},\ r\underline{\beta},\ \underline{\alpha},\ {}^{(\mathbf{F})}\underline{\beta},\ r^{-1}\check{y},\ r^{-1}\check{z},\ r^{-1}\check{b_*} \Big\}. \tag{2.20}
$$

The bootstrap assumptions for the linearized quantities are given in terms of the L^∞ norms on $S(u) \subset \Sigma_*$, i.e.

$$
\|f\|_{\infty,k}(u) = \sum_{i=0}^{k} \|\mathfrak{d}_*^i f\|_\infty(u),
$$

where $\mathfrak{d}_*$ denotes tangential derivatives to Σ_*.

Bootstrap Assumptions. *We assume that there exist positive small constants $\epsilon > 0$, $\delta > 0$ and a large enough $N > 0$ such that for any element of the set Γ_b and Γ_g the following estimates hold true on Σ_* for $0 \leqslant k \leqslant N$:*

$$
\begin{aligned}
\|\Gamma_b\|_{\infty,k} &\lesssim \frac{\epsilon}{r u^{1+\delta}}, \\
\|\Gamma_g\|_{\infty,k} &\lesssim \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}}, \\
\|\nabla_\nu \Gamma_g\|_{\infty,k-1} &\lesssim \frac{\epsilon}{r^2 u^{1+\delta}}.
\end{aligned}
\tag{2.21}
$$

Observe that, according to the above assumptions, for $r, u \gg 1$, we have

$$
r^{-1}\|\Gamma_b\|_{\infty,k} \lesssim \|\Gamma_g\|_{\infty,k}. \tag{2.22}
$$

The *dominance condition* below quantifies the largeness of r and u along Σ_*.

Dominance Condition. *We assume that on S_*, the values r_* and u_* are related by the following relation*

$$r_* = \delta_* \epsilon_0^{-1} u_*^{1+\delta},$$

where $\delta_* > 0$, $\epsilon_0 > 0$ are positive small constants satisfying $\epsilon = \epsilon_0^{\frac{2}{3}}$. This implies in particular on Σ_*

$$r \geqslant r_* \geqslant \delta_* \epsilon_0^{-1} u^{1+\delta},$$

which can be written as

$$\frac{1}{r} \lesssim \frac{\epsilon_0}{u^{1+\delta}}. \tag{2.23}$$

2.5 The Einstein-Maxwell equations

We collect here the equations satisfied by the linearized quantities on Σ_*, using the bootstrap assumptions to simplify nonlinear terms[2].

Proposition 2.10. *The following equations hold true on Σ_*:*

1. *The linearized Maxwell equations are given by:*

$$\begin{aligned}
\nabla_3 {}^{(\mathbf{F})}\beta - \nabla \widetilde{{}^{(\mathbf{F})}\rho} + {}^*\nabla^* {}^{(\mathbf{F})}\rho - \frac{\Upsilon}{r} {}^{(\mathbf{F})}\beta &= \left(\frac{2M}{r^2} - \frac{2Q^2}{r^3}\right) {}^{(\mathbf{F})}\beta + \frac{2Q}{r^2}\eta + \Gamma_b \cdot \Gamma_g,\\
\nabla_4 \widetilde{{}^{(\mathbf{F})}\rho} - \operatorname{div} {}^{(\mathbf{F})}\beta &= -\frac{2}{r} \widetilde{{}^{(\mathbf{F})}\rho},\\
\nabla_3 \widetilde{{}^{(\mathbf{F})}\rho} + \operatorname{div} {}^{(\mathbf{F})}\underline{\beta} &= \frac{2\Upsilon}{r} \widetilde{{}^{(\mathbf{F})}\rho} - \frac{Q}{r^2}\check{\underline{\kappa}} + \frac{2Q}{r^3}\check{y} + \Gamma_b \cdot \Gamma_b,\\
\nabla_4 {}^* {}^{(\mathbf{F})}\rho - \operatorname{curl} {}^{(\mathbf{F})}\beta &= -\frac{2}{r} {}^* {}^{(\mathbf{F})}\rho,\\
\nabla_3 {}^* {}^{(\mathbf{F})}\rho - \operatorname{curl} {}^{(\mathbf{F})}\underline{\beta} &= \frac{2\Upsilon}{r} {}^* {}^{(\mathbf{F})}\rho + \Gamma_b \cdot \Gamma_b.
\end{aligned}$$

2. *The linearized null structure equations are given by*

$$\begin{aligned}
\nabla_4 \widetilde{\operatorname{tr}\chi} &= \Gamma_g \cdot \Gamma_g,\\
\nabla_4 \widehat{\chi} + \frac{2}{r}\widehat{\chi} &= -\alpha,\\
\nabla_4 \zeta + \frac{2}{r}\zeta &= -\beta - \frac{Q}{r^2} {}^{(\mathbf{F})}\beta + \Gamma_g \cdot \Gamma_g,\\
\nabla_4 \widetilde{\operatorname{tr}\underline{\chi}} + \frac{1}{r}\widetilde{\operatorname{tr}\underline{\chi}} &= -2\operatorname{div}\zeta + 2\check{\rho} + \Gamma_b \cdot \Gamma_g,\\
\nabla_4 \widehat{\underline{\chi}} + \frac{1}{r}\widehat{\underline{\chi}} &= \frac{\Upsilon}{r}\widehat{\chi} - \nabla \widehat{\otimes} \zeta + \Gamma_b \cdot \Gamma_g,
\end{aligned}$$

[2]We keep in the equations only the nonlinear terms with the worst decay according to the rule (2.22).

and

$$\nabla_3 \widetilde{\mathrm{tr}\,\chi} = 2\,\mathrm{div}\,\eta + 2\check{\rho} - \frac{1}{r}\widetilde{\mathrm{tr}\,\underline{\chi}} + \frac{4}{r}\underline{\check{\omega}} + \frac{2}{r^2}\check{y} + \Gamma_b\cdot\Gamma_b,$$

$$\nabla_3 \widetilde{\mathrm{tr}\,\underline{\chi}} - \frac{2\Upsilon}{r}\widetilde{\mathrm{tr}\,\underline{\chi}} = 2\,\mathrm{div}\,\underline{\xi} + \frac{4\Upsilon}{r}\underline{\check{\omega}} - \left(\frac{2M}{r^2}+\frac{2Q^2}{r^3}\right)\widetilde{\mathrm{tr}\,\underline{\chi}} - \left(\frac{2}{r^2}-\frac{8M}{r^3}+\frac{6Q^2}{r^4}\right)\check{y} + \Gamma_b\cdot\Gamma_b,$$

$$\nabla_3 \underline{\widehat{\chi}} - \frac{2\Upsilon}{r}\underline{\widehat{\chi}} = -\underline{\alpha} - \left(\frac{2M}{r^2}-\frac{2Q^2}{r^3}\right)\underline{\widehat{\chi}} + \nabla\widehat{\otimes}\underline{\xi} + \Gamma_b\cdot\Gamma_b,$$

$$\nabla_3\zeta = -\underline{\beta} - \frac{Q}{r^2}\,{}^{(\mathbf{F})}\underline{\beta} - 2\nabla\underline{\check{\omega}} + \frac{\Upsilon}{r}(\eta+\zeta) + \frac{1}{r}\underline{\xi} + \left(\frac{2M}{r^2}-\frac{2Q^2}{r^3}\right)(\zeta-\eta) + \Gamma_b\cdot\Gamma_b,$$

$$\nabla_3\widehat{\chi} - \frac{\Upsilon}{r}\widehat{\chi} = \nabla\widehat{\otimes}\eta - \frac{1}{r}\underline{\widehat{\chi}} + \left(\frac{2M}{r^2}-\frac{2Q^2}{r^3}\right)\widehat{\chi} + \Gamma_b\cdot\Gamma_b.$$

Also,

$$\begin{aligned}
\mathrm{div}\,\widehat{\chi} &= \frac{1}{r}\zeta - \beta + \frac{Q}{r^2}\,{}^{(\mathbf{F})}\beta + \Gamma_g\cdot\Gamma_g,\\
\mathrm{div}\,\underline{\widehat{\chi}} &= \frac{1}{2}\nabla\underline{\check{\kappa}} + \frac{\Upsilon}{r}\zeta + \underline{\beta} - \frac{Q}{r^2}\,{}^{(\mathbf{F})}\underline{\beta} + \Gamma_b\cdot\Gamma_g,\\
\mathrm{curl}\,\zeta &= {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}},\\
\mathrm{curl}\,\eta &= {}^*\rho + \Gamma_b\cdot\Gamma_g,\\
\mathrm{curl}\,\underline{\xi} &= \Gamma_b\cdot\Gamma_b,
\end{aligned}$$

and

$$\check{K} = -\frac{1}{2r}\widetilde{\mathrm{tr}\,\underline{\chi}} - \check{\rho} + \frac{2Q}{r^2}\widetilde{{}^{(\mathbf{F})}\rho} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}, \tag{2.24}$$

$$\check{\mu} = -\,\mathrm{div}\,\zeta - \check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}. \tag{2.25}$$

3. *The linearized Bianchi identities are given by*

$$\begin{aligned}
\nabla_3\alpha - \frac{\Upsilon}{r}\alpha &= \nabla\widehat{\otimes}\beta + \left(\frac{4M}{r^2}-\frac{4Q^2}{r^3}\right)\alpha + \left(\frac{6M}{r^3}-\frac{6Q^2}{r^4}\right)\widehat{\chi} + \frac{2Q}{r^2}\left(\nabla\widehat{\otimes}\,{}^{(\mathbf{F})}\beta - \frac{Q}{r^2}\widehat{\chi}\right)\\
&\quad + \Gamma_b\cdot(\alpha,\beta,r^{-1}\,{}^{(\mathbf{F})}\beta) + r^{-1}\mathfrak{d}^{\leqslant 1}(\Gamma_g\cdot\Gamma_g),\\
\nabla_4\beta + \frac{4}{r}\beta &= -\,\mathrm{div}\,\alpha + \frac{Q}{r^2}\nabla_4\,{}^{(\mathbf{F})}\beta + \Gamma_g\cdot(\alpha,\beta,r^{-1}\,{}^{(\mathbf{F})}\beta),\\
\nabla_3\beta - \frac{2\Upsilon}{r}\beta &= (\nabla\rho + {}^*\nabla^*\rho) + \left(\frac{2M}{r^2}-\frac{2Q^2}{r^3}\right)\beta - \left(\frac{6M}{r^3}-\frac{6Q^2}{r^4}\right)\eta + \frac{Q}{r^2}\left(\nabla\,{}^{(\mathbf{F})}\rho - {}^*\nabla^*\,{}^{(\mathbf{F})}\rho\right)\\
&\quad + \frac{\Upsilon Q}{r^3}\,{}^{(\mathbf{F})}\beta - \frac{2Q}{r^3}\,{}^{(\mathbf{F})}\underline{\beta} + r^{-1}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g),\\
\nabla_4\check{\rho} + \frac{3}{r}\check{\rho} &= \mathrm{div}\,\beta - \frac{4Q}{r^3}\widetilde{{}^{(\mathbf{F})}\rho} + \frac{Q}{r^2}\,\mathrm{div}\,{}^{(\mathbf{F})}\beta + r^{-1}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g),\\
\nabla_3\check{\rho} - \frac{3\Upsilon}{r}\check{\rho} &= -\,\mathrm{div}\,\underline{\beta} + \left(\frac{3M}{r^3}-\frac{4Q^2}{r^4}\right)\widetilde{\mathrm{tr}\,\underline{\chi}} - \left(\frac{6M}{r^4}-\frac{8Q^2}{r^5}\right)\check{y} - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha} + \frac{4Q\Upsilon}{r^3}\widetilde{{}^{(\mathbf{F})}\rho}\\
&\quad - \frac{Q}{r^2}\,\mathrm{div}\,{}^{(\mathbf{F})}\underline{\beta} + r^{-1}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_b),\\
\nabla_4{}^*\rho + \frac{3}{r}{}^*\rho &= -\,\mathrm{curl}\,\beta - \frac{Q}{r^2}\,\mathrm{curl}\,{}^{(\mathbf{F})}\beta + r^{-1}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g),\\
\nabla_3{}^*\rho - \frac{3\Upsilon}{r}{}^*\rho &= -\,\mathrm{curl}\,\underline{\beta} - \frac{1}{2}\widehat{\chi}\cdot{}^*\underline{\alpha} - \frac{Q}{r^2}\,\mathrm{curl}\,{}^{(\mathbf{F})}\underline{\beta} + r^{-1}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Proof. The proof follows immediately from the Einstein-Maxwell equations (see for example [9], the definition of the linearized quantities, the definition of Γ_g and Γ_b, the fact that $y = e_3(r)$, and the GCM condition $\widetilde{\mathrm{tr}\,\chi} = 0$ on Σ_*. □

2.5.1 Additional elliptic relations on Σ_*

Let ξ be an arbitrary one-form and θ an arbitrary symmetric traceless 2-tensor on S. We recall the following differential operators on the spheres:

- $\nabla\!\!\!/$ denotes the covariant derivative associated to the metric g on S.
- $d\!\!\!/_1$ takes ξ into the pair of functions $(\mathrm{div}\,\xi, \mathrm{curl}\,\xi)$, where

$$\mathrm{div}\,\xi = g^{AB}\nabla\!\!\!/_A\xi_B, \qquad \mathrm{curl}\,\xi = \epsilon^{AB}\nabla\!\!\!/_A\xi_B.$$

- $d\!\!\!/_1^*$ is the formal L^2-adjoint of $d\!\!\!/_1$, and takes any pair of functions (ρ, σ) into the one-form $-\nabla\!\!\!/_A\rho + \epsilon_{AB}\nabla\!\!\!/^B\sigma$.
- $d\!\!\!/_2$ takes θ into the one-form $d\!\!\!/_2\theta = (\mathrm{div}\,\theta)_C = g^{AB}\nabla\!\!\!/_A\theta_{BC}$.
- $d\!\!\!/_2^*$ is the formal L^2-adjoint of $\mathcal{D}\!\!\!/_2$, and takes ξ into the symmetric traceless two tensor

$$(d\!\!\!/_2^*\xi)_{AB} = -\frac{1}{2}\left(\nabla\!\!\!/_B\xi_A + \nabla\!\!\!/_A\xi_B - (\mathrm{div}\,\xi)g_{AB}\right).$$

We collect here some important elliptic relations between the quantities.

Proposition 2.11. *The following identities hold true on Σ_*:*

$$\begin{aligned}
2d\!\!\!/_2^*d\!\!\!/_1^*d\!\!\!/_1d\!\!\!/_2d\!\!\!/_2^*\eta &= -\frac{4}{r}d\!\!\!/_2^*d\!\!\!/_1^*\,\mathrm{div}\,\underline{\beta} + r^{-5}\mathfrak{d}^{\leq 5}\Gamma_g + r^{-4}\mathfrak{d}^{\leq 4}(\Gamma_b\cdot\Gamma_b),\\
2d\!\!\!/_2^*d\!\!\!/_1^*d\!\!\!/_1d\!\!\!/_2d\!\!\!/_2^*\underline{\xi} &= -\frac{4}{r}d\!\!\!/_2^*d\!\!\!/_1^*\,\mathrm{div}\,\underline{\beta} + r^{-5}\mathfrak{d}^{\leq 5}\Gamma_g + r^{-4}\mathfrak{d}^{\leq 6}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Here $r^{-5}\mathfrak{d}^{\leq 5}\Gamma_g$ denote a non specified sum of terms which multiplied by r^5 belong to the set $\mathfrak{d}^{\leq 5}\Gamma_g$ in (2.19).

Proof. The proof follows the same steps as in Proposition 5.23 and Corollary 5.24 in [18]. Observe that, even in the presence of the electromagnetic components, using the bootstrap assumption ${}^{(\mathbf{F})}\underline{\beta} \in \Gamma_b$, following the same steps as in Proposition 5.22 in [18], we can still write

$$\begin{aligned}
2\nabla\underline{\check{\omega}} - \frac{1}{r}\underline{\xi} &= -\nabla_3\zeta - \underline{\beta} + \frac{1}{r}\eta + r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_b,\\
2d\!\!\!/_2d\!\!\!/_2^*\eta &= -\nabla_3\nabla\check{\kappa} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b),\\
2d\!\!\!/_2d\!\!\!/_2^*\underline{\xi} &= -\nabla_3\nabla\underline{\check{\kappa}} - \frac{2}{r}\nabla_3\zeta - \frac{2}{r}\underline{\beta} + r^{-2}\mathfrak{d}^{\leq 1}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

and also

$$\begin{aligned}
\nabla_3\underline{\widehat{\chi}} &= -\underline{\alpha} + r^{-1}\mathfrak{d}^{\leq 1}\Gamma_b + \Gamma_b\cdot\Gamma_b,\\
\nabla_3\check{\rho} &= -\,\mathrm{div}\,\underline{\beta} - \frac{1}{2}\underline{\widehat{\chi}}\cdot\underline{\alpha} + r^{-2}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b),\\
\nabla_3{}^*\rho &= -\,\mathrm{curl}\,\underline{\beta} - \frac{1}{2}\underline{\widehat{\chi}}\cdot{}^*\underline{\alpha} + r^{-2}\Gamma_g + r^{-1}\mathfrak{d}^{\leq 1}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

In view of the above, the proof follows the same steps as in [18]. □

We collect here a general lemma on commutations.

Lemma 2.12. *The following commutation formulas hold true for any tensor f on $S \subset \Sigma_*$:*

$$
\begin{aligned}
[\nabla_3, \nabla] f &= \frac{\Upsilon}{r} \nabla f + \Gamma_b \cdot \nabla_3 f + r^{-1} \Gamma_b \cdot \mathfrak{d}^{\leqslant 1} f, \\
[\nabla_4, \nabla] f &= -\frac{1}{r} \nabla f + r^{-1} \Gamma_g \cdot \mathfrak{d}^{\leqslant 1} f, \\
[\nabla_3, \Delta] f &= \frac{2\Upsilon}{r} \Delta f + r^{-1} \not{\mathfrak{d}}^{\leqslant 1} \Big(\Gamma_b \cdot \nabla_3 f + r^{-1} \Gamma_b \cdot \mathfrak{d} f \Big), \\
[\nabla_4, \Delta] f &= -\frac{2}{r} \Delta f + r^{-1} \not{\mathfrak{d}}^{\leqslant 1} \Big(\Gamma_g \cdot \mathfrak{d} f \Big), \\
[\nabla_\nu, \nabla] f &= \frac{2}{r} \nabla f + \Gamma_b \cdot \nabla_\nu f + r^{-1} \Gamma_b \cdot \mathfrak{d}^{\leqslant 1} f, \\
[\nabla_\nu, \Delta] f &= \frac{4}{r} \Delta f + r^{-1} \not{\mathfrak{d}}^{\leqslant 1} \Big(\Gamma_b \cdot \nabla_\nu f + r^{-1} \Gamma_b \cdot \mathfrak{d} f \Big).
\end{aligned}
$$

Proof. See Lemma 5.1.20 in [18]. □

2.6 Elliptic estimates on Σ_*

We collect here basic Hodge elliptic estimates. For a tensor f on S, we define the following standard weighted Sobolev norms for any integer $k \geqslant 0$

$$
\|f\|_{\mathfrak{h}_k(S)} := \sum_{j=0}^{k} \|\not{\mathfrak{d}}^j f\|_{L^2(S)},
$$

where $\not{\mathfrak{d}}$ denotes derivatives tangential to the sphere S.

Lemma 2.13. *For any sphere $S = S(u) \subset \Sigma_*$ we have:*

1. *If f is a 1-form*

$$
\begin{aligned}
\|f\|_{\mathfrak{h}_{k+1}(S)} &\lesssim r \|\not{d}_1 f\|_{\mathfrak{h}_k(S)}, \\
\|f\|_{\mathfrak{h}_{k+1}(S)} &\lesssim r \|\not{d}_2^* f\|_{\mathfrak{h}_k(S)} + r^2 \big|(\not{d}_1 f)_{\ell=1}\big|, \\
\big|(\not{d}_1 f)_{\ell=1}\big| &\lesssim r^{-1} \|\not{d}_1 f\|_{L^2(S)}.
\end{aligned}
$$

   Moreover,

$$
||\mathfrak{d}_*^k f||_{\mathfrak{h}_3(S)} \lesssim r^4 ||\not{d}_2^* \not{d}_1^* \not{d}_1 \mathfrak{d}_*^k f||_{L^\infty(S)} + r^2 \left| \Big(\not{d}_1 \nabla_\nu^{\leqslant k} f \Big)_{\ell=1} \right|. \tag{2.26}
$$

2. *If f is a symmetric traceless 2-tensor*

$$
\|v\|_{\mathfrak{h}_{k+1}(S)} \lesssim r \|\not{d}_2 v\|_{\mathfrak{h}_k(S)}.
$$

3. *If $(h, {}^*h)$ is a pair of scalars*

$$
\begin{aligned}
\|(h - \overline{h}, {}^*h - \overline{{}^*h})\|_{\mathfrak{h}_{k+1}(S)} &\lesssim r \|\not{d}_1^* (h, {}^*h)\|_{\mathfrak{h}_k(S)}, \\
\|(h - \overline{h}, {}^*h - \overline{{}^*h})\|_{\mathfrak{h}_{k+2}(S)} &\lesssim r^2 \|\not{d}_2^* \not{d}_1^* (h, {}^*h)\|_{\mathfrak{h}_k(S)} + r^3 \big|(\Delta h)_{\ell=1}\big| + r^3 \big|(\Delta {}^*h)_{\ell=1}\big|.
\end{aligned}
$$

Proof. See Lemmas 5.27 and 5.28 and Corollary 5.29 in [18]. To prove (2.26), we make use of the Hodge estimate to write

$$\begin{aligned} ||\mathfrak{d}_*^k f||_{\mathfrak{h}_3(S)} &\lesssim r||\not{d}_1\mathfrak{d}_*^k f||_{\mathfrak{h}_2(S)} \lesssim r^2||\not{d}_1^*\not{d}_1\mathfrak{d}_*^k f||_{\mathfrak{h}_1(S)} \\ &\lesssim r^3||\not{d}_2^*\not{d}_1^*\not{d}_1\mathfrak{d}_*^k f||_{L^2(S)} + r^2\left|\left(\not{d}_1\mathfrak{d}_*^{\leqslant k} f\right)_{\ell=1}\right| \\ &\lesssim r^4||\not{d}_2^*\not{d}_1^*\not{d}_1\mathfrak{d}_*^k f||_{L^\infty(S)} + r^2\left|\left(\not{d}_1\mathfrak{d}_*^{\leqslant k} f\right)_{\ell=1}\right|. \end{aligned}$$

Finally, the $\mathfrak{d}_*^{\leqslant k}$ in the second term on right hand side can be reduced to only contains $\nabla_\nu^{\leqslant k}$ since $\not{\mathfrak{d}}^{\leqslant k} J^{(p)}$ is a basis of $\ell = 1$ modes. ☐

Lemma 2.14. *We have the following estimate for ϕ on Σ_*:*

$$\|\not{\mathfrak{d}}^{\leqslant N}\phi\|_{L^\infty(S_*)} \lesssim \frac{\epsilon}{ru^{\frac{1}{2}+\delta}}.$$

The functions $J^{(p)}$ verify the following properties on Σ_*:

1. *We have*

$$\int_S J^{(p)} = O\left(\frac{\epsilon r}{u^{\frac{1}{2}+\delta}}\right), \qquad \int_S J^{(p)}J^{(q)} = \frac{4\pi}{3}r^2\delta_{pq} + O\left(\frac{\epsilon r}{u^{\frac{1}{2}+\delta}}\right).$$

2. *We have*

$$\left(\triangle + \frac{2}{r^2}\right) J^{(p)} \in r^{-1}\not{\mathfrak{d}}^{\leqslant 1}\Gamma_g, \qquad \not{d}_2^*\not{d}_1^* J^{(p)} \in r^{-1}\not{\mathfrak{d}}^{\leqslant 3}\Gamma_g.$$

Proof. By the Gauss equation $\check{K} \in r^{-1}\Gamma_g$ and so by the bootstrap assumptions

$$\left|K - \frac{1}{r^2}\right|_{L^\infty(S)} \lesssim \frac{\epsilon}{r^3u^{\frac{1}{2}+\delta}}.$$

As a consequence, by Klainerman-Szeftel's effective uniformization theorem in [16], the induced metric g on any sphere S is conformal to $\gamma_{\mathbb{S}^2}$ as in (2.3), i.e.

$$g = r^2e^{2\phi}\gamma_{\mathbb{S}^2}, \tag{2.27}$$

and moreover $\phi \in r\Gamma_g$. The identities above then follow from the relations between $J^{(p)}$ and ϕ, for example

$$\left(\triangle + \frac{2}{r^2}\right) J^{(p)} = \frac{2}{r^2}(1 - e^{-2\phi})J^{(p)}.$$

See also Lemmas 5.35, 5.36 and 5.37 in [18]. ☐

Corollary 2.15. *On a fixed sphere $S = S(u) \subset \Sigma_*$, we have for any pair of scalars $(h, {}^*h)$:*

$$\|(h, {}^*h)\|_{\mathfrak{h}_{k+2}(S)} \lesssim r^2\|\not{d}_2^*\,\not{d}_1^*(h, {}^*h)\|_{\mathfrak{h}_k(S)} + r\left|(h)_{\ell=1}\right| + r\left|({}^*h)_{\ell=1}\right| + r|\overline{h}| + r|\overline{{}^*h}|.$$

Proof. See Corollary 5.38 in [18]. ☐

Corollary 2.16. *The scalar functions $\underline{C}_0$, $\underline{C}_p$, M_0, M_p in* (2.16) *verify on Σ_**

$$\underline{C}_0 \in \Gamma_g, \qquad \underline{C}_p \in \Gamma_g, \qquad M_0 \in r^{-1}\Gamma_g, \qquad M_p \in r^{-1}\Gamma_g. \tag{2.28}$$

Proof. See Corollary 5.39 in [18]. ☐

2.7 Renormalized equations on Σ_*

Here we define and derive a set of renormalized quantities, in addition to $(\boldsymbol{C},\boldsymbol{J})$ defined in (2.13), which exhibit an improvement in the transport equation along the tangent vector to Σ_*, ν, at the level of their $\ell=1$ mode.

Definition 2.17. *We define the following quantities:*

$$
\begin{aligned}
\underline{\boldsymbol{C}} &:= \operatorname{div}\underline{\beta} - \frac{Q}{r^2}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta} - \frac{2Q}{r^3}\widecheck{{}^{(\mathbf{F})}\rho}, && (2.29)\\
(\boldsymbol{Z},\boldsymbol{Y}) &:= \not{d}_1\zeta + \frac{2Q}{r^2}\big(\widecheck{{}^{(\mathbf{F})}\rho}, {}^*{}^{(\mathbf{F})}\rho\big), && (2.30)\\
\widehat{\rho} &:= \check{\rho} - \frac{2Q}{r^2}\widecheck{{}^{(\mathbf{F})}\rho} - \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}, && (2.31)\\
{}^*\widehat{\rho} &:= {}^*\rho + \frac{2Q}{r^2}{}^*{}^{(\mathbf{F})}\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}. && (2.32)
\end{aligned}
$$

Remark 2.18. *The definition of $\widehat{\rho}$ is motivated by the Gauss equation* (2.24)*. In fact, using that*

$$
\check{\mu} = -\operatorname{div}\zeta - \check{\rho} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}, \qquad \operatorname{curl}\zeta = {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}},
$$

one can see that the renormalized quantities satisfy

$$
\check{K} = -\frac{1}{2r}\widecheck{\operatorname{tr}\underline{\chi}} - \widehat{\rho}, \qquad \boldsymbol{Z} = -\widehat{\rho} - \check{\mu}, \qquad \boldsymbol{Y} = \widehat{{}^*\rho}.
$$

We now express the transport equations of each renormalized quantity in terms of the others.

Proposition 2.19. *We have the following transport equations along Σ_*:*

$$
\begin{aligned}
\nabla_\nu\widehat{\rho} &= \frac{6}{r}\widehat{\rho} - \underline{\boldsymbol{C}} - (1+O(r^{-1}))\boldsymbol{C} + r^{-3}\mathfrak{d}^{\leqslant 1}\Gamma_b + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b\cdot\Gamma_b),\\
\nabla_\nu\boldsymbol{C} &= \Big(\frac{8}{r}+O(r^{-2})\Big)\boldsymbol{C} - \frac{4Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta} + \triangle\widehat{\rho} + \frac{2Q}{r^2}\Big(\triangle + \frac{2}{r^2}\Big)\widecheck{{}^{(\mathbf{F})}\rho} + O(1+O(r^{-1}))\operatorname{div}\operatorname{div}\alpha\\
&\quad - \Big(\frac{6M}{r^3} - \frac{4Q^2}{r^4}\Big)\operatorname{div}\eta + r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g),\\
\nabla_\nu\boldsymbol{J} &= \Big(\frac{8}{r}+O(r^{-2})\Big)\boldsymbol{J} - \triangle{}^*\widehat{\rho} - \frac{6M}{r^3}{}^*\widehat{\rho} + \frac{2Q}{r^2}\Big(\triangle + \frac{2}{r^2}\Big){}^*{}^{(\mathbf{F})}\rho + O(1+O(r^{-1}))\operatorname{curl}\operatorname{div}\alpha\\
&\quad + r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

Proof. See Appendix A. □

Remark 2.20. *Notice that the transport equation for $\widehat{\rho}$ is not sensitive to the lower order term in $\widecheck{{}^{(\mathbf{F})}\rho}$ (as they behave better than the $r^{-3}\Gamma_b$ terms on the right hand side), while the non linear term $\widehat{\chi}\cdot\widehat{\underline{\chi}}$ is important to cancel a nonfavorable nonlinear term in its transport equation.*

While in the equation for $\boldsymbol{J}$ the term curl ${}^{(\mathbf{F})}\underline{\beta}$ *gets cancelled, in the equation for $\boldsymbol{C}$ the term* div ${}^{(\mathbf{F})}\underline{\beta}$ *persists. For this reason we need to impose the vanishing of $\boldsymbol{C}$ on Σ_* in order to avoid to transport $\boldsymbol{C}$ using its equation. As a consequence, the equation for $\nabla_\nu\boldsymbol{C}$ is not used as a transport equation but rather as an algebraic relation, so we only need slower decay and we can simplify it to*

$$
\begin{aligned}
\nabla_\nu\boldsymbol{C} = {} & \frac{8}{r}\boldsymbol{C} + \Delta\widehat{\rho} - \frac{6M}{r^3}\operatorname{div}\eta - \frac{4Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta} + O(1+O(r^{-1}))\operatorname{div}\operatorname{div}\alpha\\
&+ r^{-4}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}
$$

We will now deduce the transport equations for the $\ell = 1$ modes of the renormalized quantities. We first recall the following lemma, which appeared as Corollary 5.32 in [18].

Lemma 2.21. *For any scalar function h on Σ_*, we have*

$$\nu\left(\int_S h\right) = \int_S \nu(h) - \frac{4}{r}\int_S h + r^3\Gamma_b \cdot \nu(h) + r^2\Gamma_b \cdot h \tag{2.33}$$

and $\nu(r) = \frac{rz}{2}\overline{z^{-1}(\underline{\kappa} + b_*\kappa)} = -2 + r\Gamma_b$. In particular, we have

$$\nu(\overline{h}) = \overline{\nu(h)} + r\Gamma_b\nu(h) + \Gamma_b h,$$

where $\overline{h}$ and $\overline{\nu(h)}$ denote respectively the average of h and $\nu(h)$ on the spheres of Σ_*.

Proof. See Corollary 5.32 in [18]. □

Corollary 2.22. *We have the following transport equations along Σ_*, for $p = 0, +, -$:*

$$\nu\left(\int_S \widehat{\rho} J^{(p)}\right) = O(r^{-1})\int_S \widehat{\rho} J^{(p)} - \int_S \underline{\boldsymbol{C}} J^{(p)} + r^{-1}\mathfrak{d}^{\leqslant 1}\Gamma_b + r\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_b),$$
$$\nu\left(\int_S \boldsymbol{J} J^{(p)}\right) = O(r^{-1})\int_S \boldsymbol{J} J^{(p)} + O(r^{-2})\int_S {}^*\widehat{\rho} J^{(p)} + r^{-3}\mathfrak{d}^{\leqslant 1}\Gamma_g + \mathfrak{d}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g).$$

We also have the following elliptic identity along Σ_*, for $p = 0, +, -$:

$$\frac{2}{r^2}\int_S \widehat{\rho} J^{(p)} + \frac{6M}{r^3}\int_S \mathrm{div}\,\eta J^{(p)} + \frac{4Q}{r^3}\int_S \mathrm{div}\ {}^{(\mathbf{F})}\underline{\beta} J^{(p)} = r^{-2}\mathfrak{d}^{\leqslant 1}\Gamma_g + \mathfrak{d}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g). \tag{2.34}$$

Proof. Since $\nu(J^{(p)}) = 0$, we have in view of Lemma 2.21 for any scalar function h on Σ_* and any $S \subset \Sigma_*$

$$\nu\left(\int_S hJ^{(p)}\right) = \int_S \nu(h)J^{(p)} - \frac{4}{r}\int_S hJ^{(p)} + r^3\Gamma_b \cdot \nu(h) + r^2\Gamma_b \cdot h, \tag{2.35}$$

where we also used $J^{(p)} = O(1)$, and where we recall that the notation $O(r^a)$, for $a \in \mathbb{R}$, denotes an explicit function of r which is bounded by r^a as $r \to +\infty$.

From Proposition 2.19 we deduce, using the GCM condition (2.17), that

$$\nu\left(\int_S \widehat{\rho} J^{(p)}\right) = O(r^{-1})\int_S \widehat{\rho} J^{(p)} - \int_S \underline{\boldsymbol{C}} J^{(p)} + r^{-1}\mathfrak{d}^{\leqslant 1}\Gamma_b + r\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_b),$$

as stated.

Using Proposition 2.19 and (2.35), and noticing that the terms $O(r^a)$ only depend on r and are thus constant on S, we infer

$$\begin{aligned}\nu\left(\int_S \boldsymbol{C} J^{(p)}\right) = {} & O(r^{-1})\int_S \boldsymbol{C} J^{(p)} + \int_S \Delta\widehat{\rho} J^{(p)} - \frac{6M}{r^3}\int_S \mathrm{div}\,\eta J^{(p)} - \frac{4Q}{r^3}\int_S \mathrm{div}\ {}^{(\mathbf{F})}\underline{\beta} J^{(p)} \\ & + O(1 + O(r^{-1}))\int_S \mathrm{div}\,\mathrm{div}\,\alpha J^{(p)} + r^{-2}\mathfrak{d}^{\leqslant 2}\Gamma_g + \mathfrak{d}^{\leqslant 2}(\Gamma_b \cdot \Gamma_g).\end{aligned}$$

Together with the GCM condition (2.17) and integrating by parts to use Lemma 2.14, we deduce

$$\begin{aligned} 0 = {} & -\frac{2}{r^2}\int_S \widehat{\rho} J^{(p)} - \frac{6M}{r^3}\int_S \mathrm{div}\,\eta J^{(p)} - \frac{4Q}{r^3}\int_S \mathrm{div}\ {}^{(\mathbf{F})}\underline{\beta} J^{(p)} \\ & + r\left(\left|\left(\Delta + \frac{2}{r^2}\right) J^{(p)}\right| + \left|\not{d}_2^*\not{d}_1^* J^{(p)}\right|\right)\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 2}\Gamma_g + \mathfrak{d}^{\leqslant 2}(\Gamma_b \cdot \Gamma_g) \\ = {} & -\frac{2}{r^2}\int_S \widehat{\rho} J^{(p)} - \frac{6M}{r^3}\int_S \mathrm{div}\,\eta J^{(p)} - \frac{4Q}{r^3}\int_S \mathrm{div}\ {}^{(\mathbf{F})}\underline{\beta} J^{(p)} + r^{-2}\mathfrak{d}^{\leqslant 2}\Gamma_g + \mathfrak{d}^{\leqslant 2}(\Gamma_b \cdot \Gamma_g),\end{aligned}$$

where by $\not{d}_1^* J^{(p)}$, we mean $\not{d}_1^*(J^{(p)},0)$. Similarly, using Proposition 2.19 and noticing that ${}^*\,{}^{(\mathbf{F})}\rho$ appears in the combination $\left(\triangle\, {}^*\,{}^{(\mathbf{F})}\rho + \frac{2}{r^2}\,{}^*\,{}^{(\mathbf{F})}\rho\right)$, we deduce the equation for $\boldsymbol{J}$ as

$$\begin{aligned}\nabla_\nu\left(\int_S \boldsymbol{J}J^{(p)}\right) &= O(r^{-1})\int_S \boldsymbol{J}J^{(p)} - O(r^{-2})\int_S {}^*\widehat{\rho}J^{(p)} + r\left(\left|\left(\Delta+\frac{2}{r^2}\right)J^{(p)}\right| + \left|\not{d}_2^*\not{d}_1^* J^{(p)}\right|\right)\Gamma_g\\ &\quad + O(r^{-3})\mathfrak{d}^{\leqslant 2}\Gamma_g + \mathfrak{d}^{\leqslant 2}(\Gamma_b\cdot\Gamma_g)\\ &= O(r^{-1})\int_S \boldsymbol{J}J^{(p)} - O(r^{-2})\int_S {}^*\widehat{\rho}J^{(p)} + r^{-3}\mathfrak{d}^{\leqslant 2}\Gamma_g + \mathfrak{d}^{\leqslant 2}(\Gamma_b\cdot\Gamma_g),\end{aligned}$$

as stated. □

The above transport equations in ν along Σ_* will be used to obtain estimates according to the following lemma.

Lemma 2.23. *Let f and h be two scalar functions on Σ_* satisfying*

$$\nu(f) = O(r^{-1})f + h.$$

Then, we have for any integer m

$$r^m\|f\|_{L^\infty(S(u))} \lesssim r_*^m\|f\|_{L^\infty(S_*)} + \int_u^{u_*}\|r^m h\|_{L^\infty(S(u'))}du'. \tag{2.36}$$

Proof. See Lemma 5.33 and Corollary 5.34 in [18]. □

3 Consequences of the bootstrap assumptions and the dominance condition

In this section we obtain control of the $\ell=0$ and $\ell=1$ modes of most scalar quantities and 1-forms on Σ_* solely[3] as a consequence of the bootstrap assumptions, the GCM conditions (2.14),(2.15),(2.17) and the dominance condition (2.23).

3.1 Estimates for $\ell=0$ modes on Σ_*

In this section, we control the average (i.e. the $\ell=0$ mode) of the scalar quantities involved, i.e.

$$\widecheck{\operatorname{tr}\underline{\chi}},\quad \check{\rho},\quad {}^*\rho,\quad \widecheck{{}^{(\mathbf{F})}\rho},\quad {}^{(\mathbf{F})*}\rho,\quad \check{\mu}.$$

As a consequence of definition of charge, the Maxwell equations and Stokes theorem, recall Remark 2.2, we have on Σ_*

$$\overline{\widecheck{{}^{(\mathbf{F})}\rho}} = \overline{{}^{(\mathbf{F})}\rho} - \frac{Q}{r^2} = 0, \qquad \overline{{}^*\,{}^{(\mathbf{F})}\rho} = 0, \tag{3.1}$$

so there is no need to control the $\ell=0$ mode of these two quantities any further.

In what follows, we will make use of the control of the Hawking mass function defined in (2.10), in terms of the constant auxiliary mass defined in (2.4). Notice that, by definition,

$$\frac{2m_H}{r} = 1 + \frac{1}{16\pi}\int_S \operatorname{tr}\chi\operatorname{tr}\underline{\chi} = 1+\frac{1}{16\pi}\int_S\frac{2}{r}\left(-\frac{2\Upsilon}{r}+\widecheck{\operatorname{tr}\underline{\chi}}\right) = \frac{2M}{r}-\frac{Q^2}{r^2}+2r\overline{\widecheck{\operatorname{tr}\underline{\chi}}} = \frac{2M}{r}-\frac{Q^2}{r^2}+r\Gamma_g,$$

and so $M = m_H + \frac{Q}{2r} + r^2\Gamma_g$.

[3] In particular, here we do not use the condition (2.18) of $(\operatorname{div}\underline{\xi})_{\ell=1}=0$.

Lemma 3.1. *We have on Σ_*:*

$$|m_H - m| + \left| m_H + \frac{Q^2}{2r} - M \right| \lesssim \frac{\epsilon_0}{u^{1+2\delta}}. \tag{3.2}$$

Proof. From the null structure equations, we deduce on Σ_*,

$$\begin{aligned}
\nu(\operatorname{tr}\chi \operatorname{tr}\underline{\chi}) &= e_3(\operatorname{tr}\chi \operatorname{tr}\underline{\chi}) + b_* e_4(\operatorname{tr}\chi \operatorname{tr}\underline{\chi}) \\
&= -\operatorname{tr}\chi \operatorname{tr}\underline{\chi}(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi) + 2(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi)\rho + 2\operatorname{tr}\chi \operatorname{div}\underline{\xi} + 2\operatorname{tr}\underline{\chi} \operatorname{div}\eta \\
&\quad + \operatorname{tr}\chi \left(2\underline{\xi}\cdot(\eta - 3\zeta) - |\widehat{\underline{\chi}}|^2\right) + \operatorname{tr}\underline{\chi}\left(2|\eta|^2 - \widehat{\underline{\chi}}\cdot\widehat{\chi}\right) - 2\operatorname{tr}\chi\, {}^{(\mathbf{F})}\underline{\beta}\cdot {}^{(\mathbf{F})}\underline{\beta} \\
&\quad + b_*\left(-2\operatorname{tr}\chi \operatorname{div}\zeta + \operatorname{tr}\chi\left(2|\zeta|^2 - \widehat{\chi}\cdot\widehat{\underline{\chi}}\right) - \operatorname{tr}\underline{\chi}|\widehat{\chi}|^2 - 2\operatorname{tr}\underline{\chi}\, {}^{(\mathbf{F})}\beta\cdot {}^{(\mathbf{F})}\beta\right),
\end{aligned}$$

which we rewrite

$$\begin{aligned}
\nu(\operatorname{tr}\chi \operatorname{tr}\underline{\chi}) + \operatorname{tr}\chi \operatorname{tr}\underline{\chi}(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi) &= 2(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi)\rho + \frac{4}{r}\operatorname{div}\underline{\xi} - \frac{4\Upsilon}{r}\operatorname{div}\eta \\
&\quad - \frac{4(\Upsilon - 2)}{r}\operatorname{div}\zeta + r^{-1}\slashed{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Applying Lemma 2.21, we infer

$$\begin{aligned}
\nu\left(\int_S \operatorname{tr}\chi \operatorname{tr}\underline{\chi}\right) &= z\int_S \frac{1}{z}\Big(\nu(\operatorname{tr}\chi \operatorname{tr}\underline{\chi}) + (\underline{\kappa} + b_*\kappa)\operatorname{tr}\chi \operatorname{tr}\underline{\chi}\Big) \\
&= z\int_S \frac{1}{z}\left(2(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi)\rho + \frac{4}{r}\operatorname{div}\underline{\xi} - \frac{4\Upsilon}{r}\operatorname{div}\eta - \frac{4(\Upsilon-2)}{r}\operatorname{div}\zeta + r^{-1}\slashed{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b)\right).
\end{aligned}$$

Integrating by parts the divergences, we deduce

$$\nu\left(\int_S \operatorname{tr}\chi \operatorname{tr}\underline{\chi}\right) = 2z\int_S \frac{1}{z}(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi)\rho + r\slashed{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

We infer

$$\nu\left(\frac{2m_H}{r}\right) = \frac{z}{8\pi}\int_S \frac{1}{z}(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi)\rho + r\slashed{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

On the other hand, using that $\nu(r) = \frac{rz}{2}\overline{z^{-1}(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi)}$, we have

$$\nu\left(\frac{2m_H}{r}\right) = \frac{2\nu(m_H)}{r} - \frac{2m_H}{r^2}\frac{z}{8\pi r}\int_S \frac{1}{z}(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi),$$

which yields

$$\nu(m_H) = \frac{rz}{16\pi}\int_S \frac{1}{z}(\operatorname{tr}\underline{\chi} + b_* \operatorname{tr}\chi)\left(\rho + \frac{2m_H}{r^3}\right) + r^2\slashed{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Observe that

$$\check{\rho} = \rho + \frac{2M}{r^3} - \frac{2Q^2}{r^4} = \rho + \frac{2m_H}{r^3} - \frac{Q^2}{r^4} + r^{-1}\Gamma_g,$$

and using the dominance condition to bound $\frac{1}{r^4} \lesssim \frac{\epsilon_0}{r^3 u^{1+\delta}} \lesssim r^{-1}\Gamma_g$, we deduce $\rho + \frac{2m_H}{r^3} \in r^{-1}\Gamma_g$. Plugging in the above,

$$\nu(m_H) = -\frac{1}{4\pi}\int_S \left(\rho + \frac{2m_H}{r^3}\right) + r^2\slashed{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

In order to deduce the decay for the average of ρ, recall the Gauss equation

$$K = -\rho + {}^{(\mathbf{F})}\rho^2 + {}^{(\mathbf{F})}{}^*\rho^2 - \frac{1}{4}\operatorname{tr}\chi \operatorname{tr}\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}.$$

Integrating on S, and using the definition of the Hawking mass m_H, we obtain

$$\int_S K = -\int_S \rho + \int_S {}^{(\mathbf{F})}\rho^2 + \int_S {}^{(\mathbf{F})}{}^*\rho^2 - 4\pi\left(\frac{2m_H}{r} - 1\right) + \frac{1}{2}\int_S \widehat{\chi}\cdot\widehat{\underline{\chi}}.$$

Since from Gauss–Bonnet we have $\int_S K = 4\pi$, we infer

$$\int_S \rho = -\frac{8\pi m_H}{r} + \frac{1}{2}\int_S \widehat{\chi}\cdot\widehat{\underline{\chi}} + \int_S {}^{(\mathbf{F})}\rho^2 + \int_S {}^{(\mathbf{F})}{}^*\rho^2,$$

and hence,

$$\overline{\rho} = -\frac{2m_H}{r^3} + \overline{{}^{(\mathbf{F})}\rho^2} + \Gamma_b\cdot\Gamma_g = -\frac{2m_H}{r^3} + \frac{Q^2}{r^4} + \Gamma_b\cdot\Gamma_g, \tag{3.3}$$

where we used that $\overline{{}^{(\mathbf{F})}\rho^2} = \overline{{}^{(\mathbf{F})}\rho}^2 + \Gamma_g\cdot\Gamma_g$. This gives

$$\nu(m_H) = -\frac{Q^2}{r^2} + r^2\Gamma_b\cdot\Gamma_g + r^2\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).$$

Using the bootstrap assumptions (2.21) and the dominance condition (2.23), we deduce

$$|\nu(m_H - m)| \lesssim \frac{\epsilon_0}{u^{2+2\delta}} + \frac{\epsilon^2}{ru^{\frac{3}{2}+2\delta}} + \frac{\epsilon^2}{u^{2+2\delta}} \lesssim \frac{\epsilon_0}{u^{2+2\delta}}.$$

Also, recall that we have $m = m_H$ on S_*. Integrating on Σ_* from S_*, we deduce the stated inequality for $|m_H - m|$ in (3.2). Similarly, using that $\nu(r) = -2 + r\Gamma_b$, we have

$$\begin{aligned}
\nu\left(m_H + \frac{Q}{2r} - M\right) &= -\frac{Q^2}{r^2} - \frac{Q}{2r^2}\nu(r) + r^2\Gamma_b\cdot\Gamma_g + r^2\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b)\\
&= r^{-1}\Gamma_b + r^2\Gamma_b\cdot\Gamma_g + r^2\not{\mathfrak{d}}^{\leq 1}(\Gamma_b\cdot\Gamma_b).
\end{aligned}$$

Using the bootstrap assumptions (2.21) and the dominance condition (2.23), we deduce

$$\left|\nu\left(m_H + \frac{Q}{2r} - M\right)\right| \lesssim \frac{\epsilon_0}{u^{2+2\delta}}.$$

Also, recall that we have $M = m_H + \frac{Q}{2r}$ on S_*. Integrating on Σ_* from S_*, we deduce the stated inequality for $\left|m_H + \frac{Q^2}{2r} - M\right|$ in (3.2). $\square$

Proposition 3.2. *The following estimates hold true on Σ_*:*

$$r\left|\check{\overline{\rho}}\right| + r\left|\overline{{}^*\rho}\right| + r\left|\check{\overline{\mu}}\right| + \left|\widecheck{\overline{\operatorname{tr}\underline{\chi}}}\right| \lesssim \frac{\epsilon_0}{r^2u^{1+2\delta}}. \tag{3.4}$$

Proof. From (3.3) and (3.2), we deduce

$$\begin{aligned}
\left|\check{\overline{\rho}}\right| = \left|\overline{\rho} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}\right| &\leq \left|\overline{\rho} + \frac{2m_H}{r^3} - \frac{Q}{r^4}\right| + \left|2\frac{(M - m_H - \frac{Q}{2r})}{r^3}\right|\\
&\lesssim \Gamma_b\cdot\Gamma_g + \frac{\epsilon_0}{r^3u^{1+2\delta}} \lesssim \frac{\epsilon_0}{r^3u^{1+2\delta}},
\end{aligned}$$

where we used the bootstrap assumptions (2.21). By definition (2.11) of μ, this also implies the same bound for $\overline{\widetilde{\mu}}$. Next, taking the average of $\operatorname{curl}\zeta = {}^*\rho - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}$, we infer from the bootstrap assumptions (2.21)

$$|\overline{{}^*\rho}| \lesssim |\overline{\widehat{\chi}\wedge\widehat{\underline{\chi}}}| \lesssim |\Gamma_b\cdot\Gamma_g| \lesssim \frac{\epsilon_0}{r^3 u^{\frac{3}{2}+\delta}}. \tag{3.5}$$

Next, from the definition (2.10) and the GCM condition for $\operatorname{tr}\chi$ on Σ_*, we have

$$\overline{\operatorname{tr}\underline{\chi}} = -\frac{2}{r}\left(1-\frac{2m_H}{r}\right) = -\frac{2\Upsilon}{r} + \frac{4}{r^2}\left(m_H + \frac{Q^2}{2r} - M\right).$$

Together with (3.2), we deduce

$$\left|\overline{\operatorname{tr}\underline{\chi}} + \frac{2\Upsilon}{r}\right| \lesssim \frac{\epsilon_0}{r^2 u^{1+2\delta}}.$$

This concludes the proof of Proposition 3.2. $\square$

3.2 Estimates for the $\ell = 1$ modes on Σ_*

Recall that on S_* we have the following, see (2.8), (2.14), (2.15):

$$\operatorname{tr}\chi = \frac{2}{r}, \qquad \operatorname{tr}\underline{\chi} = -\frac{2\Upsilon}{r}, \qquad \boldsymbol{C}_{\ell=1} = 0, \qquad \boldsymbol{J}_{\ell=1,\pm} = 0, \qquad \boldsymbol{J}_{\ell=1,0} = \frac{2aM}{r^5}.$$

and on Σ_* we have

$$\operatorname{tr}\chi = \frac{2}{r}, \qquad \boldsymbol{C}_{\ell=1} = 0.$$

Lemma 3.3. *The following estimates hold true on Σ_*:*

$$|\check{K}_{\ell=1}| + \left|\widehat{\rho}_{\ell=1} + \frac{1}{2r}(\widetilde{\operatorname{tr}\underline{\chi}})_{\ell=1}\right| \lesssim \frac{\epsilon_0}{r^4 u^{1+2\delta}}.$$

In particular, on S_* we have

$$|\widehat{\rho}_{\ell=1}| \lesssim \frac{\epsilon_0}{r^4 u^{1+2\delta}}.$$

Proof. Recall that by Lemma 2.14 $\phi \in r\Gamma_g$. By Klainerman-Szeftel's effective uniformization theorem in [16], the Gauss curvature satisfies

$$K = -\Delta\phi + \frac{e^{-2\phi}}{r^2}.$$

Hence, we have

$$K - \frac{1}{r^2} = -\left(\Delta + \frac{2}{r^2}\right)\phi + \frac{e^{-2\phi} - 1 + 2\phi}{r^2} = -\left(\Delta + \frac{2}{r^2}\right)\phi + \Gamma_g\cdot\Gamma_g.$$

Integrating by parts, we obtain from Lemma 2.14

$$\begin{aligned}\int_{S_*}\left(K - \frac{1}{r^2}\right)J^{(p)} &= -\int_{S_*}\left(\Delta + \frac{2}{r^2}\right)\phi J^{(p)} + r^2\Gamma_g\cdot\Gamma_g \\ &= -\int_{S_*}\phi\left(\Delta + \frac{2}{r^2}\right)J^{(p)} + r^2\Gamma_g\cdot\Gamma_g\end{aligned}$$

$$= r^2 \not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g).$$

Using the bootstrap assumptions (2.21), we infer

$$r^2(\check{K})_{\ell=1} \lesssim \frac{\epsilon^2}{r^2 u^{1+2\delta}} \lesssim \frac{\epsilon_0}{r^2 u^{1+2\delta}}.$$

Using the Gauss equation $\check{K} = -\frac{1}{2r}\widetilde{\mathrm{tr}\,\underline{\chi}} - \widehat{\rho}$ on Σ_* , we deduce

$$\left|\widehat{\rho}_{\ell=1} + \frac{1}{2r}\underline{\check{\kappa}}_{\ell=1}\right| \lesssim \frac{\epsilon_0}{r^4 u^{1+2\delta}}, \tag{3.6}$$

as stated. $\square$

Proposition 3.4. *In addition to $\boldsymbol{C}_{\ell=1} = 0$, the following estimates hold true on Σ_*:*

$$r^2 \left|\boldsymbol{Z}_{\ell=1}\right| + r\left|\widehat{\rho}_{\ell=1}\right| + r\left|\check{\mu}_{\ell=1}\right| + \left|(\widetilde{\mathrm{tr}\,\underline{\chi}})_{\ell=1}\right| \lesssim \frac{\epsilon_0}{r^2 u^{1+2\delta}}.$$

Proof. We make the following local bootstrap assumption on Σ_*:

$$\left|(\widetilde{\mathrm{tr}\,\underline{\chi}})_{\ell=1}\right| \leqslant \frac{\epsilon}{r^2 u^{1+2\delta}}. \tag{3.7}$$

Notice that this assumption is stronger than the one implied by the bootstrap assumptions in (2.21) and will be recovered in the process of the proof (in Step 4).

Step 1: Estimate for $\boldsymbol{Z}_{\ell=1}$. Differentiating with respect to div and projecting on the $\ell = 1$ modes the Codazzi equation for $\widehat{\chi}$, we infer

$$(\mathrm{div}\, \not{d}_2 \widehat{\chi})_{\ell=1} = \frac{1}{r}(\mathrm{div}\,\zeta)_{\ell=1} - (\mathrm{div}\,\beta)_{\ell=1} + \frac{Q}{r^2}(\mathrm{div}\ {}^{(\mathbf{F})}\beta)_{\ell=1} + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g).$$

Recalling the definitions (2.13) of $\boldsymbol{C}$ and (2.30) of $\boldsymbol{Z}$ we deduce

$$(\mathrm{div}\, \not{d}_2 \widehat{\chi})_{\ell=1} = \frac{1}{r}\boldsymbol{Z}_{\ell=1} - \boldsymbol{C}_{\ell=1} + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g).$$

From Lemma 2.14

$$(\not{d}_1 \not{d}_2 \widehat{\chi})_{\ell=1,p} = \frac{1}{|S|}\int_S \not{d}_1 \not{d}_2 \widehat{\chi} J^{(p)} = \frac{1}{|S|}\int_S \widehat{\chi} \cdot \not{d}_2^* \not{d}_1^* J^{(p)} = r^{-1}\Gamma_g \cdot \not{\mathfrak{d}}^{\leqslant 3}\Gamma_g.$$

We then deduce

$$\boldsymbol{Z}_{\ell=1} = r\boldsymbol{C}_{\ell=1} + \not{\mathfrak{d}}^{\leqslant 3}(\Gamma_g \cdot \Gamma_g),$$

and thus using the GCM condition $\boldsymbol{C}_{\ell=1} = 0$,

$$\left|\boldsymbol{Z}_{\ell=1}\right| \lesssim \not{\mathfrak{d}}^{\leqslant 3}(\Gamma_g \cdot \Gamma_g). \tag{3.8}$$

Using the bootstrap assumptions (2.21), we deduce

$$\left|\boldsymbol{Z}_{\ell=1}\right| \lesssim \frac{\epsilon^2}{r^4 u^{1+2\delta}} \lesssim \frac{\epsilon_0}{r^4 u^{1+2\delta}}. \tag{3.9}$$

Step 2: Estimate for $\underline{\boldsymbol{C}}_{\ell=1}$. Differentiating with respect to div and projecting on the $\ell = 1$ modes the Codazzi equation for $\widehat{\underline{\chi}}$, we infer

$$(\mathrm{div}\, \not{d}_2 \widehat{\underline{\chi}})_{\ell=1} = \frac{1}{2}(\triangle \widetilde{\mathrm{tr}\,\underline{\chi}})_{\ell=1} + \frac{\Upsilon}{r}(\mathrm{div}\,\zeta)_{\ell=1} + \mathrm{div}\left(\underline{\beta} - \frac{Q}{r^2}\,{}^{(\mathbf{F})}\underline{\beta}\right)_{\ell=1} + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g)$$

$$= \frac{1}{2}(\triangle \widetilde{\mathrm{tr}\,\underline{\chi}})_{\ell=1} + \frac{\Upsilon}{r}\boldsymbol{Z}_{\ell=1} + \underline{\boldsymbol{C}}_{\ell=1} + O(r^{-4})\,\widetilde{{}^{(\mathbf{F})}\rho} + r^{-1}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).$$

As in Step 1, we have $(\mathrm{div}\,\not{d}_2\widehat{\underline{\chi}})_{\ell=1} = r^{-1}\not{\mathfrak{d}}^{\leqslant 3}(\Gamma_b\cdot\Gamma_g)$. Also, we have from Lemma 2.14,

$$\begin{aligned}(\triangle\underline{\check{\kappa}})_{\ell=1,p} = \frac{1}{|S|}\int_S \triangle\widetilde{\mathrm{tr}\,\underline{\chi}}J^{(p)} &= -\frac{2}{r^2}\frac{1}{|S|}\int_S \widetilde{\mathrm{tr}\,\underline{\chi}}J^{(p)} + \frac{1}{|S|}\int_S \underline{\check{\kappa}}\left(\triangle + \frac{2}{r^2}\right)J^{(p)}\\ &= O(r^{-2})(\widetilde{\mathrm{tr}\,\underline{\chi}})_{\ell=1,p} + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g\cdot\Gamma_g).\end{aligned}$$

We deduce

$$|\underline{\boldsymbol{C}}_{\ell=1}| \lesssim r^{-2}|(\widetilde{\mathrm{tr}\,\underline{\chi}})_{\ell=1}| + r^{-1}|\boldsymbol{Z}_{\ell=1}| + r^{-4}\Gamma_g + r^{-1}|\not{\mathfrak{d}}^{\leqslant 3}(\Gamma_b\cdot\Gamma_g)|.$$

Together with (3.7), (3.9), the bootstrap assumptions (2.21), and the dominance condition (2.23), we obtain

$$|\underline{\boldsymbol{C}}_{\ell=1}| \lesssim \frac{\epsilon}{r^4u^{1+2\delta}} + \frac{\epsilon_0}{r^5u^{1+2\delta}} + \frac{\epsilon}{r^6u^{\frac{1}{2}+\delta}} + \frac{\epsilon^2}{r^4u^{\frac{3}{2}+2\delta}} \lesssim \frac{\epsilon_0}{r^3u^{2+3\delta}}. \tag{3.10}$$

Step 3: Estimate for $(\widehat{\rho})_{\ell=1}$. Recall from Corollary 2.22 that we have along Σ_*, for $p=0,+,-$,

$$\nu\left(\int_S \widehat{\rho}J^{(p)}\right) = O(r^{-1})\int_S \widehat{\rho}J^{(p)} + h_1, \tag{3.11}$$

where the scalar function h_1 is given by

$$h_1 = -\int_S \underline{\boldsymbol{C}}J^{(p)} + r^{-1}\Gamma_b + r\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b\cdot\Gamma_b).$$

In view of (3.10), the bootstrap assumptions (2.21) and the dominance condition (2.23), we obtain

$$|h_1| \lesssim \frac{\epsilon_0}{ru^{2+3\delta}} + \frac{\epsilon}{r^2u^{1+\delta}} + \frac{\epsilon^2}{ru^{2+2\delta}} \lesssim \frac{\epsilon_0}{ru^{2+2\delta}}.$$

By integration in u we deduce

$$\int_u^{u_*} r|h_1| \lesssim \frac{\epsilon_0}{u^{1+2\delta}}.$$

Applying Lemma 2.23 to (3.11), we deduce

$$r\left|\int_S \widehat{\rho}J^{(p)}\right| \lesssim r_*\left|\int_{S_*}\widehat{\rho}J^{(p)}\right| + \frac{\epsilon_0}{u^{1+2\delta}}.$$

Together with the control of $\widehat{\rho}_{\ell=1}$ on S_* provided by Lemma 3.3, we infer

$$|\widehat{\rho}_{\ell=1}| \lesssim \frac{\epsilon_0}{r^3u^{1+2\delta}}. \tag{3.12}$$

Step 4: Estimate for $(\widetilde{\mathrm{tr}\,\underline{\chi}})_{\ell=1}$. Using Lemma 3.3 and (3.12), we deduce

$$\left|(\widetilde{\mathrm{tr}\,\underline{\chi}})_{\ell=1}\right| \lesssim \frac{\epsilon_0}{r^3u^{1+2\delta}} + \frac{\epsilon_0}{r^2u^{1+2\delta}} \lesssim \frac{\epsilon_0}{r^2u^{1+2\delta}}.$$

which improves the local bootstrap assumption (3.7) for $\widetilde{\mathrm{tr}\,\underline{\chi}}$.

Step 5: Estimate for $\check{\mu}_{\ell=1}$. Using that $\boldsymbol{Z} = -\widehat{\rho} - \check{\mu}$, and together with (3.9), (3.12), we infer

$$\begin{aligned}|\check{\mu}_{\ell=1}| &\lesssim |\boldsymbol{Z}_{\ell=1}| + |\widehat{\rho}_{\ell=1}|\\ &\lesssim \frac{\epsilon_0}{r^4u^{1+2\delta}} + \frac{\epsilon_0}{r^3u^{1+2\delta}} \lesssim \frac{\epsilon_0}{r^3u^{1+2\delta}}.\end{aligned} \tag{3.13}$$

This completes the proof of Proposition 3.4. $\square$

4 Solving the Einstein-Maxwell equations on Σ_*

The aim of this section is to solve the Einstein–Maxwell equations on a mass-centered GCM hypersurface Σ_*, as defined in Definition 2.9. Since Σ_* is spacelike, this amounts to solving the associated constraint equations on it, that is, deriving decay estimates for the relevant geometric quantities along Σ_* in terms of suitable seed data. The introduction of seed data is essential, as the constraint equations form an underdetermined system.

4.1 The seed data and the statement of the theorem

We define on Σ_* the following quantities:

$$\mathfrak{f} := -\nabla\widehat{\otimes}\,{}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\widehat{\chi}, \tag{4.1}$$

$$\underline{\mathfrak{f}} := -\nabla\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{\beta} - {}^{(\mathbf{F})}\rho\underline{\widehat{\chi}}, \tag{4.2}$$

$$\mathfrak{b} := 2\,{}^{(\mathbf{F})}\rho\beta - 3\rho\,{}^{(\mathbf{F})}\beta, \tag{4.3}$$

$$\underline{\mathfrak{b}} := 2\,{}^{(\mathbf{F})}\rho\underline{\beta} - 3\rho\,{}^{(\mathbf{F})}\underline{\beta}, \tag{4.4}$$

and

$$\mathfrak{x} := \nabla_4\,{}^{(\mathbf{F})}\beta + 3\,\mathrm{tr}\,\chi\,{}^{(\mathbf{F})}\beta. \tag{4.5}$$

We also use the following tensors:

$$\mathfrak{p} = -2r^3 \,{\not{d}}_1^*(\rho, {}^*\rho) + 3r^2 \,{\not{d}}_1^*({}^{(\mathbf{F})}\rho, {}^{*\,(\mathbf{F})}\rho) + \text{Acceptable Error}_1, \tag{4.6}$$

$$\mathfrak{q}^{\mathbf{F}} = r^3 \,{\not{d}}_2^* {\not{d}}_1^*({}^{(\mathbf{F})}\rho, -{}^{(\mathbf{F})*}\rho) + \text{Acceptable Error}_2, \tag{4.7}$$

where we allow errors of the form

$$\text{Acceptable Error}_i = O(r^{-2}) + \mathfrak{d}^{\leqslant i}\Gamma_b + r^2\mathfrak{d}^{\leqslant i}(\Gamma_b \cdot \Gamma_g).$$

Here, $\mathfrak{q}^{\mathbf{F}}$ also satisfies[4]

$$\nabla_3(r\mathfrak{q}^{\mathbf{F}}) = r^4 \,{\not{d}}_2^* {\not{d}}_1^*(-\mathrm{div}\,{}^{(\mathbf{F})}\underline{\beta}, \mathrm{curl}\,{}^{(\mathbf{F})}\underline{\beta}) + O(r^{-2}) + r\not{\mathfrak{d}}^{\leqslant 3}\Gamma_g + r^3\mathfrak{d}^{\leqslant 3}(\Gamma_b \cdot \Gamma_g). \tag{4.8}$$

where $O(r^a)$ denotes, for $a \in \mathbb{R}$, a function of $(r, \cos\theta)$ bounded by a multiple of r^a as $r \to +\infty$.

Definition 4.1 (Seed data and its norms)**.** *We say that the set* $\{\alpha, \underline{\alpha}, \mathfrak{f}, \mathfrak{b}, \underline{\mathfrak{b}}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}\}$ *constitute the* seed data *for the constraint equations on* Σ_*.

We also define the following norms for the seed data:

$$\begin{aligned}\mathcal{F}^k[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] := &\int_{\Sigma_*(u,u_*)} u^{2+2\delta}|\mathfrak{d}_*^k\underline{\alpha}|^2 + \int_{\Sigma_*(u,u_*)} r^6 u^{2+2\delta}|\mathfrak{d}_*^{k+1}\underline{\mathfrak{b}}|^2 \\ &+ \int_{\Sigma_*} u^{2+2\delta}|\mathfrak{d}_*^k\nabla_3\mathfrak{q}^{\mathbf{F}}|^2 + \int_{\Sigma_*} |\mathfrak{d}_*^k\mathfrak{q}^{\mathbf{F}}|^2,\end{aligned} \tag{4.9}$$

$$\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] := r\|\alpha\|_{\infty,k} + r\|\mathfrak{f}\|_{\infty,k} + r^3\|\mathfrak{b}\|_{\infty,k+1} + r^{-1}\|\mathfrak{q}^{\mathbf{F}}\|_{\infty,k} + r^{-1}\|\mathfrak{p}\|_{\infty,\max(k,3)}. \tag{4.10}$$

Observe that as a consequence of the bootstrap assumptions and the dominance condition we have

$$\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] \lesssim \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}}. \tag{4.11}$$

[4]Observe that (4.8) can be deduced from (4.7) and the Einstein-Maxwell equations.

Combinations of curvature, electromagnetic tensors and Ricci coefficients such as $\alpha, \mathfrak{f}, \mathfrak{b}, \underline{\mathfrak{b}}, \mathfrak{x}$ are precisely the ones identified in previous works by the second author [9] as the gauge-invariant quantities representing electromagnetic and gravitational radiation. The tensors $\mathfrak{p}$ and $\mathfrak{q}^{\mathbf{F}}$ are obtained from $\mathfrak{b}$ and $\mathfrak{f}$ respectively through the so-called Chandrasekhar transformation, see for example [9] for the definition in perturbations of Kerr-Newman. One can deduce the form (4.7)-(4.6) for the real parts of the tensors as a consequence of the Einstein-Maxwell equations. The curvature components α and $\underline{\alpha}$ are related to $\mathfrak{f}, \underline{\mathfrak{f}}, \mathfrak{b}, \underline{\mathfrak{b}}$ through transport equations.

Observe that as a consequence of the linearizations and the definition of Γ_g, Γ_b, (4.1)–(4.5) reduce to

$$\begin{aligned}
\mathfrak{f} &= -\nabla\widehat{\otimes}\,{}^{(\mathbf{F})}\beta + \frac{Q}{r^2}\widehat{\chi} + \Gamma_g\cdot\Gamma_g,\\
\mathfrak{b} &= \frac{2Q}{r^2}\beta + \left(\frac{6M}{r^3} - \frac{6Q^2}{r^4}\right){}^{(\mathbf{F})}\beta + r^{-1}\Gamma_g\cdot\Gamma_g,\\
\underline{\mathfrak{b}} &= \frac{2Q}{r^2}\underline{\beta} + \left(\frac{6M}{r^3} - \frac{6Q^2}{r^4}\right){}^{(\mathbf{F})}\underline{\beta} + r^{-1}\Gamma_b\cdot\Gamma_g,\\
\mathfrak{x} &= \nabla_4\,{}^{(\mathbf{F})}\beta + \frac{3}{r}\,{}^{(\mathbf{F})}\beta.
\end{aligned}$$

Remark 4.2. *In the context of the nonlinear stability of charged black holes, we expect to obtain control of the seed data quantities through the analysis of the hyperbolic system of PDEs they satisfy, as in [7]. In the context of the nonlinear stability of Reissner–Nordström, [21] already establishes decay estimates for the gauge-invariant quantities $\mathfrak{p}$ and $\mathfrak{q}^{\mathbf{F}}$, valid for all $0 \leqslant k \leqslant k_*$ for some large k_*. More precisely, for $\delta_{extra} > \delta$,*

$$\|\mathfrak{q}^{\mathbf{F}}\|_{\infty,k} + \|\mathfrak{p}\|_{\infty,k+1} \lesssim \frac{\epsilon_0}{ru^{\frac{1}{2}+\delta_{extra}}}, \qquad \|\nabla_3\mathfrak{q}^{\mathbf{F}}\|_{\infty,k-1} \lesssim \frac{\epsilon_0}{ru^{1+\delta_{extra}}}.$$

In addition, one expects further bounds for $\mathfrak{b}$, $\mathfrak{f}$, and their fluxes, obtained through transport estimates from the ones of $\mathfrak{p}, \mathfrak{q}^{\mathbf{F}}$. More precisely,

$$\|\mathfrak{b}\|_{\infty,k+1} \lesssim \frac{\epsilon_0}{r^{\frac{11}{2}+\delta_{extra}}}, \qquad \|\mathfrak{f}\|_{\infty,k} \lesssim \frac{\epsilon_0}{r^{\frac{7}{2}+\delta_{extra}}},$$
$$\int_{\Sigma_*(u,u_*)} u^{2+2\delta}|\nabla_3\mathfrak{d}^k\mathfrak{q}^{\mathbf{F}}|^2 + \int_{\Sigma_*(u,u_*)} r^6u^{2+2\delta}|\mathfrak{d}_*^k\underline{\mathfrak{b}}|^2 \lesssim \epsilon_0^2,$$

see [8] for the proof in the linear case. By the definitions of the seed data norms, these estimates would imply

$$\mathcal{F}^k[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] \lesssim \epsilon_0^2, \qquad \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{extra}}}. \tag{4.12}$$

We further expect that analogous decay estimates will follow from the nonlinear generalized Regge–Wheeler system in Kerr–Newman; see [7] for the corresponding linear case.

We are now ready to state the precise version of the main theorem.

Theorem 4.3. *Let $\Sigma_* \subset \mathcal{M}$ be an electrovacuum mass-centered GCM hypersurface as in Definition 2.9, satisfying the dominance condition* (2.23)*, in a spacetime $\mathcal{M}$ solving the Einstein–Maxwell equations* (1.2)*. Assume further that the bootstrap assumptions* (2.21) *hold for Γ_g and Γ_b, and that for all $k \leqslant N$ one has*

$$\mathcal{F}^k[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] \lesssim \epsilon^2. \tag{4.13}$$

Then, for all $k \leqslant N-11$, the perturbation quantities along Σ_* satisfy the decay estimates

$$\begin{aligned}\|\Gamma_b\|_{\infty,k} &\lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{ru^{1+\delta}},\\ \|\Gamma_g\|_{\infty,k} &\lesssim \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^2u^{1+\delta}},\\ \|\nabla_\nu\Gamma_g\|_{\infty,k-1} &\lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^2u^{1+\delta}}.\end{aligned} \tag{4.14}$$

Moreover, for all $k \leqslant N-10$, we obtain the improved bounds

$$\begin{aligned}\big|\mathfrak{d}_*^{\leqslant k}\widecheck{\mathrm{tr}\,\underline{\chi}}\big| + r\big|\mathfrak{d}_*^{\leqslant k}\check{\underline{\mu}}\big| &\lesssim \frac{\epsilon_0}{r^2u^{1+\delta}},\\ r\big|\mathfrak{d}_*^{\leqslant k-1}\nabla_\nu\beta\big| + \big|\mathfrak{d}_*^{\leqslant k-1}\nabla_\nu\,{}^{(\mathbf{F})}\beta\big| &\lesssim r^{-1}\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^3u^{1+\delta}},\\ r\big|\mathfrak{d}_*^{\leqslant k-2}\nabla_\nu^2\beta\big| + \big|\mathfrak{d}_*^{\leqslant k-2}\nabla_\nu^2\,{}^{(\mathbf{F})}\beta\big| &\lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^3u^{1+\delta}}.\end{aligned} \tag{4.15}$$

We separate the proof of Theorem 4.3 in three parts: first we control the Γ_b quantities, then we use the estimates for the Γ_b quantities to deduce additional control on $\ell = 1$ modes, and finally we obtain control for the Γ_g quantities.

Remark 4.4. *If the bounds in* (4.12) *were proven to hold independently, then Theorem 4.3 would yield the following unconditional bounds for Γ_b, Γ_g on Σ_*:*

$$\begin{aligned}\|\Gamma_b\|_{\infty,k} &\lesssim \frac{\sqrt{\mathcal{F}^k[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{ru^{1+\delta}} \lesssim \frac{\epsilon_0}{ru^{1+\delta}},\\ \|\Gamma_g\|_{\infty,k} &\lesssim \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^k[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^2u^{1+\delta}}\\ &\lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta_{extra}}} + \frac{\epsilon_0}{r^2u^{1+\delta}} \lesssim \frac{\epsilon_0}{r^2u^{\frac{1}{2}+\delta}},\\ \|\mathfrak{d}_*^{\leqslant k-1}\nabla_\nu\Gamma_g\|_\infty &\lesssim \frac{\sqrt{\mathcal{F}^k[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^2u^{1+\delta}} \lesssim \frac{\epsilon_0}{r^2u^{1+\delta}},\end{aligned}$$

therefore improving the bootstrap assumptions (2.21).

Remark 4.5. *Assumption* (4.13) *on the flux norms $\mathcal{F}^k[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]$ is not implied by the bootstrap assumptions* (2.21) *but it is needed in the proof of Proposition 4.8 to improve the local bootstrap assumptions there.*

4.2 Control of Γ_b quantities on Σ_*

In this section we obtain control of the quantities in the set Γ_b as defined in (2.20). The goal of this section is to establish the following.

Proposition 4.6. *The following estimate holds true for all $k \leqslant N-7$*

$$\int_{\Sigma_*} u^{2+2\delta}\big|\mathfrak{d}_*^k\Gamma_b\big|^2 \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.16}$$

Using the trace theorem and Sobolev, we deduce that for $k \leqslant N-9$,

$$\|\Gamma_b\|_{\infty,k} \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{ru^{1+\delta}}. \tag{4.17}$$

Observe that the flux estimate (4.16) is trivially verified for $\underline{\alpha} \in \Gamma_b$ in view of the definition of $\mathcal{F}^k[\mathfrak{q}^{\mathbf{F}}, \underline{\mathfrak{b}}, \underline{\alpha}]$ in (4.9).

Proof. **Step 1: Estimate for $^{(\mathbf{F})}\underline{\beta}_{\ell\geqslant 2}$.** From (4.8), we infer, using the bootstrap assumptions (2.21) and the dominance condition (2.23) that

$$
\begin{aligned}
r^4|\mathfrak{d}_*^k \not{d}_2^* \not{d}_1^*(-\text{div}\ ^{(\mathbf{F})}\underline{\beta}, \text{curl}\ ^{(\mathbf{F})}\underline{\beta})| &\lesssim |\mathfrak{d}_*^k \nabla_3(r\mathfrak{q}^{\mathbf{F}})| + \frac{1}{r^2} + |r\not{\mathfrak{d}}^{\leqslant 3}\Gamma_g| + r^3|\mathfrak{d}^{\leqslant 3}(\Gamma_b \cdot \Gamma_g)| \\
&\lesssim |\mathfrak{d}_*^k \nabla_3(r\mathfrak{q}^{\mathbf{F}})| + \frac{\epsilon_0}{u^{2+2\delta}} + \frac{\epsilon}{ru^{\frac{1}{2}+\delta}} + \frac{\epsilon^2}{u^{\frac{3}{2}+2\delta}} \\
&\lesssim r|\mathfrak{d}_*^k \nabla_3 \mathfrak{q}^{\mathbf{F}}| + |\mathfrak{d}_*^{\leqslant k}\mathfrak{q}^{\mathbf{F}}| + \frac{\epsilon_0}{u^{\frac{3}{2}+2\delta}}.
\end{aligned}
$$

Dividing by r, squaring and integrating on Σ_* we deduce

$$
\begin{aligned}
\int_{\Sigma_*} r^6 u^{2+2\delta}|\mathfrak{d}_*^k \not{d}_2^* \not{d}_1^*\big(-\text{div}\ ^{(\mathbf{F})}\underline{\beta}, \text{curl}\ ^{(\mathbf{F})}\underline{\beta}\big)|^2 &\lesssim \int_{\Sigma_*} u^{2+2\delta}|\mathfrak{d}_*^k \nabla_3 \mathfrak{q}^{\mathbf{F}}|^2 \\
&\quad + \int_{\Sigma_*} u^{2+2\delta} r^{-2}|\mathfrak{d}_*^k \mathfrak{q}^{\mathbf{F}}|^2 + \int_{\Sigma_*} \frac{\epsilon_0^2}{r^2 u^{1+\delta}}.
\end{aligned}
$$

Using the dominance condition (2.23) in the second term on the right, we deduce for $k \leqslant N$,

$$
\int_{\Sigma_*} r^6 u^{2+2\delta}|\mathfrak{d}_*^k \not{d}_2^* \not{d}_1^*\big(-\text{div}\ ^{(\mathbf{F})}\underline{\beta}, \text{curl}\ ^{(\mathbf{F})}\underline{\beta}\big)|^2 \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2.
$$

Taking into account the commutator Lemma 2.12, this yields, for $k \leqslant N$,

$$
\int_{\Sigma_*} r^6 u^{2+2\delta}|\not{d}_1^*(\text{div}, -\text{curl})\not{d}_2^*\, \mathfrak{d}_*^k\ ^{(\mathbf{F})}\underline{\beta}|^2 \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2.
$$

Using the Hodge estimates of Lemma 2.13 for $\not{d}_1^*$ and $\not{d}_1$, we infer, for $k \leqslant N-2$,

$$
\int_{\Sigma_*} r^2 u^{2+2\delta}|\not{d}_2^*\, \mathfrak{d}_*^k\ ^{(\mathbf{F})}\underline{\beta}|^2 \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.18}
$$

Step 2: Estimate for $\underline{\beta}_{\ell\geqslant 2}$. From the definition of $\underline{\mathfrak{b}}$ in (4.4), we infer

$$
\not{d}_2^*\underline{\mathfrak{b}} = \frac{2Q}{r^2}\not{d}_2^*\underline{\beta} + \left(\frac{6M}{r^3} - \frac{6Q^2}{r^4}\right)\not{d}_2^*\ ^{(\mathbf{F})}\underline{\beta} + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g).
$$

We deduce using the bootstrap assumptions (2.21):

$$
r^2|\not{d}_2^*\mathfrak{d}_*^k\underline{\beta}| \lesssim r^4|\not{d}_2^*\mathfrak{d}_*^k\underline{\mathfrak{b}}| + r|\not{d}_2^*\mathfrak{d}_*^k\ ^{(\mathbf{F})}\underline{\beta}| + \frac{\epsilon_0}{ru^{\frac{3}{2}+2\delta}}.
$$

Squaring and integrating on Σ_* we deduce $k \leqslant N-2$

$$
\begin{aligned}
\int_{\Sigma_*} r^4 u^{2+2\delta}|\not{d}_2^*\mathfrak{d}_*^k\underline{\beta}|^2 &\lesssim \int_{\Sigma_*} r^8 u^{2+2\delta}|\not{d}_2^*\mathfrak{d}_*^k\underline{\mathfrak{b}}|^2 + \int_{\Sigma_*} r^2 u^{2+2\delta}|\not{d}_2^*\mathfrak{d}_*^k\ ^{(\mathbf{F})}\underline{\beta}|^2 + \int_{\Sigma_*} \frac{\epsilon_0^2}{r^2 u^{3+4\delta}} \\
&\lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2,
\end{aligned}
$$

where we used (4.18). Using the Hodge estimate (2.26) of Lemma 2.13, we infer for $k \leqslant N-1$,

$$
\int_{\Sigma_*} r^2 u^{2+2\delta}|\mathfrak{d}_*^k\underline{\beta}|^2 \lesssim \int_{\Sigma_*} r^4 u^{2+2\delta}\left|\left(\not{d}_1\nabla_\nu^{\leqslant k-3}\underline{\beta}\right)_{\ell=1}\right|^2 + \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.19}
$$

Step 3: Estimate for ${}^{(\mathbf{F})}\underline{\beta}_{\ell=1}$ and $\underline{\beta}_{\ell=1}$. We now need to add the control of the $\ell = 1$ mode of $\underline{\beta}$ and ${}^{(\mathbf{F})}\underline{\beta}$. Writing the Codazzi equation as

$$\not{d}_2\underline{\widehat{\chi}} = \underline{\beta} + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}\Gamma_g + \Gamma_b \cdot \Gamma_g, \tag{4.20}$$

and differentiating it with respect to $\nu^k \not{d}_1$, we infer

$$\nu^k \not{d}_1\not{d}_2\underline{\widehat{\chi}} = \nu^k \not{d}_1\underline{\beta} + r^{-2}\mathfrak{d}_*^{\leqslant k+2}\Gamma_g + r^{-1}\mathfrak{d}_*^{\leqslant k+1}(\Gamma_b \cdot \Gamma_g).$$

Taking into account Lemma 2.12 and projecting on the $\ell = 1$ modes, this yields for

$$(\not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}})_{\ell=1} = (\not{d}_1\nabla_\nu^k\underline{\beta})_{\ell=1} + r^{-2}\mathfrak{d}_*^{\leqslant k+2}\Gamma_g + r^{-1}\mathfrak{d}_*^{\leqslant k+1}(\Gamma_b \cdot \Gamma_g).$$

Next, we have from Lemma 2.14 that for

$$(\not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}})_{\ell=1,p} = \frac{1}{|S|}\int_S \not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}}J^{(p)} = \frac{1}{|S|}\int_S \nabla_\nu^k\underline{\widehat{\chi}} \cdot \not{d}_2^*\not{d}_1^* J^{(p)} = r^{-1}\mathfrak{d}^k\Gamma_b \cdot \not{\mathfrak{d}}^{\leqslant 3}\Gamma_g.$$

Using the bootstrap assumptions (2.21) and the dominance condition (2.23), we then infer for $k \leqslant N-2$

$$|(\not{d}_1\nabla_\nu^k\underline{\beta})_{\ell=1}| \lesssim |(\not{d}_1\not{d}_2\nabla_\nu^k\underline{\widehat{\chi}})_{\ell=1}| + \frac{\epsilon}{r^4 u^{\frac{1}{2}+\delta}} + \frac{\epsilon^2}{r^4 u^{\frac{3}{2}+2\delta}} \lesssim \frac{\epsilon_0}{r^3 u^{\frac{3}{2}+2\delta}}. \tag{4.21}$$

Combining (4.19) and (4.21), we obtain for $k \leqslant N-2$,

$$\int_{\Sigma_*} r^2 u^{2+2\delta}|\mathfrak{d}_*^k\underline{\beta}|^2 \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.22}$$

Using again (4.4), i.e.

$$\not{d}_1\underline{\mathfrak{b}} = \frac{2Q}{r^2}\not{d}_1\underline{\beta} + 3\left(\frac{2M}{r^3} - \frac{2Q^2}{r^4}\right)\not{d}_1\,{}^{(\mathbf{F})}\underline{\beta} + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g),$$

we deduce for $k \leqslant N-2$

$$\begin{aligned}|(\not{d}_1\nabla_\nu^k\,{}^{(\mathbf{F})}\underline{\beta})_{\ell=1}| &\lesssim r^3|(\not{d}_1\nabla_\nu^k\underline{\mathfrak{b}})_{\ell=1}| + r|(\not{d}_1\nabla_\nu^k\underline{\beta})_{\ell=1}| + \frac{\epsilon^2}{r^2 u^{\frac{3}{2}+2\delta}}\\ &\lesssim r^3|(\not{d}_1\nabla_\nu^k\underline{\mathfrak{b}})_{\ell=1}| + \frac{\epsilon_0}{r^2 u^{\frac{3}{2}+2\delta}},\end{aligned}$$

where we used (4.21). Combining (4.18) and the above we obtain for $k \leqslant N-2$,

$$\int_{\Sigma_*} u^{2+2\delta}|\mathfrak{d}_*^k\,{}^{(\mathbf{F})}\underline{\beta}|^2 \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2.$$

Step 4: Estimate for $\underline{\widehat{\chi}}$. Using once again the Codazzi equation (4.20), in view of Lemma 2.12, we have that

$$\not{d}_2\mathfrak{d}_*^k\underline{\widehat{\chi}} = (\mathfrak{d}_*^k\underline{\beta}) + r^{-1}\not{\mathfrak{d}}^{\leqslant k+2}\Gamma_g + \not{\mathfrak{d}}^{\leqslant k+1}\left(\Gamma_b \cdot \Gamma_g\right).$$

Using the bootstrap assumptions (2.21), the dominance condition (2.23), and (4.22), we infer for $k \leqslant N-2$,

$$\int_{\Sigma_*} u^{2+2\delta}|\mathfrak{d}_*^k\underline{\widehat{\chi}}|^2 \lesssim \mathcal{F}^k[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.23}$$

Step 5: Estimate for $\eta_{\ell\geqslant 2}$ and $\underline{\xi}_{\ell\geqslant 2}$. From Proposition 2.11, and using Lemma 2.12, we deduce for

$$
\begin{aligned}
2\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\eta &= r^{-4}\mathfrak{d}_*^{\leqslant k+3}\underline{\beta} + r^{-5}\mathfrak{d}_*^{\leqslant k+5}\Gamma_g + r^{-4}\mathfrak{d}_*^{\leqslant k+4}(\Gamma_b\cdot\Gamma_b),\\
2\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\underline{\xi} &= r^{-4}\mathfrak{d}_*^{\leqslant k+3}\underline{\beta} + r^{-5}\mathfrak{d}_*^{\leqslant k+5}\Gamma_g + r^{-4}\mathfrak{d}_*^{\leqslant k+6}(\Gamma_b\cdot\Gamma_b).
\end{aligned}
$$

Using the bootstrap assumptions (2.21), the dominance condition (2.23), we infer for $k\leqslant N-6$,

$$
\begin{aligned}
|\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\mathfrak{d}_*^k\eta| + |\not{d}_2^*\not{d}_1^*\not{d}_1\not{d}_2\not{d}_2^*\,\mathfrak{d}_*^k\underline{\xi}| &\lesssim r^{-4}|\mathfrak{d}_*^{\leqslant N-3}\underline{\beta}| + \frac{\epsilon}{r^7u^{\frac12+\delta}} + \frac{\epsilon^2}{r^6u^{2+2\delta}}\\
&\lesssim r^{-4}|\mathfrak{d}_*^{\leqslant N-3}\underline{\beta}| + \frac{\epsilon}{r^7u^{\frac12+\delta}} + \frac{\epsilon^2}{r^6u^{\frac32+2\delta}}.
\end{aligned}
$$

Using (4.22) and the coercivity of $\not{d}_2^*\not{d}_2$ (see Lemma 5.44 in [18]), we deduce for $k\leqslant N-6$,

$$
\int_{\Sigma_*} r^2u^{2+2\delta}\Big(|\not{d}_2^*\,\mathfrak{d}_*^k\eta|^2 + |\not{d}_2^*\,\mathfrak{d}_*^k\underline{\xi}|^2\Big) \lesssim \mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.24}
$$

Step 6: Estimate for $\eta_{\ell=1}$ and $\underline{\xi}_{\ell=1}$. Using the elliptic identity (2.34), we can estimate

$$
\begin{aligned}
\left|\int_S \operatorname{div}\eta J^{(p)}\right| &\lesssim r\left|\int_S \widehat{\rho}J^{(p)}\right| + \left|\int_S \operatorname{div}\,{}^{(\mathbf{F})}\underline{\beta}J^{(p)}\right| + r\mathfrak{d}^{\leqslant 1}\Gamma_g + r^3\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g)\\
&\lesssim \frac{\epsilon_0}{u^{1+2\delta}} + \frac{\epsilon_0}{u^{1+\delta}} + \frac{\epsilon}{ru^{\frac12+\delta}} + \frac{\epsilon^2}{u^{\frac32+2\delta}} \lesssim \frac{\epsilon_0}{u^{1+\delta}},
\end{aligned}
$$

where we used Proposition 3.4 to bound $\widehat{\rho}_{\ell=1}$, Step 3 above to bound $(\operatorname{div}\,{}^{(\mathbf{F})}\underline{\beta})_{\ell=1}$, the bootstrap assumptions (2.21) and the dominance condition (2.23). Similarly for the commuted identity with ν^k.

Also recall the GCM condition (2.18) $(\operatorname{div}\underline{\xi})_{\ell=1}=0$, from which in view of Lemma 2.21 we infer

$$
(\nu^k\operatorname{div}\underline{\xi})_{\ell=1} = r^{-2}\mathfrak{d}_*^{\leqslant k-1}\Gamma_b + \mathfrak{d}_*^{\leqslant k}(\Gamma_b\cdot\Gamma_b).
$$

Using the bootstrap assumptions (2.21) and the dominance condition (2.23), we infer for $k\leqslant N$,

$$
|(\nu^k\operatorname{div}\underline{\xi})_{\ell=1}| \lesssim \frac{\epsilon}{r^3u^{1+\delta}} + \frac{\epsilon^2}{r^2u^{2+2\delta}} \lesssim \frac{\epsilon_0}{r^2u^{2+2\delta}}. \tag{4.25}
$$

Using the null structure equations $\operatorname{curl}\eta = r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_g$ and $\operatorname{curl}\underline{\xi} = \Gamma_b\cdot\Gamma_b$ and Lemma 2.12, we have for $k\leqslant N$,

$$
\begin{aligned}
|(\not{d}_1\nabla_\nu^k\eta)_{\ell=1}| + |(\not{d}_1\nabla_\nu^k\underline{\xi})_{\ell=1}| &\lesssim |(\nu^k\operatorname{div}\eta)_{\ell=1}| + |(\nu^k\operatorname{div}\underline{\xi})_{\ell=1}| + \frac{\epsilon}{r^3u^{\frac12+\delta}} + \frac{\epsilon^2}{r^2u^{2+2\delta}}\\
&\lesssim \frac{\epsilon_0}{r^2u^{1+\delta}} + \frac{\epsilon_0}{r^2u^{2+2\delta}} + \frac{\epsilon_0}{r^2u^{2+2\delta}} \lesssim \frac{\epsilon_0}{r^2u^{1+\delta}},
\end{aligned} \tag{4.26}
$$

where we used once again the bootstrap assumptions (2.21), the dominance condition (2.23) and (4.25). Using Lemma 2.13 and (4.24) we deduce for $k\leqslant N-6$

$$
\int_{\Sigma_*} u^{2+2\delta}\Big(|\mathfrak{d}_*^k\eta|^2 + |\mathfrak{d}_*^k\underline{\xi}|^2\Big) \lesssim \mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.27}
$$

Step 7: Estimate for $\check{y}$, $\check{z}$ and $\check{b}_*$. From (2.1) and Lemma 2.12, we infer

$$
\nabla\mathfrak{d}_*^k\check{y} = -\mathfrak{d}_*^k\underline{\xi} + \mathfrak{d}_*^k\eta + \mathfrak{d}_*^{\leqslant k}\Gamma_g + r\mathfrak{d}_*^{\leqslant k}(\Gamma_b\cdot\Gamma_b), \qquad \nabla\mathfrak{d}_*^k\check{z} = -2\mathfrak{d}_*^k\eta + \mathfrak{d}_*^{\leqslant k}\Gamma_g + r\mathfrak{d}_*^{\leqslant k}(\Gamma_b\cdot\Gamma_b),
$$

Using the bootstrap assumptions (2.21) and the dominance condition (2.23), we obtain

$$
\begin{aligned}
|\nabla \mathfrak{d}_*^k \check{y}| + |\nabla \mathfrak{d}_*^k \check{z}| &\lesssim |\mathfrak{d}_*^k \underline{\xi}| + |\mathfrak{d}_*^k \eta| + \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}} + \frac{\epsilon^2}{r u^{2+2\delta}} \\
&\lesssim |\mathfrak{d}_*^k \underline{\xi}| + |\mathfrak{d}_*^k \eta| + \frac{\epsilon_0}{r u^{\frac{3}{2}+2\delta}}.
\end{aligned}
$$

Squaring and integrating on Σ_*, we infer that

$$
\int_{\Sigma_*} u^{2+2\delta}\Big(|\nabla \mathfrak{d}_*^k \check{y}|^2 + |\nabla \mathfrak{d}_*^k \check{z}|^2\Big) \lesssim \epsilon_0^2 + \int_{\Sigma_*} u^{2+2\delta}\Big(|\mathfrak{d}_*^k \eta|^2 + |\mathfrak{d}_*^k \underline{\xi}|^2\Big).
$$

Together with (4.27), this yields, for $k \leqslant N-6$,

$$
\int_{\Sigma_*} u^{2+2\delta}\Big(|\nabla \mathfrak{d}_*^k \check{y}|^2 + |\nabla \mathfrak{d}_*^k \check{z}|^2\Big) \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2.
$$

Since $b_* = -y - z$, we deduce, for $k \leqslant N-6$,

$$
\int_{\Sigma_*} u^{2+2\delta}\Big(|\nabla \mathfrak{d}_*^k \check{y}|^2 + |\nabla \mathfrak{d}_*^k \check{z}|^2 + |\nabla \mathfrak{d}_*^k \check{b}_*|^2\Big) \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.28}
$$

In view of (4.28), it remains to estimate the averages $\overline{\nu^k(\check{y})}$, $\overline{\nu^k(\check{z})}$ and $\overline{\nu^k(\check{b}_*)}$, whose proof follows identically the one in Step 7 of Proposition 5.42 in [18]. Together with (4.28), and using a Poincaré inequality, we infer, for $k \leqslant N-7$,

$$
\int_{\Sigma_*} r^{-2} u^{2+2\delta}\left(\Big|\mathfrak{d}_*^k(\check{y})\Big|^2 + \Big|\mathfrak{d}_*^k(\check{z})\Big|^2 + \Big|\mathfrak{d}_*^k(\check{b}_*)\Big|^2\right) \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2. \tag{4.29}
$$

Step 8: Estimate for $\check{\underline{\omega}}$. Finally, from the null structure equations for $\nabla_4 \widecheck{\mathrm{tr}\,\chi}$ and $\nabla_3 \widecheck{\mathrm{tr}\,\chi}$, we deduce

$$
\check{\underline{\omega}} = -\frac{r}{2}\,\mathrm{div}\,\eta - \frac{1}{2r}\check{y} + \Gamma_g + r\Gamma_b \cdot \Gamma_b.
$$

Using the bootstrap assumptions (2.21) and the dominance condition (2.23), we obtain that

$$
\begin{aligned}
|\mathfrak{d}_*^k \check{\underline{\omega}}| &\lesssim |\mathfrak{d}_*^{k+1} \eta| + r^{-1}|\mathfrak{d}_*^k \check{y}| + \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}} + \frac{\epsilon^2}{r u^{2+2\delta}} \\
&\lesssim |\mathfrak{d}_*^{k+1} \eta| + r^{-1}|\mathfrak{d}_*^k \check{y}| + \frac{\epsilon_0}{r u^{\frac{3}{2}+2\delta}}.
\end{aligned}
$$

Squaring and integrating on Σ_*, we infer that

$$
\int_{\Sigma_*} u^{2+2\delta}|\mathfrak{d}_*^k \check{\underline{\omega}}|^2 \lesssim \int_{\Sigma_*} u^{2+2\delta}|\mathfrak{d}_*^{k+1}\eta|^2 + \int_{\Sigma_*} r^{-2}u^{2+2\delta}|\mathfrak{d}_*^k \check{y}|^2 + \epsilon_0^2.
$$

Together with (4.27) for η and (4.29) for $\check{y}$, we infer, for $k \leqslant N-7$,

$$
\int_{\Sigma_*} u^{2+2\delta}|\mathfrak{d}_*^k \check{\underline{\omega}}|^2 \lesssim \mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2.
$$

This concludes the proof of Proposition 4.6. $\square$

The control of the fluxes for the Γ_b quantities allow us to have a better control of the nonlinear terms $\int \mathfrak{d}^{\leqslant k}(\Gamma_g \cdot \Gamma_b)$, that according to the bootstrap assumptions (2.21) would only decay as $u^{-\frac{1}{2}-2\delta}$. It is nevertheless necessary to improve this decay in order to control $\boldsymbol{J}_{\ell=1}$ in the subsequent section.

Corollary 4.7. *Any nonlinear term of the form $\Gamma_g \cdot \Gamma_b$ satisfies on Σ_* for $k \leqslant N-9$:*

$$\int_u^{u_*} r^3 |\mathfrak{d}^{\leqslant k}(\Gamma_g \cdot \Gamma_b)| \lesssim \frac{\epsilon}{u^{1+2\delta}} \sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}. \tag{4.30}$$

Proof. Using the bootstrap assumptions (2.21) for the Γ_g we have for $k \leqslant N$

$$|\mathfrak{d}^{\leqslant k}(\Gamma_g \cdot \Gamma_b)| \lesssim \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}} |\mathfrak{d}^{\leqslant k} \Gamma_b|.$$

By integration in u and using Sobolev and Hölder inequalities, we obtain

$$\begin{aligned}
\int_u^{u_*} r^3 |\mathfrak{d}^{\leqslant k}(\Gamma_g \cdot \Gamma_b)| &\lesssim \int_u^{u_*} \frac{\epsilon}{u'^{\frac{1}{2}+\delta}} \|\mathfrak{d}^{\leqslant k+2} \Gamma_b\|_{L^2(S)} \\
&\lesssim \epsilon \left(\int_u^{u_*} \frac{du'}{u'^{3+4\delta}} \right)^{\frac{1}{2}} \left(\int_u^{u_*} u'^{2+2\delta} \|\mathfrak{d}^{\leqslant k+2} \Gamma_b\|^2_{L^2(S)} du' \right)^{\frac{1}{2}} \\
&\lesssim \frac{\epsilon}{u^{1+2\delta}} \left(\int_{\Sigma_*} u^{2+2\delta} |\mathfrak{d}_*^{k+2} \Gamma_b|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Using Proposition 4.6, we infer the stated inequality in (4.30). □

4.3 Improved control of some $\ell = 1$ modes

We now complete the improved decay for the $\ell = 1$ modes of Proposition 3.4 with the remaining quantities.

Proposition 4.8. *Suppose $\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] \lesssim \epsilon^2$. Then the following estimates hold true on Σ_*:*

$$r\,|\boldsymbol{J}_{\ell=1,\pm}| + r\left|\boldsymbol{J}_{\ell=1,0} - \frac{2aM}{r^5}\right| + |\boldsymbol{Y}_{\ell=1,\pm}| + \left|\boldsymbol{Y}_{\ell=1,0} - \frac{2aM}{r^4}\right| \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^4 u^{1+\delta}},$$

$$|{}^*\widehat{\rho}_{\ell=1,\pm}| + \left|{}^*\widehat{\rho}_{\ell=1,0} - \frac{2aM}{r^4}\right| \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta}}.$$

Proof. We make the following local bootstrap assumptions on Σ_*:

$$|\boldsymbol{J}_{\ell=1,\pm}| + \left|\boldsymbol{J}_{\ell=1,0} - \frac{2aM}{r^5}\right| \leqslant \frac{\epsilon}{r^5 u^{1+\delta}}. \tag{4.31}$$

Notice that these assumptions are stronger than the ones implied by the bootstrap assumptions (2.21) and will be recovered in the process of the proof (in Step 3). In order to improve the local bootstrap assumptions we need to assume (4.13).

Step 1: Preliminary estimate for $\boldsymbol{Y}_{\ell=1}$. Differentiating with respect to curl and projecting on the $\ell = 1$ modes the Codazzi equation for $\widehat{\chi}$, we infer

$$(\operatorname{curl} \not{d}_2 \widehat{\chi})_{\ell=1} = \frac{1}{r}(\operatorname{curl} \zeta)_{\ell=1} - (\operatorname{curl} \beta)_{\ell=1} + \frac{Q}{r^2}(\operatorname{curl} {}^{(\mathbf{F})}\beta)_{\ell=1} + r^{-1} \not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g).$$

Recalling the definitions (2.13) of $\boldsymbol{J}$ and (2.30) of $\boldsymbol{Y}$ we deduce

$$(\operatorname{curl} \not{d}_2 \widehat{\chi})_{\ell=1} = \frac{1}{r} \boldsymbol{Y}_{\ell=1} - \boldsymbol{J}_{\ell=1} + r^{-1} \not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g).$$

Using as in Proposition 3.4 that $(\operatorname{curl} \not{d}_2 \widehat{\chi})_{\ell=1} = r^{-1} \Gamma_g \cdot \not{\mathfrak{d}}^{\leqslant 3} \Gamma_g$ we deduce

$$\boldsymbol{Y}_{\ell=1} = r \boldsymbol{J}_{\ell=1} + \not{\mathfrak{d}}^{\leqslant 3}(\Gamma_g \cdot \Gamma_g), \tag{4.32}$$

and thus using the bootstrap assumptions (2.21) and (3.7), we have

$$|\boldsymbol{Y}_{\ell=1,\pm}| + \left|\boldsymbol{Y}_{\ell=1,0} - \frac{2aM}{r^4}\right| \lesssim \frac{\epsilon}{r^4 u^{1+\delta}}. \tag{4.33}$$

Step 2: Estimate for $^*\widehat{\rho}_{\ell=1}$. Using that $\boldsymbol{Y} = \widehat{^*\rho}$, we infer from (4.33) and the dominance condition (2.23),

$$\left|(^*\widehat{\rho})_{\ell=1,\pm}\right| + \left|(^*\widehat{\rho})_{\ell=1,0} - \frac{2aM}{r^4}\right| \lesssim \frac{\epsilon}{r^4 u^{1+\delta}} \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta}}. \tag{4.34}$$

Step 3: Estimate for $\boldsymbol{J}_{\ell=1}$. Recall from Corollary 2.22 that we have along Σ_*, for $p = 0, +, -$,

$$\nu\left(\int_S \boldsymbol{J} J^{(p)}\right) = O(r^{-1})\int_S \boldsymbol{J} J^{(p)} + h_2,$$

where the scalar function h_2 is given by

$$h_2 = O(r^{-2})\int_S {}^*\widehat{\rho} J^{(p)} + r^{-3}\mathfrak{d}^{\leqslant 1}\Gamma_g + \mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).$$

In view of (4.34), the bootstrap assumptions (2.21) and the dominance condition (2.23) we obtain

$$|h_2| \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta}} + \frac{\epsilon}{r^5 u^{\frac{1}{2}+\delta}} + \mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g) \lesssim \frac{\epsilon_0}{r^3 u^{2+2\delta}} + \mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).$$

Thus, using Corollary 4.7 we infer

$$\int_u^{u_*} r^3|h_2| \lesssim \frac{\epsilon_0}{u^{1+2\delta}} + \int_u^{u_*} r^3\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g) \lesssim \frac{1}{u^{1+2\delta}}\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}.$$

Applying Lemma 2.23, we deduce for $p = \pm$, since we have $\boldsymbol{J}_{\ell=1,\pm} = 0$ on S_*,

$$|\boldsymbol{J}_{\ell=1,\pm}| \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^5 u^{1+\delta}}. \tag{4.35}$$

Next, we focus on the case $p = 0$. Using that $\nu(r) = -2 + r\Gamma_b$, we rewrite the above transport equation as

$$\begin{aligned}\nu\left(r^3\int_S \boldsymbol{J} J^{(0)}\right) &= r^3\nu\left(\int_S \boldsymbol{J} J^{(0)}\right) - 6r^2\int_S \boldsymbol{J} J^{(0)} + r^5\Gamma_b\boldsymbol{J}_{\ell=1,0}\\ &= -\frac{2}{r}r^3(1+O(r^{-1}))\int_S \boldsymbol{J} J^{(0)} + 2r(1+O(r^{-1}))\int_S {}^*\widehat{\rho} J^{(0)} + \mathfrak{d}^{\leqslant 1}\Gamma_g + r^3\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g)\end{aligned}$$

which we rewrite as

$$\nu\left(r^3\int_S \boldsymbol{J} J^{(0)} - 8\pi aM\right) = O(r^{-1})\left(r^3\int_S \boldsymbol{J} J^{(0)} - 8\pi aM\right) + h_3, \tag{4.36}$$

where the scalar function h_3 is given by

$$\begin{aligned}h_3 &= O(r^3)\left({}^*\widehat{\rho}_{\ell=1,0} - \frac{2aM}{r^4}\right) + O(r^3)(\boldsymbol{J} J^{(0)})_{\ell=1,0} + O(r^2){}^*\widehat{\rho}_{\ell=1,0}\\ &\quad + \mathfrak{d}^{\leqslant 1}\Gamma_g + r^3\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

In view of (4.34) and (4.31), the bootstrap assumptions 2.21 and the dominance condition (2.23),

$$|h_3| \lesssim \frac{\epsilon_0}{u^{2+2\delta}} + \frac{1}{r^2} + \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}} + r^3 \not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g) \lesssim \frac{\epsilon_0}{u^{2+2\delta}} + r^3 \not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g).$$

Thus, we infer using Corollary 4.7

$$\int_u^{u_*} |h_3| \lesssim \frac{\epsilon_0}{u^{1+2\delta}} + \int_u^{u_*} r^3 \mathfrak{d}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g) \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{u^{1+2\delta}}.$$

Applying Lemma 2.23 to (4.36) and recalling that there holds $\boldsymbol{J}_{\ell=1,0} = \frac{2aM}{r^5}$ on S_*, we deduce

$$r^3 \left| \int_S \boldsymbol{J} J^{(0)} - \frac{8\pi a M}{r^3} \right| \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{u^{1+2\delta}},$$

and hence, together with (4.35), we have obtained

$$|\boldsymbol{J}_{\ell=1,\pm}| + \left| \boldsymbol{J}_{\ell=1,0} - \frac{2aM}{r^5} \right| \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^5 u^{1+\delta}}, \tag{4.37}$$

which, because of assumption (4.13), improves the local bootstrap assumption (3.7) for $\boldsymbol{J}$.

Step 4: Estimate for $(\boldsymbol{Z}, \boldsymbol{Y})_{\ell=1}$. Using (4.32) and (4.37) we conclude

$$|\boldsymbol{Y}_{\ell=1,\pm}| + \left| \boldsymbol{Y}_{\ell=1,0} - \frac{2aM}{r^4} \right| \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^4 u^{1+\delta}},$$

which improves the preliminary estimate (3.9). This concludes the proof of Proposition 4.8. □

4.4 Control of Γ_g quantities

In this section we obtain control of the quantities in the set Γ_g, defined in (2.19). Notice that $\widecheck{\operatorname{tr}\chi} = 0$ in view of our GCM conditions. Moreover, the estimate is trivially verified[5] for $\alpha \in r^{-1}\Gamma_g$ in view of the definition of $\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}]$ in (4.10), while the improved estimate for $\nabla_\nu \alpha$ is obtained by the Bianchi identity for α.

Step 1: Estimate for $\widecheck{\operatorname{tr}\underline{\chi}}$ and $\check{\mu}$. Using the GCM conditions (2.16), we infer from Lemma 2.14 and Corollary 2.16 that

$$\not{d}_2^* \not{d}_1^* \widecheck{\operatorname{tr}\underline{\chi}} = \sum_p \underline{C}_p \not{d}_2^* \not{d}_1^* J^{(p)} = r^{-1} \not{\mathfrak{d}}^{\leqslant 3}(\Gamma_g \cdot \Gamma_g),$$

$$\not{d}_2^* \not{d}_1^* \check{\mu} = \sum_p M_p \not{d}_2^* \not{d}_1^* J^{(p)} = r^{-2} \not{\mathfrak{d}}^{\leqslant 3}(\Gamma_g \cdot \Gamma_g).$$

which yields for $k \leqslant N-3$

$$\|\not{d}_2^* \not{d}_1^* \widecheck{\operatorname{tr}\underline{\chi}}\|_{\mathfrak{h}_k(S)} \lesssim \|\not{\mathfrak{d}}^{\leqslant k+3}(\Gamma_g \cdot \Gamma_g)\|_{L^\infty(S)} \lesssim \frac{\epsilon^2}{r^4 u^{1+2\delta}},$$

$$\|\not{d}_2^* \not{d}_1^* \check{\mu}\|_{\mathfrak{h}_k(S)} \lesssim r^{-1} \|\not{\mathfrak{d}}^{\leqslant k+3}(\Gamma_g \cdot \Gamma_g)\|_{L^\infty(S)} \lesssim \frac{\epsilon^2}{r^5 u^{1+2\delta}}.$$

Together with Corollary 2.15, we infer from Propositions 3.4 and 3.2 that for $k \leqslant N-3$

$$\|\widecheck{\operatorname{tr}\underline{\chi}}\|_{\mathfrak{h}_{k+2}(S)} \lesssim \frac{\epsilon^2}{r^2 u^{1+2\delta}} + r|(\widecheck{\operatorname{tr}\underline{\chi}})_{\ell=1}| + r|\overline{\widecheck{\operatorname{tr}\underline{\chi}}}| \lesssim \frac{\epsilon_0}{r u^{2+3\delta}} + \frac{\epsilon_0}{r u^{1+2\delta}} \lesssim \frac{\epsilon_0}{r u^{1+2\delta}},$$

[5] In the context of linear stability of Reissner-Nordström, the estimate for α can in fact be deduced from the ones for $\mathfrak{f}$ and $\mathfrak{b}$.

$$\|\check{\mu}\|_{\mathfrak{h}_{k+2}(S)} \lesssim \frac{\epsilon^2}{r^3 u^{1+2\delta}} + r|\mu_{\ell=1}| + r|\overline{\check{\mu}}| \lesssim \frac{\epsilon_0}{r^2 u^{2+3\delta}} + \frac{\epsilon_0}{r^2 u^{1+2\delta}} \lesssim \frac{\epsilon_0}{r^2 u^{1+2\delta}},$$

where we used the dominance condition (2.23). Together with Sobolev, this implies, for $k \leqslant N-3$,

$$|\not{\mathfrak{d}}^k \widetilde{\mathrm{tr}\,\underline{\chi}}| \lesssim \frac{\epsilon_0}{r^2 u^{1+2\delta}}, \qquad |\not{\mathfrak{d}}^k \check{\mu}| \lesssim \frac{\epsilon_0}{r^3 u^{1+2\delta}}, \tag{4.38}$$

which gives the first line of (4.15).

Step 2: Estimate for $(\widetilde{{}^{(\mathbf{F})}\rho}, {}^{*\,(\mathbf{F})}\rho)$**.** From the definition of $\mathfrak{q}^{\mathbf{F}}$ in (4.7), we infer

$$\|\not{d}_2^* \not{d}_1^* (\widetilde{{}^{(\mathbf{F})}\rho}, -{}^{*\,(\mathbf{F})}\rho)\|_{\mathfrak{h}_k(S)} \lesssim r^{-2}\|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{q}^{\mathbf{F}}\|_{L^\infty(S)} + r^{-4} + r^{-2}\|\not{\mathfrak{d}}^{\leqslant k+2}\Gamma_b\|_{L^\infty(S)} + \|\not{\mathfrak{d}}^{\leqslant k+2}(\Gamma_b\cdot\Gamma_g)\|_{L^\infty(S)}.$$

Using the bootstrap assumptions (2.21) the dominance condition (2.23), and the control of Γ_b obtained in (4.17), we obtain for $k \leqslant N-11$,

$$\begin{aligned}
\|\not{d}_2^* \not{d}_1^* (\widetilde{{}^{(\mathbf{F})}\rho}, -{}^{*\,(\mathbf{F})}\rho)\|_{\mathfrak{h}_k(S)} &\lesssim r^{-2}\|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{q}^{\mathbf{F}}\|_{L^\infty(S)} + \frac{\epsilon_0}{r^3 u^{1+\delta}} + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^3 u^{1+\delta}} \\
&\lesssim r^{-2}\|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{q}^{\mathbf{F}}\|_{L^\infty(S)} + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^3 u^{1+\delta}}.
\end{aligned}$$

In view of Corollary 2.15 and (3.1), we deduce, for $k \leqslant N-11$,

$$\begin{aligned}
\|(\widetilde{{}^{(\mathbf{F})}\rho}, -{}^{*\,(\mathbf{F})}\rho)\|_{\mathfrak{h}_{k+2}(S)} \lesssim \|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{q}^{\mathbf{F}}\|_{L^\infty(S)} &+ \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r u^{1+\delta}} \\
&+ r|(\widetilde{{}^{(\mathbf{F})}\rho}, -{}^{*\,(\mathbf{F})}\rho)_{\ell=1}|.
\end{aligned} \tag{4.39}$$

We now control the $\ell = 1$ mode of $\widetilde{{}^{(\mathbf{F})}\rho}$ and ${}^{*\,(\mathbf{F})}\rho$. From the definitions (2.31) and (2.32), we have

$$\begin{aligned}
\check{\rho} &= \widehat{\rho} + r^{-2}\Gamma_g + \Gamma_g\cdot\Gamma_b, \\
{}^*\rho &= {}^*\widehat{\rho} + r^{-2}\Gamma_g + \Gamma_g\cdot\Gamma_b.
\end{aligned}$$

From Proposition 3.4 and the bootstrap assumptions, we deduce

$$|(\check{\rho})_{\ell=1}| + \left|({}^*\rho)_{\ell=1,\pm}\right| + \left|({}^*\rho)_{\ell=1,0} - \frac{2aM}{r^4}\right| \lesssim \frac{\epsilon_0}{r^3 u^{1+2\delta}} + \frac{\epsilon}{r^4 u^{\frac{1}{2}+\delta}} + \frac{\epsilon^2}{r^3 u^{1+2\delta}} \lesssim \frac{\epsilon_0}{r^3 u^{1+2\delta}},$$

where we used the dominance condition. Finally, using (4.6) we have

$$\not{d}_1 \not{d}_1^* ({}^{(\mathbf{F})}\rho, {}^{*\,(\mathbf{F})}\rho) = r^{-2}\not{d}_1 \Re(\mathfrak{p}) + r\not{d}_1\not{d}_1^*(\rho, {}^*\rho) + O(r^{-5}) + r^{-3}\mathfrak{d}^{\leqslant 1}\Gamma_b + r^{-1}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).$$

Projecting the above identity to $\ell = 1$ and using the bootstrap assumptions and dominance condition, Lemma 2.13 and Sobolev, and the control of Γ_b obtained in (4.17) we deduce

$$\begin{aligned}
\left|(\widetilde{{}^{(\mathbf{F})}\rho})\right|_{\ell=1} + |({}^{*\,(\mathbf{F})}\rho)_{\ell=1}| &\lesssim r^{-1}\|\not{\mathfrak{d}}^{\leqslant 3}\mathfrak{p}\|_{L^\infty(S)} + \frac{\epsilon_0}{r^2 u^{1+2\delta}} + \frac{1}{r^3} + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}} \\
&\lesssim r^{-1}\|\not{\mathfrak{d}}^{\leqslant 3}\mathfrak{p}\|_{L^\infty(S)} + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}}.
\end{aligned}$$

In view of Proposition 3.2 and Step 2, we infer from (4.39) for $k \leqslant N-11$,

$$\|(\widetilde{{}^{(\mathbf{F})}\rho}, -{}^{*\,(\mathbf{F})}\rho)\|_{\mathfrak{h}_{k+2}(S)} \lesssim \|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{q}^{\mathbf{F}}\|_{L^\infty(S)} + \|\not{\mathfrak{d}}^{\leqslant 3}\mathfrak{p}\|_{L^\infty(S)} + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r u^{1+\delta}}. \tag{4.40}$$

Together with Sobolev, this implies for $k \leqslant N-11$,

$$|\not{\mathfrak{d}}^k \widetilde{{}^{(\mathbf{F})}\rho}| + |\not{\mathfrak{d}}^{k} {}^{*\,(\mathbf{F})}\rho| \lesssim \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}}. \tag{4.41}$$

Step 3: Estimate for $(\check{\rho}, {}^*\rho)$. From (4.6), we infer that

$$\begin{aligned}\|\not{d}_1^*(-\check{\rho}, {}^*\rho)\|_{\mathfrak{h}_k(S)} \lesssim\ & r^{-2}\|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{p}\|_{L^\infty(S)} + r^{-1}|\not{\mathfrak{d}}^{\leqslant k}({}^{(\mathbf{F})}\rho, {}^{*\,(\mathbf{F})}\rho)| \\ &+ r^{-4} + r^{-2}\|\not{\mathfrak{d}}^{\leqslant k+2}\Gamma_b\|_{L^\infty(S)} + \|\not{\mathfrak{d}}^{\leqslant k+2}(\Gamma_b \cdot \Gamma_g)\|_{L^\infty(S)}.\end{aligned}$$

Using the bootstrap assumptions and dominance condition, and the control of Γ_b obtained in (4.17) and (4.41) we deduce for $k \leqslant N-11$,

$$\|\not{d}_1^*(-\check{\rho}, {}^*\rho)\|_{\mathfrak{h}_k(S)} \lesssim r^{-1}\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^3 u^{1+\delta}}.$$

In view of Lemma 2.13, we deduce, for $k \leqslant N-11$,

$$\|(-\check{\rho}, {}^*\rho)\|_{\mathfrak{h}_{k+1}(S)} \lesssim \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}} + r|(\overline{\check{\rho}}, \overline{{}^*\rho})|.$$

In view of Proposition 3.2, we infer for $k \leqslant N-11$,

$$\|(\check{\rho}, {}^*\rho)\|_{\mathfrak{h}_{k+2}(S)} \lesssim \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}}. \tag{4.42}$$

Together with Sobolev, this implies for $k \leqslant N-11$,

$$|\not{\mathfrak{d}}^k\check{\rho}| + |\not{\mathfrak{d}}^k {}^*\rho| \lesssim r^{-1}\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^3 u^{1+\delta}}. \tag{4.43}$$

Step 4: Estimate for ζ. Using that $\not{d}_1\zeta = (-\check{\mu} - \check{\rho}, {}^*\rho) + \Gamma_b \cdot \Gamma_g$, in view of Lemma 2.13, we infer that

$$\|\zeta\|_{\mathfrak{h}_{k+1}(S)} \lesssim r\|\check{\mu}\|_{\mathfrak{h}_k(S)} + r\|\check{\rho}\|_{\mathfrak{h}_k(S)} + r\|{}^*\rho\|_{\mathfrak{h}_k(S)} + r^2\|\not{\mathfrak{d}}^{\leqslant k}(\Gamma_b \cdot \Gamma_g)\|_{L^\infty(S)}.$$

Together with (4.38) and (4.42) and the bootstrap assumptions of Lemma 2.21, we obtain, for $k \leqslant N-11$,

$$\begin{aligned}\|\zeta\|_{\mathfrak{h}_{k+1}(S)} &\lesssim \frac{\epsilon_0}{r^2 u^{1+2\delta}} + r\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r u^{1+\delta}} + \frac{\epsilon^2}{r u^{\frac{3}{2}+2\delta}} &(4.44)\\ &\lesssim r\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r u^{1+\delta}}. &(4.45)\end{aligned}$$

Together with Sobolev, this implies, for $k \leqslant N-11$,

$$|\not{\mathfrak{d}}^k\zeta| \lesssim \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}}. \tag{4.46}$$

Step 5: Estimate for ${}^{(\mathbf{F})}\beta$. We first control the $\ell = 1$ mode of $\not{d}_1\, {}^{(\mathbf{F})}\beta$. From the definition (2.13) of $(\boldsymbol{C}, \boldsymbol{J})$ we have

$$\not{d}_1\beta = (\boldsymbol{C}, \boldsymbol{J}) + r^{-3}\mathfrak{d}^{\leqslant 1}\Gamma_g.$$

Then using Proposition 4.8, we have

$$
\begin{aligned}
|(\not{d}_1\beta)_{\ell=1}| &\lesssim |\boldsymbol{C}_{\ell=1}| + |\boldsymbol{J}_{\ell=1,\pm}| + \left|\boldsymbol{J}_{\ell=1,0} - \frac{2aM}{r^5}\right| + \frac{1}{r^5} + r^{-3}\mathfrak{d}^{\leqslant 1}\Gamma_g \\
&\lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^5 u^{1+\delta}} + \frac{1}{r^5} + \frac{\epsilon_0}{r^6 u^{\frac{1}{2}+\delta}} \lesssim \frac{\epsilon_0}{r^4 u^{1+\delta}}.
\end{aligned}
$$

Using (4.3), we have

$$
6M\not{d}_1\, {}^{(\mathbf{F})}\beta = r^3\not{d}_1\mathfrak{b} - 2Qr\not{d}_1\beta + r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g).
$$

Using the above bound, the bootstrap assumptions (2.21) and dominance condition (2.23), we deduce that

$$
\begin{aligned}
|(\not{d}_1\, {}^{(\mathbf{F})}\beta)_{\ell=1}| &\lesssim r^3|\not{d}_1\mathfrak{b}| + r|(\not{d}_1\beta)_{\ell=1}| + r^{-5}|\mathfrak{d}^{\leqslant 1}\Gamma_g| + r^{-2}|\mathfrak{d}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g)| \\
&\lesssim r^3|\not{d}_1\mathfrak{b}| + \frac{\epsilon_0}{r^3 u^{1+\delta}} + \frac{\epsilon}{r^7 u^{\frac{1}{2}+\delta}} + \frac{\epsilon_0}{r^6 u^{1+2\delta}} \\
&\lesssim r^3|\not{d}_1\mathfrak{b}| + \frac{\epsilon_0}{r^3 u^{1+\delta}}.
\end{aligned}
$$

Using (4.1), we have

$$
\not{d}_2^*\, {}^{(\mathbf{F})}\beta = -\mathfrak{f} + O(r^{-2})\Gamma_g + \Gamma_g \cdot \Gamma_g.
$$

In view of Lemma 2.13, we infer that

$$
\begin{aligned}
\|{}^{(\mathbf{F})}\beta\|_{\mathfrak{h}_{k+1}(S)} &\lesssim r\|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{f}\|_{L^2(S)} + \|\not{\mathfrak{d}}^{\leqslant k}\Gamma_g\|_{L^\infty(S)} + r^2\|\not{\mathfrak{d}}^{\leqslant k}(\Gamma_g \cdot \Gamma_g)\|_{L^\infty(S)} + r^2\big|(\not{d}_1\, {}^{(\mathbf{F})}\beta)_{\ell=1}\big| \\
&\lesssim r^2\|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{f}\|_{L^\infty(S)} + \|\not{\mathfrak{d}}^{\leqslant k}\Gamma_g\|_{L^\infty(S)} + r^2\|\not{\mathfrak{d}}^{\leqslant k}(\Gamma_g \cdot \Gamma_g)\|_{L^\infty(S)} + r^2\big|(\not{d}_1\, {}^{(\mathbf{F})}\beta)_{\ell=1}\big|.
\end{aligned}
$$

Using the bootstrap assumptions (2.21), the dominance condition (2.23), and the above we obtain

$$
\begin{aligned}
\|{}^{(\mathbf{F})}\beta\|_{\mathfrak{h}_{k+1}(S)} &\lesssim r^2\|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{f}\|_{L^\infty(S)} + r^4\|\not{\mathfrak{d}}^{\leqslant k+1}\mathfrak{b}\|_{L^\infty(S)} + \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}} + \frac{\epsilon}{r^2 u^{1+2\delta}} + \frac{\epsilon_0}{r u^{1+\delta}} \\
&\lesssim r^2\|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{f}\|_{L^\infty(S)} + r^4\|\not{\mathfrak{d}}^{\leqslant k+1}\mathfrak{b}\|_{L^\infty(S)} + \frac{\epsilon_0}{r u^{1+\delta}}.
\end{aligned}
$$

Together with Sobolev, this implies for $k \leqslant N$,

$$
|\not{\mathfrak{d}}^k\, {}^{(\mathbf{F})}\beta| \lesssim \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\epsilon_0}{r^2 u^{1+\delta}}.
$$

Step 6: Estimate for β. Using (4.3), the bootstrap assumptions (2.21) and the dominance condition (2.23), we immediately deduce that for $k \leqslant N$,

$$
\begin{aligned}
|\not{\mathfrak{d}}^k\beta| &\lesssim r^2|\not{\mathfrak{d}}^k\mathfrak{b}| + r^{-1}|\not{\mathfrak{d}}^k\, {}^{(\mathbf{F})}\beta| + r^{-5}|\not{\mathfrak{d}}^k\Gamma_g| + r^{-2}|\not{\mathfrak{d}}^k(\Gamma_g \cdot \Gamma_g)| \\
&\lesssim \|\not{\mathfrak{d}}^{\leqslant k}\mathfrak{f}\|_{L^\infty(S)} + r^2\|\not{\mathfrak{d}}^{\leqslant k+1}\mathfrak{b}\|_{L^\infty(S)} + \frac{\epsilon_0}{r^3 u^{1+\delta}} + \frac{\epsilon}{r^7 u^{\frac{1}{2}+\delta}} + \frac{\epsilon_0}{r^6 u^{1+2\delta}} \\
&\lesssim r^{-1}\mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\epsilon_0}{r^3 u^{1+\delta}}.
\end{aligned}
\tag{4.47}
$$

Step 7: Estimate for $\widehat{\chi}$. Using the Codazzi equation for $\widehat{\chi}$, from Lemma 2.13, we infer that

$$
\|\widehat{\chi}\|_{\mathfrak{h}_{k+1}(S)} \lesssim \|\zeta\|_{\mathfrak{h}_k(S)} + r\|\beta\|_{\mathfrak{h}_k(S)} + \|\not{\mathfrak{d}}^{\leqslant k}\Gamma_g\|_{L^\infty(S)} + r^2\|\not{\mathfrak{d}}^{\leqslant k}(\Gamma_g \cdot \Gamma_g)\|_{L^\infty(S)}.
$$

Together with (4.44) and (4.47) and the bootstrap assumptions (2.21), and Sobolev, we obtain for $3 \leqslant k \leqslant N-11$,

$$|\mathfrak{d}\!\!/^k \widehat{\chi}| \lesssim \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}}. \tag{4.48}$$

Step 8: Estimate for $\nabla_\nu \Gamma_g$. From the Maxwell equations, null structure equations and Bianchi identities, one observes that all quantities in Γ_g verify schematically

$$\nabla_\nu \Gamma_g = r^{-1} \mathfrak{d}\!\!/^{\leqslant 1} \Gamma_b + r^{-1} \Gamma_g + \Gamma_b \cdot \Gamma_b.$$

Combining with the control of the Γ_b quantities obtained in (4.17), the bootstrap assumptions (2.21) and dominance condition (2.23), we infer for $k \leqslant N-9$,

$$\left|\mathfrak{d}_*^{\leqslant k-1} \nabla_\nu \Gamma_g\right| \lesssim r^{-1} \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}}.$$

Using the dominance condition and (4.11), we deduce

$$\left|\mathfrak{d}_*^{\leqslant k-1} \nabla_\nu \Gamma_g\right| \lesssim \frac{\epsilon_0}{u^{1+\delta}} \frac{\epsilon}{r^2 u^{\frac{1}{2}+\delta}} + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}} \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^2 u^{1+\delta}}.$$

Step 9: Estimate for $\nabla_\nu \beta$ and $\nabla_\nu {}^{(\mathbf{F})}\beta$. In view of the Bianchi identity for $\nabla_3 \beta$ and the Maxwell equation for $\nabla_3 {}^{(\mathbf{F})}\beta$, and using the fact that $\nu = e_3 + b_* e_4$, we have

$$\begin{aligned}
\nabla_\nu \beta &= r^{-2} \mathfrak{d}\!\!/^{\leqslant 1} \Gamma_g + r^{-3} \Gamma_b + r^{-1} \mathfrak{d}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g), \\
\nabla_\nu {}^{(\mathbf{F})}\beta &= r^{-1} \mathfrak{d}\!\!/^{\leqslant 1} \Gamma_g + r^{-2} \Gamma_b + \Gamma_b \cdot \Gamma_g,
\end{aligned}$$

from which we infer for $3 \leqslant k \leqslant N-11$,

$$r\left|\mathfrak{d}_*^{\leqslant k-1} \nabla_\nu \beta\right| + \left|\mathfrak{d}_*^{\leqslant k-1} \nabla_\nu {}^{(\mathbf{F})}\beta\right| \lesssim r^{-1} \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^3 u^{1+\delta}}.$$

Similarly, applying another ∇_ν derivative one obtains

$$\begin{aligned}
\nabla_\nu^2 \beta &= r^{-2} \mathfrak{d}\!\!/^{\leqslant 1} \nabla_\nu \Gamma_g + r^{-3} \mathfrak{d}\!\!/^{\leqslant 2} \Gamma_g + r^{-3} \mathfrak{d}^{\leqslant 1} \Gamma_b + r^{-1} \mathfrak{d}^{\leqslant 2}(\Gamma_b \cdot \Gamma_g), \\
\nabla_\nu^2 {}^{(\mathbf{F})}\beta &= r^{-1} \mathfrak{d}\!\!/^{\leqslant 1} \nabla_\nu \Gamma_g + r^{-2} \mathfrak{d}\!\!/^{\leqslant 2} \Gamma_g + r^{-2} \mathfrak{d}^{\leqslant 1} \Gamma_b + \mathfrak{d}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g).
\end{aligned}$$

Using the improved decay for $\nabla_\nu \Gamma_g$ of Step 8, we obtain for $3 \leqslant k \leqslant N-11$,

$$r\left|\mathfrak{d}_*^{\leqslant k-2} \nabla_\nu^2 \beta\right| + \left|\mathfrak{d}_*^{\leqslant k-2} \nabla_\nu^2 {}^{(\mathbf{F})}\beta\right| \lesssim \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r^3 u^{1+\delta}},$$

as stated.

This concludes the proof of Theorem 4.3.

4.5 Control of the canonical one-forms

Here we collect a consequence of Theorem 4.3 regarding the control of the canonical one-forms $J^{(p)}$.

Corollary 4.9. *We have the following estimate for ϕ on Σ_* for $k \leqslant N-11$:*

$$\|\mathfrak{d}\!\!/^k \phi\|_{L^\infty(S_*)} \lesssim r \mathcal{G}^k[\alpha, \mathfrak{b}, \mathfrak{f}, \mathfrak{p}, \mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha}, \underline{\mathfrak{b}}, \mathfrak{q}^{\mathbf{F}}] + \epsilon_0^2}}{r u^{1+\delta}}.$$

The functions $J^{(p)}$ verify the following properties on Σ_*:

1. *We have*

$$\int_S J^{(p)} = r^3\mathcal{G}^0[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}] + \frac{r\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{u^{1+\delta}},$$

$$\int_S J^{(p)}J^{(q)} = \frac{4\pi}{3}r^2\delta_{pq} + r^3\mathcal{G}^0[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}] + \frac{r\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{u^{1+\delta}}.$$

2. *We have*

$$\left|\left(\triangle+\frac{2}{r^2}\right)J^{(p)}\right| \lesssim r^{-1}\mathcal{G}^0[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^3u^{1+\delta}},$$

$$|{\not{d}}_2^*{\not{d}}_1^*J^{(p)}| \lesssim r^{-1}\mathcal{G}^1[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^3u^{1+\delta}}.$$

3. *We have for all $k\leqslant N-12$*

$$\left|\mathfrak{d}_*^k\left[\widetilde{\nabla J^{(p)}}\right]\right| \lesssim \mathcal{G}^k[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^2u^{\frac{1}{2}+\delta}}.$$

Proof. The proof of the first five estimates follows from the one of Lemma 2.14 upon replacing the bootstrap assumptions for $\check{K}$ with the estimated control for $\check{K}\in r^{-1}\Gamma_g$ given by Theorem 4.3.

For the last estimate, using that on S_* (see Lemma 5.61 in [18]) we have

$$\widetilde{\nabla J^{(0)}} = -\frac{1}{r}(e^{-\phi}-1)^*f_0, \qquad \widetilde{\nabla J^{(+)}} = \frac{1}{r}(e^{-\phi}-1)f_+, \qquad \widetilde{\nabla J^{(-)}} = \frac{1}{r}(e^{-\phi}-1)f_-.$$

and using the control of ϕ above we deduce on S_*

$$|\not{\mathfrak{d}}^k\widetilde{\nabla J^{(p)}}| \lesssim \left|\mathfrak{d}_*^k\left[\widetilde{\nabla J^{(p)}}\right]\right| \lesssim \mathcal{G}^k[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^2u^{1+\delta}}.$$

On Σ_* we have (see also Lemma 5.66 in [18])

$$\nabla_\nu\left[r\widetilde{\nabla J^{(p)}}\right] = \Gamma_b\cdot\not{\mathfrak{d}}^{\leqslant 1}J^{(p)}, \tag{4.49}$$

and integrating it along Σ_* using Lemma 2.23, we deduce for $k\leqslant N-11$,

$$\begin{aligned}
r\left|\not{\mathfrak{d}}^k\left[r\widetilde{\nabla J^{(p)}}\right]\right| &\lesssim r\left|\not{\mathfrak{d}}^k\left[r\widetilde{\nabla J^{(p)}}\right]\right|_{L^\infty(S_*)} + \int_u^{u_*} r|\not{\mathfrak{d}}^{\leqslant k}\Gamma_b| \\
&\lesssim r\left|\not{\mathfrak{d}}^k\left[r\widetilde{\nabla J^{(p)}}\right]\right|_{L^\infty(S_*)} + \left(\int_u^{u_*}\frac{du'}{u'^{2+2\delta}}\right)^{\frac{1}{2}}\left(\int_u^{u_*}u'^{2+2\delta}\|\mathfrak{d}^{\leqslant k+2}\Gamma_b\|^2_{L^2(S)}du'\right)^{\frac{1}{2}} \\
&\lesssim r^2\mathcal{G}^k[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{u^{\frac{1}{2}+\delta}},
\end{aligned}$$

where we used Sobolev, Hölder inequalities and Proposition 4.6 for the control of the fluxes of Γ_b.

We finally infer that on Σ_* for $k\leqslant N-11$,

$$|\not{\mathfrak{d}}^k\widetilde{\nabla J^{(p)}}| \lesssim \mathcal{G}^k[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}] + \frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^2u^{\frac{1}{2}+\delta}}.$$

from which, using again (4.49), we deduce the stated. □

Corollary 4.10. *We have on* Σ_**:*

$$\left|\left((\boldsymbol{C},\boldsymbol{J})-\frac{3aM}{r^4}\not{d}_1 f_0\right)_{\ell=1}\right|\lesssim\frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^5u^{1+\delta}}.$$

Proof. Recall that

$$\not{d}_1 f_0=(\mathrm{div}\, f_0,\mathrm{curl}\, f_0)=(\mathrm{div}\, f_0,\frac{2}{r}\cos\theta+\widetilde{\mathrm{curl}(f_0)})=\left(\mathrm{div}\, f_0,\frac{2}{r}J^{(0)}+\widetilde{\mathrm{curl}(f_0)}\right).$$

In particular we can write

$$\left((\boldsymbol{C},\boldsymbol{J})-\frac{3aM}{r^4}\not{d}_1 f_0\right)_{\ell=1}=\left(\boldsymbol{C},\boldsymbol{J}-\frac{6aM}{r^5}J^{(0)}\right)_{\ell=1}+r^{-4}\mathcal{G}^0[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}]+\frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^6u^{\frac{1}{2}+\delta}},$$

and therefore from Proposition 3.4 and Proposition 4.8,

$$\begin{aligned}\left|\left((\boldsymbol{C},\boldsymbol{J})-\frac{3aM}{r^4}\not{d}_1 f_0)\right)_{\ell=1}\right|&\lesssim|\boldsymbol{J}_{\ell=1,\pm}|+\left|\boldsymbol{J}_{\ell=1,0}-\frac{2aM}{r^5}\right|\\&\quad+r^{-4}\mathcal{G}^0[\alpha,\mathfrak{b},\mathfrak{f},\mathfrak{p},\mathfrak{q}^{\mathbf{F}}]+\frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^6u^{\frac{1}{2}+\delta}}\\&\lesssim\frac{\sqrt{\mathcal{F}^N[\underline{\alpha},\underline{\mathfrak{b}},\mathfrak{q}^{\mathbf{F}}]+\epsilon_0^2}}{r^5u^{1+\delta}}\end{aligned}$$

where we used Corollary 4.9, and the dominance condition. □

A Proof of Proposition 2.19

Here we derive the transport equation of the renormalized quantities.

Transport equation of $\widehat{\rho}$. From Proposition 2.10, we deduce

$$\begin{aligned}\nabla_3\left(\frac{Q}{r^2}\widetilde{{}^{(\mathbf{F})}\rho}\right)&=\frac{8Q\Upsilon}{r^3}\widetilde{{}^{(\mathbf{F})}\rho}-\frac{2Q}{r^2}\mathrm{div}\,{}^{(\mathbf{F})}\underline{\beta}-\frac{2Q^2}{r^4}\underline{\check{\kappa}}+\frac{4Q^2}{r^5}\check{y}+r^{-1}\Gamma_g\cdot\Gamma_b\\&=r^{-3}\Gamma_b+r^{-1}\Gamma_g\cdot\Gamma_b,\\\nabla_3(\widehat{\chi}\cdot\underline{\widehat{\chi}})&=\widehat{\chi}\cdot\left(\frac{2\Upsilon}{r}\underline{\widehat{\chi}}-\underline{\alpha}-\left(\frac{2M}{r^2}-\frac{2Q^2}{r^3}\right)\underline{\widehat{\chi}}+\nabla\widehat{\otimes}\underline{\xi}+\Gamma_b\cdot\Gamma_b\right)\\&\quad+\left(\frac{\Upsilon}{r}\widehat{\chi}+\nabla\widehat{\otimes}\eta-\frac{1}{r}\underline{\widehat{\chi}}+\left(\frac{2M}{r^2}-\frac{2Q^2}{r^3}\right)\widehat{\chi}+\Gamma_b\cdot\Gamma_b\right)\cdot\underline{\widehat{\chi}}\\&=-\widehat{\chi}\cdot\underline{\alpha}+r^{-1}\not{\mathfrak{d}}^{\leq1}(\Gamma_b\cdot\Gamma_b),\end{aligned}$$

Combining the above with the equation for $\nabla_3\check{\rho}$ in Proposition 2.10, we obtain for $\widehat{\rho}=\check{\rho}-\frac{2Q}{r^2}\widetilde{{}^{(\mathbf{F})}\rho}-\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}$,

$$\begin{aligned}\nabla_3\widehat{\rho}&=\frac{3\Upsilon}{r}\check{\rho}-\mathrm{div}\,\underline{\beta}-\frac{1}{2}\widehat{\chi}\cdot\underline{\alpha}+\frac{1}{2}\widehat{\chi}\cdot\underline{\alpha}+r^{-3}\Gamma_b+r^{-1}\not{\mathfrak{d}}^{\leq1}(\Gamma_b\cdot\Gamma_b)\\&=\frac{3}{r}\widehat{\rho}-\underline{\boldsymbol{C}}+r^{-3}\not{\mathfrak{d}}^{\leq1}\Gamma_b+r^{-1}\not{\mathfrak{d}}^{\leq1}(\Gamma_b\cdot\Gamma_b).\end{aligned}$$

Using that $\nabla_4\left(\frac{Q}{r^2}\widetilde{{}^{(\mathbf{F})}\rho}\right)=r^{-3}\mathfrak{d}^{\leq1}\Gamma_g$ and $\nabla_4(\widehat{\chi}\cdot\underline{\widehat{\chi}})=r^{-1}\mathfrak{d}^{\leq1}(\Gamma_b\cdot\Gamma_g)$ and combining with the equation for $\nabla_4\check{\rho}$ in Proposition 2.10 we obtain

$$\begin{aligned}\nabla_4\widehat{\rho}&=-\frac{3}{r}\check{\rho}+\mathrm{div}\,\beta+r^{-3}\mathfrak{d}^{\leq1}\Gamma_g+r^{-1}\mathfrak{d}^{\leq1}(\Gamma_b\cdot\Gamma_g)\\&=-\frac{3}{r}\widehat{\rho}+\boldsymbol{C}+r^{-3}\mathfrak{d}^{\leq1}\Gamma_g+r^{-1}\mathfrak{d}^{\leq1}(\Gamma_b\cdot\Gamma_g).\end{aligned}$$

Since $\nu = e_3 + b_* e_4$ and $b_* = -1 - \frac{2M}{r} + \frac{Q^2}{r^2} + r\Gamma_b$, we infer

$$\nabla_\nu \widehat{\rho} = \frac{6}{r}\widehat{\rho} - \underline{\boldsymbol{C}} - (1 + O(r^{-1}))\boldsymbol{C} + r^{-3}\mathfrak{d}^{\leqslant 1}\Gamma_b + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_b),$$

as stated.

Transport equation of $\boldsymbol{C}$ and $\boldsymbol{J}$. Taking the divergence and the curl of the equation for $\nabla_3\beta$ of Proposition 2.10, we infer

$$\begin{aligned}
\nabla_3 \operatorname{div}\beta - \frac{2\Upsilon}{r}\operatorname{div}\beta &= [\nabla_3, \operatorname{div}]\beta + \Delta\rho + \left(\frac{2M}{r^2} - \frac{2Q^2}{r^3}\right)\operatorname{div}\beta - \left(\frac{6M}{r^3} - \frac{6Q^2}{r^4}\right)\operatorname{div}\eta \\
&\quad + \frac{Q}{r^2}\Delta\,{}^{(\mathbf{F})}\rho + \frac{\Upsilon Q}{r^3}\operatorname{div}\,{}^{(\mathbf{F})}\beta - \frac{2Q}{r^3}\operatorname{div}\,{}^{(\mathbf{F})}\underline{\beta} + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g), \\
\nabla_3 \operatorname{curl}\beta - \frac{2\Upsilon}{r}\operatorname{curl}\beta &= [\nabla_3, \operatorname{curl}]\beta - \Delta^*\rho + \left(\frac{2M}{r^2} - \frac{2Q^2}{r^3}\right)\operatorname{curl}\beta - \left(\frac{6M}{r^3} - \frac{6Q^2}{r^4}\right)\operatorname{curl}\eta \\
&\quad + \frac{Q}{r^2}\Delta\,{}^{(\mathbf{F})*}\rho + \frac{\Upsilon Q}{r^3}\operatorname{curl}\,{}^{(\mathbf{F})}\beta - \frac{2Q}{r^3}\operatorname{curl}\,{}^{(\mathbf{F})}\underline{\beta} + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g).
\end{aligned}$$

According to Lemma 2.12, using the equation for $\nabla_3\beta$,

$$\begin{aligned}
[\nabla_3, \operatorname{div}]\beta &= \frac{\Upsilon}{r}\operatorname{div}\beta + \Gamma_b \cdot \nabla_3\beta + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_b) = \frac{\Upsilon}{r}\operatorname{div}\beta + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_b) \\
[\nabla_3, \operatorname{curl}]\beta &= \frac{\Upsilon}{r}\operatorname{curl}\beta + \Gamma_b \cdot \nabla_3\beta + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_b) = \frac{\Upsilon}{r}\operatorname{curl}\beta + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_b).
\end{aligned}$$

Hence,

$$\begin{aligned}
\nabla_3 \operatorname{div}\beta = &\left(\frac{3}{r} + O(r^{-2})\right)\operatorname{div}\beta + \Delta\check{\rho} - \left(\frac{6M}{r^3} - \frac{6Q^2}{r^4}\right)\operatorname{div}\eta \\
&+ \frac{Q}{r^2}\Delta\,\widetilde{{}^{(\mathbf{F})}\rho} + \frac{Q}{r^3}\operatorname{div}\,{}^{(\mathbf{F})}\beta - \frac{2Q}{r^3}\operatorname{div}\,{}^{(\mathbf{F})}\underline{\beta} + r^{-5}\not{\mathfrak{d}}^{\leqslant 1}\Gamma_g + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g), \\
\nabla_3 \operatorname{curl}\beta = &\left(\frac{3}{r} + O(r^{-2})\right)\operatorname{curl}\beta - \Delta^*\rho - \frac{6M}{r^3}{}^*\rho \\
&+ \frac{Q}{r^2}\Delta\,{}^{(\mathbf{F})*}\rho + \frac{Q}{r^3}\operatorname{curl}\,{}^{(\mathbf{F})}\beta - \frac{2Q}{r^3}\operatorname{curl}\,{}^{(\mathbf{F})}\underline{\beta} + r^{-5}\not{\mathfrak{d}}^{\leqslant 1}\Gamma_g + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g),
\end{aligned}$$

where we used that $\operatorname{curl}\eta = {}^*\rho + \Gamma_b \cdot \Gamma_g$.

Taking the divergence and the curl of the equation for $\nabla_3\,{}^{(\mathbf{F})}\beta$ of Proposition 2.10, we infer

$$\begin{aligned}
\nabla_3 \operatorname{div}\,{}^{(\mathbf{F})}\beta - \triangle\,{}^{(\mathbf{F})}\rho &= [\nabla_3, \operatorname{div}]\,{}^{(\mathbf{F})}\beta + \left(\frac{\Upsilon}{r} + \frac{2M}{r^2} - \frac{2Q^2}{r^3}\right)\operatorname{div}\,{}^{(\mathbf{F})}\beta + \frac{2Q}{r^2}\operatorname{div}\eta + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g), \\
\nabla_3 \operatorname{curl}\,{}^{(\mathbf{F})}\beta - \triangle^*\,{}^{(\mathbf{F})}\rho &= [\nabla_3, \operatorname{curl}]\,{}^{(\mathbf{F})}\beta + \left(\frac{\Upsilon}{r} + \frac{2M}{r^2} - \frac{2Q^2}{r^3}\right)\operatorname{curl}\,{}^{(\mathbf{F})}\beta + \frac{2Q}{r^2}\operatorname{curl}\eta + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g),
\end{aligned}$$

Then recall from Lemma 2.12 that

$$\begin{aligned}
[\nabla_3, \operatorname{div}]\,{}^{(\mathbf{F})}\beta &= \frac{\Upsilon}{r}\operatorname{div}\,{}^{(\mathbf{F})}\beta + \Gamma_g \cdot \nabla_3\,{}^{(\mathbf{F})}\beta + r^{-1}\Gamma_b \cdot \mathfrak{d}^{\leqslant 1}\,{}^{(\mathbf{F})}\beta = \frac{\Upsilon}{r}\operatorname{div}\,{}^{(\mathbf{F})}\beta + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g), \\
[\nabla_3, \operatorname{curl}]\,{}^{(\mathbf{F})}\beta &= \frac{\Upsilon}{r}\operatorname{curl}\,{}^{(\mathbf{F})}\beta + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g).
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
\nabla_3 \operatorname{div}\,{}^{(\mathbf{F})}\beta - \triangle\,\widetilde{{}^{(\mathbf{F})}\rho} &= \frac{2}{r}\operatorname{div}\,{}^{(\mathbf{F})}\beta + \frac{2Q}{r^2}\operatorname{div}\eta + r^{-3}\not{\mathfrak{d}}^{\leqslant 1}\Gamma_g + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g), \\
\nabla_3 \operatorname{curl}\,{}^{(\mathbf{F})}\beta - \triangle^*\,{}^{(\mathbf{F})}\rho &= \frac{2}{r}\operatorname{curl}\,{}^{(\mathbf{F})}\beta + r^{-3}\not{\mathfrak{d}}^{\leqslant 1}\Gamma_g + r^{-1}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b \cdot \Gamma_g).
\end{aligned}$$

where we used that $\operatorname{curl}\eta = {}^*\rho + \Gamma_b\cdot\Gamma_g = r^{-1}\Gamma_g + \Gamma_b\cdot\Gamma_g$. In particular, we deduce

$$\begin{aligned}
\nabla_3\left(\frac{Q}{r^2}\operatorname{div}{}^{(\mathbf{F})}\beta\right) &= \frac{4Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\beta + \frac{Q}{r^2}\left(\triangle\widecheck{{}^{(\mathbf{F})}\rho} + \frac{2Q}{r^2}\operatorname{div}\eta\right) + r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-3}\mathfrak{d}^{\leqslant 1}\left(\Gamma_b\cdot\Gamma_g\right),\\
\nabla_3\left(\frac{Q}{r^2}\operatorname{curl}{}^{(\mathbf{F})}\beta\right) &= \frac{4Q}{r^3}\operatorname{curl}{}^{(\mathbf{F})}\beta + \frac{Q}{r^2}\triangle{}^*{}^{(\mathbf{F})}\rho + r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-3}\mathfrak{d}^{\leqslant 1}\left(\Gamma_b\cdot\Gamma_g\right).
\end{aligned}$$

On the other hand, from the equations for $\nabla_3\widecheck{{}^{(\mathbf{F})}\rho}$ and $\nabla_3{}^*{}^{(\mathbf{F})}\rho$ of Proposition 2.10, we infer

$$\begin{aligned}
\nabla_3\left(\frac{2Q}{r^3}\widecheck{{}^{(\mathbf{F})}\rho}\right) &= -\frac{6Q}{r^4}\widecheck{{}^{(\mathbf{F})}\rho}(-\Upsilon+\widecheck{y}) + \frac{2Q}{r^3}\left(-\operatorname{div}{}^{(\mathbf{F})}\underline{\beta} + \frac{2\Upsilon}{r}\widecheck{{}^{(\mathbf{F})}\rho} - \frac{Q}{r^2}\underline{\widecheck{\kappa}} + \frac{2Q}{r^3}\widecheck{y} + \Gamma_b\cdot\Gamma_b\right)\\
&= \frac{10Q}{r^4}\widecheck{{}^{(\mathbf{F})}\rho} - \frac{2Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta} + r^{-5}\Gamma_g + r^{-3}\Gamma_b\cdot\Gamma_b,\\
\nabla_3\left(\frac{2Q}{r^3}{}^*{}^{(\mathbf{F})}\rho\right) &= -\frac{6Q}{r^4}{}^*{}^{(\mathbf{F})}\rho(-\Upsilon+\widecheck{y}) + \frac{2Q}{r^3}\left(\operatorname{curl}{}^{(\mathbf{F})}\underline{\beta} + \frac{2\Upsilon}{r}{}^*{}^{(\mathbf{F})}\rho + \Gamma_b\cdot\Gamma_b\right)\\
&= \frac{10Q}{r^4}{}^*{}^{(\mathbf{F})}\rho + \frac{2Q}{r^3}\operatorname{curl}{}^{(\mathbf{F})}\underline{\beta} + r^{-5}\Gamma_g + r^{-3}\Gamma_b\cdot\Gamma_b.
\end{aligned}$$

Thus, for $\boldsymbol{C} = \operatorname{div}\beta - \frac{Q}{r^2}\operatorname{div}{}^{(\mathbf{F})}\beta + \frac{2Q}{r^3}\widecheck{{}^{(\mathbf{F})}\rho}$ and $\boldsymbol{J} = \operatorname{curl}\beta - \frac{Q}{r^2}\operatorname{curl}{}^{(\mathbf{F})}\beta + \frac{2Q}{r^3}{}^*{}^{(\mathbf{F})}\rho$, we have

$$\begin{aligned}
\nabla_3\boldsymbol{C} = {}& \big(\frac{3}{r} + O(r^{-2})\big)\operatorname{div}\beta + \Delta\widecheck{\rho} - \left(\frac{6M}{r^3} - \frac{6Q^2}{r^4}\right)\operatorname{div}\eta + \frac{Q}{r^2}\Delta\widecheck{{}^{(\mathbf{F})}\rho} + \frac{Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\beta - \frac{2Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta}\\
&- \frac{4Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\beta - \frac{Q}{r^2}\big(\triangle\widecheck{{}^{(\mathbf{F})}\rho} + \frac{2Q}{r^2}\operatorname{div}\eta\big) + \frac{10Q}{r^4}\widecheck{{}^{(\mathbf{F})}\rho} - \frac{2Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta}\\
&+ r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g)\\
={}& \big(\frac{3}{r} + O(r^{-2})\big)\operatorname{div}\beta - \frac{3Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\beta - \frac{4Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta} + \triangle\widecheck{\rho} + \frac{10Q}{r^4}\widecheck{{}^{(\mathbf{F})}\rho} - \left(\frac{6M}{r^3} - \frac{4Q^2}{r^4}\right)\operatorname{div}\eta\\
&+ r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

and

$$\begin{aligned}
\nabla_3\boldsymbol{J} = {}& \big(\frac{3}{r} + O(r^{-2})\big)\operatorname{curl}\beta - \Delta{}^*\rho - \frac{6M}{r^3}{}^*\rho + \frac{Q}{r^2}\Delta{}^{(\mathbf{F})*}\rho + \frac{Q}{r^3}\operatorname{curl}{}^{(\mathbf{F})}\beta - \frac{2Q}{r^3}\operatorname{curl}{}^{(\mathbf{F})}\underline{\beta}\\
&- \frac{4Q}{r^3}\operatorname{curl}{}^{(\mathbf{F})}\beta - \frac{Q}{r^2}\triangle{}^*{}^{(\mathbf{F})}\rho + \frac{10Q}{r^4}{}^*{}^{(\mathbf{F})}\rho + \frac{2Q}{r^3}\operatorname{curl}{}^{(\mathbf{F})}\underline{\beta} + r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g)\\
={}& \big(\frac{3}{r} + O(r^{-2})\big)\operatorname{curl}\beta - \frac{3Q}{r^3}\operatorname{curl}{}^{(\mathbf{F})}\beta - \triangle{}^*\rho + \frac{10Q}{r^4}{}^*{}^{(\mathbf{F})}\rho - \frac{6M}{r^3}{}^*\rho + r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g).
\end{aligned}$$

The above can be written as

$$\begin{aligned}
\nabla_3\boldsymbol{C} = {}& \big(\frac{3}{r} + O(r^{-2})\big)\boldsymbol{C} - \frac{4Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta} + \triangle\widehat{\rho} + \frac{2Q}{r^2}\big(\triangle + \frac{2}{r^2}\big)\widecheck{{}^{(\mathbf{F})}\rho} - \left(\frac{6M}{r^3} - \frac{4Q^2}{r^4}\right)\operatorname{div}\eta\\
&+ r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g), \qquad \text{(A.1)}\\
\nabla_3\boldsymbol{J} = {}& \big(\frac{3}{r} + O(r^{-2})\big)\boldsymbol{J} - \triangle{}^*\widehat{\rho} - \frac{6M}{r^3}{}^*\widehat{\rho} + \frac{2Q}{r^2}\big(\triangle + \frac{2}{r^2}\big){}^*{}^{(\mathbf{F})}\rho\\
&+ r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\mathfrak{d}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g), \qquad \text{(A.2)}
\end{aligned}$$

where we also wrote $\widehat{\rho} = \widecheck{\rho} - \frac{2Q}{r^2}\widecheck{{}^{(\mathbf{F})}\rho} - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}$ and ${}^*\widehat{\rho} = {}^*\rho + \frac{2Q}{r^2}{}^*{}^{(\mathbf{F})}\rho - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}$.

We now compute the ∇_4 equations. Taking the divergence and the curl of the equation for $\nabla_4\beta$ of Proposition 2.10, we infer

$$\nabla_4\operatorname{div}\beta + \frac{4}{r}\operatorname{div}\beta = [\nabla_4, \operatorname{div}]\beta - \operatorname{div}\operatorname{div}\alpha + \frac{Q}{r^2}(\nabla_4\operatorname{div}{}^{(\mathbf{F})}\beta - [\nabla_4, \operatorname{div}]{}^{(\mathbf{F})}\beta)$$

$$
\begin{aligned}
&+ r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta)),\\
\nabla_4 \operatorname{curl}\beta + \frac{4}{r}\operatorname{curl}\beta = {} & [\nabla_4, \operatorname{curl}]\beta - \operatorname{curl}\operatorname{div}\alpha + \frac{Q}{r^2}(\nabla_4 \operatorname{curl}\,{}^{(\mathbf{F})}\beta - [\nabla_4, \operatorname{curl}]\,{}^{(\mathbf{F})}\beta)\\
&+ r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta)).
\end{aligned}
$$

According to Lemma 2.12,

$$
\begin{aligned}
[\nabla_4, \operatorname{div}]\beta &= -\frac{1}{r}\operatorname{div}\beta + r^{-2}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g),\\
[\nabla_4, \operatorname{curl}]\beta &= -\frac{1}{r}\operatorname{curl}\beta + r^{-2}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot \Gamma_g),\\
[\nabla_4, \operatorname{div}]\,{}^{(\mathbf{F})}\beta &= -\frac{1}{r}\operatorname{div}\,{}^{(\mathbf{F})}\beta + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}\left(\Gamma_g \cdot \Gamma_g\right),\\
[\nabla_4, \operatorname{curl}]\,{}^{(\mathbf{F})}\beta &= -\frac{1}{r}\operatorname{curl}\,{}^{(\mathbf{F})}\beta + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}\left(\Gamma_g \cdot \Gamma_g\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\nabla_4 \operatorname{div}\beta = {} & -\frac{5}{r}\operatorname{div}\beta - \operatorname{div}\operatorname{div}\alpha + \frac{Q}{r^2}(\nabla_4 \operatorname{div}\,{}^{(\mathbf{F})}\beta + \frac{1}{r}\operatorname{div}\,{}^{(\mathbf{F})}\beta) + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta)),\\
\nabla_4 \operatorname{curl}\beta = {} & -\frac{5}{r}\operatorname{curl}\beta - \operatorname{curl}\operatorname{div}\alpha + \frac{Q}{r^2}(\nabla_4 \operatorname{curl}\,{}^{(\mathbf{F})}\beta + \frac{1}{r}\operatorname{curl}\,{}^{(\mathbf{F})}\beta) + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta)).
\end{aligned}
$$

On the other hand, from the equations for $\nabla_4 \widecheck{{}^{(\mathbf{F})}\rho}$ and $\nabla_4 {}^*\,{}^{(\mathbf{F})}\rho$ of Proposition 2.10, we infer

$$
\begin{aligned}
\nabla_4\left(\frac{2Q}{r^3}\widecheck{{}^{(\mathbf{F})}\rho}\right) &= -\frac{10Q}{r^4}\widecheck{{}^{(\mathbf{F})}\rho} + \frac{2Q}{r^3}\operatorname{div}\,{}^{(\mathbf{F})}\beta,\\
\nabla_4\left(\frac{2Q}{r^3}{}^*\,{}^{(\mathbf{F})}\rho\right) &= -\frac{10Q}{r^4}{}^*\,{}^{(\mathbf{F})}\rho + \frac{2Q}{r^3}\operatorname{curl}\,{}^{(\mathbf{F})}\beta.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\nabla_4 \boldsymbol{C} = {} & -\frac{5}{r}\operatorname{div}\beta - \operatorname{div}\operatorname{div}\alpha + \frac{Q}{r^2}(\nabla_4 \operatorname{div}\,{}^{(\mathbf{F})}\beta + \frac{1}{r}\operatorname{div}\,{}^{(\mathbf{F})}\beta) + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta))\\
& + \frac{2Q}{r^3}\operatorname{div}\,{}^{(\mathbf{F})}\beta - \frac{Q}{r^2}\nabla_4 \operatorname{div}\,{}^{(\mathbf{F})}\beta - \frac{10Q}{r^4}\widecheck{{}^{(\mathbf{F})}\rho} + \frac{2Q}{r^3}\operatorname{div}\,{}^{(\mathbf{F})}\beta\\
= {} & -\frac{5}{r}\operatorname{div}\beta + \frac{5Q}{r^3}\operatorname{div}\,{}^{(\mathbf{F})}\beta - \frac{10Q}{r^4}\widecheck{{}^{(\mathbf{F})}\rho} - \operatorname{div}\operatorname{div}\alpha + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta)),
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_4 \boldsymbol{J} = {} & -\frac{5}{r}\operatorname{curl}\beta - \operatorname{curl}\operatorname{div}\alpha + \frac{Q}{r^2}(\nabla_4 \operatorname{curl}\,{}^{(\mathbf{F})}\beta + \frac{1}{r}\operatorname{curl}\,{}^{(\mathbf{F})}\beta) + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta))\\
& + \frac{2Q}{r^3}\operatorname{curl}\,{}^{(\mathbf{F})}\beta - \frac{Q}{r^2}\nabla_4 \operatorname{curl}\,{}^{(\mathbf{F})}\beta - \frac{10Q}{r^4}{}^*\,{}^{(\mathbf{F})}\rho + \frac{2Q}{r^3}\operatorname{curl}\,{}^{(\mathbf{F})}\beta\\
= {} & -\frac{5}{r}\operatorname{curl}\beta + \frac{5Q}{r^3}\operatorname{curl}\,{}^{(\mathbf{F})}\beta - \frac{10Q}{r^4}{}^*\,{}^{(\mathbf{F})}\rho - \operatorname{curl}\operatorname{div}\alpha + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta)).
\end{aligned}
$$

The above can be written as

$$
\nabla_4 \boldsymbol{C} = -\frac{5}{r}\boldsymbol{C} - \operatorname{div}\operatorname{div}\alpha + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta)), \tag{A.3}
$$

$$
\nabla_4 \boldsymbol{J} = -\frac{5}{r}\boldsymbol{J} - \operatorname{curl}\operatorname{div}\alpha + r^{-1}\slashed{\mathfrak{d}}^{\leqslant 1}(\Gamma_g \cdot (\alpha, \beta, r^{-1}\,{}^{(\mathbf{F})}\beta)). \tag{A.4}
$$

Since $\nu = e_3 + b_* e_4$ and $b_* = -1 - \frac{2M}{r} + \frac{Q^2}{r^2} + r\Gamma_b$, combining (A.1)-(A.2) with (A.3)-(A.4) we have

$$\nabla_\nu \boldsymbol{C} = \big(\frac{8}{r} + O(r^{-2})\big)\boldsymbol{C} - \frac{4Q}{r^3}\operatorname{div}{}^{(\mathbf{F})}\underline{\beta} + \triangle\widehat{\rho} + \frac{2Q}{r^2}\big(\triangle + \frac{2}{r^2}\big)\widetilde{{}^{(\mathbf{F})}\rho} + O(1 + O(r^{-1}))\operatorname{div}\operatorname{div}\alpha$$
$$- \left(\frac{6M}{r^3} - \frac{4Q^2}{r^4}\right)\operatorname{div}\eta + r^{-5}\not{\mathfrak{d}}^{\leqslant 1}\Gamma_g + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g),$$

and

$$\nabla_\nu \boldsymbol{J} = \big(\frac{8}{r} + O(r^{-2})\big)\boldsymbol{J} - \triangle^*\widehat{\rho} - \frac{6M}{r^3}{}^*\widehat{\rho} + \frac{2Q}{r^2}\big(\triangle + \frac{2}{r^2}\big)^*{}^{(\mathbf{F})}\rho + O(1 + O(r^{-1}))\operatorname{curl}\operatorname{div}\alpha$$
$$+ r^{-5}\mathfrak{d}^{\leqslant 1}\Gamma_g + r^{-2}\not{\mathfrak{d}}^{\leqslant 1}(\Gamma_b\cdot\Gamma_g),$$

as stated.

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