

LOCALLY COMPACT STRICTLY CONVEX METRIC GROUPS ARE ABELIAN

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ABSTRACT. We show that every locally compact strictly convex metric group is abelian, thus answering one problem posed by the authors in their earlier paper. To prove this theorem we first construct the isomorphic embeddings of the real line into the strictly convex metric group using its geodesic properties and characterization of the real line as a unique not monothetic one-parametric metrizable topological group. We proceed to show that all compact subgroups in a strictly convex metric group are trivial, which combined with the classical result of Iwasawa completes the proof of the main result.

1. INTRODUCTION

The strict convexity is a fundamental geometric property of Banach spaces, ensuring the uniqueness of best approximations, playing a central role in duality theory, and underpinning applications in optimization, approximation theory, and fixed point theory (cf. [5], [6], [7], [9]). In spite of the fact that strict convexity is usually defined for normed or Banach spaces, it is a purely metric property and can be defined without involving the linear or convex structure.

Definition 1. A metric space (X, d) is defined to be *strictly convex* if for any points $x, y \in X$ and any positive real numbers a, b with $a + b = d(x, y)$, the intersection $B[x, a] \cap B[y, b]$ is a singleton.

Here we denote by $B[x, a] := \{z \in X : d(x, z) \leq a\}$ the closed ball of radius a centered at a point x of the metric space (X, d) .

In the paper [1] we proved that every strictly convex metric abelian group G admits a unique multiplication $\cdot : \mathbb{R} \times G \rightarrow G$ turning G into a normed space over the field of real numbers, and asked whether every strictly convex metric group is abelian. In this paper we give a partial answer to this problem proving that a strictly convex metric group is abelian whenever it is locally compact or finite-dimensional.

2. THE MAIN RESULT

Our study mainly considers the strict convexity in metric groups. Let us recall the definition of this mathematical structure.

Definition 2. A *group* is an algebraic structure $(G, +, 0)$, consisting of a set G , a binary operation $+: G \times G \rightarrow G$ and an identity element 0 , satisfying the following axioms:

- (1) $\forall x, y, z \in G \quad (x + y) + z = x + (y + z),$ (associativity)
- (2) $\forall x \in G \quad x + 0 = x = 0 + x,$ (identity)
- (3) $\forall x \in G \exists y \in G \quad x + y = 0 = y + x.$ (inverse)

If, in addition, $x + y = y + x$ for all $x, y \in G$, then $(G, +, 0)$ is called an *abelian group*.

Definition 3. A *metric group* is a group $(G, +, 0)$ equipped with a metric $d : G \times G \rightarrow \mathbb{R}$, which is translation invariant in the sense that $d(x + c, y + c) = d(x, y) = d(c + x, c + y)$ for all points $x, y, c \in G$. The metric d can be uniquely recovered from the *norm*

$$\|\cdot\| : G \rightarrow \mathbb{R}, \quad \|\cdot\| : x \mapsto \|x\| := d(0, x),$$

generated by this metric.

2020 *Mathematics Subject Classification.* 20K45, 46B20, 52A07.

Key words and phrases. Strictly convex metric space, normed space, abelian metric group.

Definition 4. A topological space X is called *locally compact* if for every $x \in X$ there exists an open neighborhood U of x such that the closure $\bar{U}$ is compact.

The main result of this paper is the following theorem.

Theorem 5. *Every locally compact strictly convex metric group is abelian.*

Combined with the main result in [1], this implies

Corollary 6. *Every locally compact strictly convex metric group is a finite-dimensional normed space over the field of real numbers.*

Theorem 5 will be proved in Section 7 after some preliminary work made in Sections 3–6.

3. MULTIPLICATION IN STRICTLY CONVEX METRIC GROUPS

In this section, we shall introduce the *metric multiplication* and the *algebraic multiplication* in a strictly convex metric group $(G, +, 0, d)$, investigate their properties and interplay.

Definition 7. Let $(X, d), (Y, \rho)$ be metric spaces. A map $f : X \rightarrow Y$ is an *isometry* if $\rho(f(x), f(y)) = d(x, y)$ for all points $x, y \in X$.

Definition 8. A metric space (X, d) is *geodesic* if for all $x, y \in X$ there exists a unique isometry $\gamma : [0, d(x, y)] \subseteq \mathbb{R} \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(d(x, y)) = y$.

It is known [1] that strictly convex metric spaces are geodesic. For every $x \in G$ let $\gamma : [0, \|x\|] \rightarrow G$ be the unique isometry with $\gamma(0) = 0$ and $\gamma(\|x\|) = x$. For every $t \in [0, 1] \subseteq \mathbb{R}$ put $t * x := \gamma(t\|x\|)$. The defined binary operation $* : [0, 1] \times G \rightarrow G$, $* : (t, x) \mapsto t * x$ will be called the *metric multiplication*. Witness some properties of this operation introduced in the following Lemmas.

Lemma 9. *For all $t, v \in [0, 1]$ and $x \in G$ we have $\|t * x - v * x\| = d(t * x, v * x) = |t - v| \cdot \|x\|$.*

Proof. Since $\gamma : [0, \|x\|] \rightarrow G$ is an isometry, we obtain

$$\|t * x - v * x\| = d(t * x, v * x) = d(\gamma(t\|x\|), \gamma(v\|x\|)) = |t\|x\| - v\|x\|| = |t - v| \cdot \|x\|.$$

□

Corollary 10. *For all $t \in [0, 1]$ and $x \in G$ we have $\|t * x\| = t\|x\|$.*

Lemma 11. *For all $t, v \in [0, 1]$ and $x \in G$ we have $t * (v * x) = (tv) * x$.*

Proof. For the element $t * (v * x)$ we have $\|t * (v * x)\| = t\|v * x\| = tv\|x\|$ and

$$\|t * (v * x) - v * x\| = (1 - t)\|v * x\| = (1 - t)v\|x\|,$$

by Corollary 10 and Lemma 9. Similarly, for the element $(tv) * x$ we have $\|(tv) * x\| = tv\|x\|$ and

$$\|(tv) * x - v * x\| = (v - tv)\|x\| = (1 - t)v\|x\|.$$

Since $tv\|x\| + (1 - t)v\|x\| = v\|x\| = \|v * x\| = d(0, v * x)$, the element $t * (v * x)$ coincides with $(tv) * x$ by the strict convexity of the metric space (G, d) . □

Lemma 12. *For all $x \in G$ we have $\frac{1}{2} * x + \frac{1}{2} * x = x$.*

Proof. Let $c := \|\frac{1}{2} * x\| = \frac{1}{2}\|x\| = \|x - \frac{1}{2} * x\|$ by Lemma 9 and Corollary 10. Define $z := x - \frac{1}{2} * x \in G$. Then $\|z\| = c$ and

$$\|x - z\| = d(x, z) = d(x, x - \frac{1}{2} * x) = d(0, -\frac{1}{2} * x) = \|\frac{1}{2} * x\| = c,$$

by the translation invariance of the metric. Hence, by the strict convexity of the metric space (G, d) , it follows that $\frac{1}{2} * x = z = x - \frac{1}{2} * x$, which implies $\frac{1}{2} * x + \frac{1}{2} * x = x$. □

Let us define the operation $\cdot : \mathbb{Z} \times G \rightarrow G$, $\cdot : (n, x) \mapsto n \cdot x$ by the recursive formulas: $0 \cdot x = 0$, $(n+1) \cdot x = n \cdot x + x$, and $-(n+1) \cdot x = -n \cdot x - x$ for all $n \in \mathbb{N} \cup \{0\}$. We will call this operation the *algebraic multiplication*.

Let $H = \{\frac{1}{2^n} : n \in \mathbb{N} \cup \{0\}\}$. We shall prove that the set $H * x = \{\frac{1}{2^n} * x : n \in \mathbb{N} \cup \{0\}\}$ is commutative for all $x \in G$. This will be done using the following fact for the defined multiplication operations.

Lemma 13. *For all $n, m \in \mathbb{N}, n \leq m$ and $x \in G$ we have $2^n \cdot (\frac{1}{2^m} * x) = \frac{1}{2^{m-n}} * x$.*

Proof. Fix $m \in \mathbb{N}$. We prove the claim by induction on $n \leq m$. If $n = 1$, then by Lemma 11 we have $\frac{1}{2} * (\frac{1}{2^{m-1}} * x) = \frac{1}{2^m} * x$. This implies $2 \cdot (\frac{1}{2^m} * x) = \frac{1}{2^{m-1}} * x$ by Lemma 12. Assume the statement holds for some $k \in \mathbb{N}$ with $k < m$, that is $2^k \cdot (\frac{1}{2^m} * x) = \frac{1}{2^{m-k}} * x$. Then, using the base case and the induction hypothesis, we obtain $2^{k+1} \cdot (\frac{1}{2^m} * x) = 2 \cdot (\frac{1}{2^{m-k}} * x) = \frac{1}{2^{m-k-1}} * x$, which completes the induction. $\square$

Proposition 14. *For all $x \in G$ the subset $H * x = \{\frac{1}{2^n} * x : n \in \mathbb{N} \cup \{0\}\}$ is commutative.*

Proof. Fix $x \in G$ and take $n, m \in \mathbb{N} \cup \{0\}$. We lose no generality by assuming that $n \leq m$. Then by Lemma 13, definition of the algebraic multiplication and the associativity of $+$ on G , we obtain

$$\frac{1}{2^n} * x + \frac{1}{2^m} * x = 2^{m-n} \cdot \left(\frac{1}{2^m} * x \right) + \frac{1}{2^m} * x = \frac{1}{2^m} * x + 2^{m-n} \cdot \left(\frac{1}{2^m} * x \right) = \frac{1}{2^m} * x + \frac{1}{2^n} * x.$$

$\square$

Corollary 15. *For all $x \in G$ the subgroup $H_x = \langle H * x \rangle$ is abelian.*

4. EVERY SUBGROUP H_x IS A $\mathbb{Z}[\frac{1}{2}]$ -MODULE

It is well-known that for all $x \in G$ the abelian group H_x is a $\mathbb{Z}$ -module with the algebraic multiplication. In this section, we shall endow the group H_x with a structure of a module over the ring $\mathbb{Z}[\frac{1}{2}] = \{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ of dyadic fractions.

Definition 16. An additive group $(G, +, 0)$ is (*uniquely*) *2-divisible* if for every $x \in G$ there exists a (unique) element $y \in G$ such that $y + y = x$.

Definition 17. An element x of a group $(G, +, 0)$ is defined to have *order 2* if $x + x = 0 \neq x$.

Proposition 18. *A strictly convex metric group $(G, +, 0, d)$ has no elements of order 2.*

Proof. See Lemma 2 in [1]. $\square$

Lemma 19. *For all $x \in G$ the subgroup H_x is 2-divisible.*

Proof. Take arbitrary $x \in G$ and $y \in H_x$. Since $H_x = \langle H * x \rangle$, the element y can be represented as $y = \sum_{k=0}^m a_k \cdot (\frac{1}{2^{n_k}} * x)$, where $a_k \in \mathbb{Z}, n_k \in \mathbb{N} \cup \{0\}$ for all $k \in \{0, \dots, m\}, m \in \mathbb{N}$. Consider $z := \sum_{k=0}^m a_k \cdot (\frac{1}{2^{n_k+1}} * x)$ and let us show that it satisfies the required equality using a $\mathbb{Z}$ -module properties of H_x and Lemma 13.

$$z + z = \sum_{k=0}^m a_k \cdot \left(\frac{1}{2^{n_k+1}} * x \right) + \sum_{k=0}^m a_k \cdot \left(\frac{1}{2^{n_k+1}} * x \right) = \sum_{k=0}^m a_k \cdot \left(2 \cdot \frac{1}{2^{n_k+1}} * x \right) = \sum_{k=0}^m a_k \cdot \left(\frac{1}{2^{n_k}} * x \right) = y.$$

$\square$

Proposition 20. *For all $x \in G$ the subgroup H_x is uniquely 2-divisible.*

Proof. Fix $x \in G$. The subgroup H_x is 2-divisible by Lemma 19. Since G has no elements of order 2 by Proposition 18, neither does H_x . If $z, z' \in H_x$ satisfy $2z = 2z'$, then $2(z - z') = 0$ by commutativity. The absence of elements of order 2 forces $z = z'$. Hence, H_x is uniquely 2-divisible. $\square$

Corollary 21. *For all $x \in G$ the subgroup H_x is a $\mathbb{Z}[\frac{1}{2}]$ -module.*

Proof. It follows from Proposition 20 using Proposition 2 in [1]. $\square$

As a result, for all $x \in G$ we obtained the operation $\circ_x : \mathbb{Z}[\frac{1}{2}] \times H_x \rightarrow H_x$, $\circ_x : (t, x) \mapsto t \circ_x x$ such that $(H_x, +, 0, \circ_x)$ is a $\mathbb{Z}[\frac{1}{2}]$ -module. Since H_x is a $\mathbb{Z}$ -module with the algebraic multiplication, we already know that $n \cdot y = n \circ_x y$ for all $y \in H_x$ and $n \in \mathbb{Z}$. We shall now introduce the connection between $\circ_x$ and $*$. This will be done using the following Lemma.

Lemma 22. *For all $n, m \in \mathbb{N}, n \leq 2^m$ and $x \in G$ we have $n \cdot (\frac{1}{2^m} * x) = \frac{n}{2^m} * x$.*

Proof. For element $\frac{n}{2^m} * x$ we have $\|\frac{n}{2^m} * x\| = \frac{n}{2^m} \|x\|$ and $\|\frac{n}{2^m} * x - x\| = (1 - \frac{n}{2^m}) \|x\|$, by Lemma 9 and Corollary 10. For element $n \cdot (\frac{1}{2^m} * x)$ we obtain inequality

$$\|n \cdot (\frac{1}{2^m} * x)\| \leq n \|\frac{1}{2^m} * x\| = \frac{n}{2^m} \|x\|,$$

by Corollary 10 and the triangle inequality. Similarly,

$$\begin{aligned} \|x - m \cdot (\frac{1}{2^m} * x)\| &= \|2^m \cdot (\frac{1}{2^m} * x) - m \cdot (\frac{1}{2^m} * x)\| = \|(2^m - m) \cdot (\frac{1}{2^m} * x)\| \leq \\ &\leq (2^m - m) \|\frac{1}{2^m} * x\| = (1 - \frac{m}{2^m}) \|x\|, \end{aligned}$$

by Lemma 13, Corollary 10, the triangle inequality and $\mathbb{Z}$ -module properties of H_x . Assume that at least one of the above inequalities is strict. Then the triangle inequality ensures that

$$\|x\| \leq \|m \cdot (\frac{1}{2^m} * x)\| + \|x - m \cdot (\frac{1}{2^m} * x)\| < \frac{m}{2^m} \|x\| + (1 - \frac{m}{2^m}) \|x\| = \|x\|,$$

which is a contradiction. Hence, our assumption is wrong and both inequalities for element $n \cdot (\frac{1}{2^m} * x)$ hold with equality. Then the element $\frac{n}{2^m} * x$ coincides with $n \cdot (\frac{1}{2^m} * x)$ by the strict convexity of the metric space (G, d) . $\square$

Lemma 23. *For all $x \in G$ and $t \in \mathbb{Z}[\frac{1}{2}] \cap [0, 1]$ we have $t \circ_x x = t * x$.*

Proof. Let $t = \frac{m}{2^n} \in \mathbb{Z}[\frac{1}{2}]$, $m \leq 2^n$. Lemma 13 ensures that $2^n \circ_x (\frac{1}{2^n} * x) = x$, which implies $\frac{1}{2^n} * x = \frac{1}{2^n} \circ_x x$. Then we obtain

$$\frac{m}{2^n} \circ_x x = m \circ_x (\frac{1}{2^n} \circ_x x) = m \circ_x (\frac{1}{2^n} * x) = \frac{m}{2^n} * x,$$

by the above equality, $\mathbb{Z}[\frac{1}{2}]$ -module properties of $\circ_x$ on H_x and Lemma 22. $\square$

We extend both the algebraic multiplication and the metric multiplication on G to $\cdot : \mathbb{R} \times G \rightarrow G$, $\cdot : (t, x) \mapsto tx$ by setting $tx = [t] \cdot x + \{t\} * x$ for all $t \in \mathbb{R}$ and $x \in G$. We will call this extension the *real multiplication*. Witness that the results of this section imply the following proposition.

Proposition 24. *For all $x \in G$ and $t \in \mathbb{Z}[\frac{1}{2}]$ we have $tx = t \circ_x x$.*

Proof. Since $[t] \in \mathbb{Z}$ and $\{t\} \in \mathbb{Z}[\frac{1}{2}] \cap [0, 1]$, we have

$$tx = [t] \cdot x + \{t\} * x = [t] \circ_x x + \{t\} \circ_x x = ([t] + \{t\}) \circ_x x = t \circ_x x,$$

by definition of the real multiplication, Lemma 23 and $\mathbb{Z}[\frac{1}{2}]$ -module properties of $\circ_x$ on H_x . $\square$

5. THE REAL MULTIPLICATION IS A TOPOLOGICAL GROUP HOMOMORPHISM

In this section, we start by proving some important properties of the real multiplication, introduced in the following Lemmas.

Lemma 25. *For all $t, v \in \mathbb{R}$ and $x \in G$ we have $\|tx - vx\| \leq |t - v| \cdot \|x\|$.*

Proof. We lose no generality by assuming $t < v$. If $[t] = [v]$, we immediately obtain

$$\|tx - vx\| = d([t]x + \{t\} * x, [v]x + \{v\} * x) = d(\{t\} * x, \{v\} * x) = \|\{t\} - \{v\}\| \cdot \|x\| = |t - v| \cdot \|x\|,$$

by the definition of the algebraic multiplication, translation invariance of the metric and Lemma 9. If $[t] < [v]$, we similarly obtain

$$\begin{aligned} \|tx - vx\| &\leq \|tx - [t+1]x\| + \|[t+1]x - [v]x\| + \|[v]x - vx\| = \|x - \{t\} * x\| + \|[t+1] - [v]\| \|x\| + \|\{v\} * x\| \\ &\leq (1 - \{t\}) \|x\| + ([v] - [t+1]) \|x\| + \{v\} \|x\| = (v - t) \|x\| = |t - v| \cdot \|x\|, \end{aligned}$$

by the definition of the algebraic multiplication, triangle inequality, Lemma 9 and Corollary 10. $\square$

Lemma 25 implies

Lemma 26. *For all $x \in G$ the function $\cdot_x : \mathbb{R} \rightarrow G$, $\cdot_x : t \mapsto tx$, is continuous.*

Lemma 27. *For all $t, v \in \mathbb{R}$ and $x \in G$ we have $(t + v)x = tx + vx$.*

Proof. The continuity of the functions $\cdot_x : \mathbb{R} \rightarrow G$ and $+$: $G \times G \rightarrow G$ implies that the set $F = \{(t, v) \in \mathbb{R} \times \mathbb{R} : (t + v)x = tx + vx\}$ is closed in the real plane $\mathbb{R} \times \mathbb{R}$. Then by Proposition 24 and $\mathbb{Z}[\frac{1}{2}]$ -module properties of H_x the closed set F contains the dense subset $\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{2}]$ of $\mathbb{R} \times \mathbb{R}$ and hence $F = \mathbb{R} \times \mathbb{R}$, witnessing that $(t + v)x = tx + vx$ for all $t, v \in \mathbb{R}$. $\square$

Definition 28. A map $\phi : G \rightarrow S$ between two topological groups G and S is called a *topological group homomorphism* if it is a continuous group homomorphism from G to S . If in addition, $\phi^{-1} : S \rightarrow G$ is also a topological group homomorphism, then ϕ is called a *topological group isomorphism*.

Lemma 25 and Lemma 27 imply

Proposition 29. *For all $x \in G$ the map $\cdot_x : \mathbb{R} \rightarrow G$, $\cdot_x : t \mapsto tx$ is a topological group homomorphism.*

For all $x \in G$ by $\mathbb{R}x$ we denote the image of $\mathbb{R}$ under the map $\cdot_x : \mathbb{R} \rightarrow G$.

6. EVERY SUBGROUP $\mathbb{R}x$ IS ISOMORPHIC TO $\mathbb{R}$.

In this section, we shall show that the map $\cdot_x : \mathbb{R} \rightarrow G$ is actually a topological group isomorphism. To prove it we first show that $\mathbb{R}x$ is not monothetic. This will be done by contradiction, using the following Lemmas.

Definition 30. A topological group G is called *monothetic* if it contains a dense cyclic subgroup; that is, there exists $g \in G$ such that $\langle g \rangle = \overline{\{ng : n \in \mathbb{Z}\}} = G$. The set of all such $g \in G$ is denoted by $\text{Gen}(G) = \{g \in G : \langle g \rangle = G\}$.

Lemma 31. *If $\mathbb{R}x \subseteq G$ is monothetic, then there exists dense $A \subseteq \mathbb{R}$ such that $\text{Gen}(\mathbb{R}x) = \cdot_x(A)$.*

Proof. See [2]. $\square$

Lemma 32. *If $\mathbb{R}x \subseteq G$ is monothetic, then there exists $tx \in \text{Gen}(\mathbb{R}x)$ such that $\|tx\| < |t| \cdot \|x\|$.*

Proof. Assume to the contrary that for all $tx \in \text{Gen}(\mathbb{R}x)$ we have $\|tx\| \geq |t| \cdot \|x\|$. Then Lemma 25 ensures that $\|tx\| = |t| \cdot \|x\|$. The continuity of the functions $\cdot_x$ and $\|\cdot\| : G \rightarrow \mathbb{R}$ implies that the set $F = \{t \in \mathbb{R} : \|tx\| = |t| \cdot \|x\|\}$ is closed in the real line $\mathbb{R}$. Therefore, the closed set F contains the set $\cdot_x^{-1}(\text{Gen}(\mathbb{R}x))$, which is dense subset of $\mathbb{R}$ by Lemma 31. Hence, $F = \mathbb{R}$, witnessing that $\|tx\| = |t| \cdot \|x\|$ for all $t \in \mathbb{R}$. But then for $tx \in \text{Gen}(\mathbb{R}x)$ the element $\frac{t}{2}x \notin \langle tx \rangle$, which is a contradiction. Hence, there exists $tx \in \text{Gen}(\mathbb{R}x)$ with $\|tx\| < |t| \cdot \|x\|$. $\square$

Lemma 33. *If $\mathbb{R}x \subseteq G$ is monothetic, then there exist $tx \in \text{Gen}(\mathbb{R}x)$ and distinct $a \in \mathbb{R}x$, $b \in H_{tx}$ such that $2^n a = tx = 2^n b$ for some $n \in \mathbb{N}$.*

Proof. Let $tx \in \text{Gen}(\mathbb{R}x)$ be the topological generator with $\|tx\| < |t| \cdot \|x\|$, which exists by Lemma 32. Since there exists $n \in \mathbb{N}$ such that $2^{n-1} \leq t < 2^n$, we have $\frac{t}{2^n} \in [0, 1] \subseteq \mathbb{R}$. Then for $a := \frac{t}{2^n}x \in \mathbb{R}x$ we have

$$\|a\| = \|\frac{t}{2^n}x\| = \|\frac{t}{2^n} * x\| = \frac{t}{2^n}\|x\|,$$

by definition of the real multiplication and Corollary 10. Since $\cdot_x : \mathbb{R} \rightarrow G$ is a group homomorphism, we have $2^n a = 2^n \frac{t}{2^n}x = tx$, by the definition of the algebraic multiplication. Now consider element $b := \frac{1}{2^n}(tx) \in H_{tx}$. Witness that

$$\|b\| = \|\frac{1}{2^n}(tx)\| = \|\frac{1}{2^n} * (tx)\| = \frac{1}{2^n}\|tx\| < \frac{t}{2^n}\|x\|,$$

by definition of the real multiplication, Corollary 10 and the initial strict inequality on $\|tx\|$. Lemma 13 ensures that $2^n b = 2^n(\frac{1}{2^n}(tx)) = tx$. Since $\|b\| < \|a\|$, the elements $a \in \mathbb{R}x$ and $b \in H_{tx}$ are distinct with $2^n a = tx = 2^n b$. $\square$

Proposition 34. *For all $x \in G$ the subgroup $\mathbb{R}x \subseteq G$ is not monothetic.*

Proof. Assume to the contrary that $\mathbb{R}x$ is monothetic. Lemma 33 ensures that there exists $tx \in \text{Gen}(\mathbb{R}x)$ and distinct $a \in \mathbb{R}x, b \in H_{tx}$ with $2^n a = tx = 2^n b$ for some $n \in \mathbb{N}$. Witness that H_{tx} is a $\mathbb{Z}[\frac{1}{2}]$ -module with the algebraic multiplication by Proposition 20 and hence $\langle tx \rangle \subseteq H_{tx}$. This implies $\mathbb{R}x \subseteq \overline{H_{tx}}$ by density of $\langle tx \rangle$ in $\mathbb{R}x$. It is well-known that the closure $\overline{H_{tx}}$ of the abelian group H_{tx} is abelian. Since $a \in \mathbb{R}x \subseteq \overline{H_{tx}}$ and $b \in H_{tx} \subseteq \overline{H_{tx}}$, the equality $2^n a = 2^n b$ implies $2^n(a-b) = 0$ by commutativity of $+$ on $\overline{H_{tx}}$. Since G has no elements of order 2, it has no elements of order 2^n by induction and hence the equality above implies $a = b$, which is a contradiction. $\square$

Definition 35. A topological group G is called a *one-parameter group* if there exists a topological group homomorphism $\phi : \mathbb{R} \rightarrow G$ such that $G = \phi(\mathbb{R})$.

Corollary 36. *For all $x \in G$ the metric subgroup $\mathbb{R}x \subseteq G$ is isomorphic to $\mathbb{R}$.*

Proof. It is known [2] that the real line $\mathbb{R}$ is the unique (up to topological group isomorphism) one-parameter metrizable topological group, which is not monothetic. $\mathbb{R}x$ is a metrizable topological group with topology induced by the translation-invariant metric. Moreover, $\mathbb{R}x$ is one-parametric by Proposition 29 and not monothetic by Proposition 34. Hence, $\mathbb{R}x$ must be isomorphic to $\mathbb{R}$. $\square$

Corollary 37. *For all $x \in G$ the metric subgroup $\mathbb{R}x$ is closed in G .*

Proof. Corollary 36 ensures that the metric subgroup $\mathbb{R}x$ is locally compact. It is well-known (see e.g. [8]) that locally compact subgroups of topological groups are closed. $\square$

Corollary 38. *If $K \subseteq G$ is a compact subgroup of G , then $K = \{0\}$.*

Proof. Suppose to the contrary that there exists $x \in K$ such that $x \neq 0$. Since $\langle x \rangle = \{nx : n \in \mathbb{Z}\} = \mathbb{Z}x := \cdot_x(\mathbb{Z}) \subseteq \mathbb{R}x$ is an image of $\mathbb{Z}$ under the topological group isomorphism $\cdot_x : \mathbb{R} \rightarrow \mathbb{R}x$, it is a closed subgroup of $\mathbb{R}x$. Corollary 37 ensures that $\mathbb{R}x$ is closed in G , which implies that $\mathbb{Z}x$ is closed in G . Since $\mathbb{Z}x = \langle x \rangle \subseteq K$, the group $\mathbb{Z}x$ is a closed subgroup of a compact group K and hence $\mathbb{Z}x$ is itself compact. This is a contradiction since $\mathbb{Z}x$ is an image of non-compact $\mathbb{Z} \subseteq \mathbb{R}$ under the topological group isomorphism $\cdot_x : \mathbb{R} \rightarrow \mathbb{R}x$. $\square$

7. PROOF OF THEOREM 5

To prove the main result, we plan to use the result [10] of Iwasawa, which is the following theorem.

Theorem 39. *A connected locally compact topological group contains a compact invariant neighborhood of the identity if and only if the group is compact-by-abelian, i.e., contains a compact normal subgroup whose quotient group is abelian.*

Now we finally present the proof of Theorem 5. Let us recall that it states the following: *every locally compact strictly convex metric group $(G, +, 0, d)$ is abelian.*

Proof. The strictly convex metric space G is geodesic by [1] and hence connected. Since G is locally compact we can pick an open neighborhood U of identity $0 \in G$ such that $\overline{U}$ is compact in G . Let $B \subseteq U$ be a closed ball with center in $0 \in G$. Since $B \subseteq \overline{U}$ is closed, B is compact in G . Since the metric d is translation invariant, the ball B is a compact invariant neighborhood of the identity. Then G contains a compact normal subgroup K whose quotient group is abelian by Theorem 39. Corollary 38 ensures that $K = \{0\}$. Hence $G/K = G$ is abelian. $\square$

To conclude this section, we shall consider the following corollary concerning compactly finite-dimensional metric groups.

Definition 40. A metric group G is called *compactly finite-dimensional* if

$$\text{co-dim}(G) = \sup\{\dim(K) : K \text{ is compact subspace of } G\}$$

is finite.

Corollary 41. *Every compactly finite-dimensional strictly convex metric group is abelian.*

Proof. It is known [3] that every compactly finite-dimensional path-connected topological group is locally-compact and hence satisfies Theorem 5. $\square$

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