Inequalities, identities, and bounds for divided differences of the exponential function

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Abstract

Let $\exp[x_0, x_1, \ldots, x_n]$ denote the divided difference of the exponential function.

(i) We prove that exponential divided differences are log-submodular.

(ii) We establish the four-point inequality

$$\exp[a, a, b, c]\ \exp[d, d, b, c] + \exp[b, b, a, d]\ \exp[c, c, a, d] - \exp[a, b, c, d]^2 \geq 0,$$

for all $a, b, c, d \in \mathbb{R}$.

(iii) We obtain sharp two-sided bounds for $\exp[x_0, \ldots, x_n]$ at fixed mean and variance; as a consequence, we derive their large-input asymptotics.

(iv) We present closed-form identities for divided differences of the exponential function, including a convolution identity and summation formulas for repeated arguments.

1. Introduction

Divided differences of the exponential function lie at the intersection of analysis, operator theory, probability, and computation. Analytically, the Hermite–Genocchi simplex formula represents $\exp[x_0, \ldots, x_n]$ as an average of e^x over the convex hull of the nodes, yielding positivity and a priori bounds for interpolation remainders and related extremal problems [1, 2]. In functional calculus for matrices and operators, divided differences serve as the kernels of double—and for higher variations, multiple—operator integrals, providing a concise framework for perturbation theory and operator inequalities [3, 4, 5]. Via the moment-generating function, these structures connect directly to probabilistic tail bounds [6, 7, 8].

For matrix functions $f(A)$, direct evaluators such as scaling–and–squaring with Padé and the Schur–Parlett scheme reduce to an upper (block) triangular form and fill $f(T)$ from the diagonal outward: this requires two–point divided differences, with repeated–argument limits when diagonal blocks contain repeated eigenvalues [9, 10]; for $f(x) = e^x$ this viewpoint includes block–triangular formulas of Van Loan type [11]. Sensitivity and conditioning at first order are governed by the Fréchet derivative, which in a diagonalizing basis depends entrywise on the same two–point differences [10]; multi–point divided differences enter higher

Fréchet derivatives used in second–order error estimates, Hessian-based optimization, and uncertainty quantification.

In time integration for stiff ODEs, exponential integrators such as exponential time differencing and exponential Runge–Kutta methods are built from the φ–functions, which are repeated–argument divided differences of e^z; assembling the stages typically needs only two–point differences [12, 13, 14, 15]. However, when using Newton/Leja interpolation or related multipoint polynomial/rational schemes to approximate the action of the matrix exponential or the φ-functions on a vector, both the coefficients and the error terms are naturally expressed in terms of higher–order divided differences [2, 16, 17, 18]. Higher–order divided differences also appear in multiple operator integrals and higher Fréchet derivatives used in advanced perturbation analyses [19, 20].

The problem of stable and efficient computation of exponential divided differences is as nontrivial as it is an important topic, because exponentials exacerbate conditioning issues in divided differences. Recent advances [21] allow incremental updates (add/remove) with controlled cost, well-suited to adaptive or evolving-node scenarios. Other works [1, 22, 23] focus on structuring the computations to reduce error amplification, or to permit stable use of these divided differences in exponential integrators with less restrictive time-step choices.

In computational physics, exponential divided differences play a central role in the permutation matrix representation quantum Monte Carlo (PMR–QMC) framework, where they serve as configuration weights. This framework is universal and effective for problems in many-body quantum physics and materials simulation, and have been studied and applied during last decade [24, 25, 26, 27, 28, 21, 29, 30, 31, 32, 33, 34, 35, 36, 37].

The present work identifies several properties of exponential divided differences that arise naturally in applications and, to our knowledge, have not previously been documented or proved in the literature.

The paper is organized as follows. Section 2 establishes log-submodularity and supermodularity of exponential divided differences. In Section 3 we a prove a four-point inequality and in Section 4 we derive sharp two-sided bounds at fixed mean and variance and their large-input asymptotics. Section 5 is dedicated to several closed-form identities, including a convolution identity and repeated-argument summations. We provide some concluding remarks in Section 6.

2. Log-submodularity of the divided differences of the exponential function

Fix nodes $a_1 \le \cdots \le a_n \in \mathbb{R}$ with $n \ge 0$, and integers $p, q \ge 1$. Define

$$K_n(x, y) := \exp[a_1, \ldots, a_n, x^{(p)}, y^{(q)}],$$

the divided difference of the exponential function at these $n + p + q$ nodes, with superscripts indicating multiplicities.

We prove that K_n is *log-submodular*. It follows from $K_n(x, y) > 0$ and Lemma 1 that log-submodularity is equivalent to being *totally negative of order 2* (TN_2):

$$K_n(x_1, y_1)\, K_n(x_2, y_2) \;\le\; K_n(x_1, y_2)\, K_n(x_2, y_1) \qquad \text{for all } x_1 \le x_2,\ y_1 \le y_2. \tag{1}$$

In particular, with the symmetric substitution $(x_1, y_1) = (x, x)$, $(x_2, y_2) = (y, y)$ (for $x \le y$), Eq. (1) gives

$$K_n(x, x)K_n(y, y) \le K_n(x, y)K_n(y, x).$$

Lemma 1. *Let $K : \mathbb{R}^2 \to (0,\infty)$. For $u = (u_1,u_2), v = (v_1,v_2) \in \mathbb{R}^2$ set $u \wedge v := \big(\min\{u_1,v_1\}, \min\{u_2,v_2\}\big)$, $u \vee v := \big(\max\{u_1,v_1\}, \max\{u_2,v_2\}\big)$. The following are equivalent:*

*(i) (*Log-submodularity*)*

$$\log K(u) + \log K(v) \ \geq\ \log K(u \wedge v) + \log K(u \vee v) \qquad \text{for all } u, v \in \mathbb{R}^2.$$

*(ii) (*Totally negative of order 2*)*

$$K(x_1,y_1)\,K(x_2,y_2) \ \leq\ K(x_1,y_2)\,K(x_2,y_1) \qquad \text{for all } x_1 \leq x_2,\ y_1 \leq y_2.$$

Proof. (ii) follows from (i). Fix $x_1 \leq x_2$ and $y_1 \leq y_2$, and set $u := (x_2,y_1)$, $v := (x_1,y_2)$. Then $u \wedge v = (x_1,y_1)$ and $u \vee v = (x_2,y_2)$, so (i) gives

$$\log K(x_2,y_1) + \log K(x_1,y_2) \ \geq\ \log K(x_1,y_1) + \log K(x_2,y_2),$$

which exponentiates to (ii).
Conversely, (i) follows from (ii). Fix arbitrary $u = (u_1,u_2)$ and $v = (v_1,v_2)$, and set $w := u \wedge v$ and $z := u \vee v$. There are two cases.

(a) If $u \leq v$ or $v \leq u$ (coordinatewise), then $\{w,z\} = \{u,v\}$. Hence $K(w)K(z) = K(u)K(v)$ and the desired inequality in (i) holds with equality after taking logs.

(b) Otherwise, one coordinate increases while the other decreases; without loss of generality assume $u_1 \leq v_1$ and $u_2 \geq v_2$. Then, we have $(w_1,z_2) = (u_1,u_2) = u$ and $(z_1,w_2) = (v_1,v_2) = v$. Applying (ii) with $x_1 = w_1 \leq z_1 = x_2$ and $y_1 = w_2 \leq z_2 = y_2$ yields

$$K(w_1,w_2)\,K(z_1,z_2) \ \leq\ K(w_1,z_2)\,K(z_1,w_2) = K(u)\,K(v).$$

Taking logs gives $\log K(w) + \log K(z) \leq \log K(u) + \log K(v)$, which is exactly the inequality in (i). $\square$

Lemma 2. *Let $h : [0,1] \to [0,\infty)$ be integrable and define*

$$K(x,y) \ =\ \int_0^1 h(c)\,K_0(cx,cy)\,dc, \qquad K_0(x,y) \ := \exp[x^{(p)}, y^{(q)}]. \tag{2}$$

Then K is log-submodular (TN$_2$): for all $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$K(x_1,y_1)\,K(x_2,y_2) \ \leq\ K(x_1,y_2)\,K(x_2,y_1). \tag{3}$$

Proof. We have

$$K_0(x,y) = \frac{1}{(p-1)!(q-1)!} \int_0^1 \tau^{p-1}(1-\tau)^{q-1} e^{\tau x + (1-\tau)y} d\tau. \tag{4}$$

Let $S = [0,1] \times [0,1]$ and equip it with the product measure $d\mu(c,\tau) = h(c)\ \tau^{p-1}\ (1-\tau)^{q-1}\ dc\, d\tau$. Endow S with a total order $\preceq$ as follows: for $u_1 = (c_1,\tau_1)$, $u_2 = (c_2,\tau_2) \in S$ set $u_1 \preceq u_2$ iff one of the following conditions holds:

(i) $c_1\tau_1 > c_2\tau_2$; (ii) $c_1\tau_1 = c_2\tau_2$ and $c_1 > c_2$; (iii) $c_1\tau_1 = c_2\tau_2,\ c_1 = c_2$ and $\tau_1 \geq \tau_2$.

This is a total order, and whenever $u_1 \preceq u_2$ we have $c_1\tau_1 \ge c_2\tau_2$. Write $u \vee v = \max\{u, v\}$ and $u \wedge v = \min\{u, v\}$ with respect to $\preceq$.

Fix $x_1 \le x_2$ and $y_1 \le y_2$ and define nonnegative functions on S:

$$\alpha(c,\tau) = e^{\,c(\tau x_1 + (1-\tau)y_1)}, \quad \beta(c,\tau) = e^{\,c(\tau x_2 + (1-\tau)y_2)},$$
$$\gamma(c,\tau) = e^{\,c(\tau x_1 + (1-\tau)y_2)}, \quad \delta(c,\tau) = e^{\,c(\tau x_2 + (1-\tau)y_1)}.$$

If $u_1 \preceq u_2$, then

$$\log \frac{\gamma(u_2)\,\delta(u_1)}{\alpha(u_1)\,\beta(u_2)} = (x_2 - x_1)\,\big(c_1\tau_1 - c_2\tau_2\big) \;\ge\; 0,$$

because $x_2 \ge x_1$ and the terms involving y_1, y_2 cancel. Hence the local hypothesis of the Four Functions inequality on the totally ordered space S holds:

$$\alpha(u_1)\,\beta(u_2) \;\le\; \gamma(u_2)\,\delta(u_1) \qquad \text{for all } u_1 \preceq u_2 \in S,$$

i.e., $\alpha(u)\beta(v) \le \gamma(u \vee v)\delta(u \wedge v)$ for all $u, v \in S$.

By the *integral Four Functions Theorem* (Theorem 2.1 in [38]), this integrates to

$$\Big(\int_S \alpha\, d\mu\Big)\Big(\int_S \beta\, d\mu\Big) \;\le\; \Big(\int_S \gamma\, d\mu\Big)\Big(\int_S \delta\, d\mu\Big).$$

But $\int_S \alpha\, d\mu/((p-1)!(q-1)!) = K(x_1, y_1)$, $\int_S \beta\, d\mu/((p-1)!(q-1)!) = K(x_2, y_2)$ and similarly for γ, δ, which is exactly Eq. (3). $\square$

Corollary 1. *For all $x_1 \le x_2$ and $y_1 \le y_2$,*

$$K_0(x_1, y_1)\, K_0(x_2, y_2) \;\le\; K_0(x_1, y_2)\, K_0(x_2, y_1),$$

i.e. K_0 is TN_2.

Proof. For $\varepsilon \in (0, 1)$ define

$$h_\varepsilon(c) := \frac{1}{\varepsilon}\,\mathbf{1}_{[1-\varepsilon,\,1]}(c), \qquad K_\varepsilon(x, y) := \int_0^1 h_\varepsilon(c)\, K_0(cx, cy)\, dc.$$

By Lemma 2, each K_ε is TN_2.

We have

$$\big|K_\varepsilon(x, y) - K_0(x, y)\big| = \left|\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1} \big(K_0(cx, cy) - K_0(x, y)\big)\, dc\right| \le \sup_{c\in[1-\varepsilon,1]} \big|K_0(cx, cy) - K_0(x, y)\big|.$$

Since $c \mapsto K_0(cx, cy)$ is continuous at $c = 1$, $K_\varepsilon(x, y) \to K_0(x, y)$ as $\varepsilon \to 0$. Passing to the limit in the TN_2 inequality for K_ε yields the claim for K_0. $\square$

Theorem 1 (Log-submodularity)**.** *For all $n \in \{0, 1, 2, \dots\}$, $x_1 \le x_2$ and $y_1 \le y_2$,*

$$K_n(x_1, y_1)\, K_n(x_2, y_2) \;\le\; K_n(x_1, y_2)\, K_n(x_2, y_1),$$

i.e. K_n is TN_2.

Proof. By the Hermite–Genocchi formula for divided differences, we have

$$K_n(x,y) = \int_0^1 g_n(1-c)\,K_0\big(cx,cy\big)\,dc, \tag{5}$$

where

$$g_n(t) \;:=\; \int_{\substack{\sum_{i=1}^n s_i=t\\ s_i\ge 0}} \exp\left(\sum_{i=1}^n s_i a_i\right) ds_1\dots ds_n \;\;\ge\;\; 0. \tag{6}$$

Therefore, the desired property for $n \ge 1$ follows from Lemma 2. The property for $n = 0$ is proved in Corollary 1.

$\square$

Theorem 2 (Supermodularity)**.** *Let $n \ge 0$ and integers $p, q \ge 1$. Then*

$$K_n(x_1,y_1) + K_n(x_2,y_2) \;\ge\; K_n(x_1,y_2) + K_n(x_2,y_1) \qquad \textit{for all } x_1 \le x_2,\; y_1 \le y_2.$$

Proof. Since $K_n(x,y)$ is twice continuously differentiable, supermodularity is equivalent to $\partial_{xy}K_n(x,y) \ge 0$ for all $(x,y) \in \mathbb{R}^2$. Using the differentiation identity for repeated arguments,

$$\partial_{xy}K_n(x,y) = p\,q\,\exp[a_1,\dots,a_n,x^{(p+1)},y^{(q+1)}] \;>\; 0.$$

Hence K_n is supermodular. $\square$

3. A Four-Point Inequality for Divided Differences

Let $\exp[x_0,x_1,\dots,x_n]$ denote the divided differences of the exponential function, and

$$\begin{aligned} f(a,b,c,d) &= \exp[a,a,b,c]\exp[d,d,b,c] + \exp[b,b,a,d]\exp[c,c,a,d] - \exp[a,b,c,d]^2, && (7)\\ h(x,y) &= \exp[x,x,y,y](x-y)^2 = e^x + e^y - 2\exp[x,y]. && (8)\end{aligned}$$

Equivalently,

$$h(x,y) = 2e^{(x+y)/2}\,\phi\!\left(\frac{x-y}{2}\right), \qquad \phi(u) := \cosh u - \frac{\sinh u}{u} \quad \text{with} \quad \phi(0) := 0. \tag{9}$$

We prove the four-point inequality

$$\exp[a,a,b,c]\exp[d,d,b,c] + \exp[b,b,a,d]\exp[c,c,a,d] - \exp[a,b,c,d]^2 \ge 0,$$

which holds for all $a,b,c,d \in \mathbb{R}$.

Lemma 3.

$$f(a,b,c,d) = \frac{h(a,c)h(b,d) + h(a,b)h(c,d) - h(b,c)h(a,d)}{2(a-b)(a-c)(b-d)(c-d)}. \tag{10}$$

Proof. We have

$$f(a,b,c,d) = \frac{e^a + \exp[b,c] - \exp[a,c] - \exp[a,b]}{(a-b)(a-c)} \times \frac{e^d + \exp[b,c] - \exp[b,d] - \exp[c,d]}{(b-d)(c-d)} + \\ + \frac{e^b + \exp[a,d] - \exp[a,b] - \exp[b,d]}{(a-b)(b-d)} \times \frac{e^c + \exp[a,d] - \exp[a,c] - \exp[c,d]}{(a-c)(c-d)} - \\ - \frac{\exp[a,c] - \exp[a,d] - \exp[b,c] + \exp[b,d]}{(a-b)(c-d)} \times \frac{\exp[a,b] + \exp[c,d] - \exp[a,d] - \exp[b,c]}{(a-c)(b-d)}. \tag{11}$$

It follows that

$$f(a,b,c,d)(a-b)(a-c)(b-d)(c-d) = 2\exp[a,c]\exp[b,d] - (e^a+e^c)\exp[b,d] - (e^b+e^d)\exp[a,c] + \\ + 2\exp[a,b]\exp[c,d] - (e^a+e^b)\exp[c,d] - (e^c+e^d)\exp[a,b] + e^{b+c} + e^{a+d} - \\ - \big(2\exp[a,d]\exp[b,c] - (e^b+e^c)\exp[a,d] - (e^a+e^d)\exp[b,c]\big). \tag{12}$$

Eq. (10) follows from Eq. (12). □

Lemma 4 (Triangular inequality)**.**

$$h(a,c) \geq h(a,b) + h(b,c) \qquad \textit{when} \quad a \leq b \leq c. \tag{13}$$

Proof. Since $h(x,y) = e^x + e^y - 2\exp[x,y]$, we have

$$\begin{aligned} h(a,c) - h(a,b) - h(b,c) &= 2\big[(\exp[a,b] - \exp[a,c]) + (\exp[b,c] - \exp[b,b])\big] \\ &= 2\big[(b-c)\exp[a,b,c] + (c-b)\exp[b,b,c]\big] \\ &= 2(c-b)\big(\exp[b,b,c] - \exp[a,b,c]\big) \\ &= 2(b-a)(c-b)\,\exp[a,b,b,c] \;\geq\; 0. \end{aligned}$$

Here, we used Newton's recursion for divided differences, the positivity of exponential divided differences, and $a \leq b \leq c$. □

Lemma 5.

$$\phi(x+y)\phi(y+z) \;\geq\; \phi(x)\phi(z) + \phi(y)\phi(x+y+z) \qquad \textit{for all} \quad x,y,z \geq 0. \tag{14}$$

Proof. Let $F(y) = \phi(x+y)\phi(y+z) - \phi(y)\phi(x+y+z)$. We have

$$F'(y) = G_s(y+z) - G_s(y), \tag{15}$$

where $s = x+2y+z$ and $G_s(t) := \phi'(s-t)\phi(t) + \phi(s-t)\phi'(t)$ for $t \in [0,s]$.

Since $\phi''(u) = (1+2/u^2)\phi(u)$, we have

$$G_s'(t) = \phi(s-t)\phi(t)\left(\frac{\phi''(t)}{\phi(t)} - \frac{\phi''(s-t)}{\phi(s-t)}\right) = 2\,\phi(s-t)\,\phi(t)\left(t^{-2} - (s-t)^{-2}\right).$$

Therefore G_s is strictly increasing on $[0, s/2]$, strictly decreasing on $[s/2, s]$, and attains its maximum at $t = s/2$. Moreover, G_s is symmetric about $s/2$ since $G_s(s-t) = G_s(t)$.

Recall $s = x + 2y + z$. Then

$$\left| y - \frac{s}{2} \right| = \frac{x+z}{2}, \qquad \left| y + z - \frac{s}{2} \right| = \frac{|x-z|}{2} \ \le \ \frac{x+z}{2}.$$

Thus $y + z$ lies no farther from the maximizer $s/2$ than y does. By the symmetry and unimodality of G_s, this implies

$$G_s(y+z) \ \ge \ G_s(y).$$

In view of Eq. (15), we have $F'(y) \ge 0$ for all $y \ge 0$. Since F is nondecreasing on $[0, \infty)$ and $F(0) = \phi(x)\phi(z)$, we obtain Eq. (14). □

Lemma 6. *For any $a < b < c < d$ one has*

$$h(a,c)h(b,d) \ge h(b,c)h(a,d) + h(a,b)h(c,d).$$

Proof. Let $x = (b-a)/2$, $y = (c-b)/2$, $z = (d-c)/2$. It follows from Eq. (9) that

$$h(a,c)h(b,d) - h(b,c)h(a,d) - h(a,b)h(c,d) = 4e^{(a+b+c+d)/2} Q(x,y,z), \tag{16}$$

where

$$Q(x,y,z) = \phi(x+y)\phi(y+z) - \phi(y)\phi(x+y+z) - \phi(x)\phi(z). \tag{17}$$

It follows from Lemma 5 that

$$Q(x,y,z) \ \ge \ 0,$$

and therefore the lemma follows. □

Theorem 3 (Four-point inequality)**.** *For all $a, b, c, d \in \mathbb{R}$ one has*

$$\exp[a,a,b,c] \exp[d,d,b,c] + \exp[b,b,a,d] \exp[c,c,a,d] - \exp[a,b,c,d]^2 \ge 0. \tag{18}$$

Proof. $f(a,b,c,d)$ is invariant under the eight-element subgroup G of the symmetric group S_4 acting on $\{a,b,c,d\}$: $G = \{\mathrm{id}, (ad), (bc), (ad)(bc), (ab)(cd), (ac)(bd), (abdc), (acdb)\}$. Hence the 24 permutations of (a,b,c,d) split into three G-orbits. Let $\alpha < \beta < \gamma < \delta$ be the increasing rearrangement of $\{a,b,c,d\}$. It suffices to verify that $f(\alpha,\beta,\gamma,\delta) \ge 0$, $f(\alpha,\gamma,\delta,\beta) \ge 0$, and $f(\alpha,\beta,\delta,\gamma) \ge 0$. Since $h(x,y) \ge 0$ for all $x, y \in \mathbb{R}$, the claim follows directly from Lemmas 3 and 6 in each case. If some of the points coincide, the result follows by continuity of divided differences. □

4. A Sharp Sandwich Bound for Exponential Divided Differences at Fixed Mean and Variance

Let $\Delta_n = \{\lambda \in [0,1]^{n+1} : \sum_{i=0}^n \lambda_i = 1\}$ be the standard simplex, and let $\gamma(n,z) = \int_0^z t^{n-1} e^{-t}\, dt$ denote the lower incomplete gamma function. Throughout we assume $n \ge 1$. Write

$$\mu := \frac{1}{n+1} \sum_{i=0}^n x_i, \qquad \sigma^2 := \frac{1}{n+1} \sum_{i=0}^n (x_i - \mu)^2, \qquad a_0 := \sqrt{n}\,\sigma, \qquad a := \frac{n+1}{\sqrt{n}}\,\sigma. \tag{19}$$

$$L_n(\sigma) := n\, e^{-a_0} \, (-a)^{-n} \, \gamma\big(n, -a\big), \qquad M_n(\sigma) := n\, e^{a_0} \, a^{-n} \, \gamma\big(n, a\big). \tag{20}$$

Theorem 4 (Sharp sandwich bound)**.** *For any $x_0, \ldots, x_n \in \mathbb{R}$ with $n \geq 1$, mean μ and variance σ^2 as above,*

$$e^{\mu} L_n(\sigma) \ \leq \ n! \exp[x_0, \ldots, x_n] \ \leq \ e^{\mu} M_n(\sigma). \tag{21}$$

Moreover, within the class of vectors with mean μ and variance σ^2, both inequalities are sharp: equality on the right holds for the configuration $(\mu + a_0,\ \mu - a_0/n, \ldots, \mu - a_0/n)$ (up to permutation), and equality on the left holds for the configuration $(\mu - a_0,\ \mu + a_0/n, \ldots, \mu + a_0/n)$ (up to permutation).

Proof. Write $x_i = \mu + y_i$ with $\sum_{i=0}^{n} y_i = 0$ and $\sum_{i=0}^{n} y_i^2 = (n+1)\sigma^2$. Then, $\exp[x_0, \ldots, x_n] = e^{\mu} \exp[y_0, \ldots, y_n]$. By the Hermite–Genocchi formula,

$$\exp[y_0, \ldots, y_n] = \int_{\Delta_n} \exp\Big(\sum_{i=0}^{n} \lambda_i y_i \Big)\, d\lambda, \tag{22}$$

so the map $y := (y_0, \ldots, y_n) \mapsto \Phi(y) := n! \exp[y_0, \ldots, y_n]$ is symmetric (permutation invariant) and jointly convex (an integral of convex maps). Hence it is Schur–convex (see, e.g., [39], p. 258). Let

$$\mathcal{S} \ := \ \Big\{ y \in \mathbb{R}^{n+1} : \ \sum_{i=0}^{n} y_i = 0,\ \sum_{i=0}^{n} y_i^2 = (n+1)\sigma^2 \Big\},\ y^+ := \Big(a_0, -\frac{a_0}{n}, \ldots, -\frac{a_0}{n} \Big),\ y^- := -y^+.$$

Because Φ is Schur–convex, Lemma 7 gives $\Phi(y^-) \leq \Phi(y) \leq \Phi(y^+)$ for all $y \in \mathcal{S}$, with equality precisely at the two-level vectors (up to permutation). Since $\gamma(n, a) = a^n \int_0^1 s^{n-1} e^{-as} ds$, $\exp[a_0, t] = e^{a_0} \int_0^1 e^{t - a_0} ds$, and $(n-1)! \exp\big[a_0, t, \ldots, t\big] = d^{n-1} \exp[a_0, t]/dt^{n-1}$, where the node t is repeated n times, we obtain

$$\Phi(y^+) = n! \exp\Big[a_0, -\frac{a_0}{n}, \ldots, -\frac{a_0}{n} \Big] \ = \ n e^{a_0} a^{-n} \gamma(n, a), \tag{23}$$

$$\Phi(y^-) = n! \exp\Big[-a_0, \frac{a_0}{n}, \ \ldots\ , \frac{a_0}{n} \Big] \ = \ n\, e^{-a_0}\, (-a)^{-n}\, \gamma(n, -a). \tag{24}$$

Multiplying by e^{μ} yields Eq. (21) and the equality cases. $\square$

Lemma 7. *y^+ is the majorization-maximal vector in $\mathcal{S}$ and y^- is the majorization-minimal vector in $\mathcal{S}$. Equivalently, for every $y \in \mathcal{S}$ with nonincreasing rearrangement $y_{[1]} \geq \cdots \geq y_{[n+1]}$ and $k = 1, \ldots, n$,*

$$\sum_{i=1}^{k} y_{[i]} \ \leq \ \sum_{i=1}^{k} y^+_{[i]} \quad \textit{and} \quad \sum_{i=1}^{k} y^-_{[i]} \ \leq \ \sum_{i=1}^{k} y_{[i]},$$

with equality for all k iff y is a permutation of y^+ or y^-, respectively.

Proof. Write the coordinates of $y \in \mathcal{S}$ in nonincreasing order: $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n+1]}$. Since $\sum_{i=2}^{n+1} y_{[i]} = -y_{[1]}$, the Cauchy–Schwarz inequality yields

$$\sum_{i=2}^{n+1} y_{[i]}^2 \ \geq \ \frac{\big(\sum_{i=2}^{n+1} y_{[i]} \big)^2}{n} \ = \ \frac{y_{[1]}^2}{n}.$$

Hence

$$(n+1)\sigma^2 = \sum_{i=1}^{n+1} y_{[i]}^2 \;\geq\; y_{[1]}^2 + \frac{y_{[1]}^2}{n} \;=\; y_{[1]}^2\Big(1+\frac{1}{n}\Big),$$

so

$$y_{[1]} \;\leq\; a_0 = \sqrt{n}\,\sigma. \tag{25}$$

Moreover, equality in Eq. (25) holds if and only if $y_{[2]} = \cdots = y_{[n+1]} = -a_0/n$ (equality case in Cauchy–Schwarz), i.e., y is a permutation of y^+.

Let $S_k(y) := \sum_{i=1}^{k} y_{[i]}$ be the k-th partial sum. Suppose there existed $y \in \mathcal{S}$ with $S_k(y) \geq S_k(y^+)$ for all k and strict inequality for some k. Then, in particular, $S_1(y) = y_{[1]} \geq S_1(y^+) = a_0$, which by Eq. (25) forces $y_{[1]} = a_0$ and hence (by the equality case) y is a permutation of y^+; thus all partial sums coincide and no strict improvement is possible. Therefore y^+ is majorization-maximal in $\mathcal{S}$. The argument applied to $-y$ shows y^- is majorization-minimal. □

Lemma 8.

$$L_n(\sigma) \;=\; 1 + \frac{\sigma^2}{2n} - \frac{\sigma^3}{3n^{3/2}} + O\left(\frac{1}{n^2}\right) \qquad \textit{as } n\to\infty, \tag{26}$$

$$M_n(\sigma) \;=\; 1 + \frac{\sigma^2}{2n} + \frac{\sigma^3}{3n^{3/2}} + O\left(\frac{1}{n^2}\right) \qquad \textit{as } n\to\infty. \tag{27}$$

Proof. For integer $n \geq 1$,

$$\gamma(n,z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(n+m)\,m!}\, z^{n+m},$$

hence

$$M_n(\sigma) = e^{a_0} \sum_{m=0}^{\infty} (-1)^m \frac{n}{n+m}\frac{a^m}{m!}, \qquad L_n(\sigma) = e^{-a_0} \sum_{m=0}^{\infty} \frac{n}{n+m}\frac{a^m}{m!}.$$

Use the expansion $n/(n+m) = 1 - m/n + m^2/n^2 - m^3/n^3 + O(m^4/n^4)$ and the identities $\sum_{m=0}^{\infty} m^k(-a)^m/m! = T^k(e^{-a})$ and $\sum_{m=0}^{\infty} m^k a^m/m! = T^k(e^a)$ for $T := a\cdot d/da$ and $k = 0, 1, 2, 3$.

It follows that

$$\sum_{m=0}^{\infty}(-1)^m \frac{n}{n+m}\frac{a^m}{m!} = e^{-a}\left[1 + \frac{a}{n} + \frac{a^2-a}{n^2} + \frac{a^3-3a^2+a}{n^3}\right] + O\Big(\frac{1}{n^2}\Big),$$

$$\sum_{m=0}^{\infty} \frac{n}{n+m}\frac{a^m}{m!} = e^{a}\left[1 - \frac{a}{n} + \frac{a^2+a}{n^2} - \frac{a^3+3a^2+a}{n^3}\right] + O\Big(\frac{1}{n^2}\Big),$$

Since $a_0 - a = -\sigma/\sqrt{n}$,

$$e^{\pm(a_0-a)} = 1 \pm \frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{2n} \pm \frac{\sigma^3}{6n^{3/2}} + O\Big(\frac{1}{n^2}\Big),$$

and

$$\frac{a}{n} = \frac{\sigma}{\sqrt{n}} + O\Big(\frac{1}{n^{3/2}}\Big), \quad \frac{a^2-a}{n^2} = \frac{\sigma^2}{n} + O\Big(\frac{1}{n^{3/2}}\Big), \quad \frac{a^3-3a^2+a}{n^3} = \frac{\sigma^3}{n^{3/2}} + O\Big(\frac{1}{n^2}\Big).$$

Multiplying and keeping terms up to $n^{-3/2}$ gives Eqs. (26) and (27). □

Corollary 2 (Large-input limit). *Let $(x_i)_{i=0}^{\infty}$ be a sequence of real numbers and let*

$$\mu_n := \frac{1}{n+1}\sum_{i=0}^{n} x_i, \qquad \sigma_n^2 := \frac{1}{n+1}\sum_{i=0}^{n}(x_i-\mu_n)^2.$$

Assume $\sup_{n\geq 0}\sigma_n^2 \leq C < \infty$. *Then*

$$n!\,\exp[x_0,\ldots,x_n] = \exp\Big(\mu_n + \frac{\sigma_n^2}{2n} + O\big(n^{-3/2}\big)\Big) \quad \text{as } n\to\infty.$$

5. Useful identities

In this section, we adopt the shorthand notation $e^{t[x_0,\ldots,x_q]} := f[x_0,\ldots,x_q]$ for $f(x)=e^{tx}$ and $x_0,\ldots,x_q\in\mathbb{C}$. We begin by briefly recalling several well-known results that will be used below.

The divided difference of any holomorphic function $f(x)$ can be defined over the multiset $[x_0,\ldots,x_q]$ using a contour integral [1, 2],

$$f[x_0,\ldots,x_q] \equiv \frac{1}{2\pi i}\oint\limits_{\Gamma} \frac{f(x)}{\prod_{i=0}^{q}(x-x_i)}\,dx, \tag{28}$$

for Γ a positively oriented contour enclosing all the x_i's. The divided differences can be shown to satisfy the Leibniz rule [2],

$$(f\cdot g)[x_0,\ldots,x_q] = \sum_{j=0}^{q} f[x_0,\ldots,x_j]g[x_j,\ldots,x_q] = \sum_{j=0}^{q} g[x_0,\ldots,x_j]f[x_j,\ldots,x_q]. \tag{29}$$

Replacing the variable $x\to\alpha x$ in Eq. (28), we find the rescaling relation,

$$\alpha^q e^{t[\alpha x_0,\ldots,\alpha x_q]} = e^{\alpha t[x_0,\ldots,x_q]}. \tag{30}$$

Combining Eq. (30) with Eq. (11) of Ref. [40] with $P_m(x)=1$, we find

$$\mathcal{L}\left\{\, e^{\alpha t[x_0,\ldots,x_q]} \,\right\} \;=\; \frac{\alpha^q}{\prod_{j=0}^{q}\big(s-\alpha x_j\big)}, \tag{31}$$

where $\mathcal{L}\{\cdot\}$ denote the Laplace transform in $t\to s$, and Eq. (31) holds in the right half-plane $\Re s \;>\; \max_{0\leq j\leq q}\Re(\alpha x_j)$, where the Laplace transform converges. One can alternatively derive Eq. (31) by directly performing the integration to the contour integral definition in Eq. (28), e.g. by Taylor expanding $e^{\alpha tx}$ and re-summing term-by-term, legitimate by the uniform convergence of the exponential.

Theorem 5 (Convolution). *Let $j\in\{0,\ldots,q-1\}$ and $\beta\geq 0$. Then*

$$\int_0^{\beta} e^{-\tau[x_{j+1},\ldots,x_q]}e^{-(\beta-\tau)[x_0,\ldots,x_j]}\,d\tau = -e^{-\beta[x_0,\ldots,x_q]}. \tag{32}$$

Proof. Define

$$f(t) := e^{-t[x_{j+1},\dots,x_q]}, \qquad g(t) := e^{-t[x_0,\dots,x_j]}.$$

Their convolution is

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau,$$

so Eq. (32) states that $(f * g)(\beta) = -e^{-\beta[x_0,\dots,x_q]}$. By the convolution property of the Laplace transform and Eq. (31), we find

$$\begin{aligned}\mathcal{L}\left\{(f*g)(t)\right\} &= \mathcal{L}\left\{f(t)\right\}\mathcal{L}\left\{g(t)\right\} \\ &= \left(\frac{(-1)^{q-j-1}}{\prod_{l=j+1}^{q}(s+x_l)}\right)\left(\frac{(-1)^j}{\prod_{m=0}^{j}(s+x_m)}\right) = \frac{(-1)^{q-1}}{\prod_{l=0}^{q}(s+x_l)} = \mathcal{L}\left\{-e^{-t[x_0,\dots,x_q]}\right\}.\end{aligned}$$

By the uniqueness theorem for the Laplace transform, $(f * g)(t) = -e^{-t[x_0,\dots,x_q]}$ for all $t \geq 0$. Evaluating at $t = \beta$ gives Eq. (32). □

Theorem 6 (Repeated argument sum)**.**

$$\sum_{j=0}^{q} e^{-\tau[x_0,\dots,x_q,x_j]} = -\tau e^{-\tau[x_0,\dots,x_q]} \tag{33}$$

Proof. Denote $f(\tau) \equiv e^{-\tau[x_0,\dots,x_q]}$ and recall $\mathcal{L}\left\{e^{-\tau[x_0,\dots,x_q]}\right\} = (-1)^q/(\prod_{i=0}^{q}(s+x_i))$. By the differentiation property of the Laplace transform and Eq. (31),

$$\mathcal{L}\left\{-\tau f(\tau)\right\} = \frac{\partial}{\partial s}\mathcal{L}\left\{f(\tau)\right\} = \sum_{j=0}^{q} \frac{(-1)^{q+1}}{(s+x_j)\prod_{i=0}^{q}(s+x_i)} = \mathcal{L}\left\{\sum_{j=0}^{q} e^{-\tau[x_0,\dots,x_q,x_j]}\right\}, \tag{34}$$

and the result follows by the uniqueness of the Laplace transform. □

Theorem 7 (Weighted, repeated argument sum)**.**

$$\sum_{j=0}^{q} x_j e^{-\tau[x_0,\dots,x_q,x_j]} = \left(\tau\frac{\partial}{\partial\tau} - q\right) e^{-\tau[x_0,\dots,x_q]}. \tag{35}$$

Proof. We prove this by induction.

Base case. When $q = 0$, we have $x_0 e^{-\tau[x_0,x_0]} = x_0 \partial e^{-\tau x_0}/\partial x_0 = -\tau x_0 e^{-\tau x_0}$, so Eq. (35) holds.

Induction step. By Eq. (30), we write the right-hand side as,

$$\begin{aligned}(\tau\partial_\tau - (q+1))(-\tau)^{q+1} e^{[-\tau x_0,\dots,-\tau x_{q+1}]} &= -(-\tau)^{q+2}\frac{\partial}{\partial\tau} e^{[-\tau x_0,\dots,-\tau x_{q+1}]} \\ &= -(-\tau)^{q+2}\frac{1}{2\pi i}\oint_\Gamma \frac{\partial}{\partial\tau}\frac{e^z}{\prod_{i=0}^{q+1}(z+\tau x_i)}\, dz = \sum_{j=0}^{q+1} x_j e^{-\tau[x_0,\dots,x_{q+1},x_j]}\end{aligned} \tag{36}$$

Therefore, Eq. (35) is true for all q. □

Lemma 9 (Parametric derivative)**.**

$$\frac{\partial}{\partial\tau}e^{-\tau[x_0,\ldots,x_q]} = -x_0 e^{-\tau[x_0,\ldots,x_q]} - e^{-\tau[x_1,\ldots,x_q]} \qquad \textit{for} \quad q>0. \tag{37}$$

Proof. First, define $g(x) = -xe^{-\tau x}$ for convenience. By Eq. (28),

$$\frac{\partial}{\partial\tau}e^{-\tau[x_0,\ldots,x_q]} \equiv \frac{\partial}{\partial\tau}\frac{1}{2\pi i}\oint_\Gamma \frac{e^{-\tau z}}{\prod_{i=0}^{q}(z-x_i)}\,dz = \frac{1}{2\pi i}\oint_\Gamma \frac{-ze^{-\tau z}}{\prod_{i=0}^{q}(z-x_i)}\,dz \equiv g[x_0,\ldots,x_q]. \tag{38}$$

Employing the Leibniz rule, Eq. (29), we get the desired result. □

Corollary 3.

$$\sum_{j=0}^{q} x_j e^{-\tau[x_0,\ldots,x_q,x_j]} = \begin{cases} (-x_0\tau - q)e^{-\tau[x_0,\ldots,x_q]} - \tau e^{-\tau[x_1,\ldots,x_q]}, & \text{for} \quad q>0, \\ -\tau x_0 e^{-\tau x_0}, & \text{for} \quad q=0. \end{cases} \tag{39}$$

Proof. This follows from applying Eq. (37) to Theorem 7. □

Theorem 8 (Weighted, repeated argument, double, less-than sum)**.**

$$\sum_{\substack{i,j=0\\ i\le j}}^{q} x_i e^{-\tau[x_0,\ldots,x_q,x_i,x_j]} = \left(-\frac{\tau^2}{2}\frac{\partial}{\partial\tau} - \sum_{i=0}^{q} i\cdot\frac{\partial}{\partial x_i}\right)e^{-\tau[x_0,\ldots,x_q]}. \tag{40}$$

Proof. We prove this by induction.

Base case. When $q=0$, we have

$$x_0 e^{-\tau[x_0,x_0,x_0]} = x_0\cdot\frac{1}{2}\frac{\partial^2}{\partial x_0^2}e^{-\tau x_0} = \frac{\tau^2 x_0}{2}e^{-\tau x_0} = -\frac{\tau^2}{2}\frac{\partial}{\partial\tau}e^{-\tau x_0},$$

so Eq.(40) holds.

Induction step. It follows from Newton's recursion for divided differences that

$$\begin{aligned}
\sum_{\substack{i,j=0\\ i\le j}}^{q+1} x_i e^{-\tau[x_0,\ldots,x_{q+1},x_i,x_j]} &= \frac{1}{x_0-x_1}\sum_{\substack{i,j=0\\ i\le j}}^{q+1} x_i\left(e^{-\tau[x_0,x_2,\ldots,x_{q+1},x_i,x_j]} - e^{-\tau[x_1,x_2,\ldots,x_{q+1},x_i,x_j]}\right)\\
&= \frac{1}{x_0-x_1}\Bigg(\Big(\big(-\frac{\tau^2}{2}\frac{\partial}{\partial\tau} - \sum_{i=1}^{q} i\cdot\frac{\partial}{\partial x_{i+1}}\big)e^{-\tau[x_0,x_2,\ldots,x_{q+1}]}\\
&\qquad + x_0 e^{-\tau[x_0,x_2,\ldots,x_{q+1},x_0,x_1]} + x_1\sum_{j=1}^{q+1} e^{-\tau[x_0,\ldots,x_{q+1},x_j]}\Big)\\
&\quad -\Big(\big(-\frac{\tau^2}{2}\frac{\partial}{\partial\tau} - \sum_{i=1}^{q} i\cdot\frac{\partial}{\partial x_{i+1}}\big)e^{-\tau[x_1,x_2,\ldots,x_{q+1}]} + x_0\sum_{j=0}^{q+1} e^{-\tau[x_0,x_1,\ldots,x_{q+1},x_j]}\Big)\Bigg)\\
&= \left(-\frac{\tau^2}{2}\frac{\partial}{\partial\tau} - \sum_{i=0}^{q} i\cdot\frac{\partial}{\partial x_{i+1}}\right)e^{-\tau[x_0,\ldots,x_{q+1}]} - \sum_{j=1}^{q+1} e^{-\tau[x_0,\ldots,x_{q+1},x_j]}\\
&= \left(-\frac{\tau^2}{2}\frac{\partial}{\partial\tau} - \sum_{i=0}^{q+1} i\cdot\frac{\partial}{\partial x_i}\right)e^{-\tau[x_0,\ldots,x_{q+1}]}.
\end{aligned} \tag{41}$$

Therefore, Eq. (40) is true for all q. □

Corollary 4.

$$\sum_{\substack{i,j=0\\ i\le j}}^{q} x_i e^{-\tau[x_0,\dots,x_q,x_i,x_j]}$$
$$= \begin{cases} (\tau^2 x_0/2)\, e^{-\tau[x_0,\dots,x_q]} + (\tau^2/2) e^{-\tau[x_1,\dots,x_q]} - \sum_{i=1}^{q} i \cdot e^{-\tau[x_0,\dots,x_q,x_i]}, & \text{for} \quad q>0, \\ (\tau^2 x_0/2) e^{-\tau x_0}, & \text{for} \quad q=0. \end{cases} \tag{42}$$

Proof. Eq. (42) follows from Eq. (37) and Eq. (40). □

6. Concluding remarks

The results presented here highlight several structural features of exponential divided differences that arise in a broad range of analytic and computational settings.

The log-submodularity and four-point inequality established in this work reveal a latent convexity structure underlying the exponential function's divided differences. The sharp two-sided bounds at fixed mean and variance provide quantitative control and may prove useful for stability and sensitivity analyses of exponential integrators and operator inequalities. The closed-form convolution and repeated-argument identities extend the analytic toolkit available for both theoretical investigations and numerical algorithms involving the exponential function and its divided differences, the matrix exponential, and operator analogues.

Beyond their intrinsic mathematical interest, these results suggest new connections among interpolation theory, matrix analysis, and probabilistic inequalities through the common language of exponential divided differences. Future work may explore analogous properties for other operator-convex functions, extend these inequalities to matrix- or operator-valued settings, and examine their implications for higher-order exponential integrators and quantum Monte Carlo methods.

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