SHIFTED TWISTED YANGIANS AND AFFINE GRASSMANNIAN ıSLICES

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ABSTRACT. Associated to all quasi-split Satake diagrams of type ADE and even spherical coweights μ, we introduce the shifted twisted Yangians ${}^\imath\mathcal{Y}_\mu$ and establish their PBW bases. We construct the iGKLO representations of ${}^\imath\mathcal{Y}_\mu$, which factor through quotients known as truncated shifted twisted Yangians (TSTY) ${}^\imath\mathcal{Y}_\mu^\lambda$. In type AI with μ dominant, a variant of ${}^\imath\mathcal{Y}_\mu^{N\varpi_1^\vee}$ is identified with the TSTY in another definition which are isomorphic to finite W-algebras of type BCD. We show that ${}^\imath\mathcal{Y}_\mu$ quantizes the involutive fixed point locus ${}^\imath\mathcal{W}_\mu$ arising from affine Grassmannians of type ADE, and expect that ${}^\imath\mathcal{Y}_\mu^\lambda$ quantizes a top-dimensional component of the affine Grassmannian islice ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$. We identify the islices ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ in type AI with suitable nilpotent Slodowy slices of type BCD, building on the work of Lusztig and Mirković-Vybornov in type A. We propose a framework for producing ortho-symplectic (and hybrid) Coulomb branches from split (and nonsplit) Satake framed double quivers, which are conjectured to provide a normalization of the islices ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$.

CONTENTS

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1. Introduction

A connection between truncated shifted Yangians and affine Grassmannian slices $\overline{\mathcal{W}}^\lambda_\mu$, for dominant coweights μ, was established in [KWWY14]. A key input here is what has become known as the GKLO representations of shifted Yangians, inspired by [GKLO05], whose images are known as truncated shifted Yangians Y^λ_μ. These algebras can be viewed as a vast generalization of the construction of Brundan-Kleshchev [BK06], who showed that truncated shifted Yangians Y^λ_μ in type A (with $\lambda = N\varpi_1^\vee$) are isomorphic to finite W-algebras of $\mathfrak{gl}_N$, quantizing an isomorphism between certain affine Grassmannian slices and nilpotent Slodowy slices [WWY20, MV22].

The above connection extends naturally in the framework of Coulomb branches (see [BFN19, NW23]). It is shown that a more general class of truncated shifted Yangians Y^λ_μ quantize the generalized affine Grassmannian slices $\overline{\mathcal{W}}^\lambda_\mu$, for arbitrary (not necessarily dominant) coweights μ. Moreover, the spaces $\overline{\mathcal{W}}^\lambda_\mu$ are identified as Coulomb branches of cotangent type associated with framed quiver gauge theories (with symmetrizers). This story also admits a q-deformation: it is shown in [FT19a] that shifted affine quantum groups are mapped homomorphically into the quantized K-theoretic Coulomb branches of framed quiver gauge theories.

The goal of this paper is to formulate an $\imath$fication of the above connections, a framework in which new families of algebras provide a quantization of new Poisson varieties arising from classical geometric setting. That is, we shall formulate and establish interconnections among

- ▶ shifted twisted Yangians
- ▶ iGKLO representations
- ▶ affine Grassmannian islices
- ▶ type BCD nilpotent Slodowy slices
- ▶ (conjecturally) iCoulomb branches.

Some of the main players such as affine Grassmannian *islices* and iCoulomb branches are new, and there was little expectation in literature that shifted twisted Yangians of quasi-split ADE type and their truncations admit geometric interpretations in such a generality as formulated in this paper. The connection between algebra and geometry is made through the iGKLO representations of shifted twisted Yangians which we shall construct. Let us introduce these main players and explain the main results of the paper.

1.1. **Shifted twisted Yangians.** Associated with Satake diagrams, Letzter formulated quantum symmetric pairs $(\mathbf{U}, \mathbf{U}^\imath)$ and a generalization to Kac-Moody type was given by Kolb, where $\mathbf{U}$ is a Drinfeld-Jimbo quantum group and $\mathbf{U}^\imath$ is a coideal subalgebra of $\mathbf{U}$ known as an iquantum group nowadays. The quantum group $\mathbf{U}$ appears as an iquantum group associated to the diagonal Satake diagram. The $\imath$-generalization of affine quantum groups and Yangians are known as affine iquantum groups and twisted Yangians (or iYangians). Twisted Yangians were defined earlier in [Ols92, MR02, GR16] in R-matrix form, deforming the twisted current algebras (i.e., involutive fixed point subalgebras of current algebras).

Recently, a Drinfeld presentation of the twisted Yangian of type AI has been constructed in [LWZ25b] via Gauss decomposition; such presentations have been extended to twisted Yangians of all split type [LWZ25a] and then to all quasi-split type [LZ24] via a degeneration construction from the corresponding Drinfeld presentation of (quasi-)split affine iquantum groups [LW21, LWZ24]. However, the identification of the two definitions of twisted Yangians (one via R-matrix and the other via Drinfeld presentation) is only established in type AI and AIII, and in this paper we follow the ones in Drinfeld presentation. These twisted Yangians and affine iquantum groups are determined by the underlying finite type quasi-split Satake diagrams $(\mathbb{I}, \tau)$, see Table 1 with $\tau \neq \mathrm{id}$. Here, τ is a diagram involution of the ADE Dynkin diagram $\mathbb{I}$; the cases with $\tau = \mathrm{id}$ are called split.

Associated to an arbitrary quasi-split Satake diagram $(\mathbb{I}, \tau)$, we have a symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\omega_\tau})$, where $\omega_\tau = \omega_0 \circ \tau$ for the Chevalley involution ω_0, and the corresponding twisted Yangian ${}^\imath\mathcal{Y} = {}^\imath\mathcal{Y}_0$. Modifying the twisted Yangians in Drinfeld presentation, we introduce the shifted twisted Yangians ${}^\imath\mathcal{Y}_\mu$; see Definition 2.2. Here μ is an arbitrary even spherical weight (see Definition 2.1), some natural conditions imposed by necessity of the iGKLO representations and the geometric properties of affine Grassmannian islices. Note that a dominant family of shifted twisted Yangians ${}^\imath\mathcal{Y}_\mu$ of type AI, for μ dominant and even (automatically spherical since $\tau = \mathrm{id}$), has appeared and played a fundamental role in [LPT+25].

Theorem A (PBW basis Theorem 2.16)**.** Let μ be an arbitrary even spherical coweight. Then the set of ordered monomials in the root vectors (2.22) forms a basis for ${}^\imath\mathcal{Y}_\mu$.

The shift homomorphisms between shifted twisted Yangians, $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{Y}_\mu \to {}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}$ (see Lemma 2.7), for ν anti-dominant and $\mu, \nu + \tau\nu$ even spherical, are injective as a consequence of Theorem A.

1.2. **iGKLO and truncated shifted twisted Yangians.** Recall the GKLO homomorphisms from shifted Yangians to a ring of difference operators [GKLO05, KWWY14, BFN19], with images called truncated shifted Yangians. We define a variant of such a ring of difference operators $\mathcal{A}$ as in (3.6)–(3.7). We construct iGKLO homomorphisms from shifted twisted Yangians to $\mathcal{A}$, first for ${}^\imath\mathcal{Y}_\mu(\mathfrak{gl}_n)$ in Theorem 3.4, and then for general ${}^\imath\mathcal{Y}_\mu$.

Theorem B (Theorem 3.6)**.** Let $(\mathbb{I}, \tau)$ be any quasi-split Satake diagram. Let λ be a dominant τ-invariant coweight and μ be an even spherical coweight subject to the parity constraint (3.5). Then there exists a unique homomorphism $\Phi^\lambda_\mu : {}^\imath\mathcal{Y}_\mu[z] \longrightarrow \mathcal{A}$ with prescribed formulas on generators.

Finding the explicit formulas for the images of generators of ${}^\imath\mathcal{Y}_\mu$ under Φ^λ_μ is part of the main challenges here. The proof of Theorem B is technical and will occupy the whole Section 4. It is worth noting that the same parity condition (3.5) appears naturally in the geometric setting; see Theorem G.

A truncated shifted twisted Yangian (TSTY) of type $(\mathbb{I}, \tau)$ is by definition the image of the homomorphism Φ^λ_μ and will be denoted by ${}^\imath\mathcal{Y}^\lambda_\mu$. This family of algebras will play a fundamental role in this paper, and they admit several favorable properties; here is one of them.

Proposition C (Proposition 3.12)**.** The center of the TSTY ${}^\imath\mathcal{Y}^\lambda_\mu$ is a polynomial algebra.

Associated with the Satake diagram of type AIII_{2n} with an even number of nodes, a variant of the shifted twisted Yangian based on [LZ24] is also formulated in [SSX25] though no PBW basis is established; the authors also construct a homomorphism to the ring $\mathcal{A}$, which factors through a corresponding Coulomb branch. In this case, we learned from these authors a generating function trick which allowed us to verify a Serre relation (2.9) required for the iGKLO homomorphism. The Satake diagram of type AIII_{2n} is special among all ADE Satake diagrams in that it contains no split rank one subdiagram.

1.3. **A tale of two TSTY's.** Via a parabolic generalization of the Drinfeld presentation of twisted Yangians of type AI [LWZ25b], a very different definition of shifted twisted Yangians $\mathcal{Y}_\mu$ has been given in [LPT+25], where μ is a partition assumed to be even in the sense that its associated nilpotent element gives rise to an *even* **Z**-grading on the underlying classical Lie algebra. Miraculously, this notion of evenness is compatible with the evenness condition on μ as a dominant coweight (valid for all ADE type) used in this paper. A family of truncated shifted twisted Yangians (TSTY) of type AI, denoted by $\mathcal{Y}^+_{n,\ell}(\sigma)$, was defined very differently in [LPT+25] (generalizing [BK06]) in terms of generators and relations, and one main result *loc. cit.* is an algebra isomorphism between $\mathcal{Y}^+_{n,\ell}(\sigma)$ and finite W-algebras, quantizing the corresponding Slodowy slices of type BCD.

It is a simple matter to align the combinatorial data $(n, \ell; \sigma)$ used in $\mathcal{Y}^+_{n,\ell}(\sigma)$ and a pair $(N\varpi_1^\vee, \mu)$ used in TSTY ${}^\imath\mathcal{Y}_\mu^{N\varpi_1^\vee}$ here. We introduce a variant of ${}^\imath\mathcal{Y}_\mu^{N\varpi_1^\vee}$, denoted by ${}^\imath\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee}$ in (5.22).

Theorem D (Theorem 5.8)**.** These two versions of TSTY's are isomorphic: $\mathcal{Y}^+_{n,\ell}(\sigma) \cong {}^\imath\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee}$.

This in particular gives a presentation for the TSTY ${}^\imath\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee}$. It remains an open problem to give a presentations for the TSTY ${}^\imath\mathcal{Y}_\mu^\lambda$ in general; see Conjecture 3.10.

1.4. **Twisted Yangians quantize loop symmetric spaces.** One can axiomize the notion of Yangian $\mathcal{U}_\hbar(\mathfrak{g}[z])$ which quantizes the current algebra $U(\mathfrak{g}[z])$; see Definition 6.1. Following the quantum duality principle [Dri87a, Gav02], one defines a subalgebra $\mathcal{U}_\hbar(\mathfrak{g}[z])'$ of such a Yangian. Inspired by [KWWY14, Sha16], under a general technical Assumption (6.5), one shows that $\mathcal{U}_\hbar(\mathfrak{g}[z])'$ quantizes the Poisson group $G_1[\![z^{-1}]\!]$; see Proposition 6.3.

Let ω be an involution and an isometry on $\mathfrak{g}$. It induces an involution on $\mathfrak{g}[z]$ and a Poisson involution σ in (6.11) on $G_1[\![z^{-1}]\!]$. One can axiomize a twisted Yangian $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)$, which is a coideal subalgebra of $\mathcal{U}_\hbar(\mathfrak{g}[z])$ and quantizes the twisted current algebra $U(\mathfrak{g}[z]^\omega)$ (see Definition 6.15); one also defines its subalgebra $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ following the quantum duality principle. Under a technical Assumption (6.14), by applying a general recipe of Dirac reduction (see [Xu03] or [Fer94]) at the loop group level, we obtain an isomorphism of Poisson homogeneous spaces $G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega \xrightarrow{\sim} G_1[\![z^{-1}]\!]^\sigma$, which is compatible with the natural Poisson algebra of $G_1[\![z^{-1}]\!]$; see (6.1) and Proposition 6.14.

Theorem E (Theorem 6.19)**.** $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ quantizes the Poisson symmetric space $G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega$ or equivalently the affine scheme $G_1[\![z^{-1}]\!]^\sigma$ with its (doubled) Dirac Poisson structure, and the inclusion $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \subset \mathcal{U}_\hbar(\mathfrak{g}[z])$ quantizes the map $G_1[\![z^{-1}]\!] \to G_1[\![z^{-1}]\!]^\sigma$ defined by $g \mapsto g\sigma(g)$. The left coideal structure on $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)$ quantizes the left action of $G_1[\![z^{-1}]\!]$ on $G_1[\![z^{-1}]\!]^\sigma$ given by $g \cdot p = gp\sigma(g)$.

1.5. **Shifted twisted Yangians as a quantization of ${}^\imath\mathcal{W}_\mu$.** Let G be the adjoint group of the simple Lie algebra $\mathfrak{g}$. Following [FKP+18], for any coweight μ, the closed subscheme $\mathcal{W}_\mu$ of $G(\!(z^{-1})\!)$ defined in (7.1) is quantized by the shifted Yangian $Y_\mu(\mathfrak{g})$. Associated to the quasi-split Satake diagram $(\mathbb{I}, \tau)$, the involution $\omega_\tau = \omega_0\tau$ of $\mathfrak{g}$ leads to an anti-involution σ on the loop algebra (7.16) as well as on the loop group $G(\!(z^{-1})\!)$. We show that σ preserves the subscheme $\mathcal{W}_\mu$ if and only μ is even spherical. In this case, σ restricts to a Poisson involution on $\mathcal{W}_\mu$, and hence the σ-fixed point locus of $\mathcal{W}_\mu$, denoted ${}^\imath\mathcal{W}_\mu$, inherits a Poisson structure from $\mathcal{W}_\mu$. Our next main result is that the Poisson algebra $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ is quantized by the shifted twisted Yangian ${}^\imath\mathcal{Y}_\mu$.

Theorem F (Proposition 7.7, Theorem 7.8)**.** Let μ be an even spherical coweight.

(1) For $\mu = \mu_1 + \tau\mu_1$, there is an isomorphism of $Q^\imath \times \mathbb{Z}$–graded Poisson algebras $\mathrm{gr}^{F^\bullet_{\mu_1}}\, {}^\imath\mathcal{Y}_\mu \cong \mathbb{C}[{}^\imath\mathcal{W}_\mu]$, which matches the corresponding generators.
(2) The coordinate ring $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ is the Poisson algebra generated by $h_i^{(r)}, b_i^{(s)}$ for $i \in \mathbb{I}$, $r \in \mathbb{Z}$ and $s \geqslant 1$, with explicit defining Poisson relations.
(3) $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ is a polynomial algebra with explicit PBW generators.
(4) For any antidominant weight ν such that $\nu+\tau\nu$ is even, the shift map $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{W}_{\mu+\nu+\tau\nu} \to {}^\imath\mathcal{W}_\mu$ is quantized by the shift homomorphism $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{Y}_\mu \to {}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}$.

Theorem E and Theorem F(1) complement each other, as the twisted Yangians ${}^\imath\mathcal{Y}_\mu$ are expected to be compatible with the ones used in Theorem E; see Remark 6.23.

1.6. **Affine Grassmannian islices.** For coweights $\lambda \geqslant \mu$ with λ dominant, the (generalized) affine Grassmannian slice $\overline{\mathcal{W}}^\lambda_\mu$ in (8.1) as a closed subscheme of $\mathcal{W}_\mu$ was introduced in [BFN19]. For μ dominant, $\overline{\mathcal{W}}^\lambda_\mu$ are the usual affine Grassmannian slices between spherical Schubert varieties in the affine Grassmannian. The Poisson involution σ on $\mathcal{W}_\mu$ preserves $\overline{\mathcal{W}}^\lambda_\mu$ if and only if λ is τ-invariant, and the affine Grassmannian islice ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ is defined to be the σ-fixed point locus of $\overline{\mathcal{W}}^\lambda_\mu$; cf. (8.14). Some basic properties of ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ such as Poisson structure and symplectic leaves can be derived from the counterparts for $\overline{\mathcal{W}}^\lambda_\mu$; see Theorem 8.7.

Recall the filtration on ${}^\imath\mathcal{Y}_\mu$ appearing in Theorem F(1) and (8.16), and there is also a filtration on $\mathcal{A}_{\boldsymbol{z}=0}$ as in (8.17). With [KWWY14, BFN19] in mind, one is tempted to claim that the surjective iGKLO homomorphism $\Phi^\lambda_\mu : {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}] \twoheadrightarrow {}^\imath\mathcal{Y}^\lambda_\mu$ arising from Theorem B quantizes the geometric embedding ${}^\imath\overline{\mathcal{W}}^\lambda_\mu \hookrightarrow {}^\imath\overline{\mathcal{W}}_\mu$. However, one complication arises from the fact that the islices ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ may be reducible and may not be normal (as it happens to fixed point loci of irreducible normal varieties). Such examples do occur in the setting of nilpotent Slodowy slices of type BCD, which turn out to be affine Grassmannian islices of type AI by Theorem H. Recall notation ${}^\imath\mathbb{I}$ from (2.1) and $\mathfrak{v}_i$ from (3.2).

Theorem G (Theorems 8.11 and 8.12)**.** (1) The open subscheme $U^\lambda_\mu \subset \overline{\mathcal{W}}^\lambda_\mu$ from (8.13) is invariant under σ. Its fixed point locus ${}^\imath U^\lambda_\mu$ is non-empty if and only if the parity condition (3.5) holds, and in this case we have $\dim {}^\imath U^\lambda_\mu = 2\sum_{i\in{}^\imath\mathbb{I}} \mathfrak{v}_i$.

(2) The homomorphism $\Phi^{\lambda,\boldsymbol{z}=0}_\mu : {}^\imath\mathcal{Y}_\mu \to \mathcal{A}_{\boldsymbol{z}=0}$ is filtered, and the associated graded map $\mathrm{gr}\,\Phi^{\lambda,\boldsymbol{z}=0}_\mu : \mathbb{C}[{}^\imath\mathcal{W}_\mu] \longrightarrow \mathrm{gr}\,\mathcal{A}_{\boldsymbol{z}=0}$ defines the closure $\overline{C}^\lambda_\mu \subseteq {}^\imath\overline{\mathcal{W}}^\lambda_\mu$, i.e., the kernel of this map is the defining ideal of $\overline{C}^\lambda_\mu$. Here $C^\lambda_\mu \subseteq {}^\imath U^\lambda_\mu$ is a top-dimensional irreducible component. In case when ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ is irreducible, $\overline{C}^\lambda_\mu$ is equal to ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$.

It is quite plausible that ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ are often irreducible, though we do not know of a general criterion. We expect that $\overline{C}^\lambda_\mu$ is quantized by the TSTY ${}^\imath\mathcal{Y}^\lambda_\mu$; see Conjecture 8.13. This boils down to a technical issue of identifying two filtrations on ${}^\imath\mathcal{Y}^\lambda_\mu$.

1.7. **Nilpotent Slodowy slices of type BCD.** In this subsection, we specialize to $G = \mathrm{PGL}_N$ and $\tau = \mathrm{id}$, which correspond to the Satake diagram of type AI.

Building on Lusztig's isomorphism [Lus81] between the nilpotent cone $\mathcal{N}_{\mathfrak{sl}_N}$ and the affine Grassmannian slice $\overline{\mathcal{W}}_0^{N\varpi_1^\vee}$, Mirković-Vybornov [MV22] established a general identification between nilpotent Slodowy slices and affine Grassmannian slices of type A: $\mathbb{O}_{\pi_1} \cap \mathcal{S}_{\pi_2} \cong \mathcal{W}^\lambda_\mu$, and $\overline{\mathbb{O}}_{\pi_1} \cap \mathcal{S}_{\pi_2} \cong \overline{\mathcal{W}}^\lambda_\mu$. It was mentioned in [MV22, Footnote 1] that "These observations clearly do not literally extend beyond type A". We offer a proper generalization of the MV isomorphism to classical type.

Consider an involution σ_ϵ of $\mathfrak{sl}_N$ sending $X \mapsto -J_\epsilon^{-1} X^T J_\epsilon$, where J_ϵ is any invertible $N \times N$ matrix which is symmetric if $\epsilon = +$ and skew-symmetric if $\epsilon = -$. The fixed point subalgebra $\mathfrak{sl}_N^\epsilon = (\mathfrak{sl}_N)^{\sigma_\epsilon}$ is $\mathfrak{so}_N$ if $\epsilon = +$, and $\mathfrak{sp}_N$ if $\epsilon = -$. By the classic results of Gerstenhaber and of Hesselink (see Proposition 9.1), intersecting the nilpotent cone, nilpotent orbits, and nilpotent orbit closures of $\mathfrak{sl}_N$ with $\mathfrak{sl}_N^\epsilon$ gives rise to the corresponding nilpotent cone, nilpotent orbits (if nonempty), and nilpotent orbit closures in the classical Lie algebra $\mathfrak{sl}_N^\epsilon$ compatibly.

Given an orthogonal/symplectic partition π_2 of N (for $\epsilon = +/-$), following [Top23, §4.2] we can choose J_ϵ in (9.2) for an involution σ_ϵ of $\mathfrak{sl}_N$ so that there exists an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{sl}_N^\epsilon$ (i.e., fixed pointwise by σ_ϵ) and e has Jordan form π_2. Then the involution σ_ϵ restricts to an involution on the Slodowy slice $\mathcal{S}_{\pi_2}$, and the corresponding σ_ϵ-fixed point locus can be identified with the Slodowy slice $\mathcal{S}^\epsilon_{\pi_2}$ in $\mathfrak{sl}_N^\epsilon$. In this way, we show that Mirković-Vybornov isomorphism is compatible with taking fixed point loci on both sides. We refer to (9.13) for notation $\mathrm{Par}(N)_{\leqslant n-1}$ and $\mathrm{Par}_\epsilon(N)^\diamond_{\leqslant n}$.

Theorem H (Theorem 9.6)**.** Let $\tau = \mathrm{id}$. Let $\lambda \geqslant \mu$ be dominant coweights for PGL_n with μ even, corresponding to partitions $\pi_1 \in \mathrm{Par}(N)_{\leqslant n-1}$ and $\pi_2 \in \mathrm{Par}_\epsilon(N)^\diamond_{\leqslant n}$ with $\pi_1 \trianglerighteq \pi_2$. Then we have Poisson isomorphisms

$$ {}^\imath\mathcal{W}^\lambda_\mu \cong \mathbb{O}^\epsilon_{\pi_1} \cap \mathcal{S}^\epsilon_{\pi_2}, \qquad {}^\imath\overline{\mathcal{W}}^\lambda_\mu \cong \overline{\mathbb{O}}^\epsilon_{\pi_1} \cap \mathcal{S}^\epsilon_{\pi_2}. $$

Moreover,

(1) ${}^\imath\mathcal{W}^\lambda_\mu$ is non-empty if and only if $\pi_1 \in \mathrm{Par}_\epsilon(N)$.
(2) The variety ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ is the closure of its stratum ${}^\imath\mathcal{W}^{\lambda'}_\mu \cong \mathbb{O}^\epsilon_{\pi_1'} \cap \mathcal{S}^\epsilon_{\pi_2}$, where the coweight λ' corresponds to the unique maximal element $\pi_1' \in \mathrm{Par}_\epsilon(N)$ satisfying $\pi_1 \trianglerighteq \pi_1'$.

It follows by Theorem H that the normalizations of affine Grassmannian islices ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ of type AI are always symplectic singularities; cf., e.g., [FJLS17, §1.2]; they may not always be irreducible though [KP82] (cf. [FJLS17, §1.6.2]).

A class of iquiver varieties was formulated by Li [Li19] as fixed point loci of Nakajima's quiver varieties for ADE quivers. Moreover, certain iquiver varieties (generalizing the cotangent bundles of flag varieties of type B/C; cf. [BKLW18]) are identified with involutive fixed point loci of nilpotent Slodowy slices of type A (which are claimed to be nilpotent Slodowy slices of type BCD if nonempty) and their partial resolutions [Li19]; this identification is built on the isomorphism between Nakajima quiver varieties and nilpotent Slodowy slices of type A (a conjecture of Nakajima [Nak94] proved by Maffei [Maf05]).

1.8. **iCoulomb branches.** Let $Q = (\mathbb{I}, \Omega)$ be an ADE quiver. Let $(V_i; W_i)_{i\in\mathbb{I}}$ be a representation of its framed double quiver $\overline{Q}^f$. Associated to such datum we have a symplectic vector space $E_{V,W}$ (10.1) with the action of a gauge group $GL(V)$ and a flavor symmetric group $GL(W)$. This gives rise to Coulomb branch of cotangent type $\mathcal{M}_C(V, W)$ [BFN19].

Let $(\mathbb{I}, \tau)$ be a Satake diagram as before. We impose τ-symmetry and "not-2-odd" parity conditions (10.4) on the dimension vectors of V, W, and fix a bipartite partition of $\mathbb{I}$ in (10.6). Given such datum, we associate with new vector spaces $V^\imath_i, W^\imath_i$, for $i \in {}^\imath\mathbb{I}$, in (10.8). We explain in details the $\imath$fication process on quivers and representations by reducing to the cases of the rank one and two Satake double quivers; see Tables 2 to 7 in Section 10 for diagrammatic illustrations.

Then we formulate a new symplectic vector space $E^\imath_{V,W}$ in (10.10) with the actions of a new gauge group $G^\imath(V^\imath)$ and a new flavor symmetry group $G^\imath(W^\imath)$; see Lemma 10.2, and we have a Coulomb branch $\mathcal{M}^\imath_C(V^\imath, W^\imath)$ (not of cotangent type in general) following [BDF+22]. The component groups of $G^\imath(V^\imath)$ and $G^\imath(W^\imath)$ are classical type; in case of split Satake diagrams, all component groups are orthogonal or symplectic. Recall that generalized affine Grassmannian slices $\overline{\mathcal{W}}^\lambda_\mu$ are realized as Coulomb branches $\mathcal{M}_C(V, W)$ [BFN19].

Conjecture I (Conjecture 10.9)**.**
(1) The iCoulomb branch $\mathcal{M}^\imath_C(V^\imath, W^\imath)$ is a normalization of a top-dimensional component of the affine Grassmannian islice ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$;
(2) Truncated shifted twisted Yangians are (subalgebras of) quantized Coulomb branches.

Here is some numerology: the rank vectors of the component groups of $G^\imath(V^\imath)$ and $G^\imath(W^\imath)$ are the dimension vectors $(\mathfrak{v}_i)_{i\in{}^\imath\mathbb{I}}$ and $(\mathfrak{w}_i)_{i\in{}^\imath\mathbb{I}}$ in (3.2)–(3.4) which arise from truncated shifted twisted Yangians and affine Grassmannian islices. In particular, the iCoulomb branch $\mathcal{M}^\imath_C(V^\imath, W^\imath)$ and the affine Grassmannian islice ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ have the same dimension.

1.9. **Other related works and some perspectives.** This paper provides a new framework where a family of algebras (shifted twisted Yangians and TSTY's) associated with quasi-split Satake diagrams quantize Poisson varieties which are naturally constructed. The ordinary (truncated) shifted Yangians and affine Grassmannian slices can be viewed to be associated with Satake diagrams of diagonal type. These new algebras are expected to admit rich representation theories which will be very interesting to develop. On the other hand, it will also be exciting to see if the normalizations of (top-dimensional components of) affine Grassmannian islices ${}^\imath\overline{\mathcal{W}}^\lambda_\mu$ provide new symplectic singularities. For type AI, these are indeed well-studied symplectic singularities thanks to the identification with nilpotent Slodowy slices of type BCD in Theorem H.

There are many papers in the mathematical physics literature on Coulomb branches with type A or non-type A gauge groups; see [GW09, CHK19] for samples and references therein. Finkelberg, Hanany and Nakajima [FHN25] have also been working on ortho-symplectic Coulomb branches and nilpotent Slodowy slices of classical type among other topics; we hope our $\imath$-perspectives on shifted twisted Yangians and affine Grassmannian islices can be complementary to theirs and those in math physics literature.

In [SSX25], the authors also formulated (a variant of) shifted twisted Yangians of type AIII_{2n} and defined a homomorphism from it to a Coulomb branch. Their paper provides a first example of Coulomb branches which relates precisely to shifted twisted Yangians (through iGKLO like ours), supporting our general proposal that truncated shifted twisted Yangians are intimately related to iCoulomb branches through iGKLO and affine Grassmannian islices. Note that the Coulomb branch of type AIII_{2n} *loc. cit.* is the only one among all iCoulomb branches which is of cotangent type and has a purely type A gauge group.

The $\imath$fication process often leads to type BCD features. The icanonical bases arising from iquantum groups of quasi-split type AIII have played a fundamental role in Kazhdan-Lusztig theory of type BCD [BW18], and these (affine) iquantum groups admit a geometric realization via flag varieties of classical type in [BKLW18, FLL$^+$20, SW24]; Theorem H on nilpotent Slodowy slices is a new example. The root of such type BCD phenomenon can be traced down to the split and quasi-split rank one iquantum groups. The geometric realization of twisted Yangians in this paper, which applies to all (quasi-split) ADE type, appears in very different forms. In light of the conjectural iCoulomb branch connection, the $\imath$-fication manifests itself again in the component gauge groups being type (A)BCD. This should be contrasted with the fact that the gauge groups are always of type A for the quiver gauge theories giving rise to Nakajima quiver varieties as Higgs branches and the corresponding Coulomb branches [Nak94, BFN19].

The present paper is also naturally related to integrable systems. Indeed, the algebras ${}^\imath\mathcal{Y}_\mu^\lambda$ come equipped with natural commutative subalgebras, defined in §3.4.1, which we expect to be quantum integrable systems. The classical limits of these subalgebras give integrable systems in many examples, and we conjecture that this is always the case. In particular, in the setting of Theorem H we obtain integrable systems on the varieties $\mathbb{O}_{\pi_1}^\epsilon \cap \mathcal{S}_{\pi_2}^\epsilon$. We are unaware if these have been studied previously in general, though some cases are related to work of Harada [Har06] who constructed integrable systems on generic coadjoint orbits for symplectic Lie algebras.

It is natural to expect a q-deformation of the main constructions in this paper. Drinfeld presentations of quasi-split affine iquantum groups have been constructed in [LW21, LWZ24]. We can formulate the shifted affine iquantum groups of split and quasi-split types accordingly (compare [FT19a]), construct their iGKLO representations and thus define the truncated shifted affine iquantum groups. These are then expected to be related to the K-theory of affine Grassmannian islices and iCoulomb branches.

The $\imath$-fication framework can be further enlarged. Some main constructions in this paper will be extended beyond quasi-split Satake diagrams in a sequel [LWW25]. In particular, we shall identify the affine Grassmannian islices of type AII and others with nilpotent Slodowy slices of classical types, complementary to the results in Section 9.

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Part 1. Algebra

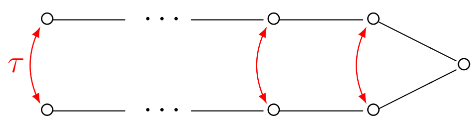
2. Shifted twisted Yangians

In this section, we introduce shifted twisted Yangians ${}^\imath\mathcal{Y}_\mu$ associated to all quasi-split ADE Satake diagrams $(\mathbb{I}, \tau)$ and even spherical coweights μ. We further establish their PBW bases and shift morphisms between shifted twisted Yangians.

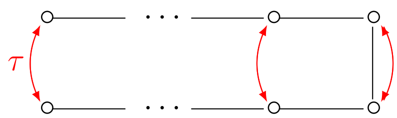
2.1. Shifted twisted Yangians of quasi-split ADE type.

Let $C = (c_{ij})_{i,j\in\mathbb{I}}$ be the Cartan matrix of type ADE, and let $\mathfrak{g}$ be the corresponding simple Lie algebra. We fix a simple system $\{\alpha_i \mid i \in \mathbb{I}\}$ with corresponding set Δ^+ of positive roots. Let τ be an involution of the Dynkin diagram of $\mathfrak{g}$, i.e., $c_{ij} = c_{\tau i,\tau j}$ such that $\tau^2 = \mathrm{id}$; note that $\tau = \mathrm{id}$ is allowed. We refer to $(\mathbb{I}, \tau)$ as quasi-split Satake diagrams and call the Satake diagrams split if $\tau = \mathrm{id}$. The split Satake diagrams formally look identical to Dynkin diagrams, and the quasi-split Satake diagrams $(\mathbb{I}, \tau)$ with $\tau \neq \mathrm{id}$ are listed in Table 1, where the index denotes the number of nodes in a given diagram.

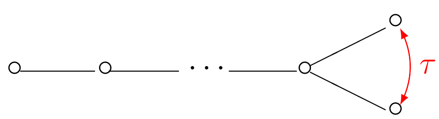
$\mathrm{AIII}_{2r-1}\ (r \geqslant 1)$



$\mathrm{AIII}_{2r}\ (r \geqslant 1)$



$\mathrm{DI}_r\ (r \geqslant 4)$



EII_6

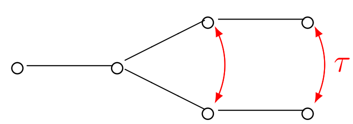


Table 1. Quasi-split Satake diagrams with $\tau \neq \mathrm{Id}$

Denote $\mathbb{I}_0$ the set of fixed points of τ in $\mathbb{I}$, i.e., $\mathbb{I}_0 = \{i \in \mathbb{I} \mid \tau i = i\}$. Let $\mathbb{I}_1$ be a set of representatives for τ-orbits in $\mathbb{I}$ of length 2 and define $\mathbb{I}_{-1} = \tau\mathbb{I}_1$; $\mathbb{I}_1$ can be conveniently chosen so that it is underlying Dynkin subdiagram is connected. Then $\mathbb{I} = \mathbb{I}_1 \sqcup \mathbb{I}_0 \sqcup \mathbb{I}_{-1}$. Set

$$ {}^\imath\mathbb{I} = \mathbb{I}_1 \sqcup \mathbb{I}_0. \tag{2.1}$$

The involution τ naturally acts on the (co)root and (co)weight lattices of $\mathfrak{g}$. A weight/coweight in this paper is always meant to be integral.

Definition 2.1. A coweight μ is called *spherical* if $\mu = \mu_1 + \tau\mu_1$ for some coweight μ_1. A coweight μ is *even* if $\mu = 2\mu'$ for some coweight μ', i.e., $\langle \mu, \alpha_i\rangle \in 2\mathbb{Z}$ for all $i \in \mathbb{I}$.

We shall need to assume that a shift coweight μ to be even spherical in the context of twisted Yangians below, and remarkably, the same condition will be need in consideration of fixed point loci of affine Grassmannian slices later on.

Denote $[A, B]_+ = AB + BA$.

Definition 2.2. Let μ be an even spherical coweight. The *shifted twisted Yangians* ${}^\imath\mathcal{Y}_\mu := {}^\imath\mathcal{Y}_\mu(\mathfrak{g})$ *of quasi-split type* is the $\mathbb{C}$-algebra with generators $H_i^{(r)}$, $B_i^{(s)}$, for $i \in \mathbb{I}$, $r \in \mathbb{Z}$, and $s \in \mathbb{Z}_{>0}$, subject to the following relations, for $r, r_1, r_2 \in \mathbb{Z}$ and $s, s_1, s_2 \in \mathbb{Z}_{>0}$:

$$H_i^{(r)} = 0 \ \text{ for } \ r < -\langle \mu, \alpha_i\rangle, \quad H_i^{(-\langle \mu,\alpha_i\rangle)} = 1, \tag{2.2}$$

$$[H_i^{(r_1)}, H_j^{(r_2)}] = 0, \tag{2.3}$$

$$\begin{aligned}[H_i^{(r+2)}, B_j^{(s)}] - [H_i^{(r)}, B_j^{(s+2)}] = \frac{c_{ij} - c_{\tau i,j}}{2}[H_i^{(r+1)}, B_j^{(s)}]_+ \\ + \frac{c_{ij} + c_{\tau i,j}}{2}[H_i^{(r)}, B_j^{(s+1)}]_+ + \frac{c_{ij}c_{\tau i,j}}{4}[H_i^{(r)}, B_j^{(s)}],\end{aligned} \tag{2.4}$$

$$[B_i^{(s_1+1)}, B_j^{(s_2)}] - [B_i^{(s_1)}, B_j^{(s_2+1)}] = \frac{c_{ij}}{2}[B_i^{(s_1)}, B_j^{(s_2)}]_+ + 2\delta_{\tau i,j}(-1)^{s_1} H_j^{(s_1+s_2)}, \tag{2.5}$$

and the Serre relations: for $c_{ij} = 0$,

$$[B_i^{(s_1)}, B_j^{(s_2)}] = (-1)^{s_1-1}\delta_{\tau i,j} H_j^{(s_1+s_2-1)}, \tag{2.6}$$

and for $c_{ij} = -1$, $i \neq \tau i \neq j$,

$$\mathrm{Sym}_{s_1,s_2}\big[B_i^{(s_1)}, [B_i^{(s_2)}, B_j^{(s)}]\big] = 0, \tag{2.7}$$

and for $c_{ij} = -1$, $i = \tau i$,

$$\mathrm{Sym}_{s_1,s_2}\big[B_i^{(s_1)}, [B_i^{(s_2)}, B_j^{(s)}]\big] = (-1)^{s_1-1}[H_i^{(s_1+s_2)}, B_j^{(s-1)}], \tag{2.8}$$

and for $c_{i,\tau i} = -1$,

$$\mathrm{Sym}_{s_1,s_2}\big[B_i^{(s_1)}, [B_i^{(s_2)}, B_{\tau i}^{(s)}]\big] = 4\,\mathrm{Sym}_{s_1,s_2}(-1)^{s_1-1}\sum_{p\geqslant 0} 3^{-p-1}[B_i^{(s_2+p)}, H_{\tau i}^{(s_1+s-p-1)}]. \tag{2.9}$$

Here, if $c_{ij} = -1$ and $s = 1$, we use the following convention in (2.8):

$$[H_i^{(r)}, B_j^{(s-1)}] = \sum_{p\geqslant 0} 2^{-2p}\big([H_i^{(r-2p-2)}, B_j^{(s+1)}] - [H_i^{(r-2p-2)}, B_j^{(s)}]_+\big),$$

which follows from (2.18) if $s > 1$, cf. [LWZ25b, Lem. 4.11].

Remark 2.3. If $\tau = \mathrm{id}$, then a spherical coweight μ is automatically even. In the quasi-split case, it is possible to define shifted twisted Yangians without the evenness condition for μ by modifying the second relation in (2.2) to $H_i^{(-\langle \mu,\alpha_i\rangle)} = (-1)^{\langle \mu_1,\alpha_{\tau i}\rangle}$, where $\mu = \mu_1 + \tau\mu_1$. Then Theorems A and B (on PBW basis and iGKLO representations) still hold. We restrict ourselves to even μ to match the geometric consideration in the second part of this paper. It is also possible to modify the affine Grassmannian setting to match with this variation, but we have refrained from taking this route.

Since τ is an involution, it follows from (2.6) that

$$H_i^{(r)} = (-1)^r H_{\tau i}^{(r)}, \tag{2.10}$$

and in particular,

$$H_i^{(2r+1)} = 0, \qquad \text{if } \tau i = i. \tag{2.11}$$

Lemma 2.4 ([LZ24, Lemma 3.4])**.** *If $c_{\tau i,j} = 0$, then the relation (2.4) is equivalent to*

$$[H_{i,r+1}, B_{j,s}] - [H_{i,r}, B_{j,s+1}] = \frac{c_{ij}}{2}[H_{i,r}, B_{j,s}]_+. \tag{2.12}$$

Lemma 2.5. *Assume that relations (2.2)–(2.5) hold in an algebra with the same set of generators as ${}^\imath\mathcal{Y}_\mu$.*

(1) *If (2.8) holds only for the special case $s = s_1 = 1$ and all $s_2 > 0$, then (2.8) holds for all $s, s_1, s_2 \in \mathbb{Z}_{>0}$.*
(2) *If (2.9) holds for the special case $s = s_1 = s_2 = 1$, then (2.9) holds for all $s, s_1, s_2 \in \mathbb{Z}_{>0}$.*

Proof. Part (1) follows by the same argument of [LWZ25b, §4.3] while Part (2) follows by the same argument of [LZ24, Proposition 3.12]. □

Definition 2.6. The Cartan doubled twisted Yangian ${}^\imath\mathcal{Y}_\infty$ of quasi-split type is the $\mathbb{C}$-algebra with generators $H_i^{(r)}$, $B_i^{(s)}$, for $i \in \mathbb{I}$, $r \in \mathbb{Z}$, and $s \in \mathbb{Z}_{>0}$ subject only to the relations (2.3)–(2.8) (i.e., excluding (2.2)).

Let Q be the root lattice for $\mathfrak{g}$. Consider the abelian group (called the $\imath$root lattice)

$$Q^\imath = Q/\langle \beta + \tau\beta \mid \beta \in Q\rangle. \tag{2.13}$$

For $\beta \in Q$, denote its image by $\overline{\beta} \in Q^\imath$. The algebra ${}^\imath\mathcal{Y}_\mu$ admits a grading by $Q^\imath$ defined by assigning degrees $\deg H_i^{(r)} = \overline{0}$ and $\deg B_i^{(s)} = \overline{\alpha_i}$. Indeed, it is not hard to see that the defining relations of ${}^\imath\mathcal{Y}_\mu$ are all homogeneous. Let $T = \operatorname{Spec}\mathbb{C}[Q]$ denote the torus whose character lattice is Q. Then the grading on ${}^\imath\mathcal{Y}_\mu$ by $Q^\imath$ corresponds to an action of a subgroup $T^\imath$ of T:

$$T^\imath = \{t \in T \mid \tau(t) = t^{-1}\} = \operatorname{Spec}\mathbb{C}[Q^\imath]. \tag{2.14}$$

Since τ permutes the basis $\{\alpha_i\}_{i\in\mathbb{I}}$ for Q, we can identify $Q^\imath \cong \mathbb{Z}^{\mathbb{I}_1} \times (\mathbb{Z}/2\mathbb{Z})^{\mathbb{I}_0}$, where the $\mathbb{Z}^{\mathbb{I}_1}$-factor has basis $\{\overline{\alpha_i}\}_{i\in\mathbb{I}_1}$ while $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{I}_0}$ is generated by $\{\overline{\alpha_i}\}_{i\in\mathbb{I}_0}$. If we quotient $Q^\imath$ by its torsion subgroup $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{I}_0}$, we obtain an induced grading on ${}^\imath\mathcal{Y}_\mu$ by $\mathbb{Z}^{\mathbb{I}_1}$. This corresponds to the adjoint action of the elements $\{H_i^{(-\langle\mu,\alpha_i\rangle+1)} \mid i \in \mathbb{I}_1\}$.

Lemma 2.7. *Let μ, ν be coweights such that both μ and $\nu + \tau\nu$ are even spherical. Suppose further that ν is anti-dominant. Then there exists a homomorphism*

$$\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{Y}_\mu \longrightarrow {}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu} \tag{2.15}$$

defined by

$$H_i^{(r)} \mapsto H_i^{(r-\langle\nu+\tau\nu,\alpha_i\rangle)}, \qquad B_i^{(s)} \mapsto \begin{cases} B_i^{(s-\langle\nu,\alpha_i\rangle)}, & \text{if } \langle\nu,\alpha_i\rangle \text{ is even},\\ \sqrt{-1}B_i^{(s-\langle\nu,\alpha_i\rangle)}, & \text{if } \langle\nu,\alpha_i\rangle \text{ is odd},\end{cases}$$

for $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{>0}$.

In particular, if $\tau = \mathrm{id}$, we call ${}^\imath\mathcal{Y}_\mu$ the *shifted twisted Yangian of split type*.

Remark 2.8. If $\mu = 0$, then ${}^\imath\mathcal{Y} := {}^\imath\mathcal{Y}_0$ is exactly the twisted Yangian introduced in [LWZ25a] (for $\tau = \mathrm{id}$) and in [LZ24] (for $\tau \neq \mathrm{id}$) via degeneration of Drinfeld presentations of affine $\imath$quantum groups of split and quasi-split types (cf. [LW21, LWZ24]). The definition of ${}^\imath\mathcal{Y}_\mu$ formally makes sense for simply-laced generalized Cartan matrix C as suggested in [LWZ25b].

If μ is dominant and spherical, i.e., $\mu = \mu_1 + \tau\mu_1$ for some coweight μ_1, then we have the homomorphism

$$\iota^\tau_{\mu,-\mu_1} : {}^\imath\mathcal{Y}_\mu \longrightarrow {}^\imath\mathcal{Y}.$$

By the PBW theorem established below, the homomorphism $\iota^\tau_{\mu,-\mu_1}$ is injective and hence identifies ${}^\imath\mathcal{Y}_\mu$ as a subalgebra of ${}^\imath\mathcal{Y}$; see Theorem 2.16.

For $i \in \mathbb{I}$, we set

$$B_i(u) = \sum_{r>0} B_i^{(r)} u^{-r}.$$

Remark 2.9. For fixed $i \in \mathbb{I}$ such that $c_{i,\tau i} = 0$, the subalgebra of ${}^\imath\mathcal{Y}_\mu$ generated by $H_i^{(r)}, B_i^{(s)}, B_{\tau i}^{(s)}$, $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{>0}$ is isomorphic (using Lemma 2.4 and Theorem 2.16) to a shifted Yangian for $\mathfrak{sl}_2$. Specifically, the identification is given as follows,

$$e(u) \to B_i(u), \quad f(u) \to B_{\tau i}(-u), \quad h(u) \to H_i(u).$$

If $i = \tau i$, then $H_i^{(r)}, B_i^{(s)}$, $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{>0}$ generate a subalgebra of ${}^\imath\mathcal{Y}_\mu$ that is isomorphic to a shifted twisted Yangian of split type A of rank 1.

For split type, Definition 2.2 greatly simplifies and it is convenient to list it separately as follows.

Definition 2.10. The *shifted twisted Yangian* ${}^\imath\mathcal{Y}_\mu = {}^\imath\mathcal{Y}_\mu(\mathfrak{g})$ *of split type* is the $\mathbb{C}$-algebra with generators $H_i^{(r)}$, $B_i^{(s)}$, for $i \in \mathbb{I}$, $r \in \mathbb{Z}$, and $s \in \mathbb{Z}_{>0}$, subject to the following relations, for $r, r_1, r_2 \in \mathbb{Z}$ and $s, s_1, s_2 \in \mathbb{Z}_{>0}$:

$$H_i^{(p)} = 0 \ \text{ for } \ p < -\langle \mu, \alpha_i \rangle \ \text{ and } \ H_i^{(-\langle \mu, \alpha_i \rangle)} = 1, \tag{2.16}$$

$$[H_i^{(r_1)}, H_j^{(r_2)}] = 0, \qquad H_i^{(2r+1)} = 0, \tag{2.17}$$

$$[H_i^{(r+2)}, B_j^{(s)}] - [H_i^{(r)}, B_j^{(s+2)}] = c_{ij}[H_i^{(r)}, B_j^{(s+1)}]_+ + \frac{1}{4}c_{ij}^2[H_i^{(r)}, B_j^{(s)}], \tag{2.18}$$

$$[B_i^{(s_1+1)}, B_j^{(s_2)}] - [B_i^{(s_1)}, B_j^{(s_2+1)}] = \frac{1}{2}c_{ij}[B_i^{(s_1)}, B_j^{(s_2)}]_+ + 2\delta_{ij}(-1)^{s_1} H_i^{(s_1+s_2)}, \tag{2.19}$$

$$[B_i^{(s_1)}, B_j^{(s_2)}] = 0, \qquad \text{if } c_{ij} = 0, \tag{2.20}$$

$$\mathrm{Sym}_{s_1,s_2}\big[B_i^{(s_1)}, [B_i^{(s_2)}, B_j^{(s)}]\big] = (-1)^{s_1-1}[H_i^{(s_1+s_2)}, B_j^{(s-1)}], \qquad \text{if } c_{ij} = -1. \tag{2.21}$$

Lemma 2.11. *Let μ be an even coweight and ν be an anti-dominant coweight. Then there exists a homomorphism $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{Y}_\mu \to {}^\imath\mathcal{Y}_{\mu+2\nu}$ defined by*

$$H_i^{(r)} \mapsto H_i^{(r-2\langle \nu, \alpha_i \rangle)}, \qquad B_i^{(s)} \mapsto \begin{cases} B_i^{(s-\langle \nu, \alpha_i \rangle)}, & \textit{if } \langle \nu, \alpha_i \rangle \textit{ is even}, \\ \sqrt{-1} B_i^{(s-\langle \nu, \alpha_i \rangle)}, & \textit{if } \langle \nu, \alpha_i \rangle \textit{ is odd}, \end{cases}$$

where $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{>0}$.

Remark 2.12. For $\mu = 0$, ${}^\imath\mathcal{Y}_0$ is exactly the twisted Yangian introduced in [LWZ25a] via degeneration of affine $\imath$quantum groups of split types. Similar to Remark 2.8, if μ is dominant, then ${}^\imath\mathcal{Y}_\mu$ can be identified as a subalgebra of ${}^\imath\mathcal{Y}_0$.

2.2. **PBW basis.** Consider a positive root β and fix an arbitrary ordered decomposition $\beta = \alpha_{i_1} + \ldots + \alpha_{i_\ell}$ into simple roots such that the elements $[e_{i_1}, [e_{i_2}, \cdots [e_{i_{\ell-1}}, e_{i_\ell}] \cdots]]$ is a nonzero element in the root subspace $\mathfrak{g}_\beta$. For any $r > 0$, we define a root vector in $\mathcal{Y}_\mu(\mathfrak{g})$

$$B_\beta^{(r)} := \Big[B_{i_1}^{(r)}, [B_{i_2}^{(1)}, \cdots [B_{i_{\ell-1}}^{(1)}, B_{i_\ell}^{(1)}] \cdots]\Big].$$

Fix a total order on elements of the following sets,

$$\begin{aligned} &\big\{B_\beta^{(r)} : \beta \in \Delta^+, r > 0\big\} \\ &\qquad \cup \big\{H_i^{(2p)} : i \in \mathbb{I}_0, 2p > -\langle \mu, \alpha_i\rangle\big\} \\ &\qquad\qquad \cup \big\{H_i^{(p)} : i \in \mathbb{I}_1, p > -\langle \mu, \alpha_i\rangle\big\}, \end{aligned} \tag{2.22}$$

and consider the corresponding ordered monomials in (2.22).

Proposition 2.13. *The shifted twisted Yangian ${}^\imath\mathcal{Y}_\mu$ is spanned by ordered monomials in the elements* (2.22)*.*

Proof. For the simplicity of notation, in this proof, we consider an order with respect to the subsets (2.22), i.e. the elements $H_i^{(s)}$ are always to the right of the elements $B_\beta^{(r)}$ in the ordered monomials.

By (2.4), it is easy to see by induction on r that $H_i^{(r)} B_j^{(s)}$ can be rewritten as $H_i^{(r)} B_j^{(s)} = B_j^{(s)} H_i^{(r)} + \sum_{p,p' :\, p+p' < r+s} m_{p,p'} B_j^{(p')} H_i^{(p)}$, for $m_{p,p'} \in \mathbb{C}$. Thus the algebra ${}^\imath\mathcal{Y}_\mu$ is spanned by elements of the form $B_{j_1}^{(n_1)} B_{j_1}^{(n_2)} \cdots B_{j_k}^{(n_k)} H_{i_1}^{(m_1)} H_{i_2}^{(m_2)} \cdots H_{i_\ell}^{(m_\ell)}$.

Next, we define a filtration on ${}^\imath\mathcal{Y}_\mu$ by setting $\deg B_j^{(s)} = 1$ and $\deg H_i^{(r)} = 0$. Let $\mathrm{gr}\, {}^\imath\mathcal{Y}_\mu$ be the associated grade and $\overline{B}_j^{(s)}$ (resp. $\overline{B}_\beta^{(r)}$) the corresponding image of $B_j^{(s)}$ (resp. $B_\beta^{(r)}$). Let $\mathrm{gr}\, {}^\imath\mathcal{Y}_\mu^{>0}$ be the subalgebra of $\mathrm{gr}\, {}^\imath\mathcal{Y}_\mu$ generated by $\overline{B}_j^{(s)}$. One can argue as in [Lu24, proof of Theorem 4.2] that the ordered monomials in $\overline{B}_\beta^{(r)}$ span $\mathrm{gr}\, {}^\imath\mathcal{Y}_\mu^{>0}$. This is because $\overline{B}_j^{(s)}$ satisfy the relations for the positive half of the corresponding ordinary Yangians and hence the claims follows from the corresponding result of the ordinary Yangians, see [Lev93]. Altogether, it proves the proposition. □

Remark 2.14. For $\mu = 0$, the set of ordered monomials in the elements (2.22) forms a basis for ${}^\imath\mathcal{Y}_0$, see [LWZ25a, Theorem 4.12] and [LZ24, Theorem 3.6].

Theorem 2.15. *Let μ be even spherical and anti-dominant. Then the set of ordered monomials in the elements* (2.22) *forms a basis for ${}^\imath\mathcal{Y}_\mu$.*

The proof of Theorem 2.15 will be given in §2.3 below. We can strengthen Theorem 2.15 as follows.

Theorem 2.16 (PBW basis)**.** *Let μ be an arbitrary even spherical coweight.*

(1) *The set of ordered monomials in the root vectors* (2.22) *forms a basis for ${}^\imath\mathcal{Y}_\mu$.*

(2) *For any anti-dominant* ν *such that* $\nu + \tau\nu$ *is even spherical, the shift homomorphism* $\iota^{\tau}_{\mu,\nu} : {}^{\imath}\mathcal{Y}_\mu \to {}^{\imath}\mathcal{Y}_{\mu+\nu+\tau\nu}$ *in* (2.15) *is injective.*

Proof. By Lemma 2.7, we have a shift homomorphism $\iota^{\tau}_{\mu,\nu} : {}^{\imath}\mathcal{Y}_\mu \to {}^{\imath}\mathcal{Y}_{\mu+\nu+\tau\nu}$. We pick ν so that $\mu + \nu + \tau\nu$ is anti-dominant. Then the root vectors of ${}^{\imath}\mathcal{Y}_\mu$ are sent to root vectors (up to a scalar multiple) of ${}^{\imath}\mathcal{Y}_{\mu+\nu+\tau\nu}$ under the shift homomorphism $\iota^{\tau}_{\mu,\nu}$. By Proposition 2.13, the set of ordered monomials in the root vectors of ${}^{\imath}\mathcal{Y}_\mu$ is a spanning set of ${}^{\imath}\mathcal{Y}_\mu$. By Theorem 2.15, the set is bijectively sent to a set of linearly independent ordered monomials (in the root vectors) in ${}^{\imath}\mathcal{Y}_{\mu+\nu+\tau\nu}$. Hence this spanning set is also linearly independent and thus form a basis of ${}^{\imath}\mathcal{Y}_\mu$, proving the first statement. The second statement follows from the first statement. □

2.3. **Proof of PBW basis Theorem 2.15.** We shall follow the strategy of [FKP+18, §3.12]. Recall ${}^{\imath}\mathcal{Y}_\infty$ from Definition 2.6.

Definition 2.17. The algebra ${}^{\imath}\widetilde{\mathcal{Y}}$ is the quotient of ${}^{\imath}\mathcal{Y}_\infty$ by the relations $H_i^{(p)} = 0$ for all $i \in \mathbb{I}$ and $p < 0$.

We use the notation $\widetilde{H}_i^{(r)}$ and $\widetilde{B}_i^{(s)}$ (also $\widetilde{B}_\beta^{(s)}$) for the generators of ${}^{\imath}\widetilde{\mathcal{Y}}$.

Lemma 2.18. *The ordered monomials in the elements of the set*

$$\big\{\widetilde{B}_\beta^{(r)} \mid \beta \in \Delta^+, r > 0\big\} \cup \big\{\widetilde{H}_i^{(2p)} \mid i \in \mathbb{I}_0, p \geqslant 0\big\} \cup \big\{\widetilde{H}_i^{(p)} \mid i \in \mathbb{I}_1, p \geqslant 0\big\}, \tag{2.23}$$

form a basis for ${}^{\imath}\widetilde{\mathcal{Y}}$.

Proof. Clearly, the elements $\widetilde{H}_i^{(0)}$ for $i \in \mathbb{I}$ are central in ${}^{\imath}\widetilde{\mathcal{Y}}$. One checks that the map

$${}^{\imath}\widetilde{\mathcal{Y}} \to {}^{\imath}\mathcal{Y} \otimes_{\mathbb{C}} \mathbb{C}[\xi_i \mid i \in \mathbb{I}], \quad \text{defined by} \quad \widetilde{H}_i^{(r)} \mapsto H_i^{(r)} \otimes \xi_i\xi_{\tau i}, \quad \widetilde{B}_i^{(r)} \mapsto B_i^{(r)} \otimes \xi_i,$$

induces an algebra homomorphism. Then proceed as Proposition 2.13, one finds that the ordered monomials in the elements of (2.23) span ${}^{\imath}\widetilde{\mathcal{Y}}$. Finally, similar to Theorem 2.16, we show that this spanning set is bijectively sent to a subset of a basis for ${}^{\imath}\mathcal{Y} \otimes_{\mathbb{C}} \mathbb{C}[\xi_i : i \in \mathbb{I}]$. Here we also need the PBW theorem for ${}^{\imath}\mathcal{Y}$ (see Remark 2.14). Thus the spanning set is linearly independent and hence a basis. □

Corollary 2.19. *If* μ *is anti-dominant, then* ${}^{\imath}\widetilde{\mathcal{Y}}$ *is free as a right module over the polynomial ring*

$$\mathcal{R} := \mathbb{C}[\widetilde{H}_i^{(2p)}, \widetilde{H}_j^{(q)} \mid i \in \mathbb{I}_0, 0 \leqslant 2p \leqslant -\langle\mu, \alpha_i\rangle, j \in \mathbb{I}_1, 0 \leqslant q \leqslant -\langle\mu, \alpha_j\rangle] \tag{2.24}$$

with a basis given by ordered monomials in the elements of the set

$$\big\{\widetilde{B}_\beta^{(r)} \mid \beta \in \Delta^+, r > 0\big\} \cup \big\{\widetilde{H}_i^{(2p)} \mid i \in \mathbb{I}_0, 2p > -\langle\mu, \alpha_i\rangle\big\} \cup \big\{\widetilde{H}_i^{(p)} \mid i \in \mathbb{I}_1, p > -\langle\mu, \alpha_i\rangle\big\}.$$

Suppose further that μ is anti-dominant, then there is a surjective homomorphism $\pi : {}^{\imath}\widetilde{\mathcal{Y}} \twoheadrightarrow {}^{\imath}\mathcal{Y}_\mu$ defined by

$$\widetilde{H}_i^{(r)} \to H_i^{(r)}, \qquad \widetilde{B}_i^{(s)} \to B_i^{(s)}.$$

The kernel of the projection π is the 2-sided ideal

$$\mathcal{I}_\mu := \big\langle \widetilde{H}_i^{(2r)} - \delta_{2r,-\langle\mu,\alpha_i\rangle}, \widetilde{H}_j^{(p)} - \delta_{p,-\langle\mu,\alpha_j\rangle} \big\rangle_{2-\text{sided}}, \tag{2.25}$$

where $i \in \mathbb{I}_0$, $0 \leqslant 2r \leqslant -\langle\mu, \alpha_i\rangle$, $j \in \mathbb{I}_1$, $0 \leqslant p \leqslant -\langle\mu, \alpha_j\rangle$.

Also denote by $\mathcal{I}_\mu^{\text{left}}$ the left ideal generated by the same elements in (2.25).

Lemma 2.20. *We have* $\mathcal{I}_\mu = \mathcal{I}_\mu^{\text{left}}$.

Proof. It is sufficient to prove that $\mathcal{I}_\mu^{\text{left}}$ is also a right ideal.

Recall that in the algebra ${}^\imath\widetilde{\mathcal{Y}}$, if $i \in \mathbb{I}_0$, then we have the following relations

$$\widetilde{H}_i^{(2r-1)} = 0, \qquad [\widetilde{H}_i^{(0)}, \widetilde{B}_j^{(s)}] = 0, \qquad [\widetilde{H}_i^{(2)}, \widetilde{B}_j^{(s)}] = 2c_{ij}B_j^{(s+1)}\widetilde{H}_i^{(0)},$$

$$[\widetilde{H}_i^{(r+2)}, \widetilde{B}_j^{(s)}] = [\widetilde{H}_i^{(r)}, \widetilde{B}_j^{(s+2)}] + c_{ij}[\widetilde{H}_i^{(r)}, \widetilde{B}_j^{(s+1)}] + \frac{1}{4}c_{ij}^2[\widetilde{H}_i^{(r)}, \widetilde{B}_j^{(s)}] + 2c_{ij}\widetilde{B}_j^{(s+1)}\widetilde{H}_i^{(r)}.$$

Thus an obvious induction on r shows that for any $r \geqslant 1$, we have

$$[\widetilde{H}_i^{(r)}, \widetilde{B}_j^{(s)}] \in \big\langle \widetilde{H}_i^{(0)}, \widetilde{H}_i^{(1)}, \ldots, \widetilde{H}_i^{(r-1)} \big\rangle^{\text{left}}.$$

Note that here $\widetilde{H}_i^{(p)} = 0$ if p is odd. Therefore, for $1 \leqslant 2r \leqslant -\langle \mu, \alpha_i \rangle$, we have

$$(\widetilde{H}_i^{(2r)} - \delta_{2r,-\langle\mu,\alpha_i\rangle})\widetilde{B}_j^{(s)} \in \widetilde{B}_j^{(s)}(\widetilde{H}_i^{(2r)} - \delta_{2r,-\langle\mu,\alpha_i\rangle}) + \mathcal{I}_\mu^{\text{left}} = \mathcal{I}_\mu^{\text{left}}.$$

A similar calculation also works for $i \in \mathbb{I}_1$. It implies that the right multiplication by $\widetilde{B}_j^{(s)}$ preserves $\mathcal{I}_\mu^{\text{left}}$. It is also clear that right multiplication by $\widetilde{H}_j^{(r)}$ preserves $\mathcal{I}_\mu^{\text{left}}$. Since the algebra ${}^\imath\widetilde{\mathcal{Y}}$ is generated by $\widetilde{B}_j^{(s)}$ and $\widetilde{H}_j^{(r)}$, we conclude that $\mathcal{I}_\mu^{\text{left}}$ is a right ideal. □

Proof of Theorem 2.15. Let $\Gamma : \mathcal{R} \to \mathbb{C}$ be the homomorphism defined by sending $\widetilde{H}_i^{(2p)} - \delta_{2p,-\langle\mu,\alpha_i\rangle}$ and $\widetilde{H}_j^{(q)} - \delta_{q,-\langle\mu,\alpha_j\rangle}$ to 0, for $i \in \mathbb{I}_0$, $0 \leqslant 2p \leqslant -\langle\mu,\alpha_i\rangle$, $j \in \mathbb{I}_1$, and $0 \leqslant q \leqslant -\langle\mu,\alpha_j\rangle$. It follows from Lemma 2.20 that ${}^\imath\mathcal{Y}_\mu$ is the base change of the right module ${}^\imath\widetilde{\mathcal{Y}}$ with respect to the homomorphism Γ,

$${}^\imath\mathcal{Y}_\mu = {}^\imath\widetilde{\mathcal{Y}} \otimes_{\mathcal{R}} \mathbb{C}.$$

By Corollary 2.19, ${}^\imath\widetilde{\mathcal{Y}}$ is a free right module over $\mathcal{R}$. Therefore, the basis from Corollary 2.19 gives rise to a basis for ${}^\imath\mathcal{Y}_\mu$ over $\mathbb{C}$, completing the proof. □

2.4. **Filtrations on shifted twisted Yangians.** We shall define filtrations on shifted twisted Yangians ${}^\imath\mathcal{Y}_\mu$ via the PBW bases introduced in Theorem 2.15. (These are analogous to the filtrations $F^\bullet_{\mu_1,\mu_2}Y_\mu$ on shifted Yangians [FKP+18, §5.4] discussed just before Proposition 7.1.)

The definition is uniform for all quasi-split cases: choose any decomposition $\mu = \mu_1 + \tau\mu_1$, where μ_1 is a coweight. Then we may define an increasing filtration $F^\bullet_{\mu_1}{}^\imath\mathcal{Y}_\mu$ on ${}^\imath\mathcal{Y}_\mu$ by defining the degrees of PBW generators as follows:

$$\deg H_i^{(r)} = r + \langle \mu, \alpha_i \rangle, \qquad \deg B_\beta^{(s)} = s + \langle \mu_1, \beta \rangle. \tag{2.26}$$

That is, for each $k \in \mathbb{Z}$ we define the kth filtered piece $F^k_{\mu_1}({}^\imath\mathcal{Y}_\mu)$ to be the span of all ordered monomials in the PBW generators whose total degree is at most k. This defines a separated and exhaustive filtration on the vector space ${}^\imath\mathcal{Y}_\mu$, i.e., $\bigcap_k F^k_{\mu_1}({}^\imath\mathcal{Y}_\mu) = 0$ and $\bigcup_k F^k_{\mu_1}({}^\imath\mathcal{Y}_\mu) = {}^\imath\mathcal{Y}_\mu$. The following nontrivial claim will be established below in Proposition 7.14:

Proposition 2.21. $F^\bullet_{\mu_1}{}^\imath\mathcal{Y}_\mu$ *is an algebra filtration, is independent of the choice of PBW basis, and the associated graded algebra* $\operatorname{gr}^{F^\bullet_{\mu_1}}{}^\imath\mathcal{Y}_\mu$ *is commutative.*

Since these filtrations are defined in terms of PBW basis vectors, we note a simple consequence of Theorem 2.16.

Lemma 2.22. *Let μ be an even spherical coweight and fix a decomposition $\mu = \mu_1 + \tau\mu_1$. For any anti-dominant coweight ν such that $\nu + \tau\nu$ is even, the shift homomorphism $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{Y}_\mu \hookrightarrow {}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}$ respects the filtrations $F^\bullet_{\mu_1}\,{}^\imath\mathcal{Y}_\mu$ and $F^\bullet_{\mu_1+\nu}{}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}$. Moreover, it is strictly filtered in the sense that*

$$\iota^\tau_{\mu,\nu}\big(F^k_{\mu_1}({}^\imath\mathcal{Y}_\mu)\big) = \iota^\tau_{\mu,\nu}({}^\imath\mathcal{Y}_\mu) \cap F^k_{\mu_1+\nu}({}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}).$$

Remark 2.23. Similarly to [FKP+18, §5.4], various choices of filtrations on ${}^\imath\mathcal{Y}_\mu$ defined above are related to one another by $\mathbb{Z}$-gradings. More precisely, if we have two decompositions $\mu = \mu_1 + \tau\mu_1 = \mu_2 + \tau\mu_2$, then we may define a $\mathbb{Z}$–grading of ${}^\imath\mathcal{Y}_\mu$ by

$$\deg H_i^{(r)} = 0, \qquad \deg B_i^{(s)} = \langle \mu_2 - \mu_1, \alpha_i\rangle. \tag{2.27}$$

The filtrations $F^\bullet_{\mu_1}\,{}^\imath\mathcal{Y}_\mu$ and $F^\bullet_{\mu_2}\,{}^\imath\mathcal{Y}_\mu$ are then related by the construction of [FKP+18, Lemma 5.1], and in particular there are (non-graded) algebra isomorphisms

$$\mathrm{Rees}^{F^\bullet_{\mu_1}}\,{}^\imath\mathcal{Y}_\mu \cong \mathrm{Rees}^{F^\bullet_{\mu_2}}\,{}^\imath\mathcal{Y}_\mu \quad \text{and} \quad \mathrm{gr}^{F^\bullet_{\mu_1}}\,{}^\imath\mathcal{Y}_\mu \cong \mathrm{gr}^{F^\bullet_{\mu_2}}\,{}^\imath\mathcal{Y}_\mu.$$

Thanks to $\tau(\mu_2-\mu_1) = -(\mu_2-\mu_1)$, the pairing of $\mu_2-\mu_1$ with the root lattice Q naturally descends to a pairing with $Q^\imath$ in (2.13), allowing us to collapse the $Q^\imath$–grading on ${}^\imath\mathcal{Y}_\mu$ to a $\mathbb{Z}$–grading defined in (2.27).

3. iGKLO representations of shifted twisted Yangians

In this section, we formulate a family of iGKLO representations of shifted twisted Yangians of arbitrary quasi-split ADE type.

3.1. **Ring of difference operators.** Fix a dominant τ-invariant coweight λ, i.e., $\tau\lambda = \lambda$, and an even spherical coweight μ such that $\lambda \geqslant \mu$. We denote

$$\lambda - \mu = \sum_{i\in\mathbb{I}} \mathbf{v}_i \alpha_i^\vee, \tag{3.1}$$

where $\mathbf{v}_i \in \mathbb{N}$ for $i \in \mathbb{I}$. Denoting $\mathbb{Z}_2 = \{\bar 0, \bar 1\}$, we set

$$\mathfrak{v}_i = \begin{cases} \mathbf{v}_i, & \text{if } \tau i \neq i \\ \lfloor \frac12 \mathbf{v}_i \rfloor, & \text{if } \tau i = i, \end{cases} \qquad \theta_i = \begin{cases} 0, & \text{if } \tau i \neq i \\ \delta_{\overline{\mathbf{v}}_i, \bar 1}, & \text{if } \tau i = i. \end{cases} \tag{3.2}$$

Introduce ϑ_i by

$$\vartheta_i = \begin{cases} \theta_i, & \text{if } \tau i \neq i, \\ \max\{\theta_j \mid j \in \mathbb{I} \text{ and } c_{ij} \neq 0\}, & \text{if } \tau i = i. \end{cases} \tag{3.3}$$

Denote $\mathbf{w}_i = \langle \lambda, \alpha_i\rangle$. We also set

$$\mathfrak{w}_i = \begin{cases} \mathbf{w}_i, & \text{if } \tau i \neq i \\ \lfloor \frac12 \mathbf{w}_i \rfloor, & \text{if } \tau i = i, \end{cases} \qquad \varsigma_i = \begin{cases} 0, & \text{if } \tau i \neq i \\ \delta_{\overline{\mathbf{w}}_i, \bar 1}, & \text{if } \tau i = i. \end{cases} \tag{3.4}$$

Note that

$$\mathfrak{v}_{\tau i} = \mathfrak{v}_i, \qquad \mathfrak{w}_{\tau i} = \mathfrak{w}_i, \qquad \text{for all } i \in \mathbb{I}.$$

Recall $C = (c_{ij})$ is the Cartan matrix. Throughout the paper, we shall impose the following fundamental parity condition on the dimension vector $\mathbf{v} = (\mathbf{v}_i)_{i\in\mathbb{I}}$; see (3.1)–(3.2):

$$c_{ij}\theta_i\theta_j = 0, \qquad \text{for } i \neq j \in \mathbb{I}, \tag{3.5}$$

that is, at least one of $\mathbf{v}_i$ and $\mathbf{v}_j$ is even when $i \neq j \in \mathbb{I}$ are connected. This condition is needed in this section and also turns out to be required for geometric constructions in Section 8; see, e.g., Theorem 8.11.

Remark 3.1. Let $i \in \mathbb{I}$. If $\theta_i = 1$, then the evenness of μ and the parity condition (3.5) imply that $\mathbf{w}_i = \langle \lambda, \alpha_i \rangle$ is even and hence $\varsigma_i = 0$.

Let $\boldsymbol{z} := (z_{i,s})_{i\in{}^\imath\mathbb{I},1\leqslant s\leqslant \mathfrak{w}_i}$ be formal variables and denote the polynomial ring

$$\mathbb{C}[\boldsymbol{z}] = \mathbb{C}[z_{i,s}]_{i\in{}^\imath\mathbb{I},1\leqslant s\leqslant \mathfrak{w}_i}$$

and define the new $\mathbb{C}$-algebra

$${}^\imath\mathcal{Y}_\mu(\mathfrak{g})[\boldsymbol{z}] := {}^\imath\mathcal{Y}_\mu(\mathfrak{g}) \otimes \mathbb{C}[\boldsymbol{z}],$$

with new central elements $z_{i,s}$. Consider the $\mathbb{C}$-algebra

$$\mathcal{A} := \mathbb{C}[\boldsymbol{z}]\langle w_{i,r}, \eth_{i,r}^{\pm 1}, (w_{i,r} \pm w_{i,r'} + m)^{-1}, (w_{i,r} + \tfrac{1}{2}m)^{-1}\rangle_{i\in{}^\imath\mathbb{I},1\leqslant r\neq r'\leqslant \mathfrak{v}_i, m\in\mathbb{Z}}, \tag{3.6}$$

subject to the relations

$$[\eth_{i,r}^{\pm 1}, w_{j,r'}] = \pm\delta_{ij}\delta_{r,r'}\eth_{i,r}^{\pm 1}, \qquad [w_{i,r}, w_{j,r'}] = [\eth_{i,r}, \eth_{j,r'}] = 0, \qquad \eth_{i,r}^{\pm 1}\eth_{i,r}^{\mp 1} = 1. \tag{3.7}$$

It is convenient to extend the notation $z_{i,s}$, $w_{i,r}$, and $\eth_{i,r}$ to all $i \in \mathbb{I}$ as follows. First set

$$z_{\tau i,s} = -z_{i,s}, \qquad \text{for } i \in \mathbb{I}_1 \text{ and } 1 \leqslant s \leqslant \mathfrak{w}_i. \tag{3.8}$$

We further set

$$w_{\tau i,r} := -w_{i,\mathfrak{v}_i+1-r}, \qquad \eth_{\tau i,r} := \eth_{i,\mathfrak{v}_i+1-r}^{-1}, \qquad \text{for } i \in \mathbb{I}_1 \text{ and } 1 \leqslant r \leqslant \mathfrak{v}_i. \tag{3.9}$$

Given a monic polynomial $f(u)$ in u, we define

$$f^-(u) := (-1)^{\deg f} f(-u) \tag{3.10}$$

to be the monic polynomial whose roots are the opposite of the roots of $f(u)$.

For each $i \in \mathbb{I}_0$, define

$$W_i(u) = \prod_{r=1}^{\mathfrak{v}_i}(u - w_{i,r}), \qquad Z_i(u) = \prod_{s=1}^{\mathfrak{w}_i}(u - z_{i,s}), \tag{3.11}$$

$$\mathbf{W}_i(u) = u^{\theta_i}\prod_{r=1}^{\mathfrak{v}_i}(u^2 - w_{i,r}^2), \qquad \mathbf{Z}_i(u) = u^{\varsigma_i}\prod_{s=1}^{\mathfrak{w}_i}(u^2 - z_{i,s}^2). \tag{3.12}$$

Then we have $\deg \mathbf{W}_i(u) = \mathbf{v}_i$ and $\mathbf{W}_i(u) = \mathbf{W}_i^-(u)$, and similarly $\deg \mathbf{Z}_i(u) = \mathbf{w}_i$ and $\mathbf{Z}_i^-(u) = \mathbf{Z}_i(u)$. We also define

$$\mathbf{W}_i^\circ(u) = \prod_{r=1}^{\mathfrak{v}_i}(u^2 - w_{i,r}^2), \quad \mathbf{W}_{i,r}(u) = u^{\theta_i}(u + w_{i,r})\prod_{s=1, s\neq r}^{\mathfrak{v}_i}(u^2 - w_{i,s}^2). \tag{3.13}$$

Introduce

$$\overline{W}_i^-(u) := u^{\theta_i} W_i^-(u), \qquad \overline{Z}_i^-(u) := u^{\varsigma_i} Z_i^-(u). \tag{3.14}$$

For simplicity, set

$$\varkappa(u) = 1 - \frac{1}{2u}, \qquad \boldsymbol{\varkappa}(u) = 1 - \frac{1}{4u^2}. \tag{3.15}$$

For each $i \in \mathbb{I}_1$, we fix a choice of $\zeta_i \in \mathbb{N}$ such that $1 \leqslant \zeta_i \leqslant \mathfrak{v}_i$ and extend it to $i \in \mathbb{I}_1 \cup \mathbb{I}_{-1}$ by

$$\zeta_{\tau i} = \mathfrak{v}_i - \zeta_i.$$

For $i \in \mathbb{I}_1 \cup \mathbb{I}_{-1}$, set

$$\begin{aligned} W_i(u) &= \prod_{r=1}^{\zeta_i} (u - w_{i,r}), \qquad & \mathbf{W}_i(u) &= \prod_{r=1}^{\mathfrak{v}_i} (u - w_{i,r}), \\ \mathbf{Z}_i(u) &= \prod_{s=1}^{\mathfrak{w}_i} (u - z_{i,s}), \qquad & \mathbf{W}_{i,r}(u) &= \prod_{s=1, s\neq r}^{\mathfrak{v}_i} (u - w_{i,s}). \end{aligned} \tag{3.16}$$

It follows by (3.8) and (3.9) that for $i \in \mathbb{I}_1 \cup \mathbb{I}_{-1}$ we have

$$\mathbf{W}_i(u) = W_i(u) W_{\tau i}^-(u), \qquad \mathbf{W}_{\tau i}(u) = \mathbf{W}_i^-(u), \qquad \mathbf{Z}_{\tau i}(u) = \mathbf{Z}_i^-(u).$$

We pick a monic polynomial $Z_i(u)$, for each $i \in \mathbb{I}_1 \cup \mathbb{I}_{-1}$, such that

$$\mathbf{Z}_i(u) = Z_i(u) Z_{\tau i}^-(u) = (-1)^{\deg Z_{\tau i}} Z_i(u) Z_{\tau i}(-u).$$

3.2. **iGKLO representations for type AI.** It is convenient to first work with the shifted version of twisted Yangian of split type A (i.e., type AI), which corresponds to $\mathfrak{gl}_n$ instead of $\mathfrak{sl}_n$. We will use the Drinfeld type presentation established in [LWZ25b] via Gauss decomposition. In this case, $\mathbb{I} = \mathbb{I}_0 = \{1, \dots, n-1\}$.

Given a Laurent series $X(u) = \sum_{r \in \mathbb{Z}} X^{(r)} u^{-r}$, denote by $(X(u))^\star$ its principal part:

$$(X(u))^\star := \sum_{r>0} X^{(r)} u^{-r}.$$

Definition 3.2. The *shifted twisted Yangian* ${}^\imath\mathcal{Y}_\mu(\mathfrak{gl}_n)$ *associated to an even coweight* μ is the associative $\mathbb{C}$-algebra generated by $D_i^{(r)}$, $\widetilde{D}_i^{(\tilde{r})}$, and $E_j^{(s)}$, where $1 \leqslant i \leqslant n$, $1 \leqslant j < n$, $r \in \mathbb{Z}_{\geqslant \langle \mu, \varepsilon_i \rangle}$, $\tilde{r} \in \mathbb{Z}_{\geqslant -\langle \mu, \varepsilon_i \rangle}$ and $s \in \mathbb{Z}_{>0}$, with the relations

$$D_i^{(\langle \varepsilon_i, \mu \rangle)} = 1, \qquad [D_i(u), D_j(v)] = 0, \qquad D_i(u)\widetilde{D}_i(u) = \widetilde{D}_i(u) D_i(u) = 1, \tag{3.17}$$

$$\widetilde{D}_i(u) D_{i+1}(u) = \widetilde{D}_i(-u+i) D_{i+1}(-u+i), \tag{3.18}$$

$$[D_i(u), E_i(v)] = \frac{D_i(u)(E_i(u) - E_i(v))}{u - v} + \frac{(E_i(v) - E_i(-u+i)) D_i(u)}{u + v - i}, \tag{3.19}$$

$$[E_i(v), D_{i+1}(u)] = \frac{D_{i+1}(u)(E_i(u) - E_i(v))}{u - v} + \frac{(E_i(v) - E_i(-u+i)) D_{i+1}(u)}{u + v - i}, \tag{3.20}$$

$$[E_i(u), E_{i+1}(v)] = \frac{-E_i(u) E_{i+1}(v) + [E_i^{(1)}, E_{i+1}(v)] - [E_i(u), E_{i+1}^{(1)}]}{u - v}, \tag{3.21}$$

$$[E_i(u), E_j(v)] = 0, \qquad \text{if } c_{ij} = 0, \tag{3.22}$$

$$[E_i(u), E_i(v)] = -\frac{(E_i(u) - E_i(v))^2}{u - v} - \frac{(\widetilde{D}_i(u)D_{i+1}(u))^\star - (\widetilde{D}_i(v)D_{i+1}(v))^\star}{u + v - i}, \tag{3.23}$$

$$\begin{aligned}&\big[B_i^{(1)}, [B_i(u), B_j^{(1)}]\big] + \big[B_i(u), [B_i^{(1)}, B_j^{(1)}]\big] \\ &\qquad = \big(B_j(-u + \tfrac{1}{2})H_i(u) - B_j(u + \tfrac{1}{2})H_i(u)\big)^\star, \quad \text{if } c_{ij} = -1,\end{aligned} \tag{3.24}$$

where

$$D_i(u) = \sum_{r \geqslant \langle \varepsilon_i, \mu \rangle} D_i^{(r)} u^{-r}, \quad \widetilde{D}_i(u) = \sum_{r \geqslant -\langle \varepsilon_i, \mu \rangle} \widetilde{D}_i^{(r)} u^{-r}, \quad E_j(u) = \sum_{s > 0} E_j^{(s)} u^{-s},$$

$$B_i(u) = E_i(u + \tfrac{i}{2}), \quad H_i(u) = \widetilde{D}_i(u + \tfrac{i}{2}) D_{i+1}(u + \tfrac{i}{2}). \tag{3.25}$$

Lemma 3.3. *For any even coweight μ, there exists a homomorphism*

$$\eta_\mu : {}^\imath\mathcal{Y}_\mu(\mathfrak{sl}_n) \longrightarrow {}^\imath\mathcal{Y}_\mu(\mathfrak{gl}_n)$$

defined by

$$B_i(u) \mapsto B_i(u), \qquad H_i(u) \mapsto H_i(u).$$

Proof. It follows from the same calculation as in [LWZ25b, §4]. Note that the Serre relation (3.24) is formulated differently from (2.21). Indeed, the Serre relation (3.24) corresponds to [LPT+25, Lemma 6.5], see [LPT+25, Remark 4.3] for more detail. □

In order to describe the iGKLO representations for shifted twisted Yangians of $\mathfrak{gl}_n$ (cf. [FPT22, Theorem. 2.35] for shifted Yangians of $\mathfrak{gl}_n$), we need additional notation. Recall the polynomials $\mathbf{W}_i(u)$ and $\mathbf{Z}_i(u)$ from §3.1 and note that $i = \tau i$ for all $i \in \mathbb{I}$. We denote

$$\vartheta_0 = 0, \quad \mathfrak{v}_0 = 0, \quad \mathfrak{v}_n = 0, \quad \mathbf{W}_0(u) = \mathbf{W}_n(u) = 1,$$

and take any

$$\mathbf{Z}_0(u) = u^{-\theta_1} \prod_{x \in \mathbb{C}} (u^2 - x^2)^{m_x},$$

that satisfies $m_x \in \mathbb{Z}$, $m_x \neq 0$ for only finitely many $x \in \mathbb{C}$, and

$$\langle \lambda, \varepsilon_1 \rangle = \theta_1 - 2 \sum_{x \in \mathbb{C}} m_x.$$

Theorem 3.4. *Let μ be an even coweight. There is a homomorphism $\Phi_\mu^\lambda : {}^\imath\mathcal{Y}_\mu(\mathfrak{gl}_n)[\boldsymbol{z}] \to \mathcal{A}$, defined by*

$$\begin{aligned}D_i(u) &\mapsto \frac{\mathbf{W}_i(u - \frac{i-1}{2})}{\mathbf{W}_{i-1}(u - \frac{i}{2})} \prod_{j=0}^{i-1} \Big(\boldsymbol{\varkappa}(u - \tfrac{j}{2})^{\vartheta_j} \mathbf{Z}_j(u - \tfrac{j}{2}) \Big), \\ E_i(u) &\mapsto -\sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(w_{i,r})^{-\vartheta_i} \mathbf{W}_{i-1}(w_{i,r} - \frac{1}{2}) \overline{W}^{-}_{i+1}(w_{i,r} - \frac{1}{2})}{(u - \frac{i-1}{2} - w_{i,r}) \mathbf{W}_{i,r}(w_{i,r})} \eth_{i,r}^{-1} \\ &\qquad - \sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(-w_{i,r})^{-\vartheta_i} \mathbf{Z}_i(w_{i,r} + \frac{1}{2}) W_{i+1}(w_{i,r} + \frac{1}{2})}{(u - \frac{i-1}{2} + w_{i,r}) \mathbf{W}_{i,r}(w_{i,r})} \eth_{i,r}\end{aligned} \tag{3.26}$$

$$+ \sqrt{(-1)^{\mathfrak{w}_i+\mathfrak{v}_{i-1}+\mathfrak{v}_{i+1}}} \frac{\theta_i Z_i(0)}{(u-\frac{i}{2})\mathbf{W}_i^\circ(\frac{1}{2})} \prod_{j\leftrightarrow i} W_j(0).$$

(The image of $\widetilde{D}_i(u)$ is the inverse of the image of $D_i(u)$.)

Theorem 3.4 is proved in §4 below.

There is some flexibility in the formula for $E_i(u)$ in (3.26), where the factor $\mathbf{Z}_i(w_{i,r}+\frac{1}{2})$ can be split into 2 factors, one for each of the first two summations. In this way, using notation (3.12) and (3.14) (also see (3.10)), the formula for $E_i(u)$ in (3.26) can be modified to be

$$\begin{aligned} E_i(u) \mapsto &- \sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(w_{i,r})^{-\vartheta_i} Z_i(w_{i,r}-\frac{1}{2})\mathbf{W}_{i-1}(w_{i,r}-\frac{1}{2})\overline{W}_{i+1}^-(w_{i,r}-\frac{1}{2})}{(u-\frac{i-1}{2}-w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \eth_{i,r}^{-1} \\ &- \sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(-w_{i,r})^{-\vartheta_i} \overline{Z}_i^-(w_{i,r}+\frac{1}{2}) W_{i+1}(w_{i,r}+\frac{1}{2})}{(u-\frac{i-1}{2}+w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \eth_{i,r} \\ &+ \sqrt{(-1)^{\mathfrak{w}_i+\mathfrak{v}_{i-1}+\mathfrak{v}_{i+1}}} \frac{\theta_i Z_i(0)}{(u-\frac{i}{2})\mathbf{W}_i^\circ(\frac{1}{2})} \prod_{j\leftrightarrow i} W_j(0). \end{aligned}$$

Fix an arbitrary orientation of the diagram $\mathbb{I}$. For $i, j \in \mathbb{I}$, we denote by $j \leftrightarrow i$ if there is an arrow $j \leftarrow i$ or $j \rightarrow i$. We give another version of iGKLO with the assumption that $\theta_i = 0$ for i even (recall the parity assumption (3.5)), whose formulas are more similar to the traditional Gelfand-Zeitlin formulas. The following theorem is not used elsewhere in this paper.

Theorem 3.5. *Let μ be an even coweight and suppose $\theta_i = 0$ for i even. Then there is a homomorphism $\Phi_\mu^\lambda : {}^\imath\mathcal{Y}_\mu(\mathfrak{gl}_n)[\boldsymbol{z}] \rightarrow \mathcal{A}$, defined by*

$$D_i(u) \mapsto \frac{\mathbf{W}_i(u-\frac{i-1}{2})}{\mathbf{W}_{i-1}(u-\frac{i}{2})} \prod_{j=0}^{i-1} \Big(\boldsymbol{\varkappa}(u-\tfrac{j}{2})^{\vartheta_j} \mathbf{Z}_j(u-\tfrac{j}{2}) \Big),$$

$$\begin{aligned} E_i(u) \mapsto &- \sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(w_{i,r})^{-\vartheta_i} Z_i(w_{i,r}-\frac{1}{2})}{(u-\frac{i-1}{2}-w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \prod_{j\rightarrow i} W_j(w_{i,r}-\tfrac{1}{2}) \prod_{j\leftarrow i} W_j^-(w_{i,r}-\tfrac{1}{2}) \eth_{i,r}^{-1} \\ &- \sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(-w_{i,r})^{-\vartheta_i} \overline{Z}_i^-(w_{i,r}+\frac{1}{2})}{(u-\frac{i-1}{2}+w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \prod_{j\rightarrow i} W_j^-(w_{i,r}+\tfrac{1}{2}) \prod_{j\leftarrow i} W_j(w_{i,r}+\tfrac{1}{2}) \eth_{i,r} \\ &+ \sqrt{(-1)^{\mathfrak{w}_i+\mathfrak{v}_{i-1}+\mathfrak{v}_{i+1}}} \frac{\theta_i Z_i(0)}{(u-\frac{i}{2})\mathbf{W}_i^\circ(\frac{1}{2})} \prod_{j\leftrightarrow i} W_j(0), \qquad \textit{for } i \textit{ odd}, \end{aligned}$$

$$\begin{aligned} E_i(u) \mapsto &- \sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(w_{i,r})^{-\vartheta_i} Z_i(w_{i,r}-\frac{1}{2})}{(u-\frac{i-1}{2}-w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \prod_{j\rightarrow i} \mathbf{W}_j(w_{i,r}-\tfrac{1}{2}) \eth_{i,r}^{-1} \\ &- \sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(-w_{i,r})^{-\vartheta_i} \overline{Z}_i^-(w_{i,r}+\frac{1}{2})}{(u-\frac{i-1}{2}+w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \prod_{j\leftarrow i} \mathbf{W}_j(w_{i,r}+\tfrac{1}{2}) \eth_{i,r}, \qquad \textit{for } i \textit{ even}. \end{aligned}$$

(The image of $\widetilde{D}_i(u)$ is the inverse of the image of $D_i(u)$.)

Proof. The proof is similar to that of Theorem 3.4. The key point is to check that Lemma 4.1 still holds for the new formulas. □

3.3. **iGKLO representations for quasi-split type.** Recall the polynomials $\mathbf{W}_i(u)$ and $\mathbf{Z}_i(u)$ from §3.1. Fix an arbitrary orientation of the diagram $\mathbb{I}$ such that for each $i \in \mathbb{I}$ with $i \neq \tau i$, if $i \to j$, then $\tau j \to \tau i$, or if $j \to i$, then $\tau i \to \tau j$. Let

$$\wp_i = \begin{cases} 1, & \text{if } \ i \leftarrow \tau i, \\ -1, & \text{if } \ i \to \tau i, \\ 0, & \text{if } \ c_{i,\tau i} = 0, 2. \end{cases} \tag{3.27}$$

We present our main result in this section on the iGKLO representations for shifted twisted Yangians ${}^\imath\mathcal{Y}_\mu$ of arbitrary quasi-split ADE types.

Theorem 3.6. *Let* $(\mathbb{I}, \tau)$ *be any quasi-split Satake diagram. Let* λ *be a dominant* τ*-invariant coweight and* μ *be an even spherical coweight subject to the constraint* (3.5) *such that* $\lambda \geqslant \mu$*. Then there exists a homomorphism*

$$\Phi^\lambda_\mu : {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}] \longrightarrow \mathcal{A}$$

such that (see (3.1)–(3.4) *and* (3.11)–(3.16) *for notations)*

$$H_i(u) \mapsto \Big(1 + \frac{\wp_i}{4u}\Big) \frac{\boldsymbol{\varkappa}(u)^{\vartheta_i}\mathbf{Z}_i(u)}{\mathbf{W}_i(u-\frac12)\mathbf{W}_i(u+\frac12)} \prod_{j \leftrightarrow i} \mathbf{W}_j(u), \qquad \textit{for } \ i \in \mathbb{I},$$

$$\begin{aligned} B_i(u) \mapsto & -\sum_{r=1}^{\zeta_i} \frac{Z_i(w_{i,r}-\frac12)}{(u+\frac12-w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \prod_{j\to i} \mathbf{W}_j(w_{i,r}-\tfrac12)\eth_{i,r}^{-1} \\ & -\sum_{r=1}^{\zeta_{\tau i}} \frac{Z_i^-(w_{\tau i,r}+\frac12)}{(u+\frac12+w_{\tau i,r})\mathbf{W}_{\tau i,r}(w_{\tau i,r})} \prod_{\tau j \leftarrow \tau i} \mathbf{W}_{\tau j}(w_{\tau i,r}+\tfrac12)\eth_{\tau i,r}, \qquad \textit{for } \ i \in \mathbb{I}\setminus\mathbb{I}_0, \end{aligned}$$

and

$$\begin{aligned} B_i(u) \mapsto & -\sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(w_{i,r})^{-\vartheta_i} Z_i(w_{i,r}-\frac12)}{(u+\frac12-w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \prod_{j\to i} \mathbf{W}_j(w_{i,r}-\tfrac12) \prod_{\substack{j\leftarrow i\\ j\in\mathbb{I}_0}} \overline{W}_j^-(w_{i,r}-\tfrac12)\eth_{i,r}^{-1} \\ & -\sum_{r=1}^{\mathfrak{v}_i} \frac{\varkappa(-w_{i,r})^{-\vartheta_i} \overline{Z}_i^-(w_{i,r}+\frac12)}{(u+\frac12+w_{i,r})\mathbf{W}_{i,r}(w_{i,r})} \prod_{\substack{j\to i\\ j\in\mathbb{I}_{\pm1}}} \mathbf{W}_j(w_{i,r}+\tfrac12) \prod_{\substack{j\leftarrow i\\ j\in\mathbb{I}_0}} W_j(w_{i,r}+\tfrac12)\eth_{i,r} \\ & + \sqrt{(-1)^{\mathfrak{w}_i + \sum_{i\leftrightarrow j \in {}^\imath\mathbb{I}} \mathfrak{v}_j}}\, \frac{\theta_i Z_i(0)}{u\mathbf{W}_i^\circ(\frac12)} \prod_{j\leftrightarrow i} W_j(0), \qquad \textit{for } \ i \in \mathbb{I}_0. \end{aligned}$$

The proof of Theorem 3.6 excluding the quasi-split type A_{2n} in §4.4 is similar to Theorem 3.4. Indeed, the verification of relations in ${}^\imath\mathcal{Y}_\mu$ under Φ^λ_μ is either similar to that of the ordinary shifted Yangians or shifted twisted Yangians of split type, see Remark 2.9.

For the quasi-split type A_{2n}, all relations except for the Serre relation (2.9) can be verified in the same way as for other types. We shall give detailed verifications for the relation (2.5) when $i = n$

and $j = n+1$ and the Serre relation (2.9) in §4.5. Note that the extra factor $1 + \frac{\wp_i}{4u}$ for $H_i(u)$ in the theorem above is compatible with [LZ24, (6.24)].

Remark 3.7. It is possible to also formulate analogous iGKLO representations for shifted twisted Yangians of split BCFG type, cf. [LWZ25a]. Proving it would require a better understanding of the complicated Serre relations.

3.4. **Truncated shifted twisted Yangians.** Inspired by the definition of ordinary truncated shifted Yangians [BFN19, §B(viii)] and [KWWY14, §4.3], we make the following.

Definition 3.8. Let λ be a dominant τ-invariant coweight and μ be an even spherical coweight with $\lambda \geqslant \mu$. The truncated shifted twisted Yangian (TSTY), denoted ${}^\imath\mathcal{Y}_\mu^\lambda$, is the $\mathbb{C}$-algebra given by the image of the iGKLO homomorphism $\Phi_\mu^\lambda : {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}] \to \mathcal{A}$.

Define a "Cartan" series $\mathsf{A}_i(u)$ in ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}][\![u^{-1}]\!]$, for $i \in \mathbb{I}$, by

$$H_i(u) = \Big(1 + \frac{\wp_i}{4u}\Big) \frac{\boldsymbol{\varkappa}(u)^{\vartheta_i} \mathbf{Z}_i(u) \prod_{j \leftrightarrow i} u^{\mathbf{v}_j}}{(u^2 - \frac{1}{4})^{\mathbf{v}_i}} \frac{\prod_{j \leftrightarrow i} \mathsf{A}_j(u)}{\mathsf{A}_i(u - \frac{1}{2}) \mathsf{A}_i(u + \frac{1}{2})}, \tag{3.28}$$

where $\wp_i$ is defined in (3.27). Expanding the series $\mathsf{A}_i(u)$ gives us a family of GKLO-type "Cartan" elements $\mathsf{A}_i^{(r)}$ in ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}]$, for $r > 0, i \in \mathbb{I}$:

$$\mathsf{A}_i(u) = 1 + \sum_{r>0} \mathsf{A}_i^{(r)} u^{-r}. \tag{3.29}$$

Then we have the following simple lemma.

Lemma 3.9. *The iGKLO homomorphism Φ_μ^λ from Theorem 3.6 sends*

$$\mathsf{A}_i(u) \mapsto u^{-\mathbf{v}_i} \mathbf{W}_i(u).$$

Define elements $\mathsf{B}_i^{(r)}$, for $i \in \mathbb{I}, r > 0$, by

$$\mathsf{B}_i(u) := u^{\mathbf{v}_i - \theta_i} (u + \tfrac{1}{2})^{\theta_i} B_i(u + \tfrac{1}{2}) \mathsf{A}_i(u) = u^{\mathbf{v}_i} \sum_{r>0} \mathsf{B}_i^{(r)} u^{-r}, \tag{3.30}$$

cf. [LPT+25, (11.5)]. It is not hard to see that $\mathsf{A}_i^{(r)}$ and $\mathsf{B}_i^{(r)}$ for $i \in \mathbb{I}, r > \mathbf{v}_i$ belong to the kernel of the homomorphism Φ_μ^λ. Motivated by the case of ordinary truncated shifted Yangians [BFN19, Remark B.21] and [KWWY14, Theorem 4.10], we propose the following.

Conjecture 3.10. *There is an isomorphism*

$${}^\imath\mathcal{Y}_\mu^\lambda \cong {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}] / \langle \mathsf{A}_i^{(r)},\ \delta_{\bar{\mathbf{v}}_j, \bar{0}} \mathsf{B}_j^{(s)} \mid i \in \mathbb{I}, j \in \mathbb{I}_0, r > \mathbf{v}_i, s > \mathbf{v}_j \rangle,$$

induced by the epimorphism Φ_μ^λ.

Remark 3.11. In the analogous (conjectural) presentation for ordinary truncated shifted Yangians, no elements like $\mathsf{B}_j^{(r)}$ are needed. In our present twisted context, since the series $\mathsf{A}_i(u)$ is always even for $i \in \mathbb{I}_0$, we need to quotient by extra elements $\mathsf{B}_i^{(r)}$ for $r > \mathbf{v}_i$ if $i \in \mathbb{I}_0$ and $\mathbf{v}_i$ is even, cf. [LPT+25, (11.6)].

3.4.1. *Commutative subalgebra and center.* Consider the subalgebra of ${}^\imath\mathcal{Y}_\mu^\lambda$ generated over $\mathbb{C}[\boldsymbol{z}]$ by the coefficients of all of the series $\mathsf{A}_i(u)$. (Equivalently, this subalgebra is generated over $\mathbb{C}[\boldsymbol{z}]$ by the coefficients of the series $H_i(u)$.) Using Lemma 3.9, one sees that this commutative subalgebra is a polynomial ring, having the following algebraically independent generators over $\mathbb{C}[\boldsymbol{z}]$:

$$\{\mathsf{A}_i^{(2r)} : i \in \mathbb{I}_0,\ 1 \leqslant 2r \leqslant \mathbf{v}_i\} \cup \{\mathsf{A}_i^{(r)} : i \in \mathbb{I}_1, 1 \leqslant r \leqslant \mathbf{v}_i\}. \tag{3.31}$$

We call this the *Gelfand-Tsetlin subalgebra* of ${}^\imath\mathcal{Y}_\mu^\lambda$, and in many cases it is a maximal commutative subalgebra of ${}^\imath\mathcal{Y}_\mu^\lambda$. We expect that it will be interesting to study corresponding categories of Gelfand-Tsetlin modules, similarly to [Web24]. We note, however, that the algebras ${}^\imath\mathcal{Y}_\mu^\lambda$ do not obviously fit into the context of [Web24], and in particular it is not clear whether ${}^\imath\mathcal{Y}_\mu^\lambda$ is generally a Galois order in the sense of Futorny-Ovsienko [FO10].

Note that $\mathbb{C}[\boldsymbol{z}]$ is a central subalgebra of ${}^\imath\mathcal{Y}_\mu^\lambda$.

Proposition 3.12. *The center of the algebra ${}^\imath\mathcal{Y}_\mu^\lambda$ is the polynomial algebra $\mathbb{C}[\boldsymbol{z}]$.*

Proof. Our proof is inspired by [FO10, Theorem 4.1(4)]. First note that every element $x \in {}^\imath\mathcal{Y}_\mu^\lambda$ can be written uniquely as a sum $x = \sum x_{\boldsymbol{a}} \eth^{\boldsymbol{a}}$, where we have used multi-index notation $\eth^{\boldsymbol{a}} = \prod_{i,r} \eth_{i,r}^{a_{i,r}}$ with all $a_{i,r} \in \mathbb{Z}$, and where the coefficients

$$x_{\boldsymbol{a}} \in \mathbb{C}[\boldsymbol{z}](w_{i,r} : i \in {}^\imath\mathbb{I}, 1 \leqslant r \leqslant \mathfrak{v}_i).$$

We'll assume from now on that x is central in ${}^\imath\mathcal{Y}_\mu^\lambda$.

We first claim that $x_{\boldsymbol{a}} = 0$ for all non-trivial $\boldsymbol{a} \neq \mathbf{0}$, so that $x = x_{\mathbf{0}} \in \mathbb{C}[\boldsymbol{z}](w_{i,r})$. Indeed, suppose that some $x_{\boldsymbol{a}} \neq 0$. By an application of [FO10, Lemma 2.1(4)], there exists an element $f \in \mathbb{C}[\boldsymbol{z}][\mathsf{A}_i^{(r)}]$ such that $\eth^{\mathbf{a}} f \neq f \eth^{\mathbf{a}}$. But then $fx \neq xf$: the coefficient of $\eth^{\boldsymbol{a}}$ on the left side is $f x_{\boldsymbol{a}}$, while the coefficient on the right side is $(\eth^{\boldsymbol{a}} f \eth^{-\boldsymbol{a}}) x_{\boldsymbol{a}} \neq f x_{\boldsymbol{a}}$.

We next claim that in fact $x \in \mathbb{C}[\boldsymbol{z}]$. To simplify notation in the proof, let us think of $x = x(w_{i,r})$ as a function of the variables $w_{i,r}$. For any $j \in {}^\imath\mathbb{I}$, consider the commutator $[\Phi_\mu^\lambda(B_j(u)), x] = 0$. Up to multiplying by some non-zero rational function, the coefficient of $\eth_{j,s}$ on the left side is $x(w_{i,r} + \delta_{i,j}\delta_{r,s}) - x(w_{i,r})$. This must be zero for all j, s, and since x is rational in the variables $w_{i,r}$ the only possibility is that x is constant. In other words, $x \in \mathbb{C}[\boldsymbol{z}]$, as claimed. ☐

Remark 3.13. A similar argument proves that

$${}^\imath\mathcal{Y}_\mu^\lambda \cap \mathbb{C}[\boldsymbol{z}](w_{i,r} : i \in {}^\imath\mathbb{I}, 1 \leqslant r \leqslant \mathfrak{v}_i)$$

is a maximal commutative subalgebra of ${}^\imath\mathcal{Y}_\mu^\lambda$, cf. [FO10, Theorem 4.1(3)].

3.4.2. *Filtrations.* Choose a coweight μ_1 such that $\mu = \mu_1 + \tau\mu_1$, and recall the corresponding filtration $F_{\mu_1}^\bullet {}^\imath\mathcal{Y}_\mu$ constructed in §2.4. This extends to a filtration $F_{\mu_1}^\bullet {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}]$, by putting all variables $z_{i,s}$ in degree 1. Since ${}^\imath\mathcal{Y}_\mu^\lambda$ is a quotient of ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}]$, it naturally inherits the quotient filtration. By abuse of notation we will denote this filtration by $F_{\mu_1}^\bullet {}^\imath\mathcal{Y}_\mu^\lambda$. Then there is an epimorphism of associated graded algebras

$$\mathrm{gr}^{F_{\mu_1}^\bullet} {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}] \twoheadrightarrow \mathrm{gr}^{F_{\mu_1}^\bullet} {}^\imath\mathcal{Y}_\mu^\lambda, \tag{3.32}$$

and it follows from Proposition 2.21 that $\mathrm{gr}^{F_{\mu_1}^\bullet} {}^\imath\mathcal{Y}_\mu^\lambda$ is commutative. We will study the underlying geometry for this commutative algebra in §8.4.

4. Proofs of the iGKLO Theorems 3.4 and 3.6

In this section, we shall prove Theorems 3.4 and 3.6. We prove Theorem 3.4 in §4.1–§4.3. Then we prove Theorem 3.6 in §4.4 excluding quasi-split type A_{2n} and finish the remaining case in §4.5.

It is clear that the image of $D_i(u)$ under Φ_μ^λ is of the form $u^{-\langle\mu,\varepsilon_i\rangle}+$(lower order terms in u). We need to verify that the relations (3.17)–(3.24) are preserved by the map Φ_μ^λ. The relations (3.17), (3.18), and (3.22) are obvious.

We shall verify the remaining relations (3.19)–(3.21), (3.23) and (3.24), respectively.

4.1. **The relations (3.19)–(3.21).** We start with the relation (3.19). To simplify the notation, we write

$$\begin{aligned} &D_i(u)\mapsto \mathbf{W}_i(u-\tfrac{i-1}{2})\varphi_i(u),\\ &E_i(u)\mapsto -\sum_{r=1}^{\mathfrak{v}_i}\frac{\xi^+_{i,r}(w_{i,r})}{u-\frac{i-1}{2}-w_{i,r}}\eth_{i,r}^{-1}-\sum_{r=1}^{\mathfrak{v}_i}\frac{\xi^-_{i,r}(w_{i,r})}{u-\frac{i-1}{2}+w_{i,r}}\eth_{i,r}+\frac{\xi_i^0}{u-\frac{i}{2}}. \end{aligned}$$

The functions $\varphi_i(u)$, $\xi^\pm_{i,r}(w_{i,r})$, and ξ_i^0 can be read off directly from formulas in Theorem 3.4 and we do not need their explicit forms. Set

$$\mathbf{W}^\diamond_{i,r}(u)=u^{\theta_i}\prod_{\substack{s=1\\ s\neq r}}^{\mathfrak{v}_i}(u^2-w_{i,s}^2). \tag{4.1}$$

Clearly, the image of $[D_i(u),E_i(v)]$ under Φ_μ^λ is given by

$$\begin{aligned} &-\sum_{r=1}^{\mathfrak{v}_i}\frac{\mathbf{W}^\diamond_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^+_{i,r}(w_{i,r})}{v-\frac{i-1}{2}-w_{i,r}}\big((u-\tfrac{i-1}{2})^2-w_{i,r}^2\big)\eth_{i,r}^{-1}\\ &-\sum_{r=1}^{\mathfrak{v}_i}\frac{\mathbf{W}^\diamond_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^-_{i,r}(w_{i,r})}{v-\frac{i-1}{2}+w_{i,r}}\big((u-\tfrac{i-1}{2})^2-w_{i,r}^2\big)\eth_{i,r}\\ &+\sum_{r=1}^{\mathfrak{v}_i}\frac{\mathbf{W}^\diamond_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^+_{i,r}(w_{i,r})}{v-\frac{i-1}{2}-w_{i,r}}\big((u-\tfrac{i-1}{2})^2-(w_{i,r}-1)^2\big)\eth_{i,r}^{-1}\\ &+\sum_{r=1}^{\mathfrak{v}_i}\frac{\mathbf{W}^\diamond_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^-_{i,r}(w_{i,r})}{v-\frac{i-1}{2}+w_{i,r}}\big((u-\tfrac{i-1}{2})^2-(w_{i,r}+1)^2\big)\eth_{i,r}\\ =\ &\sum_{r=1}^{\mathfrak{v}_i}\frac{\mathbf{W}^\diamond_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^+_{i,r}(w_{i,r})}{v-\frac{i-1}{2}-w_{i,r}}\big(2w_{i,r}-1\big)\eth_{i,r}^{-1}\\ &\qquad-\sum_{r=1}^{\mathfrak{v}_i}\frac{\mathbf{W}^\diamond_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^-_{i,r}(w_{i,r})}{v-\frac{i-1}{2}+w_{i,r}}\big(1+2w_{i,r}\big)\eth_{i,r}. \end{aligned}$$

On the other hand, the image of $\frac{1}{u-v}D_i(u)(E_i(u)-E_i(v))$ under Φ_μ^λ simplifies as

$$\sum_{r=1}^{\mathfrak{v}_i}\frac{\mathbf{W}^\diamond_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^+_{i,r}(w_{i,r})}{v-\frac{i-1}{2}-w_{i,r}}\big(u-\tfrac{i-1}{2}+w_{i,r}\big)\eth_{i,r}^{-1}$$

$$+\sum_{r=1}^{\mathfrak{v}_i}\frac{\mathbf{W}^{\diamond}_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^-_{i,r}(w_{i,r})}{v-\frac{i-1}{2}+w_{i,r}}\big(u-\tfrac{i-1}{2}-w_{i,r}\big)\eth_{i,r}-\frac{\mathbf{W}_i(u-\frac{i-1}{2})\varphi_i(u)\xi^0_i}{(u-\frac{i}{2})(v-\frac{i}{2})}$$

while the image of $\frac{1}{u+v-i}(E_i(v)-E_i(-u+i))D_i(u)$ under Φ^λ_μ simplifies as

$$-\sum_{r=1}^{a_i}\frac{\mathbf{W}^{\diamond}_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^+_{i,r}(w_{i,r})}{v-\frac{i-1}{2}-w_{i,r}}\big(u-\tfrac{i-1}{2}-(w_{i,r}-1)\big)\eth^{-1}_{i,r}$$
$$-\sum_{r=1}^{a_i}\frac{\mathbf{W}^{\diamond}_{i,r}(u-\frac{i-1}{2})\varphi_i(u)\xi^-_{i,r}(w_{i,r})}{v-\frac{i-1}{2}+w_{i,r}}\big(u-\tfrac{i-1}{2}+(w_{i,r}+1)\big)\eth_{i,r}+\frac{\mathbf{W}_i(u-\frac{i-1}{2})\varphi_i(u)\xi^0_i}{(u-\frac{i}{2})(v-\frac{i}{2})}.$$

Summing up the above formulas, we see that the image of the right-hand side of (3.19) coincides with the image of the left-hand side, proving the relation (3.19).

The proof of the relation (3.20) is very similar, which eventually reduces to the following identities:

$$\frac{1}{(u-\frac{i+1}{2})^2-w_{i,r}^2}-\frac{1}{(u-\frac{i+1}{2})^2-(w_{i,r}\pm 1)^2}$$
$$=\frac{1}{\big((u-\frac{i+1}{2})^2-w_{i,r}^2\big)\big(u-\frac{i-1}{2}\pm w_{i,r}\big)}-\frac{1}{\big((u-\frac{i+1}{2})^2-(w_{i,r}-1)^2\big)\big(u-\frac{i+1}{2}\mp w_{i,r}\big)}.$$

To simplify the notation in the next task, following [GKLO05] we set $w_{i,0}=\frac{1}{2}$ and introduce

$$\chi^+_{i,0}=\sqrt{(-1)^{\mathfrak{w}_i+\mathfrak{v}_{i-1}+\mathfrak{v}_{i+1}}}\,\frac{\theta_i Z_i(0)}{\mathbf{W}^{\circ}_i(w_{i,0})}\prod_{j\leftrightarrow i}W_j(0), \tag{4.2}$$

$$\chi^+_{i,r}=-\frac{\varkappa(w_{i,r})^{-\vartheta_i}\mathbf{W}_{i-1}(w_{i,r}-\frac{1}{2})\overline{W}^-_{i+1}(w_{i,r}-\frac{1}{2})}{\mathbf{W}_{i,r}(w_{i,r})}\eth^{-1}_{i,r},$$
$$\chi^-_{i,r}=-\frac{\varkappa(-w_{i,r})^{-\vartheta_i}W_{i+1}(w_{i,r}+\frac{1}{2})\mathbf{Z}_i(w_{i,r}+\frac{1}{2})}{\mathbf{W}_{i,r}(w_{i,r})}\eth_{i,r},\qquad \text{for } 1\leqslant r\leqslant \mathfrak{v}_i.$$

Note that $\chi^+_{i,0}=0$ when $\theta_i=0$. Then from (3.25) and (3.26) we derive that

$$\Phi^\lambda_\mu(B_i(u))=\sum_{r=0}^{\mathfrak{v}_i}\frac{1}{u+\frac{1}{2}-w_{i,r}}\chi^+_{i,r}+\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{u+\frac{1}{2}+w_{i,r}}\chi^-_{i,r}, \tag{4.3}$$
$$\Phi^\lambda_\mu(B^{(1)}_i)=\sum_{r=0}^{\mathfrak{v}_i}\chi^+_{i,r}+\sum_{r=1}^{\mathfrak{v}_i}\chi^-_{i,r}.$$

Now we proceed to prove the relation (3.21), and we shall set $j=i+1$. Note that $\theta_i\theta_j=0$ by (3.5). Without loss of generality, we assume that $\theta_j=0$.

Lemma 4.1. *Let $j=i+1$. We have*

$$(\pm w_{i,r}\mp w_{i,s}-1)\chi^\pm_{i,r}\chi^\pm_{i,s}=(\pm w_{i,r}\mp w_{i,s}+1)\chi^\pm_{i,s}\chi^\pm_{i,r},\qquad \text{if } r\neq s,$$
$$(\pm w_{i,r}\pm w_{i,s}-1)\chi^\pm_{i,r}\chi^\mp_{i,s}=(\pm w_{i,r}\pm w_{i,s}+1)\chi^\mp_{i,s}\chi^\pm_{i,r},\qquad \text{if } r\neq s,$$
$$(\pm w_{i,r}\mp w_{j,s}+\tfrac{1}{2})\chi^\pm_{i,r}\chi^\pm_{j,s}=(\pm w_{i,r}\mp w_{j,s}-\tfrac{1}{2})\chi^\pm_{j,s}\chi^\pm_{i,r},$$
$$(\pm w_{i,r}\pm w_{j,s}+\tfrac{1}{2})\chi^\pm_{i,r}\chi^\mp_{j,s}=(\pm w_{i,r}\pm w_{j,s}-\tfrac{1}{2})\chi^\mp_{j,s}\chi^\pm_{i,r},$$

where $r = 0$ is allowed in $\chi^+_{i,r}$.

Proof. Follows from a direct computation. □

Remark 4.2. Alternatively, one can also set $w_{i,0} = -\frac{1}{2}$ and introduce

$$\chi^-_{i,0} = \sqrt{(-1)^{\mathfrak{w}_i+\mathfrak{v}_{i-1}+\mathfrak{v}_{i+1}} \frac{\theta_i Z_i(0)}{\mathbf{W}^\circ_i(w_{i,0})}} \prod_{j\leftrightarrow i} W_j(0).$$

Then the above lemma also holds for $\chi^-_{i,r}$ with $r = 0$.

The relation (3.21) can be equivalently written as

$$\begin{aligned}&(u-v+\tfrac{1}{2})B_i(u)B_j(v) + B_i(u)B_j^{(1)} - B_i^{(1)}B_j(v)\\ =&(u-v-\tfrac{1}{2})B_j(v)B_i(u) + B_j^{(1)}B_i(u) - B_j(v)B_i^{(1)}.\end{aligned} \tag{4.4}$$

The image of the left-hand side of (4.4) under Φ^λ_μ simplifies as

$$\begin{aligned}&\sum_{r=0}^{\mathfrak{v}_i}\sum_{s=1}^{\mathfrak{v}_j}\Big(\frac{w_{i,r}-w_{j,s}+\frac{1}{2}}{(u+\frac{1}{2}-w_{i,r})(v+\frac{1}{2}-w_{j,s})}\chi^+_{i,r}\chi^+_{j,s} + \frac{w_{i,r}+w_{j,s}+\frac{1}{2}}{(u+\frac{1}{2}-w_{i,r})(v+\frac{1}{2}+w_{j,s})}\chi^+_{i,r}\chi^-_{j,s}\Big)\\ +&\sum_{r=1}^{\mathfrak{v}_i}\sum_{s=1}^{\mathfrak{v}_j}\Big(\frac{-w_{i,r}-w_{j,s}+\frac{1}{2}}{(u+\frac{1}{2}+w_{i,r})(v+\frac{1}{2}-w_{j,s})}\chi^-_{i,r}\chi^+_{j,s} + \frac{-w_{i,r}+w_{j,s}+\frac{1}{2}}{(u+\frac{1}{2}+w_{i,r})(v+\frac{1}{2}+w_{j,s})}\chi^-_{i,r}\chi^-_{j,s}\Big)\end{aligned}$$

while the image of the left-hand side of (4.4) under Φ^λ_μ is

$$\begin{aligned}&\sum_{r=0}^{\mathfrak{v}_i}\sum_{s=1}^{\mathfrak{v}_j}\Big(\frac{w_{i,r}-w_{j,s}-\frac{1}{2}}{(u+\frac{1}{2}-w_{i,r})(v+\frac{1}{2}-w_{j,s})}\chi^+_{j,s}\chi^+_{i,r} + \frac{w_{i,r}+w_{j,s}-\frac{1}{2}}{(u+\frac{1}{2}-w_{i,r})(v+\frac{1}{2}+w_{j,s})}\chi^-_{j,s}\chi^+_{i,r}\Big)\\ +&\sum_{r=1}^{\mathfrak{v}_i}\sum_{s=1}^{\mathfrak{v}_j}\Big(\frac{-w_{i,r}-w_{j,s}-\frac{1}{2}}{(u+\frac{1}{2}+w_{i,r})(v+\frac{1}{2}-w_{j,s})}\chi^+_{j,s}\chi^-_{i,r} + \frac{-w_{i,r}+w_{j,s}-\frac{1}{2}}{(u+\frac{1}{2}+w_{i,r})(v+\frac{1}{2}+w_{j,s})}\chi^-_{j,s}\chi^-_{i,r}\Big).\end{aligned}$$

Clearly, those two images coincide by Lemma 4.1, completing the verification of the relation (3.21).

4.2. The relation (3.23).

From now on, we set $\tilde{u}_i = u - \frac{i-1}{2}$ and $\tilde{v}_i = v - \frac{i-1}{2}$. Then we have

$$E_i(u) \mapsto \sum_{r=0}^{\mathfrak{v}_i} \frac{1}{\tilde{u}_i - w_{i,r}}\chi^+_{i,r} + \sum_{s=1}^{\mathfrak{v}_i} \frac{1}{\tilde{u}_i + w_{i,s}}\chi^-_{i,s}.$$

For simplicity, we set

$$\Omega_i(u) := \frac{\boldsymbol{\varkappa}(u-\frac{1}{2})^{\vartheta_i}\mathbf{W}_{i-1}(u-\frac{1}{2})\mathbf{W}_{i+1}(u-\frac{1}{2})\mathbf{Z}_i(u-\frac{1}{2})}{\mathbf{W}_i(u)\mathbf{W}_i(u-1)}. \tag{4.5}$$

Here $\Omega_i(u) = \Omega_i(-u+1)$. Moreover, if $\theta_i = \vartheta_i$, then

$$\Omega_i(u) = \frac{\mathbf{W}_{i-1}(u-\frac{1}{2})\mathbf{W}_{i+1}(u-\frac{1}{2})\mathbf{Z}_i(u-\frac{1}{2})}{(u-\frac{1}{2})^{2\theta_i}\mathbf{W}^\circ_i(u)\mathbf{W}^\circ_i(u-1)}. \tag{4.6}$$

To verify the relation (3.23), it is equivalent to check that

$$\begin{aligned}\Big[\sum_{r=0}^{\mathfrak{v}_i}\frac{1}{\tilde u_i-w_{i,r}}\chi_{i,r}^{+}+\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{\tilde u_i+w_{i,r}}\chi_{i,r}^{-},\sum_{s=0}^{\mathfrak{v}_i}\frac{1}{\tilde v_i-w_{i,s}}\chi_{i,s}^{+}+\sum_{s=1}^{\mathfrak{v}_i}\frac{1}{\tilde v_i+w_{i,s}}\chi_{i,s}^{-}\Big]\\ =-(\tilde u_i-\tilde v_i)\bigg(\sum_{r=0}^{\mathfrak{v}_i}\frac{1}{(\tilde u_i-w_{i,r})(\tilde v_i-w_{i,r})}\chi_{i,r}^{+}+\sum_{s=1}^{\mathfrak{v}_i}\frac{1}{(\tilde u_i+w_{i,s})(\tilde v_i+w_{i,s})}\chi_{i,s}^{-}\bigg)^2\\ -\frac{1}{\tilde u_i+\tilde v_i-1}\Big(\big(\Omega_i(\tilde u_i)\big)^\star-\big(\Omega_i(\tilde v_i)\big)^\star\Big).\end{aligned}\tag{4.7}$$

We shall move all difference operators $\chi_{i,r}^{\pm}$ to the right and then compare the terms not involving difference operators and terms containing $\chi_{i,r}^{\pm}\chi_{i,s}^{\pm}$ and $\chi_{i,r}^{\pm}\chi_{i,s}^{\mp}$.

4.2.1. *Terms involving $\chi_{i,r}^{\pm}\chi_{i,s}^{\pm}$ and $\chi_{i,r}^{\pm}\chi_{i,s}^{\mp}$ $(r\neq s)$.* Fix $0\leqslant r,s\leqslant \mathfrak{v}_i$ such that $r\neq s$. We collect terms containing $\chi_{i,r}^{+}\chi_{i,s}^{+}$ and $\chi_{i,r}^{+}\chi_{i,s}^{+}$. The terms containing $\chi_{i,r}^{+}\chi_{i,s}^{+}$ and $\chi_{i,s}^{+}\chi_{i,r}^{+}$ from the left-hand side of (4.7) are given by

$$\begin{aligned}&\frac{1}{(\tilde u_i-w_{i,r})(\tilde v_i-w_{i,s})}\chi_{i,r}^{+}\chi_{i,s}^{+}-\frac{1}{(\tilde u_i-w_{i,r})(\tilde v_i-w_{i,s})}\chi_{i,s}^{+}\chi_{i,r}^{+}\\ &\qquad+\frac{1}{(\tilde u_i-w_{i,s})(\tilde v_i-w_{i,r})}\chi_{i,s}^{+}\chi_{i,r}^{+}-\frac{1}{(\tilde u_i-w_{i,s})(\tilde v_i-w_{i,r})}\chi_{i,r}^{+}\chi_{i,s}^{+}\\ &=\frac{-(\tilde u_i-\tilde v_i)\big((w_{i,r}-w_{i,s})\chi_{i,r}^{+}\chi_{i,s}^{+}+(w_{i,s}-w_{i,r})\chi_{i,s}^{+}\chi_{i,r}^{+}\big)}{(\tilde u_i-w_{i,r})(\tilde u_i-w_{i,s})(\tilde v_i-w_{i,r})(\tilde v_i-w_{i,s})}.\end{aligned}\tag{4.8}$$

The terms containing $\chi_{i,r}^{+}\chi_{i,s}^{+}$ and $\chi_{i,s}^{+}\chi_{i,r}^{+}$ from the right-hand side of (4.7) are given by

$$\frac{-(\tilde u_i-\tilde v_i)\big(\chi_{i,r}^{+}\chi_{i,s}^{+}+\chi_{i,s}^{+}\chi_{i,r}^{+}\big)}{(\tilde u_i-w_{i,r})(\tilde u_i-w_{i,s})(\tilde v_i-w_{i,r})(\tilde v_i-w_{i,s})}.\tag{4.9}$$

To prove that (4.8) = (4.9), it is equivalent to prove that

$$\frac{-(\tilde u_i-\tilde v_i)\big((w_{i,r}-w_{i,s}-1)\chi_{i,r}^{+}\chi_{i,s}^{+}+(w_{i,s}-w_{i,r}-1)\chi_{i,s}^{+}\chi_{i,r}^{+}\big)}{(\tilde u_i-w_{i,r})(\tilde u_i-w_{i,s})(\tilde v_i-w_{i,r})(\tilde v_i-w_{i,s})}=0,\tag{4.10}$$

which follows immediately from Lemma 4.1.

To prove that the terms containing $\chi_{i,r}^{\pm}\chi_{i,s}^{\pm}$ and $\chi_{i,s}^{\pm}\chi_{i,r}^{\pm}$ (resp. $\chi_{i,r}^{\pm}\chi_{i,s}^{\mp}$ and $\chi_{i,s}^{\mp}\chi_{i,r}^{\pm}$) on both sides match, the computation is almost identical. Indeed, it essentially reduces to prove the identity (4.10) with $w_{i,r}$ and $w_{i,s}$ replaced by $\pm w_{i,r}$ and $\pm w_{i,s}$ (resp. $\pm w_{i,r}$ and $\mp w_{i,s}$), respectively. Note that we allow $r=0$ in $\chi_{i,r}^{+}$.

4.2.2. *Terms involving* $\chi^{\pm}_{i,r}\chi^{\pm}_{i,r}$ $(r \geqslant 1)$. The terms containing $\chi^{+}_{i,r}\chi^{+}_{i,r}$ from the left-hand side of (4.7) are given by

$$\begin{aligned}
&\frac{1}{\tilde{u}_i - w_{i,r}}\chi^{+}_{i,r}\frac{1}{\tilde{v}_i - w_{i,r}}\chi^{+}_{i,r} - \frac{1}{\tilde{v}_i - w_{i,r}}\chi^{+}_{i,r}\frac{1}{\tilde{u}_i - w_{i,r}}\chi^{+}_{i,r}\\
=&\Big(\frac{1}{(\tilde{u}_i - w_{i,r})(\tilde{v}_i - w_{i,r}+1)} - \frac{1}{(\tilde{v}_i - w_{i,r})(\tilde{u}_i - w_{i,r}+1)}\Big)\chi^{+}_{i,r}\chi^{+}_{i,r} \qquad (4.11)\\
=&\frac{-(\tilde{u}_i - \tilde{v}_i)}{(\tilde{u}_i - w_{i,r})(\tilde{v}_i - w_{i,r})(\tilde{u}_i - w_{i,r}+1)(\tilde{v}_i - w_{i,r}+1)}\chi^{+}_{i,r}\chi^{+}_{i,r}
\end{aligned}$$

while the terms containing $\chi^{+}_{i,r}\chi^{+}_{i,r}$ from the right-hand side of (4.7) are given by

$$-(\tilde{u}_i - \tilde{v}_i)\frac{1}{(\tilde{u}_i - w_{i,r})(\tilde{v}_i - w_{i,r})}\chi^{+}_{i,r}\frac{1}{(\tilde{u}_i - w_{i,r})(\tilde{v}_i - w_{i,r})}\chi^{+}_{i,r}. \qquad (4.12)$$

It is clear that (4.11) = (4.12) since $\chi^{+}_{i,r}$ contains $\eth^{-1}_{i,r}$.

The case for terms of $\chi^{-}_{i,r}\chi^{-}_{i,r}$ is reduced to a similar computation with $w_{i,r}$ replaced by $-w_{i,r}$.

4.2.3. *The constant terms.* We are left with comparing the terms not involving difference operators. We call such terms *constant terms*. The constant terms from the left-hand side of (4.7) are given by

$$\begin{aligned}
&\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{(\tilde{u}_i - w_{i,r})(\tilde{v}_i + w_{i,r}-1)}\chi^{+}_{i,r}\chi^{-}_{i,r} + \sum_{r=1}^{\mathfrak{v}_i}\frac{1}{(\tilde{u}_i + w_{i,r})(\tilde{v}_i - w_{i,r}-1)}\chi^{-}_{i,r}\chi^{+}_{i,r}\\
-&\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{(\tilde{v}_i - w_{i,r})(\tilde{u}_i + w_{i,r}-1)}\chi^{+}_{i,r}\chi^{-}_{i,r} - \sum_{r=1}^{\mathfrak{v}_i}\frac{1}{(\tilde{u}_i - w_{i,r}-1)(\tilde{v}_i + w_{i,r})}\chi^{-}_{i,r}\chi^{+}_{i,r}.
\end{aligned}$$

The constant terms from the right-hand side of (4.7) are

$$\begin{aligned}
-(\tilde{u}_i - \tilde{v}_i)\Big(&\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{(\tilde{u}_i - w_{i,r})(\tilde{v}_i - w_{i,r})(\tilde{u}_i + w_{i,r}-1)(\tilde{v}_i + w_{i,r}-1)}\chi^{+}_{i,r}\chi^{-}_{i,r}\\
&+\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{(\tilde{u}_i + w_{i,r})(\tilde{v}_i + w_{i,r})(\tilde{u}_i - w_{i,r}-1)(\tilde{v}_i - w_{i,r}-1)}\chi^{-}_{i,r}\chi^{+}_{i,r}\\
&+\frac{1}{(\tilde{u}_i - \frac{1}{2})^2(\tilde{v}_i - \frac{1}{2})^2}\chi^{+}_{i,0}\chi^{+}_{i,0}\Big) - \frac{1}{\tilde{u}_i + \tilde{v}_i - 1}\Big(\big(\Omega_i(\tilde{u}_i)\big)^{\star} - \big(\Omega_i(\tilde{v}_i)\big)^{\star}\Big),
\end{aligned}$$

which can be further simplified as

$$\begin{aligned}
&\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{(2w_{i,r}-1)}\Big(\frac{1}{(\tilde{u}_i - w_{i,r})(\tilde{v}_i + w_{i,r}-1)}\chi^{+}_{i,r}\chi^{-}_{i,r} - \frac{1}{(\tilde{u}_i + w_{i,r}-1)(\tilde{v}_i - w_{i,r})}\chi^{+}_{i,r}\chi^{-}_{i,r}\Big)\\
+&\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{(2w_{i,r}+1)}\Big(\frac{1}{(\tilde{u}_i - w_{i,r}-1)(\tilde{v}_i + w_{i,r})}\chi^{-}_{i,r}\chi^{+}_{i,r} - \frac{1}{(\tilde{u}_i + w_{i,r})(\tilde{v}_i - w_{i,r}-1)}\chi^{-}_{i,r}\chi^{+}_{i,r}\Big)\\
&-\frac{1}{\tilde{u}_i + \tilde{v}_i - 1}\Big(\big(\Omega_i(\tilde{u}_i)\big)^{\star} - \big(\Omega_i(\tilde{v}_i)\big)^{\star} - \frac{1}{(\tilde{u}_i - \frac{1}{2})^2}\chi^{+}_{i,0}\chi^{+}_{i,0} + \frac{1}{(\tilde{v}_i - \frac{1}{2})^2}\chi^{+}_{i,0}\chi^{+}_{i,0}\Big).
\end{aligned}$$

Hence, we need to prove that

$$\begin{aligned}
&-\frac{1}{\tilde{u}_i+\tilde{v}_i-1}\Big(\big(\Omega_i(\tilde{u}_i)\big)^\star-\big(\Omega_i(\tilde{v}_i)\big)^\star-\frac{1}{(\tilde{u}_i-\frac12)^2}\chi_{i,0}^+\chi_{i,0}^+ +\frac{1}{(\tilde{v}_i-\frac12)^2}\chi_{i,0}^+\chi_{i,0}^+\Big)\\
&=\sum_{r=1}^{\mathfrak{v}_i}\frac{2(w_{i,r}-1)}{(2w_{i,r}-1)}\Big(\frac{1}{(\tilde{u}_i-w_{i,r})(\tilde{v}_i+w_{i,r}-1)}\chi_{i,r}^+\chi_{i,r}^- -\frac{1}{(\tilde{u}_i+w_{i,r}-1)(\tilde{v}_i-w_{i,r})}\chi_{i,r}^+\chi_{i,r}^-\Big)\\
&+\sum_{r=1}^{\mathfrak{v}_i}\frac{2(w_{i,r}+1)}{(2w_{i,r}+1)}\Big(\frac{1}{(\tilde{u}_i+w_{i,r})(\tilde{v}_i-w_{i,r}-1)}\chi_{i,r}^-\chi_{i,r}^+ -\frac{1}{(\tilde{u}_i-w_{i,r}-1)(\tilde{v}_i+w_{i,r})}\chi_{i,r}^-\chi_{i,r}^+\Big).
\end{aligned}$$

Set

$$\mathbf{W}_{i,r}^\circ=(u+w_{i,r})\prod_{1\leqslant s\leqslant\mathfrak{v}_i}^{s\neq r}(u^2-w_{i,s}^2).$$

Note that, for $r\geqslant 1$,

$$\eth_{i,r}^{\pm1}\mathbf{W}_{i,r}^\circ(w_{i,r})=\mathbf{W}_{i,r}^\circ(w_{i,r}\pm1)\frac{2(w_{i,r}\pm1)}{2w_{i,r}\pm1}\eth_{i,r}^{\pm1},$$

and thus we have

$$\chi_{i,r}^+\chi_{i,r}^-=-\frac{(2w_{i,r}-1)}{2(w_{i,r}-1)}\mathrm{Res}_{u=w_{i,r}}\Omega_i(u)=\frac{(2w_{i,r}-1)}{2(w_{i,r}-1)}\mathrm{Res}_{u=-w_{i,r}+1}\Omega_i(u), \tag{4.13}$$

$$\chi_{i,r}^-\chi_{i,r}^+=-\frac{(2w_{i,r}+1)}{2(w_{i,r}+1)}\mathrm{Res}_{u=-w_{i,r}}\Omega_i(u)=\frac{(2w_{i,r}+1)}{2(w_{i,r}+1)}\mathrm{Res}_{u=w_{i,r}+1}\Omega_i(u), \tag{4.14}$$

where $\mathrm{Res}_{u=a}f(u)$ denotes the residue of $f(u)$ at $u=a$.

Note that by (4.2), we have

$$\chi_{i,0}^+\chi_{i,0}^+=\frac{\theta_i\mathbf{W}_{i-1}(0)\mathbf{W}_{i+1}(0)\mathbf{Z}_i(0)}{\mathbf{W}_i^\circ(\frac12)\mathbf{W}_i^\circ(\frac12)}. \tag{4.15}$$

Hence, to show that the constant terms from both sides of (4.7) match, it suffices to prove that

$$\big(\Omega_i(\tilde{v}_i)\big)^\star-\big(\Omega_i(\tilde{u}_i)\big)^\star$$

is equal to

$$\begin{aligned}
&\theta_i\Big(\frac{1}{(\tilde{v}_i-\frac12)^2}-\frac{1}{(\tilde{u}_i-\frac12)^2}\Big)\frac{\mathbf{W}_{i-1}(0)\mathbf{W}_{i+1}(0)\mathbf{Z}_i(0)}{\mathbf{W}_i^\circ(\frac12)\mathbf{W}_i^\circ(-\frac12)}\\
&+\sum_{r=1}^{\mathfrak{v}_i}\mathrm{Res}_{u=-w_{i,r}+1}\Omega_i(u)\Big(\frac{1}{\tilde{u}_i-w_{i,r}}+\frac{1}{\tilde{v}_i+w_{i,r}-1}-\frac{1}{\tilde{u}_i+w_{i,r}-1}-\frac{1}{\tilde{v}_i-w_{i,r}}\Big)\\
&+\sum_{r=1}^{\mathfrak{v}_i}\mathrm{Res}_{u=w_{i,r}+1}\Omega_i(u)\Big(\frac{1}{\tilde{u}_i+w_{i,r}}+\frac{1}{\tilde{v}_i-w_{i,r}-1}-\frac{1}{\tilde{u}_i-w_{i,r}-1}-\frac{1}{\tilde{v}_i+w_{i,r}}\Big);
\end{aligned}$$

this follows from the following standard result, cf. [KWWY14, proof of Theorem 4.5] and [BFN19, Appendix B.5.2].

Lemma 4.3. (1) *For any rational function $\gamma(u)$ with simple poles $\{a_k\}\subset\mathbb{C}$ and a possible pole of higher order at $u=\infty$, we have*

$$\gamma(u)^\star=\sum_k\frac{1}{u-a_k}\mathrm{Res}_{u=a_k}\gamma(u).$$

(2) *Let $\theta_i\in\{0,1\}$ be as before. For any even rational function $\gamma(u)$ with simple poles $\{a_k\}\subset\mathbb{C}^\times$ and a possible pole of higher order at $u=\infty$, we have*

$$\big(u^{-2\theta_i}\gamma(u)\big)^\star=\frac{1}{u^2}\theta_i\gamma(0)+\sum_k\frac{1}{u-a_k}\mathrm{Res}_{u=a_k}u^{-2\theta_i}\gamma(u).$$

Indeed, we have 2 cases. If $\theta_i=\vartheta_i$, then the desired equality follows from Lemma 4.3 and (4.6). If $\theta_i\neq\vartheta_i$, then it only happens if $\theta_i=0$ and $\vartheta_i=1$. Thus $\max\{\theta_{i-1},\theta_{i+1}\}=1$ and at least one of $\mathbf{W}_{i-1}(u)$ and $\mathbf{W}_{i+1}(u)$ are odd. It follows that $\Omega_i(u)$ has at most a simple pole at $u=\frac12$. However, $\Omega_i(u+\frac12)$ is even and hence $\Omega_i(u)$ must be regular at $u=\frac12$.

4.3. **The Serre relation** (3.24)**.** We prove it for the case $j=i+1$ as the other case $j=i-1$ is similar. In the remainder of this section, sometimes we write $\frac{p\chi^+_{i,r}}{q}$ for $\frac{p}{q}\chi^+_{i,r}$ to shorten formulas, where p,q are polynomials in u and $w_{j,s}$, and similarly for others. We understand these ratio as follows: the terms $\chi^\pm_{i,r}$ involving difference operators are always to the right of the scalar rational functions. It is also convenient to set $\chi^-_{i,0}=\chi^-_{j,0}=0$.

We start with the left-hand side of (3.24). We have

$$\begin{aligned}
&\big[\Phi^\lambda_\mu(B_i^{(1)}),[\Phi^\lambda_\mu(B_i(u)),\Phi^\lambda_\mu(B_j^{(1)})]\big]+\big[\Phi^\lambda_\mu(B_i(u)),[\Phi^\lambda_\mu(B_i^{(1)}),\Phi^\lambda_\mu(B_j^{(1)})]\big] \qquad (4.16)\\
=&\Big[\sum_{r_1=0}^{\mathfrak{v}_i}\big(\chi^+_{i,r_1}+\chi^-_{i,r_1}\big),\Big[\sum_{r_2=0}^{\mathfrak{v}_i}\Big(\frac{\chi^+_{i,r_2}}{u+\frac12-w_{i,r_2}}+\frac{\chi^-_{i,r_2}}{u+\frac12+w_{i,r_2}}\Big),\sum_{s=0}^{\mathfrak{v}_j}\big(\chi^+_{j,s}+\chi^-_{j,s}\big)\Big]\Big]\\
&+\Big[\sum_{r_1=0}^{\mathfrak{v}_i}\Big(\frac{\chi^+_{i,r_1}}{u+\frac12-w_{i,r_1}}+\frac{\chi^-_{i,r_2}}{u+\frac12+w_{i,r_1}}\Big),\Big[\sum_{r_2=0}^{\mathfrak{v}_i}\big(\chi^+_{i,r_2}+\chi^-_{i,r_2}\big),\sum_{s=0}^{\mathfrak{v}_j}\big(\chi^+_{j,s}+\chi^-_{j,s}\big)\Big]\Big].
\end{aligned}$$

We first consider the case $r_1\neq r_2$. By Lemma 4.1, we have

$$\begin{aligned}
&\Big[\frac{1}{u+\frac12-w_{i,r_1}}\chi^+_{i,r_1},\big[\chi^+_{i,r_2},\chi^+_{j,s}\big]\Big]+\Big[\chi^+_{i,r_2},\Big[\frac{1}{u+\frac12-w_{i,r_1}}\chi^+_{i,r_1},\chi^+_{j,s}\Big]\Big]\\
=&\Big[\frac{1}{u+\frac12-w_{i,r_1}}\chi^+_{i,r_1},\frac{-1}{w_{i,r_2}-w_{j,s}-\frac12}\chi^+_{i,r_2}\chi^+_{j,s}\Big]\\
&+\Big[\chi^+_{i,r_2},\frac{-1}{(u+\frac12-w_{i,r_1})(w_{i,r_1}-w_{j,s}-\frac12)}\chi^+_{i,r_1}\chi^+_{j,s}\Big]\\
=&\frac{-1}{(u+\frac12-w_{i,r_1})(w_{i,r_2}-w_{j,s}-\frac12)}\Big(\chi^+_{i,r_1}\chi^+_{i,r_2}\chi^+_{j,s}-\frac{w_{i,r_1}-w_{j,s}+\frac12}{w_{i,r_1}-w_{j,s}-\frac12}\chi^+_{i,r_2}\chi^+_{i,r_1}\chi^+_{j,s}\Big)\\
&+\frac{-1}{(u+\frac12-w_{i,r_1})(w_{i,r_1}-w_{j,s}-\frac12)}\Big(\chi^+_{i,r_2}\chi^+_{i,r_1}\chi^+_{j,s}-\frac{w_{i,r_2}-w_{j,s}+\frac12}{w_{i,r_2}-w_{j,s}-\frac12}\chi^+_{i,r_1}\chi^+_{i,r_2}\chi^+_{j,s}\Big)
\end{aligned}$$

$$=\frac{-1}{(u+\frac12-w_{i,r_1})(w_{i,r_1}-w_{j,s}-\frac12)(w_{i,r_2}-w_{j,s}-\frac12)}\Big((w_{i,r_1}-w_{i,r_2}-1)\chi^+_{i,r_1}\chi^+_{i,r_2}\chi^+_{j,s}$$
$$+(w_{i,r_2}-w_{i,r_1}-1)\chi^+_{i,r_2}\chi^+_{i,r_1}\chi^+_{j,s}\Big)=0.$$

The same type of computation implies that

$$\Big[\frac{1}{u+\frac12-w_{i,r_1}}\chi^{*_1}_{i,r_1},\big[\chi^{*_2}_{i,r_2},\chi^{*_3}_{j,s}\big]\Big]+\Big[\chi^{*_2}_{i,r_2},\Big[\frac{1}{u+\frac12-w_{i,r_1}}\chi^{*_1}_{i,r_1},\chi^{*_3}_{j,s}\Big]\Big]=0$$

for $*_i\in\{\pm\}$, $i=1,2,3$, provided $r_1\neq r_2$. Specifically, one only needs to change w_{i,r_1}, w_{i,r_2}, $w_{j,s}$ to $*_1w_{i,r_1}$, $*_2w_{i,r_2}$, $*_3w_{j,s}$, respectively, in the above calculation.

Similarly, one finds that

$$\Big[\frac{1}{u+\frac12-w_{i,r}}\chi^{*}_{i,r},\big[\chi^{*}_{i,r},\chi^{\pm}_{j,s}\big]\Big]+\Big[\chi^{*}_{i,r},\Big[\frac{1}{u+\frac12-w_{i,r}}\chi^{*}_{i,r},\chi^{\pm}_{j,s}\Big]\Big]=0$$

for $*\in\{\pm\}$ for $r\geqslant 1$. Thus the right-hand side of (4.16) is equal to

$$\Big[\frac{1}{u+\frac12-w_{i,0}}\chi^{+}_{i,0},\big[\chi^{+}_{i,0},\chi^{*}_{j,s}\big]\Big]+\Big[\chi^{+}_{i,0},\Big[\frac{1}{u+\frac12-w_{i,0}}\chi^{+}_{i,0},\chi^{*}_{j,s}\Big]\Big]\tag{4.17}$$

plus

$$\sum_{*\in\{\pm\}}\sum_{s=0}^{\mathfrak{v}_j}\sum_{r=1}^{\mathfrak{v}_i}\Big(\Big[\frac{1}{u+\frac12-w_{i,r}}\chi^{+}_{i,r},\big[\chi^{-}_{i,r},\chi^{*}_{j,s}\big]\Big]+\Big[\chi^{-}_{i,r},\Big[\frac{1}{u+\frac12-w_{i,r}}\chi^{+}_{i,r},\chi^{*}_{j,s}\Big]\Big]$$
$$+\Big[\frac{1}{u+\frac12+w_{i,r}}\chi^{-}_{i,r},\big[\chi^{+}_{i,r},\chi^{*}_{j,s}\big]\Big]+\Big[\chi^{+}_{i,r},\Big[\frac{1}{u+\frac12+w_{i,r}}\chi^{-}_{i,r},\chi^{*}_{j,s}\Big]\Big]\Big).\tag{4.18}$$

We shall prove the Serre relation (3.24) by compare the terms containing the same $\chi^+_{j,s}$. The case for $\chi^-_{j,s}$ is similar.

Recall our assumption (3.5), $\chi^+_{i,0}$ and $\chi^+_{j,0}$ cannot be both nonzero. Hence if $s\neq 0$, it follows from Lemma 4.1 and (4.15) that the part (4.17) can be transformed to

$$\frac{2\theta_i\mathbf{W}_{i-1}(0)\mathbf{W}_j(0)\mathbf{Z}_i(0)}{(u+\frac12-w_{i,0})w_{j,s}^2\mathbf{W}_i^\circ(\frac12)\mathbf{W}_i^\circ(\frac12)}\chi^+_{j,s}.\tag{4.19}$$

Again using Lemma 4.1, the part (4.18) can be rewritten as

$$\sum_{r=1}^{\mathfrak{v}_i}\Big(\Big[\frac{1}{u+\frac12-w_{i,r}}\chi^{+}_{i,r},\big[\chi^{-}_{i,r},\chi^{+}_{j,s}\big]\Big]+\Big[\chi^{-}_{i,r},\Big[\frac{1}{u+\frac12-w_{i,r}}\chi^{+}_{i,r},\chi^{+}_{j,s}\Big]\Big]$$
$$+\Big[\frac{1}{u+\frac12+w_{i,r}}\chi^{-}_{i,r},\big[\chi^{+}_{i,r},\chi^{+}_{j,s}\big]\Big]+\Big[\chi^{+}_{i,r},\Big[\frac{1}{u+\frac12+w_{i,r}}\chi^{-}_{i,r},\chi^{+}_{j,s}\Big]\Big]\Big)$$
$$=\sum_{r=1}^{\mathfrak{v}_i}\Big(\Big[\frac{\chi^+_{i,r}}{u+\frac12-w_{i,r}},\frac{\chi^-_{i,r}\chi^+_{j,s}}{w_{i,r}+w_{j,s}+\frac12}\Big]+\Big[\chi^-_{i,r},\frac{-\chi^+_{i,r}\chi^+_{j,s}}{(u+\frac12-w_{i,r})(w_{i,r}-w_{j,s}-\frac12)}\Big]$$
$$+\Big[\frac{\chi^-_{i,r}}{u+\frac12+w_{i,r}},\frac{-\chi^+_{i,r}\chi^+_{j,s}}{w_{i,r}-w_{j,s}-\frac12}\Big]+\Big[\chi^+_{i,r},\frac{\chi^-_{i,r}\chi^+_{j,s}}{(u+\frac12+w_{i,r})(w_{i,r}+w_{j,s}+\frac12)}\Big]\Big)$$

$$
\begin{aligned}
=\sum_{r=1}^{\mathfrak{v}_i}\Big(&\frac{\chi_{i,r}^{+}\chi_{i,r}^{-}\chi_{j,s}^{+}}{(u+\frac12-w_{i,r})(w_{i,r}+w_{j,s}-\frac12)}-\frac{(w_{i,r}-w_{j,s}+\frac32)\chi_{i,r}^{-}\chi_{i,r}^{+}\chi_{j,s}^{+}}{(u-\frac12-w_{i,r})((w_{i,r}+\frac12)^2-w_{j,s}^2)}\\
&-\frac{\chi_{i,r}^{-}\chi_{i,r}^{+}\chi_{j,s}^{+}}{(u-\frac12-w_{i,r})(w_{i,r}-w_{j,s}+\frac12)}+\frac{(w_{i,r}+w_{j,s}-\frac32)\chi_{i,r}^{+}\chi_{i,r}^{-}\chi_{j,s}^{+}}{(u+\frac12-w_{i,r})((w_{i,r}-\frac12)^2-w_{j,s}^2)}\\
&-\frac{\chi_{i,r}^{-}\chi_{i,r}^{+}\chi_{j,s}^{+}}{(u+\frac12+w_{i,r})(w_{i,r}-w_{j,s}+\frac12)}+\frac{(w_{i,r}+w_{j,s}-\frac32)\chi_{i,r}^{+}\chi_{i,r}^{-}\chi_{j,s}^{+}}{(u-\frac12+w_{i,r})((w_{i,r}-\frac12)^2-w_{j,s}^2)}\\
&+\frac{\chi_{i,r}^{+}\chi_{i,r}^{-}\chi_{j,s}^{+}}{(u-\frac12+w_{i,r})(w_{i,r}+w_{j,s}-\frac12)}-\frac{(w_{i,r}-w_{j,s}+\frac32)\chi_{i,r}^{-}\chi_{i,r}^{+}\chi_{j,s}^{+}}{(u+\frac12+w_{i,r})((w_{i,r}+\frac12)^2-w_{j,s}^2)}\Big)\\
=\sum_{r=1}^{\mathfrak{v}_i}\Big(&\frac{2(w_{i,r}-1)\chi_{i,r}^{+}\chi_{i,r}^{-}\chi_{j,s}^{+}}{(u+\frac12-w_{i,r})((w_{i,r}-\frac12)^2-w_{j,s}^2)}-\frac{2(w_{i,r}+1)\chi_{i,r}^{-}\chi_{i,r}^{+}\chi_{j,s}^{+}}{(u-\frac12-w_{i,r})((w_{i,r}+\frac12)^2-w_{j,s}^2)}\\
&-\frac{2(w_{i,r}+1)\chi_{i,r}^{-}\chi_{i,r}^{+}\chi_{j,s}^{+}}{(u+\frac12+w_{i,r})((w_{i,r}+\frac12)^2-w_{j,s}^2)}+\frac{2(w_{i,r}-1)\chi_{i,r}^{+}\chi_{i,r}^{-}\chi_{j,s}^{+}}{(u-\frac12+w_{i,r})((w_{i,r}-\frac12)^2-w_{j,s}^2)}\Big).
\end{aligned}
$$

Recall $\mathbf{W}_{j,s}^{\diamond}(u)$ from (4.1) and set (cf. (4.5))

$$
\Omega_{i;j,s}(u)=\frac{\boldsymbol{\varkappa}(u-\frac12)^{\vartheta_i}\mathbf{W}_{i-1}(u-\frac12)\mathbf{W}_{j,s}^{\diamond}(u-\frac12)\mathbf{Z}_i(u-\frac12)}{\mathbf{W}_i(u)\mathbf{W}_i(u-1)}.
$$

Note that $\big((u-\frac12)^2-w_{j,s}^2\big)\Omega_{i;j,s}(u)=\Omega_i(u)$.

We rewrite the above formula using (4.13)–(4.14) in two cases, i.e., $s=0$ and $s\geqslant 1$. If $s=0$, then the above formula is equal to

$$
\begin{aligned}
\sum_{r=1}^{\mathfrak{v}_i}\Big(&\frac{(1-2w_{i,r})}{(w_{i,r}-\frac12)^2-w_{j,s}^2}\mathrm{Res}_{u=-w_{i,r}+\frac12}\Omega_i(u+\tfrac12)\Big(\frac{1}{u+\frac12-w_{i,r}}+\frac{1}{u-\frac12+w_{i,r}}\Big)\\
&-\frac{(2w_{i,r}+1)}{(w_{i,r}+\frac12)^2-w_{j,s}^2}\mathrm{Res}_{u=w_{i,r}+\frac12}\Omega_i(u+\tfrac12)\Big(\frac{1}{u-\frac12-w_{i,r}}+\frac{1}{u+\frac12+w_{i,r}}\Big)\Big)\chi_{j,s}^{+}.
\end{aligned}
$$

If $s\geqslant 1$, then the above formula is equal to

$$
\begin{aligned}
\sum_{r=1}^{\mathfrak{v}_i}\Big(&\big(1-2w_{i,r}\big)\mathrm{Res}_{u=-w_{i,r}+\frac12}\Omega_{i;j,s}(u+\tfrac12)\Big(\frac{1}{u+\frac12-w_{i,r}}+\frac{1}{u-\frac12+w_{i,r}}\Big)\\
&-\big(2w_{i,r}+1\big)\mathrm{Res}_{u=w_{i,r}+\frac12}\Omega_{i;j,s}(u+\tfrac12)\Big(\frac{1}{u-\frac12-w_{i,r}}+\frac{1}{u+\frac12+w_{i,r}}\Big)\Big)\chi_{j,s}^{+}.
\end{aligned}
$$

Now we consider the right-hand side of (3.24). Then

$$
\begin{aligned}
&\Phi_\mu^\lambda\big(B_j(-u+\tfrac12)H_i(u)-B_j(u+\tfrac12)H_i(u)\big)\\
=&\sum_{s=0}^{\mathfrak{v}_j}\Big(\frac{\chi_{j,s}^{+}}{-u+1-w_{j,s}}+\frac{\chi_{j,s}^{-}}{-u+1+w_{j,s}}\\
&\qquad-\frac{1}{u+1-w_{j,s}}\chi_{j,s}^{+}-\frac{1}{u+1+w_{j,s}}\chi_{j,s}^{-}\Big)\Omega_i(u+\tfrac12)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{2u\Omega_i(u+\frac12)}{u^2-w_{j,0}^2}\chi_{j,0}^+ \\
&\quad + \sum_{s=1}^{\mathfrak{v}_j}\Omega_{i;j,s}(u+\tfrac12)\Big(\Big(\frac{u^2-(w_{j,s}-1)^2}{-u+1-w_{j,s}}-\frac{u^2-(w_{j,s}-1)^2}{u+1-w_{j,s}}\Big)\chi_{j,s}^+ \\
&\qquad\qquad + \Big(\frac{u^2-(w_{j,s}+1)^2}{-u+1+w_{j,s}}-\frac{u^2-(w_{j,s}+1)^2}{u+1+w_{j,s}}\Big)\chi_{j,s}^-\Big) \\
&= -\frac{2u\Omega_i(u+\frac12)}{u^2-w_{j,0}^2}\chi_{j,0}^+ - \sum_{s=1}^{\mathfrak{v}_j}2u\Omega_{i;j,s}(u+\tfrac12)(\chi_{j,s}^+ + \chi_{j,s}^-).
\end{aligned}
$$

Comparing the coefficients of $\chi_{j,0}^+$ and $\chi_{j,s}^+$ for $s \geqslant 1$, the Serre relation (3.24) follows from the formulas established above and Lemma 4.3 (see the end of the previous subsection where this lemma was applied similarly). Again, we need a case-by-case study. We only remark the following.

- The term (4.19) contributes only if $\theta_i = 1$, in which case $\chi_{j,0}^+ = 0$ by the parity assumption (3.5).
- If $\chi_{j,0}^+ \neq 0$, then $\mathbf{W}_j(u)$ is an odd function and by (3.3) we have $\vartheta_i = 1$. It is easy to see that $2u\Omega_i(u+\frac12)$ is regular at $u=0$ and has simple zeros at $u = \pm w_{j,0}$. Similarly, $2u\Omega_{i;j,s}(u+\frac12)$ is regular at $u=0$.

This completes the proof of Theorem 3.4.

4.4. **Completing the proof except quasi-split type $\mathbf{A}_{2n}$.** It remains to prove Theorem 3.6.

In this subsection, we exclude the quasi-split type A_{2n}. Note that $c_{i,\tau i} \neq -1$ for all $i \in \mathbb{I}$, the Serre relation (2.9) does not show up and $\wp_i = 0$ for all $i \in \mathbb{I}$. According to Lemma 2.4 and Remark 2.9, it is not hard to see that the verification of all the relations will be either similar to the case for the ordinary shifted Yangians, or the case for shifted twisted Yangians of split type A. Here we only discuss for example the Serre relation for $c_{ij} = -1$ as the relation for $c_{ij} = 0$ reduces to the case of shifted Yangian.

Suppose $c_{ij} = -1$. Note that $j \neq \tau i$, then we have several cases. First, one needs to verify that similar identities hold as in Lemma 4.1. We proceed case-by-case.

(1) If both i and j are not fixed by τ, then this is the case of ordinary shifted Yangian as verified in [BFN19, Appendix B].
(2) If both i and j are fixed by τ, then this is the split case as verified in §4.3.
(3) If i is fixed by τ and j is not, i.e., $i \in \mathbb{I}_0$ and $j \notin \mathbb{I}_0$, then we need to verify the relation (2.8). Then the detail is parallel to that of §4.3. Again one needs to carefully deal with the terms containing $\chi_{i,0}^+\chi_{i,0}^+$ and $\chi_{i,r}^\pm\chi_{i,r}^\mp$ (those are scalar functions) while all other terms cancel due to Lemma 4.1.
(4) If j is fixed by τ and i is not, i.e., $i \notin \mathbb{I}_0$ and $j \in \mathbb{I}_0$, then we need to verify the relation (2.7). Again, we argue as in §4.3. Since i is not fixed by τ and $j \neq \tau i$, there is no way to obtain constant terms (scalar rational functions without difference operators). Thus this case essentially corresponds to the beginning of §4.3.

4.5. **Completing the proof for the quasi-split type A_{2n}.** In this subsection, we complete the proof of Theorem 3.6 for the case of quasi-split type A_{2n}. We shall only verify the most complicated relations (2.5) and (2.9) for the case $j=\tau i$ and $c_{i,\tau i}=-1$ for the iGKLO homomorphism in Theorem 3.6. We prove the case $i\to j$ (as the other case $i\leftarrow j$ is similar). Then $\wp_i=-1$ and $\wp_j=1$.

4.5.1. *The relation* (2.5). We work with the corresponding relations in terms of generating series,

$$(u-v)[B_i(u),B_j(v)]=-\frac{1}{2}[B_i(u),B_j(v)]_+ +\big([B_i^{(1)},B_j(v)]-[B_i(u),B_j^{(1)}]\big)\\ -\Big(\frac{2u}{u+v}\big(H_i(u)\big)^\star+\frac{2v}{u+v}\big(H_j(v)\big)^\star\Big),$$

see [LZ24, (3.21)]. It can be equivalently written as

$$\begin{aligned}&(u-v+\tfrac{1}{2})B_i(u)B_j(v)+B_i(u)B_j^{(1)}-B_i^{(1)}B_j(v)\\ &=(u-v-\tfrac{1}{2})B_j(v)B_i(u)+B_j^{(1)}B_i(u)-B_j(v)B_i^{(1)}\\ &\qquad-\Big(\frac{2u}{u+v}\big(H_i(u)\big)^\star+\frac{2v}{u+v}\big(H_j(v)\big)^\star\Big),\end{aligned}\tag{4.20}$$

cf. (4.4). We proceed as in §4.1 by introducing $\chi_{i,r}$ and $\chi_{j,s}$ for $1\leqslant r\leqslant\mathfrak{v}_i$ and $1\leqslant s\leqslant\mathfrak{v}_j$ such that

$$\Phi^\lambda_\mu(B_i(u))=\sum_{r=1}^{\mathfrak{v}_i}\frac{1}{u+\frac{1}{2}-w_{i,r}}\chi_{i,r},\qquad \Phi^\lambda_\mu(B_j(u))=\sum_{s=1}^{\mathfrak{v}_j}\frac{1}{u+\frac{1}{2}-w_{j,s}}\chi_{j,s}.\tag{4.21}$$

Note that we have used (3.9). It is convenient to write $r':=\mathfrak{v}_i+1-r$ for $1\leqslant r\leqslant\mathfrak{v}_i$.

Lemma 4.4. *If $r\neq s$, then we have*

$$(w_{i,r}-w_{i,s}-1)\chi_{i,r}\chi_{i,s}=(w_{i,r}-w_{i,s}+1)\chi_{i,s}\chi_{i,r},$$
$$(w_{i,r}-w_{j,s'}+\tfrac{1}{2})\chi_{i,r}\chi_{j,s'}=(w_{i,r}-w_{j,s'}-\tfrac{1}{2})\chi_{j,s'}\chi_{i,r}.$$

Proof. Follows by a direct calculation. □

Thus it follows from the same calculation as in §4.1 that all $\chi_{i,r}\chi_{j,s'}$ (or $\chi_{j,s'}\chi_{i,r}$) with $r\neq s$ in the LHS and RHS of (4.20) cancel. Hence we are left with terms involving $\chi_{i,r}\chi_{j,r'}$ and $\chi_{j,r'}\chi_{i,r}$ summed over r and the scalar series from $H_i(u)$ and $H_j(v)$. Note that by (3.9), we have $w_{i,r}=-w_{j,r'}$ and $\eth_{i,r}=\eth_{j,r'}^{-1}$ for $1\leqslant r\leqslant\mathfrak{v}_i$. Thus these terms do not involve the difference operators.

The constant terms from the LHS of (4.20) are given by

$$\frac{1}{u+v}\sum_{r=1}^{\mathfrak{v}_i}\Big(\frac{2w_{i,r}-\frac{1}{2}}{u+\frac{1}{2}-w_{i,r}}-\frac{2w_{j,r'}+\frac{1}{2}}{v-\frac{1}{2}-w_{j,r'}}\Big)\chi_{i,r}\chi_{j,r'},$$

while the constant terms from $(u-v-\frac{1}{2})B_j(v)B_i(u)+B_j^{(1)}B_i(u)-B_j(v)B_i^{(1)}$ is given by

$$\frac{1}{u+v}\sum_{r=1}^{\mathfrak{v}_i}\Big(\frac{2w_{i,r}+\frac{1}{2}}{u-\frac{1}{2}-w_{i,r}}-\frac{2w_{j,r'}-\frac{1}{2}}{v+\frac{1}{2}-w_{j,r'}}\Big)\chi_{j,r'}\chi_{i,r}.$$

Note that $i \to j$. Expressing $\chi_{i,r}\chi_{j,r'}$ and $\chi_{j,r'}\chi_{i,r}$ explicitly, we have

$$\begin{aligned}\chi_{i,r}\chi_{j,r'} &= \frac{(2w_{i,r}-\frac{3}{2})\mathbf{Z}_i(w_{i,r}-\frac{1}{2})}{(2w_{i,r}-\frac{1}{2})\mathbf{W}_{i,r}(w_{i,r})\mathbf{W}_{i,r}(w_{i,r}-1)}\prod_{k\leftrightarrow i}\mathbf{W}_k(w_{i,r}-\tfrac{1}{2}),\\ &= \frac{(2w_{j,r'}+\frac{3}{2})\mathbf{Z}_j(w_{j,r'}+\frac{1}{2})}{(2w_{j,r'}+\frac{1}{2})\mathbf{W}_{j,r'}(w_{j,r'})\mathbf{W}_{j,r'}(w_{j,r'}+1)}\prod_{k\leftrightarrow j}\mathbf{W}_k(w_{j,r'}+\tfrac{1}{2}).\end{aligned} \tag{4.22}$$

The formulas for $\chi_{j,r'}\chi_{i,r}$ are similar but without the extra ratios (due to the choice of $i \to j$):

$$\begin{aligned}\chi_{j,r'}\chi_{i,r} &= \frac{\mathbf{Z}_j(w_{j,r'}-\frac{1}{2})}{\mathbf{W}_{j,r'}(w_{j,r'})\mathbf{W}_{j,r'}(w_{j,r'}-1)}\prod_{k\leftrightarrow j}\mathbf{W}_k(w_{j,r'}-\tfrac{1}{2}),\\ &= \frac{\mathbf{Z}_i(w_{i,r}+\frac{1}{2})}{\mathbf{W}_{i,r}(w_{i,r})\mathbf{W}_{i,r}(w_{i,r}+1)}\prod_{k\leftrightarrow i}\mathbf{W}_k(w_{i,r}+\tfrac{1}{2}).\end{aligned} \tag{4.23}$$

The rest follows the same type of argument at the end of §4.2, now with the help of Lemma 4.3. We remark that the extra factor $1 \pm \frac{1}{4u}$ accounts for the factors

$$2u\big(1 \pm \tfrac{1}{4u}\big)\Big|_{u=w_{i,r}\pm\frac{1}{2}} = 2w_{i,r} \pm \tfrac{3}{2}, \qquad 2u\big(1 \pm \tfrac{1}{4u}\big)\Big|_{u=w_{i,r}\mp\frac{1}{2}} = 2w_{i,r} \mp \tfrac{1}{2},$$

and similar factors for $w_{j,r'}$.

4.5.2. *The relation* (2.9). By Lemma 2.5, it suffices to verify the relation (2.9) is preserved by Φ^λ_μ for the case $s_1 = s_2 = s = 1$. As first pointed out by [SSX25], it is actually easier to prove the following more general relation in generating function form

$$\big[B_i^{(1)}, [B_i^{(1)}, B_j(u)]\big] = \big(4u[B_i(3u), H_j(u)]\big)^\star, \tag{4.24}$$

which implies that the relation (2.9) is preserved by Φ^λ_μ for $s_1 = s_2 = 1$ and arbitrary $s > 0$.

Recall the image of $B_i(u)$ from (4.21) and note that $w_{j,r'} = -w_{i,r}$. It follows from the identity

$$\begin{aligned}\frac{1}{3u+\frac{1}{2}-w_{i,r}}\Big(\frac{u-w_{i,r}+1}{(u-\frac{1}{2}+w_{i,r})(u-\frac{3}{2}+w_{i,r})} - \frac{u-w_{i,r}}{(u+\frac{1}{2}+w_{i,r})(u-\frac{1}{2}+w_{i,r})}\Big)\\ = \frac{1}{(u+\frac{1}{2}+w_{i,r})(u-\frac{1}{2}+w_{i,r})(u-\frac{3}{2}+w_{i,r})}\end{aligned}$$

that

$$\Phi^\lambda_\mu\Big(4u[B_i(3u), H_j(u)]\Big) = \sum_{r=1}^{\mathfrak{v}_i} \Xi_{i,r}(u)\chi_{i,r}, \tag{4.25}$$

where

$$\Xi_{i,r}(u) := \frac{(4u+1)\mathbf{Z}_j(u)\mathbf{W}_{i,r}(u)}{(u-\frac{3}{2}+w_{i,r})\mathbf{W}_j(u+\frac{1}{2})\mathbf{W}_j(u-\frac{1}{2})}\prod_{\substack{k\leftrightarrow j\\ k\neq i}}\mathbf{W}_k(u). \tag{4.26}$$

On the other hand, the image of $\big[B_i^{(1)},[B_i^{(1)},B_j(u)]\big]$ under the map Φ_μ^λ is

$$\Bigg[\sum_{r_1=1}^{\mathfrak{v}_i}\chi_{i,r_1},\Big[\sum_{r_2=1}^{\mathfrak{v}_i}\chi_{i,r_2},\sum_{s=1}^{\mathfrak{v}_i}\frac{1}{u+\frac12-w_{j,s'}}\chi_{j,s'}\Big]\Bigg]. \tag{4.27}$$

If $r_1\neq s$ and $r_2\neq s$, then a similar calculation as in §4.3 using Lemma 4.4 shows that

$$\mathrm{Sym}_{r_1,r_2}\bigg[\chi_{i,r_1},\Big[\chi_{i,r_2},\frac{1}{u+\frac12-w_{j,s'}}\chi_{j,s'}\Big]\bigg]=0.$$

Thus only the terms with either $r_1=s$ or $r_2=s$ in (4.27) survive and will make a nontrivial contribution to $\Phi_\mu^\lambda\big(\big[B_i^{(1)},[B_i^{(1)},B_j(u)]\big]\big)$. Denote by X_r the sum of the surviving terms containing $\chi_{i,r}$ from (4.27), for $1\leqslant r\leqslant \mathfrak{v}_i$. From the discussion above, we have

$$\Phi_\mu^\lambda\Big(\big[B_i^{(1)},[B_i^{(1)},B_j(u)]\big]\Big)=\sum_{r=1}^{\mathfrak{v}_i}X_r, \tag{4.28}$$

where

$$X_r=X_r^\circ+\sum_{s=1,s\neq r}^{\mathfrak{v}_i}(X_s'+X_s''), \tag{4.29}$$

with

$$X_r^\circ=\bigg[\chi_{i,r},\Big[\chi_{i,r},\frac{1}{u+\frac12-w_{j,r'}}\chi_{j,r'}\Big]\bigg],\qquad X_s'=\bigg[\chi_{i,r},\Big[\chi_{i,s},\frac{1}{u+\frac12-w_{j,s'}}\chi_{j,s'}\Big]\bigg],$$

$$X_s''=\bigg[\chi_{i,s},\Big[\chi_{i,r},\frac{1}{u+\frac12-w_{j,s'}}\chi_{j,s'}\Big]\bigg].$$

The summand X_r° is equal to

$$X_r^\circ=\frac{1}{u-\frac32+w_{i,r}}\chi_{i,r}\chi_{i,r}\chi_{j,r'}-\frac{2}{u-\frac12+w_{i,r}}\chi_{i,r}\chi_{j,r'}\chi_{i,r}+\frac{1}{u+\frac12+w_{i,r}}\chi_{j,r'}\chi_{i,r}\chi_{i,r}.$$

It follows from (4.22)–(4.23) (and recall $w_{j,r'}=-w_{i,r}$ and (4.26) here) that it can be rewritten as

$$\begin{aligned}X_r^\circ=\Big(\frac{1}{u-\frac32+w_{i,r}}\mathrm{Res}_{u=-w_{i,r}+\frac32}+\frac{1}{u-\frac12+w_{i,r}}\mathrm{Res}_{u=w_{j,r'}+\frac12}\\+\frac{1}{u+\frac12+w_{i,r}}\mathrm{Res}_{u=w_{j,r'}-\frac12}\Big)\,\Xi_{i,r}(u)\chi_{i,r}.\end{aligned} \tag{4.30}$$

Now let $1\leqslant s\leqslant\mathfrak{v}_i$ with $s\neq r$. We have

$$\begin{aligned}X_s'&=\bigg[\chi_{i,r},\Big[\chi_{i,s},\frac{1}{u+\frac12-w_{j,s'}}\chi_{j,s'}\Big]\bigg]\\&=\frac{\chi_{i,r}\chi_{i,s}\chi_{j,s'}-\chi_{i,s}\chi_{j,s'}\chi_{i,r}}{u-\frac12-w_{j,s'}}+\frac{\chi_{j,s'}\chi_{i,s}\chi_{i,r}-\chi_{i,r}\chi_{j,s'}\chi_{i,s}}{u+\frac12-w_{j,s'}}.\end{aligned}$$

On the other hand, it follows from Lemma 4.4 that

$$X_s''=\bigg[\chi_{i,s},\Big[\chi_{i,r},\frac{1}{u+\frac12-w_{j,s'}}\chi_{j,s'}\Big]\bigg]$$

$$= \left[\chi_{i,s}, \frac{-1}{(u+\frac{1}{2}-w_{j,s'})(w_{i,r}-w_{j,s'}+\frac{1}{2})}\chi_{j,s'}\chi_{i,r}\right]$$
$$= \frac{-\chi_{i,s}\chi_{j,s'}\chi_{i,r}}{(u-\frac{1}{2}-w_{j,s'})(w_{i,r}-w_{j,s'}-\frac{1}{2})} + \frac{\chi_{i,r}\chi_{j,s'}\chi_{i,s}}{(u+\frac{1}{2}-w_{j,s'})(w_{i,r}-w_{j,s'}-\frac{1}{2})}$$

Summing up the above two expressions, we have

$$X'_s + X''_s = \frac{1}{u-\frac{1}{2}-w_{j,s'}}\Big(\chi_{i,r}\chi_{i,s}\chi_{j,s'} - \frac{w_{i,r}-w_{j,s'}+\frac{1}{2}}{w_{i,r}-w_{j,s'}-\frac{1}{2}}\chi_{i,s}\chi_{j,s'}\chi_{i,r}\Big)$$
$$+ \frac{1}{u+\frac{1}{2}-w_{j,s'}}\Big(\chi_{j,s'}\chi_{i,s}\chi_{i,r} - \frac{w_{i,r}-w_{j,s'}-\frac{3}{2}}{w_{i,r}-w_{j,s'}-\frac{1}{2}}\chi_{i,r}\chi_{j,s'}\chi_{i,s}\Big).$$

Using (4.22)–(4.23) along with the identities

$$\frac{(w_{j,s'}+w_{i,r}+1)(w_{j,s'}+\frac{3}{2}-w_{i,r})}{w_{j,s'}+w_{i,r}-1} + (w_{i,r}-w_{j,s'}+\tfrac{1}{2}) = \frac{4w_{j,s'}+1}{w_{j,s'}-1+w_{i,r}},$$
$$\frac{w_{j,s'}-\frac{1}{2}-w_{i,r}}{(w_{j,s'}+w_{i,r})(w_{j,s'}-1+w_{i,r})} + \frac{w_{i,r}-\frac{3}{2}-w_{j,s'}}{(w_{j,s'}-1+w_{i,r})(w_{j,s'}-2+w_{i,r})}$$
$$= \frac{4w_{j,s'}-1}{(w_{j,s'}+w_{i,r})(w_{j,s'}-1+w_{i,r})(w_{j,s'}-2+w_{i,r})},$$

we can rewrite the previous equation as

$$X'_s + X''_s = \Big(\frac{1}{u-\frac{1}{2}-w_{j,s'}}\mathrm{Res}_{u=w_{j,s'}+\frac{1}{2}} + \frac{1}{u+\frac{1}{2}-w_{j,s'}}\mathrm{Res}_{u=w_{j,s'}-\frac{1}{2}}\Big)\,\Xi_{i,r}(u)\chi_{i,r}. \tag{4.31}$$

Plugging (4.30)–(4.31) into (4.29) and applying Lemma 4.3 we obtain that $X_r = \big(\Xi_{i,r}(u)\big)^{\star}\chi_{i,r}$. Now comparing (4.25) with (4.28) completes the proof that the relation (2.9) is preserved by Φ^{λ}_{μ}.

This completes the proof of Theorem 3.6.

5. Identification of two definitions of TSTY

In this section, we make precise the connections between the TSTY for split type A (type AI) here with a distinguished family of TSTY defined in a very different way in [LPT+25, Section 11]; that version of TSTY was motivated by its connection to finite W-algebras of classical type.

5.1. **Shifted twisted Yangians and TSTY from** [LPT+25]**.** Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\tau = \mathrm{id}$. We consider the special case $\lambda = N\varpi_1^\vee$ where $\varpi_1^\vee$ is the first fundamental coweight. Let μ be a dominant even coweight such that $N\varpi_1^\vee \geqslant \mu$. Recall $\mathbf{v}_i \in \mathbb{N}$ from (3.1) and θ_i from (3.2). Define $p_i \in \mathbb{N}$ by

$$p_1 = \mathbf{v}_{n-1}, \quad p_i = \mathbf{v}_{n-i} - \mathbf{v}_{n-i+1}, \quad p_n = N - \mathbf{v}_1, \quad (1 < i < n). \tag{5.1}$$

Then $(p_n, p_{n-1}, \ldots, p_1)$ form a partition of N such that $p_{i+1} - p_i = \langle \mu, \alpha_{n-i}\rangle$. It follows that all p_i have the same parity. The parity assumption defined in [LPT+25, (11.1)] then is equivalent to the evenness condition of μ. This partition determines a pair (σ, ℓ), where $\ell = p_n$ and $\sigma = (\mathfrak{s}_{i,j})_{1\leqslant i,j\leqslant n}$ is a symmetric matrix satisfying $\mathfrak{s}_{i,i+1} = \frac{1}{2}(p_{i+1} - p_i)$ and $\mathfrak{s}_{i,j} + \mathfrak{s}_{j,m} = \mathfrak{s}_{i,m}$ provided $|i-j| + |j-m| = |i-m|$. We call σ a shift matrix.

The shifted twisted Yangian of type AI, denoted by $\mathcal{Y}_n^+(\sigma)$, can be defined in parabolic presentation in [LPT+25, §8.1] with a composition of n that is admissible to the shift matrix σ, or alternatively, in Drinfeld presentation in [LPT+25, §8.3]. It has been proved in [LPT+25, Proposition 8.9] that the two shifted twisted Yangians are isomorphic. Here we only recall the Drinfeld presentation.

Let $C = (c_{ij})_{1\leqslant i,j<n}$ be the Cartan matrix of type A_{n-1}, and set $c_{0j} = -\delta_{j,1}$.

Definition 5.1 ([LWZ25b, Theorem 5.3])**.** The twisted Yangian of type AI is the algebra $\mathcal{Y}_n^+$ over $\mathbb{C}$ generated by $\{\mathcal{H}_i^{(r)}\}_{r>0}$, $\{\mathcal{B}_j^{(r)}\}_{r>0}$, for $0 \leqslant i < n$ and $1 \leqslant j < n$, subject to the following relations, for $r_1, r_2, r, s \in \mathbb{Z}_{>0}$:

$$[\mathcal{H}_i^{(r)}, \mathcal{H}_j^{(s)}] = 0, \qquad \mathcal{H}_i^{(2r-1)} = 0, \tag{5.2}$$

$$[\mathcal{H}_i^{(r+1)}, \mathcal{B}_j^{(s)}] - [\mathcal{H}_i^{(r-1)}, \mathcal{B}_j^{(s+2)}] = c_{ij}[\mathcal{H}_i^{(r)}, \mathcal{B}_j^{(s+1)}]_+ + \frac{1}{4}c_{ij}^2[\mathcal{H}_i^{(r-1)}, \mathcal{B}_j^{(s)}], \tag{5.3}$$

$$[\mathcal{B}_i^{(r+1)}, \mathcal{B}_j^{(s)}] - [\mathcal{B}_i^{(r)}, \mathcal{B}_j^{(s+1)}] = \frac{c_{ij}}{2}[\mathcal{B}_i^{(r)}, \mathcal{B}_j^{(s)}]_+ - 2\delta_{ij}(-1)^r\mathcal{H}_i^{(r+s)}, \tag{5.4}$$

$$[\mathcal{B}_i^{(r)}, \mathcal{B}_j^{(s)}] = 0, \qquad \text{for } |i-j| > 1, \tag{5.5}$$

$$\begin{aligned} &\mathrm{Sym}_{r_1,r_2}[\mathcal{B}_i^{(r_1)}, [\mathcal{B}_i^{(r_2)}, \mathcal{B}_j^{(r)}]] = \\ &(-1)^{r_1}\sum_{p\geqslant 0} 2^{-2p}\big([\mathcal{H}_i^{(r_1+r_2-2p-1)}, \mathcal{B}_j^{(r+1)}] - [\mathcal{H}_i^{(r_1+r_2-2p-1)}, \mathcal{B}_j^{(r)}]_+\big), \quad \text{if } c_{i,j} = -1, \end{aligned} \tag{5.6}$$

for all admissible indices i, j, r, s. Here by convention, $\mathcal{H}_i^{(0)} = 1$.

Definition 5.2 ([LPT+25, Definition 8.7])**.** The (*dominantly*) *shifted Drinfeld twisted Yangian* associated to the shift matrix σ is the algebra $\mathcal{Y}_n^+(\sigma)$ over $\mathbb{C}$ generated by $\{\mathcal{H}_i^{(r)}\}_{r>0}$, $\{\mathcal{B}_j^{(r)}\}_{r>\mathfrak{s}_{j+1,j}}$, for $0 \leqslant i < n$ and $1 \leqslant j < n$, subject to the relations (5.2)–(5.6), for all admissible indices $r_1, r_2, r, s \in \mathbb{Z}_{>0}$.

The shifted twisted Yangian $\mathcal{Y}_n^+(\sigma)$ can be naturally identified as a subalgebra of $\mathcal{Y}_n^+$ by identifying elements with the same symbols.

The twisted Yangian $\mathcal{Y}_n^+$ also contains mutually commuting elements $\mathcal{D}_i^{(r)}$ for $1 \leqslant i \leqslant n$ and $r > 0$ defined as follows

$$\mathcal{H}_0(u) = \mathcal{D}_1(u), \qquad \mathcal{H}_i(u) = \big(\mathcal{D}_i(u - \tfrac{i}{2})\big)^{-1}\mathcal{D}_{i+1}(u - \tfrac{i}{2}), \qquad 1 \leqslant i < n,$$

where

$$\mathcal{H}_i(u) := 1 + \sum_{r>0}\mathcal{H}_i^{(r)}u^{-r}, \qquad \mathcal{D}_j(u) := 1 + \sum_{r>0}\mathcal{D}_j^{(r)}u^{-r}, \quad 0 \leqslant i < n,\ 1 \leqslant j \leqslant n,$$

see [LPT+25, §3.2] or [LWZ25b, §3].

Introduce $\mathcal{Q}_i(u)$ for $1 \leqslant i \leqslant n$ by the rule

$$\mathcal{Q}_i(u) := \mathcal{D}_1(u + \tfrac{i-1}{2})\mathcal{D}_2(u + \tfrac{i-3}{2})\cdots\mathcal{D}_{i-1}(u - \tfrac{i-3}{2})\mathcal{D}_i(u - \tfrac{i-1}{2}).$$

Then

$$\mathcal{H}_0(u) = \mathcal{D}_1(u) = \mathcal{Q}_1(u), \qquad \mathcal{H}_i(u) = \frac{\mathcal{Q}_{i-1}(u)\mathcal{Q}_{i+1}(u)}{\mathcal{Q}_i(u-\frac{1}{2})\mathcal{Q}_i(u+\frac{1}{2})}, \qquad 1 \leqslant i < n. \tag{5.7}$$

Moreover, the series $\mathcal{Q}_n(u)$ corresponds to the Sklyanin determinant $\mathrm{sdet}\, S(u+\frac{n-1}{2})$ which is an even series, see [Mol07, Theorems 2.5.3, 2.12.1].

Let $\mathcal{SY}_n^+$ be the subalgebra of $\mathcal{Y}_n^+$ generated by $\{\mathcal{H}_i^{(r)}, \mathcal{B}_i^{(r)}\}_{r>0}$, for $0<i<n$. Denote by $\mathcal{ZY}_n^+$ the center of $\mathcal{Y}_n^+$. Then it is known [LWZ25b, Theorem 5.1] that $\mathcal{SY}_n^+$ is isomorphic to the algebra generated by $\{\mathcal{H}_i^{(r)}, \mathcal{B}_i^{(r)}\}_{r>0}$ for $0<i<n$ subject to the relations (5.2)–(5.6). Define $\mathcal{Z}^{(r)}$ for $r>0$ by

$$\mathcal{Q}_n(u) = 1 + \sum_{r>0} \mathcal{Z}^{(r)} u^{-2r}. \tag{5.8}$$

Then it is known [Mol07, Theorem 2.8.2] that the center of $\mathcal{Y}_n^+$ is freely generated by the elements $\mathcal{Z}^{(r)}$ for $r>0$. Thus it follows from [Mol07, Theorem 2.9.2] that

$$\mathcal{Y}_n^+ = \mathcal{SY}_n^+ \otimes \mathcal{ZY}_n^+ = \mathcal{SY}_n^+ \otimes \mathbb{C}[\mathcal{Z}^{(r)} \mid r>0]. \tag{5.9}$$

Similarly, let $\mathcal{SY}_n^+(\sigma)$ be the subalgebra of $\mathcal{Y}_n^+$ generated by $\{\mathcal{H}_i^{(r)}\}_{r>0}$, $\{\mathcal{B}_i^{(r)}\}_{r>\mathfrak{s}_{i,i+1}}$, for $0<i<n$. Denote by $\mathcal{ZY}_n^+(\sigma)$ the center of $\mathcal{Y}_n^+(\sigma)$. By passing to the associated graded (the loop filtration), one obtains that $\mathcal{ZY}_n^+(\sigma) = \mathcal{ZY}_n^+$ (see e.g. [LPT+25, Remark 8.14]) and

$$\mathcal{Y}_n^+(\sigma) = \mathcal{SY}_n^+(\sigma) \otimes \mathcal{ZY}_n^+ = \mathcal{SY}_n^+(\sigma) \otimes \mathbb{C}[\mathcal{Z}^{(r)} \mid r>0]. \tag{5.10}$$

Lemma 5.3. *The map $\kappa : {}^\imath\mathcal{Y}_\mu \to \mathcal{SY}_n^+(\sigma)$ defined by the rule*

$$H_{n-i}^{(r)} \mapsto \mathcal{H}_i^{(r+2\mathfrak{s}_{i,i+1})}, \qquad B_{n-i}^{(s)} \mapsto \sqrt{(-1)^{\mathfrak{s}_{i,i+1}}}\mathcal{B}_i^{(s+\mathfrak{s}_{i,i+1})},$$

for $0<i<n$, $r>-2\mathfrak{s}_{i,i+1}$, $s>0$, uniquely induces an isomorphism of algebras.

Proof. It is clear that the map induces an epimorphism. Then it follows from comparing the PBW bases in Theorem 2.16 and [LWZ25b, Proposition 3.14] that it is indeed an isomorphism. □

Then we recall the truncation from [LPT+25, §11.2] in terms of the Drinfeld presentation rather than the parabolic presentation, thanks to [LPT+25, Proposition 11.5]. In this case, the composition ν is simply $(1,\dots,1)$, the element $H_{1;1,1}^{(r)}$ therein corresponds to $\mathcal{H}_0^{(r)}$ for $r>0$ and $B_{1;1,1}^{(s)}$ corresponds to $\mathcal{B}_1^{(s)}$ for $s>\mathfrak{s}_{1,2}$. Introduce elements $\widetilde{\mathcal{B}}_i^{(r+\mathfrak{s}_{1,2})}$ for $r>0$ by the rule

$$\mathcal{B}_1(u) = \sum_{r>0} \mathcal{B}_1^{(r+\mathfrak{s}_{1,2})} u^{-r}, \qquad \widetilde{\mathcal{B}}_1(u) := \mathcal{B}_1(u+\tfrac12)\mathcal{H}_0(u) = \sum_{r>0} \widetilde{\mathcal{B}}_i^{(r+\mathfrak{s}_{1,2})} u^{-r}. \tag{5.11}$$

Thus the element $\widetilde{B}_{1;1,1}^{(\mathfrak{s}_{1,2};p_1+1)}$ corresponds to $\widetilde{\mathcal{B}}_1^{(\mathfrak{s}_{1,2}+p_1+1)}$. Then the truncated shifted twisted Yangian (TSTY) $\mathcal{Y}_{n,\ell}^+(\sigma)$ in [LPT+25] is defined to be the quotient

$$\mathcal{Y}_{n,\ell}^+(\sigma) := \mathcal{Y}_n^+(\sigma)/\mathcal{I}_\ell, \tag{5.12}$$

where $\mathcal{I}_\ell$ is the 2-sided ideal of $\mathcal{Y}_n^+(\sigma)$ generated by

$$\{\mathcal{H}_0^{(r)} \mid r>p_1\} \cup \{\delta_{\bar{p}_1,\bar{0}} \widetilde{\mathcal{B}}_1^{(\mathfrak{s}_{1,2}+p_1+1)}\}. \tag{5.13}$$

It has been proved that in [LPT+25, Theorem F] that $\mathcal{Y}_n^+(\sigma)$ are isomorphic to finite W-algebra of classical types. Let $k = \lfloor \frac{N}{2} \rfloor$. The precise type is determined as follows:

$$\begin{cases} \mathrm{C}_k, & \text{if all } p_i \text{ are even,} \\ \mathrm{B}_k, & \text{if all } p_i \text{ and } n \text{ are odd,} \\ \mathrm{D}_k, & \text{if all } p_i \text{ are odd and } n \text{ are even.} \end{cases} \tag{5.14}$$

We shall slightly modify the iGKLO representations and establish the connections between the TSTY ${}^{\imath}\mathcal{Y}_\mu^{N\varpi_1^\vee}$ and $\mathcal{Y}_n^+(\sigma)$.

Under the choice of the data, we have $\lambda = N\varpi_1^\vee$ and hence $\mathbf{w}_1 = N$ and $\mathbf{w}_i = 0$ for $i > 1$. Thus by the conventions from §3.1–§3.2, only for $i = 1$, the parameters $\mathfrak{w}_i, \varsigma_i$ and the polynomials $\mathbf{Z}_i(u)$ are nontrivial. For that reason we shall drop the index $i = 1$. Let $\varsigma = N - 2k$. We have

$$\mathbf{Z}(u) := \mathbf{Z}_1(u) = u^{\varsigma} \prod_{s=1}^{k} (u - z_s^2), \qquad \mathbf{Z}_i(u) = 1, \tag{5.15}$$

for $i > 1$.

Recall that all p_i have the same parity, which we denote by θ. It is easy to see from (3.2)–(3.3) that $\vartheta_i = \theta$ for all $i \in \mathbb{I}$. Hence the action of $H_i(u)$ in the iGKLO representation $\Phi_\mu^{N\varpi_1^\vee}$ is given as follows

$$H_i(u) \mapsto \frac{\boldsymbol{\varkappa}(u)^{\theta} \mathbf{Z}_i(u)}{\mathbf{W}_i(u - \frac{1}{2}) \mathbf{W}_i(u + \frac{1}{2})} \prod_{j \leftrightarrow i} \mathbf{W}_j(u), \qquad 1 \leqslant i < n.$$

We introduce some variant of the algebra ${}^{\imath}\mathcal{Y}_\mu[\boldsymbol{z}]$:

$${}^{\imath}\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k} := {}^{\imath}\mathcal{Y}_\mu \otimes \mathbb{C}[z_1^2, \ldots, z_k^2]^{S_k}, \qquad {}^{\imath}\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee} := \Phi_\mu^{N\varpi_1^\vee}\big({}^{\imath}\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}\big),$$

where $\mathbb{C}[z_1^2, \ldots, z_k^2]^{S_k}$ is the algebra of symmetric polynomials in $z_1^2, \ldots, z_k^2$.

Introduce formal variables $\mathsf{Z}^{(r)}$ for $r > 0$ and consider the tensor product of algebras

$${}^{\imath}\mathcal{Y}_\mu[\mathsf{Z}] := {}^{\imath}\mathcal{Y}_\mu \otimes \mathbb{C}[\mathsf{Z}^{(r)} \mid r > 0]. \tag{5.16}$$

Set

$$\mathsf{Z}(u) = 1 + \sum_{r>0} \mathsf{Z}^{(r)} u^{-2r}. \tag{5.17}$$

Now we have obtained four algebras $\mathcal{Y}_n^+(\sigma)$, ${}^{\imath}\mathcal{Y}_\mu[\mathsf{Z}]$, ${}^{\imath}\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}$, ${}^{\imath}\widetilde{\mathcal{Y}}_\mu^{N\varpi_1}$. Since both ${}^{\imath}\mathcal{Y}_\mu[\mathsf{Z}]$ and ${}^{\imath}\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}$ are extensions of ${}^{\imath}\mathcal{Y}_\mu$ by tensoring with a polynomial algebra, it is easy to see we have a surjective homomorphism ${}^{\imath}\mathcal{Y}_\mu[\mathsf{Z}] \twoheadrightarrow {}^{\imath}\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}$ via specializing $\mathsf{Z}(u)$ to $u^{-N}\mathbf{Z}(u)$, where $\mathbf{Z}(u)$ is given in (5.15).

Introduce elements $\mathsf{A}_i^{(r)}$ and $\mathsf{B}_i^{(r)}$ in ${}^{\imath}\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}$ for $r > 0$, respectively, by (3.28) and (3.30). Also, introduce $\mathsf{A}_i^{(r)}$ (we use the same notation as they are canonically identified under the quotient map ${}^{\imath}\mathcal{Y}_\mu[\mathsf{Z}] \twoheadrightarrow {}^{\imath}\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}$) in ${}^{\imath}\mathcal{Y}_\mu[\mathsf{Z}]$ for $r > 0$ by

$$H_i(u) = \frac{\boldsymbol{\varkappa}(u)^{\theta} (u^N \mathsf{Z}(u))^{\delta_{1,i}} \prod_{j \leftrightarrow i} u^{\mathbf{v}_j}}{(u^2 - \frac{1}{4})^{\mathbf{v}_i}} \frac{\prod_{j \leftrightarrow i} \mathsf{A}_j(u)}{\mathsf{A}_i(u - \frac{1}{2}) \mathsf{A}_i(u + \frac{1}{2})}, \tag{5.18}$$

where $\mathsf{A}_i^{(r)}$ are the coefficients of $\mathsf{A}_i(u)$ as in (3.29). Recall the numbers $\mathbf{v}_i$ from (3.1). Define a family of new numbers q_i for $0 \leqslant i < n$ by

$$q_i := (\mathbf{v}_i - \theta) + 2\sum_{j=1}^{n-i-1}(-1)^j(\mathbf{v}_{i+j} - \theta), \tag{5.19}$$

where by convention $\mathbf{v}_0 = N$. If $\theta = 0$, then all $\theta_i = 0$. If $\theta = 1$, then θ_i is equal to the parity of $n - i$.

Lemma 5.4. *There exists a unique isomorphism $\tilde{\kappa} : {}^{\imath}\mathcal{Y}_\mu[\mathsf{Z}] \to \mathcal{Y}_n^+(\sigma)$ which extends κ from Lemma 5.3 by letting*

$$\mathsf{Z}(u) \mapsto \mathcal{Q}_n(u)\prod_{i=1}^{n-1}\big(1 - (\tfrac{i}{2u})^2\big)^{q_i}.$$

Moreover, under the isomorphism $\tilde{\kappa}$, we have

$$\mathsf{A}_{n-i}(u) \mapsto \mathcal{Q}_i(u)\prod_{j=1}^{i-1}\big(1 - (\tfrac{i-j}{2u})^2\big)^{q_{n-j}}, \qquad 1 \leqslant i < n.$$

Proof. The first statement follows from Lemma 5.3, (5.8), (5.10), and (5.16)–(5.17).

Now we prove the second statement. Denote

$$\mathfrak{P}_i(u) := \prod_{j=1}^{i-1}\big(1 - (\tfrac{i-j}{2u})^2\big)^{q_{n-j}}, \qquad 1 \leqslant i \leqslant n. \tag{5.20}$$

Recall the definition of $\mathsf{A}_i(u)$ from (5.18) and the relation

$$\kappa(H_i(u)) = u^{\langle\mu,\alpha_i\rangle}\mathcal{H}_{n-i}(u) = u^{\langle\mu,\alpha_i\rangle}\frac{\mathcal{Q}_{n-i-1}(u)\mathcal{Q}_{n-i+1}(u)}{\mathcal{Q}_{n-i}(u-\frac{1}{2})\mathcal{Q}_{n-i}(u+\frac{1}{2})}$$

from (5.7). Thus it suffices to verify that

$$\big(1 - (\tfrac{1}{2u})^2\big)^{\theta - \mathbf{v}_i}\frac{\mathfrak{P}_{n-i-1}(u)\mathfrak{P}_{n-i+1}(u)}{\mathfrak{P}_{n-i}(u-\frac{1}{2})\mathfrak{P}_{n-i}(u+\frac{1}{2})} = 1,$$

which follows from a straightforward calculation using (5.19) and (5.20). □

Remark 5.5. If all p_i are even, then $\theta_i = 0$ and the polynomial

$$\mathcal{P}_n(u) := u^{q_0}\prod_{j=1}^{n-1}\big(u^2 - (\tfrac{i}{2})^2\big)^{q_i} \tag{5.21}$$

coincides with the polynomial in [LPT$^+$25, Proposition 12.5] for ℓ even (associated to the symmetric pyramid determined by the partition $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$).

Similarly, if all p_i are odd, then the polynomial $(u + \frac{1}{2})\mathcal{P}_n(u)$ coincides with the polynomial in [LPT$^+$25, Proposition 12.5] for ℓ odd. Note that it is not hard to prove that the extra factor $u + \rho_0 = u + \frac{1}{2}$ there (for ℓ odd) can be removed so that the RHS of [LPT$^+$25, Proposition 12.5] remains to be a polynomial in u.

5.2. **Identifying two TSTY's.** We introduced a variant of the TSTY ${}^\imath\mathcal{Y}_\mu^{N\varpi_1^\vee}$, by restricting the allowed elements coming from $\mathbb{C}[\boldsymbol{z}]$:

$$ {}^\imath\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee} := \Phi_\mu^{N\varpi_1^\vee}\big({}^\imath\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}\big). \tag{5.22} $$

Proposition 5.6. *There exists an epimorphism $\mathcal{Y}_{n,\ell}^+(\sigma) \twoheadrightarrow {}^\imath\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee}$.*

Proof. By the discussion above, the following composition,

$$ \mathcal{Y}_n^+(\sigma) \xrightarrow{\sim} {}^\imath\mathcal{Y}_\mu[\mathbf{Z}] \twoheadrightarrow {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k} \twoheadrightarrow {}^\imath\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee}, $$

is an epimorphism, where the first isomorphism is from Lemma 5.4, the second epimorphism is obtained by specializing $\mathbf{Z}(u)$ to $u^{-N}\mathbf{Z}(u)$, and the last one by definition.

Note that the two-sided ideal generated by $\mathsf{A}_i^{(r)}$, $\delta_{\bar{\mathbf{v}}_i,\bar{0}}\mathsf{B}_i^{(r)}$ for $i \in \mathbb{I}$, $r > \mathbf{v}_i$ is contained in the kernel of the quotient ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k} \twoheadrightarrow {}^\imath\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee}$. By Lemma 5.4, we find that $\mathcal{H}_0(u) = \mathcal{Q}_1(u)$ in $\mathcal{Y}_n^+(\sigma)$ is sent to $\mathsf{A}_{n-1}(u)$ in ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}$. If $\mathbf{v}_{n-1}$ is even, then it follows from Lemma 5.3, (3.30), and (5.11) that $\widetilde{\mathcal{B}}_1^{(\mathfrak{s}_{1,2}+p_1+1)}$ in $\mathcal{Y}_n^+(\sigma)$ is sent to $\mathsf{B}_{n-1}^{(p_1+1)} = \mathsf{B}_{n-1}^{(\mathbf{v}_{n-1}+1)}$ in ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}$. Thus it follows from (5.12)-(5.13) that the ideal $\mathcal{I}_\ell$ is sent to zero. Thus we have an epimorphism $\mathcal{Y}_{n,\ell}^+(\sigma) \twoheadrightarrow {}^\imath\widetilde{\mathcal{Y}}_\mu^{N\varpi_1^\vee}$. ☐

Recall from [LPT+25, Corollary 11.11], that $\mathcal{Y}_{n,\ell}^+(\sigma)$ quantizes the Slodowy slice $\mathcal{S}_\pi^\epsilon$ for classical Lie algebra $\mathfrak{sl}_N^\epsilon$ in (9.3), where π is the partition $(p_n, \ldots, p_1)$ of N; cf. (5.1).

Lemma 5.7. *The number of PBW generators for $\mathcal{Y}_{n,\ell}^+(\sigma)$ is equal to $\dim \mathcal{S}_\pi^\epsilon$, which is given by $2\sum_{i=1}^{n-1}\mathfrak{v}_i + k$.*

Proof. The first equality holds by [LPT+25, Corollary 11.11]. By [LPT+25, Corollary 11.9] (and taking the admissible shape there to be $(1, 1, \ldots, 1)$), this number of PBW generators is given by

$$ \sum_{a=1}^{n-1}(n-a)p_a + \sum_{a=1}^{n}\lfloor \tfrac{p_a}{2} \rfloor. \tag{5.23} $$

By (5.1), we have

$$ \sum_{a=1}^{b} p_a = \mathbf{v}_{n-b}, \qquad \text{for } 1 \leqslant b \leqslant n-1, \tag{5.24} $$

and thus by (3.2) we have

$$ \sum_{a=1}^{n-1}(n-a)p_a = \sum_{i=1}^{n-1}\mathbf{v}_i = 2\sum_{i=1}^{n-1}\mathfrak{v}_i + \sum_{i=1}^{n-1}\theta_i. \tag{5.25} $$

Recall that all p_a have the same parity, $\sum_{a=1}^{n} p_a = N$, and $k = \lfloor \frac{N}{2} \rfloor$. Calculating θ_i in (3.2) using (5.24) case-by-case, we have

$$ \sum_{i=1}^{n-1}\theta_i + \sum_{a=1}^{n}\lfloor \tfrac{p_a}{2} \rfloor = \begin{cases} 0 + N/2 = k, & \text{if all } p_i \text{ are even,} \\ (n-1)/2 + (N-n)/2 = k, & \text{if all } p_i \text{ and } n \text{ are odd,} \\ n/2 + (N-n)/2 = k, & \text{if all } p_i \text{ are odd and } n \text{ are even.} \end{cases} $$

The lemma now follows by plugging this last formula and (5.25) into (5.23). ☐

Theorem 5.8. *The two TSTY's* $\mathcal{Y}^+_{n,\ell}(\sigma)$ *and* ${}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu$ *are isomorphic.*

Proof. In the proof, we will heavily use the notion of *Gelfand-Kirillov dimension* $\mathrm{GKdim}\, A$ of an algebra A. We refer the reader to [KL00] for a detailed overview.

We shall prove that the epimorphism $\mathcal{Y}^+_{n,\ell}(\sigma) \twoheadrightarrow {}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu$ from Proposition 5.6 is an isomorphism.

By [LPT+25, Corollary 11.11], there is an isomorphism $\mathrm{gr}'\ \mathcal{Y}^+_{n,\ell}(\sigma) \cong \mathbb{C}[\mathcal{S}^\epsilon_\pi]$ where $\mathcal{S}^\epsilon_\pi$ is a Slodowy slice. Since $\mathbb{C}[\mathcal{S}^\epsilon_\pi]$ is a polynomial ring, it follows that $\mathcal{Y}^+_{n,\ell}(\sigma)$ is a domain. The algebra $\mathcal{Y}^+_{n,\ell}(\sigma)$ has a PBW basis in [LPT+25, Corollary 11.9], and thus its Gelfand-Kirillov dimension can be computed by counting the number of PBW generators, which by Lemma 5.7 is equal to

$$\mathrm{GKdim}\,\mathcal{Y}^+_{n,\ell}(\sigma) = \dim \mathcal{S}^\epsilon_\pi = 2\sum_{i=1}^{n-1} \mathfrak{v}_i + k.$$

Claim. We have $\mathrm{GKdim}\,{}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu \geqslant 2\sum_{i=1}^{n-1}\mathfrak{v}_i + k$.

Assuming this claim for the moment, we deduce that

$$\mathrm{GKdim}\,\mathcal{Y}^+_{n,\ell}(\sigma) \leqslant \mathrm{GKdim}\,{}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu. \tag{5.26}$$

Since $\mathcal{Y}^+_{n,\ell}(\sigma)$ is a domain, by [KL00, Proposition 3.15] the Gelfand-Kirillov dimension of any of its proper quotients is strictly smaller. From the inequality (5.26), we conclude that the epimorphism $\mathcal{Y}^+_{n,\ell}(\sigma) \twoheadrightarrow {}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu$ must be an isomorphism.

It remains to prove the claim. Consider the quotient filtration $F^\bullet_{\mu_1}\,{}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu$ inherited from ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k}$ as in §3.4.2. Since Gelfand-Kirillov dimensions can only decrease upon passing to associated graded algebras [KL00, Lemma 6.5], we have $\mathrm{GKdim}\,{}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu \geqslant \mathrm{GKdim}\,\mathrm{gr}\,{}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu$. It is also a general fact that if $f : A \to B$ is a filtered map of filtered algebras with associated graded $\mathrm{gr} f : \mathrm{gr} A \to \mathrm{gr} B$, then $\mathrm{Im}(f)$ inherits a quotient filtration from A, and there is a surjection of graded algebras $\mathrm{gr}\,\mathrm{Im}(f) \twoheadrightarrow \mathrm{Im}(\mathrm{gr} f)$. Applied to the map $\Phi^{N\varpi_1^\vee}_\mu : {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}^2]^{S_k} \to \mathcal{A}$, we obtain

$$\mathrm{gr}\,{}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu = \mathrm{gr}\,\mathrm{Im}(\Phi^{N\varpi_1^\vee}_\mu) \twoheadrightarrow \mathrm{Im}(\mathrm{gr}\ \Phi^{N\varpi_1^\vee}_\mu).$$

It follows from Theorem 8.12 that the Gelfand-Kirillov dimension of $\mathrm{Im}(\mathrm{gr}\ \Phi^\lambda_\mu)$ is exactly $2\sum_i \mathfrak{v}_i + k$. (Recall that for a commutative algebra the Gelfand-Kirillov dimension is equal to the Krull dimension [KL00, Proposition 7.9].) Altogether we have

$$\mathrm{GKdim}\,{}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu \geqslant \mathrm{GKdim}\,\mathrm{gr}\,{}^\imath\widetilde{\mathcal{Y}}^{N\varpi_1^\vee}_\mu \geqslant \mathrm{GKdim}\,\mathrm{Im}(\mathrm{gr}\ \Phi^{N\varpi_1^\vee}_\mu) = 2\sum_{i=1}^{n-1}\mathfrak{v}_i + k,$$

proving the claim. This completes the proof of the theorem. □

Part 2. Geometry

6. Twisted Yangians via quantum duality principle and Dirac reduction

In this section, we establish a general relationship between twisted Yangians and fixed point loci in loop groups by a Poisson involution σ, by applying the quantum duality principle and Dirac reduction. This serves as motivation for our constructions and results in subsequent sections, but it may be of independent interest.

6.1. **Yangians and quantum duality principle.** Let G be an affine algebraic group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$. Then we may consider the corresponding loop groups such as $G(\!(z^{-1})\!), G[z]$ and $G[\![z^{-1}]\!]$, whose Lie algebras are $\mathfrak{g}(\!(z^{-1})\!)$, $\mathfrak{g}[z]$ and $\mathfrak{g}[\![z^{-1}]\!]$, respectively. More precisely, these Lie algebras are defined via base change, such as $\mathfrak{g}(\!(z^{-1})\!) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}(\!(z^{-1})\!)$, while the loop groups are defined as (ind-)schemes via their functors of points. For example, $G(\!(z^{-1})\!)$ represents the functor sending each $\mathbb{C}$–algebra R to the group $G\big(R(\!(z^{-1})\!)\big)$. Finally, we will denote by $G_1[\![z^{-1}]\!]$ the kernel of the evaluation $G[\![z^{-1}]\!] \to G$ at $z^{-1} = 0$. Then $G_1[\![z^{-1}]\!]$ is an affine group scheme with Lie algebra $z^{-1}\mathfrak{g}[\![z^{-1}]\!]$, and the exponential map defines an isomorphism of affine schemes

$$\exp : z^{-1}\mathfrak{g}[\![z^{-1}]\!] \xrightarrow{\sim} G_1[\![z^{-1}]\!]. \tag{6.1}$$

Suppose that $\mathfrak{g}$ is equipped with a non-degenerate invariant symmetric bilinear form $(\cdot,\cdot)_{\mathfrak{g}}$. In the standard way [CP94, Example 1.3.9] we may form a Manin triple $\big(\mathfrak{g}(\!(z^{-1})\!), \mathfrak{g}[z], z^{-1}\mathfrak{g}[\![z^{-1}]\!]\big)$, equipped with the non-degenerate symmetric invariant bilinear form

$$(x, y) = -\operatorname{Res}_{z=0}(x, y)_{\mathfrak{g}}, \qquad x, y \in \mathfrak{g}(\!(z^{-1})\!). \tag{6.2}$$

This endows $\mathfrak{g}(\!(z^{-1})\!)$ with a Lie bialgebra structure, with $\mathfrak{g}[z]$ as a sub-Lie bialgebra. The Lie algebra $\mathfrak{g}[z]$ is $\mathbb{Z}$–graded with respect to degree in z, and its Lie cobracket is homogeneous of degree -1. Its dual $z^{-1}\mathfrak{g}[\![z^{-1}]\!]$ is also naturally a Lie bialgebra, corresponding to the Manin triple $\big(\mathfrak{g}(\!(z^{-1})\!), z^{-1}\mathfrak{g}[\![z^{-1}]\!], \mathfrak{g}[z]\big)$ with respect to the same bilinear form (6.2). The groups $G(\!(z^{-1})\!)$, $G[z]$ and $G_1[\![z^{-1}]\!]$ each carry corresponding Poisson structures.

We now turn to quantization. Consider $\mathbb{C}[\hbar]$ as a graded ring with $\deg \hbar = 1$.

Definition 6.1. A *Yangian* for $(\mathfrak{g}, (\cdot,\cdot)_{\mathfrak{g}})$ is a $\mathbb{Z}$-graded Hopf algebra $\mathcal{U}_\hbar(\mathfrak{g}[z])$ over $\mathbb{C}[\hbar]$, which is free as a graded module over $\mathbb{C}[\hbar]$ and quantizes the Lie bialgebra $\mathfrak{g}[z]$ in the sense that

$$\mathcal{U}_\hbar(\mathfrak{g}[z])/\hbar\mathcal{U}_\hbar(\mathfrak{g}[z]) \cong U(\mathfrak{g}[z]) \tag{6.3}$$

as graded co-Poisson-Hopf algebras.

For each integer $n \geqslant 0$ define a map $\Delta^n : \mathcal{U}_\hbar(\mathfrak{g}[z]) \to \mathcal{U}_\hbar(\mathfrak{g}[z])^{\otimes n}$, where $\Delta^0 = \varepsilon$ is the counit, $\Delta^1 = \mathrm{Id}$, and $\Delta^n = (\Delta \otimes \mathrm{Id}^{\otimes n-2}) \circ \Delta^{n-1}$ for $n \geqslant 2$. The *Quantum Duality Principle* [Dri87b, Gav02] (also called *Drinfeld-Gavarini Duality*) defines a subalgebra:

$$\mathcal{U}_\hbar(\mathfrak{g}[z])' = \big\{a \in \mathcal{U}_\hbar(\mathfrak{g}[z]) \mid (\mathrm{Id} - \varepsilon)^{\otimes n} \circ \Delta^n(a) \in \hbar^n\mathcal{U}_\hbar(\mathfrak{g}[z])^{\otimes n} \text{ for all } n \geqslant 0\big\}. \tag{6.4}$$

Then $\mathcal{U}_\hbar(\mathfrak{g}[z])'$ is a graded sub-Hopf algebra over $\mathbb{C}[\hbar]$, which is almost-commutative in the sense that for any $a, b \in \mathcal{U}_\hbar(\mathfrak{g}[z])'$ we have $[a, b] \in \hbar\mathcal{U}_\hbar(\mathfrak{g}[z])'$. Following [FT19b, §A(ii)], it will be useful to make the following technical assumption:

$$\begin{gathered}\text{There is a subset } \{x_\gamma\} \subset \mathcal{U}_\hbar(\mathfrak{g}[z]) \text{ consisting of homogeneous elements,}\\ \text{which lifts a basis } \{\overline{x_\gamma}\} \text{ for } \mathfrak{g}[z], \text{ and such that all } \hbar x_\gamma \in \mathcal{U}_\hbar(\mathfrak{g}[z])'.\end{gathered} \tag{6.5}$$

Under this assumption, any homogeneous basis for $\mathfrak{g}[z]$ admits a similar lift. Moreover, an induction on degree shows that $\mathcal{U}_\hbar(\mathfrak{g}[z])$ has a PBW basis over $\mathbb{C}[\hbar]$ consisting of ordered monomials in the set $\{x_\gamma\}$, with respect to any given total order.

Remark 6.2. If $\mathfrak{g}$ is a simple Lie algebra then the Yangian $\mathcal{U}_\hbar(\mathfrak{g}[z])$ is unique up to isomorphism [Dri85, Theorem 2], and thus Assumption (6.5) holds by [FT19b, Lemma A.20], see also [Wen24,

§5.2]. If $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathcal{U}_\hbar(\mathfrak{g}[z])$ is the usual Yangian defined in RTT presentation as in [Mol07], then Assumption (6.5) holds by [FT19b, Proposition A.25].

In fact, based on related results for formal quantizations of finite-dimensional Lie bialgebras from [Gav02], it seems likely that Assumption (6.5) holds for *any* Lie algebra $\mathfrak{g}$ with a non-degenerate invariant symmetric bilinear form and corresponding Yangian $\mathcal{U}_\hbar(\mathfrak{g}[z])$. We will not address this question here.

We have the following generalization of [KWWY14, Theorem 3.9] and [Sha16, Remark 1.12]:

Proposition 6.3. *Let $\mathcal{U}_\hbar(\mathfrak{g}[z])$ be a Yangian satisfying Assumption (6.5). Then:*

(1) *$\mathcal{U}_\hbar(\mathfrak{g}[z])'$ is free over $\mathbb{C}[\hbar]$, with a PBW basis consisting of ordered monomials in the elements $\{\hbar x_\gamma\}$ with respect to any given total order.*
(2) *$\mathcal{U}_\hbar(\mathfrak{g}[z])'$ is a quantization of the Poisson algebraic group $G_1[\![z^{-1}]\!]$. In other words, there is an isomorphism of Poisson-Hopf algebras*

$$\mathcal{U}_\hbar(\mathfrak{g}[z])'/\hbar\mathcal{U}_\hbar(\mathfrak{g}[z])' \;\cong\; \mathbb{C}\big[G_1[\![z^{-1}]\!]\big].$$

In particular, these results hold for Yangian $\mathcal{U}_\hbar(\mathfrak{g}[z])$ when $\mathfrak{g}$ is simple, and for the (RTT) Yangian of $\mathfrak{g} = \mathfrak{gl}_n$.

Proof. Part (1) is an application of [FT19b, Proposition A.5].

For Part (2), first note that the Hopf algebras $\mathbb{C}[G_1[\![z^{-1}]\!]]$ and $U(z^{-1}\mathfrak{g}[z^{-1}])$ are graded dual to one another, via the perfect Hopf pairing defined by $(X, f) = (X \cdot f)(e)$ for any $X \in U(z^{-1}\mathfrak{g}[z^{-1}])$ and $f \in \mathbb{C}\big[G_1[\![z^{-1}]\!]\big]$. Here $X \cdot f$ denotes the action of X thought of as a left-invariant differential operator and $e \in G_1[\![z^{-1}]\!]$ denotes the identity element. Meanwhile, the Quantum Duality Principle provides a non-degenerate graded Hopf pairing between $U(z^{-1}\mathfrak{g}[z^{-1}])$ and the classical limit $\mathcal{U}_\hbar(\mathfrak{g}[z])'/\hbar\mathcal{U}_\hbar(\mathfrak{g}[z])'$. As in [KWWY14, Corollary 3.4], this pairing is uniquely determined by the property that

$$\langle y, \hbar x_\gamma + \hbar\mathcal{U}_\hbar(\mathfrak{g}[z])' \rangle = (y, \overline{x_\gamma}) \tag{6.6}$$

for all $y \in z^{-1}\mathfrak{g}[z^{-1}]$ and generators $\hbar x_\gamma \in U_\hbar(\mathfrak{g}[z])'$. This pairing defines an injective map $\mathcal{U}_\hbar(\mathfrak{g}[z])'/\hbar\mathcal{U}_\hbar(\mathfrak{g}[z])' \to \mathbb{C}\big[G_1[\![z^{-1}]\!]\big]$ of graded Hopf algebras, which is an isomorphism (cf. the proof of [KWWY14, Theorem 3.9]) since both have Hilbert series $\prod_{r\geqslant 1}(1-q^r)^{-\dim\mathfrak{g}}$. Finally, the Quantum Duality Principle [Gav02] identifies the Poisson structures. □

Remark 6.4. We may also specialize $\hbar = 1$ and work with the filtered $\mathbb{C}$–algebra $\mathcal{U}_{\hbar=1}(\mathfrak{g}[z])'$. Then $\mathbb{C}\big[G_1[\![z^{-1}]\!]\big]$ is the associated graded algebra, and we recover $\mathcal{U}_\hbar(\mathfrak{g}[z])'$ via the Rees algebra construction, see for example [FT19b, §A(iii)].

6.2. **Dirac reduction and fixed points.** Let $X = \operatorname{Spec} A$ be an affine Poisson scheme over $\mathbb{C}$. In other words, A is a commutative algebra over $\mathbb{C}$ which is equipped with a Poisson bracket. Let σ be an algebra involution of A, which respects the Poisson structure; one may equivalently think of σ as defining an action of the group $\mathbb{Z}/2\mathbb{Z}$ on X by Poisson automorphisms. The corresponding *fixed point scheme*

$$X^\sigma = X^{\mathbb{Z}/2\mathbb{Z}} = \operatorname{Spec} R(A,\sigma), \tag{6.7}$$

is defined via the quotient algebra $R(A,\sigma) = A/\langle \sigma(a) - a : a \in A\rangle$. In particular $X^\sigma \subseteq X$ is a closed subscheme.

Remark 6.5. Recall that schemes can also be described by their functors of points (see e.g. [GW10, §4.1]). For any test scheme S, an element $x \in X(S)$ is simply a morphism $x : S \to X$. The functor corresponding to X^σ has a very simple description [Fog73]:

$$X^\sigma(S) = \big\{ x \in X(S) \mid \sigma(x) = x \big\}.$$

Here $\sigma(x)$ denotes the composition $S \xrightarrow{x} X \xrightarrow{\sigma} X$. In particular, this provides a definition of X^σ even when X is not affine.

Lemma 6.6 ([Edi92, Prop. 3.4])**.** *If X is smooth, then X^σ is smooth.*

The fixed-point scheme X^σ has a natural Poisson structure defined via *Dirac reduction*, see [Xu03, §4.1] or [Top23, §2.1]. To describe this structure, let $A = A(1) \oplus A(-1)$ denote the eigenspace decomposition for the action of σ. Then the inclusion $A(1) \subset A$ induces a natural identification

$$R(A, \sigma) \equiv A/\langle A(-1)\rangle \cong A(1)/A(-1)^2. \tag{6.8}$$

It is easy to see that $A(1) \subset A$ is a Poisson subalgebra, and that $A(-1)^2 \subset A(1)$ is a Poisson ideal. In this way the quotient $A(1)/A(-1)^2$ inherits a Poisson structure.

Remark 6.7. Explicitly, given two functions f, g on X^σ, their Poisson bracket is defined by choosing σ–invariant extensions $\tilde{f}, \tilde{g}$ to functions on X, computing the Poisson bracket $\{\tilde{f}, \tilde{g}\}$ on X, and then restricting the result to X^σ.

Lemma 6.8. *Assume that X has finitely many symplectic leaves. Then X^σ also has finitely many symplectic leaves, i.e., the connected components of the fixed point loci $\mathcal{S}^\sigma$, for those leaves $\mathcal{S}$ of X stable under the action of σ.*

Proof. Since X has finitely many symplectic leaves, each leaf $\mathcal{S} \subseteq X$ is a smooth locally-closed subvariety by [BG03, Theorem 3.7]. Its image $\sigma(\mathcal{S})$ is also a leaf, and thus the involution σ permutes the set of all symplectic leaves. It follows that the fixed point locus X^σ is the union of fixed point loci $\mathcal{S}^\sigma$, this union being taken over the set of all leaves $\mathcal{S}$ satisfying $\sigma(\mathcal{S}) = \mathcal{S}$. For each leaf satisfying $\sigma(\mathcal{S}) = \mathcal{S}$, since $\mathcal{S}$ is smooth, $\mathcal{S}^\sigma$ is also smooth by Lemma 6.6. Moreover the symplectic leaves of $\mathcal{S}^\sigma$ are precisely its (finitely many) connected components, by [Xu03, Theorem 2.3]. Since the inclusions $\mathcal{S}^\sigma \subset X^\sigma$ are Poisson by construction, the claim follows. □

We conclude with two results for later use. For simplicity we ignore Poisson structures.

Lemma 6.9 ([Top23, Lemma 2.2].)**.** *Let V be a vector space equipped with an involution σ, with corresponding eigenspaces $V = V(1) \oplus V(-1)$. Extend σ to an involution of the symmetric algebra $S(V)$ in the natural way. If $U \subset V$ is a subspace complementary to $V(-1)$, then the composition*

$$S(U) \hookrightarrow S(V) \twoheadrightarrow R\big(S(V), \sigma\big)$$

is an isomorphism of algebras. In particular, this applies to $U = V(1)$.

Lemma 6.10. *Let X, Y, Z be schemes, each equipped with an involution σ, and let $\alpha : X \to Z$ and $\beta : Y \to Z$ be σ–equivariant maps. Then there is an equality of schemes:*

$$(X \times_Z Y)^\sigma = X^\sigma \times_{Z^\sigma} Y^\sigma.$$

Proof. Recall the description of a fiber product as a functor of points, see e.g. [GW10, Chapter 4]:

$$(X \times_Z Y)(S) = \{(x, y) \in X(S) \times Y(S) : \alpha(x) = \beta(y) \in Z(S)\}.$$

Using Remark 6.5, we see that:

$$(X \times_Z Y)^\sigma(S) = \{(x, y) \in X(S) \times Y(S) : \alpha(x) = \beta(y) \in Z(S),\ \sigma(x) = x,\ \sigma(y) = y\}.$$

For any pair (x, y) on the right side, since α, β are σ-equivariant the image $z = \alpha(x) = \beta(y)$ satisfies $\sigma(z) = z$, so $z \in Z^\sigma(S)$. Therefore $(X^\sigma \times_{Z^\sigma} Y^\sigma)(S) = (X \times_Z Y)^\sigma(S)$. □

6.3. **Loop symmetric spaces.** Now let ω be an involution of G, and denote the corresponding involution of $\mathfrak{g}$ by the same notation. We will assume that the bilinear form $(\cdot, \cdot)_\mathfrak{g}$ on $\mathfrak{g}$ is ω-invariant:

$$(\omega(x), \omega(y))_\mathfrak{g} = (x, y)_\mathfrak{g}, \qquad \forall x, y \in \mathfrak{g}. \tag{6.9}$$

Extend ω to an involution, again denoted by ω, of $\mathfrak{g}(\!(z^{-1})\!)$ via

$$\omega(x \otimes z^k) = \omega(x) \otimes (-z)^k. \tag{6.10}$$

This involution preserves the subalgebras $\mathfrak{g}[z]$ and $z^{-1}\mathfrak{g}[\![z^{-1}]\!]$, and there is a natural corresponding involution ω of the group $G(\!(z^{-1})\!)$, preserving its subgroups $G[z]$ and $G_1[\![z^{-1}]\!]$. Viewed as an involution of any of these groups ω is anti-Poisson, since ω is anti-symmetric for the bilinear form (6.2) on $\mathfrak{g}(\!(z^{-1})\!)$.

Lemma 6.11. *The fixed-point subgroup $G_1[\![z^{-1}]\!]^\omega$ is coisotropic, i.e., its defining ideal $J \subset \mathbb{C}\big[G_1[\![z^{-1}]\!]\big]$ satisfies $\{J, J\} \subseteq J$. Consequently, the quotient*

$$G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega$$

has the structure of a Poisson homogeneous space.

Proof. On the Lie algebra level the orthogonal complement of $(z^{-1}\mathfrak{g}[\![z^{-1}]\!])^\omega$ is $\mathfrak{g}[z]^\omega \subset \mathfrak{g}[z]$. Since this orthogonal complement is a Lie subalgebra, it follows that the subgroup $G_1[\![z^{-1}]\!]^\omega$ is coisotropic and the corresponding quotient space is a Poisson homogeneous space, see e.g. [CG06, §1.3]. □

Remark 6.12. We may think of $G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega$ as a symmetric space for $G_1[\![z^{-1}]\!]$, and thus an infinite-dimensional *Poisson symmetric space* in the sense of [Xu03, §5.3] or [Fer94, §4].

This homogeneous space admits a useful alternative description as a fixed point scheme. To this end, define a map on the group $G(\!(z^{-1})\!)$ by

$$\sigma = \mathrm{inv} \circ \omega = \omega \circ \mathrm{inv}, \tag{6.11}$$

where inv denotes the inverse map. Clearly σ is an involution and a group anti-homomorphism. We use the same notation for the restriction of σ to the subgroups $G[z]$ and $G_1[\![z^{-1}]\!]$.

Lemma 6.13. *The map σ is a Poisson involution of $G(\!(z^{-1})\!)$, $G[z]$ and $G_1[\![z^{-1}]\!]$.*

Proof. Follows by definition (6.11) since both ω and inv are anti-Poisson maps. □

We will be interested in the fixed-point locus

$$G_1[\![z^{-1}]\!]^\sigma = \big\{ g \in G_1[\![z^{-1}]\!] \mid \sigma(g) = g \big\} = \big\{ g \in G_1[\![z^{-1}]\!] \mid \omega(g) = g^{-1} \big\}. \tag{6.12}$$

Since σ is a Poisson involution, this fixed-point locus carries a natural Poisson structure via Dirac reduction as in §6.2. If we take this Poisson structure and rescale it by 2, we obtain another Poisson structure. We will always consider this doubled Poisson structure on $G_1[\![z^{-1}]\!]^\sigma$, because of the following result. Let $e \in G_1[\![z^{-1}]\!]$ denote the identity element.

Proposition 6.14. (1) *Group multiplication provides an isomorphism of affine schemes*

$$G_1[\![z^{-1}]\!]^\sigma \times G_1[\![z^{-1}]\!]^\omega \to G_1[\![z^{-1}]\!].$$

(2) *There is a transitive Poisson action of $G_1[\![z^{-1}]\!]$ on $G_1[\![z^{-1}]\!]^\sigma$ (with its doubled Poisson structure) defined by*

$$g \cdot p = gp\sigma(g) = gp\omega(g)^{-1}.$$

(3) *There is an induced isomorphism of Poisson homogeneous spaces*

$$G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega \xrightarrow{\ \sim\ } G_1[\![z^{-1}]\!]^\sigma, \qquad [g] \mapsto g \cdot e = g\sigma(g).$$

In particular, there is a corresponding isomorphism

$$\mathbb{C}\big[G_1[\![z^{-1}]\!]^\sigma\big] \xrightarrow{\ \sim\ } \mathbb{C}\big[G_1[\![z^{-1}]\!]\big]^{G_1[\![z^{-1}]\!]^\omega}$$

of graded Poisson algebras with left co-actions of $\mathbb{C}\big[G_1[\![z^{-1}]\!]\big]$.

Proof. We remark that every $g \in G_1[\![z^{-1}]\!]$ admits a unique square root $h \in G_1[\![z^{-1}]\!]$ satisfying $g = h^2$, as can be seen for example via the exponential map (6.1). Moreover if $g \in G_1[\![z^{-1}]\!]^\omega$ then $h \in G_1[\![z^{-1}]\!]^\omega$, and similarly if $g \in G_1[\![z^{-1}]\!]^\sigma$ then $h \in G_1[\![z^{-1}]\!]^\sigma$.

Given $g \in G_1[\![z^{-1}]\!]$ we can thus uniquely find $p \in G_1[\![z^{-1}]\!]^\sigma$ such that $p^2 = g\sigma(g)$. Defining $k = p^{-1}g$, we have $k \in G_1[\![z^{-1}]\!]^\omega$ and $g = pk$, and this decomposition is easily seen to be unique. Since these constructions are all algebraic, as is the multiplication map, this proves Part (1).

Next, consider the group action of $G_1[\![z^{-1}]\!]$ on $G_1[\![z^{-1}]\!]^\sigma$ given by $g \cdot p = gp\sigma(g)$. If $p \in G_1[\![z^{-1}]\!]^\sigma$, then there is a unique $h \in G_1[\![z^{-1}]\!]^\sigma \subset G_1[\![z^{-1}]\!]$ such that $p = h^2$. Since

$$h \cdot e = h\sigma(h) = h^2 = p,$$

this group action is transitive, and the stabilizer of the identity element $e \in G_1[\![z^{-1}]\!]^\sigma$ is precisely $G_1[\![z^{-1}]\!]^\omega$. The action is Poisson by [Xu03, Theorem 5.9], the application of which requires doubling the Poisson structure on $G_1[\![z^{-1}]\!]^\sigma$ coming from Dirac reduction, which proves (2).

On the level of sets and Poisson structures, Part (3) follows from Part (2). Regarding the ring isomorphism in the end, first consider the simpler map $G_1[\![z^{-1}]\!]^\sigma \to G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega$ defined by $p \mapsto [p]$. This is a bijection of sets by Part (1), and in fact of affine schemes, via a sequence of ring isomorphisms:

$$\begin{aligned}\mathbb{C}\big[G_1[\![z^{-1}]\!]^\sigma\big] &\cong \mathbb{C}\big[G_1[\![z^{-1}]\!]^\sigma\big] \otimes \mathbb{C} \\ &= \mathbb{C}\big[G_1[\![z^{-1}]\!]^\sigma\big] \otimes \mathbb{C}\big[G_1[\![z^{-1}]\!]^\omega\big]^{G_1[\![z^{-1}]\!]^\omega} \cong \mathbb{C}\big[G_1[\![z^{-1}]\!]\big]^{G_1[\![z^{-1}]\!]^\omega}.\end{aligned}$$

Now, composing this simpler map with the claimed isomorphism of homogeneous spaces, one obtains the map from $G_1[\![z^{-1}]\!]^\sigma$ to itself defined by $p \mapsto p^2$. This is an automorphism of this affine scheme, which proves the desired ring isomorphism. □

6.4. **Twisted Yangians as quantizations.** We now turn to quantization of this story. Fix a Yangian $\mathcal{U}_\hbar(\mathfrak{g}[z])$ for $(\mathfrak{g}, (\cdot,\cdot)_\mathfrak{g})$ as in Definition 6.1.

Definition 6.15. We say that a graded left coideal subalgebra $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \subset \mathcal{U}_\hbar(\mathfrak{g}[z])$ over $\mathbb{C}[\hbar]$ is a *twisted Yangian* for $(\mathfrak{g}, (\cdot,\cdot)_\mathfrak{g}, \omega)$, if the following conditions hold:

(1) $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)$ is free as a graded module over $\mathbb{C}[\hbar]$,
(2) $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \cap \hbar\mathcal{U}_\hbar(\mathfrak{g}[z]) = \hbar\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)$,
(3) $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \subset \mathcal{U}_\hbar(\mathfrak{g}[z])$ quantizes $\mathfrak{g}[z]^\omega \subset \mathfrak{g}[z]$, in the sense that the isomorphism (6.3) restricts to an isomorphism of graded algebras
$$\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)/\hbar\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \cong U(\mathfrak{g}[z]^\omega).$$

Remark 6.16. In fact $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)/\hbar\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \cong U(\mathfrak{g}[z]^\omega)$ are isomorphic as Hopf algebras. Indeed, if we lift $\overline{x} \in \mathfrak{g}[z]^\omega$ to an element $x \in \mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)$, then since $\mathcal{U}_\hbar(\mathfrak{g}[z])$ quantizes $U(\mathfrak{g}[z])$ we see that
$$\Delta(x) \in \ x \otimes 1 + 1 \otimes x + \hbar\, \mathcal{U}_\hbar(\mathfrak{g}[z]) \otimes \mathcal{U}_\hbar(\mathfrak{g}[z]^\sigma).$$
The classical limit is thus the usual coproduct on $U(\mathfrak{g}[z]^\omega)$, and similarly for the counit and antipode.

Remark 6.17. By imposing constraints on the coproduct $\Delta(x) \pmod{\hbar^2}$, a related (but stricter) notion of twisted Yangian is given in [BR17, Definition 5.3].

In [CG06, CG14], the quantum duality principle was extended to incorporate (coisotropic) subgroups and homogeneous spaces. Following their approach, define
$$\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)' = \left\{c \in \mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \mid (\mathrm{Id} - \varepsilon)^{\otimes n} \circ \Delta^n(c) \in \hbar^n \mathcal{U}_\hbar(\mathfrak{g}[z])^{\otimes(n-1)} \otimes \mathcal{U}_\hbar(\mathfrak{g}[z]^\omega), \forall n \geqslant 0\right\}. \tag{6.13}$$
(Notation $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)^{\urcorner}$ was used *loc. cit.*) Then $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)' \subset \mathcal{U}_\hbar(\mathfrak{g}[z])'$ is a graded left coideal subalgebra over $\mathbb{C}[\hbar]$. Similarly to (6.5), it will be useful to make the following technical assumption:

$$\begin{gathered}\text{There is a subset } \{y_\eta\} \subset \mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \text{ consisting of homogeneous elements,}\\ \text{which lifts a basis } \{\overline{y_\eta}\} \text{ for } \mathfrak{g}[z]^\omega, \text{ and such that all } \hbar y_\eta \in \mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'.\end{gathered} \tag{6.14}$$

In this case any homogeneous basis for $\mathfrak{g}[z]^\omega$ admits a similar lift, and $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)$ admits a PBW basis over $\mathbb{C}[\hbar]$ consisting of ordered monomials in the set $\{y_\eta\}$, with respect to any total order.

Theorem 6.18. *Let $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \subset \mathcal{U}_\hbar(\mathfrak{g}[z])$ be a twisted Yangian such that Assumptions (6.5) and (6.14) both hold. Then:*

(1) *$\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ is free over $\mathbb{C}[\hbar]$, with PBW basis consisting of ordered monomials in the elements $\{\hbar y_\eta\}$.*
(2) *$\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)' \cap \hbar\mathcal{U}_\hbar(\mathfrak{g}[z])' = \hbar\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$.*
(3) *$\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ is a quantization of the Poisson homogeneous space $G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega$. More precisely, the isomorphism $\mathcal{U}_\hbar(\mathfrak{g}[z])'/\hbar\mathcal{U}_\hbar(\mathfrak{g}[z])' \cong \mathbb{C}\big[G_1[\![z^{-1}]\!]\big]$ from Proposition 6.3 restricts to an isomorphism of graded Poisson left coideal subalgebras*
$$\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'/\hbar\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)' \ \cong\ \mathbb{C}\big[G_1[\![z^{-1}]\!]\big]^{G_1[\![z^{-1}]\!]^\omega}.$$

Proof. By Assumption (6.14) we have lifts $\{y_\eta\}$ of some homogeneous basis $\{\overline{y_\eta}\}$ for $\mathfrak{g}[z]^\omega$. If we enlarge $\{\overline{y_\eta}\}$ to a homogeneous basis $\{\overline{x_\gamma}\}$ for $\mathfrak{g}[z]$, then using Assumption (6.5) we may also enlarge $\{y_\eta\}$ to a set of lifts $\{x_\gamma\}$. Choose compatible total orders on these sets. By Proposition 6.3(1), along with the PBW property for $\mathcal{U}_\hbar(\mathfrak{g}[z])^\omega)$, it follows that $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \cap \mathcal{U}_\hbar(\mathfrak{g}[z])'$ has a basis consisting of ordered monomials in $\{\hbar y_\eta\}$. But by assumption these monomials all lie in the subset $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$. Therefore $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)' = \mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \cap \mathcal{U}_\hbar(\mathfrak{g}[z])'$, and this discussion proves (1) and (2).

Part (3) formally follows from the Quantum Duality Principle from [CG06, CG14]. However, since we work in an infinite-dimensional graded situation, we give a detailed direct argument.

First, the classical limit of $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ certainly corresponds to some graded Poisson left coideal subalgebra $C \subset \mathbb{C}\big[G_1[\![z^{-1}]\!]\big]$. Let $C^\perp \subset U(z^{-1}\mathfrak{g}[z^{-1}])$ be its orthogonal complement, under the duality discussed in the proof of Proposition 6.3. Then $C^\perp$ is a left ideal, and a two-sided coideal. Since $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ is a left coideal spanned by monomials in $\{\hbar y_\eta\}$, and the classical limits $\overline{y_\eta} \in \mathfrak{g}[z]^\omega$ are orthogonal to $(z^{-1}\mathfrak{g}[z^{-1}])^\omega$, using the pairing (6.6) it is not hard to see that $(z^{-1}\mathfrak{g}[z^{-1}])^\omega \subset C^\perp$. Thus $C^\perp$ contains the left ideal J generated by $(z^{-1}\mathfrak{g}[z^{-1}])^\omega$. We claim that in fact $C^\perp = J$.

Indeed, decompose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into (± 1)-eigenspaces for the involution ω. The quotient $U(z^{-1}\mathfrak{g}[z^{-1}])/J$ is isomorphic as a vector space to the symmetric algebra on the (-1)–eigenspace for ω acting on $z^{-1}\mathfrak{g}[z^{-1}]$, which is $z\mathbb{C}[z^2]\mathfrak{k} \oplus \mathbb{C}[z^2]\mathfrak{p}$. Thus this quotient has Hilbert series

$$\frac{1}{\prod_{\substack{r\geqslant 1,\\ r \text{ odd}}} (1-q^{-r})^{\dim \mathfrak{k}} \cdot \prod_{\substack{r\geqslant 1,\\ r \text{ even}}} (1-q^{-r})^{\dim \mathfrak{p}}}.$$

But the PBW basis for $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ from Part (1) implies that C has exactly the same Hilbert series, up to replacing q by q^{-1} (i.e., up to taking the graded dual vector space). This is only possible if $C^\perp = J$, proving the claim.

Finally, we show that the orthogonal complement of the left ideal $J = C^\perp$ is precisely the coordinate ring of $G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega$. A containment in one direction is straightforward, since the Lie algebra of $G_1[\![z^{-1}]\!]^\sigma$ contains $(z^{-1}\mathfrak{g}[z^{-1}])^\sigma$. Equality follows from Proposition 6.14(3), since the Hilbert series of the affine scheme $G_1[\![z^{-1}]\!]^\sigma \cong (z^{-1}\mathfrak{g}[\![z^{-1}]\!])^\sigma$ is given once again by the Hilbert series above, up to replacing q by q^{-1}. This proves Part (3). ☐

Combining Proposition 6.14 and Theorem 6.18, we obtain the main result of this section.

Theorem 6.19. *Let $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \subset \mathcal{U}_\hbar(\mathfrak{g}[z])$ be a twisted Yangian such that Assumptions (6.5) and (6.14) both hold. Then $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ quantizes the affine scheme $G_1[\![z^{-1}]\!]^\sigma$ with its (doubled) Dirac Poisson structure, and the inclusion $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega) \subset \mathcal{U}_\hbar(\mathfrak{g}[z])$ quantizes the map $G_1[\![z^{-1}]\!] \to G_1[\![z^{-1}]\!]^\sigma$ defined by $g \mapsto g\sigma(g)$. The left coideal structure on $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)$ corresponds to the left action of $G_1[\![z^{-1}]\!]$ on $G_1[\![z^{-1}]\!]^\sigma$ given by $g \cdot p = gp\sigma(g)$.*

Remark 6.20. As in Remark 6.4 we may consider instead the filtered algebra $\mathcal{U}_{\hbar=1}(\mathfrak{g}[z]^\omega)'$, from which we recover $\mathcal{U}_\hbar(\mathfrak{g}[z]^\omega)'$ via the Rees algebra construction. This filtered algebra perspective is in keeping with the approach to (shifted) twisted Yangians taken in the main body of this paper, and much of the literature.

Assumptions (6.5) and (6.14) hold for many of the twisted Yangians studied in the literature.

Example 6.21. The classic twisted Yangians were first introduced mathematically by Olshanski [Ols92]. These algebras fit the framework of this section. More precisely, equip $\mathbb{C}^n$ with a nondegenerate bilinear form which we assume to be either orthogonal or symplectic, encoded by a matrix Q. As in [Mol07, Definition 2.1.1], we may associate a corresponding algebra denoted $Y_Q(\mathfrak{g}_n)$. It follows from the results of [Mol07, §2] that these algebras are twisted Yangians in the conventions of this section (specialized at $\hbar = 1$). In this case we take $G = GL_n$, and the anti-involution σ of $G_1[\![z^{-1}]\!]$ is defined by $g(z) \mapsto Q^{-1}g(-z)^T Q$ where $A \mapsto A^T$ denotes matrix transpose. By the above results, it follows that $Y_Q(\mathfrak{g}_n)$ quantizes a Poisson structure on the fixed point locus

$$G_1[\![z^{-1}]\!]^\sigma = \left\{ g(z) \in G_1[\![z^{-1}]\!] \mid g(z) = Q^{-1}g(-z)^T Q \right\}.$$

This locus is naturally a homogeneous space for $G_1[\![z^{-1}]\!]$, quantized by the coideal structure on the twisted Yangian.

Remark 6.22. In the setting of the previous example, the symplectic case was previously studied by Harada [Har06, §4.1, 4.2] in connection with integrable systems and Gelfand-Tsetlin bases for symplectic Lie algebras. Harada views the twisted Yangian as a quantization of the homogeneous space $G_1[\![z^{-1}]\!]/G_1[\![z^{-1}]\!]^\omega$, which is isomorphic to $G_1[\![z^{-1}]\!]^\sigma$ by Proposition 6.14. Thus, we can think of Theorem 6.18 and Theorem 6.19 as a generalization of aspects of Harada's work.

Remark 6.23. We conjecture that a variant of the algebra ${}^\imath\mathcal{Y}_0$ over the ring $\mathbb{C}[\hbar]$ in Drinfeld presentation considered in other sections is always a twisted Yangian in the sense of Definition 6.15. This conjecture holds for type AI and quasi-split type AIII. In these cases, the twisted Yangians in Drinfeld presentation [LWZ25b, LZ24] are isomorphic to the ones defined via R-matrix [Ols92, MR02]; hence they are coideal subalgebras of the type A Yangian. It is nontrivial but possible to verify more cases.

7. Shifted twisted Yangians and fixed point loci

We show that a Poisson involution σ descending from the loop group preserves the closed subscheme $\mathcal{W}_\mu$ exactly when μ is even spherical. In this case, we give a Poisson algebra presentation for the coordinate algebra of the σ-fixed point locus ${}^\imath\mathcal{W}_\mu$, and show that it is isomorphic to the associated graded of the shifted twisted Yangian ${}^\imath\mathcal{Y}_\mu$ with respect to a natural filtration.

7.1. Spaces $\mathcal{W}_\mu$ from loop groups.

We briefly recall the notion of presentations of Poisson algebras by generators and relations, following [Top23, §3.1]. The free Poisson algebra generated by a set X is the symmetric algebra $S(L_X)$, where L_X is the free Lie algebra generated by X. Here $S(L_X)$ is endowed with a Poisson structure in the standard way. For any Poisson algebra A, we say that A is *Poisson generated* by its subset $X \subset A$ if there is a surjective homomorphism of Poisson algebras $S(L_X) \twoheadrightarrow A$. Given any set $R \subset S(L_X)$ which we refer to as *Poisson relations*, we may consider the Poisson ideal $\langle R \rangle_{Poisson} \subset S(L_X)$ which they generate. If the homomorphism $S(L_X) \twoheadrightarrow A$ induces an isomorphism $S(L_X)/\langle R \rangle_{Poisson} \cong A$, then we refer to this as presentation of A as a Poisson algebra by generators and relations.

For any affine algebraic group H over $\mathbb{C}$, there is a loop group $H(\!(z^{-1})\!)$ and its subgroups $H_1[\![z^{-1}]\!]$ and $H[z]$; see §6.1.

Recall that $\mathfrak{g}$ is a simple Lie algebra of type ADE, and let G be a corresponding connected algebraic group of adjoint type[1]. Fix Chevalley generators $\{e_i, h_i, f_i\}_{i\in\mathbb{I}}$ for $\mathfrak{g}$, yielding a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, and let $U^+, U^-, T \subset G$ be the corresponding unipotent subgroups and maximal torus.

Each coweight $\mu : \mathbb{C}^\times \to T$ yields a point $z^\mu \in T(\!(z^{-1})\!)$. Following [FKP+18, §5.9], we define a closed subscheme of $G(\!(z^{-1})\!)$ by:

$$\mathcal{W}_\mu \;=\; U_1^+[\![z^{-1}]\!]T_1[\![z^{-1}]\!]z^\mu U_1^-[\![z^{-1}]\!]. \tag{7.1}$$

This is an affine scheme and is the product of its factors. In particular, $\mathcal{W}_0 = G_1[\![z^{-1}]\!]$. Following [FKP+18, §5.9], for antidominant coweights ν_1, ν_2, we define a (surjective) shift map

$$\iota_{\mu,\nu_1,\nu_2} : \mathcal{W}_{\mu+\nu_1+\nu_2} \longrightarrow \mathcal{W}_\mu, \qquad g \mapsto \pi(z^{-\nu_1} g z^{-\nu_2}). \tag{7.2}$$

Here π is the surjective map from $U^+(\!(z^{-1})\!)T_1[\![z^{-1}]\!]z^\mu U^-(\!(z^{-1})\!)$ onto $U_1^+[\![z^{-1}]\!]T_1[\![z^{-1}]\!]z^\mu U_1^-[\![z^{-1}]\!]$ defined by projecting along the appropriate factors in the decompositions $U^\star(\!(z^{-1})\!) = U^\star[z] \times U_1^\star[\![z^{-1}]\!]$ (for $\star = +$ or $-$) induced by multiplication.

There is a natural action of $\mathbb{C}^\times$ on $G(\!(z^{-1})\!)$ by loop rotation, however this action does not preserve the subscheme $\mathcal{W}_\mu$ unless $\mu = 0$. Following [FKP+18, §5.9] we can correct this via a slight modification: pick any decomposition $\mu = \mu_1 + \mu_2$ into coweights μ_1, μ_2, and define an action of $\mathbb{C}^\times$ on $\mathcal{W}_\mu$ by

$$s \cdot g(z) = s^{-\mu_1} g(sz) s^{-\mu_2}. \tag{7.3}$$

For each $i \in \mathbb{I}$ there are homomorphisms $\psi_i^\pm : U^\pm \to \mathbb{G}_a$, normalized by $\psi_i^+(\exp(te_i)) = t$ and $\psi_i^-(\exp(tf_i)) = t$. We also have homomorphisms $\alpha_i : T \to \mathbb{G}_m$ corresponding to the simple roots. Define distinguished functions on $\mathcal{W}_\mu$ as follows: given $g = n^+ h z^\mu n^- \in \mathcal{W}_\mu$, we have

$$e_i^{(r)}(g) = [z^r]\; \psi_i^+(n^+), \qquad h_i^{(r)}(g) = [z^r]\; \alpha_i(hz^\mu), \qquad f_i^{(r)}(g) = [z^r]\; \psi_i^-(n^-). \tag{7.4}$$

Here, for a series $F(z) = \sum F_n z^n$ we denote by $[z^r]\; F(z) = F_r$ the coefficient of z^r. We note that $e_i^{(r)} = f_i^{(r)} = 0$ for $r \leqslant 0$, $h_i^{(s)} = 0$ for $s < -\langle \mu, \alpha_i\rangle$, and $h_i^{(-\langle \mu, \alpha_i\rangle)} = 1$. Under the $\mathbb{Z}$–grading on $\mathbb{C}[\mathcal{W}_\mu]$ corresponding to a $\mathbb{C}^\times$–action as in (7.3), we have

$$\deg h_i^{(r)} = r + \langle \mu, \alpha_i\rangle, \qquad \deg e_i^{(s)} = s + \langle \mu_1, \alpha_i\rangle, \qquad \deg f_i^{(s)} = s + \langle \mu_2, \alpha_i\rangle. \tag{7.5}$$

The torus T acts on $\mathcal{W}_\mu$ by conjugation, inducing a grading on $\mathbb{C}[\mathcal{W}_\mu]$ by the root lattice Q of $\mathfrak{g}$. For each $\beta \in \Delta^+$ and $s \geqslant 1$ we fix arbitrary decompositions $\beta = \alpha_{i_1} + \ldots + \alpha_{i_\ell}$ as in §2.2, and set

$$e_\beta^{(s)} = \{e_{i_1}^{(s)}, \{e_{i_2}^{(1)}, \cdots \{e_{i_{\ell-1}}^{(1)}, e_{i_\ell}^{(1)}\} \cdots\}\}, \qquad f_\beta^{(s)} = \{f_{i_1}^{(s)}, \{f_{i_2}^{(1)}, \cdots \{f_{i_{\ell-1}}^{(1)}, f_{i_\ell}^{(1)}\} \cdots\}\}.$$

Following [BFN19, Definition B.2] and [FKP+18, Definition 3.5], one defines algebras $Y_\mu = Y_\mu(\mathfrak{g})$ called *shifted Yangians*. The algebra Y_μ has a PBW basis in elements $H_i^{(r)}, E_\alpha^{(s)}, F_\alpha^{(s)}$ by [FKP+18, Corollary 3.15]. There are shift homomorphisms $\iota_{\mu,\nu_1,\nu_1} : Y_\mu \to Y_{\mu+\nu_1+\nu_2}$ by [FKP+18, Proposition 3.8], and filtrations $F^\bullet_{\mu_1,\mu_2} Y_\mu$ by [FKP+18, §5.4] (compare with degrees in (7.5)). There is also a grading of Y_μ by the root lattice, defined by $\deg E_\alpha^{(s)} = \alpha$, $\deg H_i^{(r)} = 0$ and $\deg F_\alpha^{(s)} = -\alpha$, which respects the filtrations $F^\bullet_{\mu_1,\mu_2} Y_\mu$.

[1]We make the assumption that G is adjoint type in order that it has all fundamental coweights. However this assumption is largely unnecessary, see [MW24, Remark 2.10].

As a preparation for Proposition 7.1, we list the Poisson relations satisfied by generators $h_i^{(r)}, e_i^{(s)}, f_i^{(s)}$ in $\mathbb{C}[\mathcal{W}_\mu]$, for $i \in \mathbb{I}$, $r \in \mathbb{Z}$ and $s \geqslant 1$:

$$h_i^{(r)} = 0 \text{ for } r < -\langle \mu, \alpha_i \rangle, \quad h_i^{(-\langle \mu, \alpha_i \rangle)} = 1, \tag{7.6}$$

$$\{h_i^{(r_1)}, h_j^{(r_2)}\} = 0, \tag{7.7}$$

$$\{e_i^{(s_1)}, f_j^{(s_2)}\} = \delta_{ij} h_i^{(s_1+s_2-1)}, \tag{7.8}$$

$$\{h_i^{(r)}, e_j^{(s)}\} = c_{ij} \sum_{p \geqslant 0} h_i^{(r-p-1)} e_j^{(s+q)}, \tag{7.9}$$

$$\{h_i^{(r)}, f_j^{(s)}\} = -c_{ij} \sum_{p \geqslant 0} h_i^{(r-p-1)} f_j^{(s+p)}, \tag{7.10}$$

$$\{e_i^{(r_1+1)}, e_j^{(r_2)}\} - \{e_i^{(r_1)}, e_j^{(r_2+1)}\} = c_{ij} e_i^{(r_1)} e_j^{(r_2)}, \tag{7.11}$$

$$\{f_i^{(r_1+1)}, f_j^{(r_2)}\} - \{f_i^{(r_1)}, f_j^{(r_2+1)}\} = -c_{ij} f_i^{(r_1)} f_j^{(r_2)}, \tag{7.12}$$

$$i \neq j, N = 1 - c_{ij} \Longrightarrow \mathrm{Sym}_{r_1,\ldots,r_N} \{e_i^{(r_1)}, \{e_i^{(r_2)}, \cdots \{e_i^{(r_N)}, e_j^{(s)}\} \cdots\}\} = 0, \tag{7.13}$$

$$i \neq j, N = 1 - c_{ij} \Longrightarrow \mathrm{Sym}_{r_1,\ldots,r_N} \{f_i^{(r_1)}, \{f_i^{(r_2)}, \cdots \{f_i^{(r_N)}, f_j^{(s)}\} \cdots\}\} = 0. \tag{7.14}$$

The next result is an extension of the results of [FKP+18], inspired by the definition of the semi-classical shifted Yangian from [Top23, §3.3] and [TT24, Proposition 3.4].

Proposition 7.1. (1) *For each decomposition $\mu = \mu_1 + \mu_2$ there is an isomorphism of $Q \times \mathbb{Z}$–graded algebras*

$$\mathrm{gr}^{F^\bullet_{\mu_1,\mu_2}} Y_\mu \cong \mathbb{C}[\mathcal{W}_\mu],$$

where the $\mathbb{Z}$–grading on $\mathbb{C}[\mathcal{W}_\mu]$ corresponds to the $\mathbb{C}^\times$–action (7.3). This isomorphism identifies the classes of $E_i^{(s)}, H_i^{(r)}, F_i^{(s)}$ with the elements $e_i^{(s)}, h_i^{(r)}, f_i^{(s)}$. It endows $\mathcal{W}_\mu$ with a Poisson structure, independent of μ_1, μ_2, with the Poisson bracket having degree $(0, -1)$.

(2) *The coordinate ring $\mathbb{C}[\mathcal{W}_\mu]$ is the Poisson algebra generated by $h_i^{(r)}, e_i^{(s)}, f_i^{(s)}$ for $i \in \mathbb{I}$, $r \in \mathbb{Z}$ and $s \geqslant 1$, with the defining Poisson relations* (7.6)–(7.14).

(3) *$\mathbb{C}[\mathcal{W}_\mu]$ is a polynomial algebra on the PBW generators*

$$\{e_\beta^{(s)} : \beta \in \Delta^+, s \geqslant 1\} \cup \{h_i^{(r)} : i \in \mathbb{I}, r > -\langle \mu, \alpha_i \rangle\} \cup \{f_\beta^{(s)} : \beta \in \Delta^+, s \geqslant 1\}.$$

(4) *The shift map (7.2) is Poisson, and it is quantized by the shift homomorphism ι_{μ,ν_1,ν_2} of shifted Yangians. On the Poisson generators the corresponding map $\iota_{\mu,\nu_1,\nu_2} : \mathbb{C}[\mathcal{W}_\mu] \to \mathbb{C}[\mathcal{W}_{\mu+\nu_1+\nu_2}]$ is given by*

$$e_i^{(r)} \mapsto e_i^{(r-\langle \nu_1, \alpha_i \rangle)}, \qquad h_i^{(r)} \mapsto h_i^{(r-\langle \nu_1+\nu_2, \alpha_i \rangle)}, \qquad f_i^{(r)} \mapsto f_i^{(r-\langle \nu_2, \alpha_i \rangle)}.$$

Proof. Parts (1), (3) and (4) follow from [FKP+18, Theorem 5.15], which also implies that $\mathbb{C}[\mathcal{W}_\mu]$ is Poisson generated by the elements in Part (2). Furthermore it follows that the claimed relations from (2) hold in $\mathbb{C}[\mathcal{W}_\mu]$, since these are simply the classical limits of the defining relations of Y_μ.

Let us temporarily denote by $\mathcal{Y}_\mu$ the Poisson algebra defined by generators and relations as in Part (2), so we have $\mathcal{Y}_\mu \twoheadrightarrow \mathbb{C}[\mathcal{W}_\mu]$. To complete the proof, it remains to prove the claim that $\mathcal{Y}_\mu$ is spanned by the same PBW monomials from Part (3). For μ dominant, this follows from

[Top23, Theorem 3.4]. (Although that result is stated for $\mathfrak{gl}_n$, it holds for any simple $\mathfrak{g}$ with minor modifications.) For general μ, consider the subalgebras $\mathcal{Y}_\mu^>, \mathcal{Y}_\mu^0, \mathcal{Y}_\mu^<$ Poisson generated by the elements $e_i^{(q)}$ (resp. $h_i^{(p)}$, resp. $f_i^{(q)}$). Then $\mathcal{Y}_\mu^0$ is a quotient of the polynomial ring generated by the $h_i^{(p)}$. Similarly to [FKP+18, Remark 3.3] there are surjections $\mathcal{Y}_0^> \twoheadrightarrow \mathcal{Y}_\mu^>$ and $\mathcal{Y}_0^< \twoheadrightarrow \mathcal{Y}_\mu^<$, so the spanning set for $\mathcal{Y}_0^>$ yields one for $\mathcal{Y}_\mu^>$, and similarly for $\mathcal{Y}_\mu^<$. Finally using the relations one can see that $\mathcal{Y}_\mu^< \otimes \mathcal{Y}_\mu^0 \otimes \mathcal{Y}_\mu^> \twoheadrightarrow \mathcal{Y}_\mu$, cf. the first half of the proof of Lemma 7.12, which proves the claim. □

Remark 7.2. $\mathcal{W}_\mu$ inherits a Poisson structure from its quantization Y_μ, by Proposition 7.1(1). If μ is antidominant, then this Poisson structure admits an alternate description by [KPW22, Theorem A.8]: it is restricted from the Poisson structure on the group $G(\!(z^{-1})\!)$ corresponding to a Manin triple as in §6.1.

7.2. **Fixed point loci of the spaces $\mathcal{W}_\mu$.** Recall that τ is a (possibly trivial) involution of the Dynkin diagram of $\mathfrak{g}$. Define a corresponding involution ω_τ of the Lie algebra $\mathfrak{g}$, given on Chevalley generators by $e_i \mapsto -f_{\tau i}$, $h_i \mapsto -h_{\tau i}$ and $f_i \mapsto -e_{\tau i}$. In other words, ω_τ is the composition of the Chevalley involution ω_0 on $\mathfrak{g}$ and the involution induced by τ on $\mathfrak{g}$.

We shall work from now on with this explicit involution ω_τ in place of ω in Section 6. We define an anti-involution σ of the loop algebra $\mathfrak{g}(\!(z^{-1})\!)$ by

$$\sigma(x \otimes z^k) = -\omega_\tau(x) \otimes (-z)^k. \tag{7.15}$$

Explicitly, this anti-involution σ is determined by

$$e_i \otimes z^k \mapsto (-1)^k f_{\tau i} \otimes z^k, \qquad h_i \otimes t^k \mapsto (-1)^k h_{\tau i} \otimes z^k, \qquad f_i \otimes z^k \mapsto (-1)^k e_{\tau i} \otimes z^k. \tag{7.16}$$

The corresponding anti-involution σ of the group $G(\!(z^{-1})\!)$ (see §6.3 and (6.11)) satisfies $\sigma(T_1[\![z^{-1}]\!]) = T_1[\![z^{-1}]\!]$ and $\sigma(U_1^\pm[\![z^{-1}]\!]) = U_1^\mp[\![z^{-1}]\!]$, as well as

$$\sigma(z^\mu) = (-z)^{\tau\mu} = (-1)^{\tau\mu} z^{\tau\mu}. \tag{7.17}$$

Similarly, if $s \in \mathbb{C}^\times$ then the element $s^\mu \in T$ satisfies $\sigma(s^\mu) = s^{\tau\mu}$.

Inside the maximal torus $T \subset G$, recall from (2.14) the following subgroup

$$T^\imath = \{t \in T \mid \sigma(t) = t^{-1}\} = \{t \in T \mid \tau(t) = t^{-1}\}.$$

The coordinate ring of $T^\imath$ is the group algebra $\mathbb{C}[Q^\imath]$; see (2.13).

Example 7.3. Take $\mathfrak{g} = \mathfrak{sl}_n$ and $\tau = \mathrm{id}$, then ω_τ is given by negative transpose $\omega_\tau(x) = -x^T$. For any $g(z) \in \mathrm{PGL}_n(\!(z^{-1})\!)$ we then have $\sigma\big(g(z)\big) = g(-z)^T$.

Recall the notion of spherical and even coweights from Definition 2.1.

Lemma 7.4. *The following are equivalent, for any coweight μ:*

(1) *σ preserves the subscheme $\mathcal{W}_\mu \subset G(\!(z^{-1})\!)$;*
(2) *$\sigma(z^\mu) = z^\mu$;*
(3) *μ is even and $\tau\mu = \mu$;*
(4) *μ is even spherical.*

Proof. Each $g \in \mathcal{W}_\mu$ has a unique Gauss decomposition $g = n^+ h z^\mu n^-$ in (7.1). Then

$$\sigma(g) = \sigma(n^-)\sigma(h)\sigma(z^\mu)\sigma(n^+) \in U_1^+[\![z^{-1}]\!]T_1[\![z^{-1}]\!]\sigma(z^\mu)U_1^-[\![z^{-1}]\!].$$

Since Gauss decompositions are unique, we have $\sigma(g) \in \mathcal{W}_\mu$ if and only if $\sigma(z^\mu) = z^\mu$. Comparing with (7.17), this holds exactly when $(-1)^{\tau\mu} = 1$ and $\tau\mu = \mu$. But $(-1)^{\tau\mu} = 1$ means precisely that $\tau\mu$ is even, which is equivalent to μ itself being even.

The equivalence of (3) and (4) is also clear. □

Lemma 7.5. *Suppose that μ is an even spherical coweight.*

1. *Consider a $\mathbb{C}^\times$–action on $\mathcal{W}_\mu$ as in (7.3), such that $\mu_2 = \tau\mu_1$, and hence $\mu = \mu_1 + \tau\mu_1$. Then this $\mathbb{C}^\times$–action commutes with σ.*
2. *Under the adjoint action of the group T on $\mathcal{W}_\mu$, the subgroup $T^\imath$ commutes with σ.*
3. *Let ν be an even antidominant coweight. Then the shift map $\iota_{\mu,\nu,\tau\nu} : \mathcal{W}_{\mu+\nu+\tau\nu} \to \mathcal{W}_\mu$ from (7.2) intertwines the respective involutions $\sigma|_{\mathcal{W}_\mu} \circ \iota_{\mu,\nu,\tau\nu} = \iota_{\mu,\nu,\tau\nu} \circ \sigma|_{\mathcal{W}_{\mu+\nu+\tau\nu}}$.*
4. *The restriction $\sigma|_{\mathcal{W}_\mu}$ is a Poisson involution, and given on the Poisson generators by*

$$e_i^{(r)} \mapsto (-1)^r f_{\tau i}^{(r)}, \qquad h_i^{(s)} \mapsto (-1)^s h_{\tau i}^{(s)}, \qquad f_i^{(r)} \mapsto (-1)^r e_{\tau i}^{(r)}.$$

Proof. For Part (1), let $g = g(z) \in \mathcal{W}_\mu$. Then we compute:

$$\begin{aligned}\sigma(s\cdot g(z)) = \sigma(s^{-\mu_1}g(sz)s^{-\tau\mu_1}) &= \sigma(s^{-\tau\mu_1})\sigma(g(sz))\sigma(s^{-\mu_1}) \\ &= s^{-\mu_1}\sigma(g(sz))s^{-\tau\mu_1} = s\cdot\sigma(g(z)).\end{aligned}$$

Parts (2) and (3) follow from similar direct calculations.

It remains to prove Part (4). It follows from the results of §6.3 that σ is a Poisson involution of $G(\!(z^{-1})\!)$ in a natural way. For μ antidominant, $\mathcal{W}_\mu \subset G(\!(z^{-1})\!)$ is a Poisson subscheme by [KPW22, Theorem A.8], and therefore the restriction $\sigma|_{\mathcal{W}_\mu}$ is Poisson. For general μ, pick an even antidominant weight ν such that $\mu + \nu + \tau\nu$ is antidominant, and by (3) the corresponding Poisson embedding $\mathbb{C}[\mathcal{W}_\mu] \hookrightarrow \mathbb{C}[\mathcal{W}_{\mu+\nu+\tau\nu}]$ intertwines the action of σ, which proves that $\sigma|_{\mathcal{W}_\mu}$ is Poisson. The explicit description in coordinates in (4) follows from the formula $\sigma(n^+hz^\mu n^-) = \sigma(n^-)\sigma(h)z^\mu\sigma(n^+)$ together with equations (7.4) and (7.16). □

Recall an element in $\mathcal{W}_\mu$ (7.1) is written naturally as $uhz^\mu u_-$.

Definition 7.6. For any even spherical coweight μ, we define the fixed point locus

$${}^\imath\mathcal{W}_\mu \;:=\; (\mathcal{W}_\mu)^\sigma = \{uhz^\mu u_- \in \mathcal{W}_\mu \mid \sigma(u) = u_-,\ \sigma(h) = h\}.$$

Since σ defines a Poisson involution of $\mathcal{W}_\mu$, the fixed point locus ${}^\imath\mathcal{W}_\mu$ inherits a Poisson structure via Dirac reduction as in §6.2. We may rescale this Poisson structure by a factor of 2, i.e., $\{a,b\} := 2\{a,b\}_{Dirac}$; see Proposition 6.14 for an explanation. From now on, we will always equip ${}^\imath\mathcal{W}_\mu$ with this doubled Poisson bracket.

Proposition 7.7. *${}^\imath\mathcal{W}_\mu$ has a natural Poisson structure inherited from $\mathcal{W}_\mu$ via (doubled) Dirac reduction. The group $T^\imath$ acts on ${}^\imath\mathcal{W}_\mu$ by conjugation, and for each choice of decomposition $\mu = \mu_1 + \tau\mu_1$ there is a $\mathbb{C}^\times$–action on ${}^\imath\mathcal{W}_\mu$ by $s \cdot g(z) = s^{-\mu_1}g(sz)s^{-\tau\mu_1}$ such that the Poisson bracket has degree -1. Finally, for each antidominant weight ν such that $\nu + \tau\nu$ is even, the shift map $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{W}_{\mu+\nu+\tau\nu} \to {}^\imath\mathcal{W}_\mu$, $g \mapsto \pi(z^{-\nu}gz^{-\tau\nu})$, is surjective and Poisson.*

Proof. This follows from Lemma 7.5 and Definition 7.6: the subscheme ${}^\imath\mathcal{W}_\mu$ is preserved by the $\mathbb{C}^\times$–action on $\mathcal{W}_\mu$ corresponding to $\mu = \mu_1 + \mu_2$ with $\mu_2 = \tau\mu_1$, as well as the action of the group $T^\imath$, and finally the map $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{W}_{\mu+\nu+\tau\nu} \to {}^\imath\mathcal{W}_\mu$ is the restriction of the Poisson map $\iota_{\mu,\nu,\tau\nu} : \mathcal{W}_{\mu+\nu+\tau\nu} \to \mathcal{W}_\mu$. □

We introduce natural functions on ${}^\imath\mathcal{W}_\mu$ by restriction from $\mathcal{W}_\mu$:

$$h_i^{(r)} := h_i^{(r)}\big|_{{}^\imath\mathcal{W}_\mu} = (-1)^{(r)} h_{\tau i}^{(r)}\big|_{{}^\imath\mathcal{W}_\mu}, \qquad b_i^{(s)} := e_i^{(s)}\big|_{{}^\imath\mathcal{W}_\mu} = (-1)^s f_{\tau i}^{(s)}\big|_{{}^\imath\mathcal{W}_\mu} \tag{7.18}$$

Consider the action of $\mathbb{C}^\times$ on ${}^\imath\mathcal{W}_\mu$ associated with a decomposition $\mu = \mu_1 + \tau\mu_1$. Under the corresponding $\mathbb{Z}$–grading of $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ we have degrees

$$\deg h_i^{(r)} = r + \langle \mu, \alpha_i \rangle, \qquad \deg e_i^{(s)} = s + \langle \mu_1, \alpha_i \rangle$$

as follows from (7.5); compare with the filtration $F^\bullet_{\mu_1}\mathcal{Y}_\mu$ defined in (2.26). At the same time, the adjoint action of $T^\imath$ induces a grading on $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ by $Q^\imath$, with $h_i^{(r)}$ in degree $\overline{0}$ and $b_j^{(s)}$ in degree $\overline{\alpha_i}$. Since the actions of $\mathbb{C}^\times$ and $T^\imath$ commute, we obtain a grading on $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ by $Q^\imath \times \mathbb{Z}$.

As a preparation for Theorem 7.8 below, we list the Poisson relation (7.19)–(7.26) satisfied by generators $h_i^{(r)}, b_i^{(s)}$ for $i \in \mathbb{I}$, $r \in \mathbb{Z}$ and $s \geqslant 1$:

$$h_i^{(r)} = 0 \text{ for } r < -\langle \mu, \alpha_i \rangle, \quad h_i^{(-\langle \mu, \alpha_i \rangle)} = 1, \tag{7.19}$$

$$\{h_i^{(r_1)}, h_j^{(r_2)}\} = 0, \tag{7.20}$$

$$\{h_i^{(r)}, b_j^{(s)}\} = \sum_{p\geqslant 0} \big(c_{ij} + (-1)^{p+1} c_{\tau i, j}\big) h_i^{(r-p-1)} b_j^{(s+p)}, \tag{7.21}$$

$$\{b_i^{(s_1+1)}, b_j^{(s_2)}\} - \{b_i^{(s_1)}, b_j^{(s_2+1)}\} = c_{ij} b_i^{(s_1)} b_j^{(s_2)} + 2\delta_{\tau i, j} (-1)^{s_1} h_j^{(s_1+s_2)} \tag{7.22}$$

as well as the Serre relations:

$$\{b_i^{(s_1)}, b_{\tau i}^{(s_2)}\} = (-1)^{s_1-1} h_{\tau i}^{(s_1+s_2-1)}, \quad \text{for } c_{ij} = 0,\ \tau i \neq j, \tag{7.23}$$

$$\mathrm{Sym}_{s_1,s_2} \{b_i^{(s_1)}, \{b_i^{(s_2)}, b_j^{(s)}\}\} = 0, \qquad \text{for } c_{ij} = -1,\ i \neq \tau i \neq j, \tag{7.24}$$

$$\mathrm{Sym}_{s_1,s_2} \{b_i^{(s_1)}, \{b_i^{(s_2)}, b_j^{(s)}\}\} = 2c_{ij}(-1)^{s_1-1} \sum_{p\geqslant 0} h_i^{(s_1+s_2-2p-1)} b_j^{(s+2p-1)}, \quad \text{for } c_{ij} = -1,\ i = \tau i, \tag{7.25}$$

$$\mathrm{Sym}_{s_1,s_2} \{b_i^{(s_1)}, \{b_i^{(s_2)}, b_{\tau i}^{(s)}\}\} = 4\, \mathrm{Sym}_{s_1,s_2} (-1)^{s_1-1} \sum_{p\geqslant 0} h_{\tau i}^{(s_1+s-2p-2)} b_i^{(s_2+2p)}, \quad \text{for } c_{i,\tau i} = -1. \tag{7.26}$$

Since τ is an involution, it follows from (7.23) that

$$h_i^{(r)} = (-1)^r h_{\tau i}^{(r)}, \tag{7.27}$$

and in particular,

$$h_i^{(2r+1)} = 0, \qquad \text{if } \tau i = i. \tag{7.28}$$

For each $\beta \in \Delta^+$ and $s \geqslant 1$ we choose an element $b_\beta^{(s)} \in \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ similarly to §2.2:

$$b_\beta^{(s)} = \Big\{ b_{i_1}^{(s)}, \{ b_{i_2}^{(1)}, \cdots \{ b_{i_{\ell-1}}^{(1)}, b_{i_\ell}^{(1)} \} \cdots \} \Big\}.$$

Now we can formulate the main result of this section.

Theorem 7.8. *Let μ be an even spherical coweight.*

(1) *For each decomposition $\mu = \mu_1 + \tau\mu_1$, there is an isomorphism of $Q^\imath \times \mathbb{Z}$–graded Poisson algebras*
$$\mathrm{gr}^{F^\bullet_{\mu_1}} {}^\imath\mathcal{Y}_\mu \cong \mathbb{C}[{}^\imath\mathcal{W}_\mu],$$
which identifies the classes $\bar{H}_i^{(r)}, \bar{B}_j^{(s)}$ with the elements $h_i^{(r)}, b_j^{(s)}$.

(2) *The coordinate ring $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ is the Poisson algebra generated by $h_i^{(r)}, b_i^{(s)}$ for $i \in \mathbb{I}$, $r \in \mathbb{Z}$ and $s \geqslant 1$, with defining Poisson relations (7.19)–(7.26).*

(3) *$\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ is a polynomial algebra on PBW generators*
$$\{b_\beta^{(s)} : \beta \in \Delta^+, s \geqslant 1\} \cup \{h_i^{(2p)} : i \in \mathbb{I}_0, 2p > -\langle \mu, \alpha_i \rangle\} \cup \{h_i^{(s)} : i \in \mathbb{I}_1, s > -\langle \mu, \alpha_i \rangle\}.$$

(4) *For any antidominant weight ν such that $\nu + \tau\nu$ is even, the shift map $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{W}_{\mu+\nu+\tau\nu} \to {}^\imath\mathcal{W}_\mu$ is quantized by the shift homomorphism $\iota^\tau_{\mu,\nu} : {}^\imath\mathcal{Y}_\mu \to {}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}$ from Lemmas 2.7 and 2.11.*

The proof of this theorem will be given in §7.3 below.

Remark 7.9. Assume that ${}^\imath\mathcal{Y}_0$ are identified with the twisted Yangians (specialized at $\hbar = 1$) in Definition 6.15; this is conjectured to be always true and known in type AI and AIII, see Remark 6.23. Then Theorem 6.19 is compatible with Theorem 7.8(1), stating that the twisted Yangian ${}^\imath\mathcal{Y}_0$ quantizes $G_1[\![z^{-1}]\!]^\sigma = {}^\imath\mathcal{W}_0$. We expect that Theorem 7.8 could then be proven along the lines of [FKP⁺18, Theorem 5.15]. Note that our proof below is not reliant on any such assumptions.

7.3. **Proof of quantization Theorem 7.8 of ${}^\imath\mathcal{W}_\mu$.** We will break the proof into several steps.

Lemma 7.10. *The elements $h_i^{(r)}$ and $b_i^{(s)}$ of $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ satisfy the Poisson relations (7.19)–(7.26).*

Proof. We remind the reader that we use the *doubled* Dirac reduction Poisson bracket on ${}^\imath\mathcal{W}_\mu$. By Remark 6.7, this means that for $f, g \in \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ we choose σ–invariant lifts $\widetilde{f}, \widetilde{g} \in \mathbb{C}[\mathcal{W}_\mu]$, and then define $\{f, g\} = 2\{\widetilde{f}, \widetilde{g}\}\big|_{{}^\imath\mathcal{W}_\mu}$. A similar formula applies for iterated Poisson brackets. With this in mind, the claim is a direct calculation using the relations (7.6)–(7.13); similar calculations also appeared in the proof of [Top23, Theorem 3.7], except for the analogues of (7.24) and (7.26) which are not included since $\tau = \mathrm{id}$ in [Top23].

We illustrate this process with (7.26). Lift each $b_i^{(s)} \in \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ to the σ–invariant element $\widetilde{b}_i^{(s)} = \frac{1}{2}(e_i^{(s)} + (-1)^s f_{\tau i}^{(s)}) \in \mathbb{C}[\mathcal{W}_\mu]$. Then $\{b_i^{(s_1)}, \{b_i^{(s_2)}, b_{\tau i}^{(s)}\}\}$ is determined by a calculation in $\mathbb{C}[\mathcal{W}_\mu]$:

$$\begin{aligned}
&2\Big\{\widetilde{b}_i^{(s_1)}, 2\Big\{\widetilde{b}_i^{(s_2)}, \widetilde{b}_{\tau i}^{(s)}\Big\}\Big\} \\
&= \tfrac{1}{2}\Big\{ e_i^{(s_1)} + (-1)^{s_1} f_{\tau i}^{(s_1)}, \Big\{ e_i^{(s_2)} + (-1)^{s_2} f_{\tau i}^{(s_2)}, e_{\tau i}^{(s)} + (-1)^s f_i^{(s)} \Big\}\Big\}
\end{aligned}$$

$$= \tfrac{1}{2}\Big\{ e_i^{(s_1)} + (-1)^{s_1} f_{\tau i}^{(s_1)}, \\ \{e_i^{(s_2)}, e_{\tau i}^{(s)}\} + (-1)^s h_i^{(s_2+s-1)} - (-1)^{s_2} h_{\tau i}^{(s_2+s-1)} + (-1)^{s_2+s}\{f_{\tau i}^{(s_2)}, f_i^{(s)}\}\Big\}.$$

We will expand this Poisson bracket. Observe that the term $\{e_i^{(s_1)}, \{e_i^{(s_2)}, e_{\tau i}^{(s)}\}\}$ will vanish once we symmetrize in s_1 and s_2 due to (7.13), and similarly for the term $(-1)^{s_1+s_2+s}\{f_{\tau i}^{(s_1)}, \{f_{\tau i}^{(s_2)}, f_i^{(s)}\}\}$. We thus omit these terms below. Using the relations (7.8)–(7.10), the remaining terms are seen to be:

$$\begin{aligned} & -(-1)^s \sum_{p\geqslant 0} h_i^{(s_2+s-p-2)} e_i^{(s_1+p)} - (-1)^{s_2}\tfrac{1}{2} \sum_{p\geqslant 0} h_{\tau i}^{(s_2+s-p-2)} e_i^{(s_1+p)} \\ & + (-1)^{s_2+s}\tfrac{1}{2}\{f_{\tau i}^{(s_2)}, h_i^{(s_1+s-1)}\} + (-1)^{s_1}\tfrac{1}{2}\{e_i^{(s_2)}, -h_{\tau i}^{(s_1+s-1)}\} \\ & -(-1)^{s_1+s}\tfrac{1}{2}\sum_{p\geqslant 0} h_i^{(s_2+s-p-2)} f_{\tau i}^{(s_1+p)} - (-1)^{s_1+s_2}\sum_{p\geqslant 0} h_{\tau i}^{(s_2+s-p-2)} f_{\tau i}^{(s_1+p)} \\ = \; & -(-1)^s \sum_{p\geqslant 0} h_i^{(s_2+s-p-2)} e_i^{(s_1+p)} - (-1)^{s_2}\tfrac{1}{2} \sum_{p\geqslant 0} h_{\tau i}^{(s_2+s-p-2)} e_i^{(s_1+p)} \\ & -(-1)^{s_2+s}\tfrac{1}{2}\sum_{p\geqslant 0} h_i^{(s_1+s-p-2)} f_{\tau i}^{(s_2+p)} - (-1)^{s_1}\tfrac{1}{2}\sum_{p\geqslant 0} h_{\tau i}^{(s_1+s-p-2)} e_i^{(s_2+p)} \\ & -(-1)^{s_1+s}\tfrac{1}{2}\sum_{p\geqslant 0} h_i^{(s_2+s-p-2)} f_{\tau i}^{(s_1+p)} - (-1)^{s_1+s_2}\sum_{p\geqslant 0} h_{\tau i}^{(s_2+s-p-2)} f_{\tau i}^{(s_1+p)}. \end{aligned}$$

Restricting to ${}^\imath\mathcal{W}_\mu$ and applying (7.18) yields

$$\sum_{p\geqslant 0}\left(2(-1)^{s_2-p-1} + (-1)^{s_2-1}\right) h_{\tau i}^{(s_2+s-p-2)} b_i^{(s_1+p)} + \sum_{p\geqslant 0}(-1)^{s_1-1} h_{\tau i}^{(s_1+s-p-2)} b_i^{(s_2+p)}.$$

Finally, after symmetrizing in s_1 and s_2 we get

$$\mathrm{Sym}_{s_1,s_2}\sum_{p\geqslant 0}\left(2(-1)^{s_1-p-1} + 2(-1)^{s_1-1}\right) h_{\tau i}^{(s_1+s-p-2)} b_i^{(s_2+p)}.$$

The coefficient appearing inside the sum vanishes unless p is even, yielding (7.26). □

The next result establishes Part (3) of Theorem 7.8.

Lemma 7.11. *$\mathbb{C}[{}^\imath\mathcal{W}_\mu]$ is Poisson generated by the elements $h_i^{(r)}$ and $b_i^{(s)}$ for $i\in\mathbb{I}$, $r\in\mathbb{Z}$ and $s\geqslant 1$, and is a polynomial ring in PBW generators*

$$\{b_\beta^{(s)} : \beta\in\Delta^+, s\geqslant 1\}\cup\{h_i^{(2p)} : i\in\mathbb{I}_0, 2p > -\langle\mu,\alpha_i\rangle\}\cup\{h_i^{(r)} : i\in\mathbb{I}_1, r > -\langle\mu,\alpha_i\rangle\}.$$

Proof. Inside $\mathbb{C}[\mathcal{W}_\mu]$, consider the subspace V spanned by the PBW generators

$$\{e_\beta^{(s)} : \beta\in\Delta^+, s\geqslant 1\}\cup\{h_i^{(r)} : i\in\mathbb{I}, r > -\langle\mu,\alpha_i\rangle\}\cup\{f_\beta^{(s)} : \beta\in\Delta^+, s\geqslant 1\}.$$

In the definition of the PBW generators in $\mathbb{C}[\mathcal{W}_\mu]$, we may choose them is such a way that $\sigma(e_\beta^{(s)}) = (-1)^{s+\mathrm{ht}\,\beta-1} f_{\tau\beta}^{(s)}$. Then σ acts on V, with invariants $V(1)\subset V$ having as basis the set

$$\{e_\beta^{(s)} + (-1)^{s+\mathrm{ht}\,\beta-1} f_{\tau\beta}^{(s)} : \beta\in\Delta^+, s\geqslant 1\}$$

$$\cup \{h_i^{(2p)} : i \in \mathbb{I}_0, 2p > -\langle \mu, \alpha_i \rangle\} \cup \{h_i^{(s)} + (-1)^s h_{\tau i}^{(r)} : i \in \mathbb{I}_1, r > -\langle \mu, \alpha_i \rangle\}.$$

Proposition 7.1(3) says that $\mathbb{C}[\mathcal{W}_\mu] = S(V)$ is a symmetric algebra on V. By Lemma 6.9 the composed map $S\big(V(1)\big) \hookrightarrow \mathbb{C}[\mathcal{W}_\mu] \twoheadrightarrow \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ induces an isomorphism $S\big(V(1)\big) \xrightarrow{\sim} \mathbb{C}[{}^\imath\mathcal{W}_\mu]$. To complete the proof, it suffices to prove that the images of the elements $e_\beta^{(s)} + (-1)^{s+\mathrm{ht}\,\beta-1} f_\beta^{(s)}$ of $V(1)$ can all be written in terms of the PBW generators from the statement of the lemma. This follows by an induction on $\mathrm{ht}\,\beta$, similarly to the proof of [LWZ25a, Lemma 2.3]. $\square$

For the remainder of this section, we denote by ${}^\imath\mathcal{Y}_\mu$ the Poisson algebra with generators $h_i^{(r)}$ and $b_i^{(s)}$ for $i \in \mathbb{I}$, $r > -\langle \mu, \alpha_i \rangle$, and $s \geqslant 1$, with relations (7.19)–(7.26). By the previous two lemmas, there is a natural surjective Poisson map ${}^\imath\mathcal{Y}_\mu \twoheadrightarrow \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ by sending $h_i^{(r)}, b_i^{(s)}$ to the same-named elements.

We can now prove Part (2) of Theorem 7.8.

Lemma 7.12. *The map ${}^\imath\mathcal{Y}_\mu \twoheadrightarrow \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ is an isomorphism of Poisson algebras.*

Proof. It is enough to show that ${}^\imath\mathcal{Y}_\mu$ is spanned by monomials in the same PBW generators from Lemma 7.11. This is essentially a Poisson algebra variation on the proof of Proposition 2.13.

First, we claim that ${}^\imath\mathcal{Y}_\mu$ is spanned by elements of the form $b_1 \cdots b_k h_{i_1}^{(m_1)} h_{i_2}^{(m_2)} \cdots h_{i_\ell}^{(m_\ell)}$, where $b_1, \ldots, b_k$ are each nested commutators of the elements $b_i^{(s)}$. It suffices to prove that the subalgebra $S \subseteq {}^\imath\mathcal{Y}_\mu$ generated by these elements is closed under Poisson brackets. By the Leibniz rule, it is in turn sufficient to show the Poisson bracket of any two generators of S is still in S. The only interesting case is to show that $\{h_i^{(r)}, b\} \in S$, where b is any nested commutator in the $b_i^{(s)}$. This follows by a straightforward induction on the length of b, using the relation (7.21) in ${}^\imath\mathcal{Y}_\mu$.

Next, consider the filtration on ${}^\imath\mathcal{Y}_\mu$ defined by setting $\deg b_j^{(s)} = 1$ and $\deg h_i^{(r)} = 0$. More precisely, these degrees define a grading on the free Lie algebra L generated by symbols $h_i^{(r)}$ and $b_i^{(s)}$, and thus on the symmetric algebra $S(L)$. There is a surjection $S(L) \twoheadrightarrow {}^\imath\mathcal{Y}_\mu$ and the grading on $S(L)$ thus induces a filtration on ${}^\imath\mathcal{Y}_\mu$ (a similar and fuller discussion can be found in the paragraph right after this proof below). Note that the Poisson bracket is degree zero.

Similarly to the proof of Proposition 2.13, one can argue that the associated graded algebra $\mathrm{gr}\, {}^\imath\mathcal{Y}_\mu$ is spanned by monomials in the images of the PBW generators from Lemma 7.11. This is because the images $\overline{b_i^{(s)}} \in \mathrm{gr}\, {}^\imath\mathcal{Y}_\mu$ satisfy the Poisson relations (7.11) and (7.13) which define the positive half of the algebra $\mathbb{C}[\mathcal{W}_\mu]$. This completes the proof. $\square$

To complete the proof of the remaining parts (1) and (4) of Theorem 7.8, we turn to the shifted twisted Yangian ${}^\imath\mathcal{Y}_\mu$ and its filtrations. We take an indirect approach, inspired by the proof of [Top23, Theorem 3.4]. Consider the set

$$X = \Big\{ H_i^{(r)}, B_i^{(s)} \mid i \in \mathbb{I}, r \in \mathbb{Z}, s \geqslant 1 \Big\} \tag{7.29}$$

and the free Lie algebra L on this set. Then there is a surjection $U(L) \cong \mathbb{C}\langle X \rangle \twoheadrightarrow {}^\imath\mathcal{Y}_\mu$. Define a grading on $L = \bigoplus_{n\in\mathbb{Z}} L_n$ by declaring that $\deg H_i^{(r)} = r + \langle \mu, \alpha_i \rangle - 1$ and $\deg B_i^{(s)} = s + \langle \mu_1, \alpha_i \rangle - 1$. Then we may define a filtration on the enveloping algebra $U(L)$, such that L_n lives in filtered degree $n+1$. The PBW theorem for $U(L)$ ensures that $\mathrm{gr}\ U(L) \cong S(L)$ with its standard Poisson structure. In particular, $\mathrm{gr}\, U(L)$ is Poisson generated by (the images of) the set X. Via the surjection $U(L) \twoheadrightarrow {}^{\imath}\mathcal{Y}_\mu$ we induce a corresponding (quotient) filtration $G^{\bullet}_{\mu_1}{}^{\imath}\mathcal{Y}_\mu$ on ${}^{\imath}\mathcal{Y}_\mu$. Note that $\mathrm{gr}^{G^{\bullet}_{\mu_1}}{}^{\imath}\mathcal{Y}_\mu$ is commutative and identified with a quotient of $S(L)$.

Lemma 7.13. *There is a surjective homomorphism of graded algebras ${}^{\imath}\mathscr{Y}_\mu \twoheadrightarrow \mathrm{gr}^{G^{\bullet}_{\mu_1}}{}^{\imath}\mathcal{Y}_\mu$, sending the generators $h_i^{(r)}, b_i^{(s)}$ to the classes $\bar{H}_i^{(r)}, \bar{B}_i^{(s)}$.*

Proof. Looking at the top degree parts of the defining relations (2.2)–(2.9) of ${}^{\imath}\mathcal{Y}_\mu$, one can see that the above assignment on generators defines a homomorphism ${}^{\imath}\mathscr{Y}_\mu \to \mathrm{gr}^{G^{\bullet}_{\mu_1}}{}^{\imath}\mathcal{Y}_\mu$. Let us denote the image of generators of ${}^{\imath}\mathcal{Y}_\mu$ in $\mathrm{gr}^{G^{\bullet}_{\mu_1}}{}^{\imath}\mathcal{Y}_\mu$ by $\bar{H}_i^{(r)}$, $\bar{B}_j^{(s)}$ and so on. For example, modulo lower terms the relation (2.4) becomes

$$[\bar{H}_i^{(r+2)}, \bar{B}_j^{(s)}] - [\bar{H}_i^{(r)}, \bar{B}_j^{(s+2)}] = (c_{ij} - c_{\tau i,j})\bar{H}_i^{(r+1)}\bar{B}_j^{(s)} + (c_{ij} + c_{\tau i,j})\bar{H}_i^{(r)}\bar{B}_j^{(s+1)}. \tag{7.30}$$

This is simply an inductive version of the relation (7.21). Similarly (2.8) reduces to (7.25), cf. [LWZ25a, (6.18)]. Now we show that (2.9) reduces to (7.26). Indeed, with the help of (7.21) (corresponding to (7.30)), we rewrite (2.9) in the associated graded after cancellation of many summands as

$$\begin{aligned}
&\mathrm{Sym}_{s_1,s_2}[\bar{B}_i^{(s_1)}, [\bar{B}_i^{(s_2)}, \bar{B}_{\tau i}^{(s)}]] \\
&= \frac{4}{3}\mathrm{Sym}_{s_1,s_2}(-1)^{s_1-1}\sum_{p=0}^{s_1+s-2} 3^{-p}\Big(\sum_{q\geqslant 0} 3\bar{H}_{\tau i}^{(s_1+s-p-2-2q)}\bar{B}_j^{(s_2+p+2q)} \\
&\qquad\qquad - \sum_{q\geqslant 0}\bar{H}_{\tau i}^{(s_1+s-p-3-2q)}\bar{B}_j^{(s_2+p+2q+1)}\Big) \\
&= \frac{4}{3}\mathrm{Sym}_{s_1,s_2}(-1)^{s_1-1}\sum_{q\geqslant 0} 3\bar{H}_{\tau i}^{(s_1+s-2-2q)}\bar{B}_j^{(s_2+2q)}.
\end{aligned}$$

This matches (7.26). Finally, this homomorphism ${}^{\imath}\mathscr{Y}_\mu \to \mathrm{gr}^{G^{\bullet}_{\mu_1}}{}^{\imath}\mathcal{Y}_\mu$ is surjective since $\mathrm{gr}\, U(L)$ is Poisson generated by the set X in (7.29), and thus so is its quotient $\mathrm{gr}\, U(L) \twoheadrightarrow \mathrm{gr}^{G^{\bullet}_{\mu_1}}\, {}^{\imath}\mathcal{Y}_\mu$. ☐

Recall that in §2.4, we defined a *vector space* filtration $F^{\bullet}_{\mu_1}{}^{\imath}\mathcal{Y}_\mu$, which a priori depends on a choice of PBW basis for ${}^{\imath}\mathcal{Y}_\mu$. The following technical result is key:

Proposition 7.14. *The filtrations $F^{\bullet}_{\mu_1}{}^{\imath}\mathcal{Y}_\mu$ and $G^{\bullet}_{\mu_1}{}^{\imath}\mathcal{Y}_\mu$ coincide.*

In particular, this result establishes Proposition 2.21, since by construction $G^{\bullet}_{\mu_1}{}^{\imath}\mathcal{Y}_\mu$ is an algebra filtration and does not depend on any choice of PBW basis.

Proof. We adopt a general proof strategy used in [FKP+18, KPW22]: the $\mu = 0$ case first, then antidominant μ, and finally all remaining μ.

For each $k \in \mathbb{Z}$ there is a natural inclusion $F^k_{\mu_1}({}^{\imath}\mathcal{Y}_\mu) \subseteq G^k_{\mu_1}({}^{\imath}\mathcal{Y}_\mu)$, so we must establish the opposite inclusion. Similarly to Remark 2.23, for each μ is suffices to prove this claim for a single choice of decomposition $\mu = \mu_1 + \tau\mu_1$.

First, consider the case of $\mu = 0$, with $\mu_1 = 0$. In this case all associated filtrations and gradings are non-negative: $F^k_0({}^{\imath}\mathcal{Y}_0) = G^k_0({}^{\imath}\mathcal{Y}_0) = ({}^{\imath}\mathscr{Y}_0)_k = 0$ for $k < 0$. By definition $F^k_{\mu_1}({}^{\imath}\mathcal{Y})$ has a basis consisting of those PBW basis elements which have total degree $\leqslant k$. There are finitely many such monomials. In addition, analogous monomials label a basis for the degree $\leqslant k$ part of ${}^\imath\mathcal{Y}_0 \cong \mathbb{C}[{}^\imath\mathcal{W}_0]$. Due to the surjection of graded algebras ${}^\imath\mathcal{Y}_0 \twoheadrightarrow \mathrm{gr}^{G_0^\bullet}{}^\imath\mathcal{Y}_0$, the dimension of $\dim G_0^k({}^\imath\mathcal{Y}_0)$ is at most that of the degree $\leqslant k$ part of ${}^\imath\mathcal{Y}_0$. Put together, we obtain the estimate

$$\dim F_0^k({}^\imath\mathcal{Y}_0) \;\geqslant\; \dim G_0^k({}^\imath\mathcal{Y}_0).$$

As $F_0^k({}^\imath\mathcal{Y}_0) \subseteq G_0^k({}^\imath\mathcal{Y}_0)$, we must have $F_0^k({}^\imath\mathcal{Y}_0) = G_0^k({}^\imath\mathcal{Y}_0)$ as claimed.

Next, let μ be antidominant. Since μ is even and $\tau\mu = \mu$, we can write $\mu = 2\mu_1$ with $\tau\mu_1 = \mu_1$. Recall the algebra ${}^\imath\widetilde{\mathcal{Y}}$ from §2.3, and define a filtration $F_{\mu_1}^k({}^\imath\widetilde{\mathcal{Y}})$ as the span of PBW monomials of degree $\leqslant k$ where

$$\deg \widetilde{H}_i^{(r)} = r + \langle \mu, \alpha_i\rangle, \qquad \deg \widetilde{B}_\beta^{(s)} = s + \langle \mu_1, \beta\rangle.$$

Similarly, we may define a filtration $G_{\mu_1}^\bullet({}^\imath\widetilde{\mathcal{Y}})$ analogously to the definition of $G_{\mu_1}^\bullet({}^\imath\mathcal{Y}_\mu)$. Recall also from the proof of Lemma 2.23 that there is an algebra embedding ${}^\imath\widetilde{\mathcal{Y}} \hookrightarrow {}^\imath\mathcal{Y}_0 \otimes \mathbb{C}[\xi_i \mid i \in \mathbb{I}]$. If we consider the tensor product of the filtration $F_0^\bullet({}^\imath\mathcal{Y}_0) = G_0^\bullet({}^\imath\mathcal{Y}_0)$ with the filtration on $\mathbb{C}[\xi_i \mid i \in \mathbb{I}]$ defined by giving $\deg \xi_i = \langle \mu_1, \alpha_i\rangle$, then one can see that our algebra embedding is strictly filtered, for both filtrations. It follows that $F_{\mu_1}^\bullet({}^\imath\widetilde{\mathcal{Y}}) = G_{\mu_1}^\bullet({}^\imath\widetilde{\mathcal{Y}})$.

Finally, for general μ we may again write $\mu = 2\mu_1$. Choose any shift homomorphism $\iota_{\mu,\nu}^\tau : {}^\imath\mathcal{Y}_\mu \hookrightarrow {}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}$ as in Lemma 2.7, such that $\mu+\nu+\tau\nu$ is antidominant. Then it is easy to see that the image of $G_{\mu_1}^k({}^\imath\mathcal{Y}_\mu)$ lies in $G_{\mu_1+\nu}^k({}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu})$. Since $\mu+\nu+\tau\nu$ is antidominant, we have already proven that $F_{\mu_1+\nu}^k({}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}) = G_{\mu_1+\nu}^k({}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu})$. Therefore:

$$\begin{aligned}\iota_{\mu,\nu}^\tau\big(F_{\mu_1}^k({}^\imath\mathcal{Y}_\mu)\big) \subseteq \iota_{\mu,\nu}^\tau\big(G_{\mu_1}^k({}^\imath\mathcal{Y}_\mu)\big) &\subseteq \iota_{\mu,\nu}^\tau({}^\imath\mathcal{Y}_\mu) \cap G_{\mu_1+\nu}^k({}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu})\\ &= \iota_{\mu,\nu}^\tau({}^\imath\mathcal{Y}_\mu) \cap F_{\mu_1+\nu}^k({}^\imath\mathcal{Y}_{\mu+\nu+\tau\nu}).\end{aligned}$$

But by Lemma 2.22 the right hand side above is exactly $\iota_{\mu,\nu}^\tau\big(F_{\mu_1}^\bullet({}^\imath\mathcal{Y}_\mu)\big)$. As $\iota_{\mu,\nu}^\tau$ is injective we conclude that $F_{\mu_1}^k({}^\imath\mathcal{Y}_\mu) = G_{\mu_1}^k({}^\imath\mathcal{Y}_\mu)$, which completes the proof. □

We can now prove the remaining Parts (1) and (4) of Theorem 7.8. Since $F_{\mu_1}^\bullet {}^\imath\mathcal{Y}_\mu$ is defined in terms of a PBW basis, the algebra $\mathrm{gr}^{F_{\mu_1}^\bullet}{}^\imath\mathcal{Y}_\mu$ inherits a PBW basis. On the other hand, Lemmas 7.11 and 7.12 show that $\mathbb{C}[{}^\imath\mathcal{W}_\mu] \cong {}^\imath\mathcal{Y}_\mu$ has an analogous PBW basis. These are mapped onto one another under the surjection (cf. Proposition 7.14)

$$\mathbb{C}[{}^\imath\mathcal{W}_\mu] \cong {}^\imath\mathcal{Y}_\mu \twoheadrightarrow \mathrm{gr}^{G_{\mu_1}^\bullet}{}^\imath\mathcal{Y}_\mu = \mathrm{gr}^{F_{\mu_1}^\bullet}{}^\imath\mathcal{Y}_\mu.$$

Therefore this map is a $Q^\imath$–grading preserving isomorphism, which proves Part (1). For Part (4), note that there are two shift homomorphisms $\mathbb{C}[{}^\imath\mathcal{W}_{\mu+\nu+\tau\nu}] \to \mathbb{C}[{}^\imath\mathcal{W}_\mu]$: one defined geometrically in Corollary 7.7, and the other inherited from Lemma 2.11 by taking associated graded. They are both Poisson, and have the same effect on the Poisson generators of $\mathbb{C}[{}^\imath\mathcal{W}_\mu]$. Therefore they are equal, which proves (4). This completes the proof of Theorem 7.8.

8. Fixed point loci of affine Grassmannian slices

In this section, for any even spherical coweight μ, we show that the Poisson involution σ preserves the generalized affine Grassmannian slice $\overline{\mathcal{W}}_\mu^\lambda$ exactly when λ is τ-invariant. Then we study the affine Grassmannian *islices*, i.e., the fixed point loci ${}^\imath\overline{\mathcal{W}}_\mu^\lambda = (\overline{\mathcal{W}}_\mu^\lambda)^\sigma$, and connect them to iGKLO.

8.1. **Generalized affine Grassmannian slices.** Let $\lambda \geqslant \mu$ be coweights with λ dominant, and recall the space $\mathcal{W}_\mu$ from (7.1). Define the *generalized affine Grassmannian slice* as a closed subscheme of $\mathcal{W}_\mu$:

$$\overline{\mathcal{W}}_\mu^\lambda \;=\; \mathcal{W}_\mu \cap \overline{G[z]z^\lambda G[z]}. \tag{8.1}$$

This is an irreducible affine variety over $\mathbb{C}$ of dimension $\langle 2\rho, \lambda - \mu\rangle$ [BFN19], and is a closed Poisson subvariety of $\mathcal{W}_\mu$. The $\mathbb{C}^\times$–actions from (7.3) all preserve $\overline{\mathcal{W}}_\mu^\lambda \subset \mathcal{W}_\mu$, as does the action of T by conjugation. The shift maps (7.2) restrict to birational Poisson maps $\iota_{\mu,\nu_1,\nu_2} : \overline{\mathcal{W}}_{\mu+\nu_1+\nu_2}^{\lambda+\nu_1+\nu_2} \to \overline{\mathcal{W}}_\mu^\lambda$ for any dominant coweight $\lambda \geqslant \mu$ such that $\lambda + \nu_1 + \nu_2$ is also dominant [KPW22, Proposition 4.11].

Remark 8.1. The varieties $\overline{\mathcal{W}}_\mu^\lambda$ admit an alternate description as a moduli space of G-bundles on $\mathbb{P}^1$, see [BFN19, §2] for details.

For μ dominant, $\overline{\mathcal{W}}_\mu^\lambda$ are the usual affine Grassmannian slices between spherical Schubert varieties in the affine Grassmannian Gr_G. For general μ, generalized affine Grassmannian slices are Coulomb branches for quiver gauge theories [BFN19]. They have symplectic singularities, as proven for μ dominant by [KWWY14, Theorem 2.7] and in general by [Zho20](and also see [Bel23]). In general, $\overline{\mathcal{W}}_\mu^\lambda$ is a union of finitely many symplectic leaves [MW23]:

$$\overline{\mathcal{W}}_\mu^\lambda = \bigsqcup_{\substack{\mu \leqslant \nu \leqslant \lambda,\\ \nu \text{ dominant}}} \mathcal{W}_\mu^\nu. \tag{8.2}$$

Example 8.2. Let $G = \mathrm{PGL}_2$. Given coweights $\lambda \geqslant \mu$ with λ dominant, we extract non-negative integers $\mathbf{w} = \langle \lambda, \alpha\rangle$ and $\mathbf{v} = \langle \lambda - \mu, \varpi\rangle$. The slice $\overline{\mathcal{W}}_\mu^\lambda$ has an explicit description as matrices with polynomial entries due to Kamnitzer, see [BFN19, Proposition 2.17]:

$$\overline{\mathcal{W}}_\mu^\lambda \cong \left\{ \begin{pmatrix} \mathsf{d} & \mathsf{b} \\ \mathsf{c} & \mathsf{a} \end{pmatrix} : \begin{array}{cl} (i) & \mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d} \in \mathbb{C}[z], \\ (ii) & \mathsf{a} \text{ is monic of degree } \mathbf{v}, \\ (iii) & \mathsf{b},\mathsf{c} \text{ have degree } < \mathbf{v}, \\ (iv) & \mathsf{ad} - \mathsf{bc} = z^{\mathbf{w}}. \end{array} \right\}. \tag{8.3}$$

Note that the condition (iv) entails that a divides $z^{\mathbf{w}} + \mathsf{bc}$ with quotient d, and in particular d is uniquely determined by the other matrix entries.

The matrix entries from the previous example admit generalizations to all types, which we consider in the next section.

8.2. **Classical GKLO homomorphism.** For each fundamental weight ϖ_i of $\mathfrak{g}$ let $V(-\varpi_i)$ denote the corresponding irreducible representation of $\mathfrak{g}$ with *lowest weight* $-\varpi_i$. Let $s_i \in W$ denote the simple reflection for $i \in \mathbb{I}$. Choose nonzero extremal weight vectors $v_{-\varpi_i} \in V(-\varpi_i)_{-\varpi_i}$ and $v_{-s_i\varpi_i} \in V(-\varpi_i)_{-s_i\varpi_i}$, with corresponding dual vectors $v_{-\varpi_i}^*, v_{-s_i\varpi_i}^* \in V(-\varpi_i)^*$.

Let $\lambda \geqslant \mu$ be coweights with λ dominant, and write $\lambda - \mu = \sum_i \mathbf{v}_i \alpha_i^\vee$ as in §3.1. For any element $g = n^+ h z^\mu n^- \in \overline{\mathcal{W}}_\mu^\lambda$, we can extract various formal Laurent series in z^{-1}:

$$\mathsf{a}_i(z) = z^{\langle \lambda, \varpi_i\rangle} \langle v_{-\varpi_i}^*, g v_{-\varpi_i}\rangle = z^{\mathbf{v}_i} \langle v_{-\varpi_i}^*, h v_{-\varpi_i}\rangle, \tag{8.4}$$

$$\mathsf{b}_i(z) = z^{\langle \lambda, \varpi_i\rangle} \langle v_{-s_i\varpi_i}^*, g v_{-\varpi_i}\rangle = z^{\mathbf{v}_i} \langle v_{-s_i\varpi_i}^*, n^+ h v_{-\varpi_i}\rangle, \tag{8.5}$$

$$\mathsf{c}_i(z) = z^{\langle \lambda, \varpi_i \rangle} \langle v^*_{-\varpi_i}, g v_{-s_i\varpi_i} \rangle = z^{\mathbf{v}_i} \langle v^*_{-\varpi_i}, h n^- v_{-s_i\varpi_i} \rangle. \tag{8.6}$$

In fact, these are all polynomials in z, with $\mathsf{a}_i(z)$ being monic of degree $\mathbf{v}_i$, while $\mathsf{b}_i(z)$ and $\mathsf{c}_i(z)$ have degree strictly less than $\mathbf{v}_i$, see [MW24, §2.2.1]. Writing $\mathsf{a}_i(z) = \sum_s z^{\mathbf{v}_i - s} \mathsf{a}_i^{(s)}$, and defining $\mathsf{b}_i^{(s)}$ and $\mathsf{c}_i^{(s)}$ similarly, these coefficients define regular functions on $\overline{\mathcal{W}}^\lambda_\mu$. In fact, the coefficients $\mathsf{a}_i^{(s)}$ and $\mathsf{b}_i^{(s)}$ for $i \in \mathbb{I}$ form a system of birational coordinates on $\overline{\mathcal{W}}^\lambda_\mu$ by the results of [FKMM99] and [BFN19, §2]. These functions satisfy the following fundamental equation, which stems from a relation for generalized minors due to Fomin-Zelevinsky [FZ99, Theorem 1.17], see also [MW24, (2.70)]: for each $i \in \mathbb{I}$, recalling that $\mathbf{w}_i = \langle \lambda, \alpha_i \rangle$, we have

$$\mathsf{a}_i(z)\mathsf{d}_i(z) - \mathsf{b}_i(z)\mathsf{c}_i(z) = z^{\mathbf{w}_i} \prod_{j \leftrightarrow i,\ j \in \mathbb{I}} \mathsf{a}_j(z). \tag{8.7}$$

Here $\mathsf{d}_i(z) = z^{\langle \lambda, \varpi_i \rangle} \langle v^*_{-s_i\varpi_i}, g v_{-s_i\varpi_i} \rangle$ which is again a polynomial in z.

Remark 8.3. We have defined G to be the *adjoint* group associated to $\mathfrak{g}$, however $V(-\varpi_i)$ is only a representation of the *simply-connected* group associated to $\mathfrak{g}$. Nevertheless, the functions (8.4)–(8.6) are well defined on the variety $\overline{\mathcal{W}}^\lambda_\mu$. One can see this via zastava spaces, as in [BFN19, §2]. Alternatively, following [MW24, Remark 2.10], note that $U_1^\pm[\![z^{-1}]\!]$ and $T_1[\![z^{-1}]\!]$ are naturally independent of the choice of group, so the factors $n^\pm, h$ of $g = n^+ h z^\mu n^-$ all act on $V(-\varpi_i)(\!(z^{-1})\!)$. Letting D denote the determinant of the Cartan matrix, then the factor z^μ also has a well-defined action on $V(-\varpi_i)(\!(z^{-\frac{1}{D}})\!)$. Thus (8.4)–(8.6) are all well-defined as Laurent series in $z^{-\frac{1}{D}}$, and in fact are Laurent series in z^{-1}.

Let $\mathbb{A}^{(\lambda-\mu)}$ denote the variety of tuples $\big(\mathsf{a}_i(z)\big)_{i\in\mathbb{I}}$ of monic polynomials $\mathsf{a}_i(z)$ of degree $\mathbf{v}_i$. Then $\mathbb{A}^{(\lambda-\mu)}$ is an affine space with coordinates $\mathsf{a}_i^{(s)}$ for $i \in \mathbb{I}$ and $1 \leqslant s \leqslant \mathbf{v}_i$. The functions (8.4) define a natural map $\overline{\mathcal{W}}^\lambda_\mu \to \mathbb{A}^{(\lambda-\mu)}$ which is faithfully flat by [BFN19, Lemma 2.7]. We may identify $\mathbb{A}^{(\lambda-\mu)} = \mathbb{A}^{|\lambda-\mu|}/S_{\lambda-\mu}$ as the quotient of an affine space $\mathbb{A}^{|\lambda-\mu|} = \prod_i \mathbb{A}^{\mathbf{v}_i}$ by the product of symmetric groups $S_{\lambda-\mu} = \prod_i S_{\mathbf{v}_i}$. Indeed, we denote natural coordinates on $\mathbb{A}^{|\lambda-\mu|}$ by $w_{i,r}$ for $i \in \mathbb{I}$ and $1 \leqslant r \leqslant \mathbf{v}_i$, defined so that

$$\mathsf{a}_i(z) = \prod_{r=1}^{\mathbf{v}_i} (z - w_{i,r}) \tag{8.8}$$

with $S_{\mathbf{v}_i}$ acting by permuting these roots. Define also

$$y^-_{i,r} = \mathsf{b}_i(w_{i,r}) \quad \text{and} \quad y^+_{i,r} = \mathsf{c}_i(w_{i,r}), \tag{8.9}$$

and observe that because of the relation (8.7) these must satisfy

$$y^+_{i,r} y^-_{i,r} = -w_{i,r}^{\mathbf{w}_i} \prod_{j \leftrightarrow i,\ j \in \mathbb{I}} \mathsf{a}_j(w_{i,r}). \tag{8.10}$$

The polynomials $\mathsf{b}_i(z), \mathsf{c}_i(z)$ can be recovered from the $y^\pm_{i,r}$ by Lagrange interpolation:

$$\mathsf{b}_i(z) = \sum_{r=1}^{\mathbf{v}_i} \Big(\prod_{s=1, s\neq r}^{\mathbf{v}_i} \frac{z - w_{i,s}}{w_{i,r} - w_{i,s}} \Big) y^-_{i,r}, \qquad \mathsf{c}_i(z) = \sum_{r=1}^{\mathbf{v}_i} \Big(\prod_{s=1, s\neq r}^{\mathbf{v}_i} \frac{z - w_{i,s}}{w_{i,r} - w_{i,s}} \Big) y^+_{i,r}. \tag{8.11}$$

Remark 8.4. The functions from (7.4) can also be expressed in terms of the above coordinates:

$$\sum_r e_i^{(r)} z^{-r} = \frac{\mathsf{b}_i(z)}{\mathsf{a}_i(z)}, \quad \sum_r h_i^{(r)} z^{-r} = z^{\mathbf{w}_i} \prod_{j\in\mathbb{I}} \mathsf{a}_j(z)^{-c_{ji}}, \quad \sum_r f_i^{(r)} z^{-r} = \frac{\mathsf{c}_i(z)}{\mathsf{a}_i(z)}. \tag{8.12}$$

This leads to variations on the Lagrange interpolation formulas above, such as

$$\sum_r e_i^{(r)} z^{-r} = \sum_{r=1}^{\mathbf{v}_i} \frac{1}{(z-w_{i,r})\prod_{s=1,s\neq r}^{\mathbf{v}_i}(w_{i,r}-w_{i,s})} y_{i,r}^-,$$

where the right side is expanded as a Laurent series in z^{-1}.

In a slight variation on the notation of [BFN19, §2], we define an open subset $\mathring{\mathbb{A}}^{|\lambda-\mu|} \subset \mathbb{A}^{|\lambda-\mu|}$ as the complement to all diagonals $w_{i,r} = w_{j,s}$ for pairs $(i,r) \neq (j,s)$ with $c_{ij} \neq 0$. Let $\mathring{\mathbb{A}}^{(\lambda-\mu)} = \mathring{\mathbb{A}}^{|\lambda-\mu|}/S_{\lambda-\mu}$. Then $\mathring{\mathbb{A}}^{(\lambda-\mu)} \subset \mathbb{A}^{(\lambda-\mu)}$ is open, consisting of tuples $(\mathsf{a}_i(z))_{i\in\mathbb{I}}$ where each $\mathsf{a}_i(z)$ has no multiple roots, and where $\mathsf{a}_i(z)$ and $\mathsf{a}_j(z)$ have no common roots whenever $c_{ij} = -1$. Define an open subset $U_\mu^\lambda \subseteq \overline{\mathcal{W}}_\mu^\lambda$ as the fiber product

$$U_\mu^\lambda = \overline{\mathcal{W}}_\mu^\lambda \times_{\mathbb{A}^{(\lambda-\mu)}} \mathring{\mathbb{A}}^{(\lambda-\mu)}. \tag{8.13}$$

The following definition provides an étale covering of U_μ^λ.

Definition 8.5. Consider an affine space $\mathbb{A}^{3|\lambda-\mu|}$ with coordinates $w_{i,r}, y_{i,r}^+, y_{i,r}^-$ for $i \in \mathbb{I}$ and $1 \leqslant r \leqslant \mathbf{v}_i$. Let $X_\mu^\lambda \subseteq \mathbb{A}^{3|\lambda-\mu|}$ be the locally closed subvariety defined by imposing the relations (8.10) for all pairs (i,r), and by also imposing that $w_{i,r} - w_{j,s} \neq 0$ for all pairs $(i,r) \neq (j,s)$ where $c_{ij} \neq 0$.

There is a natural map $X_\mu^\lambda \to \mathbb{A}^{|\lambda-\mu|}$ defined by projecting onto the coordinates $w_{i,r}$. Note that the image of X_μ^λ actually lands in the open set $\mathring{\mathbb{A}}^{|\lambda-\mu|}$. The following result is a variant of results from [BFN19, §2] describing the birational geometry of $\overline{\mathcal{W}}_\mu^\lambda$. The Poisson structure on X_μ^λ is related to the Poisson structure for zastava spaces in the étale coordinates of [FKMM99]. Denote

$$\delta_{i\leftrightarrow j} = \begin{cases} 1, & \text{if } c_{ij} = -1, \\ 0, & \text{else.} \end{cases}$$

Theorem 8.6 (cf. [BFN19])**.** (1) *The variety X_μ^λ is equipped with a Poisson structure defined by:*

$$\{w_{i,r}, w_{j,s}\} = 0, \quad \{w_{i,r}, y_{j,s}^\pm\} = \mp\delta_{i,j}\delta_{r,s} y_{j,s}^\pm,$$
$$\{y_{i,r}^-, y_{j,s}^+\} = \delta_{i,j}\delta_{r,s}\frac{\partial}{\partial w_{i,r}}\Big(w_{i,r}^{\mathbf{w}_i} \prod_{j\leftrightarrow i,\ j\in\mathbb{I}} \mathsf{a}_i(w_{i,r})\Big),$$
$$\{y_{i,r}^\pm, y_{j,s}^\pm\} = \pm\frac{\delta_{i\leftrightarrow j}}{w_{i,r}-w_{j,s}} y_{i,r}^\pm y_{j,s}^\pm.$$

(2) *There is a Poisson map $X_\mu^\lambda \to U_\mu^\lambda$, uniquely determined by*

$$\mathsf{a}_i(z) = \prod_r (z - w_{i,r}),$$

$$\mathsf{b}_i(z) = \sum_r \Big(\prod_{s\neq r} \frac{z - w_{i,s}}{w_{i,r} - w_{i,s}} \Big) y_{i,r}^-, \qquad \mathsf{c}_i(z) = \sum_r \Big(\prod_{s\neq r} \frac{z - w_{i,s}}{w_{i,r} - w_{i,s}} \Big) y_{i,r}^+.$$

This map is finite and faithfully flat, and identifies

$$X_\mu^\lambda = U_\mu^\lambda \times_{\mathbb{A}^{(\lambda-\mu)}} \mathbb{A}^{|\lambda-\mu|} = U_\mu^\lambda \times_{\mathring{\mathbb{A}}^{(\lambda-\mu)}} \mathring{\mathbb{A}}^{|\lambda-\mu|}.$$

Proof. An isomorphism $\overline{\mathcal{W}}_\mu^\lambda \cong \mathcal{M}_C(\mathrm{GL}(V), \mathbf{N})$ with the Coulomb branch for a quiver gauge theory is established in [BFN19, Theorem 3.10]. It follows that the base change $\overline{\mathcal{W}}_\mu^\lambda \times_{\mathbb{A}^{(\lambda-\mu)}} \mathring{\mathbb{A}}^{|\lambda-\mu|}$ can be described using Coulomb branch techniques from [BFN18, §5 and §6]. Indeed, fix a maximal torus $\mathbf{T} \subset \mathrm{GL}(V)$ and let $\mathbf{N}' = \bigoplus_{i\in\mathbb{I}} \mathrm{Hom}(W_i, V_i)$ be the sub-representation of $\mathbf{N}$ where only framing arrows are left (see [BFN19, §3(iii)] for the definition of $\mathbf{N}$), there is an isomorphism:

$$\overline{\mathcal{W}}_\mu^\lambda \times_{\mathbb{A}^{(\lambda-\mu)}} \mathring{\mathbb{A}}^{|\lambda-\mu|} \cong \mathcal{M}_C(\mathbf{T}, \mathbf{N}') \times_{\mathbb{A}^{|\lambda-\mu|}} \mathring{\mathbb{A}}^{|\lambda-\mu|}.$$

Denoting this space by $\widetilde{X}_\mu^\lambda$, there is a natural map $\widetilde{X}_\mu^\lambda \to U_\mu^\lambda$. It is finite and faithfully flat since it arises via base change along $\mathbb{A}^{|\lambda-\mu|} \to \mathbb{A}^{(\lambda-\mu)}$. It is étale by an application of [Wee22, Theorem 10].

To complete the proof, it suffices to show that $\widetilde{X}_\mu^\lambda = X_\mu^\lambda$. In fact, both can be identified with the same open subset of a product of Kleinian singularities of type A. For $\mathcal{M}_C(\mathbf{T}, \mathbf{N}')$ this follows from [BFN18, §3(vii)(a)] and [BFN18, §4(iv)], since $\mathbf{T}$ is a product of multiplicative groups $\mathbb{C}_{i,r}^\times$ each acting independently by scaling a space $\mathrm{Hom}(W_i, \mathbb{C}_{i,r})$. Meanwhile, in Definition 8.5 note that all linear factors of $\mathsf{a}_j(z)$ in the defining relation (8.10) are invertible. Thus up to renormalizing $y_{i,r}^\pm$ this relation is also simply the Kleinian singularity $y_{i,r}^+ y_{i,r}^- = w_{i,r}^{\mathbf{w}_i}$. ☐

8.3. **Affine Grassmannian islices $^\imath\overline{\mathcal{W}}_\mu^\lambda$.** Recall from Lemma 7.5 a Poisson involution σ of $\mathcal{W}_\mu$, for any even spherical coweight μ. We are interested in the fixed point loci of the subvarieties $\overline{\mathcal{W}}_\mu^\lambda \subset \mathcal{W}_\mu$ under the action of σ:

$$^\imath\overline{\mathcal{W}}_\mu^\lambda := (\overline{\mathcal{W}}_\mu^\lambda)^\sigma. \tag{8.14}$$

We shall refer to $^\imath\overline{\mathcal{W}}_\mu^\lambda$ as an affine Grassmannian *islice*.

Theorem 8.7. *Let μ be an even spherical coweight, and let λ be a dominant coweight with $\lambda \geqslant \mu$. Then σ preserves the subvariety $\overline{\mathcal{W}}_\mu^\lambda \subset \mathcal{W}_\mu$ if and only if $\tau\lambda = \lambda$. In this case, the islice $^\imath\overline{\mathcal{W}}_\mu^\lambda$ inherits a Poisson structure via (doubled) Dirac reduction, and $^\imath\overline{\mathcal{W}}_\mu^\lambda \subset {}^\imath\mathcal{W}_\mu$ is a Poisson embedding. There is a decomposition into smooth strata:*

$$^\imath\overline{\mathcal{W}}_\mu^\lambda = \bigsqcup_{\substack{\mu\leqslant\nu\leqslant\lambda,\\ \nu \text{ dominant},\\ \tau\nu=\nu}} {}^\imath\mathcal{W}_\mu^\nu, \qquad \textit{where } {}^\imath\mathcal{W}_\mu^\nu := (\mathcal{W}_\mu^\nu)^\sigma.$$

The symplectic leaves of $^\imath\overline{\mathcal{W}}_\mu^\lambda$ consist of the connected components of the strata $^\imath\mathcal{W}_\mu^\nu$. Finally, the shift maps restrict to Poisson maps $\iota_{\mu,\nu}^\tau : {}^\imath\overline{\mathcal{W}}_{\mu+\nu+\tau\nu}^{\lambda+\nu+\tau\nu} \longrightarrow {}^\imath\overline{\mathcal{W}}_\mu^\lambda$.

Proof. For any dominant coweight ν, using (7.17) we have

$$\sigma(G[z]z^\nu G[z]) = \sigma(G[z])\sigma(z^\nu)\sigma(G[z]) = G[z](-1)^{\tau\nu} z^{\tau\nu} G[z] = G[z] z^{\tau\nu} G[z].$$

Since $\tau\nu$ is dominant, we have $G[z]z^{\nu}G[z] \cap G[z]z^{\tau\nu}G[z] = \varnothing$ unless $\tau\nu = \nu$. It follows that σ maps $\mathcal{W}_\mu^\nu$ isomorphically onto $\mathcal{W}_\mu^{\tau\nu}$, and preserves $\mathcal{W}_\mu^\nu$ if and only if $\tau\nu = \nu$. Similarly σ preserves $\overline{\mathcal{W}}_\mu^\lambda$ if and only if $\tau\lambda = \lambda$. In this case the fixed point locus ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ inherits a Poisson structure by Dirac reduction, and ${}^\imath\overline{\mathcal{W}}_\mu^\lambda \subset {}^\imath\mathcal{W}_\mu$ is Poisson since $\overline{\mathcal{W}}_\mu^\lambda \subset \mathcal{W}_\mu$ is Poisson. The restricted shift maps are therefore Poisson by Theorem 7.8(4). Finally, the decomposition of ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ into symplectic leaves follows from Lemma 6.8 and (8.2). □

Remark 8.8. Similarly to Remark 8.1, the islice ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ admits a modular description as a moduli space of G-bundles. More precisely, using the symmetric definition $\overline{\mathcal{W}}_{\mu_1,\tau\mu_1}^\lambda$ of the slice $\overline{\mathcal{W}}_\mu^\lambda$ from [BFN19, §2(v)], we can cut out ${}^\imath\overline{\mathcal{W}}_\mu^\lambda \subset \overline{\mathcal{W}}_{\mu_1,\tau\mu_1}^\lambda$ by imposing that the bundles denoted $\mathcal{P}_\pm$ are related by the involution σ. We leave the details to the interested reader.

Example 8.9. Let $G = \mathrm{PGL}_2$ and $\tau = \mathrm{id}$. The involution σ on $G(\!(z^{-1})\!)$ is given by $g(z) \mapsto g(-z)^T$. Take coweights $\lambda \geqslant \mu$ for G, with λ dominant. We assume that μ is even; note that it is equivalent to assume that λ is even, since $\langle \lambda - \mu, \alpha \rangle = \langle \lambda - \mu, 2\varpi \rangle = 2\mathbf{v}$. Recall the explicit model for $\overline{\mathcal{W}}_\mu^\lambda$ from Example 8.2. After renormalizing by a multiple of the identity matrix (this is harmless since $G = \mathrm{PGL}_2$), in this model the involution takes the form, cf. (8.21):

$$\begin{pmatrix} \mathsf{d}(z) & \mathsf{b}(z) \\ \mathsf{c}(z) & \mathsf{a}(z) \end{pmatrix} \longmapsto (-1)^{\mathbf{v}} \begin{pmatrix} \mathsf{d}(-z) & \mathsf{c}(-z) \\ \mathsf{b}(-z) & \mathsf{a}(-z) \end{pmatrix}$$

The fixed point locus can thus be described analogously to Example 8.2:

$$ {}^\imath\overline{\mathcal{W}}_\mu^\lambda \cong \left\{ \begin{pmatrix} \mathsf{d} & \mathsf{b} \\ \mathsf{c} & \mathsf{a} \end{pmatrix} : \begin{array}{cl} (i) & \mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d} \in \mathbb{C}[z], \\ (ii) & \mathsf{a} \text{ is monic of degree } \mathbf{v}, \\ (iii) & \mathsf{b},\mathsf{c} \text{ have degree } < \mathbf{v}, \\ (iv) & \mathsf{ad} - \mathsf{bc} = z^{\mathbf{w}}, \\ (v) & \mathsf{a}(z) = (-1)^{\mathbf{v}}\mathsf{a}(-z),\ \mathsf{c}(z) = (-1)^{\mathbf{v}}\mathsf{b}(z). \end{array} \right\} \tag{8.15}$$

Similarly to Example 8.2, the coordinate d is uniquely determined by the others by condition (iv). Note that it automatically satisfies $\mathsf{d}(z) = (-1)^{\mathbf{v}}\mathsf{d}(-z)$, because of (v), which also determines c uniquely from b. So in the most simple terms, the variety ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ consists of pairs (a,b) of polynomials satisfying the degree conditions (ii) and (iii), such that $\mathsf{a}(z) = (-1)^{\mathbf{v}}\mathsf{a}(-z)$ and $\mathsf{a}(z)$ divides $z^{\mathbf{w}} + (-1)^{\mathbf{v}}\mathsf{b}(z)\mathsf{b}(-z)$.

In this case ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ is normal and a complete intersection. When μ is dominant, we will give an alternate description of ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ in terms of Slodowy slices, see Example 9.7. In particular these cases are all symplectic singularities.

8.4. **Classical iGKLO for ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$.** Let μ be an even spherical coweight, and let $\tau\lambda = \lambda$ be a dominant coweight with $\lambda \geqslant \mu$. Recall the iGKLO homomorphism $\Phi_\mu^\lambda : {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}] \to \mathcal{A}$ from Theorem 3.6. This homomorphism depends on a choice of orientation of the Dynkin diagram $\mathbb{I}$, which we fix from now on.

8.4.1. *Classical iGKLO.* We show that the classical limit of Φ_μ^λ (or more precisely, its specialization at $\boldsymbol{z}=0$) is related to the fixed point scheme ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$. We first define appropriate filtrations on these algebras.

On one hand, we consider a filtration $F_{\mu_1}^\bullet({}^\imath\mathcal{Y}_\mu)$ as in §2.4. Recall that $\mathrm{gr}^{F_{\mu_1}^\bullet}\,{}^\imath\mathcal{Y}_\mu \cong \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ by Theorem 7.8. For our purposes here, we define the coweight μ_1 for $\mathfrak{g}$ as follows:

$$\langle \mu_1, \alpha_i\rangle = \begin{cases} \deg Z_i - \mathbf{v}_i + \sum_{\substack{j\leftrightarrow i,\\ j\in\mathbb{I}}} \mathbf{v}_j, & \text{if } i\in\mathbb{I}_{\pm1},\\ \mathfrak{w}_i - 2\mathfrak{v}_i + \sum_{\substack{j\leftrightarrow i,\\ j\in{}^\imath\mathbb{I}}}\mathfrak{v}_j + \frac12\Big(\varsigma_i + 2\theta_i + \sum_{\substack{j\leftrightarrow i,\\ j\in\mathbb{I}_0}}\theta_j\Big), & \text{if } i\in\mathbb{I}_0.\end{cases} \tag{8.16}$$

The following result is necessary so that the filtration $F_{\mu_1}^\bullet({}^\imath\mathcal{Y}_\mu)$ is defined.

Lemma 8.10. *μ_1 is an integral coweight, and $\mu=\mu_1+\tau\mu_1$.*

Proof. Both parts follow from direct calculation using the definitions from §3.1. In particular note that for $i\in\mathbb{I}_0$ we have:

$$\langle\mu,\alpha_i\rangle = \mathbf{w}_i - 2\mathbf{v}_i + \sum_{\substack{j\leftrightarrow i,\\ j\in\mathbb{I}}}\mathbf{v}_j = (2\mathfrak{w}_i+\varsigma_i) - 2(2\mathfrak{v}_i+\theta_i) + 2\sum_{\substack{j\leftrightarrow i,\\ j\in\mathbb{I}_1}}\mathfrak{v}_j + \sum_{\substack{j\leftrightarrow i,\\ j\in\mathbb{I}_0}}(2\mathfrak{v}_j+\theta_j).$$

This is an even integer since μ is even, and $\langle\mu_1,\alpha_i\rangle$ is forced to be half of this value to ensure that $\mu=\mu_1+\tau\mu_1$. □

On the other hand, in the algebra $\mathcal{A}$ we specialize all (central) variables $z_{i,s}$ to 0, obtaining its quotient algebra $\mathcal{A}_{\boldsymbol{z}=0}$. Similarly to [BFN19, §B(ii)], define a filtration on $\mathcal{A}_{\boldsymbol{z}=0}$ by setting $\deg w_{i,r}=1$ for all pairs (i,r), and setting

$$\deg \eth_{i,r}^{\pm1} = \begin{cases} 0, & \text{if } i\in\mathbb{I}_{\pm1},\\ \pm\sum_{\substack{j\to i,\\ j\in\mathbb{I}_0}}\mathfrak{v}_j \pm \frac12\Big(-\varsigma_i + \sum_{\substack{j\leftrightarrow i,\\ j\in\mathbb{I}_0}}\theta_j\Big), & \text{if } i\in\mathbb{I}_0.\end{cases} \tag{8.17}$$

Similarly to the previous lemma, we note that $\deg\eth_{i,r}^{\pm1}$ is always an integer. By the same argument as [KWWY14, Proposition 4.4], the associated graded algebra is a localized polynomial ring:

$$\mathrm{gr}\,\mathcal{A}_{\boldsymbol{z}=0} = \mathbb{C}[w_{i,r},\eth_{i,r}^{\pm1},(w_{i,r}\pm w_{i,r'})^{-1},w_{i,r}^{-1}\ :\ i\in{}^\imath\mathbb{I}, 1\leqslant r\neq r'\leqslant\mathfrak{v}_i]. \tag{8.18}$$

Its Poisson structure is determined by $\{\eth_{i,r}^{\pm1},w_{j,s}\}=\pm\delta_{i,j}\delta_{r,s}\eth_{i,r}^{\pm1}$ and $\{w_{i,r},w_{j,s}\}=\{\eth_{i,r},\eth_{j,s}\}=0$. As in §3.1 we extend the notation $\eth_{i,r}$ to $i\in\mathbb{I}_{-1}$ by $\eth_{i,r}=-\eth_{\tau i,\mathfrak{v}_i+1-r}$.

The next theorem is the main result of this section.

Theorem 8.11. *Retain the notation for filtrations on ${}^\imath\mathcal{Y}_\mu$ and $\mathcal{A}_{\boldsymbol{z}=0}$ as above.*

(1) *The open subscheme $U_\mu^\lambda\subset\overline{\mathcal{W}}_\mu^\lambda$ from (8.13) is invariant under σ. Its fixed point locus ${}^\imath U_\mu^\lambda$ is non-empty if and only if the parity condition (3.5) holds, and in this case we have*

$$\dim{}^\imath U_\mu^\lambda = 2\sum_{i\in{}^\imath\mathbb{I}}\mathfrak{v}_i.$$

(2) *The homomorphism $\Phi_\mu^{\lambda,\boldsymbol{z}=0}:{}^\imath\mathcal{Y}_\mu\to\mathcal{A}_{\boldsymbol{z}=0}$ is filtered, and the associated graded map $\mathrm{gr}\,\Phi_\mu^{\lambda,\boldsymbol{z}=0}:\mathbb{C}[{}^\imath\mathcal{W}_\mu]\longrightarrow\mathrm{gr}\,\mathcal{A}_{\boldsymbol{z}=0}$ defines the closure $\overline{C}_\mu^\lambda\subseteq{}^\imath\overline{\mathcal{W}}_\mu^\lambda$, i.e., the kernel of this map is the defining ideal of $\overline{C}_\mu^\lambda$. Here $C_\mu^\lambda\subseteq{}^\imath U_\mu^\lambda$ is a top-dimensional irreducible component.*

The proof of this theorem will be given in §8.5. By Theorem 8.11, if ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ is irreducible then $\mathrm{gr}\,\Phi_\mu^{\lambda,\boldsymbol{z}=0}$ defines ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ itself, and hence it is important to determine when ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ is irreducible.

8.4.2. *Deformed iGKLO.* Although we specialize the parameters $\boldsymbol{z}=0$ above, this is for simplicity and not necessity. The variety $\overline{\mathcal{W}}_\mu^\lambda$ admits a Beilinson-Drinfeld deformation $\overline{\mathcal{W}}_\mu^{\underline{\lambda}}$ defined in [BFN19, §2(x)]. It is a closed subvariety $\overline{\mathcal{W}}_\mu^{\underline{\lambda}} \subset \mathcal{W}_\mu \times \mathbb{A}^{|\lambda|}$, where $\mathbb{A}^{|\lambda|} = \mathbb{A}^{\sum_i \mathbf{w}_i}$ is the affine space having points $\boldsymbol{z} = (z_{i,s})_{i\in\mathbb{I},1\leqslant s\leqslant \mathbf{w}_i}$. By definition, the fiber over $\boldsymbol{z}\in\mathbb{A}^{|\lambda|}$ is

$$\overline{\mathcal{W}}_\mu^{\underline{\lambda},\boldsymbol{z}} = \mathcal{W}_\mu \cap \overline{G[z]z^{\lambda,\boldsymbol{z}}G[z]}, \tag{8.19}$$

where $z^{\lambda,\boldsymbol{z}} = \prod_{i,s}(z-z_{i,s})^{\varpi_i^\vee} \in G(\!(z^{-1})\!)$. In particular, the fiber over $0\in\mathbb{A}^{|\lambda|}$ is exactly $\overline{\mathcal{W}}_\mu^\lambda$.

Now suppose that $\tau\lambda=\lambda$ and μ is even spherical. Then the involution σ of $G(\!(z^{-1})\!)$ induces an involution of $\overline{\mathcal{W}}_\mu^{\underline{\lambda}}$, permuting the above fibers with $\sigma(\overline{\mathcal{W}}_\mu^{\underline{\lambda},\boldsymbol{z}}) = \overline{\mathcal{W}}_\mu^{\underline{\lambda},-\boldsymbol{z}}$. Define an involution σ of $\mathbb{A}^{|\lambda|}$ by

$$\sigma(z_{i,s}) = \begin{cases} -z_{i,\mathbf{w}_i-s+1}, & \text{if } i\in\mathbb{I}_0,\\ -z_{\tau i,s}, & \text{if } i\in\mathbb{I}\setminus\mathbb{I}_0.\end{cases}$$

Then the map $\overline{\mathcal{W}}_\mu^{\underline{\lambda}} \to \mathbb{A}^{|\lambda|}$ is σ-equivariant and there is an embedding of fixed point loci

$${}^\imath\overline{\mathcal{W}}_\mu^{\underline{\lambda}} \subset {}^\imath\mathcal{W}_\mu \times {}^\imath\mathbb{A}^{|\lambda|}. \tag{8.20}$$

Similarly to (8.26) below, one sees that the fixed point locus ${}^\imath\mathbb{A}^{|\lambda|}$ has coordinates $\{z_{i,s} : i\in{}^\imath\mathbb{I}, 1\leqslant s\leqslant \mathfrak{w}_i\}$. Thus, a point $\boldsymbol{z} = (z_{i,s})_{i\in{}^\imath\mathbb{I},1\leqslant s\leqslant\mathfrak{w}_i} \in {}^\imath\mathbb{A}^{|\lambda|}$ is exactly as in §3.1. The fixed point locus ${}^\imath U_\mu^{\underline{\lambda}}$ is an open subvariety of ${}^\imath\overline{\mathcal{W}}_\mu^{\underline{\lambda}}$.

Let the filtration on ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}]$ be defined as above for ${}^\imath\mathcal{Y}_\mu$, and with the variables $z_{i,s}$ in degree 1. We can now formulate a deformed version of Theorem 8.11.

Theorem 8.12. *The map $\Phi_\mu^\lambda : {}^\imath\mathcal{Y}_\mu[\boldsymbol{z}] \to \mathcal{A}$ is filtered, and its associated graded map*

$$\mathrm{gr}\,\Phi_\mu^\lambda : \mathbb{C}[{}^\imath\mathcal{W}_\mu \times {}^\imath\mathbb{A}^{|\lambda|}] \longrightarrow \mathrm{gr}\,\mathcal{A}$$

defines the closure $\overline{C}_\mu^{\underline{\lambda}} \subseteq {}^\imath\overline{\mathcal{W}}_\mu^{\underline{\lambda}}$. Here $C_\mu^{\underline{\lambda}} \subseteq {}^\imath U_\mu^{\underline{\lambda}}$ is a top-dimensional irreducible component of dimension $\sum_{i\in{}^\imath\mathbb{I}}(2\mathfrak{v}_i+\mathfrak{w}_i)$.

Recall from Definition 3.8 that we define the truncated shifted twisted Yangian ${}^\imath\mathcal{Y}_\mu^\lambda$ as the image of the map Φ_μ^λ. Recall also from §3.4.2 that we define a filtration $F_{\mu_1}^\bullet{}^\imath\mathcal{Y}_\mu^\lambda$ as the quotient filtration from ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}]$.

Then ${}^\imath\mathcal{Y}_\mu^\lambda$ inherits natural filtrations: firstly as a quotient of ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}]$, and second as a subalgebra of $\mathcal{A}$. Since Φ_μ^λ is filtered, the first filtration is contained in the second. There is a canonical surjection, cf. the proof of Theorem 5.8:

$$\mathrm{gr}\,{}^\imath\mathcal{Y}_\mu^\lambda = \mathrm{gr}\,\mathrm{Im}(\Phi_\mu^\lambda) \twoheadrightarrow \mathrm{Im}(\mathrm{gr}\,\Phi_\mu^\lambda) \cong \mathbb{C}[\overline{C}_\mu^{\underline{\lambda}}].$$

We expect that it is an isomorphism:

Conjecture 8.13. *The truncated shifted twisted Yangian ${}^\imath\mathcal{Y}_\mu^\lambda$ quantizes the variety $\overline{C}_\mu^{\underline{\lambda}} \subseteq {}^\imath\overline{\mathcal{W}}_\mu^{\underline{\lambda}}$. In particular, if ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ is irreducible then ${}^\imath\mathcal{Y}_\mu^\lambda$ quantizes ${}^\imath\overline{\mathcal{W}}_\mu^{\underline{\lambda}}$.*

Proving this conjecture would be a question of comparing two filtrations on ${}^\imath\mathcal{Y}_\mu^\lambda$: the quotient filtration from ${}^\imath\mathcal{Y}_\mu[\boldsymbol{z}]$ as defined above, and the subspace filtration inherited from $\mathcal{A}$. The former filtration is contained within the latter, and if they are equal, then the conjecture holds.

For truncated shifted (untwisted) Yangians, the analogous conjecture is proven for μ dominant in [BFN19, Corollary B.28] and for general μ in [Wee19, Theorem 3.13].

Remark 8.14. A weak form of the above conjecture states that ${}^\imath\mathcal{Y}_\mu^\lambda$ quantizes a scheme supported on $\overline{C}_\mu^\lambda$. A similar result was proven for truncated shifted Yangians in [KWWY14, Theorem 4.8].

8.5. **Proof of Theorem 8.11 on ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$.** First, we observe that the involution σ on $\overline{\mathcal{W}}_\mu^\lambda$ maps

$$\begin{aligned} \sigma\big(\mathsf{a}_i(z)\big) &= (-1)^{\mathbf{v}_i}\mathsf{a}_{\tau i}(-z), \qquad & \sigma\big(\mathsf{b}_i(z)\big) &= (-1)^{\mathbf{v}_i}\mathsf{c}_{\tau i}(-z),\\ \sigma\big(\mathsf{c}_i(z)\big) &= (-1)^{\mathbf{v}_i}\mathsf{b}_{\tau i}(-z), \qquad & \sigma\big(\mathsf{d}_i(z)\big) &= (-1)^{\mathbf{v}_i}\mathsf{d}_{\tau i}(-z). \end{aligned} \tag{8.21}$$

Indeed, one sees this from the identity $\sigma(n^+hz^\mu n^-) = \sigma(n^-)\sigma(h)z^\mu\sigma(n^+)$ for $n^+hz^\mu n^- \in {}^\imath\mathcal{W}_\mu$, together with (7.16) and the definitions (8.4)–(8.6). See also [FZ99, Proposition 2.7]. Therefore, viewed as functions on ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ by restriction:

$$\mathsf{a}_i(z) = (-1)^{\mathbf{v}_i}\mathsf{a}_{\tau i}(-z), \quad \mathsf{c}_i(z) = (-1)^{\mathbf{v}_i}\mathsf{b}_{\tau i}(-z), \quad \mathsf{d}_i(z) = (-1)^{\mathbf{v}_i}\mathsf{d}_{\tau i}(-z). \tag{8.22}$$

Remark 8.15. The algebra $\mathbb{C}[{}^\imath\overline{\mathcal{W}}_\mu^\lambda]$ is Poisson generated by the coefficients of $\mathsf{a}_i(z)$ and $\mathsf{b}_i(z)$ for $i \in \mathbb{I}$. This follows from Theorem 7.8(2) together with equations (7.18) and (8.12).

Consequently, if we define an involution σ on $\mathbb{A}^{(\lambda-\mu)}$ by $\sigma\big(\mathsf{a}_i(z)\big) = (-1)^{\mathbf{v}_i}\mathsf{a}_{\tau i}(-z)$, then the map $\overline{\mathcal{W}}_\mu^\lambda \to \mathbb{A}^{(\lambda-\mu)}$ from §8.2 is σ-equivariant and induces a map on fixed point loci

$${}^\imath\overline{\mathcal{W}}_\mu^\lambda \longrightarrow {}^\imath\mathbb{A}^{(\lambda-\mu)}. \tag{8.23}$$

Applying Lemma 6.9, we note that ${}^\imath\mathbb{A}^{(\lambda-\mu)}$ is an affine space with coordinates

$$\{\mathsf{a}_i^{(s)} : i \in \mathbb{I}_0, 1 \leqslant s \leqslant \mathbf{v}_i \text{ with } s \text{ even}\} \cup \{\mathsf{a}_i^{(s)} : i \in \mathbb{I}_1, 1 \leqslant s \leqslant \mathbf{v}_i\}. \tag{8.24}$$

Here, we view the $\mathsf{a}_i^{(s)}$ as functions on ${}^\imath\mathbb{A}^{(\lambda-\mu)}$ by restriction from $\mathbb{A}^{(\lambda-\mu)}$. Recalling the notation $\mathfrak{v}_i$ from (3.2), we see that ${}^\imath\mathbb{A}^{(\lambda-\mu)}$ has dimension $\sum_{i\in{}^\imath\mathbb{I}}\mathfrak{v}_i$.

We will next extend σ to an involution of $\mathbb{A}^{|\lambda-\mu|}$, such that the map $\mathbb{A}^{|\lambda-\mu|} \to \mathbb{A}^{(\lambda-\mu)}$ defined by (8.8) is σ-equivariant and well behaved (in the sense of Lemma 8.16 and Corollary 8.17). For $i \in \mathbb{I}$ and $1 \leqslant r \leqslant \mathbf{v}_i$, we set

$$\sigma(w_{i,r}) = -w_{\tau i,\tau r}, \qquad \text{where} \qquad \tau r := \mathbf{v}_i + 1 - r. \tag{8.25}$$

This defines an involution σ on the space $\mathbb{A}^{|\lambda-\mu|}$.

Using Lemma 6.9, there are various choices of coordinates on the fixed point scheme ${}^\imath\mathbb{A}^{|\lambda-\mu|}$ defined by restriction from $\mathbb{A}^{|\lambda-\mu|}$. For our next result, a good choice is to use coordinates

$$\{w_{i,r} : i \in {}^\imath\mathbb{I}, 1 \leqslant r \leqslant \mathfrak{v}_i\}. \tag{8.26}$$

The restriction of any other $w_{i,r}$ to ${}^\imath\mathbb{A}^{|\lambda-\mu|}$ can be expressed in terms of these coordinates, using (8.25). Note that for any $i \in \mathbb{I}_0$ such that $\mathbf{v}_i = 2\mathfrak{v}_i + 1$ is odd (or equivalently if $\theta_i = 1$, in the notation of §3.1) the restriction $w_{i,\mathfrak{v}_i+1} = 0$. Hence we have $\dim {}^\imath\mathbb{A}^{|\lambda-\mu|} = \sum_{i\in{}^\imath\mathbb{I}} \mathfrak{v}_i$. Consider now the following group:

$$ {}^\imath S_{\lambda-\mu} = \prod_{i\in\mathbb{I}_0} \big(S_{\mathfrak{v}_i} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\mathfrak{v}_i}\big) \times \prod_{i\in\mathbb{I}_1} S_{\mathfrak{v}_i}, \tag{8.27} $$

Then there is an action of ${}^\imath S_{\lambda-\mu}$ on the space ${}^\imath\mathbb{A}^{|\lambda-\mu|}$: each symmetric group factor $S_{\mathfrak{v}_i}$ permutes the corresponding coordinates $\{w_{i,r} : 1 \leqslant r \leqslant \mathfrak{v}_i\}$ in (8.26), while each $\mathbb{Z}/2\mathbb{Z}$ factor acts on its corresponding coordinate by $w_{i,r} \mapsto -w_{i,r}$.

Lemma 8.16. *The map $\mathbb{A}^{|\lambda-\mu|} \to \mathbb{A}^{(\lambda-\mu)}$ from (8.8) is σ-equivariant. The induced map ${}^\imath\mathbb{A}^{|\lambda-\mu|} \to {}^\imath\mathbb{A}^{(\lambda-\mu)}$ on fixed point loci identifies*

$$ {}^\imath\mathbb{A}^{(\lambda-\mu)} = {}^\imath\mathbb{A}^{|\lambda-\mu|}/{}^\imath S_{\lambda-\mu}. $$

The open subsets $\mathring{\mathbb{A}}^{|\lambda-\mu|}$ and $\mathring{\mathbb{A}}^{(\lambda-\mu)}$ are preserved by σ, and their fixed point loci ${}^\imath\mathring{\mathbb{A}}^{|\lambda-\mu|}$ and ${}^\imath\mathring{\mathbb{A}}^{(\lambda-\mu)}$ are non-empty if and only if the parity condition (3.5) holds.

Proof. The map $\mathbb{A}^{|\lambda-\mu|} \to \mathbb{A}^{(\lambda-\mu)}$ is determined by (8.8). This map is σ-equivariant, since

$$ \sigma\Big(\prod_r (z-w_{i,r})\Big) = \prod_r (z-\sigma(w_{i,r})) = \prod_r (z+w_{\tau i,\tau r}) = \prod_r (z+w_{\tau i,r}) $$

which agrees with the action $\sigma\big(\mathsf{a}_i(z)\big) = (-1)^{\mathbf{v}_i}\mathsf{a}_{\tau i}(-z)$ on $\mathbb{A}^{(\lambda-\mu)}$.

Now consider the restriction ${}^\imath\mathbb{A}^{|\lambda-\mu|} \to {}^\imath\mathbb{A}^{(\lambda-\mu)}$ of this map to fixed point loci. We use the coordinates (8.26) on ${}^\imath\mathbb{A}^{|\lambda-\mu|}$. If $i\in\mathbb{I}_1$, then we have $\mathsf{a}_i(z) = \prod_{r=1}^{\mathfrak{v}_i}(z-w_{i,r})$. The coefficients of this polynomial generate the invariants for the action of $S_{\mathfrak{v}_i}$ on $\mathbb{C}[w_{i,1},\ldots,w_{i,\mathfrak{v}_i}]$. Meanwhile, if $i\in\mathbb{I}_0$ then we have

$$ \mathsf{a}_i(z) = z^{\theta_i}\prod_{r=1}^{\mathfrak{v}_i}(z^2-w_{i,r}^2). $$

The coefficients of this polynomial generate the invariants for the action of $S_{\mathfrak{v}_i}\ltimes(\mathbb{Z}/2\mathbb{Z})^{\mathfrak{v}_i}$ on $\mathbb{C}[w_{i,1},\ldots,w_{i,\mathfrak{v}_i}]$. It follows that ${}^\imath\mathbb{A}^{(\lambda-\mu)} = {}^\imath\mathbb{A}^{|\lambda-\mu|}/{}^\imath S_{\lambda-\mu}$.

Finally, observe that if there exist $i,j\in\mathbb{I}_0$ which are connected and have $\theta_i\theta_j = 1$, then this means that $\mathsf{a}_i(z)$ and $\mathsf{a}_j(z)$ both have odd degree. Since $\mathsf{a}_i(z) = -\mathsf{a}_i(-z)$ and $\mathsf{a}_j(z) = -\mathsf{a}_j(-z)$, this means that these two polynomials must have a common zero $z=0$. Therefore ${}^\imath\mathring{\mathbb{A}}^{(\lambda-\mu)}$ is empty. If no such i,j exist, then there is no obstruction. The case of ${}^\imath\mathring{\mathbb{A}}^{|\lambda-\mu|}$ is similar. □

Corollary 8.17. *The map ${}^\imath\mathbb{A}^{|\lambda-\mu|} \to {}^\imath\mathbb{A}^{(\lambda-\mu)}$ is finite and faithfully flat.*

Proof. The group ${}^\imath S_{\lambda-\mu}$ is a product of Weyl groups of types B/C and A, acting on the product $\mathbb{A}^{|\lambda-\mu|}$ of affine spaces in the standard ways. Thus the quotient map ${}^\imath\mathbb{A}^{|\lambda-\mu|} \to {}^\imath\mathbb{A}^{|\lambda-\mu|}/{}^\imath S_{\lambda-\mu}$ has the desired properties, by Chevalley's Theorem on Weyl group invariants. The claim now follows from the previous lemma. □

Recall that we define the open subset $U_\mu^\lambda \subseteq \overline{\mathcal{W}}_\mu^\lambda$ as the preimage of $\mathring{\mathbb{A}}^{(\lambda-\mu)} \subseteq \mathbb{A}^{(\lambda-\mu)}$. It follows from the previous lemma that U_μ^λ is preserved by the involution σ. We will study its fixed point locus ${}^\imath U_\mu^\lambda$, via the simpler space X_μ^λ from Theorem 8.6.

Lemma 8.18. *There is a Poisson involution σ of X_μ^λ defined by*

$$\sigma(w_{i,r}) = -w_{\tau i,\tau r}, \qquad \sigma(y_{i,r}^{\pm}) = (-1)^{\mathbf{v}_i} y_{\tau i,\tau r}^{\mp} \tag{8.28}$$

such that the map $X_\mu^\lambda \to \mathbb{A}^{|\lambda-\mu|}$ is σ-equivariant.

Proof. Follows by Definition 8.5 and Theorem 8.6. □

Let ${}^\imath X_\mu^\lambda \subseteq X_\mu^\lambda$ denote the corresponding fixed point locus. Viewed as functions on ${}^\imath X_\mu^\lambda$ by restriction, we thus have

$$w_{i,r} = -w_{\tau i,\tau r}, \quad y_{i,r}^{+} = (-1)^{\mathbf{v}_i} y_{\tau i,\tau r}^{-}. \tag{8.29}$$

Note that as σ is a Poisson involution on X_μ^λ, its fixed point locus ${}^\imath X_\mu^\lambda$ inherits a Poisson structure by (doubled) Dirac reduction. The same is true of ${}^\imath U_\mu^\lambda$, and the map ${}^\imath X_\mu^\lambda \to {}^\imath U_\mu^\lambda$ is Poisson since Dirac reduction is functorial.

Recalling the notations of §3.1, observe that for any $i \in \mathbb{I}$, when restricted to ${}^\imath X_\mu^\lambda$ we have

$$\mathsf{a}_i(z) = \mathbf{W}_i(z). \tag{8.30}$$

Proposition 8.19. *Retain the Assumption (3.5). Then the space ${}^\imath X_\mu^\lambda$ has coordinates*

$$\{w_{i,r}, y_{i,r}^- \mid i \in \mathbb{I}, 1 \leqslant r \leqslant \mathbf{v}_i\},$$

and is defined scheme-theoretically in these coordinates by the relations (1)–(4):

(1) $w_{\tau i,\tau r} = -w_{i,r}$, *for $i \in \mathbb{I}$ and $1 \leqslant r \leqslant \mathbf{v}_i$;*

(2)

$$y_{i,r}^- y_{\tau i,\tau r}^- = (-1)^{\mathbf{v}_i+1} w_{i,r}^{\mathbf{w}_i} \prod_{j\leftrightarrow i,\ j\in\mathbb{I}} \mathbf{W}_j(w_{i,r}), \qquad \textit{for } i \in {}^\imath\mathbb{I},\ 1 \leqslant r \leqslant \mathfrak{v}_i; \tag{8.31}$$

(3) *or each $i \in \mathbb{I}_0$ with $\theta_i = 1$:*

$$(y_{i,\mathfrak{v}_i+1}^-)^2 = \begin{cases} (-1)^{\sum_{j\leftrightarrow i, j\in{}^\imath\mathbb{I}} \mathfrak{v}_j} \prod_{j\leftrightarrow i,\ j\in{}^\imath\mathbb{I}} \prod_{s=1}^{\mathfrak{v}_j} w_{j,s}^2, & \textit{if } \mathbf{w}_i = 0 \\ 0, & \textit{if } \mathbf{w}_i > 0; \end{cases} \tag{8.32}$$

(4) $w_{i,r} - w_{j,s} \neq 0$, *for all pairs $(i,r) \neq (j,s)$ with $i,j \in \mathbb{I}$ and $c_{ij} \neq 0$.*

In terms of these coordinates, the Poisson structure on ${}^\imath X_\mu^\lambda$ is determined by

$$\{w_{i,r}, w_{j,s}\} = 0, \qquad \{w_{i,r}, y_{j,s}^-\} = (\delta_{(i,r),(j,s)} - \delta_{(\tau i,\tau r),(j,s)}) y_{i,r}^-,$$

$$\{y_{i,r}^-, y_{j,s}^-\} = \frac{-\delta_{i\leftrightarrow j}}{w_{i,r} - w_{j,s}} y_{i,r}^- y_{j,s}^- + \delta_{(i,r),(\tau j,\tau s)} (-1)^{\mathbf{v}_i} \frac{\partial}{\partial x}\Big(x^{\mathbf{w}_i} \prod_{j\leftrightarrow i, j\in\mathbb{I}} \mathbf{W}_j(x)\Big)\Big|_{x=w_{i,r}}.$$

Proof. The same map from Lemma 8.18 defines an involution of the space $\mathbb{A}^{3|\lambda-\mu|}$ from Definition 8.5, and its fixed point locus ${}^\imath\mathbb{A}^{3|\lambda-\mu|}$ is defined simply by imposing (8.29). Put differently, the fixed point locus is an affine space with coordinates $w_{i,r}, y_{i,r}^-$ for $i \in \mathbb{I}$, $1 \leqslant r \leqslant \mathbf{v}_i$, with the relation (1). Now consider the open subset $\mathring{\mathbb{A}}^{3|\lambda-\mu|} \subseteq \mathbb{A}^{3|\lambda-\mu|}$ where $w_{i,r} - w_{j,s} \neq 0$ for all pairs $(i,r) \neq (j,s)$ with $c_{ij} \neq 0$. Then its fixed point variety ${}^\imath\mathring{\mathbb{A}}^{3|\lambda-\mu|} \subseteq {}^\imath\mathbb{A}^{3|\lambda-\mu|}$ is also open, defined by imposing condition (4). Now observe by Definition 8.5 that $X_\mu^\lambda \subseteq \mathring{\mathbb{A}}^{3|\lambda-\mu|}$ is a closed subscheme. It follows that ${}^\imath X_\mu^\lambda \subseteq {}^\imath\mathring{\mathbb{A}}^{3|\lambda-\mu|}$ is also closed and is defined by imposing the defining relations (8.10) of X_μ^λ. Keeping in mind the relations (8.29), this is equivalent to conditions (2) and (3) above, proving the first claim.

The Poisson structure is calculated using (doubled) Dirac reduction via Remark 6.7, cf. the proof of Lemma 7.10. For example, let us compute the bracket $\{y_{i,r}^-, y_{j,s}^-\}$ in $\mathbb{C}[{}^\imath X_\mu^\lambda]$. We lift $y_{i,r}^- \in \mathbb{C}[{}^\imath X_\mu^\lambda]$ to the σ–invariant element $\frac{1}{2}(y_{i,r}^- + (-1)^{\mathbf{v}_i} y_{\tau i,\tau r}^+) \in \mathbb{C}[X_\mu^\lambda]$, and compute the Poisson bracket of these lifts using Theorem 8.6:

$$\begin{aligned} &2\left\{\tfrac{1}{2}(y_{i,r}^- + (-1)^{\mathbf{v}_i} y_{\tau i,\tau r}^+), \tfrac{1}{2}(y_{j,s}^- + (-1)^{\mathbf{v}_j} y_{\tau j,\tau s})\right\} \\ &= \tfrac{1}{2}\frac{-\delta_{i\leftrightarrow j}}{w_{i,r}-w_{j,s}} y_{i,r}^- y_{j,s}^- + \tfrac{1}{2}\delta_{(i,r),(\tau j,\tau s)}(-1)^{\mathbf{v}_i}\frac{\partial}{\partial x}\Big(x^{\mathbf{w}_i}\prod_{k\leftrightarrow i,k\in\mathbb{I}}\mathbf{W}_k(x)\Big)\Big|_{x=w_{i,r}} \\ &\qquad - \tfrac{1}{2}\delta_{(\tau i,\tau r),(j,s)}(-1)^{\mathbf{v}_j}\frac{\partial}{\partial v}\Big(v^{\mathbf{w}_j}\prod_{\ell\leftrightarrow j,\ell\in\mathbb{I}}\mathbf{W}_\ell(v)\Big)\Big|_{v=w_{j,s}} \\ &\qquad + \tfrac{1}{2}(-1)^{\mathbf{v}_i+\mathbf{v}_j}\frac{\delta_{\tau i\leftrightarrow\tau j}}{w_{\tau i,\tau r}-w_{\tau j,\tau s}} y_{\tau i,\tau r}^+ y_{\tau j,\tau s}^+. \end{aligned}$$

Upon restricting to ${}^\imath X_\mu^\lambda$, the first and last terms above become equal because of (8.29). Let us show that the second and third terms also become equal upon restriction. If $(\tau i,\tau r) = (j,s)$ then we know that $w_{j,s} = -w_{i,r}$, $\mathbf{v}_j = \mathbf{v}_i$, and $\mathbf{w}_i = \mathbf{w}_j$. There is also a bijection between the sets $\{\ell \in \mathbb{I} : \ell \leftrightarrow j\}$ and $\{k \in \mathbb{I} : k \leftrightarrow i\}$ defined by $k = \tau\ell$, under which the corresponding polynomials satisfy $\mathbf{W}_\ell(v) = (-1)^{\mathbf{v}_k}\mathbf{W}_k(-v)$. With this in mind, and making the substitution $x = -v$ of dummy variables, we get:

$$\begin{aligned} &-(-1)^{\mathbf{v}_j}\frac{\partial}{\partial v}\Big(v^{\mathbf{w}_j}\prod_{\ell\leftrightarrow j,\ell\in\mathbb{I}}\mathbf{W}_\ell(v)\Big)\Big|_{v=w_{j,s}} \\ &\qquad = (-1)^{\mathbf{v}_i}\frac{\partial}{\partial x}\Big((-x)^{\mathbf{w}_i}\prod_{k\leftrightarrow i,k\in\mathbb{I}}(-1)^{\mathbf{v}_k}\mathbf{W}_k(x)\Big)\Big|_{x=w_{i,r}}. \end{aligned}$$

Within the derivative on the right side there is a sign of $(-1)^{\mathbf{w}_i+\sum_{k\leftrightarrow i}\mathbf{v}_k} = (-1)^{\mu_i} = 1$ since μ is even. This gives the claimed equality of the second and third terms. □

As in §3.1, for each $i \in \mathbb{I}_1$ we fix a choice of $\zeta_i \in \mathbb{N}$ with $1 \leqslant \zeta_i \leqslant \mathfrak{v}_i = \mathbf{v}_i$, and extend it to $i \in \mathbb{I}_1 \cup \mathbb{I}_{-1}$ by $\zeta_{\tau i} = \mathfrak{v}_i - \zeta_i$. Recall also our conventions from §8.4. The following result defines the classical iGKLO homomorphism.

Lemma 8.20. *Retain the Assumption (3.5) and fix an orientation of the Dynkin diagram $\mathbb{I}$ satisfying the conditions from §3.3. Then there is a Poisson homomorphism*

$$\mathbb{C}[{}^\imath X_\mu^\lambda] \longrightarrow \operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0} \tag{8.33}$$

defined by (1)–(3) below:

(1) $w_{i,r} \mapsto w_{i,r}$, *for $i \in \mathbb{I}$ and $1 \leqslant r \leqslant \mathbf{v}_i$;*

(2)

$$y_{i,r}^- \mapsto \begin{cases} -w_{i,r}^{\mathfrak{w}_i}\prod_{j\to i}\mathbf{W}_j(w_{i,r})\eth_{i,r}^{-1} & \text{if } 1 \leqslant r \leqslant \zeta_i, \\ -w_{i,r}^{\mathfrak{w}_i}\prod_{\tau j\leftarrow\tau i}\mathbf{W}_{\tau j}(w_{\tau i,r})\eth_{\tau i,r}, & \text{if } \zeta_i < r \leqslant \mathbf{v}_i, \end{cases}$$

for $i \in \mathbb{I}_1 \cup \mathbb{I}_{-1}$ and $1 \leqslant r \leqslant \mathbf{v}_i$;

(3)

$$y_{i,r}^{-} \mapsto \begin{cases} -w_{i,r}^{\mathfrak{w}_i} \prod_{j\to i} \mathbf{W}_j(w_{i,r}) \prod_{\substack{j\leftarrow i,\\ j\in\mathbb{I}_0}} \overline{W}_j^{-}(w_{i,r})\eth_{i,r}^{-1}, & \textit{if } 1\leqslant r\leqslant \mathfrak{v}_i,\\ -w_{i,r}^{\mathfrak{w}_i+\varsigma_i} \prod_{\substack{j\to i,\\ j\in\mathbb{I}_{\pm 1}}} \mathbf{W}_j(w_{i,\mathfrak{v}_i-r}) \prod_{\substack{j\leftarrow i,\\ j\in\mathbb{I}_0}} W_j(w_{i,\mathfrak{v}_i-r})\eth_{i,\mathfrak{v}_i-r}, & \textit{if } \mathbf{v}_i-\mathfrak{v}_i< r\leqslant \mathbf{v}_i,\\ \sqrt{(-1)^{\sum_{j\leftrightarrow i, j\in{}^\imath\mathbb{I}}\mathfrak{v}_i}}\prod_{j\leftrightarrow i} W_j(0), & \textit{if } r=\mathfrak{v}_i+1,\ \theta_i=1 \textit{ and } \mathbf{w}_i=0,\\ 0, & \textit{if } r=\mathfrak{v}_i+1,\ \theta_i=1 \textit{ and } \mathbf{w}_i>0, \end{cases}$$

for $i\in\mathbb{I}_0$ *and* $1\leqslant r\leqslant \mathbf{v}_i$.

Proof. After unwinding the definitions it is straightforward to see that this assignment satisfies the relations (1)–(4) from Proposition 8.19, and thus defines an algebra homomorphism $\mathbb{C}[{}^\imath X_\mu^\lambda]\to \operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0}$. One proves that it is a Poisson map by a direct case-by-case calculation, essentially an application of the following basic identity: denoting $P=(w_{i,r}-w_{j,s})^A(w_{i,r}+w_{j,s})^B\eth_{i,r}^{-1}$ and $Q=(w_{j,s}-w_{i,r})^C(w_{j,s}+w_{i,r})^D\eth_{j,s}^{-1}$ where $A,B,C,D\in\mathbb{N}$, one finds that

$$\{P,Q\}=\frac{-A-C}{w_{i,r}-w_{j,s}}PQ+\frac{B-D}{w_{i,r}+w_{j,s}}PQ.$$

We leave the details to the reader. ☐

Proposition 8.21. *The space* ${}^\imath X_\mu^\lambda$ *is non-empty if and only if the parity condition* (3.5) *holds. In this case* $\dim {}^\imath X_\mu^\lambda = 2\sum_{i\in{}^\imath\mathbb{I}}\mathfrak{v}_i$ *and the map* $\mathbb{C}[{}^\imath X_\mu^\lambda]\to\operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0}$ *from* (8.33) *defines a top-dimensional irreducible component of* ${}^\imath X_\mu^\lambda$*, i.e., the kernel of this map is its defining ideal.*

Proof. If the condition (3.5) does not hold, then ${}^\imath X_\mu^\lambda$ is empty by Lemma 8.16. On the other hand, if (3.5) holds, then the map $\mathbb{C}[{}^\imath X_\mu^\lambda]\to\operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0}$ from (8.33) shows that ${}^\imath X_\mu^\lambda$ is non-empty. Moreover, this ring homomorphism becomes surjective if we localize at all $w_{i,r}$ and $w_{i,r}\pm w_{j,s}$, which shows that $\dim {}^\imath X_\mu^\lambda\geqslant \dim\operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0}=2\sum_{i\in{}^\imath\mathbb{I}}\mathfrak{v}_i$.

Next, we prove the opposite inequality $\dim{}^\imath X_\mu^\lambda\leqslant 2\sum_{i\in{}^\imath\mathbb{I}}\mathfrak{v}_i$. Recall the coordinates on ${}^\imath X_\mu^\lambda$ from Proposition 8.19. Because of the relations among the $w_{i,r}$, we need only those elements from (8.26), of which there are $\sum_{i\in{}^\imath\mathbb{I}}\mathfrak{v}_i$. In the relation (8.31) observe that either $y_{i,r}^-\neq 0$, in which case we can uniquely solve for $y_{\tau i,\tau r}^-$, or else $y_{i,r}^-=0$, in which case there is no constraint on $y_{\tau i,\tau r}^-$. This dichotomy applies for each $y_{i,r}^-$ with $i\in{}^\imath\mathbb{I}$ and $1\leqslant r\leqslant\mathfrak{v}_i$, and in each case only one of $y_{i,r}^-$ or $y_{\tau i,\tau r}^-$ is needed as a coordinate. Moreover, the relation (8.32) shows that any remaining coordinates $y_{i,\mathfrak{v}_i+1}^-$ are finite over the $w_{i,r}$, and thus may ignored for computing dimension. This shows that ${}^\imath X_\mu^\lambda$ is covered by finitely many locally-closed subvarieties, each of which has dimension at most $2\sum_{i\in{}^\imath\mathbb{I}}\mathfrak{v}_i$. This gives the claimed upper bound on $\dim{}^\imath X_\mu^\lambda$.

Finally, since $\operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0}$ is a domain of the correct dimension, we conclude that it corresponds to an irreducible component of ${}^\imath X_\mu^\lambda$ of top dimension. ☐

Armed with this understanding of ${}^\imath X_\mu^\lambda$, we now return to studying the open subvariety ${}^\imath U_\mu^\lambda\subseteq{}^\imath\overline{\mathcal{W}}_\mu^\lambda$.

Lemma 8.22. *The map $X^\lambda_\mu \to U^\lambda_\mu$ is σ-equivariant. The corresponding map ${}^\imath X^\lambda_\mu \to {}^\imath U^\lambda_\mu$ on fixed point loci identifies*

$$ {}^\imath X^\lambda_\mu = {}^\imath U^\lambda_\mu \times_{{}^\imath \mathbb{A}^{(\lambda-\mu)}} {}^\imath \mathbb{A}^{|\lambda-\mu|} = {}^\imath U^\lambda_\mu \times_{{}^\imath \mathring{\mathbb{A}}^{(\lambda-\mu)}} {}^\imath \mathring{\mathbb{A}}^{|\lambda-\mu|}. $$

In particular, the map ${}^\imath X^\lambda_\mu \to {}^\imath U^\lambda_\mu$ is finite and faithfully flat.

Proof. Under the action of σ on X^λ_μ, we have

$$ \sigma\left(\sum_r \Big(\prod_{s\neq r} \frac{z-w_{i,s}}{w_{i,r}-w_{i,s}}\Big) y^\pm_{i,r}\right) = (-1)^{\mathrm{v}_i} \sum_r \Big(\prod_{s\neq r} \frac{-z-w_{\tau i,\tau s}}{w_{\tau i,\tau r}-w_{\tau i,\tau s}}\Big) y^\mp_{\tau i,\tau r}. $$

This matches with $\sigma(\mathsf{b}_i(z)) = (-1)^{\mathrm{v}_i}\mathsf{c}_{\tau i}(-z)$ and $\sigma(\mathsf{c}_i(z)) = (-1)^{\mathrm{v}_i}\mathsf{b}_{\tau i}(-z)$ from (8.21). Using Theorem 8.6(2) we conclude that $X^\lambda_\mu \to U^\lambda_\mu$ is σ-equivariant.

Recall from Theorem 8.6 that $X^\lambda_\mu = U^\lambda_\mu \times_{\mathbb{A}^{(\lambda-\mu)}} \mathbb{A}^{|\lambda-\mu|}$. The maps defining this fiber product are all σ-equivariant, so by Lemma 6.10 we conclude that the fixed point scheme ${}^\imath X^\lambda_\mu$ is the fiber product ${}^\imath U^\lambda_\mu \times_{{}^\imath\mathbb{A}^{(\lambda-\mu)}} {}^\imath\mathbb{A}^{|\lambda-\mu|}$ of corresponding fixed point loci. Since the map ${}^\imath\mathbb{A}^{|\lambda-\mu|} \to {}^\imath\mathbb{A}^{(\lambda-\mu)}$ is finite and faithfully-flat by Lemma 8.16, its base change ${}^\imath X^\lambda_\mu \to {}^\imath U^\lambda_\mu$ is also finite and faithfully flat. □

Proof of Theorem 8.11. For Part (1), first note that the locus U^λ_μ is preserved by σ because of Lemma 8.16. Since the map ${}^\imath X^\lambda_\mu \to {}^\imath U^\lambda_\mu$ is finite and faithfully flat by Lemma 8.22, the rest of Part (1) follows from Proposition 8.21.

For Part (2), it is a tedious but straightforward calculation to show that the filtered degrees of the generators $B_i^{(r)}, H_i^{(r)}$ are preserved under $\Phi^{\lambda,\boldsymbol{z}=0}_\mu$. The map $\Phi^{\lambda,\boldsymbol{z}=0}_\mu$ is therefore filtered by the same argument as in [BFN19, §B(vii)]. Next we look at the composed map

$$ {}^\imath X^\lambda_\mu \to {}^\imath U^\lambda_\mu \hookrightarrow {}^\imath\overline{\mathcal{W}}^\lambda_\mu \hookrightarrow {}^\imath\mathcal{W}_\mu, $$

which corresponds to a Poisson algebra homomorphism $\mathbb{C}[{}^\imath\mathcal{W}_\mu] \to \mathbb{C}[{}^\imath X^\lambda_\mu]$. Composed with Lemma 8.20, we obtain a Poisson map $\mathbb{C}[{}^\imath\mathcal{W}_\mu] \to \operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0}$. Its effect on the Poisson generators $b_i^{(r)}, h_i^{(s)} \in \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ is given by substituting Lemma 8.20 into the formulas from Remark 8.4.

We can obtain a second Poisson map $\mathbb{C}[{}^\imath\mathcal{W}_\mu] \to \operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0}$ from the iGKLO homomorphism: the map Φ^λ_μ is filtered by Part (1), and we know that $\operatorname{gr}^{F^\bullet_{\mu_1}} {}^\imath\mathcal{Y}_\mu \cong \mathbb{C}[{}^\imath\mathcal{W}_\mu]$ by Theorem 7.8. Inspecting leading terms in Theorem 3.6, we see that both homomorphisms agree on the Poisson generators $b_i^{(r)}, h_i^{(s)} \in \mathbb{C}[{}^\imath\mathcal{W}_\mu]$. Thus, the two Poisson maps are equal.

Finally, recall that the map from Proposition 8.21 defines a top-dimensional irreducible component of ${}^\imath X^\lambda_\mu$, and thus a top-dimensional irreducible component $C^\lambda_\mu \subseteq {}^\imath U^\lambda_\mu$ under the finite and faithfully flat map ${}^\imath X^\lambda_\mu \to {}^\imath U^\lambda_\mu$. The map $\mathbb{C}[{}^\imath\mathcal{W}_\mu] \to \operatorname{gr}\mathcal{A}_{\boldsymbol{z}=0}$ therefore defines its closure $\overline{C}^\lambda_\mu \subseteq {}^\imath\overline{\mathcal{W}}^\lambda_\mu$, completing the proof of Part (2). □

9. Affine Grassmannian islices of type AI and nilpotent Slodowy slices

In this section, we restrict ourselves to affine Grassmannian slices for G of PGL type, and the diagram automorphism $\tau = \mathrm{id}$, corresponding to the Satake diagram of type AI. We identify the affine Grassmannian islices with specific nilpotent Slodowy slices of type BCD.

9.1. **Nilpotent orbits and transversal slices ABC$^+$.** Let $\mathfrak{a}$ be a reductive Lie algebra over $\mathbb{C}$, and let $\mathcal{N}_\mathfrak{a}$ denote its nilpotent cone. Then $\mathcal{N}_\mathfrak{a} \subset \mathfrak{a}$ is an irreducible Poisson subvariety. It decomposes as a union of finitely many nilpotent orbits under the action of the adjoint type group corresponding to $\mathfrak{a}$, which are also precisely the symplectic leaves of $\mathcal{N}_\mathfrak{a}$. We recall aspects of this theory in classical types, following [CM93].

For $\mathfrak{sl}_N$, nilpotent orbits are parametrized by the set $\mathrm{Par}(N)$ of partitions of N: each $\pi \in \mathrm{Par}(N)$ corresponds to the orbit $\mathbb{O}_\pi \subset \mathcal{N}_{\mathfrak{sl}_N}$ through any nilpotent element $e_\pi \in \mathfrak{sl}_N$ with Jordan type π. Write $\overline{\mathbb{O}}_\pi$ for the closure. Then there are decompositions

$$\mathcal{N}_{\mathfrak{sl}_N} = \bigsqcup_{\pi\in\mathrm{Par}(N)} \mathbb{O}_\pi, \qquad \overline{\mathbb{O}}_\pi = \bigsqcup_{\substack{\pi'\in\mathrm{Par}(N),\\ \pi\trianglerighteq\pi'}} \mathbb{O}_{\pi'}. \tag{9.1}$$

Here $\pi \trianglerighteq \pi'$ denotes the dominance order on partitions.

To treat the classical types uniformly, we shall denote $\epsilon = +$ for orthogonal types and $\epsilon = -$ for symplectic. Let J_ϵ be an invertible $N\times N$ matrix which is symmetric if $\epsilon = +$ and skew-symmetric if $\epsilon = -$, and define an involution σ_ϵ of $\mathfrak{sl}_N$ by

$$\sigma_\epsilon : X \mapsto -J_\epsilon^{-1} X^T J_\epsilon. \tag{9.2}$$

Denoting by $\mathfrak{sl}_N^\epsilon = (\mathfrak{sl}_N)^{\sigma_\epsilon}$ the corresponding fixed point subalgebra, we thus have

$$\mathfrak{sl}_N^\epsilon = \begin{cases} \mathfrak{so}_N, & \text{if } \epsilon = +,\\ \mathfrak{sp}_N, & \text{if } \epsilon = -. \end{cases} \tag{9.3}$$

The involution σ_ϵ preserves each nilpotent orbit $\mathbb{O}_\pi$ of $\mathfrak{sl}_N$. We will consider the corresponding fixed point loci:

$$\mathbb{O}_\pi^\epsilon := (\mathbb{O}_\pi)^{\sigma_\epsilon}, \qquad \overline{\mathbb{O}}_\pi^\epsilon := (\overline{\mathbb{O}}_\pi)^{\sigma_\epsilon}. \tag{9.4}$$

Recall that a partition is *orthogonal* (resp. *symplectic*) if all its even (resp. odd) parts occur with even multiplicity. Denote by $\mathrm{Par}_\epsilon(N)$ the set of orthogonal (resp. symplectic) partitions of N, for $\epsilon = +$ (resp. $-$). The following summarizes results of Gerstenhaber and of Hesselink, see [CM93, Theorems 5.1.6, 6.2.5 & 6.3.3]:

Proposition 9.1. *The fixed point locus $\mathbb{O}_\pi^\epsilon$ is non-empty if and only if $\pi \in \mathrm{Par}_\epsilon(N)$, and in this case, $\mathbb{O}_\pi^\epsilon$ is an O_N-orbit for $\epsilon = +$ and a Sp_N-orbit for $\epsilon = -$. There are decompositions*

$$(\mathcal{N}_{\mathfrak{sl}_N})^{\sigma_\epsilon} = \mathcal{N}_{\mathfrak{sl}_N^\epsilon} = \bigsqcup_{\pi\in\mathrm{Par}_\epsilon(N)} \mathbb{O}_\pi^\epsilon, \qquad \overline{\mathbb{O}}_\pi^\epsilon = \bigsqcup_{\substack{\pi_1\in\mathrm{Par}_\epsilon(N),\\ \pi_1\trianglelefteq\pi}} \mathbb{O}_{\pi_1}^\epsilon.$$

Moreover, for any $\pi \in \mathrm{Par}(N)$, the variety $\overline{\mathbb{O}}_\pi^\epsilon$ is the closure of the orbit $\mathbb{O}_{\pi'}^\epsilon$, where $\pi' \in \mathrm{Par}_\epsilon(N)$ is the unique maximal element satisfying $\pi' \trianglelefteq \pi$. In particular, if $\pi \in \mathrm{Par}_\epsilon(N)$ then $\pi' = \pi$.

Remark 9.2. For $\pi \in \mathrm{Par}_+(N)$, the O_N-orbit $\mathbb{O}_\pi^+$ is connected except for π *very even*, in which case it has two components (each of which is an SO_N-orbit). See [CM93, Theorem 5.1.4] for details.

Fix a partition $\pi \in \mathrm{Par}_\epsilon(N)$ where $\epsilon \in \{\varnothing, +, -\}$, and choose an element $e \in \mathbb{O}_\pi^\epsilon$; it is understood that the superscript $\varnothing$ denotes type A and can be dropped. By the Jacobson-Morozov Theorem there exists a corresponding $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{sl}_N^\epsilon$, which is unique up to conjugation by a result of Kostant [CM93, §3]. We consider the corresponding Slodowy slice in $\mathfrak{sl}_N^\epsilon$, defined by:

$$\mathcal{S}_\pi^\epsilon = e + \mathrm{Ker}(\mathrm{ad}\, f). \tag{9.5}$$

Then $\mathcal{S}_\pi^\epsilon$ is a transversal slice to the nilpotent orbit $\mathbb{O}_\pi^\epsilon$ at $e \in \mathfrak{sl}_N^\epsilon$, and more generally it intersects all nilpotents in $\mathfrak{sl}_N^\epsilon$ transversely [GG02, §2.2]. It also follows that for any inclusion $\mathbb{O}_{\pi_2}^\epsilon \subseteq \overline{\mathbb{O}}_{\pi_1}^\epsilon$, there are non-empty intersections

$$\mathbb{O}_{\pi_1}^\epsilon \cap \mathcal{S}_{\pi_2}^\epsilon \neq \varnothing, \qquad \overline{\mathbb{O}}_{\pi_1}^\epsilon \cap \mathcal{S}_{\pi_2}^\epsilon \neq \varnothing. \tag{9.6}$$

All of these spaces inherit Poisson structures from $\mathfrak{sl}_N^\epsilon$ by [GG02, §3]. Note that $\mathcal{S}_\pi^\epsilon$ is uniquely determined from $\mathbb{O}_\pi^\epsilon$, up to conjugation.

9.2. **Results of Lusztig and Mirković-Vybornov.** Throughout this section we will consider affine Grassmannian slices $\overline{\mathcal{W}}_\mu^\lambda$ where G is of PGL type, and their relation to nilpotent orbits in $\mathfrak{sl}_N$.

First consider $G = \mathrm{PGL}_N$. Lusztig [Lus81, §2] constructed an isomorphism

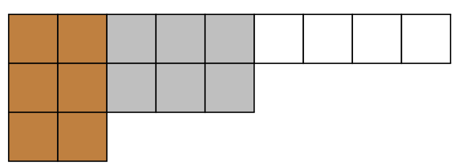
$$\mathcal{N}_{\mathfrak{sl}_N} \cong \overline{\mathcal{W}}_0^{N\varpi_1^\vee}, \tag{9.7}$$

given explicitly by $X \in \mathcal{N}_{\mathfrak{sl}_N} \mapsto I + z^{-1}X \in G_1[\![z^{-1}]\!] = \mathcal{W}_0$. Moreover, for each partition $\pi = (\lambda_1, \ldots, \lambda_N) \in \mathrm{Par}(N)$ this map restricts to isomorphisms of strata $\mathbb{O}_\pi \cong \mathcal{W}_0^\lambda$ and their closures $\overline{\mathbb{O}}_\pi \cong \overline{\mathcal{W}}_0^\lambda$, where π and λ are related by $\lambda = \lambda_1\varepsilon_1^\vee + \ldots + \lambda_N\varepsilon_N^\vee$.

More generally, let $G = \mathrm{PGL}_n$ and let $\lambda \geqslant \mu$ be dominant coweights. Label the nodes of the Dynkin diagram by $\mathbb{I} = \{1, \ldots, n-1\}$ in the standard way. Recall our notation $\mathbf{w}_i = \langle \lambda, \alpha_i \rangle$ for $i \in \mathbb{I}$, and define an integer $N := \sum_{i=1}^{n-1} i\mathbf{w}_i$. Define a partition $\pi_1 \in \mathrm{Par}(N)$ in exponential notation followed by a transpose t by letting

$$\pi_1 = \left(1^{\mathbf{w}_1} 2^{\mathbf{w}_2} \cdots (n-1)^{\mathbf{w}_{n-1}}\right)^t. \tag{9.8}$$

Note that its length $\ell(\pi_1) \leqslant n-1$. For example, for $n = 4$, $\mathbf{w}_1 = 4, \mathbf{w}_2 = 3$, $\mathbf{w}_3 = 2$, the Young diagram of π_1 can be depicted as follows:



(9.9)

A second partition $\pi_2 = (p_n \geqslant p_{n-1} \geqslant \cdots \geqslant p_1) \in \mathrm{Par}(N)$ may be defined uniquely by requiring

$$\langle \mu, \alpha_{n-i} \rangle = p_{i+1} - p_i, \qquad \text{for } 1 \leqslant i \leqslant n-1. \tag{9.10}$$

The dominance condition $\lambda \geqslant \mu$ translates into the partition dominance condition $\pi_1 \trianglerighteq \pi_2$. Conversely, given a pair of partitions $\pi_1, \pi_2 \in \mathrm{Par}(N)$ such that $\ell(\pi_1) \leqslant n-1$, $\ell(\pi_2) \leqslant n$ and $\pi_1 \trianglerighteq \pi_2$, we can reconstruct a pair of dominant coweights λ, μ for $G = \mathrm{PGL}_n$ such that $\lambda \geqslant \mu$.

Proposition 9.3. (Mirković-Vybornov [MV22, Proposition 4.3.4]) *With notation as above, there is an isomorphism*

$$\mathcal{N}_{\mathfrak{sl}_N} \cap \mathcal{S}_{\pi_2} \cong \overline{\mathcal{W}}_\mu^{N\varpi_1^\vee}.$$

It restricts to isomorphisms of strata and their closures:

$$\mathbb{O}_{\pi_1} \cap \mathcal{S}_{\pi_2} \cong \mathcal{W}_\mu^\lambda, \qquad \overline{\mathbb{O}}_{\pi_1} \cap \mathcal{S}_{\pi_2} \cong \overline{\mathcal{W}}_\mu^\lambda.$$

Strictly speaking, we will use an alternate formulation of this isomorphism by Cautis-Kamnitzer [CK18, §3.3]. Note that the above isomorphisms are Poisson by [WWY20, Theorem A].

Remark 9.4. In fact, the above works use transverse slices which are generally different from Slodowy's $\mathcal{S}_\pi$. Thankfully this is not important: all reasonable slices ("MV slices") are Poisson isomorphic by [WWY20, Theorem 5.5], and these isomorphisms are easily seen to restrict to isomorphisms of intersections with nilpotent orbits and their closures. Similarly the precise choice of $\mathfrak{sl}_2$-triple $\{e, h, f\} \in \mathfrak{sl}_N^\epsilon$ used to define $\mathcal{S}_{\pi_2}^\epsilon$ is irrelevant, up to an isomorphism induced by conjugation.

9.3. i**Slices and O_N-orbit closures.** Let $G = \mathrm{PGL}_N$, and on the loop group $G(\!(z^{-1})\!)$ we have $\sigma(g(z)) = g(-z)^T$ as in Example 7.3. Transporting σ via the isomorphism $\mathcal{N}_{\mathfrak{sl}_N} \cong \overline{\mathcal{W}}_0^{N\varpi_1^\vee}$ from (9.7), we obtain the involution $\sigma : X \mapsto -X^T$ on $\mathcal{N}_{\mathfrak{sl}_N}$. Its fixed point locus is $\mathcal{N}_{\mathfrak{so}_N}$, as follows from [CM93, Proposition 1.1.3]. By restriction, we thus obtain an isomorphism of fixed point loci:

$$\mathcal{N}_{\mathfrak{so}_N} \cong {}^\imath\overline{\mathcal{W}}_0^{N\varpi_1^\vee}.$$

Recall that the isomorphism $\mathcal{N}_{\mathfrak{sl}_N} \cong \overline{\mathcal{W}}_0^{N\varpi_1^\vee}$ restricts to isomorphisms of strata and their closures. Since our involutions preserve strata, we obtain isomorphisms of fixed point loci:

$$\mathbb{O}_\pi^\epsilon \cong {}^\imath\mathcal{W}_0^\lambda, \qquad \overline{\mathbb{O}}_\pi^\epsilon \cong {}^\imath\overline{\mathcal{W}}_0^\lambda. \tag{9.11}$$

By Proposition 9.1 the locus $\mathbb{O}_\pi^\epsilon$ is non-empty if and only if the partition π is orthogonal. Recall also that $\overline{\mathbb{O}}_\pi^\epsilon$ is the closure of an orbit $\mathbb{O}_{\pi'}^\epsilon$ (with $\pi = \pi'$ if π is orthogonal, but $\pi \neq \pi'$ otherwise), and thus ${}^\imath\overline{\mathcal{W}}_0^\lambda$ is the closure of a corresponding stratum ${}^\imath\mathcal{W}_0^{\lambda'}$. We also deduce that the variety ${}^\imath\overline{\mathcal{W}}_0^\lambda$ is reducible for some λ (see Remark 9.2), or even non-normal since this is true of classical nilpotent orbits by [KP82, Theorem 1].

9.4. i**Slices and nilpotent Slodowy slices.** Next, we move on to the setting of Proposition 9.3 with $G = \mathrm{PGL}_n$. First, we focus on the special case where $\lambda = N\varpi_1^\vee$. In this case, from Lemma 9.3 we have

$$\mathcal{N}_{\mathfrak{sl}_N} \cap \mathcal{S}_{\pi_2} \cong \overline{\mathcal{W}}_\mu^{N\varpi_1^\vee}. \tag{9.12}$$

To consider the fixed point locus ${}^\imath\overline{\mathcal{W}}_\mu^{N\varpi_1^\vee}$, recall from Lemma 7.5 that we need to assume that μ is even (and spherical, but this is automatic since $\tau = \mathrm{id}$). Inspecting Equation (9.10), we observe that μ being even corresponds to the partition π_2 having all parts of the same parity. Following [Top23, §4.1], we let $\epsilon = +$ if all parts of π_2 are odd, and $\epsilon = -$ if all parts of π_2 are even. Observe that π_2 is orthogonal when $\epsilon = +$, and symplectic when $\epsilon = -$, just as in §9.1.

Define the following subsets of $\mathrm{Par}_\epsilon(N)$:

$$\mathrm{Par}_\epsilon(N)^\diamond := \{\pi \in \mathrm{Par}_\epsilon(N) \mid \text{all parts are odd (resp. even) for } \epsilon = + \text{ (resp. } \epsilon = -)\}, \tag{9.13}$$

$$\mathrm{Par}_\epsilon(N)_{\leqslant n} := \{\pi \in \mathrm{Par}_\epsilon(N) \mid \ell(\pi) \leqslant n\}, \tag{9.14}$$

$$\mathrm{Par}_\epsilon(N)_{\leqslant n}^\diamond := \mathrm{Par}_\epsilon(N)^\diamond \cap \mathrm{Par}_\epsilon(N)_{\leqslant n}. \tag{9.15}$$

Given $\pi_2 \in \mathrm{Par}_\epsilon(N)$, we choose J_ϵ in (9.2) for the involution σ_ϵ of $\mathfrak{sl}_N$ exactly as in [Top23, §4.2] (the assumption that $\pi_2 \in \mathrm{Par}_\epsilon(N)^\diamond$ can be relaxed for now). This allows Topley [Top23, §4.2] to choose $e \in \mathfrak{sl}_N^\epsilon$ of Jordon form π_2 and construct an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ which is fixed pointwise by σ_ϵ, i.e., lies inside $\mathfrak{sl}_N^\epsilon$. We shall use this $\mathfrak{sl}_2$-triple throughout the rest of this section. Then we see that the involution σ_ϵ on $\mathfrak{sl}_N$ in (9.2) restricts to an involution on the Slodowy slice $\mathcal{S}_{\pi_2}$, and the corresponding fixed point locus is the Slodowy slice $\mathcal{S}^\epsilon_{\pi_2}$ in the classical Lie algebra $\mathfrak{sl}_N^\epsilon$:

$$(\mathcal{S}_{\pi_2})^{\sigma_\epsilon} = \mathcal{S}^\epsilon_{\pi_2}. \tag{9.16}$$

While we did not find this formulation in literature, an isomorphism (instead of equality) of this form seems implicit in [Top23, (4.24)] by passing through the Dirac reduction.

Proposition 9.5. *Let μ be the even dominant coweight for $G = \mathrm{PGL}_n$ such that $N\varpi_1^\vee \geqslant \mu$, with corresponding partition $\pi_2 \in \mathrm{Par}_\epsilon(N)^\diamond_{\leqslant n}$ via (9.10). Then the isomorphism $\overline{\mathcal{W}}^{N\varpi_1^\vee}_\mu \cong \mathcal{N}_{\mathfrak{sl}_N} \cap \mathcal{S}_{\pi_2}$ in (9.12) is σ_ϵ-equivariant, giving rise to a Poisson isomorphism of fixed-point loci:*

$$^\imath\overline{\mathcal{W}}^{N\varpi_1^\vee}_\mu \cong \mathcal{N}_{\mathfrak{sl}_N^\epsilon} \cap \mathcal{S}^\epsilon_{\pi_2}. \tag{9.17}$$

Proof. In brief, by [WWY20, Theorem 4.3(d)] the isomorphism (9.12) can be understood as the classical limit of the Brundan-Kleshchev isomorphism [BK06, Theorem 10.1] between a finite W-algebra and a quotient of a shifted Yangian. Topley has proven [Top23, Proposition 4.7] that the classical Brundan-Kleshchev isomorphism is σ_ϵ-equivariant. Putting these two pieces together proves our claim.

More precisely, following the conventions of [WWY20, Theorem 4.3] the Brundan-Kleshchev isomorphism is an isomorphism of filtered algebras:

$$Y^{N\varpi_1^\vee}_\mu \xrightarrow{\sim} W(\mathfrak{gl}_N, \pi_2). \tag{9.18}$$

The left side is a truncated shifted Yangian for $\mathfrak{sl}_n$, while the right side is a finite W-algebra for $\mathfrak{gl}_N$. These algebras have non-trivial centers, and we should pass to central quotients as in [WWY20, Theorem 4.9]. (The precise choice of central quotient is not important, since all choices yield the same associated graded algebras.) Taking a central quotient on the left side, we obtain an algebra $Y^{N\varpi_1^\vee}_\mu(\mathbf{R})$ whose associated graded algebra is the coordinate ring $\mathbb{C}[\overline{\mathcal{W}}^{N\varpi_1^\vee}_\mu]$ by [WWY20, Theorem 2.5]. On the right side, we obtain a central quotient $W(\mathfrak{gl}_N, \pi_2)_\mathbf{R}$ defined as in [WWY20, §3.3.2].

Now consider the classical finite W-algebra $S(\mathfrak{gl}_N, \pi_2) = \mathrm{gr}\, W(\mathfrak{gl}_N, \pi_2)$. By [GG02, Theorem 4.1], there is an isomorphism $S(\mathfrak{gl}_N, \pi_2) \cong \mathbb{C}[\mathcal{S}'_{\pi_2}]$, where $\mathcal{S}'_{\pi_2}$ denotes the Slodowy slice defined in $\mathfrak{gl}_N$. Consider also the quotient algebra $S(\mathfrak{gl}_N, \pi_2)_0 = S(\mathfrak{gl}_N, \pi_2)/J$, where J is the ideal generated by the coefficients of the characteristic polynomial. Then we have $S(\mathfrak{gl}_N, \pi_2)_0 = \mathrm{gr}\, W(\mathfrak{gl}_N, \pi_2)_\mathbf{R}$, as follows from [WWY20, (4.5)] combined with [BK08, Lemma 3.7]. There is an isomorphism

$$S(\mathfrak{gl}_N, \pi_2)_0 \cong \mathbb{C}[\mathcal{N}_{\mathfrak{sl}_N} \cap \mathcal{S}_{\pi_2}]$$

as in [WWY20, Remark 3.18]. This is related to the well-known fact that if $K \subset \mathbb{C}[\mathfrak{gl}_N]$ denotes the ideal generated by the coefficients of the characteristic polynomial, then $\mathbb{C}[\mathcal{N}_{\mathfrak{sl}_N}] = \mathbb{C}[\mathfrak{gl}_N]/K$.

Altogether, the isomorphism (9.12) is induced by a composition of algebra isomorphisms:

$$\mathbb{C}[\overline{\mathcal{W}}_{\mu}^{N\varpi_1^\vee}] \xrightarrow{\sim} S(\mathfrak{gl}_N, \pi_2)_0 \xrightarrow{\sim} \mathbb{C}[\mathcal{N}_{\mathfrak{sl}_N} \cap \mathcal{S}_{\pi_2}]. \tag{9.19}$$

Finally, (9.2) naturally induces an involution on $S(\mathfrak{gl}_N, \pi_2)$, see [Top23, §2.3]. By [Top23, Proposition 4.7], the classical limit of (9.18) is σ_ϵ-equivariant. Note that Topley works with classical shifted Yangians for $\mathfrak{gl}_n$, whereas we work with $\mathfrak{sl}_n$. These algebras are easily compared, as in (the classical limit) of [WWY20, §4.2]. Tracing through conventions, one can see that [Top23, (3.58)] matches with Lemma 7.5(4). As a result, we see that the first map in (9.19) is σ_ϵ-equivariant. By construction, the second isomorphism in (9.19) is also σ_ϵ-equivariant. This completes the proof. □

We now return to the case of general dominant coweights $\lambda \geqslant \mu$ for PGL_n. Using equations (9.8) and (9.10) we extract corresponding partitions $\pi_1 \in \mathrm{Par}(N)_{\leqslant n-1}$ and $\pi_2 \in \mathrm{Par}(N)_{\leqslant n}$ such that $\pi_1 \trianglerighteq \pi_2$. Recall that this correspondence is reversible. Since we wish to consider fixed points we must assume that μ is even, which corresponds to $\pi_2 \in \mathrm{Par}_\epsilon(N)^\diamond_{\leqslant n}$. Note however that there is no need to impose any additional condition on λ or π_1.

The following is the main result of this section.

Theorem 9.6. *Let $\tau = \mathrm{id}$. Let $\lambda \geqslant \mu$ be dominant coweights for PGL_n with μ even, corresponding to partitions $\pi_1 \in \mathrm{Par}(N)_{\leqslant n-1}$ and $\pi_2 \in \mathrm{Par}_\epsilon(N)^\diamond_{\leqslant n}$ with $\pi_1 \trianglerighteq \pi_2$. Then the isomorphism ${}^\imath\overline{\mathcal{W}}_{\mu}^{N\varpi_1^\vee} \cong \mathcal{N}^\epsilon_{\mathfrak{sl}_N} \cap \mathcal{S}^\epsilon_{\pi_2}$ from (9.17) restricts to Poisson isomorphisms*

$${}^\imath\mathcal{W}_\mu^\lambda \cong \mathbb{O}^\epsilon_{\pi_1} \cap \mathcal{S}^\epsilon_{\pi_2}, \qquad {}^\imath\overline{\mathcal{W}}_\mu^\lambda \cong \overline{\mathbb{O}}^\epsilon_{\pi_1} \cap \mathcal{S}^\epsilon_{\pi_2}.$$

Moreover,

1. *${}^\imath\mathcal{W}_\mu^\lambda$ is non-empty if and only if $\pi_1 \in \mathrm{Par}_\epsilon(N)$.*
2. *The variety ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ is the closure of its stratum ${}^\imath\mathcal{W}_\mu^{\lambda'} \cong \mathbb{O}^\epsilon_{\pi_1'} \cap \mathcal{S}^\epsilon_{\pi_2}$, where λ' is the unique coweight corresponding to the maximal element $\pi_1' \in \mathrm{Par}_\epsilon(N)$ satisfying $\pi_1 \trianglerighteq \pi_1'$.*

Proof. By Proposition 9.5, the isomorphism $\overline{\mathcal{W}}_{\mu}^{N\varpi_1^\vee} \cong \mathcal{N}_{\mathfrak{sl}_N} \cap \mathbb{O}_{\pi_2}$ is σ_ϵ-equivariant. We also know from Proposition 9.3 that this isomorphism restricts to isomorphisms of strata and their closures. Since these strata and closures are preserved by σ_ϵ, by restriction we obtain isomorphisms of their respective fixed point loci. This proves the first claim, and the remaining claims then follow from Proposition 9.1. □

Example 9.7. We return once again to $G = \mathrm{PGL}_2$ and $\tau = \mathrm{id}$, as in Examples 8.2 and 8.9. Let $\lambda \geqslant \mu$ be dominant coweights for G, with μ even (or equivalently, with λ even). We extract partitions $\pi_1 = (N) = (\mathbf{w})$ and $\pi_2 = (\mathbf{w} - \mathbf{v}, \mathbf{v})$. We have $\epsilon = +$ for $\mathbf{v}$ odd, and $\epsilon = -$ for $\mathbf{v}$ even, and an isomorphism ${}^\imath\overline{\mathcal{W}}_\mu^\lambda \cong \mathcal{N}_{\mathfrak{sl}^\epsilon_N} \cap \mathcal{S}^\epsilon_{\pi_2}$, where the right-hand side is identified with the space of matrices from Example 8.9.

10. iCoulomb branches and affine Grassmannian islices

In this section, starting with quiver gauge theory data under some mild τ-symmetry/parity conditions suggested in earlier sections, we produce a new symplectic representation with new gauge group and flavor symmetry group. We speculate that the resulting iCoulomb branch (typically not of cotangent type) provides a normalization of the affine Grassmannian islices constructed earlier.

10.1. **Quiver datum.** Let $Q = (\mathbb{I}, \Omega)$ be an ADE quiver, and an arrow h in Ω sends a vertex h' in $\mathbb{I}$ to another vertex h'' in $\mathbb{I}$, that is, we denote $h' \xrightarrow{h} h''$. We have

▶ the double quiver $\overline{Q} = (\mathbb{I}, \Omega \cup \overline{\Omega})$, with a reversed arrow $\overline{h}$ for each $h \in \Omega$;

▶ the framed quiver $Q^f = (\mathbb{I} \cup \mathbb{I}', \Omega \cup \Omega_{\mathbb{I}})$, where $\mathbb{I}' = \{i' | i \in \mathbb{I}\}$ is an isomorphic copy of $\mathbb{I}$ and $\Omega_{\mathbb{I}}$ consists of arrows from i' to i, one for each $i \in \mathbb{I}$;

▶ the framed double quiver $\overline{Q}^f = (\mathbb{I} \cup \mathbb{I}', \Omega \cup \overline{\Omega} \cup \Omega_{\mathbb{I}} \cup \overline{\Omega}_{\mathbb{I}})$.

Let $(V_i; W_i)_{i \in \mathbb{I}}$ be a representation of the framed double quiver $\overline{Q}^f$, with dimension vectors $\mathbf{v} = (\mathbf{v}_i)_{i \in \mathbb{I}}$ and $\mathbf{w} = (\mathbf{w}_i)_{i \in \mathbb{I}}$; it is understood that W_i is attached to the vertex i', for $i \in \mathbb{I}$. We have a symplectic vector space:

$$E_{V,W} = \bigoplus_{h \in \Omega \cup \overline{\Omega}} \mathrm{Hom}(V_{h'}, V_{h''}) \bigoplus \bigoplus_{i \in \mathbb{I}} \big(\mathrm{Hom}(W_i, V_i) \oplus \mathrm{Hom}(V_i, W_i)\big), \tag{10.1}$$

which can be identified with the cotangent space $T^*(\oplus_{h \in \Omega} \mathrm{Hom}(V_{h'}, V_{h''}) \bigoplus \oplus_{i \in \mathbb{I}} \mathrm{Hom}(W_i, V_i))$. Denote the gauge group by $GL_{\mathbf{v}} = GL(V) = \prod_{i \in \mathbb{I}} GL(\mathbf{v}_i)$ and the flavor symmetry group by $GL_{\mathbf{w}} = GL(W) = \prod_{i \in \mathbb{I}} GL(\mathbf{w}_i)$.

Associated to such data, one attaches a Nakajima quiver variety $\mathcal{M}_H(\mathbf{v}, \mathbf{w})$ as well a Coulomb branch $\mathcal{M}_C(\mathbf{v}, \mathbf{w})$ (of cotangent type) corresponding to type A quiver gauge theories [BFN19].

10.2. **iCoulomb branches.** Following earlier sections, we let τ be a bijection on $\mathbb{I}$ such that $\tau^2 = \mathrm{id}$. This induces a bijection $\ddot{\tau}$ on $\Omega \cup \overline{\Omega}$ such that, for $h \in \Omega \cup \overline{\Omega}$,

(1) $(\ddot{\tau} h)' = \tau(h'')$ and $(\ddot{\tau} h)'' = \tau(h')$;
(2) $\ddot{\tau} h = \overline{h}$ whenever both ends of h are fixed by τ, i.e., $\tau(h') = h', \tau(h'') = h''$;
(3) $\ddot{\tau} h = h$ whenever $\tau\big(h'\big) = h''$.

Note that $\ddot{\tau}$ is always a non-identity involution, and this makes the double quiver equipped with $\ddot{\tau}$, denoted by $(\overline{Q}, \tau, \ddot{\tau})$ or $(\overline{Q}, \ddot{\tau})$, a symmetric quiver à la [DW02]. Examples of symmetric quivers with two neighboring vertices fixed by τ, see Case (2) above (e.g., $\tau = \mathrm{id}$), were not explicitly considered *loc. cit.*, as they do not occur if one considers ADE quivers Q (instead of $\overline{Q}$); but they occur frequently and form an important family in our setting (see Example 10.5) as we consider symmetric quivers on the double quivers $\overline{Q}$.

Applying the above construction to the double quiver $\overline{Q}^f$, we obtain a symmetric quiver $(\overline{Q}^f, \ddot{\tau})$.

Recall $\mathbb{I}_0 = \{i \in \mathbb{I} \mid \tau i = i\}$, and the partition $\mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_0 \cup \mathbb{I}_{-1}$ so that $\mathbb{I}_1$ (and resp. $\mathbb{I}_{-1}$) a set of representatives of τ-orbits in $\mathbb{I}$ of length 2. We shall choose $\mathbb{I}_1$ such that the underlying Dynkin subdiagram on $\mathbb{I}_1$ is connected. Recall ${}^{\imath}\mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_0$. Introduce the following distinguished subsets of the arrow set Ω:

$$\begin{aligned} \Omega_0 &= \{h \in \Omega \mid h', h'' \in \mathbb{I}_0\} \\ \Omega_{1\text{-}0} &= \{h \in \Omega \mid h' \in \mathbb{I}_1, h'' \in \mathbb{I}_0\}, \\ \Omega_{\mathsf{qs}} &= \{h \in \Omega \mid \tau\big(h'\big) = h''\}. \end{aligned} \tag{10.2}$$

Denote

$$\mathbb{I}_1^{\mathtt{qs}} = \{i \in \mathbb{I}_1 \mid i, \tau i \text{ are connected by an edge } h \in \Omega\}. \tag{10.3}$$

There is a bijection $\mathbb{I}_1^{\mathtt{qs}} \to \Omega_{\mathtt{qs}}$, sending i to the edge h connecting i and τi.

Throughout this section, we make the following τ-symmetry and "not-2-odd" parity assumption:

$$\begin{cases} \dim V_i = \dim V_{\tau i}, \quad \dim W_i = \dim W_{\tau i}, & \text{for } i \in \mathbb{I}_1 \cup \mathbb{I}_{-1}; \\ \dim V_i \text{ or } \dim W_i \text{ must be even}, & \text{for } i \in \mathbb{I}_0; \\ \dim V_{h'} \text{ or } \dim V_{h''} \text{ must be even}, & \text{for } h \in \Omega_0. \end{cases} \tag{10.4}$$

The third assumption here is a rephrasing of the assumption (3.5), while the second one comes from Remark 3.1.

Denote by $\overline{Q}_1^f$ the full subquiver of $\overline{Q}^f$ with vertex set $\mathbb{I}_1 \cup \mathbb{I}_1'$, where $\mathbb{I}_1'$ is the subset of $\mathbb{I}'$ which is in natural bijection with $\mathbb{I}_1$. Let

$$(V^{\mathtt{A}}, W^{\mathtt{A}}) := (V_i; W_i)_{i \in \mathbb{I}_1} \tag{10.5}$$

be the representation of $\overline{Q}_1^f$, and $E_{V^{\mathtt{A}}, W^{\mathtt{A}}}$ be the corresponding symplectic vector space; cf. (10.1). Note that the gauge group for the subquiver $\overline{Q}_1^f$ is type A.

We fix a bipartite partition of $\mathbb{I}_0$:

$$\mathbb{I}_0 = \mathbb{I}_0^{\oplus} \cup \mathbb{I}_0^{\ominus} \tag{10.6}$$

such that each arrow in Q connects a vertex in $\mathbb{I}_0^{\oplus}$ and a vertex in $\mathbb{I}_0^{\ominus}$. (There are exactly two bipartite partitions for $\mathbb{I}$ of ADE type and rank $\geqslant 2$.) We shall denote

$$\epsilon_i = \begin{cases} +, & \text{if } i \in \mathbb{I}_0^{\oplus} \\ -, & \text{if } i \in \mathbb{I}_0^{\ominus}. \end{cases} \tag{10.7}$$

For $U = V$ or $U = W$, U_i^+ denotes an orthogonal vector space (i.e., equipped with a non-degenerate symmetric bilinear form) of the same dimension as U_i, and U_i^- denotes a symplectic vector space of dimension $2\lfloor \frac{1}{2} \dim U_i \rfloor$. For $i \in \mathbb{I}_0$, we define

$$V_i^\imath := V_i^{\epsilon_i} = \begin{cases} V_i^+, & \text{for } i \in \mathbb{I}_0^{\oplus} \\ V_i^-, & \text{for } i \in \mathbb{I}_0^{\ominus}, \end{cases} \qquad W_i^\imath := W_i^{\overline{\epsilon}_i} = \begin{cases} W_i^-, & \text{for } i \in \mathbb{I}_0^{\oplus} \\ W_i^+, & \text{for } i \in \mathbb{I}_0^{\ominus}. \end{cases} \tag{10.8}$$

We also define $G^{\epsilon_i}(V_i^\imath)$ to be the ϵ_i-isometry group of $V^\imath$, which means special orthogonal for $\epsilon_i = +$ and symplectic for $\epsilon_i = -$. Similarly, we define $G^{\overline{\epsilon}_i}(W_i^\imath)$ to be the $\overline{\epsilon}_i$-isometry group of $W^\imath$.

Define the new gauge group and flavor symmetry group

$$\begin{aligned} G^\imath(V^\imath) &:= \prod_{i \in \mathbb{I}_1} GL(V_i) \times \prod_{i \in \mathbb{I}_0} G^{\epsilon_i}(V_i^\imath), \\ G^\imath(W^\imath) &:= \prod_{i \in \mathbb{I}_1} GL(W_i) \times \prod_{i \in \mathbb{I}_0} G^{\overline{\epsilon}_i}(W_i^\imath). \end{aligned} \tag{10.9}$$

Given a quiver representation (V, W) of $\overline{Q}^f$ subject to the τ-symmetry and parity condition (10.4), with the above preparation we can now construct a new vector space

$$\begin{aligned} E^\imath_{V,W} = E_{V^{\mathtt{A}},W^{\mathtt{A}}} \bigoplus \big(\oplus_{h\in\Omega_{1\text{-}0}} \mathrm{Hom}(V_{h'}, V^\imath_{h''}) \oplus \oplus_{h\in\overline{\Omega}_{1\text{-}0}} \mathrm{Hom}(V^\imath_{h'}, V_{h''})\big) \\ \bigoplus \big(\oplus_{h\in\Omega_0} \mathrm{Hom}(V^\imath_{h'}, V^\imath_{h''}) \oplus \oplus_{i\in\mathbb{I}_0} \mathrm{Hom}(W^\imath_i, V^\imath_i)\big) \\ \bigoplus_{i\in\mathbb{I}_1^{\mathsf{qs}}} (\wedge^2 V_i \oplus \wedge^2 V_i^*). \end{aligned} \tag{10.10}$$

Remark 10.1. If $\mathbb{I}_1^{\mathsf{qs}} \neq \varnothing$, then $\mathbb{I}_1^{\mathsf{qs}}$ is a singleton and this only occurs in the Satake diagram $(\mathbb{I}, \tau)$ of type AIII_{2r}; see Table 1. This case corresponds to the last rank one framed iquiver in Table 4 with $\wedge^2 V_i \oplus \wedge^2 V_i^*$ attached. We learned how to include the last summand in (10.10) when $\mathbb{I}_1^{\mathsf{qs}} \neq \varnothing$ from [SSX25].

Given (natural) representations M_a of groups G_a, for $a = 1, 2$, $G_1 \times G_2$ acts on $\mathrm{Hom}(M_1, M_2)$ naturally by $\big((g_1, g_2).f\big)(m_1) = g_2.f(g_1^{-1} m_1)$, for $g_a \in G_a$ and $m_1 \in M_1$. All the smaller components of the first three big summands of $E^\imath_{V,W}$ in (10.10) are of the form $\mathrm{Hom}(M_1, M_2)$ and so the (suitable components of) groups $G^\imath(V^\imath), G^\imath(W^\imath)$ act on them naturally. Only $GL(V_i)$ for $i \in \mathbb{I}_1^{\mathsf{qs}}$ acts nontrivially on the ith component $\wedge^2 V_i \oplus \wedge^2 V_i^*$ of the fourth summand in (10.10). In this way we have defined the action of the groups $G^\imath(V^\imath)$ and $G^\imath(W^\imath)$ on $E^\imath_{V,W}$.

Lemma 10.2. *The vector space $E^\imath_{V,W}$ in* (10.10) *is naturally symplectic and endowed with actions of $G^\imath(V^\imath)$ and $G^\imath(W^\imath)$.*

Proof. Clearly the first and fourth summands $E_{V^{\mathtt{A}},W^{\mathtt{A}}}$ (see (10.5)) and $\wedge^2 V_i \oplus \wedge^2 V_i^* \cong T^*(\wedge^2 V_i)$ of the vector space $E^\imath_{V,W}$ are symplectic. The second summand $\big(\oplus_{h\in\Omega_{1\text{-}0}} \mathrm{Hom}(V_{h'}, V^\imath_{h''}) \oplus \oplus_{h\in\overline{\Omega}_{1\text{-}0}} \mathrm{Hom}(V^\imath_{h'}, V_{h''})\big)$ is isomorphic to $T^*\big(\oplus_{h\in\Omega_{1\text{-}0}} \mathrm{Hom}(V_{h'}, V^\imath_{h''})\big)$, and hence symplectic too.

By the $\imath$fication rule, exactly one of the two vector spaces $W^\imath_i$ and $V^\imath_i$ in (10.8), for $i \in \mathbb{I}_0$, is symplectic while the other one is orthogonal. Hence $\mathrm{Hom}(W^\imath_i, V^\imath_i)$ is naturally symplectic. By the bipartite partition requirement (10.6), exactly one of the two vector spaces $V^\imath_{h'}$ and $V^\imath_{h''}$, for $h \in \Omega_0$, is symplectic while the other one is orthogonal. Hence, $\mathrm{Hom}(V^\imath_{h'}, V^\imath_{h''})$ is symplectic as well. Therefore, the third summand $\big(\oplus_{h\in\Omega_0} \mathrm{Hom}(V^\imath_{h'}, V^\imath_{h''}) \oplus \oplus_{i\in\mathbb{I}_0} \mathrm{Hom}(W^\imath_i, V^\imath_i)\big)$ of $E^\imath_{V,W}$ is naturally symplectic. This proves that $E^\imath_{V,W}$ is symplectic. □

By Lemma 10.2, we can define a Coulomb branch $\mathcal{M}^\imath_C(V^\imath, W^\imath)$. Note that this Coulomb branch is not of cotangent type in most cases, and thus for its definition one should follow [BDF+22, Tel22, BF24]. Here we gloss over verifying necessary homotopy vanishing conditions in these constructions, and we wonder if the τ-symmetry and not-2-odd condition (10.4) helps. To distinguish from $\mathcal{M}_C(V, W)$ arising from type A quiver guage theories, we shall refer to $\mathcal{M}^\imath_C(V^\imath, W^\imath)$ as *iCoulomb branches*.

Remark 10.3. We reinterpret (3.2)–(3.4) as defining new dimension vectors $(\mathfrak{v}_i)_{i\in{}^\imath\mathbb{I}}$ and $(\mathfrak{w}_i)_{i\in{}^\imath\mathbb{I}}$ from $\mathbf{v} = (\mathbf{v}_i)_{i\in\mathbb{I}}$ and $\mathbf{w} = (\mathbf{w}_i)_{i\in\mathbb{I}}$, respectively. Then the ranks of the component groups of the gauge group $G^\imath(V^\imath)$ and the flavor symmetry group $G^\imath(W^\imath)$ (see (10.8)–(10.9)) are exactly given by these dimension vectors $(\mathfrak{v}_i)_{i\in{}^\imath\mathbb{I}}$ and $(\mathfrak{w}_i)_{i\in{}^\imath\mathbb{I}}$, respectively. See Remark 10.11 for more numerology!

Remark 10.4. The discussions and the formulation of $E^{\imath}_{V,W}$ in this section are valid for general quivers Q without loops under some mild assumptions; for example, they are valid for affine quivers (except the cyclic quiver $A^{(1)}_{2r}$ due to (10.6)). We have restricted ourselves to ADE type in order to match with earlier sections and to be more specific with the subsets defined in (10.2).

Example 10.5. The construction of $E^{\imath}_{V,W}$ is much simplified in two cases.

(1) (Split type, i.e., $\tau = \mathrm{id}$). In this case, we have $\mathbb{I}_0 = \mathbb{I}$, $\Omega_0 = \Omega$ and thus

$$E^{\imath}_{V,W} = \bigoplus_{h\in\Omega} \mathrm{Hom}(V^{\imath}_{h'}, V^{\imath}_{h''}) \bigoplus \bigoplus_{i\in\mathbb{I}} \mathrm{Hom}(W^{\imath}_i, V^{\imath}_i).$$

In this case, the gauge group is purely ortho-symplectic.

(2) In another extreme case with $\mathbb{I}_0 = \varnothing$, we have

$$E^{\imath}_{V,W} = E_{V^{\mathtt{A}},W^{\mathtt{A}}} \bigoplus \bigoplus_{i\in\mathbb{I}_1^{\mathsf{qs}}} (\wedge^2 V_i \oplus \wedge^2 V_i^*).$$

In and only in this case, the gauge group is purely type A and $E^{\imath}_{V,W}$ is of cotangent type; the new case with $\mathbb{I}_1^{\mathsf{qs}} \neq \varnothing$ is studied in [SSX25].

10.3. **$\imath$fication on (framed) Satake double quivers.** We can visualize the construction of $E^{\imath}_{V,W}$ in (10.10) via a new iquiver ${}^{\imath}\overline{Q} = ({}^{\imath}\mathbb{I}, {}^{\imath}\Omega)$ and the corresponding framed iquiver ${}^{\imath}\overline{Q}^f = ({}^{\imath}\mathbb{I} \cup {}^{\imath}\mathbb{I}', {}^{\imath}\Omega \cup {}^{\imath}\Omega_{{}^{\imath}\mathbb{I}})$, by applying $\imath$fication rules to $\overline{Q}$ and $\overline{Q}^f$, respectively. We first focus on $\imath$fications of quivers.

Recall the genuinely quasi-split ADE Satake diagrams $(\mathbb{I}, \tau)$ from Table 1; the split Satake diagrams (with $\tau = \mathrm{id}$) formally look the same as ADE Dynkin diagrams. By Satake double quivers, we simply mean double quivers enhanced with a diagram involution τ. Rank one framed Satake double quivers, which are building blocks of general framed Satake double quivers, are listed in Table 2. In the first diagram where no τ is visible, it is understood that $\tau = \mathrm{id}$. We denote nodes in the quiver Q by $\bigcirc$ and nodes on the framing by $\square$.

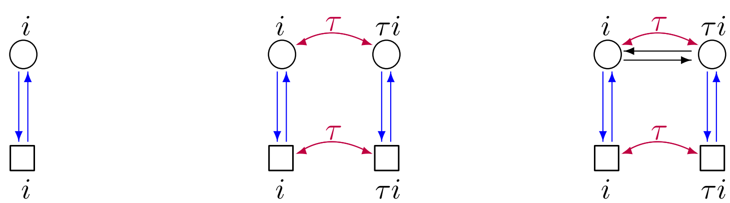


Table 2. Rank one framed Satake double quivers

Rank two Satake double quivers are listed in Table 3. To build rank two *framed* Satake double quivers, we simply need to copy and paste suitable frames from Table 2 to Table 3.

In Tables 4–5, we present the iquivers obtained by $\imath$fication of Satake double quivers from Tables 2–3, respectively. Here and below we set

$$\epsilon \in \{\pm\}, \quad \text{and} \quad \bar{\epsilon} := -\epsilon,$$

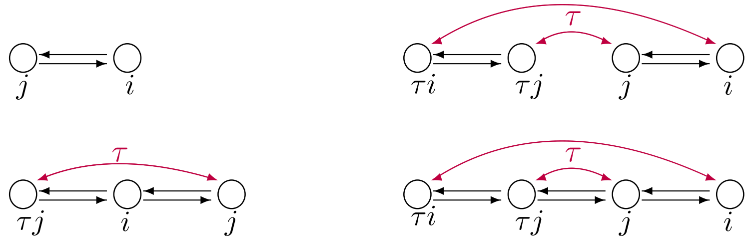


TABLE 3. Rank two Satake double quivers

and we add signs to (framed or not) nodes, e.g., ⊕, ⊖, ⊞, or ⊟, in iquivers obtained from modifying a τ-fixed node in Satake double quivers. Note that the signs are always opposite in neighboring connected nodes.

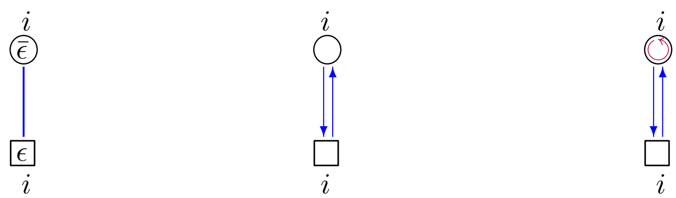


TABLE 4. Rank one framed iquivers

TABLE 5. Rank two iquivers

Following the $\imath$fication rules on rank one and rank two Satake double quivers, we can assemble them together naturally to obtain modifications of arbitrary Satake double quivers of split and quasi-split ADE type.

Now we illustrate diagrammatically how we modify the representations (V, W) of framed double quivers to become $(V^\imath, W^\imath)$ of the corresponding frame iquivers; this provides the input for the symplectic vector space $E^\imath_{V,W}$ in (10.10). It suffices to do so on rank one framed Satake double quivers. By attaching vector spaces V_i, W_i to the nodes in Table 2 and $V_i^\imath, W_i^\imath$ defined in (10.8) to the nodes in Table 4, we obtain Table 6 and Table 7 below. The ↺ inside ◯ in Table 7 is used to indicate the summand $\wedge^2 V_i \oplus \wedge^2 V_i^*$ in $E^\imath_{V,W}$ in (10.10) which corresponds to the case when $\mathbb{I}_1^{\mathrm{qs}} = \{i\}$ in Remark 10.1. Applying the $\imath$fication rule to Table 6 yields Table 7. Note that no arrow is indicated in the first diagram in Table 7 thanks to a natural identification $\mathrm{Hom}(V_i^\imath, W_i^\imath) \cong \mathrm{Hom}(W_i^\imath, V_i^\imath)$.

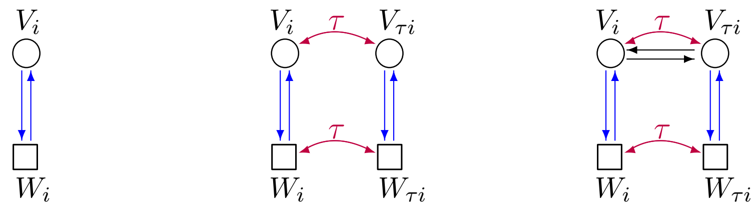


TABLE 6. Representations of rank one framed Satake double quivers

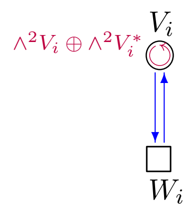


TABLE 7. Representations of rank one framed iquivers

10.4. **Discussions and examples.**

Example 10.6. In Table 8, we specify the dimension vectors $\mathbf{v}$ on ◯ and $\mathbf{w}$ on □ on framed double quiver of type A_{N-1}. It is well known that the corresponding Coulomb branch is identified with the nilpotent cone $\mathcal{N}_{\mathfrak{sl}_N}$ for $\mathfrak{sl}_N$.

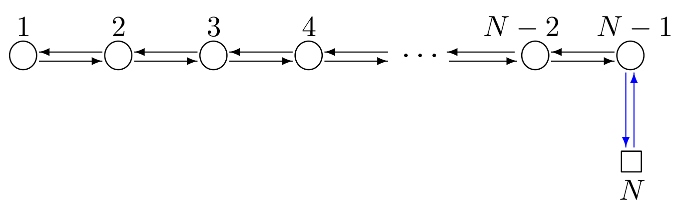


TABLE 8. Nilpotent cone for $\mathfrak{sl}_N$

Example 10.7. Fix $N = 2n$ in Table 8, and specify the bipartite partition of $\mathbb{I}$ with the leftmost node to be in $\mathbb{I}^\ominus$. Note that the leftmost node ◯ got removed under the $\imath$fication (see (10.8)–(10.9)) since the corresponding $V_i^\imath = 0$. Then applying the $\imath$fication rules, we obtain the (orth-symplectic) iquiver in Table 9, where the new orthogonal/symplectic (or $\oplus/\ominus$, $\boxplus/\boxminus$ diagramatically) vector spaces $V_i^\imath$ and $W_i^\imath$ and their dimensions are specified.

The corresponding iCoulomb branch $\mathcal{M}_C^\imath(V^\imath, W^\imath)$ is indeed the nilpotent cone $\mathcal{N}_{\mathfrak{so}_{2n}}$ for $\mathfrak{so}_{2n}$; cf. [FHN25].

Example 10.8. Fix $N = 2n + 1$ in Table 8, and specify the bipartite partition of $\mathbb{I}$ with the leftmost node to be in $\mathbb{I}^\ominus$. The leftmost node ◯ is again removed under $\imath$fication (see (10.8)–(10.9)) since the corresponding $V_i^\imath = 0$. Then applying the $\imath$fication rules, we obtain the (ortho-symplectic) iquiver in Table 10, where the new orthogonal/symplectic vector spaces $V_i^\imath$ and $W_i^\imath$ and their dimensions are specified.

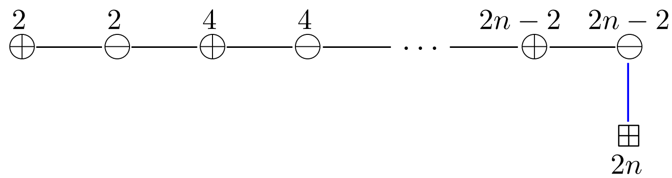


TABLE 9. Nilpotent cone for $\mathfrak{so}_{2n}$

The corresponding iCoulomb branch $\mathcal{M}_C^\imath(V^\imath, W^\imath)$ is indeed the nilpotent cone $\mathcal{N}_{\mathfrak{so}_{2n+1}}$; cf. [FHN25].

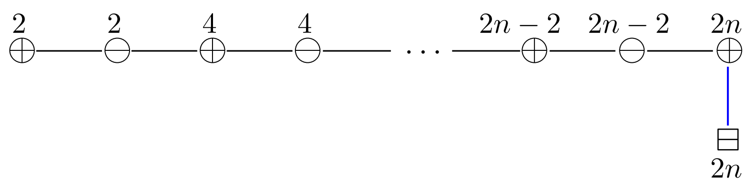


TABLE 10. Nilpotent cone for $\mathfrak{so}_{2n+1}$

We continue to assume that V, W satisfy the τ-symmetry/parity conditions (10.4).

By [BFN19, Theorem 3.10], generalized affine Grassmannian slices $\overline{\mathcal{W}}_\mu^\lambda$ are realized as Coulomb branches $\mathcal{M}_C(V, W)$ for specified dimension vectors $\mathbf{v}, \mathbf{w}$ or equivalently for specific groups $GL(V), GL(W)$. Such group datum give rise to new groups $G^\imath(V^\imath)$ and $G^\imath(W^\imath)$ as in (10.9) and then the corresponding iCoulomb branch $\mathcal{M}_C^\imath(V^\imath, W^\imath)$.

Conjecture 10.9. (1) *The iCoulomb branch $\mathcal{M}_C^\imath(V^\imath, W^\imath)$ is a normalization of a top-dimensional component of the affine Grassmannian islice ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$.*
(2) *Truncated shifted twisted Yangians are (subalgebras of) quantized iCoulomb branches.*
(3) *Truncated shifted affine iquantum groups are (subalgebras of) K-theoretic iCoulomb branch algebras.*

Remark 10.10. For type AI, in light of Theorem 9.6, Conjecture 10.9(1) can be rephrased as a connection between iCoulomb branches and a family of nilpotent Slodowy slices of type BCD.

Remark 10.11. By Theorem 8.11(2) and Remark 10.3, the dimension of ${}^\imath\overline{\mathcal{W}}_\mu^\lambda$ is equal to twice the rank of the group $G^\imath(V^\imath)$, which is the known dimension of the iCoulomb branch $\mathcal{M}_C^\imath(V^\imath, W^\imath)$. In addition, the deformation space ${}^\imath\mathbb{A}^{|\lambda|}$ in (8.20) has dimension equal to $\sum_{i \in {}^\imath\mathbb{I}} \mathfrak{w}_i$, the rank of the group $G^\imath(W^\imath)$ by Remark 10.3 again. This is consistent with Conjecture 10.9.

Conjecture 10.9 as formulated is a bit ambiguous and perhaps too general to take it literally; there is a sign ambiguity arising from the choice of bipartite partition (10.6). There is also a subtle issue of choosing $G^+(V_i^\imath)$ or $G^+(W_i^\imath)$ to be orthogonal or special orthogonal. Nevertheless, we hope that it points toward a meaningful connection between seemingly unrelated subjects which can be made rigorous.

A natural sufficient condition for the third assumption of (10.4) is that $\mathbf{v} = (\mathbf{v}_i)_{i\in\mathbb{I}}$ is parity-alternating. Under such a parity-alternating condition, we can fix the ambiguity of (10.6) by letting $\mathbb{I}^{\oplus}$ (resp. $\mathbb{I}^{\ominus}$) correspond to even (resp. odd) $\mathbf{v}_i$. In this case, the type B group does not appear as a component group of $G^\imath(V^\imath)$.

Let us specialize to the type A_{N-1} quiver with $\mathbb{I} = \{1, \dots, N-1\}$. We denote $\lambda = \sum_{a=1}^N \lambda_a \epsilon_a^\vee$, $\mu = \sum_{a=1}^N \mu_a \epsilon_a^\vee$, and write $\lambda - \mu = \sum_{i\in\mathbb{I}} \mathbf{v}_i \alpha_i^\vee = \sum_{a=l}^u c_a \epsilon_a^\vee$, with $c_a \in \mathbf{Z}$ and $c_l, c_u \neq 0$. Then one checks that

$$\text{(a) } \mathbf{v} \text{ is parity-alternating} \Leftrightarrow \text{ (b) } \begin{cases} c_a \, (l < a < u) \text{ are odd, and} \\ \#\{l \leqslant a \leqslant u \mid c_a \text{ is odd}\} \text{ is even.} \end{cases}$$

Actually the set of $\mathbf{v}$ satisfying (a) is in bijection with the set of $(c_a \mid 1 \leqslant a \leqslant N)$ satisfying (b) together with $\sum_a c_a = 0$.

For example, let $\lambda = (N)$. In this case, if $\mu \vdash N$ has all parts odd, then $\mathbf{v}$ is parity-alternating.

Take another example with $\mu = (1^N)$. In this case, the partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash N$ satisfies that λ_i are even for all $i \geqslant 2$ if and only if $\mathbf{v}$ is parity-alternating. In Examples 10.6, 10.7 and 10.8, we have $\lambda = (N)$ and $\mu = (1^N)$.

We end with raising a question of geometric symmetric pairs.

Problem 10.12. *Is $\mathcal{M}_C(V^\imath, W^\imath)$ a normalization of a Dirac reduction of $\mathcal{M}_C(V, W)$?*

We may refer to $\big(\mathcal{M}_C(V, W),\ \mathcal{M}_C^\imath(V^\imath, W^\imath)\big)$ as a geometric symmetric pair if the answer to the above problem is affirmative. It might be of interest to study geometric symmetric pairs under symplectic duality. If Conjecture 10.9 holds for the islices corresponding to the nilpotent Slodowy slices of type BCD covered by Theorem 9.6, then we obtain a large class of geometric symmetric pairs. Already in this case, understanding the Poisson involution σ within the framework of iCoulomb branches could be a good starting point.

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