Hardy spaces for the Lamé equation *†

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Abstract

We study, for $1 \leq p \leq \infty$, the Hardy space $\boldsymbol{h}_e^p(\mathbb{B})$, the elastic analogue of the classical Hardy spaces of harmonic functions in the unit ball of $\mathbb{R}^3$. The space consists of vector-field solutions of the Lamé system satisfying the standard integrability condition on concentric spheres centered at the origin. Using the elastic Poisson kernel, we establish a Fatou-type theorem and show that $\boldsymbol{h}_e^p(\mathbb{B})$ is isomorphic to the $\mathbb{R}^3$-valued Lebesgue space L^p on the unit sphere for $1 < p \leq \infty$, while $\boldsymbol{h}_e^1(\mathbb{B})$ corresponds to the space of $\mathbb{R}^3$-valued Borel measures on the unit sphere.

For $1 < p < \infty$, we prove that $\boldsymbol{h}_e^p(\mathbb{B})$ decomposes as the direct sum of three subspaces. The main contribution of this paper is to describe each of these subspaces along with the corresponding spaces of boundary values. In particular, two of these spaces consist of solutions of the Lamé equation for all eligible choices of the Lamé constants: one of them is the space of Riesz fields (solutions of the generalized Cauchy–Riemann equations) in $\boldsymbol{h}_e^p(\mathbb{B})$; the second is the space of fields given by the cross product of x with such Riesz fields. The results rely on the classical decomposition of L^2 vector fields on the sphere into the direct sum of three spaces of vector spherical harmonics, which we extend to L^p.

1 Introduction

The purpose of this paper is to study the structure of Hardy spaces of solutions of the Lamé equation

$$\Delta^* \boldsymbol{u} = 0$$

in the unit ball $\mathbb{B}$ in $\mathbb{R}^3$, where

$$\Delta^* = \mu\Delta + (\lambda + \mu)\nabla div,$$

and the Lamé coefficients λ and μ satisfy $\mu > 0$ and $2\mu + \lambda > 0$, making Δ^* an elliptic operator acting on vector fields of $\mathbb{R}^3$ (see for example [10]. These

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spaces, which we will denote by $\boldsymbol{h}_e^p(\mathbb{B})$, are endowed with the classical norm from the Hardy space theory, consisting of the supremum of the $\boldsymbol{L}^p$ norms on concentric spheres centered at the origin and radius less than one. Since for $\lambda = -\mu$ the solution of the Lamé equations are identical to the vector-valued harmonic functions, then these spaces generalize the Hardy spaces of harmonic functions (see [3, 4, 1]). As expected, the space of boundary limits of elements in $\boldsymbol{h}_e^p(\mathbb{B})$ is the $\mathbb{R}^3$-valued Lebesgue $\boldsymbol{L}^p$ space in the sphere for $1 < p \leq \infty$, and for $\boldsymbol{h}_e^1(\mathbb{B})$, it corresponds to the space of borel measures in the unit sphere with values in $\mathbb{R}^3$.

For the Lamé equation and other applications it is natural to use of certain vector spherical harmonics (see [7]) that provide a decomposition of the vector $\boldsymbol{L}^2(\mathbb{S})$ space into three orthogonal subspaces. We will prove that for $1 < p < \infty$, a decomposition of the $\boldsymbol{L}^p$ space in the sphere holds as well, leading via the elastic Poisson transform to a decomposition of $\boldsymbol{h}_e^p(\mathbb{B})$ as a Banach direct product of three subspaces. The aim of this paper is the precise description of each of these spaces. This will provide insight of the local structure of the Lamé equation.

Remarkably, one of them will consist of Riesz fields, namely fields $\boldsymbol{u} \in \boldsymbol{h}_e^p(\mathbb{B})$ that satisfy the generalized Cauchy-Riemann equations $div\, \boldsymbol{u} = 0$ and $\nabla \times \boldsymbol{u} = 0$. It is well known the relevance of these fields in harmonic analysis. These vector fields are common elements of $\boldsymbol{h}_e^p(\mathbb{B})$ for every admissible values of λ and μ. Moreover in this case, the vector field $x \times \boldsymbol{u} \in \boldsymbol{h}_e^p(\mathbb{B})$ for such λ and μ.

The paper is organized as follows: In Sections 2 and 3 we will be dedicated to give definitions and to state all the preliminary results needed in the paper like Sobolev space and distributions in the sphere. In Section 4 we will study the basic structure of the elastic Hardy spaces including the elastic Poisson transform and we will prove Fatou theorems for these Hardy spaces. In Section 5 we will review the mentioned vector-valued spherical harmonics decomposition and we will prove the decompositions of the vector $\boldsymbol{L}^p$ spaces of the sphere. In Section 6 we complete the aim of this paper: to characterize the elastic extensions of vector fields on each of the $\boldsymbol{L}^p$ subspaces defined in Section 5. This will be achieved after describing in detail the three boundary $\boldsymbol{L}^p$ subspaces.

2 Hardy spaces for the Lamé operator

In this section we define the elastic Hardy spaces $\boldsymbol{h}_e^p(\mathbb{B})$. If $\boldsymbol{u} : \mathbb{B} \to \mathbb{R}^3$, we say that $\boldsymbol{u} \in \boldsymbol{h}_e^p(\mathbb{B})$, with $1 \leq p < \infty$, if it is a solution of the Lamé system

$$\Delta^* \boldsymbol{u} = 0, \tag{1}$$

and satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{h}_e^p} = \sup_{0 \leq r < 1} \left(\int_{\mathbb{S}} |\boldsymbol{u}(rw)|^p d\sigma(w) \right)^{1/p} < \infty,$$

where $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$. We say that $\boldsymbol{u} \in \boldsymbol{h}_e^\infty(\mathbb{B})$ if it is a bounded solution of (1).

The maximal elastic Hardy spaces are defined as follows: for $\xi \in \mathbb{S}$ we define

$$\Gamma_\xi = \mathbb{B} \cap conv\left(\{\xi\} \cup \{x \in \mathbb{R}^3 : \ |x| \leq 1/2\}\right),$$

where for A a subset of $\mathbb{R}^3$, $conv(A)$ is the convex hull of A.

We say that a solution $\boldsymbol{u} : \mathbb{B} \to \mathbb{R}^3$ of (1) belongs to $\boldsymbol{H}^p_e(\mathbb{B})$, $1 \leq p \leq \infty$, if

$$\mathcal{N}\boldsymbol{u}(\xi) = \sup_{x \in \Gamma_\xi} |\boldsymbol{u}(x)| \tag{2}$$

belongs to $L^p(\mathbb{S})$. Obviously $\boldsymbol{H}^p_e(\mathbb{B}) \subset \boldsymbol{h}^p_e(\mathbb{B})$.

The harmonic Hardy spaces in $\mathbb{B}$ are defined for scalar harmonic functions and are denoted by $h^p(\mathbb{B})$ and $H^p(\mathbb{B})$ same norms as before.

3 Preliminaries

The following notation will be used throughout the paper: for non-negative quantities X and Y we will use $X \lesssim Y$ ($X \gtrsim Y$) to denote the existence of a positive constant c such that $X \leq cY$ ($X \geq cY$). We will write $X \sim Y$ if both $X \lesssim Y$ and $X \gtrsim Y$.

The following obvious estimate will be used throughout the paper.

$$(1-r)\int_0^1 \frac{1}{(1-rt)^2}\,dt \leq C, \tag{3}$$

for some $C > 0$ and all $r \in [0, 1)$.

Given a function f on $\mathbb{B}$ we denote for every $r \in [0, 1)$ the function f_r defined on $\mathbb{S}$ by $f_r(\eta) = f(r\eta)$, $\eta \in \mathbb{S}$.

We denote by $\boldsymbol{M}(\mathbb{S})$ the space of $\mathbb{R}^3$-valued Borel measures in $\mathbb{S}$. Every $\boldsymbol{\mu} \in \boldsymbol{M}(\mathbb{S})$ can be represented by $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$, where each μ_i is a signed Borel measure in $\mathbb{S}$. We define

$$|\boldsymbol{\mu}| = \sum_1^3 |\mu_i|$$

where $|\mu_i|$ is the variation of the signed measure μ_i. Actually, $|\boldsymbol{\mu}|$ is the variation of the vector measure $\boldsymbol{\mu}$ (see[5]) with respect to the norm $|\boldsymbol{x}|_1 = |x_1|+|x_2|+|x_3|$ in $\mathbb{R}^3$.

Let $\{Y_{\ell,m}\}_{\ell=0,1,\cdots,\ |m|\leq\ell}$, be an orthonormal basis of real spherical harmonics for $L^2(\mathbb{S})$ and let $\mathcal{H}^\ell$ denote the space of all spherical harmonics of degree ℓ, which is generated by $\{Y_{\ell,m}\}_{|m|\leq\ell}$. For any function f on $L^1(\mathbb{S})$ we denote

$$\hat{f}_{\ell,m} = \int_{\mathbb{S}} f(\eta) Y_{\ell,m}(\eta)\,d\sigma(\eta).$$

We denote by Δ_σ and ∇_σ, the Laplacian (the Laplace-Beltrami operator) and the gradient on $\mathbb{S}$. For a function f defined on $\mathbb{B}$ we write $\Delta_\sigma f(x)$, where $x =$

$|x|\frac{x}{|x|} = rx'$, for $\Delta_\sigma f(r\cdot)(x')$ and the same for the spherical gradient $\nabla_\sigma f(rx') = \nabla_\sigma f(r\cdot)(x')$.

Recall that $\mathcal{H}^\ell$ is the eigenspace of $-\Delta_\sigma$ corresponding the eigenvalue $\ell(\ell+1)$. We have the estimates in ℓ, (see for example [12, page 225]),

$$\|Y_{\ell,m}\|_\infty = O(\ell^{1/2}) \text{ and } \|\nabla_\sigma Y_{\ell,m}\|_\infty = O(\ell^{3/2}). \tag{4}$$

For every $Y \in \mathcal{H}^\ell$ we denote its harmonic extension to $\mathbb{B}$ by $Y(x) = r^\ell Y(x')$. If f is a real function we define the vector function in $\mathbb{S}$,

$$\boldsymbol{f}^\vee(\xi) = f(\xi)\xi.$$

Then

$$\nabla Y(x) = r^{\ell-1}\left(\ell \boldsymbol{Y}^\vee(x') + \nabla_\sigma Y(x')\right), \tag{5}$$

where ∇ denotes the gradient in $\mathbb{R}^3$.

For any sequence $\{\beta_\ell\}_{\ell\geq 0}$ in $\mathbb{R}$ we define the zonal multiplier

$$\mathcal{M}_{\beta_\ell}(f) = \sum_{\ell\geq 0}\sum_{|m|\leq \ell} \beta_\ell a_{\ell,m} Y_{\ell,m}, \tag{6}$$

for $f = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m}Y_{\ell,m}$ in the linear span of $\cup_{\ell\geq 0}\mathcal{H}^\ell$.

Denote by $\boldsymbol{C}^\infty(\mathbb{S})$ the space of C^∞-vector fields defined in the sphere, and by $\mathcal{D}'(\mathbb{S})$ the space of distributions in $\mathbb{S}$, namely the space of continuous $\mathbb{R}$-linear functionals $f : C^\infty(\mathbb{S}) \to \mathbb{R}$. Likewise, denote by $\boldsymbol{D}'(\mathbb{S})$, the space of continuous $\mathbb{R}^3$-valued linear functions on $\boldsymbol{C}^\infty(\mathbb{S})$.

Every $f \in \mathcal{D}'(\mathbb{S})$ can be written as

$$f = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m}Y_{\ell,m}$$

where $a_{\ell,m} = f(Y_{\ell,m})$ and the convergence is in the weak* topology. The same holds for $\boldsymbol{f} \in \boldsymbol{D}'(\mathbb{S})$. In either case, there exist $N > 0$ such that, $|a_{\ell,m}| \lesssim \ell^N$, uniformly for $|m| \leq \ell$, and this characterizes $\mathcal{D}'(\mathbb{S})$ and $\boldsymbol{D}'(\mathbb{S})$.

The action of such f on $\varphi \in C^\infty(\mathbb{S})$ is given by

$$\begin{aligned} f(\varphi) &= a_{0,0}\int_{\mathbb{S}}\varphi d\sigma + \int_{\mathbb{S}}\left(\sum_{\ell>0}\sum_{|m|\leq\ell}\frac{a_{\ell,m}}{(\ell(\ell+1))^M}Y_{\ell,m}\right)(-\Delta_\sigma)^M\varphi d\sigma \\ &= a_{0,0}\int_{\mathbb{S}}\varphi d\sigma + \sum_{\ell>0}\sum_{|m|\leq\ell}\frac{a_{\ell,m}}{(\ell(\ell+1))^M}\int_{\mathbb{S}}Y_{\ell,m}(-\Delta_\sigma)^M\varphi d\sigma. \end{aligned}$$

for M large enough to assure the convergence the series.

If the sequence $\{\beta_\ell\}_{\ell\geq 0}$ has polynomial growth then $\mathcal{M}_{\beta_\ell} : \mathcal{D}'(\mathbb{S}) \to \mathcal{D}'(\mathbb{S})$. Let $E \subset \mathcal{D}'(\mathbb{S})$ be a topological linear space containing all the zonal spaces $\mathcal{H}^\ell$. We say that $\mathcal{M}_{\beta_\ell}$ is a zonal multiplier for E if $\mathcal{M}_{\beta_\ell}$ has a continuous extension to E. In particular, for $M > 0$, $(-\Delta_\sigma)^M$ is a zonal multiplier for $\mathcal{D}'(\mathbb{S})$ with the weak* topology, associated with the sequence $\left\{\ell^M(\ell+1)^M\right\}_{\ell\geq 0}$.

We have the following result by R. S. Strichartz [14].

Theorem 1. *Let $1 < p < \infty$. If $h : (0, \infty) \to \mathbb{R}$ satisfies*

$$\left| x^k h^{(k)}(x) \right| \leq A, \qquad x \in (0, \infty), \quad k = 0, 1, \cdots, N, \tag{7}$$

then $\mathcal{M}_{h_\ell}$ is bounded in $L^p(\mathbb{S})$ for $\left| \frac{1}{2} - \frac{1}{p} \right| < \frac{N}{2}$, where $h_\ell = h(\ell), \ell \in \mathbb{N}$.

The following result, (see [2]), it will be used throughout the paper.

Theorem 2. *(D. Bakry) Let $1 < p < \infty$ and $f \in C^1(\mathbb{S})$. Then*

$$\| \left(-\Delta_\sigma \right)^{1/2} f \|_{L^p} \sim \| \nabla_\sigma f \|_{\boldsymbol{L}^p}. \tag{8}$$

Define the Sobolev space $W^{1,p}(\mathbb{S})$ as the space of the functions $f \in L^p(\mathbb{S})$ such that $(-\Delta_\sigma)^{1/2} f \in L^p(\mathbb{S})$.

Lemma 3. *Let $1 < p < \infty$ and $f \in L^p(\mathbb{S})$, then $f \in W^{1,p}(\mathbb{S})$ if and only if there exists a sequence $\varphi_n \in C^\infty(\mathbb{S})$ such that $\varphi_n \to f$ in $L^p(\mathbb{S})$ and $\nabla_\sigma \varphi_n \to \boldsymbol{F}$ in $\boldsymbol{L}^p(\mathbb{S})$ for some vector field $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{S})$. In this case we denote $\nabla_\sigma f = \boldsymbol{F}$ (which is independent of the sequence φ_n).*

Also

$$\|(-\Delta_\sigma)^{1/2} f\|_{L^p} \sim \|\nabla_\sigma f\|_{\boldsymbol{L}^p}. \tag{9}$$

for $f \in W^{1,p}(\mathbb{S})$.

Proof. Let $f \in W^{1,p}(\mathbb{S})$. There exists a sequence $\varphi_n \in C^\infty(\mathbb{S})$ such that $(-\Delta_\sigma)^{1/2}\varphi_n \to (-\Delta_\sigma)^{1/2} f$ in $L^p(\mathbb{S})$ (take for example $\varphi_n = P_{r_n} f$ where P_r is the Poisson transform in the sphere (see (15) in Section 4) and $r_n \nearrow 1$). By (8) $\nabla_\sigma \varphi_n$ is a Cauchy sequence in $\boldsymbol{L}^p(\mathbb{S})$ converging to a vector field $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{S})$. Conversely if $\varphi_n \in C^\infty(\mathbb{S})$ converges to f in $L^p(\mathbb{S})$ and $\nabla_\sigma \varphi_n \to \boldsymbol{F}$ in $\boldsymbol{L}^p(\mathbb{S})$ norm, then by (8) $(-\Delta_\sigma)^{1/2}\varphi_n$ converges in $L^p(\mathbb{S})$ and converges to $(-\Delta_\sigma)^{1/2} f$ in $\mathcal{D}'(\mathbb{S})$, implying that $(-\Delta_\sigma)^{1/2} f \in L^p(\mathbb{S})$. It follows that (9) holds for $f \in W^{1,p}(\mathbb{S})$. $\square$

Lemma 4. *Let $g \in W^{1,p}(\mathbb{S})$, $1 < p < \infty$, with $g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}$ in $\mathcal{D}'(\mathbb{S})$. Then*

$$\nabla_\sigma g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} \nabla_\sigma Y_{\ell,m} \quad \text{in } \boldsymbol{D}'(\mathbb{S}).$$

Proof. First notice that the lemma is equivalent to prove that

$$\langle \nabla_\sigma g, \boldsymbol{h} \rangle = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} \langle \nabla_\sigma Y_{\ell,m}, \boldsymbol{h} \rangle.$$

for any $\boldsymbol{h} \in \boldsymbol{C}^\infty(\mathbb{S})$.

If $g \in C^\infty(\mathbb{S})$ then $g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}$ in $C^\infty(\mathbb{S})$ and by (8)

$$\|\nabla_\sigma g - \sum_{\ell=0}^{L} \sum_{|m| \leq \ell} a_{\ell,m} \nabla_\sigma Y_{\ell,m}\|_{\boldsymbol{L}^p} \leq C \|(-\Delta_\sigma)^{1/2}(g - \sum_{\ell=0}^{L} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m})\|_{L^p} \to 0$$

and the lemma follows in this case. Now let $g \in W^{1,p}(\mathbb{S})$ and a sequence $g_k \in C^\infty(\mathbb{S})$ converging to g in $W^{1,p}(\mathbb{S})$. Then

$$\nabla_\sigma g_k \to \nabla_\sigma g \text{ in } \boldsymbol{L}^p(\mathbb{S}). \tag{10}$$

Let

$$g_k = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a^k_{\ell,m} Y_{\ell,m},$$

and

$$\nabla_\sigma g_k = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a^k_{\ell,m} \nabla_\sigma Y_{\ell,m}.$$

Also, since $g_k \to g$ in $L^p(\mathbb{S})$, then $lim_{k\to\infty} a^k_{\ell,m} = a_{\ell,m}$.

Now let $\boldsymbol{h} \in \boldsymbol{C}^\infty(\mathbb{S})$ a vector field. Denote $\boldsymbol{h}_T$ the tangential component of $\boldsymbol{h}$, namely

$$\boldsymbol{h}_T(x') = \pi_{x'^\perp} \boldsymbol{h}(x') \quad x' \in \mathbb{S},$$

where $\pi_{x'^\perp}$ is the orthogonal projection of $\boldsymbol{h}$ in the tangent space of $\boldsymbol{h}$ at x'.

Let

$$I_k = \sum_{\ell > 0} \sum_{|m| \leq \ell} a^k_{\ell,m} \int_{\mathbb{S}} \nabla_\sigma Y_{\ell,m} \cdot \boldsymbol{h} d\sigma = \sum_{\ell > 0} \sum_{|m| \leq \ell} a^k_{\ell,m} \int_{\mathbb{S}} \nabla_\sigma Y_{\ell,m} \cdot \boldsymbol{h}_T d\sigma$$

and

$$I = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} \int_{\mathbb{S}} \nabla_\sigma Y_{\ell,m} \cdot \boldsymbol{h} d\sigma = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} \int_{\mathbb{S}} \nabla_\sigma Y_{\ell,m} \cdot \boldsymbol{h}_T d\sigma.$$

Then by the divergence theorem in $\mathbb{S}$,

$$I_k = -\sum_{\ell > 0} \sum_{|m| \leq \ell} a^k_{\ell,m} \int_{\mathbb{S}} Y_{\ell,m}\, div_\sigma \boldsymbol{h}_T d\sigma,$$

$$= -\sum_{\ell > 0} \sum_{|m| \leq \ell} \frac{a^k_{\ell,m}}{\ell(\ell+1)} \int_{\mathbb{S}} Y_{\ell,m} \left(-\Delta_\sigma\right) div_\sigma \boldsymbol{h}_T d\sigma$$

and

$$I = -\sum_{\ell > 0} \sum_{|m| \leq \ell} \frac{a_{\ell,m}}{\ell(\ell+1)} \int_{\mathbb{S}} Y_{\ell,m} \left(-\Delta_\sigma\right) div_\sigma \boldsymbol{h}_T d\sigma,$$

where div_σ denotes the divergence in the sphere. By (4) we have that

$$\left|a^k_{\ell,m}\right| = \left| \int_{\mathbb{S}} g\, Y_{\ell,m} d\sigma \right| = O(\ell^{1/2}).$$

Then I_k converges to I by the dominated convergence theorem. This together with (10) implies

$$\langle \nabla_\sigma g, \boldsymbol{h} \rangle = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} \langle \nabla_\sigma Y_{\ell,m}, \boldsymbol{h} \rangle.$$

$\square$

4 Elastic Poisson kernel in the unit ball and boundary functions

Following Kupradze (see [9, Ch. XIV, 1.26]), the elastic Poisson kernel is the matrix-valued function

$$\mathcal{P}_e : \mathbb{B} \times \mathbb{S} \to M_{3\times 3}(\mathbb{R}),$$

which can be seen as a perturbation of the Poisson kernel for harmonic functions, $P(x,\eta)$, in $\mathbb{B}$, that is,

$$\mathcal{P}_e(x,\eta) = P(x,\eta)\mathcal{I} + \mathcal{L}(x,\eta), \tag{11}$$

where $\mathcal{I}$ is the identity matrix,

$$P(x,\eta) = \frac{1}{4\pi}\frac{1-|x|^2}{|x-\eta|^3}, \tag{12}$$

and

$$\mathcal{L}_{ij}(x,\eta) = \beta(1-|x|^2)\frac{\partial^2\Phi}{\partial x_i \partial x_j}(x,\eta), \quad i,j \in \{1,2,3\}, \tag{13}$$

with

$$\Phi(x,\eta) = \frac{1}{4\pi}\int_0^1 \left(\frac{1-t^2|x|^2}{(1-2tx\cdot\eta + t^2|x|^2)^{3/2}} - 1 - 3tx\cdot\eta\right)\frac{dt}{t^{1+\alpha}} \tag{14}$$

and

$$\beta = \frac{\lambda+\mu}{2\lambda+6\mu}, \qquad \alpha = \frac{\lambda+2\mu}{\lambda+3\mu}.$$

Notice that for x and η fixed, we have $4\pi P(tx,\eta) - 1 - 3tx\cdot\eta = O(t)$ so, since $\alpha \in (0,1)$, Φ is well defined. Furthermore, by (44) and (45) of the appendix, $\frac{\partial^2\Phi}{\partial x_i\partial x_j}(x,\eta)$ is also well defined.

The Poisson transform P is defined by

$$Pf(x) = \int_{\mathbb{S}} P(t\xi,\eta)f(\eta)d\sigma(\eta), \quad f \in L^1(\mathbb{S}), \quad x = t\xi \in \mathbb{B}. \tag{15}$$

We define for $t \in (0,1)$

$$P_t f(\xi) = Pf(t\xi). \tag{16}$$

It is known that, (see for example [1, page 99]),

$$P(t\xi,\eta) = \sum_{\ell\geq 0}\sum_{|m|\leq \ell} t^\ell Y_{\ell,m}(\xi)Y_{\ell,m}(\eta), \tag{17}$$

which allows us to see P_t as a zonal multiplier associated to the sequence $\{t^\ell\}_{\ell=0,1,2,\cdot}$. It is well known that P_t is bounded in $L^p(\mathbb{S})$ for $1 \leq p \leq \infty$.

The Hardy-Littlewood maximal function of $f \in L^p(\mathbb{S})$, denoted by $M[f]$, is the function on $\mathbb{S}$ defined by

$$M[f](\xi) = \sup_{\delta>0} \frac{1}{\sigma(\kappa(\xi,\delta))} \int_{\kappa(\xi,\delta)} |f(\eta)| d\sigma(\eta),$$

where $\kappa(\xi,\delta) = \{\eta \in \mathbb{S} : \ |\eta - \xi| < \delta\}$.

M is bounded on $L^p(\mathbb{S})$ for $p \in (1,\infty)$ and weak type $(1,1)$. It is a standard fact (see [13, Ch VI, Sect 1]) that if $f \in L^1(\mathbb{S})$ then

$$\mathcal{N}(Pf)(\zeta) \leq M[f](\zeta), \ \ \zeta \in \mathbb{S}. \tag{18}$$

The elastic Poisson transform $\boldsymbol{P}_e$ is defined by

$$\boldsymbol{P}_e \boldsymbol{f}(x) = \int_{\mathbb{S}} \mathcal{P}_e(x,\eta) \boldsymbol{f}(\eta) d\sigma(\eta), \quad \boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{S}), \ \ 1 \leq p \leq \infty.$$

We have that

$$\begin{aligned} \boldsymbol{P}_e \boldsymbol{f}(x) &= P\boldsymbol{f}(x) + \boldsymbol{L}\boldsymbol{f}(x) \\ &= \int_{\mathbb{S}} P(x,\eta)\boldsymbol{f}(\eta) d\sigma(\eta) + \int_{\mathbb{S}} \mathcal{L}(x,\eta) \boldsymbol{f}(\eta) d\sigma(\eta). \end{aligned}$$

For $\boldsymbol{\mu} \in \boldsymbol{M}(\mathbb{S})$, we define likewise

$$\boldsymbol{P}_e \boldsymbol{\mu}(x) = \int_{\mathbb{S}} \mathcal{P}_e(x,\eta) d\boldsymbol{\mu}(\eta) = \int_{\mathbb{S}} \mathcal{P}_e(x,\eta)(d\mu_1(\eta), d\mu_2(\eta), d\mu_3(\eta))^t,$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$.

Proposition 5. *Let $\boldsymbol{f} \in \boldsymbol{L}^1(\mathbb{S})$. We have*

a) The matrix function $\partial_x^\alpha \mathcal{P}_e(x,\eta)$ is uniformly bounded in $K \times \mathbb{S}$ for any compact $K \subset \mathbb{B}$ and every partial derivative ∂_x^α.

b) $|\boldsymbol{L}\boldsymbol{f}(x)| \lesssim (1-|x|^2) \int_0^1 \frac{P|\boldsymbol{f}|(tx)}{(1-t^2|x|^2)^2} t^{1-\alpha} dt$.

c) $\mathcal{N}(\boldsymbol{P}_e \boldsymbol{f})(\xi) \lesssim \mathcal{N}(P|\boldsymbol{f}|)(\xi) \lesssim M[|\boldsymbol{f}|](\xi)$.

Proof. See the appendix. $\square$

We have the standard result on Hardy spaces for the case of the Lamé equation:

Theorem 6. *1. Let $1 < p \leq \infty$. $\boldsymbol{u} \in \boldsymbol{h}_e^p(\mathbb{B})$ if and only if $\boldsymbol{u}(x) = \boldsymbol{P}_e \boldsymbol{f}(x)$ for some $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{S})$.*

2. $\boldsymbol{u} \in \boldsymbol{h}_e^1(\mathbb{B})$ if and only if $\boldsymbol{u}(x) = \boldsymbol{P}_e \boldsymbol{\mu}(x)$ for some $\boldsymbol{\mu} \in \boldsymbol{M}(\mathbb{S})$.

Proof. 1. Suppose that $\boldsymbol{u} \in \boldsymbol{h}_e^p(\mathbb{B})$, with $p > 1$. We proceed in the standard way: let $\boldsymbol{u}_r(\eta) = \boldsymbol{u}(r\eta)$, $\eta \in \mathbb{S}$ and $r \in (0,1)$. Notice that since $\Delta_x^* \boldsymbol{u}(rx) = 0$ on $\overline{\mathbb{B}}$, then $\boldsymbol{u}(rx) = \int_{\mathbb{S}} \mathcal{P}_e(x,\eta)\boldsymbol{u}_r(\eta)d\sigma(\eta)$.

We have that $\mathcal{A} = \{\boldsymbol{u}_r\}_{r\in(0,1)}$ is a bounded set in $\boldsymbol{L}^p(\mathbb{S}) = \boldsymbol{L}^q(\mathbb{S})^*$, where q is the conjugate exponent of p with the duality

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle = \int_{\mathbb{S}} \boldsymbol{f}(\theta)\cdot \boldsymbol{g}(\theta)d\sigma(\theta), \quad \boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{S}),\ \boldsymbol{g} \in \boldsymbol{L}^q(\mathbb{S}).$$

Then $\mathcal{A}$ is a w^* relatively compact by the Banach-Alouglou theorem. Hence there exist a sequence $r_n \nearrow 1$ and $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{S})$ such that $\mathbf{u}_{r_n} \to \boldsymbol{f}$ in the weak* topology. Since the rows of $\mathcal{P}_e(x,\cdot)$ are bounded functions (see Proposition 5) in $\mathbb{S}$ and hence they are in $\boldsymbol{L}^q(\mathbb{S})$, it follows that

$$\boldsymbol{u}(x) = \lim_{n\to\infty} \boldsymbol{u}(r_n x) = \lim_{n\to\infty} \int_{\mathbb{S}} \mathcal{P}_e(x,\eta)\boldsymbol{u}_{r_n}(\eta)d\sigma(\eta) = \int_{\mathbb{S}} \mathcal{P}_e(x,\eta)\boldsymbol{f}(\eta)d\sigma(\eta).$$

Conversely, let $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{S})$. Then the standard properties of the Poisson kernel imply that

$$\|(P\boldsymbol{f})_r\|_{\boldsymbol{L}^p} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^p}.$$

Also, by Proposition 5 b), if $\eta \in \mathbb{S}$,

$$|(\boldsymbol{L}\boldsymbol{f})_r(\eta)| \lesssim (1-r^2)\int_0^1 \frac{P|\boldsymbol{f}|(tr\eta)}{(1-t^2r^2)^2} t^{1-\alpha}dt.$$

so that using Minkowskii's inequalty,

$$\|(\boldsymbol{L}\boldsymbol{f})_r\|_{\boldsymbol{L}^p} \lesssim (1-r^2)\int_0^1 \frac{\|(P|\boldsymbol{f}|)_{rt}\|_{\boldsymbol{L}^p(\mathbb{S})}}{(1-t^2r^2)^2} t^{1-\alpha}dt$$

$$\le \|\boldsymbol{f}\|_{\boldsymbol{L}^p} \int_0^1 \frac{(1-r^2)}{(1-t^2r^2)^2} t^{1-\alpha}dt \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^p},$$

and this holds for $1 \le p < \infty$. This completes the proof of *1.*

Now we prove the representation of elements in $\boldsymbol{h}_e^1(\mathbb{B})$. This is done as in the previous case using the duality $\boldsymbol{C}(\mathbb{S})^* = \boldsymbol{M}(\mathbb{S})$, where $\boldsymbol{C}(\mathbb{S})$ is the Banach space of continuous functions in $\mathbb{S}$ with values in $\mathbb{R}^3$. Finally, we will prove that $\boldsymbol{P}_e\boldsymbol{\mu} \in \boldsymbol{h}_e^1(\mathbb{B})$ for $\boldsymbol{\mu} \in \boldsymbol{M}(\mathbb{S})$. First recall that if μ is a Borel signed measure in $\mathbb{S}$ then

$$\left|\int_{\mathbb{S}} \varphi d\mu\right| \le \int_{\mathbb{S}} |\varphi| d|\mu|, \tag{19}$$

for every $\varphi \in C(\mathbb{S})$. This implies that for any $0 \le r < 1$

$$|(P\boldsymbol{\mu})_r| \le (P|\boldsymbol{\mu}|)_r.$$

Hence

$$\|(P\boldsymbol{\mu})_r\|_{\boldsymbol{L}^1} \le \|(P|\boldsymbol{\mu}|)_r\|_{L^1} \le |\boldsymbol{\mu}|(\mathbb{S}), \tag{20}$$

by the theory of harmonic Hardy spaces. Similarly by (48) and (19) we have

$$\|(\boldsymbol{L}\boldsymbol{\mu})_r\|_{\boldsymbol{L}^1} \lesssim (1-r^2)\int_0^1 \frac{\|(P\boldsymbol{\mu})_{rt}\|_{\boldsymbol{L}^1}}{(1-t^2r^2)^2} t^{1-\alpha} dt \leq |\boldsymbol{\mu}|(\mathbb{S}). \tag{21}$$

Then, by (20) and (21) we have that $\boldsymbol{P}_e\boldsymbol{\mu} \in \boldsymbol{h}_e^1(\mathbb{B})$.

$\square$

Remark 7. *By the open mapping theorem* $\boldsymbol{L}^p(\mathbb{S}) \cong \boldsymbol{h}_e^p(\mathbb{B})$ *for* $1 < p \leq \infty$ *and* $\boldsymbol{M}(\mathbb{S}) \cong \boldsymbol{h}_e^1(\mathbb{B})$ *via the elastic Poisson transformation* $\boldsymbol{P}_e$*.*

Theorem 8. $\boldsymbol{h}_e^p(\mathbb{B}) = \boldsymbol{H}_e^p(\mathbb{B})$ *for* $1 < p < \infty$*.*

Proof. Let $\boldsymbol{u} \in \boldsymbol{h}_e^p(\mathbb{B})$, and $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{S})$ such that $\boldsymbol{u}(x) = \boldsymbol{P}_e\boldsymbol{f}(x)$. Then by proposition 5,

$$|\boldsymbol{u}(x)| \leq C\mathcal{M}(|\boldsymbol{f}|)(\eta)$$

for $x \in \Gamma_\eta$. Then since $M[|\boldsymbol{f}|](\eta) \in \boldsymbol{L}^p(\mathbb{S})$, we have that

$$\mathcal{N}\boldsymbol{u} \in L^p(\mathbb{S}).$$

$\square$

We say that $\boldsymbol{u} : \mathbb{B} \to \mathbb{R}^3$ has non-tangential limits almost everywhere if the limit

$$\lim_{x\in\Gamma_\xi,\, x\to\xi} \boldsymbol{u}(x).$$

exists for almost all $\xi \in \mathbb{S}$.

We have the Fatou theorem for $\boldsymbol{h}_e^p(\mathbb{B})$. (See [11] for very general results on Fatou theorems for elliptic systems).

Theorem 9. *For any* $\boldsymbol{f} \in \boldsymbol{L}^1(\mathbb{S})$, $\boldsymbol{u} = \boldsymbol{P}_e\boldsymbol{f}$ *converges non-tangentially to* $\boldsymbol{f}$ *almost everywhere. In particular this holds for every* $\boldsymbol{u} \in \boldsymbol{h}_e^p(\mathbb{B})$, $1 < p \leq \infty$*. Moreover, every* $\boldsymbol{u} \in \boldsymbol{h}_e^1(\mathbb{B})$ *has non-tangential limits almost everywhere.*

Proof. First notice that if $\boldsymbol{f} \in \boldsymbol{C}(\mathbb{S})$, then by elliptic regularity we have that $\boldsymbol{P}_e(\boldsymbol{f}) \in \boldsymbol{C}(\overline{\mathbb{B}})$, We start with $\boldsymbol{f} = (f_1, f_2, f_3) \in \boldsymbol{C}^\infty(\mathbb{S})$. On the other hand, by Proposition 5c), $\mathcal{N}\boldsymbol{P}_e\boldsymbol{f}$ is weak $(1,1)$. This together with the fact that $\boldsymbol{C}(\mathbb{S})$ is dense in $\boldsymbol{L}^1(\mathbb{S})$, implies that $\boldsymbol{P}_e\boldsymbol{f}$ has non-tangential limits almost everywhere for $\boldsymbol{f} \in \boldsymbol{L}^1(\mathbb{S})$.

Now, to handle the case $\boldsymbol{u} \in \boldsymbol{h}_e^1(\mathbb{B})$, consider first $\boldsymbol{\mu} \in \boldsymbol{M}(\mathbb{S})$, singular with respect to the Lebesgue measure σ. Then $|\boldsymbol{\mu}|$ is also singular so that (see [1, Th. 6.42])

$$\lim_{x\in\Gamma_\xi,\, x\to\xi} P|\boldsymbol{\mu}|(x) = 0,$$

for ξ in a measurable set A with $\sigma(\mathbb{S} - A) = 0$. We claim that the same holds for $\boldsymbol{P}_e\boldsymbol{\mu}$.

Since

$$|\boldsymbol{P}_e\boldsymbol{\mu}(x)| \leq |P\boldsymbol{\mu}(x)| + |\boldsymbol{L}\boldsymbol{\mu}(x)|,$$

it remains to prove that

$$\lim_{x\in\Gamma_\xi,\, x\to\xi} \boldsymbol{L}\boldsymbol{\mu}(x) = 0. \tag{22}$$

In fact, let $\epsilon > 0$, $\xi \in A$ and $0 < \delta < 1$ such that $P|\boldsymbol{\mu}|(y) < \epsilon$ for $y \in \Gamma_\xi$ with $|\xi - y| < \delta$. We write

$$|\boldsymbol{L}\boldsymbol{\mu}(x)| \lesssim (1 - |x|^2) \left(\int_0^{\delta/2} + \int_{\delta/2}^1 \right) \frac{P|\boldsymbol{\mu}|(tx)}{(1 - |tx|^2)^2} t^{1-\alpha} dt.$$

Notice that if $x \in \Gamma_\xi$ with $|x - \xi| < \delta/2$ then $tx \in \Gamma_\xi$ and $|tx - \xi| < \delta$ for $0 < 1 - t < \delta/2$. This implies by (3) that

$$(1 - |x|^2) \int_{\delta/2}^1 \frac{P|\boldsymbol{\mu}|(tx)}{(1 - |tx|^2)^2} t^{1-\alpha} dt \leq C\epsilon.$$

On the other side, since $P|\boldsymbol{\mu}|(tx)$ is bounded for $t \in [0, \delta]$ and $x \in \mathbb{B}$, it is clear that

$$(1 - |x|^2) \int_0^{\delta/2} \frac{P|\boldsymbol{\mu}|(tx)}{(1 - |tx|^2)^2} t^{1-\alpha} dt \to 0$$

as $x \to \xi$. Hence (22) holds.

Now if $\boldsymbol{\mu} \in \boldsymbol{M}(\mathbb{S})$ then by the Radon Nikodym theorem $d\boldsymbol{\mu} = f d\sigma + d\boldsymbol{\mu}_s$, $f \in L^1(S)$ and $\boldsymbol{\mu}_s$ singular, so that the Theorem follows from the previous analysis.

□

Remark 10. *Notice that if $\boldsymbol{u} \in \boldsymbol{H}^1_e(\mathbb{B})$, then since $\mathcal{N}\boldsymbol{u} \in \boldsymbol{L}^1(\mathbb{S})$, its boundary limit $\boldsymbol{f}$ according to Theorem 9 belongs to $\boldsymbol{L}^1(\mathbb{S})$ by the dominated convergence theorem and $\boldsymbol{u}_r \to \boldsymbol{f}$ as $r \to 1-$ in $\boldsymbol{L}^1(\mathbb{S})$, hence $\boldsymbol{u} = \boldsymbol{P}_e\boldsymbol{f}$.*

Recall that $\boldsymbol{u}$ is a Riesz field such that $div\, \boldsymbol{u} = 0$ and $\nabla \times \boldsymbol{u} = 0$. This is equivalent to $\boldsymbol{u} = \nabla h$, where h is a harmonic function, and also to the condition $\nabla \boldsymbol{u}$ symmetric with null trace.

Riesz fields and solutions of $\Delta^* u = 0$ independent of μ and λ. There are fields like $\boldsymbol{u}(x) = x$, that are solutions of the Lamé equation for any value of λ and μ. In general, any harmonic field $\boldsymbol{u} : \mathbb{B} \to \mathbb{R}^3$ such that $\nabla div\, \boldsymbol{u} = 0$ is a solution to the Lamé equation for any μ and λ. An example of this are the Riesz fields.

If $\boldsymbol{u}$ is a Riesz field then $\Delta \boldsymbol{u} = 0$ and $\nabla div\, \boldsymbol{u} = \nabla \times (\nabla \times \boldsymbol{u}) = 0$, hence $\Delta^* \boldsymbol{u} = 0$ for any eligible λ, μ. An easy calculation shows that if $\boldsymbol{u}$ is a Riesz field then $\boldsymbol{v}(x) = x \times \boldsymbol{u}(x)$ is harmonic and $div\, \boldsymbol{v} = 0$. So that it is a solution of the Lamé equation for any λ and μ.

Notice that if $\boldsymbol{u}$ is a solution of $\Delta^* \boldsymbol{u} = 0$ for all μ and λ and has boundary value $\boldsymbol{f}$, then

$$\boldsymbol{u} = \boldsymbol{P}_e \boldsymbol{f} = P\boldsymbol{f}. \tag{23}$$

5 Vector spherical harmonics and vector-valued $\boldsymbol{L}^p$ spaces

For $\ell \geq 0$ we define the real vector functions on $\mathbb{S}$

$$
\begin{aligned}
\boldsymbol{E}^{+}_{\ell,m}(\eta) &= (\ell+1)\boldsymbol{Y}^{\vee}_{\ell,m}(\eta) - \nabla_\sigma Y_{\ell,m}(\eta), && |m| \leq \ell,\\
\boldsymbol{E}^{-}_{\ell,m}(\eta) &= \ell\boldsymbol{Y}^{\vee}_{\ell,m}(\eta) + \nabla_\sigma Y_{\ell,m}(\eta), && |m| \leq \ell,\\
\boldsymbol{E}^{0}_{\ell,m}(\eta) &= \eta \times \nabla_\sigma Y_{\ell,m}(\eta), && |m| \leq \ell,
\end{aligned} \tag{24}
$$

and the following vector spaces of vector fields on $\mathbb{S}$

$$
\begin{aligned}
\boldsymbol{E}^{+}_{\ell}(\mathbb{S}) &= \bigvee \left\{\boldsymbol{E}^{+}_{\ell,m},\ |m| \leq \ell\right\},\\
\boldsymbol{E}^{-}_{\ell}(\mathbb{S}) &= \bigvee \left\{\boldsymbol{E}^{-}_{\ell,m},\ \ |m| \leq \ell\right\},\\
\boldsymbol{E}^{0}_{\ell}(\mathbb{S}) &= \bigvee \left\{\boldsymbol{E}^{0}_{\ell,m},\ \ |m| \leq \ell\right\}.
\end{aligned} \tag{25}
$$

We note that $\boldsymbol{E}^{-}_{0}(\mathbb{S}) = \boldsymbol{E}^{0}_{0}(\mathbb{S}) = \{0\}$.

It can be checked that for $m, m' \in \{-\ell, \cdots, \ell\}$, (see [7]),

$$
\begin{aligned}
&\int_{\mathbb{S}} \boldsymbol{E}^{+}_{\ell,m}(\eta) \cdot \boldsymbol{E}^{+}_{\ell,m'}(\eta) d\sigma(\eta) = (\ell+1)(2\ell+1)\delta_{mm'},\\
&\int_{\mathbb{S}} \boldsymbol{E}^{-}_{\ell,m}(\eta) \cdot \boldsymbol{E}^{-}_{\ell,m'}(\eta) d\sigma(\eta) = \ell(2\ell+1)\delta_{mm'},\ \ \ell \neq 0,\\
&\int_{\mathbb{S}} \boldsymbol{E}^{0}_{\ell,m}(\eta) \cdot \boldsymbol{E}^{0}_{\ell,m'}(\eta) d\sigma(\eta) = \ell(\ell+1)\delta_{mm'},\ \ \ell \neq 0,
\end{aligned} \tag{26}
$$

If $\mu^{+}_{\ell} = \sqrt{(\ell+1)(2\ell+1)}$, $\mu^{-}_{\ell} = \sqrt{\ell(2\ell+1)}$ and $\mu^{0}_{\ell} = \sqrt{\ell(\ell+1)}$, then $\left\{\frac{1}{\mu^{+}_{\ell}}\boldsymbol{E}^{+}_{\ell,m}\right\}_{|m|\leq\ell}$, $\left\{\frac{1}{\mu^{-}_{\ell}}\boldsymbol{E}^{-}_{\ell,m}\right\}_{|m|\leq\ell}$ and $\left\{\frac{1}{\mu^{0}_{\ell}}\boldsymbol{E}^{0}_{\ell,m}\right\}_{|m|\leq\ell}$ are an orthonormal basis of $\boldsymbol{E}^{+}_{\ell}(\mathbb{S})$, $\boldsymbol{E}^{-}_{\ell}(\mathbb{S})$ and $\boldsymbol{E}^{0}_{\ell}(\mathbb{S})$ respectively.

Theorem 11. *The spaces $\boldsymbol{E}^{+}_{\ell}(\mathbb{S})$, $\boldsymbol{E}^{-}_{\ell}(\mathbb{S})$ and $\boldsymbol{E}^{0}_{\ell}(\mathbb{S})$ are mutually orthogonal and their elements are the restriction of vector harmonic homogeneous polynomials of degree $\ell+1$, $\ell-1$ and ℓ respectively, (in the latter two cases $\ell \geq 1$).*

We have the orthogonal decomposition

$$
\boldsymbol{L}^2(\mathbb{S}) = \boldsymbol{L}^2_{+}(\mathbb{S}) \oplus \boldsymbol{L}^2_{-}(\mathbb{S}) \oplus \boldsymbol{L}^2_{0}(\mathbb{S}), \tag{27}
$$

where

$$
\boldsymbol{L}^2_{+}(\mathbb{S}) := \bigoplus_{\ell\geq 0} \boldsymbol{E}^{+}_{\ell}(\mathbb{S}), \tag{28}
$$

$$
\boldsymbol{L}^2_{-}(\mathbb{S}) := \bigoplus_{\ell\geq 1} \boldsymbol{E}^{-}_{\ell}(\mathbb{S}), \tag{29}
$$

$$
\boldsymbol{L}^2_{0}(\mathbb{S}) := \bigoplus_{\ell\geq 1} \boldsymbol{E}^{0}_{\ell}(\mathbb{S}). \tag{30}
$$

Proof. Let $\ell, \ell' \geq 0,\ \ell \neq \ell' \neq 0$, $|m| \leq \ell$ and $|m'| \leq \ell'$.

$$\begin{aligned}
\langle \boldsymbol{E}^{+}_{\ell,m}, \boldsymbol{E}^{-}_{\ell',m'} \rangle &= \ell'(\ell+1)\langle Y_{\ell,m}, Y_{\ell',m'}\rangle - \langle \nabla_\sigma Y_{\ell,m}, \nabla_\sigma Y_{\ell',m'}\rangle \\
&= \ell'(\ell+1)\delta_{\ell\ell'}\delta_{mm'} + \int_{\mathbb{S}} Y_{\ell,m}(\eta)\Delta_\sigma Y_{\ell',m'} d\sigma(\eta) \\
&= \ell'(\ell+1)\delta_{\ell\ell'}\delta_{mm'} - \ell'(\ell'+1)\int_{\mathbb{S}} Y_{\ell,m}(\eta) Y_{\ell',m'} d\sigma(\eta) \\
&= \ell'(\ell+1)\delta_{\ell\ell'}\delta_{mm'} - \ell'(\ell'+1)\delta_{\ell\ell'}\delta_{mm'} = 0,
\end{aligned}$$

which proves that the spaces $\boldsymbol{E}^{+}_{\ell}(\mathbb{S})$ and $\boldsymbol{E}^{-}_{\ell'}(\mathbb{S})$ are orthogonal. It is clear that $\boldsymbol{E}^{0}_{\ell'}(\mathbb{S})$ is orthogonal to $\boldsymbol{E}^{+}_{\ell}(\mathbb{S})$ and $\boldsymbol{E}^{-}_{\ell}(\mathbb{S})$.

Now we show that the elements of the spaces $\boldsymbol{E}^{+}_{\ell}(\mathbb{S})$, $\boldsymbol{E}^{-}_{\ell}(\mathbb{S})$ and $\boldsymbol{E}^{0}_{\ell}(\mathbb{S})$ are the restriction of vector harmonic homogeneous polynomials of degree $\ell+1$, $\ell-1$ and ℓ respectively. Let $Y \in \mathcal{H}^\ell$, then by (5) and for $x = |x|x'$ we have

$$\begin{aligned}
\nabla Y(x)\big|_{x=x'} &= \ell \boldsymbol{Y}^{\vee}(x') + \nabla_\sigma Y(x'),\quad \ell \neq 0, \\
((2\ell+1)Y(x)x - |x|^2\nabla Y(x))\big|_{x=x'} &= (\ell+1)\boldsymbol{Y}^{\vee}(x') - \nabla_\sigma Y(x'), \\
x \times \nabla Y(x)\big|_{x=x'} &= x' \times \nabla_\sigma Y(x'),\quad \ell \neq 0,
\end{aligned} \tag{31}$$

and the result follows from the fact that $(2\ell+1)Y(x)x - |x|^2\nabla Y(x)$, $x \times \nabla Y(x)$ and $\nabla Y(x)$ are vector harmonic homogeneous polynomials of degree $\ell+1, \ell$ and $\ell-1$ respectively.

Finally to see (27) we note that

$$\boldsymbol{E}^{+}_{\ell-1}(\mathbb{S}) \oplus \boldsymbol{E}^{-}_{\ell+1}(\mathbb{S}) \oplus \boldsymbol{E}^{0}_{\ell}(\mathbb{S}) \subset (\mathcal{H}^\ell)^3$$

and by considering the dimension of the spaces in the above inclusion we have that they are equal and hence (27) holds.

□

Remark 12. *The above theorem tells us that*

$$\left\{\frac{1}{\mu^{+}_{\ell}}\boldsymbol{E}^{+}_{\ell,m}\right\}_{\ell\geq 0,\ |m|\leq \ell}, \left\{\frac{1}{\mu^{-}_{\ell}}\boldsymbol{E}^{-}_{\ell,m}\right\}_{\ell\geq 1,\ |m|\leq \ell} \quad \text{and} \quad \left\{\frac{1}{\mu^{0}_{\ell}}\boldsymbol{E}^{0}_{\ell,m}\right\}_{\ell\geq 1,\ |m|\leq \ell},$$

are an orthonormal basis of $\boldsymbol{L}^2_{+}(\mathbb{S})$*,* $\boldsymbol{L}^2_{-}(\mathbb{S})$ *and* $\boldsymbol{L}^2_{0}(\mathbb{S})$ *respectively.*

We will use this systems throughout this section. We also notice that from (4) we have the estimate in ℓ

$$\|\boldsymbol{E}^{\#}_{\ell,m}\|_{\boldsymbol{L}^\infty(\mathbb{S})} = O(\ell^{3/2}),\quad \# \in \{+,-,0\}. \tag{32}$$

Denote by $\boldsymbol{\pi}_+, \boldsymbol{\pi}_-, \boldsymbol{\pi}_0$ the orthogonal projection of $\boldsymbol{L}^2(\mathbb{S})$ onto $\boldsymbol{L}^2_{+}(\mathbb{S})$, $\boldsymbol{L}^2_{-}(\mathbb{S})$ and $\boldsymbol{L}^2_{0}(\mathbb{S})$ respectively.

Theorem 13. *For any $1 < p < \infty$ the projections $\boldsymbol{\pi}_+, \boldsymbol{\pi}_-, \boldsymbol{\pi}_0$ defined on $\boldsymbol{C}^\infty(\mathbb{S})$ can be extended to continuous projections in $\boldsymbol{L}^p(\mathbb{S})$, and*

$$\boldsymbol{L}^p(\mathbb{S}) = \boldsymbol{L}^p_+(\mathbb{S}) \oplus \boldsymbol{L}^p_-(\mathbb{S}) \oplus \boldsymbol{L}^p_0(\mathbb{S}), \tag{33}$$

where $\boldsymbol{L}^p_+(\mathbb{S})$, $\boldsymbol{L}^p_-(\mathbb{S})$ and $\boldsymbol{L}^p_0(\mathbb{S})$ are the image of $\boldsymbol{L}^p(\mathbb{S})$ under $\boldsymbol{\pi}_+, \boldsymbol{\pi}_-$ and $\boldsymbol{\pi}_0$ respectively.

Proof. See the appendix. □

We define the operator acting on functions on $\mathbb{S}$,

$$\mathcal{L}_- g = (\boldsymbol{\mathcal{M}_\ell g})^\vee + \nabla_\sigma g. \tag{34}$$

By (5),

$$\nabla(PY) = P(\mathcal{L}_- Y), \;\; Y \in \mathcal{H}^\ell.$$

Theorem 14. *Let $1 < p < \infty$. Then*

a) The operator $\mathcal{L}_-$ maps continuously $W^{1,p}(\mathbb{S})$ onto $\boldsymbol{L}^p_-(\mathbb{S})$ and its kernel are the constant functions.

b) For $g \in W^{1,p}(\mathbb{S})$ we have

$$P(\mathcal{L}_- g)(x) = \nabla(Pg)(x) \qquad x \in \mathbb{B}. \tag{35}$$

c) $\boldsymbol{f} \in \boldsymbol{L}^p_-(\mathbb{S})$ if and only if there exists $g \in W^{1,p}(\mathbb{S})$ such that

$$\boldsymbol{f}(\eta) = \lim_{r \to 1^-} \nabla(Pg)(r\eta).$$

Proof. *a)* Let $g \in W^{1,p}(\mathbb{S})$ with expansion $g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}$ in $\mathcal{D}'(\mathbb{S})$. Then

$$\mathcal{M}_\ell g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} \ell a_{\ell,m} Y_{\ell,m} = \mathcal{M}_{\beta_\ell} (-\Delta_\sigma)^{1/2} g \in L^p(\mathbb{S}),$$

where $\beta_\ell = \frac{\ell}{\sqrt{\ell(\ell+1)}}$, if $\ell > 0$ and $\beta_0 = 0$, and by Theorem 1 and Lemma 3 we have that $\mathcal{L}_- : W^{1,p}(\mathbb{S}) \to \boldsymbol{L}^p(\mathbb{S})$ is continuous.

Since $\mathcal{L}_- Y_{\ell,m} = \boldsymbol{E}^-_{\ell,m}$, then if g is smooth,

$$\mathcal{L}_- g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} \boldsymbol{E}^-_{\ell,m} \in \boldsymbol{L}^p_-(\mathbb{S}).$$

Finally approximating a $g \in W^{1,p}(\mathbb{S})$ by a sequence of smooth functions, it follows that $\mathcal{L}_- g \in \boldsymbol{L}^p_-(\mathbb{S})$.

Notice that for $g \in W^{1,p}(\mathbb{S})$, $(\boldsymbol{\mathcal{M}_\ell g})^\vee(x')$ and $\nabla_\sigma g(x')$ are orthogonal for almost every $x' \in \mathbb{S}$. From this it follows that $\mathcal{L}_- g = 0$ if and only if $\nabla_\sigma g = 0$, namely if and only if g is a constant.

Now we prove that $\mathcal{L}_-$ is onto.

If $\boldsymbol{f} \in \boldsymbol{L}^p_-(\mathbb{S})$ is smooth then we can write

$$\boldsymbol{f} = \sum_{\ell \geq 1} \sum_{|m| \leq \ell} a_{\ell,m} \boldsymbol{E}^-_{\ell,m}.$$

Then $\boldsymbol{f} = \mathcal{L}_- g$, with $g = \sum_{\ell \geq 1} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}$.

Again by the orthogonality of $(\boldsymbol{\mathcal{M}_\ell g})^\vee(x')$ and $\nabla_\sigma g(x')$ implies that

$$|\mathcal{L}_- g(x')| = |(\boldsymbol{\mathcal{M}_\ell g})^\vee(x') + \nabla_\sigma g(x')| \sim \left(|(\boldsymbol{\mathcal{M}_\ell g})^\vee(x')|^p + |\nabla_\sigma g(x')|^p\right)^{1/p},$$

uniformly in x', hence

$$\|(-\Delta)^{1/2} g\|_{L^p} \lesssim \|\nabla_\sigma g\|_{\boldsymbol{L}^p} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^p}.$$

Moreover, since $\int_{\mathbb{S}} g(x')dx' = 0$, we can write $g = \gamma_\ell(-\Delta_\sigma)^{1/2}$, with $\gamma_\ell = \frac{\ell}{\sqrt{\ell(\ell+1)}}$, if $\ell > 0$ and $\gamma_0 = 0$.

Hence

$$\|g\|_{L^p} \lesssim \|(-\Delta_\sigma)^{1/2} g\|_{L^p} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^p},$$

that is,

$$\|g\|_{W^{1,p}} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^p}.$$

Now, if $\boldsymbol{f}$ is any element of $\boldsymbol{L}^p_-(\mathbb{S})$, let $\boldsymbol{f}_n \in \boldsymbol{L}^p_-(\mathbb{S}) \cap \boldsymbol{C}^\infty(\mathbb{S})$ such that $\boldsymbol{f}_n$ converges to $\boldsymbol{f}$ in $\boldsymbol{L}^p(\mathbb{S})$ and $g_n \in C^\infty(\mathbb{S})$ such that $\mathcal{L}_-(g_n) = \boldsymbol{f}_n$. The previous argument shows that

$$\|g_n - g_m\|_{W^{1,p}} \leq C\|\boldsymbol{f}_n - \boldsymbol{f}_m\|_{\boldsymbol{L}^p},$$

and the limit g of g_n in $W^{1,p}(\mathbb{S})$ satisfies $\mathcal{L}_- g = \boldsymbol{f}$.

b) Let $g \in C^\infty(\mathbb{S})$ such that $g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}$ in $W^{1,p}(\mathbb{S})$. For $x \in \mathbb{B}$ fixed by (5) (noticing that all the series below converge uniformly on compact subsets of $\mathbb{B}$)

$$P(\mathcal{L}_- g)(x) = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} P(\mathcal{L}_- Y_{\ell,m})(x) = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} \nabla(PY_{\ell,m})(x)$$

$$= \nabla\left(\sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} PY_{\ell,m}\right)(x) = \nabla\left(P \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}\right)(x) = \nabla(Pg)(x).$$

Hence the result holds in this case. Now let $g \in W^{1,p}(\mathbb{S})$ and $g_n \in C^\infty(\mathbb{S})$ such that $g_n \to g$ in $W^{1,p}(\mathbb{S})$. If $x \in \mathbb{B}$ we have $P(\mathcal{L}_- g_n)(x) \to P(\mathcal{L}_- g)(x)$. On the other hand $\nabla(Pg_n)(x) \to \nabla(Pg)(x)$ and $\nabla(Pg_n)(x) = P(\mathcal{L}_- g_n)(x) \to P(\mathcal{L}_- g)(x)$, then $\nabla(Pg) = P(\mathcal{L}_- g)$.

c) This follows directly from a) and b).

□

Now we consider the operator acting on functions on $\mathbb{S}$,

$$\mathcal{L}_0 g(\eta) = \eta \times \nabla_\sigma g(\eta). \tag{36}$$

Theorem 15. *Let $1 < p < \infty$. Then*

a) The operator $\mathcal{L}_0$ maps continuously $W^{1,p}(\mathbb{S})$ onto $\boldsymbol{L}_0^p(\mathbb{S})$ and its kernel are the constant functions.

b) For $g \in W^{1,p}(\mathbb{S})$ we have

$$P(\mathcal{L}_0 g)(x) = x \times \nabla(Pg)(x), \quad x \in \mathbb{B}. \tag{37}$$

c) $\boldsymbol{f} \in \boldsymbol{L}_0^p(\mathbb{S})$ if and only if there exists $g \in W^{1,p}(\mathbb{S})$ such that

$$\boldsymbol{f}(\eta) = \lim_{r \to 1^-} r\eta \times \nabla(Pg)(r\eta).$$

Proof. *a)* Let $g \in W^{1,p}(\mathbb{S})$ with expansion $g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}$ in $\mathcal{D}'(\mathbb{S})$ and $\tilde{g} = \sum_{\ell \geq 1} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}$. By using Lemma 3 we have

$$\|\mathcal{L}_0 g\|_{\boldsymbol{L}^p} = \|\nabla_\sigma \tilde{g}\|_{\boldsymbol{L}^p} \lesssim \|(-\Delta_\sigma)^{\frac{1}{2}} \tilde{g}\|_{L^p} \lesssim \|g\|_{W^{1,p}},$$

implying that $\mathcal{L}_0$ maps continuously $W^{1,p}(\mathbb{S})$ onto $\boldsymbol{L}_0^p(\mathbb{S})$.

Let us see now that $\mathcal{L}_0$ is onto: let $\boldsymbol{f} \in \boldsymbol{L}_0^p(\mathbb{S})$ be and $\boldsymbol{f}_n \in \boldsymbol{C}^\infty(\mathbb{S}) \cap \boldsymbol{L}_0^p(\mathbb{S})$ converging to $\boldsymbol{f}$ in $\boldsymbol{L}^p(\mathbb{S})$. We have that

$$\begin{aligned}
\boldsymbol{f}_n(\eta) &= \sum_{\ell \geq 1} \sum_{|m| \leq \ell} a_{\ell,m}^n \boldsymbol{E}_{\ell,m}^0(\eta) = \sum_{\ell \geq 1} \sum_{|m| \leq \ell} a_{\ell,m}^n \eta \times \nabla_\sigma Y_{\ell,m}(\eta) \\
&= \eta \times \nabla_\sigma(g_n) = \mathcal{L}_0(g_n),
\end{aligned}$$

where $g_n(\eta) = \sum_{\ell \geq 1} \sum_{|m| \leq \ell} a_{\ell,m}^n Y_{\ell,m}(\eta)$. Since $\int_{\mathbb{S}} g_n(\eta) d\sigma(\eta) = 0$, it follows that

$$\|g_n\|_{W^{1,p}} \sim \|\nabla_\sigma g_n\|_{L^p} \lesssim \|\boldsymbol{f}_n\|_{\boldsymbol{L}^p},$$

proving that $g_n \in W^{1,p}(\mathbb{S})$.

In a similar way

$$\|g_n - g_m\|_{W^{1,p}} \lesssim \|\boldsymbol{f}_n - \boldsymbol{f}_m\|_{\boldsymbol{L}^p},$$

so there is $g \in W^{1,p}(\mathbb{S})$ such that g_n converges to $g \in W^{1,p}(\mathbb{S})$. As $\mathcal{L}_0$ is continuous, $\mathcal{L}_0(g_n) = \boldsymbol{f}_n$ converges to $\mathcal{L}_0(g)$ and it must be verified that $\mathcal{L}_0(g) = \boldsymbol{f}$.

Finally, we will study the kernel of $\mathcal{L}_0$. Let $g \in W^{1,p}(\mathbb{S})$ be such that $\mathcal{L}_0 g(\eta) = \eta \times \nabla_\sigma g(\eta) = 0$ implying that $\nabla_\sigma g = 0$ in $\boldsymbol{L}^p(\mathbb{S})$, and by Lemma 3 $(-\Delta)^{\frac{1}{2}} g = 0$ in $L^p(\mathbb{S})$. If $g = \sum_{\ell \geq 0} \sum_{|m| \leq \ell} a_{\ell,m} Y_{\ell,m}$ in $\mathcal{D}'(\mathbb{S})$, then $(-\Delta)^{\frac{1}{2}} g = \sum_{\ell \geq 1} \sum_{|m| \leq \ell} \ell(\ell+1) a_{\ell,m} Y_{\ell,m} = 0$ in $\mathcal{D}'(\mathbb{S})$. This implies that $a_{\ell,m} = 0$, $\ell \geq 1$ and $|m| \leq \ell$ and therefore g is a constant.

b) Note that if $g = Y$, $Y \in \mathcal{H}^\ell$, then it follows from (31) that $P(\mathcal{L}_0 g)(x) = x \times \nabla Y(x)$. Then the proof of b) for $g \in W^{1,p}$ follows as the proof of part b) of Theorem 14.

c) This follows directly from a) and b).

□

Finally we define the operator acting on functions on S,

$$\mathcal{L}_+ g = (\boldsymbol{\mathcal{M}_{\ell+1} g})^\vee - \nabla_\sigma g. \tag{38}$$

We have

Theorem 16. *Let $1 < p < \infty$. Then*

a) The operator $\mathcal{L}_+$ is an isomorphism of $W^{1,p}(\mathbb{S})$ onto $\boldsymbol{L}^p_+(\mathbb{S})$.

b) For $g \in W^{1,p}(\mathbb{S})$ we have

$$P(\mathcal{L}_+ g)(x) = [2x \cdot \nabla(Pg)(x) + Pg(x)]x - |x|^2 \nabla(Pg)(x), \quad x \in \mathbb{B}. \tag{39}$$

c) $\boldsymbol{f} \in \boldsymbol{L}^p_+(\mathbb{S})$ if and only if there exists $g \in W^{1,p}(\mathbb{S})$ such that

$$\boldsymbol{f}(\eta) = \lim_{r\to 1^-} \left([2r\eta \cdot \nabla(Pg)(r\eta) + Pg(r\eta)]r\eta - r^2 \nabla(Pg)(r\eta)\right).$$

Proof. a) The proof is analogous to the proof of Theorem 14 a).

b) First consider $g = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m} Y_{\ell,m} \in C^\infty(\mathbb{S})$. Define

$$h(x) = Pg(x) = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m} Y_{\ell,m}(x), \quad x \in \mathbb{B}.$$

then by (5),

$$x \cdot \nabla h(x) = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} \ell a_{\ell,m} Y_{\ell,m}(x).$$

Hence if we let

$$\boldsymbol{u}(x) = \{2x \cdot \nabla h(x) + h(x)\}x - |x|^2 \nabla h(x),$$

it is easy to see that $\boldsymbol{u}$ is harmonic on $\mathbb{B}$ and

$$\boldsymbol{u}(x) = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} \{(2\ell+1)a_{\ell,m} Y_{\ell,m}(x)\}x - \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m} \nabla Y_{\ell,m}(x).$$

Furthermore by (5) for $x = r\eta$ this simplifies to

$$\begin{aligned}\boldsymbol{u}(r\eta) &= \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m} r^{\ell+1}\{(\ell+1)Y_{\ell,m}(\eta)\eta - \nabla_\sigma Y_{\ell,m}(\eta)\} \\ &= \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m} r^{\ell+1} \mathcal{L}_+(Y_{\ell,m})(\eta),\end{aligned}$$

and the result follows in this case, noting that $\lim_{r\to 1^-} \boldsymbol{u}(r\eta) = \mathcal{L}_+ g(\eta)$.

The proof for $g \in W^{1,p}(\mathbb{S})$ follows as the proof of part b) of Theorem 14.

c) follows directly from *a)* and *b)*. □

The following result follows at once from Theorems 14,15 and 16.

Corollary 17. *Let $1 < p < \infty$. Then $\boldsymbol{u}$ belongs to the harmonic vector-valued Hardy space $\boldsymbol{h}^p(\mathbb{B})$ if and only if for some $g, \widetilde{g}, h \in W^{1,p}(\mathbb{S})$*

$$\boldsymbol{u}(x) = \nabla(Pg)(x) + x \times \nabla(P\widetilde{g})(x) + [2x \cdot \nabla(Ph)(x) + Ph(x)]x - |x|^2 \nabla(Ph)(x).$$

6 The spaces $\boldsymbol{h}^p_+(\mathbb{B})$, $\boldsymbol{h}^p_-(\mathbb{B})$ and $\boldsymbol{h}^p_0(\mathbb{B})$

In this section we study the elastic extensions of the vector L^p spaces defined in section 5.

In [7, lemma 10.3.6] it is proved the following

Lemma 18. *The solutions to the Dirichlet problem*

$$\begin{aligned} \Delta^* \boldsymbol{u} &= 0 \quad \text{in } \mathbb{B}, \\ \boldsymbol{u} &= \boldsymbol{f} \quad \text{in } \mathbb{S}, \end{aligned}$$

are given as follows:

a) If $\boldsymbol{f} = \boldsymbol{E}^+_{\ell,m}$ then

$$\boldsymbol{u}^+_{\ell,m}(x) = (2\ell+1)\big(Y_{\ell,m}(x)x + \alpha_\ell(|x|^2-1)\nabla Y_{\ell,m}(x)\big) - \nabla Y_{\ell,m}(x),$$

where $\alpha_\ell = -\frac{(\ell+3)\tau+2}{2(\ell(\tau+2)+1)}$ and $\tau = \frac{\lambda+\mu}{\mu}$.

b) If $\boldsymbol{f} = \boldsymbol{E}^-_{\ell,m}$ then $\boldsymbol{u}^-_{\ell,m}(x) = \nabla Y_{\ell,m}(x)$.

c) If $\boldsymbol{f} = \boldsymbol{E}^0_{\ell,m}$ then $\boldsymbol{u}^0_{\ell,m}(x) = x \times \nabla Y_{\ell,m}(x)$.

We will denote by $\boldsymbol{h}^p_+(\mathbb{B})$, $\boldsymbol{h}^p_-(\mathbb{B})$ and $\boldsymbol{h}^p_0(\mathbb{B})$ the image of $\boldsymbol{L}^p_+(\mathbb{S})$, $\boldsymbol{L}^p_-(\mathbb{S})$ and $\boldsymbol{L}^p_0(\mathbb{S})$ respectively, under the elastic Poisson transform $\boldsymbol{P}_e$. Notice that the space $\boldsymbol{h}^p_-(\mathbb{B})$ is the Hardy space of Riesz fields. The space $\boldsymbol{L}^2_-(\mathbb{S})$ of boundary values of $\boldsymbol{h}^2_-(\mathbb{B})$ was characterized in [8] as a subspace of $\boldsymbol{L}^2(\mathbb{S})$ with reproducing kernel. By (33) it follows that for $1 < p < \infty$

$$\boldsymbol{h}^p_e(\mathbb{B}) = \boldsymbol{h}^p_+(\mathbb{B}) \oplus \boldsymbol{h}^p_-(\mathbb{B}) \oplus \boldsymbol{h}^p_0(\mathbb{B}).$$

The following is a restatement of Theorems 14 and 15:

Theorem 19. *1) Let $1 < p < \infty$. Then $\boldsymbol{u} \in \boldsymbol{h}^p_-(\mathbb{B})$ if and only if there exist $g \in W^{1,p}(\mathbb{S})$ such that $\boldsymbol{u}(x) = \nabla(Pg)(x) = P(\mathcal{L}_- g)(x)$.*

2) Let $1 < p < \infty$. Then $\boldsymbol{u} \in \boldsymbol{h}^p_0(\mathbb{B})$ if and only if there exists $g \in W^{1,p}(\mathbb{S})$ such that $\boldsymbol{u}(x) = x \times \nabla(Pg)(x) = P(\mathcal{L}_0 g)(x)$.

Corollary 20. *Let $1 < p < \infty$. Then $\boldsymbol{h}^p_-(\mathbb{B})$ is isomorphic to $\boldsymbol{h}^p_0(\mathbb{B})$.*

Theorem 21. *Let $1 < p < \infty$, $\boldsymbol{u} \in \boldsymbol{h}^p_+(\mathbb{B})$ with $\boldsymbol{u}$ harmonic if and only if $\boldsymbol{u}(x) = cx$ for some constant c.*

Proof. Let $\boldsymbol{u} \in \boldsymbol{h}^p_+(\mathbb{B})$ with $\boldsymbol{u}$ harmonic. Then there is $g \in W^{1,p}(\mathbb{S})$ be such that $\boldsymbol{u} = P_e(\mathcal{L}_+ g) = P(\mathcal{L}_+ g)$. Now by Theorem 16 we have that

$$\boldsymbol{u}(x) = [2x \cdot \nabla(Pg)(x) + Pg(x)]x - |x|^2 \nabla(Pg)(x).$$

It follows that

$$div\, \boldsymbol{u}(x) = 3Pg(x) + 5x \cdot \nabla(Pg)(x) + 2x \cdot \nabla(x \cdot \nabla(Pg)(x)). \tag{40}$$

Let $g = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m}Y_{\ell,m}$ in $\mathcal{D}'(\mathbb{S})$, so that

$$Pg(x) = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m}Y_{\ell,m}(x), \quad x\in\mathbb{B}.$$

It is easy to see from (40) that

$$div\,\boldsymbol{u}(x) = \sum_{\ell\geq 0}\sum_{|m|\leq\ell}(2\ell^2+5\ell+3)a_{\ell,m}Y_{\ell,m}(x). \tag{41}$$

Now since $\Delta^*\boldsymbol{u} = \Delta\boldsymbol{u} = 0$, we have that $\nabla div\,\boldsymbol{u}(x) = 0$, namely $div\,\boldsymbol{u}(x)$ is constant. From (41) we can conclude that $a_{\ell,m} = 0$ for $\ell > 0$ and then that $g = c$. This is that $\boldsymbol{u}(x) = cx$. The sufficient condition is clear. □

Corollary 22. *Let $1 < p < \infty$, $\boldsymbol{u}\in\boldsymbol{h}^p_e(\mathbb{B})$ with $\boldsymbol{u}$ harmonic if and only if*

$$\boldsymbol{u}(x) = \nabla(Pg)(x) + x\times\nabla(P\widetilde{g})(x) + cx,$$

for some $g,\widetilde{g}\in W^{1,p}(\mathbb{S})$ and c constant. Moreover, these are the unique solutions of the Lamé system belonging to $\boldsymbol{h}^p_e(\mathbb{B})$ for all λ and μ.

Proof. It follows at once from Theorems 19 and 21. □

Now we will characterize the vector fields in $\boldsymbol{h}^p_+(\mathbb{B})$. To motive the next Theorem recall from Lemma 18 (and the notation there) that

$$\boldsymbol{P}_e(\boldsymbol{E}^+_{\ell,m})(x) = (2\ell+1)\big(Y_{\ell,m}(x)x + \alpha_\ell(|x|^2-1)\nabla Y_{\ell,m}(x)\big) - \nabla Y_{\ell,m}(x).$$

From Theorem 11, this can be written as

$$\boldsymbol{P}_e(\boldsymbol{E}^+_{\ell,m})(x) = P(\boldsymbol{E}^+_{\ell,m})(x) + (|x|^2-1)\nabla P(\beta_\ell Y_{\ell,m})(x).$$

where $\beta_\ell = (2\ell+1)\alpha_\ell + 1$. We are going to consider the multiplier $\mathcal{M}_{\beta_\ell}$.

Lemma 23. *For $1 < p < \infty$, the multiplier $\mathcal{M}_{\beta_\ell}: W^{1,p}(\mathbb{S})\to L^p(\mathbb{S})$ is bounded.*

Proof. The proof follows directly from Theorem 1. □

Theorem 24. *Let $1 < p < \infty$. If $\boldsymbol{u}\in\boldsymbol{h}^p_+(\mathbb{B})$, there exist $g\in W^{1,p}(\mathbb{S})$ such that*

$$\boldsymbol{u}(x) = P(\mathcal{L}_+g)(x) + (|x|^2-1)\nabla(P(\mathcal{M}_{\beta_\ell}g))(x) \tag{42}$$

Conversely, any function $\boldsymbol{u}$ as in (42) for $g\in W^{1,p}(\mathbb{S})$, belongs to $\boldsymbol{h}^p_+(\mathbb{B})$.

Proof. Let $\boldsymbol{u}\in\boldsymbol{h}^p_+(\mathbb{B})$ by Theorem 16 a) there is $g\in W^{1,p}(\mathbb{S})$ be such that $\boldsymbol{u}(x) = \boldsymbol{P}_e(\mathcal{L}_+g)(x)$. Let $g = \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m}Y_{\ell,m}$ in $\mathcal{D}'(\mathbb{S})$, first note that by Lemma 23, $\mathcal{M}_{\beta_\ell}: W^{1,p}(\mathbb{S})\to L^p(\mathbb{S})$ is bounded. Then

$$\begin{aligned}\boldsymbol{u}(x) = \boldsymbol{P}_e(\mathcal{L}_+g)(x) &= \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m}\boldsymbol{P}_e(E^+_{\ell,m})(x) = \\ &= \sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m}P(E^+_{\ell,m})(x) + (|x|^2-1)\sum_{\ell\geq 0}\sum_{|m|\leq\ell} a_{\ell,m}\nabla(P(\beta_\ell Y_{\ell,m}))(x) \\ &= P(\mathcal{L}_+g)(x) + (|x|^2-1)\nabla(P(\mathcal{M}_{\beta_\ell}g))(x).\end{aligned}$$

The sufficient condition follows by reversing the steps above, to prove that $\boldsymbol{u} = \boldsymbol{P}_e(\mathcal{L}_+ g)(x)$. $\square$

Corollary 25. *Let $1 < p < \infty$, $\boldsymbol{u} \in \boldsymbol{h}_e^p(\mathbb{B})$, if and only if*

$$\boldsymbol{u}(x) = \nabla(Pg)(x) + x \times \nabla(P\widetilde{g})(x) + P(\mathcal{L}_+ h)(x) + (|x|^2 - 1)\nabla(P(\mathcal{M}_{\beta_\ell} h))(x)$$

for some $g, \widetilde{g}, h \in W^{1,p}(\mathbb{S})$.

7 Appendix

7.1 Estimates for $P_e(x, \eta)$

Proof of Proposition 5.

a): Let K be a compact in $\mathbb{B}$. There exists $\gamma > 0$, depending only on K, such that

$$\gamma \le |tx - \eta| \le 2, \quad t \in [0,1],\ \eta \in \mathbb{S},\ x \in K. \tag{43}$$

From (12) and (14) we have

$$4\pi\Phi(x,\eta) = \int_0^1 (4\pi P(tx,\eta) - 1 - 3tx \cdot \eta)\, \frac{dt}{t^{1+\alpha}},$$

then

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,\eta) = \int_0^1 \partial_i \partial_j P(tx,\eta) t^{1-\alpha} dt. \tag{44}$$

Now,

$$\begin{aligned} 4\pi \partial_i P_i(x,\eta) &= \frac{-2x_i}{|x-\eta|^3} - 3\frac{(1-|x|^2)(x_i - \eta_i)}{|x-\eta|^5}, \\ 4\pi \partial_j \partial_i P(x,\eta) &= \frac{-2\delta_{ij}|x-\eta|^3 + 6x_i|x-\eta|(x_j-\eta_j)}{|x-\eta|^6} \\ &\quad + 3\frac{|x-\eta|^5\left((1-|x|^2)\delta_{ij} + 2x_j(x_i-\eta_i)\right)}{|x-\eta|^{10}} \\ &\quad + 15\frac{(1-|x|^2)(x_i-\eta_i)(x_j-\eta_j)|x-\eta|^3}{|x-\eta|^{10}}. \end{aligned} \tag{45}$$

By using (11)-(13), (43) and (45) it is easy to see that $\partial_x^\alpha \mathcal{P}_e(x,\eta)$ is uniformly bounded in $K \times \mathbb{S}$ for any compact $K \subset \mathbb{B}$.

b): By (45) and the fact that

$$|tx - \eta| \ge 1 - t|x| \ge 1 - |x|, \quad t \in [0,1],\ \eta \in \mathbb{S},\ x \in \mathbb{B}, \tag{46}$$

we have

$$\left|\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,\eta)\right| \lesssim \int_0^1 \frac{t^{1-\alpha}}{|tx-\eta|^4} dt. \tag{47}$$

Then by (46) and (47), the kernel $\mathcal{L}$ of L satisfies

$$|\mathcal{L}_{i,j}(x,\eta)| \lesssim (1-|x|^2)\int_0^1 \frac{t^{1-\alpha}P(tx,\eta)}{|tx-\eta|^2}dt. \tag{48}$$

Then *b)* follows immediately.

c): Let $\xi \in \mathbb{S}$ and $x \in \Gamma_\xi$. Then

$$|\boldsymbol{P}_e\boldsymbol{f}(x)| \le |P\boldsymbol{f}(x)| + |\boldsymbol{L}\boldsymbol{f}(x)|. \tag{49}$$

By (18),

$$|P\boldsymbol{f}(x)| \le P|\boldsymbol{f}|(x) \le \mathcal{N}P|\boldsymbol{f}|(\xi) \lesssim M[|\boldsymbol{f}|](\xi). \tag{50}$$

Since $tx \in \Gamma_\eta$ if $x \in \Gamma_\xi$ and $t \in [0,1]$, then by *b)*, (50) and (3) we have

$$\begin{aligned}|\boldsymbol{L}\boldsymbol{f}(x)| &\lesssim \mathcal{N}P|\boldsymbol{f}|(\xi)(1-|x|^2)\int_0^1 \frac{t^{1-\alpha}}{(1-t^2|x|^2)^2}dt \\ &\lesssim M[|\boldsymbol{f}|](\xi). \end{aligned}\tag{51}$$

Then the proof of *(c)* follows from (49), (50) and (51). □

7.2 Projections

Proof of Theorem 13. Let $1 < p < \infty$. The projections $\boldsymbol{\pi}_+$, $\boldsymbol{\pi}_-$ and $\boldsymbol{\pi}_0$ on $\boldsymbol{C}^\infty(\mathbb{S})$ are defined by

$$\begin{aligned}\boldsymbol{\pi}_+(\boldsymbol{f})(\xi) &= \sum_{\ell\ge 0}\sum_{|m|\le\ell}\langle \boldsymbol{f}, \frac{\boldsymbol{E}^+_{\ell,m}}{\mu^+_\ell}\rangle \frac{\boldsymbol{E}^+_{\ell,m}(\xi)}{\mu^+_\ell}, \\ \boldsymbol{\pi}_-(\boldsymbol{f})(\xi) &= \sum_{\ell\ge 1}\sum_{|m|\le\ell}\langle \boldsymbol{f}, \frac{\boldsymbol{E}^-_{\ell,m}}{\mu^-_\ell}\rangle \frac{\boldsymbol{E}^-_{\ell,m}(\xi)}{\mu^-_\ell}, \\ \boldsymbol{\pi}_0(\boldsymbol{f})(\xi) &= \sum_{\ell\ge 1}\sum_{|m|\le\ell}\langle \boldsymbol{f}, \frac{\boldsymbol{E}^0_{\ell,m}}{\mu^0_\ell}\rangle \frac{\boldsymbol{E}^0_{\ell,m}(\xi)}{\mu^0_\ell}. \end{aligned}$$

Boundedness of $\boldsymbol{\pi}_+$:

We have

$$\boldsymbol{\pi}_+(\boldsymbol{f})(\xi) = \sum_{i=1}^4 \boldsymbol{T}_i(\mathbf{f})(\xi), \tag{52}$$

where

$$\boldsymbol{T}_1(\boldsymbol{f})(\xi) = \sum_{\ell\geq 0}\sum_{|m|\leq \ell} \frac{\ell+1}{2\ell+1}\langle \boldsymbol{f}, \boldsymbol{Y}^{\vee}_{\ell,m}\rangle \boldsymbol{Y}^{\vee}_{\ell,m}(\xi),$$

$$\boldsymbol{T}_2(\boldsymbol{f})(\xi) = -\sum_{\ell\geq 0}\sum_{|m|\leq \ell} \frac{1}{2\ell+1}\langle \boldsymbol{f}, \nabla_\sigma Y_{\ell,m}\rangle \boldsymbol{Y}^{\vee}_{\ell,m}(\xi),$$

$$\boldsymbol{T}_3(\boldsymbol{f})(\xi) = -\sum_{\ell\geq 0}\sum_{|m|\leq \ell} \frac{1}{2\ell+1}\langle \boldsymbol{f}, \boldsymbol{Y}^{\vee}_{\ell,m}\rangle \nabla_\sigma Y_{\ell,m}(\xi),$$

$$\boldsymbol{T}_4(\boldsymbol{f})(\xi) = \sum_{\ell\geq 0}\sum_{|m|\leq \ell} \frac{1}{(\ell+1)(2\ell+1)}\langle \boldsymbol{f}, \nabla_\sigma Y_{\ell,m}\rangle \nabla_\sigma Y_{\ell,m}(\xi).$$

Notice that for $\boldsymbol{f}\in \boldsymbol{C}^\infty(\mathbb{S})$, we can write

$$\boldsymbol{T}_1(\boldsymbol{f}) = \mathcal{M}_{\frac{\ell+1}{2\ell+1}}(\boldsymbol{f}\cdot\boldsymbol{\nu})^{\vee}.$$

Since by Theorem 1 the operator $\mathcal{M}_{\frac{\ell+1}{2\ell+1}}$ is continuous in $L^p(\mathbb{S})$, it follows immediately that $\boldsymbol{T}_1$ has a bounded extension in $\boldsymbol{L}^p-$norm.

Next, for $\boldsymbol{f}\in \boldsymbol{C}^\infty(\mathbb{S})$, we can write

$$-\boldsymbol{T}_3(\boldsymbol{f})(\xi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\widehat{(\boldsymbol{f}\cdot\boldsymbol{\nu})}_{\ell,m}\nabla_\sigma Y_{\ell,m}(\xi),$$

Since $\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\widehat{(\boldsymbol{f}\cdot\boldsymbol{\nu})}_{l,m}Y_{\ell,m}(\xi)$ converges in the Sobolev space $W^{1,p}(\mathbb{S})$, we have that

$$-\boldsymbol{T}_3(\boldsymbol{f})(\xi) = \nabla_\sigma\mathcal{M}_{\frac{1}{2\ell+1}}(\boldsymbol{f}\cdot\boldsymbol{\nu}).$$

Then, using Theorems 1 and 2,

$$\|\boldsymbol{T}_3(\boldsymbol{f})\|_{\boldsymbol{L}^p} \lesssim \|(-\Delta)^{1/2}(\boldsymbol{f}\cdot\boldsymbol{\nu})\|_{L^p} = \|\mathcal{M}_{\alpha_\ell}(\boldsymbol{f}\cdot\boldsymbol{\nu})\|_{L^p} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^p},$$

where

$$\alpha_\ell = \frac{\ell^{1/2}\,(\ell+1)^{1/2}}{2\ell+1}.$$

It also follows that $\boldsymbol{T}_2$ is $\boldsymbol{L}^p-$bounded since its adjoint is $\boldsymbol{T}_3$, which is $\boldsymbol{L}^q-$bounded for any $1<q<\infty$.

Finally, to study $\boldsymbol{T}_4$, we observe that for $\boldsymbol{f}\in \boldsymbol{C}^\infty(\mathbb{S})$,

$$\boldsymbol{T}_4(\boldsymbol{f}) = \nabla_\sigma\left(\sum_{\ell\geq 1}\sum_{|m|\leq \ell}\frac{1}{(\ell+1)(2\ell+1)}\langle \boldsymbol{f}, \nabla_\sigma Y_{\ell,m}\rangle Y_{\ell,m}\right).$$

By Theorem 2,

$$
\begin{aligned}
&\int_{\mathbb{S}} |\boldsymbol{T}_4(\boldsymbol{f})(\xi)|^p \, d\sigma(\xi) \\
&\lesssim \int_{\mathbb{S}} \left| (-\Delta)^{1/2} \left(\sum_{\ell \geq 1} \sum_{|m| \leq \ell} \frac{1}{(\ell+1)(2\ell+1)} \langle \boldsymbol{f}, \nabla_\sigma Y_{\ell k} \rangle Y_{\ell k}(\cdot) \right)(\xi) \right|^p d\sigma(\xi) \\
&= \int_{\mathbb{S}} \left| \sum_{\ell \geq 1} \sum_{|m| \leq \ell} \frac{\ell^{1/2}\,(\ell+1)^{1/2}}{(\ell+1)(2\ell+1)} \langle \boldsymbol{f}, \nabla_\sigma Y_{\ell,m} \rangle Y_{\ell,m}(\xi) \right|^p d\sigma(\xi) \\
&= \int_{\mathbb{S}} \left| \sum_{\ell \geq 1} \sum_{|m| \leq \ell} \frac{\ell^{1/2}\,(\ell+1)^{1/2}}{(\ell+1)(2\ell+1)} \langle \boldsymbol{f}, \nabla_\sigma Y_{\ell,m} \rangle \boldsymbol{Y}^{\vee}_{\ell,m}(\xi) \right|^p d\sigma(\xi). \qquad (53)
\end{aligned}
$$

Let $\widetilde{\boldsymbol{T}}_2$ be the operator defined by

$$
\widetilde{\boldsymbol{T}}_2(\xi) = \sum_{\ell \geq 1} \sum_{|m| \leq \ell} \frac{\ell^{1/2}\,(\ell+1)^{1/2}}{(\ell+1)(2\ell+1)} \langle \mathbf{f}, \nabla_\sigma Y_{\ell,m} \rangle \boldsymbol{Y}^{\vee}_{\ell,m}(\xi),
$$

and its adjoint

$$
\widetilde{\boldsymbol{T}}_3(\xi) = \sum_{\ell \geq 1} \sum_{|m| \leq \ell} \frac{\ell^{1/2}\,(\ell+1)^{1/2}}{(\ell+1)(2\ell+1)} \langle \mathbf{f}, \boldsymbol{Y}^{\vee}_{\ell,m} \rangle \nabla_\sigma Y_{\ell,m}(\xi).
$$

As done for $\boldsymbol{T}_3$, it can proved that $\widetilde{\boldsymbol{T}}_3$ is $\boldsymbol{L}^p-$bounded for any $1 < p < \infty$, and hence the same holds for $\widetilde{\boldsymbol{T}}_2$. Thus from (53),

$$
\|\boldsymbol{T}_4(\boldsymbol{f})\|_{\boldsymbol{L}^p} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^p}.
$$

This completes the proof that $\boldsymbol{\pi}_+$ is bounded in $\boldsymbol{L}^p(\mathbb{S})$. The proof of the L^p-continuity of $\boldsymbol{\pi}_-$ is completely analogous, and since $\boldsymbol{\pi}_0 = Id - \boldsymbol{\pi}_+ - \boldsymbol{\pi}_-$ on $\boldsymbol{C}^\infty(\mathbb{S})$, the proof of the theorem is complete.

□

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References

[1] Axler, S., Bourdon, P., Ramey, W., *Harmonic Functions Theory.* Springer, Graduate texts in mathematics. Second edition, 2001.

[2] Bakry, D., Étude des tranformations de Riesz dans les variétes riemannienes à courbure de Ricci minorãe, *Séminaire de Probabilités*, Lecture Notes in Math. 1247, 1987.

[3] Colzani, L., *Hardy and Lipschitz spaces on unit spheres*, Thesis (Ph.D.)–Washington University in St. Louis, 1982.

[4] Colzani, L., Hardy spaces on unit spheres, Boll. Un. Mat. Ital. C (6) **4**(1985), no. 1, 219–244.

[5] Diestel, J., Uhl, J. J., *Vector measures*, Math. Surveys 15, American Mathematical Society, Providence, RI, 1977.

[6] Duoandikoetxea, J., *Fourier analysis*, Grad. Stud. Math., 29, American Mathematical Society, Providence, RI, 2001.

[7] Freeden, W., Schreiner, M., *Spherical functions of mathematical geosciences: a scalar, vectorial, and tensorial setup*, Second Edition, Birkhauser, 2022.

[8] Korányi, A., Vági, S., Cauchy-Szegö integrals for systems of harmonic functions, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **26**(3)(1972), 181–196.

[9] Kupradze, V.D., *Three dimensional problems of mathematical theory of elasticity and thermoelasticity*, North-Holland, Amsterdam, 1979.

[10] Mitrea, D. *Distributions, partial differential equations, and harmonic analysis.* Universitext. Springer, New York, 2013.

[11] Martell, J.M, Mitrea D., Mitrea, I and Mitrea, M., A Fatou theorem and Poisson's integral representation formula for elliptic systems in the upper half space, *Topics in Clifford Analysis: Special Volume in Honor of Wolfgang Sprößig*, Swanhild Bernstein editor, Trends in Mathematics (TM), Birkhäuser Cham, Ch. 5, p. 104-124, 2019.

[12] Neri, U., *Singular integrals*, Springer-Verlag, Lectures notes 200, 1971.

[13] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970.

[14] Strichartz, R. S., Multipliers for spherical harmonic expansions, *Trans. Amer. Math. Soc.* **167**(1972), 115–124.

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