OPTIMAL GRADIENT ESTIMATES FOR CONDUCTIVITY PROBLEMS WITH IMPERFECT LOW-CONDUCTIVITY INTERFACES

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Abstract. This paper studies field concentration between two nearly touching conductors separated by imperfect low-conductivity interfaces, modeled by Robin boundary conditions. It is known that for any sufficiently small interfacial bonding parameter $\gamma > 0$, the gradient remains uniformly bounded with respect to the separation distance ε. In contrast, for the perfect bonding case ($\gamma = 0$, corresponding to the perfect conductivity problem), the gradient may blow up as $\varepsilon \to 0$ at a rate depending on the dimension. In this work, we establish optimal pointwise gradient estimates that explicitly depend on both γ and ε in the regime where these parameters are small. These estimates provide a unified framework that encompasses both the previously known bounded case ($\gamma > 0$) and the singular blow-up scenario ($\gamma = 0$), thus furnishing a complete and continuous characterization of the gradient behavior throughout the transition in γ. The key technical achievement is the derivation of new regularity results for elliptic equations as $\gamma \to 0$, along with a case dichotomy based on the relative sizes of γ and a distance function $\delta(x')$. Our results hold for strictly relatively convex conductors in all dimensions $n \geq 2$.

1 Introduction and main results

1.1 Introduction

When two inclusions in a composite material approach each other while exhibiting material properties in high contrast to those of the surrounding matrix, the narrow inter-inclusion region may develop significant electric field or mechanical stress concentrations, as documented in the seminal works of Budiansky and Carrier [11], Keller [27, 28], Markenscoff [41], and Babuška et.al. [5]. Such extreme field concentrations can precipitate material failure and substantially alter the effective properties of the composite medium [19]. The analysis of this singular behavior is not only important in composite material science but also represents a fundamental challenge in the theory of partial differential equations, owing to the mathematical difficulties introduced by vanishing inter-inclusion distances; see Li and Vogelius [36] and Li and Nirenberg [35]. Over the past three decades, substantial advances have been made in the quantitative analysis of field concentration for perfectly bonded inclusions, yielding optimal estimates for local fields in narrow regions and rigorous asymptotic characterizations of field enhancement.

An alternative model featuring non-ideal bonding conditions is of considerable practical significance, as it accounts for interfacial contact resistance resulting from surface roughness

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and possible debonding, while also providing approximations for membrane structures in biological systems. Motivated by influential physics studies of Torquato and Rintoul [42] and recent pioneering mathematical work of Fukushima, Ji, Kang, and Li [20], this paper presents an investigation of such imperfect interface conditions. The profound influence of interfacial effects on effective material properties is well-established in various systems; see Lipton and Vernescu [40], Hashin [22], and Benveniste [8]. In particular, [40, 42] established rigorous bounds for the effective conductivity of random composites containing equisized spherical inclusions with imperfect interfaces through sophisticated constructions of trial fields and applications of minimum energy principles. Nevertheless, these studies did not provide precise estimates for local field behavior or characterize field concentration in the nearly-touching regime, as explicitly noted in [42]: "the complexity of the microstructure prohibits one from obtaining the local fields exactly." The present work addresses this fundamental problem by establishing optimal estimates for field concentration within narrow regions between inclusions in the regime of small bonding parameter ($\gamma \ll 1$), thereby bridging a crucial gap in the existing literature.

We now describe the mathematical framework. Since our primary interest lies in the narrow region between two nearly touching inclusions, we formulate the problem with exactly two inclusions. Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary, containing two strictly relatively convex inclusions D_1 and D_2. The boundaries ∂D_1 and ∂D_2 are of class $C^{2,\alpha}$, and their principal curvatures are bounded below by a positive constant κ_0. Denote the matrix domain exterior to the inclusions by $\tilde{\Omega} = \Omega \setminus \overline{D_1 \cup D_2}$. The distance between the inclusions

$$\varepsilon := \operatorname{dist}(D_1, D_2) \ll 1$$

is small, while their distance to the outer boundary $\partial\Omega$ remains of order one. If ε is bounded away from zero, estimates for the gradient and derivatives follow from standard elliptic theory.

When the bonding between the inclusions and the matrix is perfect, with conductivities k inside $D_1 \cup D_2$ and 1 in the matrix $\tilde{\Omega}$, the electric potential u satisfies the classical transmission problem:

$$\begin{cases} \operatorname{div}\left(a_k(x)\nabla u\right) = 0 & \text{in } \Omega, \\ \qquad\qquad u = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the piecewise constant coefficient $a_k(x) = k\chi_{D_1\cup D_2} + \chi_{\tilde{\Omega}}$ represents the conductivity distribution. A fundamental feature of perfect bonding is the continuity of both the potential and the flux across the interfaces:

$$u|_+ = u|_- \quad \text{and} \quad \left.\frac{\partial u}{\partial \nu}\right|_+ = k \left.\frac{\partial u}{\partial \nu}\right|_- \quad \text{on } \partial D_1 \cup \partial D_2. \tag{1.2}$$

Here and throughout the paper, the subscripts $\pm$ denote limits taken from the outside and inside of the inclusion, respectively, and ν is the unit inward normal vector to D_j on ∂D_j, $j = 1, 2$.

When the contrast k is bounded away from 0 and ∞, it was proved that the gradient of the solution remains bounded uniformly in ε, see, for instance, Bonnetier and Vogelius [10], Li and Vogelius [36] and Li and Nirenberg [35]. Furthermore, higher-order derivative estimates for solutions to (1.1) with piecewise coefficients and closely spaced inclusions have been derived by Dong and Zhang in [13], Dong and Li [12], Ji and Kang [23], Dong and Yang [14], Kim [29], and Dong and Xu [17].

However, if k degenerates to either ∞ or 0, the gradient ∇u may blow up as $\varepsilon \to 0$. Specifically, in the case of perfect conductors ($k = \infty$), problem (1.1) reduces to

$$\begin{cases} \Delta u = 0 & \text{in } \widetilde{\Omega}, \\ u = K_j & \text{on } \partial D_j, j = 1, 2, \\ u = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the constants K_j are uniquely determined by the flux conditions $\int_{\partial D_j} \partial_\nu u \, dS = 0$, $j = 1, 2$. The optimal blow-up rates of ∇u for this limiting problem are now well-established:

— $\varepsilon^{-1/2}$ in two dimensions, by Ammari, Kang, and collaborators [1–3] and Yun [44];
— $|\varepsilon \log \varepsilon|^{-1}$ in three dimensions, by Bao, Li, and Yin [6, 7] and Lim and Yun [38];
— ε^{-1} in dimensions four and higher, by Bao, Li, and Yin [6].

Subsequent works by Kang, Lim, and Yun [25, 26] and Li et al. [31–33] have further refined these results by establishing precise asymptotics that capture the singular behavior of ∇u near the origin.

In the insulated case ($k = 0$), problem (1.1) becomes

$$\begin{cases} \Delta u = 0 & \text{in } \widetilde{\Omega}, \\ \partial_\nu u = 0 & \text{on } \partial D_j, j = 1, 2, \\ u = \varphi(x) & \text{on } \partial\Omega. \end{cases}$$

While the optimal blow-up rate for ∇u in two dimensions was established about two decades ago through a dual argument [1, 3], the higher-dimensional case has been resolved only recently in Dong, Li, and Yang [16] under an asymptotically radially symmetry condition. See also earlier work by Li and Yang [37] and Weinkove [43]. Subsequent work [15, 34] extended this result to strictly relatively convex insulators. The optimal blow-up rate in higher dimensions is given by $\varepsilon^{-\frac{1}{2}+\alpha(\lambda_1)}$, where $\alpha(\lambda_1)$ is related to the first nonzero eigenvalue λ_1 of an elliptic operator on $\mathbb{S}^{n-2}$, which in turn is determined by the principal curvatures of the inclusions. Thus, the study of field concentration phenomena—particularly the optimal blow-up rates—is relatively complete for the case of perfect bonding. For a comprehensive review, we refer to Kang's ICM survey [24].

When the bonding is non-ideal, at least one of transmission conditions in (1.2) fails to hold. We consider imperfect bonding interfaces of low-conductivity type (LC-type), which arises as the limiting case of each inclusion surrounded by a low-conductivity shell as its thickness tends to zero. In this limit, the transmission conditions become

$$\frac{\partial u}{\partial \nu}\bigg|_+ = k \, \frac{\partial u}{\partial \nu}\bigg|_- = -\gamma^{-1}(u|_+ - u|_-),$$

where $\gamma > 0$ is the bonding parameter; we then say the inclusion has an LC-type imperfect bonding interface. Under such conditions, the continuity of the potential is no longer guaranteed. Another type of imperfect bonding, of high-conductivity type (HC-type), violates the continuity of flux. In two dimensions, the HC-type problem is dual to the LC-type problem. For detailed derivations, we refer to [20] and Benveniste and Miloh [9].

In this paper, we assume that D_1 and D_2 possess LC-type imperfect bonding interfaces and take $k \to \infty$. The conductivity problem then reduces to the following Robin-type boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \widetilde{\Omega}, \\ u + \gamma\partial_\nu u = K_j & \text{on } \partial D_j,\ j = 1, 2, \\ \displaystyle\int_{\partial D_j} \partial_\nu u \, dS = 0 & j = 1, 2, \\ u = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\varphi \in C^2(\partial\Omega)$, $\gamma > 0$, ν denotes the inward unit normal to D_j on ∂D_j, and the constant K_j is uniquely determined by the the third line of (1.4). This configuration models suspensions of metallic particles in epoxy matrices with Kapitza-type interfacial resistance, as studied in [4,42]. The solution $u \in H^1(\widetilde{\Omega})$ to problem (1.4) is indeed the unique minimizer of the functional

$$I_\gamma[u] = \min_{v \in \mathscr{A}} I[v], \quad I_\gamma[v] := \int_{\widetilde{\Omega}} |\nabla v|^2 + \frac{1}{\gamma} \sum_{i=1}^{2} \int_{\partial D_i} |v - (v)_{\partial D_i}|^2,$$

where $(v)_{\partial D_i} := \fint_{\partial D_i} v \, dS,\ i = 1, 2$, over the admissible set

$$\mathscr{A} := \{ v \in H^1(\widetilde{\Omega}) :\ v = \varphi \text{ on } \partial\Omega \}.$$

A proof of this variational characterization can be found in Lemma 2.2 of Dong, Yang, and Zhu [18].

Consequently, the study of conductivity problems with imperfect bonding interfaces—especially those of low-conductivity type—is of fundamental importance in both applied mathematics and materials science. Such interfaces commonly arise in composite materials, where non-ideal bonding between phases creates thin interfacial regions with markedly reduced conductivity. These imperfect contacts significantly influence macroscopic material properties, including effective electrical and thermal conductivity, mechanical stress distribution, and fracture behavior.

From a mathematical standpoint, these problems are characterized by singularly perturbed Robin-type boundary conditions involving a small parameter γ that governs the interface conductivity. The central challenge is to characterize the blow-up behavior of the electric field (the gradient of the solution) as the inter-inclusion distance $\varepsilon \to 0$ and the interface parameter $\gamma \to 0$. The present work provides a complete and sharp quantitative description of this regime, thereby bridging the gap to the well-understood perfect conductor case. Moreover, the mathematical framework developed here applies directly to analogous physical phenomena, such as anti-plane elasticity and ideal flow past obstacles, rendering the results broadly impactful across disciplines.

As mentioned above, when $\gamma = 0$, problem (1.4) reduces to (1.3), which has been extensively studied. While, for $\gamma > 0$, a striking observation from [20] is that in two dimensions for circular inclusions, any fixed $\gamma > 0$ can completely suppress gradient blow-up for certain boundary data. This contrasts sharply with the well-known result [3] that for $\gamma = 0$ (the perfect conductivity), $|\nabla u|$ blows up as $\varepsilon \to 0$. This demonstrates that even a thin, low-conductivity coating can effectively inhibit gradient singularity formation. Motivated by the biological principle that living systems cannot tolerate excessive stress, the authors of [20] conjectured that this "stress shielding" phenomenon should persist for arbitrary inclusion shapes and in all dimensions, provided γ is bounded away from zero. In [18], this conjecture was confirmed under certain symmetry conditions or when γ is sufficiently small.

However, for large γ, it was demonstrated in [18] that gradient blow-up may occur. The present paper further resolves this conjecture by establishing sharp quantitative estimates that simultaneously account for both small ε and small γ.

1.2 Main results

To state our results precisely, we begin by parametrizing the boundaries of the inclusions and introducing necessary notation. Working in coordinates $x = (x', x_n)$, we position the inclusions such that $D_1 = D_1^* + (0', \varepsilon)$ and $D_2 = D_2^*$, where D_1^* and D_2^* are two open sets touching at the origin with the inner normal of ∂D_1^* aligned with the positive x_n-axis. Near the origin, the boundaries ∂D_1 and ∂D_2 are described by

$$x_n = \varepsilon + f_1(x'), \quad x_n = f_2(x'), \quad |x'| \le 2R \tag{1.5}$$

where $f_1, f_2 \in C^{2,\alpha}$ satisfy

$$f_1(0') = f_2(0') = 0, \quad D_{x'} f_1(0') = D_{x'} f_2(0') = \vec{0}, \quad D^2(f_1 - f_2)(0') \ge \kappa > 0. \tag{1.6}$$

Define the separation function and a reference function by

$$\delta(x') := \varepsilon + f_1(x') - f_2(x') \quad \text{and} \quad \eta(x') := \varepsilon + |x'|^2.$$

By (1.5)–(1.6) and Taylor's theorem, there exists $\kappa > 0$ such that

$$\kappa |x'|^2 \le f_1(x') - f_2(x') \le \frac{1}{\kappa}|x'|^2, \quad \text{for } 0 \le |x'| \le 2R \text{ and some } \kappa > 0,$$

which implies $\eta(x') \sim \delta(x')$. Here, $A \sim B$ denotes the relation $\frac{1}{C}A < B < CA$ for a constant C depending only on n, R, α, κ, and the upper bound of $\|f_1\|_{C^{2,\alpha}}$ and $\|f_2\|_{C^{2,\alpha}}$. For $|x_0'| \le R$, we introduce the narrow region centered at x_0',

$$\Omega_t(x_0) := \{|x' - x_0'| \le t,\ f_2(x') \le x_n \le \varepsilon + f_1(x')\}, \tag{1.7}$$

with top and bottom boundaries

$$\Gamma_{1,t}(x_0) := \{|x' - x_0'| \le t, x_n = \varepsilon + f_1(x')\} \quad \text{and} \quad \Gamma_{2,t}(x_0) := \{|x' - x_0'| \le t, x_n = f_2(x')\}.$$

We write $\Omega_t := \Omega_t(0)$ and $\Gamma_{i,t} := \Gamma_{i,t}(0')$, $i = 1, 2$, when centered at the origin.

Recall the Frobenius norm for the Hessian of $f_1 - f_2$ is given by

$$|D^2(f_1 - f_2)(x')| = \left(\sum_{i,j=1}^{n-1} |\partial_{ij}(f_1 - f_2)(x')|^2 \right)^{1/2}.$$

We define the uniform bound of this norm over the ball $\mathtt{B}_R := \{|x'| < R\}$ as $\|D^2(f_1 - f_2)\|_{L^\infty(\mathtt{B}_R)} = \sup_{x' \in \mathtt{B}_R} |D^2(f_1 - f_2)(x')|$. Using this notation, we introduce the parameter γ_0 by

$$\gamma_0 := \frac{2}{(n+1)\|D^2(f_1 - f_2)\|_{L^\infty(\mathtt{B}_R)}}. \tag{1.8}$$

The main results of this paper are the following.

Theorem 1.1. *Let u be the solution of* (1.4) *in dimension $n \ge 2$ with boundaries ∂D_1 and ∂D_2 satisfying* (1.5)–(1.6)*. Let γ_0 be the constant specified by* (1.8)*. Then for $\varepsilon \in (0, \frac{1}{4})$ and*

$\gamma \in (0, \gamma_0)$, we have for $x \in \Omega_{R/2}$,

$$|Du(x)| \leq \begin{cases} \dfrac{C}{\sqrt{\gamma+\varepsilon+|x'|^2}}\|\varphi\|_{C^2(\partial\Omega)} & \textit{for } n=2, \\ \dfrac{C}{(\gamma+\varepsilon+|x'|^2)|\ln(\varepsilon+\gamma)|}\|\varphi\|_{C^2(\partial\Omega)} & \textit{for } n=3, \\ \dfrac{C}{\gamma+\varepsilon+|x'|^2}\|\varphi\|_{C^2(\partial\Omega)} & \textit{for } n\geq 4, \end{cases}$$

and

$$\|Du\|_{L^\infty(\tilde{\Omega}\setminus\Omega_{R/2})} \leq C\|\varphi\|_{C^2(\partial\Omega)} \quad \textit{for } n \geq 2,$$

where the constant $C>0$ depends only on n, R, α, κ, a lower bound of $\gamma_0-\gamma$ and upper bounds of $\|f_1\|_{C^{2,\alpha}}$ and $\|f_2\|_{C^{2,\alpha}}$.

This result demonstrates that the gradient in the narrow region exhibits a dimension-dependent blow-up rate, characterized explicitly by the bonding parameter γ and the gap ε, with validity over the precisely quantified range $\gamma \in (0, \gamma_0)$. Moreover, the blow-up rate is shown to recover that of the perfect conductivity problem (1.3) in the limit as $\gamma \to 0$.

Corollary 1.2. *Under the hypotheses of Theorem 1.1, for $\varepsilon \in (0, \frac{1}{4})$ and $\gamma \in (0, \gamma_0)$, the following global estimates hold*

$$\|Du\|_{L^\infty(\tilde{\Omega})} \leq \begin{cases} \dfrac{C}{\sqrt{\varepsilon+\gamma}}\|\varphi\|_{C^2(\partial\Omega)} & \textit{for } n=2, \\ \dfrac{C}{(\varepsilon+\gamma)|\ln(\varepsilon+\gamma)|}\|\varphi\|_{C^2(\partial\Omega)} & \textit{for } n=3, \\ \dfrac{C}{\varepsilon+\gamma}\|\varphi\|_{C^2(\partial\Omega)} & \textit{for } n\geq 4, \end{cases}$$

with the same dependence of the constants as above.

These upper bounds are shown to be optimal via the following lower-bound example.

Theorem 1.3. *Let $n \geq 2$. Let Ω be the ball of radius 6 centered at $(0', \frac{\varepsilon}{2})$, and let D_1, D_2 be the unit balls centered at $(0', 1+\varepsilon)$ and $(0', -1)$, respectively. Let $u \in H^1(\widetilde{\Omega})$ be the solution to (1.4) with boundary data $\varphi = x_n - \frac{\varepsilon}{2}$. Then there exist sufficiently small positive constants $\bar{\varepsilon}$ and $\bar{\gamma}$, depending only on n, such that for all $\varepsilon \in (0, \bar{\varepsilon})$ and $\gamma \in (0, \bar{\gamma})$,*

$$\|Du\|_{L^\infty(\tilde{\Omega})} \geq \begin{cases} \dfrac{C}{\sqrt{\varepsilon+\gamma}} & \textit{for } n=2, \\ \dfrac{C}{(\varepsilon+\gamma)|\ln(\varepsilon+\gamma)|} & \textit{for } n=3, \\ \dfrac{C}{\varepsilon+\gamma} & \textit{for } n\geq 4, \end{cases}$$

where C is a constant depending only on n.

This lower bound confirms that the estimates in Theorem 1.1 are sharp—the gradient indeed blows up at the predicted rate as $\varepsilon + \gamma \to 0$.

The proof of Theorem 1.1 proceeds by flattening the boundary of the narrow region (see the transformation (4.4) and the resulting equation (4.5) below). For a point $x_0 \in \mathbb{R}^n$, denote by $B_r(x_0)$ the ball of radius r centered at x_0. Let

$$B_r^+ := B_r(0) \cap \{x_n > 0\}, \quad \Gamma_r^0 := B_r(0) \cap \{x_n = 0\}, \quad \Gamma_r^+ := \partial B_r(0) \cap \{x_n > 0\}.$$

We need to consider the following boundary-value problem on the half unit ball

$$\begin{cases} \partial_i(A^{ij}\partial_j U) = \partial_i F^i + G & \text{in } B_1^+, \\ h\,U + \hat{\gamma} A^{ij}\partial_j U \nu_i = \hat{\gamma} F^i \nu_i + \phi & \text{on } \Gamma_1^0, \end{cases} \tag{1.9}$$

where $\nu = (0', -1)$ is the unit outward normal to B_1^+ on Γ_1^0, $\hat{\gamma}$ is a constant, $h \in C^\alpha(B_1^+)$, and the coefficient matrix $A^{ij} \in C^\alpha(B_1^+)$ is uniformly elliptic and bounded:

$$\lambda|\xi|^2 \leq A^{ij}(x)\xi_i\xi_j, \ \forall \xi \in \mathbb{R}^n, \quad |A^{ij}(x)| \leq \Lambda, \tag{1.10}$$

with positive constants λ and Λ. Here and henceforth, we adopt the Einstein summation convention for repeated indices. The following result is of independent interest.

Theorem 1.4. *Let $\lambda, \hat{\gamma} > 0$ and $U \in H^1(B_1^+)$ be a solution of (1.9) with $A^{ij} \in C^\alpha(B_1^+)$ satisfying (1.10) in B_1^+. Suppose that $F^i \in C^\alpha(B_1^+)$, $G \in L^\infty(B_1^+)$, and h satisfies*

$$h \in C^\alpha(B_1^+), \quad |h| > \lambda \ \text{on } B_1^+, \ \text{and} \ h(x)\hat{\gamma} > 0 \ \text{in } \Gamma_1^0,$$

with $\phi/h \in C^{1,\alpha}(B_1^+)$. Then

$$\|U\|_{C^{1,\alpha}(B_{1/4}^+)} \leq C\left(\|U\|_{L^2(B_{3/4}^+)} + \|F\|_{C^\alpha(B_{3/4}^+)} + \|G\|_{L^\infty(B_{3/4}^+)} + \|\frac{\phi}{h}\|_{C^{1,\alpha}(B_{3/4}^+)}\right), \tag{1.11}$$

where C depends on $\lambda, \Lambda, n, \|A^{ij}\|_{C^\alpha(B_1^+)}$, and the upper bound of $\|h\|_{C^\alpha(B_1^+)}$, but is independent of $\hat{\gamma}$.

In the limiting case when $\hat{\gamma} = 0$, the boundary condition in (1.9) reduces to a Dirichlet condition, and the estimate (1.11) is classical. A standard technique for analyzing elliptic equations with Robin boundary conditions is to transform them into equivalent problems with homogeneous Neumann conditions [39]. For sufficiently large $\hat{\gamma}$, specifically when $\hat{\gamma} \geq \hat{\gamma}_0$ for some fixed $\hat{\gamma}_0 \gg 1$, the boundary condition approximates a conormal condition as $\hat{\gamma} \to \infty$, and classical theory yields a bound of order $O(1/\hat{\gamma}_0)$. However, the case of small $\hat{\gamma}$ ($0 < \hat{\gamma} < \hat{\gamma}_0$) has remained open. Theorem 1.4 establishes the uniform bound (1.11), which is independent of $\hat{\gamma}$, demonstrating that the gradient remains bounded even as $\hat{\gamma} \to 0^+$. This uniform estimate is essential for the analysis in narrow regions, as it provides $\hat{\gamma}$-independent control over the regularity of the solution near the Robin boundary. We further remark that analogous estimates hold for higher order norms of the solution, given appropriate smoothness of the data.

The main technical contributions of this paper, which culminate in Theorem 4.1, are developed in Sections 2–4. In Section 2, we establish uniform Hölder estimates for solutions to a class of elliptic equations with Robin boundary conditions as the parameter $\hat{\gamma}$, which scales the normal derivative, tends to zero. Section 3 is devoted to investigating the regularity of solutions for another class of elliptic equations whose lower-order coefficients are of order $O(\frac{1}{\gamma})$, particularly for small γ. In Section 4, we prove Theorem 4.1 via a dichotomy based on the relative sizes of the bonding parameter γ and a distance function $\delta(x')$. Specifically, for a sufficiently large constant $\mu > 0$, we treat the two complementary cases $\gamma \leq \mu\delta$ and $\gamma > \mu\delta$ by applying the regularity theories from Sections 2 and 3, respectively. The results from both cases are then synthesized to complete the proof. Having established these ingredients, in Section 5 we adapt the decomposition technique similar as in the perfect conductivity problem and construct a new and suitable auxiliary function to prove Theorem 1.1. The optimality of these bounds (Theorem 1.3) is established in Section 6 by constructing a subsolution under a specific symmetry condition. Finally, Appendix A demonstrates the convergence of solutions to the perfect conductivity problem as $\gamma \to 0$.

2 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. We begin by normalizing the Robin boundary condition in (1.9) so that the coefficient of the conormal derivative becomes unity, yielding

$$\frac{h}{\hat{\gamma}}u + A^{ij}\partial_j u\nu_i = F^i\nu_i + \frac{\phi}{\hat{\gamma}}.$$

Observe that as $\hat{\gamma} \to 0$, the term $|\phi|_{L^\infty(\Gamma_1^0)}/\hat{\gamma}$ may become unbounded. The key insight is to consider the function $u - \phi/h$ instead of u itself, which effectively reduces the problem to the case $\phi = 0$ without loss of generality. The proof follows Campanato's method, proceeding in two stages: first, we establish higher-order derivative estimates for constant-coefficient equations in Subsection 2.1; then, using a perturbation argument and an iterative technique, we complete the proof of Theorem 1.4 in Subsection 2.2.

We now introduce notation used throughout this section. For $k \geq 1$, let $D_{x'}^k$ denote any kth-order tangential derivative with respect to $x' = (x_1, \dots, x_{n-1})$. For $x_0 \in \mathbb{R}^n$ and $r > 0$, define

$$\begin{aligned} B_r^+(x_0) &:= \{y = (y', y_n) \in \mathbb{R}^n \mid y \in B_r(x_0),\ y_n > 0\}, \\ \Gamma_r^0(x_0) &:= \{y = (y', y_n) \in \mathbb{R}^n \mid y \in \partial B_r^+(x_0), |y - x_0| < r,\ y_n = 0\}, \\ \Gamma_r^+(x_0) &:= \{y = (y', y_n) \in \mathbb{R}^n \mid y \in \partial B_r^+(x_0),\ |y - x_0| = r,\ y_n > 0\}. \end{aligned}$$

When $x_0 = 0$, we write B_r, B_r^+, Γ_r^0, and Γ_r^+. The average of a function U over a domain $\mathcal{D}$ is denoted by $(U)_{\mathcal{D}}$.

2.1 Constant coefficient equation

Lemma 2.1. *Let $U_1 \in H^1(B_1^+)$ be a solution of the constant-coefficient elliptic problem*

$$\begin{cases} \partial_i(\bar{A}^{ij}\partial_j U_1) = \partial_i \bar{F}^i & \text{in } B_1^+, \\ \bar{h}\, U_1 - \hat{\gamma}\, \bar{A}^{nj}\partial_j U_1 = -\hat{\gamma}\bar{F}^n & \text{on } \Gamma_1^0, \end{cases} \tag{2.1}$$

where $\bar{h}$ and $\bar{F}^i$ are constants, and $\bar{A}^{ij}$ is a constant matrix satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \bar{A}^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall\ \xi \in \mathbb{R}^n. \tag{2.2}$$

If $\hat{\gamma}\bar{h} > 0$, then for any $0 < \rho < \frac{1}{4}$, the following estimates hold:

$$\|U_1\|_{C^k(B_{1/2}^+)} \leq C\|U_1\|_{L^2(B_1^+)} + C\|\bar{F}\|_{L^2(B_1^+)}, \quad \forall k \geq 0, \tag{2.3}$$

$$\|DU_1 - (DU_1)_{B_\rho^+}\|_{L^2(B_\rho^+)}^2 \leq C\rho^{n+2}\|DU_1 - (DU_1)_{B_1^+}\|_{L^2(B_1^+)}^2, \tag{2.4}$$

where C depends only on λ, Λ, n, and k.

Proof. Without loss of generality, we assume $\bar{h} = 1$; the general case follows by dividing the boundary condition by $\bar{h}$ and replacing $\hat{\gamma}$ with $\hat{\gamma}/\bar{h}$.

Proof of (2.3)**.** Multiply (2.1) by $U_1\eta^2$ where $\eta \in C_c^\infty(B_1)$ satisfies $\eta \equiv 1$ in $B_{1/2}^+$ and $|\nabla\eta| \leq C$ in B_1^+. Integration by parts yields

$$\int_{B_1^+} (\bar{A}^{ij}\partial_j U_1 - \bar{F}^i)\partial_i(U_1\eta^2)\, dx + \frac{1}{\hat{\gamma}}\int_{\Gamma_1^0} U_1^2\eta^2\, dx' = 0.$$

Using ellipticity and Cauchy's inequality, we obtain the energy estimate

$$\int_{B_{1/2}^+} |DU_1|^2\,dx \le C\int_{B_1^+} U_1^2\,dx + \int_{B_1^+} |\bar F|^2\,dx. \tag{2.5}$$

Since the coefficients $\bar A^{ij}$ are constant, we may differentiate the equation tangentially. The tangential derivatives $D_{x'}^k U_1$ ($k \ge 1$) still satisfy the same equation and boundary conditions as in (2.1). By iterating estimate (2.5) for these derivatives, we obtain

$$\int_{B_{1/2}^+} |DD_{x'}^k U_1|^2\,dx \le C\int_{B_1^+} |U_1|^2\,dx + \int_{B_1^+} |\bar F|^2\,dx.$$

To estimate the normal derivatives, we use the equation to express $\partial_n^2 U_1$:

$$\bar A^{nn}\partial_n^2 U_1 = -\sum_{i+j<2n} \bar A^{ij}\partial_{ij}U_1, \quad \text{in } B_1^+. \tag{2.6}$$

Since $\bar A^{nn} \ge \lambda > 0$, it follows that

$$\|\partial_n^2 U_1\|_{L^2(B_{1/2}^+)} \le C\sum_{i+j<2n}\|\partial_{x_ix_j}U_1\|_{L^2(B_{1/2}^+)} \le C\|U_1\|_{L^2(B_1^+)} + C\|\bar F\|_{L^2(B_1^+)}.$$

By induction, for all $k \ge 0$,

$$\|D_{x'}^k U_1\|_{L^2(B_{1/2}^+)} + \sum_{l=1}^{2}\|\partial_n^l D_{x'}^k U_1\|_{L^2(B_{1/2}^+)} \le C\|U_1\|_{L^2(B_1^+)} + C\|\bar F\|_{L^2(B_1^+)}.$$

By repeatedly differentiating the equation in x_n, we can estimate higher-order normal derivatives (e.g., $\partial_n^3 U_1$, $\partial_n^4 U_1$, etc.). Combining these estimates and using the Sobolev embedding $H^\ell(B_{1/2}^+) \hookrightarrow C^k(\overline{B_{1/2}^+})$ for $\ell > k + n/2$, we conclude the proof of (2.3).

Proof of (2.4)**.** Since $D_{x_i}U_1$ $(1 \le i \le n-1)$ satisfies (2.1) with $\bar F^i = 0$, we apply (2.3) to $D_{x_i}U_1$ to obtain

$$\|D_{x'}U_1\|_{C^1(B_{1/4}^+)} \le C\|D_{x'}U_1\|_{L^2(B_{1/2}^+)}.$$

From (2.6), it follows that

$$\|D^2U_1\|_{L^\infty(B_{1/4}^+)} \le C\|DD_{x'}U_1\|_{L^\infty(B_{1/4}^+)} \le C\|D_{x'}U_1\|_{L^2(B_{1/2}^+)}. \tag{2.7}$$

Estimate (2.7) can be generalized as follows:

$$\|D^2U_1\|_{L^\infty(B^+_{\frac12-\frac{1}{2^{k+2}}})} \le C2^{(k+3)(1+\frac n2)}\|DU_1\|_{L^2(B^+_{\frac12-\frac{1}{2^{k+3}}})}, \quad \text{for } k \ge 0. \tag{2.8}$$

Indeed, for $y \in \Gamma_1^0$ with $|y'| \le \frac12 - \frac{1}{2^{k+2}}$, a scaling argument applied to U_1 in $B^+_{\frac{1}{2^{k+3}}}(y)$, along with estimate (2.7), yields the following L^∞-bound on D^2U_1 over $B^+_{\frac{1}{2^{k+4}}}(y)$:

$$\|D^2U_1\|_{L^\infty(B^+_{\frac{1}{2^{k+4}}}(y))} \le C2^{(k+3)(1+\frac n2)}\|DU_1\|_{L^2(B^+_{\frac{1}{2^{k+3}}}(y))} \le C2^{(k+3)(1+\frac n2)}\|DU_1\|_{L^2(B^+_{\frac12-\frac{1}{2^{k+3}}})}. \tag{2.9}$$

For $y \in B^+_{\frac{1}{2}-\frac{1}{2^{k+2}}}$ with $y_n > \frac{1}{2^{k+4}}$, standard interior estimates for elliptic equations give an L^∞-estimate for D^2U_1 in $B_{\frac{1}{2^{k+5}}}(y)$:

$$\|D^2U_1\|_{L^\infty(B_{\frac{1}{2^{k+5}}}(y))} \le C2^{(k+4)(1+\frac{n}{2})}\|DU_1\|_{L^2(B_{\frac{1}{2^{k+4}}}(y))} \le C2^{(k+4)(1+\frac{n}{2})}\|DU_1\|_{L^2(B^+_{\frac{1}{2}-\frac{1}{2^{k+3}}})}. \tag{2.10}$$

Hence, (2.9) and (2.10) together imply (2.8).

From (2.8) and the interpolation inequality, we obtain, for $k \ge 0$,

$$\begin{aligned}&\|D^2U_1\|_{L^\infty(B^+_{\frac{1}{2}-\frac{1}{2^{k+2}}})}\\ &\le \frac{1}{3^{2(1+\frac{n}{2})}}\|D^2U_1\|_{L^2(B^+_{\frac{1}{2}-\frac{1}{2^{k+3}}})} + C2^{2(k+3)(1+\frac{n}{2})}\|U_1-(U_1)_{B^+_{1/2}}\|_{L^2(B^+_{\frac{1}{2}-\frac{1}{2^{k+3}}})}.\end{aligned}$$

Multiplying both sides by $\frac{1}{3^{2(k+3)(1+\frac{n}{2})}}$ and summing over $k = 0, 1, 2, \dots$, we derive

$$\begin{aligned}&\sum_{k=0}^{+\infty}\frac{1}{3^{2(k+3)(1+\frac{n}{2})}}\|D^2U_1\|_{L^\infty(B^+_{2^{-1}-2^{-(k+2)}})}\\ &\le \sum_{k=0}^{+\infty}\frac{1}{3^{2(k+4)(1+\frac{n}{2})}}\|D^2U_1\|_{L^\infty(B^+_{2^{-1}-2^{-(k+3)}})} + C\sum_{k=0}^{+\infty}(\frac{2}{3})^{(k+3)(n+2)}\|U_1-(U_1)_{B^+_{1/2}}\|_{L^2(B^+_{1/2})}.\end{aligned}$$

Since $\|D^2U_1\|_{L^\infty(B^+_{1/2})} < +\infty$, the summation on the right-hand side can be absorbed into the left-hand side, which implies

$$\|D^2U_1\|_{L^\infty(B^+_{1/4})} \le C\|U_1-(U_1)_{B^+_{1/2}}\|_{L^2(B^+_{1/2})}. \tag{2.11}$$

Using the interpolation inequality and (2.11), we have

$$\|DU_1\|_{L^\infty(B^+_{1/4})} \le C\|U_1-(U_1)_{B^+_{1/2}}\|_{L^2(B^+_{1/2})}. \tag{2.12}$$

Applying (2.12) to $D_{x_i}U_1 (1 \le i \le n-1)$ and using (2.6), we conclude

$$\|D^2U_1\|_{L^\infty(B^+_{1/4})} \le C\|DD_{x'}U_1\|_{L^\infty(B^+_{1/4})} \le C\|D_{x'}U_1-(D_{x'}U_1)_{B^+_{1/2}}\|_{L^2(B^+_{1/2})}. \tag{2.13}$$

Estimate (2.4) follows directly from (2.13). □

Applying a scaling argument to Lemma 2.1 and combining it with standard elliptic interior estimates yields the following corollary, which will be useful later.

Corollary 2.2. *Let $U_1 \in H^1(B_1^+)$ satisfy the constant-coefficient problem*

$$\begin{cases} \partial_i(\bar{A}^{ij}\partial_j U_1) = \partial_i \bar{F}^i & \text{in } B_1^+,\\ \bar{h}\,U_1 - \hat{\gamma}\,\bar{A}^{nj}\partial_j U_1 = -\hat{\gamma}\bar{F}^n & \text{on } \Gamma_1^0,\end{cases}$$

with $\bar{A}^{ij}$ satisfying the uniform ellipticity condition (2.2). If $\hat{\gamma}\bar{h} > 0$, then for $x_0 \in B^+_{1/2}$ and $0 < \rho < r \le \frac{1}{2}$, it holds that

$$\begin{aligned}&\|D^kU_1\|_{L^\infty(B^+_{r/2})} \le Cr^{-n/2-k}\Big(\|U_1\|_{L^2(B_r^+)} + \|\bar{F}\|_{L^2(B_r^+)}\Big), \quad \forall\ k \ge 0,\\ &\|DU_1-(DU_1)_{B^+_\rho(x_0)}\|^2_{L^2(B^+_\rho(x_0))} \le C(\frac{\rho}{r})^{n+2}\|DU_1-(DU_1)_{B^+_r(x_0)}\|^2_{L^2(B^+_r(x_0))},\end{aligned}$$

where $C > 0$ depends only on λ, Λ, and n.

2.2 Proof of Theorem 1.4

Proof of Theorem 1.4. By considering the function $U - \frac{\phi}{h}$, we may assume without loss of generality that $\phi = 0$. Throughout the proof, we define the quantity

$$\mathcal{Q} := \|U\|_{L^2(B^+_{1/2})} + \|F\|_{C^\alpha(B^+_{1/2})} + \|G\|_{L^\infty(B^+_{1/2})}.$$

Step 1. Decomposition of U. Let $x_0 \in B^+_{1/4}$ and $r \in (0, r_0)$, where $r_0 \in (0, \frac{1}{8})$ will be chosen sufficiently small later. We decompose the solution via the method of frozen coefficients:

$$U := U_1 + U_2 \quad \text{in } B^+_r(x_0).$$

This decomposition separates the problem into a constant-coefficient part, which is tractable via classical methods, and a variable-coefficient correction that will be shown to be small.

If $\Gamma^0_r(x_0)$ is nonempty, then U_1 solves the following constant–coefficient problem

$$\begin{cases} \partial_i(\bar{A}^{ij}\partial_j U_1) = \partial_i \bar{F}^i & \text{in } B^+_r(x_0), \\ \bar{h}\, U_1 - \hat{\gamma}\, \bar{A}^{nj}\partial_j U_1 = -\hat{\gamma}\bar{F}^n & \text{on } \Gamma^0_r(x_0), \\ U_1 = U & \text{on } \Gamma^+_r(x_0), \end{cases} \tag{2.14}$$

and U_2 satisfies the variable–coefficient problem

$$\begin{cases} \partial_i(\bar{A}^{ij}\partial_j U_2) = -\partial_i \tilde{F}^i + \partial_i(F^i - \bar{F}^i) + G & \text{in } B^+_r(x_0), \\ \bar{h}U_2 - \hat{\gamma}\left(A^{nj}\partial_j U_2 + \tilde{F}^n - F^n + \bar{F}^n\right) = (\bar{h} - h)U & \text{on } \Gamma^0_r(x_0), \\ U_2 = 0 & \text{on } \Gamma^+_r(x_0), \end{cases} \tag{2.15}$$

where

$$\bar{A}^{ij} = A^{ij}(x_0),\ \bar{F}^i = F^i(x_0', 0),\ \bar{h} = h(x_0', 0), \ \text{ and } \tilde{F}^i := (A^{ij} - \bar{A}^{ij})\partial_j U,\ i = 1, 2, \ldots, n.$$

If $\Gamma^0_r(x_0)$ is empty, then $U_1 \in H^1(B_r(x_0))$ with $U_1 = U$ on $\partial B_r(x_0)$ and $U_2 \in H^1_0(B_r(x_0))$ satisfy the first line of (2.14) and (2.15), respectively.

Step 2. Energy estimate for U_2. We now derive an energy estimate to control the remainder term U_2. Multiply the equation in (2.15) by U_2 and integrate by parts over $B^+_r(x_0)$:

$$\int_{B^+_r(x_0)} A^{ij}\partial_j U_2 \partial_i U_2 + \left(\tilde{F}^i - F^i + \bar{F}^i\right)\partial_i U_2 + GU_2\, dx = \int_{\Gamma^0_r(x_0)} \frac{1}{\hat{\gamma}} U_2\Big((\bar{h} - h)U - \bar{h}U_2\Big)\, dx'. \tag{2.16}$$

We estimate each term. By uniform ellipticity,

$$\int_{B^+_r(x_0)} A^{ij}\partial_j U_2 \partial_i U_2\, dx \geq \lambda \int_{B^+_r(x_0)} |DU_2|^2\, dx.$$

For the next term, we apply Cauchy's inequality

$$\begin{aligned} &\Big|\int_{B^+_r(x_0)} \left(\tilde{F}^i + \bar{F}^i - F^i\right)\partial_i U_2\, dx\Big| \\ &\leq \frac{\lambda}{8}\int_{B^+_r(x_0)} |DU_2|^2\, dx + C\Big(\|DU\|^2_{L^\infty(B^+_r(x_0))} + \mathcal{Q}^2\Big) r^{n+2\alpha}. \end{aligned}$$

For the term involving G, since $U_2 = 0$ on $\Gamma_r^+(x_0)$, we use the boundary Poincaré inequality and Cauchy's inequality:

$$
\begin{aligned}
\Big|\int_{B_r^+(x_0)} G U_2\,dx\Big| &\le \frac{\lambda}{C}\int_{B_r^+(x_0)} \frac{U_2^2}{r^2}\,dx + C\int_{B_r^+(x_0)} G^2 r^2\,dx \\
&\le \frac{\lambda}{8}\int_{B_r^+(x_0)} |DU_2|^2\,dx + C\mathcal{Q}^2 r^{n+2}.
\end{aligned}
$$

Finally, we tackle the boundary integral in (2.16). Using the boundary condition $\frac{U}{\hat{\gamma}} = \frac{A^{nj}\partial_j U - F^n}{h}$ on Γ_r^0 along with the trace theorem and Poincaré inequality to U_2 in B_r^+ again, we have

$$
\begin{aligned}
&\int_{\Gamma_r^0(x_0)} \frac{1}{\hat{\gamma}} U_2((\bar{h}-h)U - \bar{h}U_2)\,dx' \le \int_{\Gamma_r^0(x_0)} \frac{U}{\hat{\gamma}}(\bar{h}-h)U_2\,dx' \\
&\le C\int_{\Gamma_r^0(x_0)} \Big(\|DU\|_{L^\infty(B_r^+(x_0))} + \mathcal{Q}\Big)|(\bar{h}-h)U_2|\,dx' \\
&\le Cr^{n+2\alpha}\Big(\|DU\|^2_{L^\infty(B_r^+(x_0))} + \mathcal{Q}^2\Big) + \frac{\lambda}{Cr}\int_{\Gamma_r^0(x_0)} U_2^2\,dx' \\
&\le Cr^{n+2\alpha}\Big(\|DU\|^2_{L^\infty(B_r^+(x_0))} + \mathcal{Q}^2\Big) + \frac{\lambda}{8}\int_{B_r^+} |DU_2|^2\,dx.
\end{aligned}
$$

Substituting all these estimates into (2.16), we obtain

$$
\int_{B_r^+(x_0)} |DU_2|^2\,dx \le C\Big(\|DU\|^2_{L^\infty(B_r^+(x_0))} + \mathcal{Q}^2\Big) r^{n+2\alpha}. \tag{2.17}
$$

Step 3. Iterative argument for the gradient estimate. By Corollary 2.2 and inequality (2.17), for any $0 < \rho < r \le r_0$, we have

$$
\begin{aligned}
&\int_{B_\rho^+(x_0)} |DU - (DU)_{B_\rho^+(x_0)}|^2\,dx \\
&\le C\int_{B_\rho^+(x_0)} |DU_1 - (DU_1)_{B_\rho^+(x_0)}|^2\,dx + C\int_{B_\rho^+(x_0)} |DU_2 - (DU_2)_{B_\rho^+(x_0)}|^2\,dx \\
&\le C\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r^+(x_0)} |DU_1 - (DU_1)_{B_r^+(x_0)}|^2\,dx + C\int_{B_r^+(x_0)} |DU_2|^2\,dx \\
&\le C\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r^+(x_0)} |DU - (DU)_{B_r^+(x_0)}|^2\,dx + C\Big(\|DU\|^2_{L^\infty(B_r^+(x_0))} + \mathcal{Q}^2\Big) r^{n+2\alpha}.
\end{aligned} \tag{2.18}
$$

Define the averaged quantities:

$$
\Phi_\rho(x_0) := \fint_{B_\rho^+(x_0)} |DU - (DU)_{B_\rho^+(x_0)}|^2\,dx, \quad \Psi_\rho(x_0) := \fint_{B_\rho^+(x_0)} |DU|\,dx.
$$

From (2.18), we derive

$$
\Phi_{\sigma r}(x_0) \le C_0\sigma^2\Phi_r(x_0) + C\sigma^{-n} r^{2\alpha}\Big(\|DU\|^2_{L^\infty(B_r^+(x_0))} + \mathcal{Q}^2\Big).
$$

Choose σ such that $C_0\sigma^2 \le \frac{1}{4}$. Iterating this inequality yields

$$
\Phi_{\sigma^j r}(x_0) \le \frac{1}{4^j}\Phi_r(x_0) + C\sum_{i=1}^{j} \frac{1}{4^{j-i}}(\sigma^i r)^{2\alpha}\Big(\|DU\|^2_{L^\infty(B_r^+(x_0))} + \mathcal{Q}^2\Big). \tag{2.19}
$$

By the triangle inequality and Hölder's inequality,

$$\begin{aligned}|\Psi_{\sigma r}(x_0)-\Psi_r(x_0)| &\le \fint_{B^+_{\sigma r}(x_0)} |DU-(DU)_{B^+_r(x_0)}|\,dx\\ &\le C\fint_{B^+_r(x_0)} |DU-(DU)_{B^+_r(x_0)}|\,dx \le C\Phi_r(x_0)^{1/2}.\end{aligned}\tag{2.20}$$

Using (2.19) and (2.20), we obtain

$$\begin{aligned}\big|\Psi_{\sigma^j r}(x_0)-\Psi_{\sigma^{j-1}r}(x_0)\big| &\le C\Phi_{\sigma^{j-1}r}(x_0)^{1/2}\\ &\le C2^{-j+1}\Phi_r(x_0)^{1/2}+C\sum_{i=1}^{j-1}\frac{1}{2^{j-i}}(\sigma^i r)^\alpha\Big(\|DU\|_{L^\infty(B^+_r(x_0))}+\mathcal{Q}\Big)\end{aligned}$$

Summing from $j=1$ to k,

$$\begin{aligned}\Psi_{\sigma^k r}(x_0) &\le \Psi_r(x_0)+\sum_{j=1}^{k}\big|\Psi_{\sigma^j r}(x_0)-\Psi_{\sigma^{j-1}r}(x_0)\big|\\ &\le \Psi_r(x_0)+C\Phi_r(x_0)^{1/2}+C\sum_{j=1}^{k}\sum_{i=1}^{j-1}\frac{1}{2^{j-i}}(\sigma^i r)^\alpha\Big(\|DU\|_{L^\infty(B^+_r(x_0))}+\mathcal{Q}\Big)\\ &\le \Psi_r(x_0)+C\Phi_r(x_0)^{1/2}+C\sum_{i=1}^{k-1}(\sigma^i r)^\alpha\Big(\|DU\|_{L^\infty(B^+_r(x_0))}+\mathcal{Q}\Big).\end{aligned}\tag{2.21}$$

Because Du is continuous, letting $k\to\infty$ in (2.21) yields, for $r\in(0,r_0)$

$$|DU(x_0)|\le \Psi_r(x_0)+C\Phi_r(x_0)^{1/2}+C_1r^\alpha\Big(\|DU\|_{L^\infty(B^+_r(x_0))}+\mathcal{Q}\Big).\tag{2.22}$$

Choosing r_0 small such that $C_1r_0^\alpha\le\frac{1}{3^n}$ in (2.22). Applying Hölder's inequality, we then obtain

$$|DU(x_0)|\le Cr^{-\frac{n}{2}}\Big(\|DU\|_{L^2(B^+_{3/8})}+\mathcal{Q}\Big)+\frac{1}{3^n}\|DU\|_{L^\infty(B^+_r(x_0))}.\tag{2.23}$$

Now take $x_0\in B^+_{\frac14-\frac{r_0}{4^{k+2}}}$ and $r=\frac{1}{4^{k+3}}r_0$ in (2.23). Since x_0 is arbitrary, it follows that

$$\|DU\|_{L^\infty(B^+_{\frac14-\frac{r_0}{4^{k+2}}})}\le C2^{n(k+3)}\Big(\|DU\|_{L^2(B^+_{3/8})}+\mathcal{Q}\Big)+\frac{1}{3^n}\|DU\|_{L^\infty(B^+_{\frac14-\frac{r_0}{4^{k+3}}})}.$$

Multiply both sides by $\frac{1}{3^{n(k+3)}}$ and sum over $k=0,1,2,\ldots$,

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{3^{n(k+3)}}\|DU\|_{L^\infty(B^+_{\frac14-\frac{r_0}{4^{k+2}}})}\\ &\le C\sum_{k=0}^{\infty}\big(\frac23\big)^{n(k+3)}\Big(\|DU\|_{L^2(B^+_{3/8})}+\mathcal{Q}\Big)+\sum_{k=0}^{\infty}\frac{1}{3^{n(k+4)}}\|DU\|_{L^\infty(B^+_{\frac14-\frac{r_0}{4^{k+3}}})}.\end{aligned}$$

Since $\|DU\|_{L^\infty(B^+_{1/2})}<+\infty$, we absorb the right-hand summation to obtain

$$\|DU\|_{L^\infty(B^+_{1/8})}\le C\Big(\|DU\|_{L^2(B^+_{3/8})}+\mathcal{Q}\Big).\tag{2.24}$$

Recall $\phi = 0$. Multiply (1.9) by $U\xi^2$ where $\xi \in C_c^\infty(B_{1/2})$ satisfies $\xi = 1$ in $B_{3/8}^+$ and $|D\xi| \le C$ in $B_{1/2}^+$. Integrating by parts and Cauchy's inequality yield $\|DU\|_{L^2(B_{3/8}^+)} \le C\mathcal{Q}$. Substituting into (2.24) gives

$$\|DU\|_{L^\infty(B_{1/8}^+)} \le C\mathcal{Q}.$$

From this, (2.18), and Campanato's characterization of Hölder continuous functions, we conclude

$$[DU]_{C^\alpha(B_{1/16}^+)} \le C\mathcal{Q}.$$

Finally, by interpolation inequalities, we use $\|U\|_{L^2(B_{1/8}^+)}$ and $\|DU\|_{L^\infty(B_{1/8}^+)}$ to estimate $\|U\|_{L^\infty(B_{1/8}^+)}$. This completes the proof of Theorem 1.4. $\square$

Lemma 2.3. *Let $U \in H^1(B_1^+)$ solve* (1.9) *with $A^{ij} \in C^\alpha(B_1^+)$ satisfying* (1.10) *in B_1^+. Suppose $F^i \in C^\alpha(B_1^+)$, $G \in L^\infty(B_1^+)$, $h \in C^{1,\alpha}(B_1^+)$, $\phi \in C^\alpha(B_1^+)$, and $|\hat{\gamma}| > 1$. Then it holds that*

$$\|DU\|_{C^\alpha(B_{1/4}^+)} \le C\Big(\|DU\|_{L^2(B_{3/4}^+)} + \|\frac{U}{\hat{\gamma}}\|_{L^2(B_{3/4}^+)} + \|\frac{\phi}{\hat{\gamma}}\|_{C^\alpha(B_{3/4}^+)} + \|F\|_{C^\alpha(B_{3/4}^+)} + \|G\|_{L^\infty(B_{3/4}^+)}\Big),$$

where C depends on $\lambda, \Lambda, n, \|A^{ij}\|_{C^\alpha(B_1^+)}$ and $\|h\|_{C^\alpha(B_1^+)}$, but is independent of $\hat{\gamma}$.

Proof. Define $\widehat{U} = U - (U)_{B_{3/4}^+}$. Then $\widehat{U}$ solves

$$\begin{cases} \partial_i(A^{ij}\partial_j\widehat{U}) = \partial_i F^i + G & \text{in } B_1^+, \\ \dfrac{h}{\hat{\gamma}}\widehat{U} - A^{nj}\partial_j\widehat{U} = -F^n + \dfrac{1}{\hat{\gamma}}(\phi - h\,(U)_{B_{3/4}^+}) & \text{on } \Gamma_1^0. \end{cases}$$

By using [39, Theorem 5.54],

$$\|D\widehat{U}\|_{C^\alpha(B_{1/2}^+)} \le C\Big(\|\widehat{U}\|_{L^2(B_{3/4}^+)} + \|\frac{1}{\hat{\gamma}}(\phi - h\,(U)_{B_{3/4}^+})\|_{C^\alpha(B_{3/4}^+)} + \|F\|_{C^\alpha(B_{3/4}^+)} + \|G\|_{L^\infty(B_{3/4}^+)}\Big).$$

The Poincaré inequality gives $\|\widehat{U}\|_{L^2(B_{3/4}^+)} \le C\|D\widehat{U}\|_{L^2(B_{3/4}^+)}$. Using Hölder inequality, we have

$$|(U)_{B_{3/4}^+}| \le C\|U\|_{L^2(B_{3/4}^+)}.$$

Since $DU = D\widehat{U}$, the proof is finished. $\square$

3 Gradient estimate in a reduced dimensional space for small γ

In this section, we study the elliptic equation $L\mathtt{U} = \partial_i\mathtt{F}^i + \mathtt{G}$ in $\mathbb{R}^{n-1}$, where the operator L is defined by

$$L\mathtt{U} = \partial_i(A^{ij}(x')\partial_j\mathtt{U}) + c(x')\mathtt{U}$$

with $c(x') = -\frac{1}{\gamma}$ for a small positive constant γ. Our primary objective is to characterize the behavior of the solution and its dependence on γ in the small-γ regime. In Subsection 3.1, we consider the case where the principal coefficients A^{ij} are uniformly elliptic. By combining

Agmon's method with the weak Harnack inequality, we derive an iterative formula for the L^∞-norm of $\mathtt{U}$, which ultimately establishes the regularity of $\mathtt{U}$ as $\gamma \to 0$. Subsection 3.2 addresses the degenerate case, employing the comparison principle, iterative techniques, and the regularity results from Subsection 3.1. We remark that, through an argument building on Subsection 3.1, the assumption that $c(x')$ is constant can be relaxed to the case where $c(x')$ is measurable and satisfies $\frac{\lambda}{\gamma} < -c(x') \leq \frac{\Lambda}{\gamma}$ for positive constants λ and Λ. Consequently, the constant-coefficient assumption in the theorem of Subsection 3.2 can be similarly extended within this measurable and bounded framework.

To distinguish from the full-space $\mathbb{R}^n$ notation, we denote by $\mathtt{B}_r(x_0')$ the ball in $\mathbb{R}^{n-1}$ of radius r centered at x_0', writing $\mathtt{B}_r$ when centered at the origin. Functions on $\mathbb{R}^{n-1}$ are denoted by $\mathtt{U}$, $\mathtt{F}$, $\mathtt{G}$, etc.

3.1 Interior and global estimates for uniform elliptic equations

Proposition 3.1. *Let $n \geq 2$, and let $\mathtt{U} \in H^1(\mathtt{B}_1)$ be a solution of*

$$\sum_{i,j=1}^{n-1} \partial_i\Big(A^{ij}(x')\partial_j \mathtt{U}\Big) - \frac{1}{\gamma}\mathtt{U} = \sum_{i=1}^{n-1} \partial_i \mathtt{F}^i + \mathtt{G} \quad \text{in } \mathtt{B}_1 \subset \mathbb{R}^{n-1}, \tag{3.1}$$

where the coefficients $A^{ij} \in L^\infty(\mathtt{B}_1)$ satisfy the uniformly elliptic condition (1.10) *in $\mathtt{B}_1$ and $\gamma > 0$.*

(i) Assume $\mathtt{F} \in L^\infty(\mathtt{B}_1)$ and $\mathtt{G} \in L^\infty(\mathtt{B}_1)$. Then there exists $\bar\gamma_0 = \bar\gamma_0(\lambda, \Lambda, n, \|A^{ij}\|_{L^\infty(\mathtt{B}_1)})$ such that for all $\gamma \in (0, \bar\gamma_0)$,

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/4})} \leq C\Big(\frac{1}{2}\Big)^{\frac{1}{8C_0\sqrt{\gamma}}} \|\mathtt{U}\|_{L^2(\mathtt{B}_{3/4})} + C\sqrt{\gamma}\|\mathtt{F}\|_{L^\infty(\mathtt{B}_{3/4})} + C\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{3/4})}, \tag{3.2}$$

where C_0 and C depend only on λ, Λ, n and $\|A^{ij}\|_{L^\infty(\mathtt{B}_1)}$, and are independent of $\frac{1}{\gamma}$.

(ii) Assume $A^{ij} \in C^\alpha(\mathtt{B}_1)$, $\mathtt{F} \in C^\alpha(\mathtt{B}_1)$, and $\mathtt{G} \in L^\infty(\mathtt{B}_1)$. Then there exists $\bar\gamma_0 = \bar\gamma_0(\lambda, \Lambda, n, \|A^{ij}\|_{C^\alpha(\mathtt{B}_1)})$ such that for all $\gamma \in (0, \bar\gamma_0)$,

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/4})} \leq C\Big(\frac{1}{2}\Big)^{\frac{1}{8C_0\sqrt{\gamma}}} \|\mathtt{U}\|_{L^2(\mathtt{B}_{1/2})} + C\gamma^{\frac{1+\alpha}{2}}[\mathtt{F}]_{C^\alpha(\mathtt{B}_{1/2})} + C\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{1/2})}, \tag{3.3}$$

and

$$\begin{aligned} &\|D\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/4})} + \gamma^{\frac{\alpha}{2}}[D\mathtt{U}]_{C^\alpha(\mathtt{B}_{1/4})} \\ &\leq C\Big(\frac{1}{2}\Big)^{\frac{1}{8C_0\sqrt{\gamma}}}\gamma^{-1/2}\|\mathtt{U}\|_{L^2(\mathtt{B}_{3/4})} + C\gamma^{\frac{\alpha}{2}}[\mathtt{F}]_{C^\alpha(\mathtt{B}_{3/4})} + C\gamma^{1/2}\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{3/4})}, \end{aligned} \tag{3.4}$$

where C_0 and C depend only on λ, Λ, n and $\|A^{ij}\|_{C^\alpha(\mathtt{B}_1)}$, and are independent of $\frac{1}{\gamma}$.

Proof. Here we adapt Agmon's idea (see [30, Lemma 5.5]) to extend the problem into a higher-dimensional space where the singular term $-\frac{1}{\gamma}\mathtt{U}$ is transformed into a regular derivative by considering an auxiliary function on a higher-dimensional space, thereby enabling the use of standard elliptic estimates.

We define the cylindrical domain $Q_R \subset \mathbb{R}^n = \{(x', x_n) \,|\, x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}\}$ by $Q_R := \mathtt{B}_R \times (-R, R)$, and introduce the elliptic operator

$$L_0 W := \sum_{i,j=1}^{n-1} \partial_i\Big(A^{ij}(x')\partial_j W\Big) + \partial_{nn} W.$$

Suppose $\mathtt{U}(x')$ is a solution to equation (3.1). Define the function $W(x',x_n)=\mathtt{U}(x')\cos\frac{x_n}{\sqrt{\gamma}}$. Then W satisfies

$$L_0W=\sum_{i=1}^{n}\partial_i\tilde{F}^i,$$

where the modified right-hand side terms are given by

$$\tilde{F}^i(x',x_n)=\mathtt{F}^i(x')\cos\frac{x_n}{\sqrt{\gamma}},\ 1\le i\le n-1,\quad \tilde{F}^n(x',x_n)=\sqrt{\gamma}\mathtt{G}(x')\sin\frac{x_n}{\sqrt{\gamma}}.$$

These terms are explicitly defined and remain controlled under the assumptions on $\mathtt{F}$ and $\mathtt{G}$. By classical elliptic theory, we obtain the interior estimate

$$\|W\|_{L^\infty(Q_{1/8})}\le C\Big(\|W\|_{L^2(Q_{1/4})}+\|\tilde{F}\|_{L^\infty(Q_{1/4})}\Big). \tag{3.5}$$

Returning to the original function $\mathtt{U}$, and noting that $|W(x',t)|\le|\mathtt{U}(x')|$, we deduce

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/8})}\le C\Big(\|\mathtt{U}\|_{L^2(\mathtt{B}_{1/4})}+\|\mathtt{F}\|_{L^\infty(\mathtt{B}_{1/4})}+\sqrt{\gamma}\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{1/4})}\Big). \tag{3.6}$$

(i) The proof of estimate (3.2) relies on an application of the weak Harnack inequality to W. For any $r\in(0,\frac{1}{8})$, applying the weak Harnack inequality (see Theorem 8.18 in [21]) to the functions $W-\inf_{Q_{4r}}W$ and $\sup_{Q_{4r}}W-W$ in Q_{4r}, we obtain

$$\operatorname*{osc}_{Q_r}W\le\beta\operatorname*{osc}_{Q_{4r}}W+Cr\|\tilde{F}\|_{L^\infty(Q_{4r})}, \tag{3.7}$$

where $\beta\in(0,1)$ is a universal constant and C depends only on λ,Λ,n.

Define $r_0=\frac{\sqrt{\gamma}\pi}{2}$ and set $r_{i+1}=4r_i$, for $i\ge0$. Then inequality (3.7) implies

$$\operatorname*{osc}_{Q_{r_i}}W\le\beta\operatorname*{osc}_{Q_{r_{i+1}}}W+Cr_{i+1}\|\tilde{F}\|_{L^\infty(Q_{r_{i+1}})}. \tag{3.8}$$

Let k_0 be the smallest integer such that $\beta^{k_0}\le\frac{1}{4}$, and set $C_0=4^{k_0}\pi$. Iterating (3.8) from $i=0$ to k_0-1 yields

$$\operatorname*{osc}_{Q_{\frac{\sqrt{\gamma}\pi}{2}}}W\le\frac{1}{4}\operatorname*{osc}_{Q_{C_0\sqrt{\gamma}}}W+C\sqrt{\gamma}\|\tilde{F}\|_{L^\infty(Q_{C_0\sqrt{\gamma}})}. \tag{3.9}$$

Note that $\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{\frac{\sqrt{\gamma}\pi}{2}})}\le\operatorname*{osc}_{Q_{\frac{\sqrt{\gamma}\pi}{2}}}W$ and $\operatorname*{osc}_{Q_{C_0\sqrt{\gamma}}}W\le2\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{C_0\sqrt{\gamma}})}$. Therefore, (3.9) implies

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{\frac{\sqrt{\gamma}\pi}{2}})}\le\frac{1}{2}\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{C_0\sqrt{\gamma}})}+C\sqrt{\gamma}\|\mathtt{F}\|_{L^\infty(\mathtt{B}_{C_0\sqrt{\gamma}})}+C\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{C_0\sqrt{\gamma}})}.$$

This provides an estimate for $\mathtt{U}$ at the scale $\sqrt{\gamma}$. To establish the global estimate over $\mathtt{B}_{1/4}$ we iterate this inequality along a chain of balls. Specifically, for any $x_0\in\mathtt{B}_{C_0k\sqrt{\gamma}}$, applying the above estimate to the ball $\mathtt{B}_{C_0\sqrt{\gamma}}$ yields

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{C_0k\sqrt{\gamma}})}\le\frac{1}{2}\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{C_0(k+1)\sqrt{\gamma}})}+C\sqrt{\gamma}\|\mathtt{F}\|_{L^\infty(\mathtt{B}_{C_0(k+1)\sqrt{\gamma}})}+C\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{C_0(k+1)\sqrt{\gamma}})}. \tag{3.10}$$

Choose $\bar{\gamma}_0>0$ such that $C_0\sqrt{\bar{\gamma}_0}=\frac{1}{16}$. For $\gamma\in(0,\bar{\gamma}_0)$, let k_γ be the largest integer less than $\frac{1}{8C_0\sqrt{\gamma}}-1$. Iterating (3.10) from $k=1$ to k_γ gives

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{\frac{1}{16}})}\le C\Big(\frac{1}{2}\Big)^{\frac{1}{8C_0\sqrt{\gamma}}}\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/8})}+C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{L^\infty(\mathtt{B}_{1/8})}+\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{1/8})}\Big). \tag{3.11}$$

Substituting the interior estimate (3.6) into (3.11), we obtain

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{\frac{1}{16}})} \leq C\Big(\frac{1}{2}\Big)^{\frac{1}{8C_0\sqrt{\gamma}}}\|\mathtt{U}\|_{L^2(\mathtt{B}_{1/4})} + C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{L^\infty(\mathtt{B}_{1/4})} + \gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{1/4})}\Big).$$

(ii) Under the higher regularity assumptions in (ii), the same general strategy applies, but we now employ Schauder estimates. Specifically, in (3.5), the term $\|\tilde{F}\|_{L^\infty(Q_{1/4})}$ may be replaced with $C(\sum_{i=1}^{n-1}[\tilde{F}^i]_{C^\alpha_{x'}(Q_{\frac{1}{4}})} + [\tilde{F}^n]_{C^\alpha_{x_n}(Q_{\frac{1}{4}})})$, and in (3.7), the term $Cr\|\tilde{F}\|_{L^\infty(Q_{4r})}$ is replaced with $Cr^{1+\alpha}(\sum_{i=1}^{n-1}[\tilde{F}^i]_{C^\alpha_{x'}(Q_{4r})} + [\tilde{F}^n]_{C^\alpha_{x_n}(Q_{4r})})$. Here, we employ the notation for a partial Hölder semi-norm of the function f with respect to x' over domain $\mathcal{D}$ as :

$$[f]_{C^\alpha_{x'}(\mathcal{D})} = \sup_{\substack{(z',x_n),(y',x_n)\in\mathcal{D}\\ z'\neq y'}} \frac{|f(z',x_n) - f(y',x_n)|}{|z'-y'|^\alpha},$$

and a partial Hölder semi-norm of the function f with respect to x_n over domain $\mathcal{D}$ as:

$$[f]_{C^\alpha_{x_n}(\mathcal{D})} = \sup_{\substack{(x',z_n),(x',y_n)\in\mathcal{D}\\ z_n\neq y_n}} \frac{|f(x',z_n) - f(x',y_n)|}{|z_n-y_n|^\alpha}.$$

To prove the gradient estimate (3.4), we perform a change of variables: set $x' = \sqrt{\gamma}y'$ with $y' \in \mathtt{B}_1$ and define $\widetilde{\mathtt{U}}(y') = \mathtt{U}(\sqrt{\gamma}y') = \mathtt{U}(x')$. This rescaling absorbs the parameter γ into the spatial variable, resulting in a uniformly elliptic equation with order-one coefficients. Specifically, $\widetilde{\mathtt{U}} \in H^1(\mathtt{B}_1)$ satisfies

$$\partial_i(\widetilde{A}^{ij}\partial_j\widetilde{\mathtt{U}}) - \tilde{\mathtt{U}} = \mathrm{div}(\sqrt{\gamma}\widetilde{\mathtt{F}}) + \gamma\widetilde{\mathtt{G}}, \quad \text{in } \mathtt{B}_1,$$

where $\widetilde{A}^{ij}(y') = A^{ij}(\sqrt{\gamma}y')$, $\widetilde{\mathtt{F}}(y') = \mathtt{F}(\sqrt{\gamma}y')$ and $\widetilde{\mathtt{G}}(y') = \mathtt{G}(\sqrt{\gamma}y')$. Applying standard elliptic estimates yields

$$\|D\widetilde{\mathtt{U}}\|_{C^\alpha(\mathtt{B}_{1/2})} \leq C\|\widetilde{\mathtt{U}}\|_{L^2(\mathtt{B}_1)} + C[\sqrt{\gamma}\widetilde{\mathtt{F}}]_{C^\alpha(\mathtt{B}_1)} + C\|\gamma\widetilde{\mathtt{G}}\|_{L^\infty(\mathtt{B}_1)}.$$

Reverting to the original variables, we obtain

$$\gamma^{1/2}\|D\mathtt{U}\|_{L^\infty(\mathtt{B}_{\frac{\sqrt{\gamma}}{2}})} + \gamma^{\frac{1+\alpha}{2}}[D\mathtt{U}]_{C^\alpha(\mathtt{B}_{\frac{\sqrt{\gamma}}{2}})} \leq C\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{\sqrt{\gamma}})} + C\gamma^{\frac{1+\alpha}{2}}[\mathtt{F}]_{C^\alpha(\mathtt{B}_{\sqrt{\gamma}})} + C\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{\sqrt{\gamma}})}. \tag{3.12}$$

Finally, using estimate (3.3) to bound $\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{\sqrt{\gamma}})}$ in (3.12) yields the desired gradient estimate (3.4). □

For the solution to equation (3.1), standard elliptic theory yields the corresponding estimates for $\gamma \geq \bar{\gamma}_0$. Combined with the results of Proposition 3.1 for $\gamma \in (0, \bar{\gamma}_0)$, we obtain the following corollary.

Corollary 3.2. *Let* $\mathtt{U} \in H^1(\mathtt{B}_1)$ *be a solution of* (3.1). *Suppose the coefficients* $A^{ij} \in L^\infty(\mathtt{B}_1)$ *satisfy the uniformly elliptic condition* (1.10) *in* $\mathtt{B}_1$, $\mathtt{F} \in L^\infty(\mathtt{B}_1)$ *and* $\mathtt{G} \in L^\infty(\mathtt{B}_1)$.

(i) For any $\gamma \in (0, +\infty)$,

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/4})} \leq C\|\mathtt{U}\|_{L^2(\mathtt{B}_{3/4})} + C\sqrt{\gamma}\|\mathtt{F}\|_{L^\infty(\mathtt{B}_{3/4})} + C\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{3/4})},$$

where the constant C *depends only on* λ, Λ, n *and* $\|A^{ij}\|_{L^\infty(\mathtt{B}_1)}$.

(ii) If, in addition, $A^{ij} \in C^\alpha(\mathtt{B}_1)$ $\mathtt{F} \in C^\alpha(\mathtt{B}_1)$, *then for any* $\gamma \in (0, +\infty)$,

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/4})} \leq C\|\mathtt{U}\|_{L^2(\mathtt{B}_{1/2})} + C\gamma^{\frac{1+\alpha}{2}}[\mathtt{F}]_{C^\alpha(\mathtt{B}_{1/2})} + C\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{1/2})},$$

and

$$\|D\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/4})} \le C\|\mathtt{U}\|_{L^2(\mathtt{B}_{3/4})} + C\gamma^{\frac{\alpha}{2}}[\mathtt{F}]_{C^\alpha(\mathtt{B}_{3/4})} + C\gamma^{1/2}\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{3/4})},$$

where C depends only on $\lambda, \Lambda, n,$ and $\|A^{ij}\|_{C^\alpha(\mathtt{B}_1)}$.

Combining Corollary 3.2 (i) with a standard extension argument, we obtain the following estimate near the boundary.

Corollary 3.3. *For $n \ge 2$, let $\mathtt{U} \in H^1(\mathtt{B}_1^+)$ satisfy*

$$\begin{cases} \sum_{i,j=1}^{n-1} \partial_i(A^{ij}\partial_j \mathtt{U}) - \frac{1}{\gamma}\mathtt{U} = \sum_{i=1}^{n-1} \partial_i \mathtt{F}^i + \mathtt{G} & \text{in} \quad \mathtt{B}_1^+ \subset \mathbb{R}^{n-1}, \\ \mathtt{U} = 0 & \text{on} \quad \Gamma_1^0 := \mathtt{B}_1 \cap \{x_{n-1} = 0\}, \end{cases}$$

with $A^{ij} \in L^\infty(\mathtt{B}_1^+)$ satisfying (1.10) in $\mathtt{B}_1^+$. Assume $\mathtt{F}, \mathtt{G} \in L^\infty(\mathtt{B}_1^+)$. Then for $\gamma \in (0, +\infty)$,

$$\|\mathtt{U}\|_{L^\infty(\mathtt{B}_{1/4}^+)} \le C\|\mathtt{U}\|_{L^2(\mathtt{B}_{3/4}^+)} + C\sqrt{\gamma}\|\mathtt{F}\|_{L^\infty(\mathtt{B}_{3/4}^+)} + C\gamma\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{3/4}^+)},$$

where C depends only on λ, Λ, n and $\|A^{ij}\|_{L^\infty(\mathtt{B}_1)}$.

3.2 Pointwise estimates for degenerate equations

For $l, s > 0$, $\tilde{\varepsilon} \in (0, \frac{1}{4})$, we define the weighted norm

$$\|\mathtt{F}\|_{l,\mathtt{B}_s} := \sup_{x' \in \mathtt{B}_s} \frac{|\mathtt{F}(x')|}{(\tilde{\varepsilon} + |x'|^2)^l}.$$

Proposition 3.4. *Let $n \ge 2$, $\tilde{\lambda}, \tilde{\Lambda} > 0$, and suppose that $v \in H^1(\mathtt{B}_1)$ satisfies*

$$\sum_{i=1}^{n-1} \partial_i\Big(\big(\tilde{\varepsilon} + t(x')\big)\partial_i v(x')\Big) - \frac{1}{\gamma}v(x') = \sum_{i=1}^{n-1} \partial_i \mathtt{F}^i + \mathtt{G} \quad \text{for } x' \in \mathtt{B}_1 \subset \mathbb{R}^{n-1}, \tag{3.13}$$

where $t \in C^1(\mathtt{B}_1)$ satisfies

$$\tilde{\lambda}|x'|^2 \le t(x') \le \tilde{\Lambda}|x'|^2 \quad \text{and } |Dt(x')| \le 2\tilde{\Lambda}|x'|. \tag{3.14}$$

Assume $\mathtt{F}, G \in L^\infty(\mathtt{B}_1)$. Then, for any $\sigma > 0$, if we define

$$\tilde{\gamma}_0 = \Big(\tilde{\Lambda}\sigma\big(n + 1 + 2(\frac{\sigma}{2} - 1)_+\big)\Big)^{-1}, \tag{3.15}$$

the estimate

$$|v(x')| \le C(\tilde{\varepsilon} + |x'|^2)^{\sigma/2}\left(\|v\|_{L^\infty(\partial\mathtt{B}_1)} + \sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_1} + \gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_1}\right)$$

holds whenever $0 < \gamma < \tilde{\gamma}_0$. The constant $C > 0$ depends only on n, $\tilde{\lambda}$, $\tilde{\Lambda}$, σ and a lower bound of $\tilde{\gamma}_0 - \gamma$.

For notational simplicity, we define the operator

$$\mathcal{L}_\gamma v := \sum_{i=1}^{n-1} \partial_i\Big(\big(\tilde{\varepsilon} + t(x')\big)\partial_i v(x')\Big) - \frac{1}{\gamma}v(x').$$

The proof relies on a comparison principle with a barrier function that captures the decay near the origin. We decompose the solution into a homogeneous part, controlled via the maximum principle, and an inhomogeneous part, estimated via energy methods and a blow-up argument.

Specifically, for $0 < r < 1$, we write

$$v := v_1 + v_2, \quad \text{in } \mathtt{B}_r,$$

where $v_1 \in H^1(\mathtt{B}_r)$ satisfies the homogeneous equation

$$\begin{cases} \mathcal{L}_\gamma v_1 = 0 & \text{in } \mathtt{B}_r, \\ \quad v_1 = v & \text{on } \partial\mathtt{B}_r, \end{cases}$$

and $v_2 \in H_0^1(\mathtt{B}_r)$ solves the inhomogeneous problem

$$\begin{cases} \mathcal{L}_\gamma v_2 = \displaystyle\sum_{i=1}^{n-1} \partial_i \mathtt{F}^i + \mathtt{G} & \text{in } \mathtt{B}_r, \\ \quad v_2 = 0 & \text{on } \partial\mathtt{B}_r. \end{cases}$$

We need the following two lemmas.

Lemma 3.5. *For any $\tilde{\sigma} > 0$, let $\gamma(\tilde{\sigma})$ be defined as*

$$\gamma(\tilde{\sigma}) := \frac{1}{2\tilde{\Lambda}\tilde{\sigma}(n+1) + 4\tilde{\Lambda}\tilde{\sigma}(\tilde{\sigma}-1)_+} \tag{3.16}$$

Then for any $\gamma \in (0, \gamma(\tilde{\sigma})]$ and $r \in (0,1)$,

$$|v_1(x')| \le \Big(\frac{\tilde{\varepsilon} + \tilde{\Lambda}|x'|^2}{\tilde{\varepsilon} + \tilde{\Lambda} r^2}\Big)^{\tilde{\sigma}} \|v_1\|_{L^\infty(\partial\mathtt{B}_r)}.$$

Proof. A direct computation shows that for $\tilde{\eta}(x') = \tilde{\varepsilon}/\tilde{\Lambda} + |x'|^2$ and any $\tilde{\sigma} > 0$,

$$\partial_i \tilde{\eta}^{\tilde{\sigma}} = 2\tilde{\sigma}\tilde{\eta}^{\tilde{\sigma}-1} x_i, \quad \partial_{ij}\tilde{\eta}^{\tilde{\sigma}} = 4\tilde{\sigma}(\tilde{\sigma}-1) x_i x_j \tilde{\eta}^{\tilde{\sigma}-2} + 2\tilde{\sigma}\tilde{\eta}^{\tilde{\sigma}-1}\delta_i^j,$$

where δ_i^j is the Kronecker symbol. Using the assumptions on t (3.14), we obtain

$$\begin{aligned} \mathcal{L}_\gamma \tilde{\eta}^{\tilde{\sigma}} &= \sum_{i=1}^{n-1} \partial_i\big(\tilde{\varepsilon} + t(x')\big)\partial_i \tilde{\eta}(x')^{\tilde{\sigma}} + \big(\tilde{\varepsilon} + t(x')\big)\Delta\tilde{\eta}(x')^{\tilde{\sigma}} - \frac{1}{\gamma}\tilde{\eta}(x')^{\tilde{\sigma}} \\ &\le 4\tilde{\Lambda}\tilde{\sigma}\tilde{\eta}(x')^{\tilde{\sigma}} + 4\tilde{\Lambda}\tilde{\sigma}(\tilde{\sigma}-1)_+\tilde{\eta}(x')^{\tilde{\sigma}} + 2\tilde{\Lambda}(n-1)\tilde{\sigma}\tilde{\eta}(x')^{\tilde{\sigma}} - \frac{1}{\gamma}\tilde{\eta}(x')^{\tilde{\sigma}} \\ &= \Big(2\tilde{\Lambda}\tilde{\sigma}(n+1) + 4\tilde{\Lambda}\tilde{\sigma}(\tilde{\sigma}-1)_+ - \frac{1}{\gamma}\Big)\tilde{\eta}(x')^{\tilde{\sigma}}. \end{aligned}$$

Therefore, for all $\gamma \in (0, \gamma(\tilde{\sigma})]$, we have $\mathcal{L}_\gamma \tilde{\eta}^{\tilde{\sigma}} \le 0$.

Thus, for any $0 < r < 1$,

$$\begin{cases} \mathcal{L}_\gamma\Big(v_1 + \dfrac{\tilde{\eta}(x')^{\tilde{\sigma}}}{(\tilde{\varepsilon}/\tilde{\Lambda} + r^2)^{\tilde{\sigma}}}\|v_1\|_{L^\infty(\partial\mathtt{B}_r)}\Big) \le 0 & \text{in } \mathtt{B}_r, \\ \qquad v_1 + \dfrac{\tilde{\eta}(x')^{\tilde{\sigma}}}{(\tilde{\varepsilon}/\tilde{\Lambda} + r^2)^{\tilde{\sigma}}}\|v_1\|_{L^\infty(\partial\mathtt{B}_r)} \ge 0 & \text{on } \partial\mathtt{B}_r, \end{cases}$$

and

$$\begin{cases} \mathcal{L}_\gamma\Big(v_1 - \dfrac{\tilde{\eta}(x')^{\tilde{\sigma}}}{(\tilde{\varepsilon}/\tilde{\Lambda} + r^2)^{\tilde{\sigma}}}\|v_1\|_{L^\infty(\partial\mathtt{B}_r)}\Big) \ge 0 & \text{in } \mathtt{B}_r, \\ \qquad v_1 - \dfrac{\tilde{\eta}(x')^{\tilde{\sigma}}}{(\tilde{\varepsilon}/\tilde{\Lambda} + r^2)^{\tilde{\sigma}}}\|v_1\|_{L^\infty(\partial\mathtt{B}_r)} \le 0 & \text{on } \partial\mathtt{B}_r. \end{cases}$$

The result follows by applying the maximum principle to $v_1 \pm \frac{\tilde{\eta}(x')^{\tilde{\sigma}}}{(\tilde{\varepsilon}/\tilde{\Lambda}+r^2)^{\tilde{\sigma}}}\|v_1\|_{L^\infty(\partial\mathtt{B}_r)}$. $\square$

Lemma 3.6. *Suppose that* $\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}<\infty$. *Let* $\tilde{\gamma}_0$ *be defined as* (3.15). *Then for any* $\gamma\in(0,\tilde{\gamma}_0)$ *and* $r\in(0,1)$,

$$\|v_2\|_{L^\infty(\mathtt{B}_r)}\leq C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)(\tilde{\varepsilon}+r^2)^{\sigma/2}.$$

where C *depends only on* $\tilde{\lambda}$, $\tilde{\Lambda}$, n *and* σ, *and is independent of* $\tilde{\varepsilon}$ *and* γ.

Proof. **Step 1.** We show that for $\rho\in(\sqrt{\tilde{\varepsilon}},r]$,

$$\|v_2\|_{L^\infty(\mathtt{B}_\rho\setminus\mathtt{B}_{\rho/8})}\leq\|v_2\|_{L^\infty(\partial\mathtt{B}_\rho)}+C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)(\tilde{\varepsilon}+\rho^2)^{\sigma/2}. \tag{3.17}$$

To prove this, we decompose

$$v_2:=v_{21}+v_{22}\quad\text{in }\mathtt{B}_\rho,$$

where

$$\begin{cases}\mathcal{L}_\gamma v_{21}=0 & \text{in }\mathtt{B}_\rho,\\ v_{21}=v_2 & \text{on }\partial\mathtt{B}_\rho,\end{cases}\quad\text{and}\quad\begin{cases}\mathcal{L}_\gamma v_{22}=\displaystyle\sum_{i=1}^{n-1}\partial_i\mathtt{F}^i+\mathtt{G} & \text{in }\mathtt{B}_\rho,\\ v_{22}=0 & \text{on }\partial\mathtt{B}_\rho.\end{cases}$$

The claim then follows from the maximum principle and a rescaling argument.

First, the maximum principle gives

$$\|v_{21}\|_{L^\infty(\mathtt{B}_\rho)}\leq\|v_2\|_{L^\infty(\partial\mathtt{B}_\rho)}. \tag{3.18}$$

For v_{22}, we rescale by setting $x'=\rho y'$ for $y'\in\mathtt{B}_1\subset\mathbb{R}^{n-1}$ and define

$$\tilde{v}_{22}(y')=v_{22}(\rho y'),\quad\hat{\varepsilon}=\frac{\tilde{\varepsilon}}{\rho^2},\quad\tilde{t}(y')=\frac{1}{\rho^2}t(\rho y'),\quad\tilde{\mathtt{F}}(y')=\frac{1}{\rho}\mathtt{F}(\rho y'),\quad\text{and }\tilde{\mathtt{G}}(y')=\mathtt{G}(\rho y').$$

Then $\tilde{v}_{22}\in H_0^1(\mathtt{B}_1)$ satisfies

$$\sum_{i=1}^{n-1}\partial_i\Big((\hat{\varepsilon}+\tilde{t}(y'))\partial_i\tilde{v}_{22}\Big)-\frac{1}{\gamma}\tilde{v}_{22}=\operatorname{div}\tilde{\mathtt{F}}+\tilde{\mathtt{G}}\quad\text{in }\mathtt{B}_1\subset\mathbb{R}^{n-1}.$$

Multiplying both sides by $\tilde{v}_{22}$ and integrating by parts gives

$$\int_{\mathtt{B}_1}(\hat{\varepsilon}+\tilde{t}(y'))|D\tilde{v}_{22}|^2+\frac{1}{\gamma}\tilde{v}_{22}^2dy'=\int_{\mathtt{B}_1}\tilde{\mathtt{F}}^i\partial_i\tilde{v}_{22}-\tilde{\mathtt{G}}\tilde{v}_{22}dy'.$$

By Hölder's inequality,

$$\int_{\mathtt{B}_1}\Big((\hat{\varepsilon}+\tilde{t}(y'))|D\tilde{v}_{22}|^2+\frac{1}{\gamma}\tilde{v}_{22}^2\Big)dy'\leq C\int_{\mathtt{B}_1}\frac{|\tilde{\mathtt{F}}|^2}{\hat{\varepsilon}+\tilde{t}(y')}+\gamma\tilde{\mathtt{G}}^2dy'.$$

Multiplying both sides by γ and using the bounds on t, we have

$$\|\tilde{v}_{22}\|_{L^2(\mathtt{B}_1)}\leq C\Big(\sqrt{\gamma}\Big\|\frac{\tilde{\mathtt{F}}}{\sqrt{\hat{\varepsilon}+|y'|^2}}\Big\|_{L^\infty(\mathtt{B}_1)}+\gamma\,\|\tilde{\mathtt{G}}\|_{L^\infty(\mathtt{B}_1)}\Big). \tag{3.19}$$

By using a flattening transform, and Corollary 3.3 and the corresponding interior estimates on a standard ball, together with (3.19), we obtain

$$\begin{aligned}\|\tilde{v}_{22}\|_{L^\infty(\mathtt{B}_1\setminus\mathtt{B}_{1/8})}&\leq C\|\tilde{v}_{22}\|_{L^2(\mathtt{B}_1\setminus\mathtt{B}_{1/16})}+C\big(\sqrt{\gamma}\|\tilde{\mathtt{F}}\|_{L^\infty(\mathtt{B}_1\setminus\mathtt{B}_{1/16})}+\gamma\|\tilde{\mathtt{G}}\|_{L^\infty(\mathtt{B}_1\setminus\mathtt{B}_{1/16})}\big)\\&\leq C\Big(\sqrt{\gamma}\Big\|\frac{\tilde{\mathtt{F}}}{\sqrt{\hat{\varepsilon}+|y'|^2}}\Big\|_{L^\infty(\mathtt{B}_1)}+\gamma\,\|\tilde{\mathtt{G}}\|_{L^\infty(\mathtt{B}_1)}\Big).\end{aligned}$$

Returning to v_{22}, we have

$$\|v_{22}\|_{L^\infty(\mathtt{B}_\rho\setminus\mathtt{B}_{\rho/8})}\le C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)(\tilde\varepsilon+\rho^2)^{\sigma/2}.$$

Combining this with (3.18) proves (3.17).

Step 2. Using a covering argument and iteration, we prove

$$\|v_2\|_{L^\infty(\mathtt{B}_r\setminus\mathtt{B}_{\sqrt{\tilde\varepsilon}}))}\le C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)(\tilde\varepsilon+r^2)^{\sigma/2}.$$

Since $v_2\in H_0^1(\mathtt{B}_r)$, (3.17) gives

$$\|v_2\|_{L^\infty(\mathtt{B}_r\setminus\mathtt{B}_{r/4})}\le C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)(\tilde\varepsilon+r^2)^{\sigma/2}. \tag{3.20}$$

From the classical theory for elliptic equations, $v_2\in C^\beta(\mathtt{B}_r)$ for some $\beta\in(0,1)$. For any $\rho\in(\sqrt{\tilde\varepsilon},\frac{r}{4})$, we choose $\rho_0=\rho$, $\rho_i=2^i\rho$, $i=1,2,\dots,k$, and k such that $\frac{r}{4}\le 2^{k-1}\rho_0<2^k\rho_0<r$. By (3.17),

$$\|v_2\|_{L^\infty(\partial\mathtt{B}_{\rho_i})}\le\|v_2\|_{L^\infty(\partial\mathtt{B}_{\rho_{i+1}})}+C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)\rho_{i+1}^\sigma. \tag{3.21}$$

Iterating (3.21) from $i=0$ to $k-1$, and using (3.20), we obtain

$$\begin{aligned}\|v_2\|_{L^\infty(\partial\mathtt{B}_{\rho_0})}&\le\|v_2\|_{L^\infty(\partial\mathtt{B}_{2^k\rho_0})}+C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)\sum_{i=1}^k(2^i\rho_0)^\sigma\\&\le Cr^\sigma\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big).\end{aligned}$$

Combining this with (3.20) gives

$$\|v_2\|_{L^\infty(\mathtt{B}_r\setminus\mathtt{B}_{\sqrt{\tilde\varepsilon}})}\le C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)(\tilde\varepsilon+r^2)^{\sigma/2}. \tag{3.22}$$

Step 3. We estimate v_2 in $\mathtt{B}_{\sqrt{\tilde\varepsilon}}$ via a similar decomposition and scaling.
Decompose

$$v_2:=v_{21}+v_{22}\quad\text{in }\mathtt{B}_{2\sqrt{\tilde\varepsilon}},$$

as in Step 1, with $\rho=2\sqrt{\tilde\varepsilon}$. For v_{21}, we apply the maximum principle. For v_{22}, we utilize Corollary 3.3, the corresponding interior estimates on a standard ball, and a scaling argument. By combining the estimates for v_{21} and v_{22}, we derive

$$\|v_2\|_{L^\infty(\mathtt{B}_{\sqrt{\tilde\varepsilon}})}\le\|v_2\|_{L^\infty(\partial\mathtt{B}_{2\sqrt{\tilde\varepsilon}})}+C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)\tilde\varepsilon^{\sigma/2}.$$

By using (3.22),

$$\|v_2\|_{L^\infty(\mathtt{B}_{\sqrt{\tilde\varepsilon}})}\le C\Big(\sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_r}+\gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_r}\Big)\tilde\varepsilon^{\sigma/2}.$$

This, together with (3.22), completes the proof. □

Proof of Proposition 3.4. Without loss of generality, we assume $\|v\|_{L^\infty(\partial\mathtt{B}_1)}+\sqrt{\gamma}\|\mathtt{F}\|_{\mathtt{B}_1,\frac{\sigma+1}{2}}+\gamma\|\mathtt{G}\|_{\mathtt{B}_1,\sigma/2}\le 1$.

Let $\tilde\gamma_0$ and $\gamma(\tilde\sigma)$ be defined as in (3.15) and (3.16), respectively. One can easily verify that $\tilde\gamma_0=\gamma(\frac{\sigma}{2})$ and that $\gamma(\tilde\sigma)$ is strictly decreasing with respect to $\tilde\sigma$. Therefore, for any $\gamma\in(0,\tilde\gamma_0)$, there exists $\bar\sigma>\frac{\sigma}{2}$ such that $\gamma\in(0,\gamma(\bar\sigma)]$.

Then by Lemmas 3.5 and 3.6, for $0<\rho<r<1$, we obtain

$$\|v\|_{L^\infty(\partial \mathtt{B}_\rho)} \le \|v_1\|_{L^\infty(\partial \mathtt{B}_\rho)} + \|v_2\|_{L^\infty(\partial \mathtt{B}_\rho)} \le \Big(\frac{\tilde\varepsilon+\tilde\Lambda\rho^2}{\tilde\varepsilon+\tilde\Lambda r^2}\Big)^{\bar\sigma}\|v\|_{L^\infty(\partial \mathtt{B}_r)} + C(\tilde\varepsilon+r^2)^{\sigma/2}. \tag{3.23}$$

For $\rho\in(\sqrt{\tilde\varepsilon},r)$, this implies

$$\|v\|_{L^\infty(\partial \mathtt{B}_\rho)} \le \Big(\frac{(\tilde\Lambda+1)\rho^2}{\tilde\Lambda r^2}\Big)^{\bar\sigma}\|v\|_{L^\infty(\partial \mathtt{B}_r)} + Cr^\sigma. \tag{3.24}$$

Fix $\mu\in(0,1)$ such that $\frac{\tilde\Lambda+1}{\tilde\Lambda}\mu^{2\bar\sigma}\le\frac12\mu^\sigma$. Iterating (3.24) along the scales $\rho=\mu^{i+1}$ and $r=\mu^i$ in (3.24) from $i=0$ to $k-1$, with $\rho_k:=\mu^k\in(\sqrt{\tilde\varepsilon},1)$, yields

$$\|v\|_{L^\infty(\partial \mathtt{B}_{\rho_k})} \le \frac{1}{2^k}\mu^{k\sigma}\|v\|_{L^\infty(\partial \mathtt{B}_1)} + C\sum_{i=1}^{k}(\frac12\mu^\sigma)^{i-1}(\mu^{k-i})^\sigma \le C\mu^{k\sigma}. \tag{3.25}$$

For $|x'|\in(\sqrt{\tilde\varepsilon},1)$, choose k such that $\mu^{k+1}<|x'|\le\mu^k$, then by (3.24) and (3.25),

$$|v(x')| \le \|v\|_{L^\infty(\partial \mathtt{B}_{|x'|})} \le C\|v\|_{L^\infty(\partial \mathtt{B}_{\mu^k})} + C\mu^{k\sigma} \le C|x'|^\sigma \le C(\tilde\varepsilon+|x'|^2)^{\sigma/2}. \tag{3.26}$$

For $x'\in\mathtt{B}_{\sqrt{\tilde\varepsilon}}$, we use (3.23) and (3.26) to obtain

$$\|v\|_{L^\infty(\mathtt{B}_{\sqrt{\tilde\varepsilon}})} \le C\sup_{|x'|\le\sqrt{\tilde\varepsilon}}\|v\|_{L^\infty(\partial \mathtt{B}_{|x'|})} \le \|v\|_{L^\infty(\partial \mathtt{B}_{2\sqrt{\tilde\varepsilon}})} + C\tilde\varepsilon^{\sigma/2} \le C\tilde\varepsilon^{\sigma/2}.$$

Combining these estimates completes the proof of Proposition 3.4. □

3.3 Pointwise gradient estimates for degenerate equations

Proposition 3.7. *Let $v\in H^1(\mathtt{B}_1)$ be a solution to (3.13), and let $t(x')$ satisfy (3.14). Suppose $\mathtt{F}\in C^\alpha(\mathtt{B}_1)$, $\mathtt{G}\in L^\infty(\mathtt{B}_1)$, and $\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_1}+\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_1}<\infty$. For any $\sigma>0$, let $\tilde\gamma_0$ be as defined in (3.15). Then, for all $x'\in\mathtt{B}_{\frac12}$, the estimate*

$$\begin{aligned}|Dv(x')| \le{}& C\tilde\eta(x')^{\frac{\sigma-1}{2}}\Big(\|v\|_{L^\infty(\partial\mathtt{B}_1)} + \sqrt{\gamma}\|\mathtt{F}\|_{\frac{\sigma+1}{2},\mathtt{B}_1} + \gamma\|\mathtt{G}\|_{\frac{\sigma}{2},\mathtt{B}_1}\Big)\\ &+ C\tilde\eta(x')^{-1+\frac{\alpha}{2}}\gamma^{\frac{\alpha}{2}}[\mathtt{F}]_{C^\alpha(\mathtt{B}_{\frac18\sqrt{\tilde\eta(x')}}(x'))} + C\gamma^{1/2}\tilde\eta(x')^{-1/2}\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{\frac18\sqrt{\tilde\eta(x')}}(x'))}\end{aligned}$$

holds for all $0<\gamma<\tilde\gamma_0$. Here, $\tilde\eta(x')=\tilde\varepsilon/\tilde\Lambda+|x'|^2$ and the constant C depends only on n, $\tilde\lambda$, $\tilde\Lambda$, σ and a lower bound of $\tilde\gamma_0-\gamma$.

Proof. For any $x_0'\in\mathtt{B}_{1/2}$, we use a change of variables in $\mathtt{B}_{\frac18\sqrt{\tilde\eta(x_0')}}(x_0)$ by setting

$$x'=x_0'+\frac18\sqrt{\tilde\eta(x_0')}y',\quad y'\in\mathtt{B}_1,$$

and set $\tilde v(y)=v(x_0'+\frac18\sqrt{\tilde\eta(x_0')}y')$. Then $\tilde v(y')$ solves

$$\sum_{i=1}^{n-1}\partial_i(\tilde a^{ii}\partial_i\tilde v)-\frac1\gamma\tilde v=\sum_{i=1}^{n-1}\partial_i\tilde{\mathtt{F}}^i+\tilde{\mathtt{G}}\quad\text{in }\mathtt{B}_1,$$

where $\tilde a^{ii}(y')=\frac{64}{\tilde\eta(x_0')}\delta(x_0'+\frac18\sqrt{\tilde\eta(x_0')}y')$, and

$$\tilde{\mathtt{F}}^i(y')=\frac{8}{\sqrt{\tilde\eta(x_0')}}\mathtt{F}^i(x_0'+\frac18\sqrt{\tilde\eta(x_0')}y'),\quad \tilde{\mathtt{G}}(y')=\mathtt{G}(x_0'+\frac18\sqrt{\tilde\eta(x_0')}y').$$

Applying Corollary 3.2 (ii) to $\tilde{v}$ and rescaling back, we have

$$\begin{aligned}|Dv(x_0')| \le\, & C\tilde{\eta}(x_0')^{-1/2}\|v\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\tilde{\eta}(x_0')}}(x_0'))}\\ & + C\tilde{\eta}(x_0')^{-1+\frac{\alpha}{2}}\gamma^{\frac{\alpha}{2}}[\mathtt{F}]_{C^\alpha(\mathtt{B}_{\frac{1}{8}\sqrt{\tilde{\eta}(x_0')}}(x_0'))} + C\gamma^{1/2}\tilde{\eta}(x_0')^{-1/2}\|\mathtt{G}\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\tilde{\eta}(x_0')}}(x_0'))}.\end{aligned}$$

Combining with Proposition 3.4, we complete the proof of Proposition 3.7. $\square$

4 Gradient estimates in the narrow region

In this section, we establish gradient estimates for solutions to the nonhomogeneous equation in the narrow region Ω_{2R}, subject to nonvanishing boundary conditions:

$$\begin{cases} \Delta w = \mathfrak{R}(x) & \text{in } \Omega_{2R},\\ w + \gamma\partial_\nu w = \Psi_1(x') & \text{on } \Gamma_{1,2R},\\ w + \gamma\partial_\nu w = \Psi_2(x') & \text{on } \Gamma_{2,2R}, \end{cases} \tag{4.1}$$

where ν is the unit normal vector, pointing upward on $\Gamma_{1,2R}$ and downward on $\Gamma_{2,2R}$. The boundary profiles f_1 and f_2 satisfy assumptions (1.5)–(1.6). We show that under suitable growth conditions on the right-hand side and boundary data, the gradient remains bounded by a constant independent of both ε and γ.

The proof of gradient boundedness proceeds via a case analysis based on the conductivity regime. In the low-conductivity regime $\gamma \le \mu\eta(x')$, we combine an iterative estimate for w with a rescaled version of Theorem 1.4, directly yielding a gradient bound. In the high-conductivity regime ($\gamma > \mu\eta(x')$), a more refined approach is adopted. Inspired by studies of Neumann boundary conditions [16], we employ a dimension reduction technique, examining the vertical average $\bar{w}$ of w. This average satisfies an elliptic equation with degenerate principal coefficients and lower-order terms of order $1/\gamma$ for small γ. The regularity theory from Section 3 then provides a gradient estimate for $\bar{w}$ in this context. The final boundedness of Dw in Ω_R is achieved by synthesizing these case-specific estimates. This systematic division into regimes, along with the tailored methods applied in each case, yields the global gradient bound. Here, μ is a fixed large constant, to be specified in Subsection 4.3.

To streamline notation, we introduce a weighted Hölder norm. For a domain $\mathcal{D} \subset \mathbb{R}^n$ with diameter $d = \operatorname{diam}\mathcal{D}$, define

$$\begin{aligned}\|\Psi\|^*_{C^\alpha(\mathcal{D})} &:= \|\Psi\|_{L^\infty(\mathcal{D})} + d^\alpha[\Psi]_{C^\alpha(\mathcal{D})},\\ \|\Psi\|^*_{C^{1,\alpha}(\mathcal{D})} &:= \|\Psi\|_{L^\infty(\mathcal{D})} + d\,\|\nabla\Psi\|_{L^\infty(\mathcal{D})} + d^{1+\alpha}\sup_{1\le i\le n}[\partial_i\Psi]_{C^\alpha(\mathcal{D})}.\end{aligned}$$

We recall several facts and fix notation. Let $\Omega_r(x_0)$ be as in (1.7), and set $\eta(x') := \varepsilon + |x'|^2$ and $\delta(x') := \varepsilon + f_1(x') - f_2(x')$. Under assumptions (1.5)–(1.6), we have $\eta(x') \sim \delta(x')$ and

$$|Df_1(x')| + |Df_2(x')| \le C\eta(x')^{1/2}$$

for all $x \in \Omega_R$. We may assume without loss of generality that $R \le 1/32$. Under these conditions, for $\varepsilon \le \frac{1}{4}R^2$, the following inclusions and comparisons hold:

— $\Omega_{\eta(x')}(x) \subset \Omega_R$ for all $x \in \Omega_{\frac{3}{4}R}$;
— $\Omega_{\eta(x')}(x) \subset \Omega_{\frac{1}{16}\sqrt{\eta(x')}}(x)$ for all $x \in \Omega_{\frac{3}{4}R}$;
— $\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x) \subset \Omega_{\frac{3}{4}R}$ for all $x \in \Omega_{\frac{R}{2}}$;
— for any $x \in \Omega_R$ and $x_0 \in \Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x)$, we have $\eta(x') \sim \eta(x_0')$.

Theorem 4.1. *Let $w \in H^1(\Omega_{2R})$ be a solution to* (4.1) *with boundary profiles f_1 and f_2 satisfying* (1.6)*. Assume the normalization conditions :*

$$\|w\|_{L^\infty(\Omega_{2R})} \le 1,$$

and

$$|\mathfrak{R}(x_0)| \le \frac{1}{\gamma + \delta(x_0')}, \quad \|\Psi_i\|^*_{C^{1,\alpha}(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \le \frac{\gamma\delta(x_0')}{\gamma + \delta(x_0')}, \quad i = 1, 2, \ \forall x_0 \in \Omega_R. \tag{4.2}$$

Let γ_0 be given by (1.8)*. Then, for any $\varepsilon \in (0, \varepsilon_0)$ and $\gamma \in (0, \gamma_0)$, there holds*

$$\|Dw\|_{L^\infty(\Omega_{R/2})} \le C. \tag{4.3}$$

Here, $\varepsilon_0 > 0$ and $C > 0$ depend only on n, R, α, κ, a lower bound of $\gamma_0 - \gamma$ and upper bounds of $\|f_1\|_{C^{2,\alpha}}$ and $\|f_2\|_{C^{2,\alpha}}$.

Remark 4.2. Under the hypotheses of Theorem 4.1, for a sufficiently large constant μ, the quantity $\eta(x_0')^{-1}\sum_{i=1}^2 \|\Psi_i\|^*_{C^{1,\alpha}(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + \eta(x_0')\|\mathfrak{R}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))}$ on the right-hand side of (4.8) below remains bounded when $\gamma \le \mu\eta(x_0')$. Conversely, when $\gamma > \mu\eta(x_0')$, the expression $\frac{1}{\gamma}\sum_{i=1}^2 \|\Psi_i\|^*_{C^\alpha(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + \eta(x_0')\|\mathfrak{R}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \sim \frac{\eta(x_0')}{\gamma}$ on the right-hand side of (4.20) is much smaller than 1. This dichotomy is essential for establishing the uniform gradient bound in Theorem 4.1.

In the next two subsections, we flatten the boundary via the coordinate transformation

$$\begin{cases} y' = x', \\ y_n = 2\eta(x_0')\Big(\dfrac{x_n - f_2(x')}{\varepsilon + f_1(x') - f_2(x')} - \dfrac{1}{2}\Big), \end{cases} \tag{4.4}$$

which maps the domain $\Omega_s(x_0)$ to the cylinder $Q_{s,\eta(x_0')}(x_0)$ of height $\eta(x_0')$, where

$$Q_{s,t}(x_0) := \{y = (y', y_n) \in \mathbb{R}^n \ \big| \ |y' - x_0'| < s, \ |y_n| < t \ \}.$$

Denote the upper and lower boundaries of the cylinder by $\Gamma^\pm_{s,t}(x_0)$. If w is a solution to (4.1), then the function $\tilde{w}(y) = w(x)$ satisfies

$$\begin{cases} \partial_i(a^{ij}(y)\partial_j\tilde{w}(y)) = \tilde{\mathfrak{R}}(y) & \text{in } Q_{s,\eta(x_0')}(x_0), \\ h_1(y')\tilde{w}(y) + \gamma a^{nj}(y)\partial_j\tilde{w}(y) = \tilde{\Psi}_1(y') & \text{on } \Gamma^+_{s,\eta(x_0')}(x_0), \\ h_2(y')\tilde{w}(y) - \gamma a^{nj}(y)\partial_j\tilde{w}(y) = \tilde{\Psi}_2(y') & \text{on } \Gamma^-_{s,\eta(x_0')}(x_0), \end{cases} \tag{4.5}$$

where the coefficients and data are given by:

$$a^{ij}(y) = \sum_{k=1}^n \frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k}(\det \partial_x y)^{-1}, \quad \tilde{\mathfrak{R}}(y) = \mathfrak{R}(x)(\det \partial_x y)^{-1},$$

$$h_i(y') = \sqrt{1 + |\nabla_{y'} f_i(y')|^2}, \quad \text{and } \tilde{\Psi}_i(y') = \sqrt{1 + |\nabla_{y'} f_i(y')|^2}\,\Psi_i(y') \text{ for } i = 1, 2.$$

The coefficients $a^{ij}(y)$ can be computed explicitly as:

$$a^{ii}(y) = \frac{\delta(y')}{2\eta(x_0')} \quad \text{for } 1 \le i \le n-1; \qquad a^{ij} = 0 \quad \text{for } 1 \le i, j \le n-1,\ i \ne j;$$

$$a^{in}(y) = a^{ni}(y) = \partial_i f_2(y') \frac{y_n - \eta(x_0')}{2\eta(x_0')} - \partial_i f_1(y') \frac{y_n + \eta(x_0')}{2\eta(x_0')} \quad \text{for } 1 \le i \le n-1;$$

$$a^{nn}(y) = \frac{2\eta(x_0')}{\delta(y')} + \sum_{i=1}^{n-1} \frac{((\partial_i f_2)(y_n - \eta(x_0')) - (\partial_i f_1)(y_n + \eta(x_0')))^2}{2\eta(x_0')\delta(y')}.$$

A direct computation shows that for $y \in Q_{\frac{1}{4}\sqrt{\eta(x_0')}, \eta(x_0')}(x_0)$:

$$\begin{gathered} \frac{1}{C} \le a^{ii}(y) \le C \text{ for } 1 \le i \le n; \quad |a^{in}(y)| = |a^{ni}(y)| \le C\eta(x_0')^{1/2} \text{ for } 1 \le i \le n-1; \\ |D\tilde{w}(y)| \sim |Dw(x)|; \qquad |\tilde{\mathfrak{R}}(y)| \sim |\mathfrak{R}(x)|; \end{gathered} \tag{4.6}$$

and

$$\begin{aligned} &[a^{ij}]_{C^\alpha(Q_{\frac{1}{4}\sqrt{\eta(x_0')}, \eta(x_0')}(x_0))} \le C\eta(x_0')^{-\alpha/2}; \\ &\|h_i\|_{C^\alpha(Q_{\frac{1}{4}\sqrt{\eta(x_0')}, \eta(x_0')}(x_0))} \le C \quad \text{for } i = 1, 2; \\ &\|\Psi_i\|^*_{C^{1,\alpha}(\Omega_{s,\eta(x_0')}(x_0))} \sim \|\tilde{\Psi}_i\|^*_{C^{1,\alpha}(Q_{s,\eta(x_0')}(x_0))} \quad \text{for } 0 < s < \frac{1}{4}\sqrt{\eta(x_0')}. \end{aligned} \tag{4.7}$$

For further details, we refer to Page 20 of [18].

4.1 Proof of Theorem 4.1 for the case $\gamma \le \mu\eta(x')$.

The proof in this regime relies on the following two lemmas. For any domain $\mathcal{D}$, we denote the averaged L^p norm of u over $\mathcal{D}$ by

$$\|u\|_{L^p(\mathcal{D})} := \left(\fint_{\mathcal{D}} |u|^p \right)^{1/p}.$$

Lemma 4.3. *Let $w \in H^1(\Omega_{\eta(x_0')}(x_0))$ be a solution to the boundary value problem*

$$\begin{cases} \Delta w = \mathfrak{R}(x) & \text{in } \Omega_{\eta(x_0')}(x_0), \\ w + \gamma\partial_\nu w = \Psi_1(x) & \text{on } \Gamma_{1,\eta(x_0')}(x_0), \\ w + \gamma\partial_\nu w = \Psi_2(x) & \text{on } \Gamma_{2,\eta(x_0')}(x_0), \end{cases}$$

with $\gamma > 0$ and f_1, f_2 satisfying (1.5)–(1.6). Assume $\mathfrak{R} \in L^\infty(\Omega_{\eta(x_0')}(x_0))$ and $\Psi_i \in C^{1,\alpha}(\Omega_{\eta(x_0')}(x_0))$, $i = 1, 2$. Then

$$\begin{aligned} \|Dw\|_{L^\infty(\Omega_{\frac{1}{2}\eta(x_0')}(x_0))} \le C\Big(&\eta(x_0')^{-1} \|w\|_{L^2(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + \eta(x_0')^{-1} \sum_{i=1}^{2} \|\Psi_i\|^*_{C^{1,\alpha}(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \\ &+ \eta(x_0')\|\mathfrak{R}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))}\Big), \end{aligned} \tag{4.8}$$

where the constant C depends only on $n, \alpha, \kappa, \|f_1\|_{C^{2,\alpha}(B_R')}$, and $\|f_2\|_{C^{2,\alpha}(B_R')}$.

Proof. We flatten the domain via the change of variables given in (4.4). Let $\tilde{w}(y) = w(x)$. Then $\tilde{w}$ satisfies (4.5). Introduce the rescaled variables

$$\begin{cases} y' = x_0' + \eta(x_0')z', \\ y_n = \eta(x_0')z_n, \end{cases}$$

which maps $Q_{\eta(x_0'),\eta(x_0')}(x_0)$ onto $Q_{1,1}(0)$. The rescaled function $\hat{w}(z) = \tilde{w}(y)$ satisfies

$$\begin{cases} \partial_i(\hat{a}^{ij}(z)\partial_j \hat{w}(z)) = \hat{\mathfrak{R}}(z) & \text{in } Q_{1,1}(0), \\ \hat{h}_1(z)\hat{w}(z) + \hat{\gamma}\hat{a}^{nj}(z)\partial_j\hat{w}(z) = \hat{\Psi}_1(z') & \text{on } \Gamma^+_{1,1}(0), \\ \hat{h}_2(z)\hat{w}(z) - \hat{\gamma}\hat{a}^{nj}(z)\partial_j\hat{w}(z) = \hat{\Psi}_2(z') & \text{on } \Gamma^-_{1,1}(0), \end{cases}$$

where

$$\hat{a}^{ij}(z') = a^{ij}(y'), \quad \hat{\gamma} = \frac{\gamma}{\eta(x_0')}, \quad \hat{h}_i(z') = h_i(y') \text{ for } i = 1,2,$$
$$\hat{\mathfrak{R}}(z) = \eta(x_0')^2\tilde{\mathfrak{R}}(y), \qquad \text{and } \hat{\Psi}_i(z') = \tilde{\Psi}_i(y') \text{ for } i = 1,2.$$

For any $z_0 \in \Gamma^{\pm}_{\frac{1}{2},1}(0)$, applying the uniform estimate from Theorem 1.4 to $\hat{w}$ in $B_{\frac{1}{4}}(z_0) \cap Q_{1,1}(0)$ and combining it with standard interior estimates for elliptic equations, we obtain

$$\|\nabla\hat{w}\|_{L^\infty(Q_{\frac{1}{2},1}(0))} \le C\|\hat{w}\|_{L^2(Q_{\frac{3}{4},1}(0))} + C\|\hat{\mathfrak{R}}\|_{L^\infty(Q_{\frac{3}{4},1}(0))} + \sum_{i=1}^{2}\|\hat{\Psi}_i\|_{C^{1,\alpha}(Q_{\frac{3}{4},1}(0))}.$$

Rescaling back to $\tilde{w}$ yields

$$\begin{aligned} \eta(x_0')\|\nabla\tilde{w}\|_{L^\infty(Q_{\eta(x_0')/2,\eta(x_0')}(x_0))} &\le C \nparallel\tilde{w}\|_{L^2(Q_{3\eta(x_0')/4,\eta(x_0')}(x_0))} \\ &\quad + C\eta(x_0')^2\|\tilde{\mathfrak{R}}\|_{L^\infty(Q_{3\eta(x_0')/4,\eta(x_0')}(x_0))} + \sum_{i=1}^{2}\|\tilde{\Psi}_i\|^*_{C^{1,\alpha}(Q_{(Q_{3\eta(x_0')/4,\eta(x_0')}(x_0))}(x_0))}. \end{aligned}$$

Returning to w and using (4.6) and (4.7), we obtain the desired estimate (4.8). $\square$

Lemma 4.4. *Under the assumptions of Theorem 4.1, let $x_0 \in \Omega_R$. If $\gamma \le \mu\eta(x_0')$ for some constant $\mu > 0$, then the following estimates hold*

$$\begin{aligned} &\nparallel w\|_{L^2(\Omega_{\eta(x_0')}(x_0))} \le C\eta(x_0')\Big(\int_{\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)} w^2\,dx\Big)^{1/2} + C\eta(x_0'), \\ &\textit{and} \\ &\|\nabla w\|_{L^\infty(\Omega_{\eta(x_0')/2}(x_0))} \le C, \end{aligned} \tag{4.9}$$

where C depends on n, κ, α, $\|f_1\|_{C^{2,\alpha}}$ and $\|f_2\|_{C^{2,\alpha}}$, and the upper bound of μ, but is independent of γ and ε.

Proof. Multiply the equation in (4.1) by $w\xi^2$ and integrate by parts, where the cut-off function $\xi \in C^\infty(\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0))$ satisfies

$$\xi = 1 \text{ in } \Omega_s(x_0), \quad \xi = 0 \text{ in } \Omega_t(x_0)\backslash\Omega_s(x_0),\ 0 < t < s < \frac{1}{16}\sqrt{\eta(x_0')}, \quad \text{and } |D\xi| \le \frac{C}{t-s}.$$

This yields

$$\begin{aligned}\int_{\Omega_t(x_0)} |Dw|^2\xi^2\,dx + \frac{1}{\gamma}\int_{\Gamma_{1,t}(x_0)\cup\Gamma_{2,t}(x_0)} w^2\xi^2 dS \\ = \int_{\Omega_t(x_0)} \Big(-\mathfrak{R}w\xi^2 - 2w\xi DwD\xi\Big)\,dx + \frac{1}{\gamma}\sum_{i=1}^{2}\int_{\Gamma_{i,t}(x_0)} \Psi_i w\xi^2 dS.\end{aligned}$$

Applying Young's inequality to the right–hand side gives

$$\begin{aligned}&\int_{\Omega_t(x_0)} |Dw|^2\xi^2\,dx + \frac{1}{\gamma}\int_{\Gamma_{1,t}(x_0)\cup\Gamma_{2,t}(x_0)} w^2\xi^2 dS \\ &\le C\left(\int_{\Omega_t(x_0)} w^2|D\xi|^2 + |\mathfrak{R}w\xi^2|\,dx + \frac{1}{\gamma}\sum_{i=1}^{2}\int_{\Gamma_{i,t}(x_0)} \Psi_i^2\xi^2 dS\right).\end{aligned} \tag{4.10}$$

Define the boundary trace

$$w_{\Gamma_1}(x) = w(x',\varepsilon + f_1(x')), \quad w_{\Gamma_2}(x) = w(x', f_2(x')). \tag{4.11}$$

For $0 < s < t < \frac{1}{16}\sqrt{\eta(x')}$, using (4.10) and the assumption $\gamma \le \mu\eta(x_0')$, we obtain

$$\begin{aligned}\int_{\Omega_s(x_0)} |w|^2\,dx &\le C\int_{\Omega_s(x_0)} |w - w_{\Gamma_2}|^2\,dx + C\int_{\Omega_s(x_0)} |w_{\Gamma_2}|^2\,dx \\ &\le C\eta(x_0')^2\int_{\Omega_s(x_0)} |Dw|^2\,dx + C\eta(x_0')\int_{\Gamma_{2,s}(x_0)} w^2 dS \\ &\le C\Big(\eta(x_0')^2 + \eta(x_0')\gamma\Big)\left(\int_{\Omega_s(x_0)} |Dw|^2\,dx + \frac{1}{\gamma}\int_{\Gamma_{2,s}(x_0)} w^2 dS\right) \\ &\le C\eta(x_0')^2\left(\int_{\Omega_t(x_0)} w^2|D\xi|^2 + |\mathfrak{R}w\xi^2|\,dx + \frac{1}{\gamma}\sum_{i=1}^{2}\int_{\Gamma_{i,t}(x_0)} \Psi_i^2\xi^2 dS\right).\end{aligned}$$

Let C_0 and C_1 be fixed universal constants. From (4.2) and the above estimate, it follows that

$$\int_{\Omega_s(x_0)} |w|^2\,dx \le C_0\left(\frac{\eta(x_0')^2}{(t-s)^2} + \frac{1}{16C_0}\right)\int_{\Omega_t(x_0)} |w|^2\,dx + C_1\eta(x_0')^3 t^{n-1}. \tag{4.12}$$

The remainder of the proof is split into two cases.

Case 1. If $\eta(x_0') + 4\sqrt{C_0}\eta(x_0') > \frac{1}{16}\sqrt{\eta(x_0')}$, then $\eta(x_0') \ge \frac{1}{16^2(4\sqrt{C_0}+1)^2}$, and the estimates in (4.9) follow immediately from Theorem 1.4.

Case 2. If $\eta(x_0') + 4\sqrt{C_0}\eta(x_0') \le \frac{1}{16}\sqrt{\eta(x_0')}$, define a sequence t_i by

$$t_0 = \eta(x_0'), \quad \text{and } t_{i+1} = t_i + 4\sqrt{C_0}\eta(x_0').$$

Let $\omega(s) = \int_{\Omega_s(x_0)} |w|^2\,dx$. Then we have the iteration formula

$$\omega(t_i) \le \frac{1}{8}\omega(t_{i+1}) + C\eta(x_0')^3 t_{i+1}^{n-1}.$$

Iterating from $i = 0$ to $k(x_0) - 1$, where $k(x_0) = \left[\frac{\frac{1}{16}\sqrt{\eta(x_0')} - \eta(x_0')}{4\sqrt{C_0}\eta(x_0')}\right]$, and using $(\frac{1}{8})^{k(x_0)} \leq C\eta(x_0')^{n+2}$, we obtain

$$\begin{aligned}\omega(t_0) &\leq (\frac{1}{8})^{k(x_0)}\omega(t_{k(x_0)}) + \sum_{i=1}^{k(x_0)} \frac{C\eta(x_0')^3}{8^{i-1}}\Big(\eta(x_0') + 4i\sqrt{C_0}\eta(x_0')\Big)^{n-1} \\ &\leq C\eta(x_0')^{n+2}\int_{\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)} |w|^2\,dx + C\eta(x_0')^{n+2}.\end{aligned} \tag{4.13}$$

This establishes the first estimate in (4.9). The gradient estimate follows by combining this result with Lemma 4.3. □

4.2 Theorem 4.1 for the case $\gamma > \mu\eta(x')$.

In this subsection, we estimate $|\nabla w(x)|$ in the regime $\gamma > \mu\eta(x')$. The key idea is to analyze the vertical average of the solution, defined by

$$\overline{w}(x') := \fint_{f_2(x')}^{\varepsilon + f_1(x')} w(x', x_n)\,dx_n. \tag{4.14}$$

Taking average to equation (4.5) in the y_n-variable and applying the coordinate transformation from y to x, we find that $\overline{w}$ satisfies the degenerate elliptic equation in the $n-1$ dimensional variable x':

$$\sum_{i=1}^{n-1}\partial_i(\delta(x')\partial_i\overline{w}) - \frac{2}{\gamma}\overline{w} = \sum_{i=1}^{n-1}\partial_i F^i + G, \quad \text{for } x \in \mathtt{B}_R, \tag{4.15}$$

where

$$F^i(x') := \int_{f_2(x')}^{\varepsilon + f_1(x')}\Big(\frac{x_n - f_2(x')}{\delta(x')}\partial_{x_i} f_1(x') - \frac{x_n - \varepsilon - f_1(x')}{\delta(x')}\partial_{x_i} f_2(x')\Big)\partial_{x_n} w(x)\,dx_n,$$

$$G(x') := -\frac{2}{\gamma}\overline{w}(x') + \frac{1}{\gamma}\sum_{i=1}^{2}\Big(w_{\Gamma_i}(x') - \Psi_i(x')\Big)\sqrt{1 + |D_{x'}f_i|^2} + \int_{f_2(x')}^{\varepsilon + f_1(x')}\mathfrak{R}(x)\,dx_n,$$

$\delta(x') = \varepsilon + f_1(x') - f_2(x')$, and w_{Γ_i} is the boundary trace given by (4.11).

Under assumptions (1.5)–(1.6), the following bounds hold for $i = 1, 2$:

$$\begin{aligned}&\|f_i\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} \leq C\eta(x'), \\ &\|Df_i\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} \leq C\eta(x')^{1/2}, \quad \|D^2 f_i\|_{L^\infty(\Omega_R)} \leq C, \\ &[f_i]_{C^\alpha(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} \leq C\|Df_i\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))}\eta(x')^{\frac{1-\alpha}{2}} \leq C\eta(x')^{1-\frac{\alpha}{2}},\end{aligned} \tag{4.16}$$

For convenience, define

$$\mathcal{F}^i(x) := \frac{x_n - f_2(x')}{\delta(x')}\partial_{x_i} f_1(x') - \frac{x_n - \varepsilon - f_1(x')}{\delta(x')}\partial_{x_i} f_2(x').$$

Then, using (4.16), we obtain

$$\begin{aligned}&\|\mathcal{F}^i\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} \leq C\eta(x')^{1/2}, \\ &[\mathcal{F}^i]_{C^\alpha_{x'}(\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x))} \leq C\|D_{x'}\mathcal{F}^i\|_{L^\infty(\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x))}\eta(x')^{\frac{1-\alpha}{2}} \leq C\eta(x')^{\frac{1-\alpha}{2}}.\end{aligned} \tag{4.17}$$

From (4.16) and (4.17), we estimate the Hölder norm of F^i:

$$|F^i(x')| \leq \, C\eta(x')^{\frac{3}{2}}\|\partial_n w\|_{L^\infty(\Omega_R)},$$

and

$$\begin{aligned}
\big[F^i\big]_{C^\alpha(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} \leq & \, C\sum_{j=1}^{2}[f_j]_{C^\alpha(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))}\|\mathcal{F}^i\partial_n w\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} \\
& + C\eta(x')[\mathcal{F}^i]_{C^\alpha_{x'}(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))}\|\partial_n w\|_{L^\infty(\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x))} \\
& + C\eta(x')[\partial_n w]_{C^\alpha(\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x'))}\|\mathcal{F}^i\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} \\
\leq & \, C\eta(x')^{\frac{3}{2}-\alpha}\|\partial_n w\|_{L^\infty(\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x))} + C\eta(x')^{\frac{3}{2}}[\partial_n w]_{C^\alpha(\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x'))}.
\end{aligned} \tag{4.18}$$

Under the assumption (4.2), the term G satisfies

$$\begin{aligned}
|G(x')| \leq & \, C\sum_{i=1}^{2}\frac{1}{\gamma}|\bar{w}(x') - w_{\Gamma_i}(x')|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} \\
& + C\sum_{i=1}^{2}\frac{1}{\gamma}|\Psi_i(x')| + C\eta(x')\sup_{f_2(x')\leq x_n\leq\varepsilon+f_1(x')}|\mathfrak{R}(x', x_n)| \\
\leq & \, C\frac{\eta(x')}{\gamma}\|Dw\|_{L^\infty(\Omega_R)} + C\frac{\eta(x')}{\gamma+\eta(x')}.
\end{aligned} \tag{4.19}$$

4.2.1 Pointwise estimate for $\overline{w}(x')$

Using a scaling argument and Lemma 2.3, we obtain the following estimate.

Lemma 4.5. *Let $w \in H^1(\Omega_{\eta(x_0')}(x_0))$, $x_0 \in \Omega_{\frac{3}{4}R}$, be a solution to*

$$\begin{cases}
\Delta w = \mathfrak{R}(x) & \text{in } \Omega_{\eta(x_0')}(x_0), \\
w + \gamma\partial_\nu w = \Psi_1(x) & \text{on } \Gamma_{1,\eta(x_0')}(x_0), \\
w + \gamma\partial_\nu w = \Psi_2(x) & \text{on } \Gamma_{2,\eta(x_0')}(x_0),
\end{cases}$$

with $\gamma > 0$ and f_1, f_2 satisfying (1.5)–(1.6). Assume $\mathfrak{R} \in L^\infty(\Omega_{\eta(x_0')}(x_0))$ and $\Psi_i \in C^{1,\alpha}(\Omega_{\eta(x_0')}(x_0))$, $i = 1, 2$. If $\gamma > \eta(x_0')$, then

$$\begin{aligned}
& \|Dw\|_{L^\infty(\Omega_{\frac{1}{2}\eta(x_0')}(x_0))} + \eta(x_0')^\alpha[Dw]_{C^\alpha(\Omega_{\frac{1}{2}\eta(x_0')}(x_0))} \\
& \leq C\Bigg(\frac{1}{\gamma}\not{\|}w\|_{L^2(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + \not{\|}Dw\|_{L^2(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + \frac{1}{\gamma}\sum_{i=1}^{2}\|\Psi_i\|^*_{C^\alpha(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \\
& \quad + \eta(x_0')\|\mathfrak{R}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))}\Bigg),
\end{aligned} \tag{4.20}$$

where C depends only on n, α, κ, and the upper bound of $\|f_i\|_{C^{2,\alpha}(\mathtt{B}_R)}$ for $i = 1, 2$.

We now apply the pointwise estimates for degenerate equations from Section 3 (Propositions 3.4 and 3.7) to equation (4.15), which yields control over $\overline{w}$ and its gradient in terms of $\|Dw\|_{L^\infty(\Omega_R)}$.

Proposition 4.6. *Under the assumptions of Theorem 4.1, let $w \in H^1(\Omega_R)$ be a solution to (4.1), with $\overline{w}$ defined as in (4.14) and γ_0 given by (1.8). Then, for any $\varepsilon \in (0, \varepsilon_0]$ and $\gamma \in (0, \gamma_0)$, the following estimates hold:*

(i) *For $|x'| \le R$,*

$$|\overline{w}(x')| \le C\,\eta(x')\left(\|Dw\|_{L^\infty(\Omega_R)} + 1\right). \tag{4.21}$$

(ii) *For all $x \in \Omega_{R/2}$ with $\eta(x') < \gamma$,*

$$|D\overline{w}(x')| \le C\left(\gamma^{\frac{\alpha}{2}}\eta(x')^{\frac{1-\alpha}{2}} + \left(\frac{\eta(x')}{\gamma}\right)^{1/2}\right)\left(\|Dw\|_{L^\infty(\Omega_R)} + 1\right). \tag{4.22}$$

The constants ε_0 and $C > 0$ depend only on n, α, κ, R, a lower bound of $\gamma_0 - \gamma$ and upper bounds of $\|f_1\|_{C^{2,\alpha}}$ and $\|f_2\|_{C^{2,\alpha}}$.

Proof. (i) Using (4.18) and (4.19), we obtain

$$\begin{aligned}\sqrt{\gamma}\|F\|_{\frac{3}{2},\mathtt{B}_R} + \gamma\|G\|_{1,\mathtt{B}_R} &\le C\|Dw\|_{L^\infty(\Omega_R)} + C,\\ \gamma^{1/2}\eta(x')^{-1/2}\|G\|_{L^\infty(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} &\le C\Big(\frac{1}{\gamma}\eta(x')\Big)^{1/2}\Big(\|Dw\|_{L^\infty(\Omega_R)} + 1\Big).\end{aligned} \tag{4.23}$$

By (1.6), for $x \in \Omega_R$ we have

$$\begin{aligned}|f_1(x') - f_2(x')| &\le \frac{1}{2}\|D^2(f_1 - f_2)\|_{L^\infty(\Omega_R)}|x'|^2,\\ |Df_1(x') - Df_2(x')| &\le \|D^2(f_1 - f_2)\|_{L^\infty(\Omega_R)}|x'|.\end{aligned} \tag{4.24}$$

Let $x' = Ry'$ for $y' \in \mathtt{B}_1$, and define $t(y') = (f_1(Ry') - f_2(Ry'))/(2R^2)$. Then from (4.24), we choose $\tilde{\Lambda} = \frac{1}{4}\|D^2(f_1 - f_2)\|_{L^\infty(\Omega_R)}$, which allows us to apply Proposition 3.4 to $\bar{w}(Ry')$ with $\tilde{\varepsilon} = \varepsilon/(2R^2)$ and $\sigma = 2$, thereby obtaining (4.21) for all $\gamma \in (0, \gamma_0)$.

(ii) For $x_0 \in \Omega_{3R/4}$, we use the assumption

$$\frac{1}{\gamma}\sum_{i=1}^{2}\|\Psi_i\|^*_{C^\alpha(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + \eta(x_0')\|\mathfrak{R}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \le C\frac{\eta(x_0')}{\gamma}.$$

Then, by Lemma 4.5 and (4.21), we obtain

$$\begin{aligned}\|Dw\|^*_{C^\alpha(\Omega_{\frac{1}{2}\eta(x_0')}(x_0))} &\le C\|Dw\|_{L^\infty(\Omega_{\frac{1}{2}\eta(x_0')}(x_0))} + C\eta(x_0')^\alpha[Dw]_{C^\alpha(\Omega_{\frac{1}{2}\eta(x_0')}(x_0))}\\ &\le C\frac{\eta(x_0')}{\gamma}\Big(\|Dw\|_{L^\infty(\Omega_R)} + 1\Big) + C\|Dw\|_{L^\infty(\Omega_R)}.\end{aligned}$$

Since $\eta(x_0') \sim \eta(x')$ for $x_0 \in \Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x')$, it follows that for $x \in \Omega_{R/2}$ with $\eta(x') < \gamma$,

$$\sup_{x_0 \in \Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x)}\|Dw\|^*_{C^\alpha(\Omega_{\frac{1}{2}\eta(x_0')}(x_0))} \le C\|Dw\|_{L^\infty(\Omega_R)} + C.$$

Substituting this into (4.18) gives

$$\begin{aligned}\left[F^i\right]_{C^\alpha(\mathtt{B}_{\frac{1}{8}\sqrt{\eta(x')}}(x'))} &\le C\eta(x')^{\frac{3}{2}-\alpha}\|\partial_n w\|_{L^\infty(\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x))} + C\eta(x')^{\frac{3}{2}}[\partial_n w]_{C^\alpha(\Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x'))}\\ &\le C\eta(x')^{\frac{3}{2}-\alpha}\sup_{x_0 \in \Omega_{\frac{1}{8}\sqrt{\eta(x')}}(x)}\|Dw\|^*_{C^\alpha(\Omega_{\frac{1}{2}\eta(x_0')}(x_0))}\\ &\le C\eta(x')^{\frac{3}{2}-\alpha}\Big(\|Dw\|_{L^\infty(\Omega_R)} + 1\Big).\end{aligned} \tag{4.25}$$

Estimate (4.22) now follows from Proposition 3.7 together with (4.23) and (4.25). ∎

4.2.2 Energy estimate for $w-\overline{w}$

Lemma 4.7. *Under the assumptions of Theorem 4.1, let w be a solution to (4.1) and let $\overline{w}$ be defined as in (4.14). For $x_0 \in \Omega_{\frac{R}{2}}$,*

$$\int_{\Omega_{\eta(x_0')}(x_0)} |D(w-\overline{w})|^2\,dx \le C\eta(x_0')^n \left(1+Q(x_0)^2\right),$$

where

$$Q(x_0) := \sup_{x\in\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)} \left(|D\overline{w}(x)| + \frac{1}{\gamma}|\overline{w}(x)|\right).$$

The constant C depends only on n, α, κ, $\|f_1\|_{C^{2,\alpha}}$, $\|f_2\|_{C^{2,\alpha}}$, and R.

Proof. We multiply (4.1) by $(w-\overline{w})\xi^2$ and integrate by parts, where $\xi(x)=\xi(x')$ satisfies $\xi \in C_c^\infty(\Omega_t(x_0))$, $\xi = 1$ in $\Omega_s(x_0)$, where $0<s<t<\frac{1}{16}\sqrt{\eta(x_0')}$, and $|D\xi| \le \frac{C}{|s-t|}$. This gives

$$\int_{\Omega_t(x_0)} Dw\cdot D\big((w-\overline{w})\xi^2\big) + \mathfrak{R}(w-\overline{w})\xi^2\,dx = \sum_{i=1}^{2}\int_{\Gamma_{i,t}(x_0)} \frac{\tilde{\Psi}_i - w}{\gamma}(w-\overline{w})\xi^2 dS. \tag{4.26}$$

Let w_{Γ_i} be the boundary trace given by (4.11). Note that

$$\partial_i\overline{w}(x') = \fint_{f_2(x')}^{\varepsilon+f_1(x')} \partial_i w(x',x_n)\,dx_n + \frac{\partial_i f_1(x')}{\delta(x')}(w_{\Gamma_1}-\overline{w})(x') - \frac{\partial_i f_2(x')}{\delta(x')}(w_{\Gamma_2}-\overline{w})(x'),$$

and $\int_{f_2(x')}^{\varepsilon+f_1(x')}(w-\overline{w})\,dx_n = 0$, $\partial_n\overline{w}(x) = \partial_n\overline{w}(x') = 0$. Hence,

$$\begin{aligned}
&\int_{\Omega_t(x_0)} D\overline{w}\cdot D((w-\overline{w})\xi^2)\,dx\\
&= \int_{|x'-x_0'|\le t} (D_{x'}\overline{w}\cdot D_{x'}\xi^2)\Big(\int_{f_2(x')}^{\varepsilon+f_1(x')}(w-\overline{w})\,dx_n\Big)\,dx' + \int_{\Omega_t(x_0)} \xi^2 D_{x'}\overline{w}\cdot D_{x'}(w-\overline{w})\,dx\\
&= \int_{\Omega_t(x_0)} \xi^2\frac{\overline{w}-w_{\Gamma_1}}{\delta(x')}D_{x'}\overline{w}\cdot D_{x'}f_1 + \xi^2\frac{w_{\Gamma_2}-\overline{w}}{\delta(x')}D_{x'}\overline{w}\cdot D_{x'}f_2\,dx \qquad (4.27)\\
&:= Q_t[f_1,f_2].
\end{aligned}$$

Subtracting (4.27) from (4.26) yields

$$\begin{aligned}
&\int_{\Omega_t(x_0)} D(w-\overline{w})\cdot D\big((w-\overline{w})\xi^2\big)\,dx + Q_t[f_1,f_2] + \int_{\Omega_t(x_0)} \mathfrak{R}(w-\overline{w})\xi^2\,dx\\
&= \frac{1}{\gamma}\sum_{i=1}^{2}\int_{\Gamma_{i,t}(x_0)} (\tilde{\Psi}_i - w)(w-\overline{w})\xi^2 dS.
\end{aligned}$$

We now estimate each term.

Step 1. Estimating the first term on the left-hand side. Using Cauchy inequality,

$$\begin{aligned}
&\int_{\Omega_t(x_0)} D(w-\overline{w})\cdot D\big((w-\overline{w})\xi^2\big)\,dx\\
&=\int_{\Omega_t(x_0)} |D(w-\overline{w})|^2\xi^2+2\xi(w-\overline{w})D\xi\cdot D(w-\overline{w})\,dx\\
&\geq \int_{\Omega_t(x_0)} |D(w-\overline{w})|^2\xi^2\,dx-\frac{1}{8}\int_{\Omega_t(x_0)} |D(w-\overline{w})|^2\xi^2\,dx-C\int_{\Omega_t(x_0)}(w-\overline{w})^2|D\xi|^2\,dx.
\end{aligned}\tag{4.28}$$

By Hölder's inequality,

$$\begin{aligned}
|w(x)-\overline{w}(x)|&\leq \int_{f_2(x')}^{\varepsilon+f_1(x')}|\partial_n w(x',x_n)|\,dx_n\leq\, C\eta(x_0')^{1/2}\left(\int_{f_2(x')}^{\varepsilon+f_1(x')}|\partial_n w(x',x_n)|^2\,dx_n\right)^{1/2}\\
&\leq C\eta(x_0')^{1/2}\left(\int_{f_2(x')}^{\varepsilon+f_1(x')}|D(w-\overline{w})(x',x_n)|^2\,dx_n\right)^{1/2}.
\end{aligned}\tag{4.29}$$

Substituting (4.29) into (4.28), we obtain

$$\begin{aligned}
&\int_{\Omega_t(x_0)} D(w-\overline{w})\cdot D\big((w-\overline{w})\xi^2\big)\,dx\\
&\geq\frac{7}{8}\int_{\Omega_t(x_0)}|D(w-\overline{w})|^2\xi^2\,dx-C\eta(x_0')^2\int_{\Omega_t(x_0)}|D(w-\overline{w})|^2|D\xi|^2\,dx.
\end{aligned}$$

Step 2. Estimating the second and third terms on the left-hand side. For $x\in\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)$, we have $|Df_1(x')|+|Df_2(x')|\leq C\eta(x_0')^{1/2}$. Then, for $0\leq t\leq\frac{1}{16}\sqrt{\eta(x_0')}$,

$$\begin{aligned}
|Q_t[f_1,f_2]|&\leq\int_{\Omega_t(x_0)}\xi^2\Big(|D_{x'}\overline{w}\cdot D_{x'}f_1|+|D_{x'}\overline{w}\cdot D_{x'}f_2|\Big)\fint_{f_2(x')}^{\varepsilon+f_1(x')}|\partial_n w|\,dx_n\,dx\\
&=\int_{\Omega_t(x_0)}\xi^2\Big(|D_{x'}\overline{w}\cdot D_{x'}f_1|+|D_{x'}\overline{w}\cdot D_{x'}f_2|\Big)|\partial_n w|\,dx\\
&\leq\frac{1}{16}\int_{\Omega_t(x_0)}\xi^2|D(w-\overline{w})|^2\,dx+C\eta(x_0')\int_{\Omega_t(x_0)}\xi^2|D\overline{w}|^2\,dx.
\end{aligned}$$

Step 3. Estimating the remaining terms. Using (4.29) and Hölder's inequality again,

$$\int_{\Omega_t(x_0)}\mathfrak{R}(w-\overline{w})\xi^2\,dx\leq\frac{1}{16}\int_{\Omega_t(x_0)}|D(w-\overline{w})|^2\xi^2\,dx+C\eta(x_0')^2\int_{\Omega_t(x_0)}\mathfrak{R}^2\,dx.$$

Similarly, for the boundary integrals on the right-hand side,

$$\begin{aligned}
&\frac{1}{\gamma}\int_{\Gamma_{1,t}(x_0)}(\tilde{\Psi}_1-w)(w-\overline{w})\xi^2dS\\
&=\frac{1}{\gamma}\int_{\Gamma_{1,t}(x_0)}\tilde{\Psi}_1(w-\overline{w})\xi^2-(w-\bar{w})^2\xi^2-\bar{w}(w-\bar{w})\xi^2dS\\
&\leq\frac{1}{\gamma}\int_{\Gamma_{1,t}(x_0)}|\tilde{\Psi}_1||w-\overline{w}|\xi^2+|\overline{w}(w-\overline{w})|\,dS\\
&\leq\frac{1}{16}\int_{\Omega_t(x_0)}|D(w-\overline{w})|^2\xi^2\,dx+C\frac{1}{\gamma^2}\eta(x_0')\int_{\Gamma_{1,t}(x_0)}\Big(\tilde{\Psi}_1^2+\overline{w}^2\Big)\,dS,
\end{aligned}$$

and the same estimate holds on $\Gamma_{2,t}(x_0)$ with $\tilde\Psi_2$ in place of $\tilde\Psi_1$.

Combining these estimates, we arrive at

$$\begin{aligned}&\int_{\Omega_t(x_0)}|D(w-\overline{w})|^2\xi^2\,dx\\&\leq\frac{1}{2}\int_{\Omega_t(x_0)}|D(w-\overline{w})|^2\xi^2\,dx+C_0\eta(x_0')^2\int_{\Omega_t(x_0)}|D(w-\overline{w})|^2|D\xi|^2\,dx\\&\quad+C_0\eta(x_0')\int_{\Omega_t(x_0)}|D\overline{w}|^2\,dx+C\int_{\Omega_t(x_0)}\Big(\frac{1}{\gamma^2}(|\tilde\Psi_1|^2+|\tilde\Psi_2|^2+|\overline{w}|^2)+\eta(x_0')^2\mathfrak{R}^2\Big)\,dx,\end{aligned}\tag{4.30}$$

where C_0 is a fixed universal constant. Let $\tilde C_0:=2C_0$. From (4.30) and the assumption (4.2), we obtain

$$\int_{\Omega_s(x_0)}|D(w-\overline{w})|^2\,dx\leq\tilde C_0\frac{\eta(x_0')^2}{(s-t)^2}\int_{\Omega_t(x_0)}|D(w-\overline{w})|^2\,dx+Ct^{n-1}\eta(x_0')(Q(x_0)+1)^2.$$

Step 4. Iteration and conclusion. Proceeding with an iteration argument (as in Lemma 4.4, from (4.12) to (4.13)), and noting that

$$\int_{\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)}|D(w-\overline{w})|^2\,dx\leq\int_{\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)}|Dw|^2\,dx,$$

we obtain

$$\int_{\Omega_{\eta(x_0')}(x_0)}|D(w-\overline{w})|^2\,dx\leq C\eta(x_0')^n\Big(\int_{\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)}|Dw|^2\,dx+Q(x_0)^2+1\Big).\tag{4.31}$$

Finally, we estimate the integral on the right-hand side of (4.31). Multiply the equation in (4.1) by $w\xi^2$ and integrate by parts, where

$$\xi\in C_c^\infty(\Omega_R),\quad \xi=1\text{ in }\Omega_{\frac{3}{4}R},\quad\text{and }|D\xi|\leq\frac{C}{R},$$

using Hölder's inequality, we have

$$\begin{aligned}&\int_{\Omega_{\frac{3}{4}R}}|Dw|^2\xi^2\,dx+\frac{1}{\gamma}\int_{\Gamma_{1,\frac{3}{4}R}\cup\Gamma_{2,\frac{3}{4}R}}w^2\xi^2dS\\&\leq C\left(\int_{\Omega_R}w^2|D\xi|^2+|\mathfrak{R}w\xi^2|\,dx+\frac{1}{\gamma}\sum_{i=1}^2\int_{\Gamma_{i,R}}\Psi_i^2\xi^2dS\right).\end{aligned}$$

By above inequality, the assumption that $\|w\|_{L^\infty(\Omega_R)}\leq 1$ and the fact that $\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)\subset\Omega_{\frac{3}{4}R}$ for $x_0\in\Omega_{\frac{R}{2}}$, we have

$$\int_{\Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)}|Dw|^2\,dx\leq\int_{\Omega_{\frac{3}{4}R}}|Dw|^2\xi^2\,dx\leq C.$$

Substituting this into (4.31) completes the proof. □

4.3 Proof of Theorem 4.1

Proof of Theorem 4.1. We now complete the proof by decoupling the estimates. Using the energy estimate from Lemma 4.7, we compare the gradient of w with that of its vertical average $\overline{w}$. Combining this with estimates for $\overline{w}$ yields a pointwise bound on $Dw(x_0)$ in terms of the global norm $\|Dw\|_{L^\infty(\Omega_R)}$. A subsequent iteration over points then establishes the uniform boundedness of $\|Dw\|_{L^\infty(\Omega_R)}$.

We first prove that for every $x_0 \in \Omega_{R/2}$ satisfying $\eta(x_0') < \gamma$ the following estimate holds:

$$|Dw(x_0)| \le C_1 \left(\eta(x_0')^{\frac{1-\alpha}{2}} + \big(\frac{1}{\gamma}\eta(x_0')\big)^{1/2} \right) \Big(\|Dw\|_{L^\infty(\Omega_R)} + 1 \Big) + C_1, \tag{4.32}$$

where $C_1 > 1$ is a universal constant independent of ε, γ, and x_0.

To show this, we apply Lemma 4.5 together with the triangle inequality and the bounds $|\mathfrak{R}(x)| \le \frac{C}{\delta(x')}$, and $\|\Psi_i\|^*_{C^\alpha(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \le C\gamma$ from (4.2), obtaining

$$\begin{aligned}
|Dw(x_0)| \le\, & C \left(\frac{1}{\gamma} \|w\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + \eta(x_0')^{-n/2} \|Dw\|_{L^2(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \right) + C \\
\le\, & \frac{C}{\gamma} \left(\|w - \overline{w}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + \|\overline{w}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \right) \\
& + C\eta(x_0')^{-n/2} \|D(w-\overline{w})\|_{L^2(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + C\|D\overline{w}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} + C.
\end{aligned}$$

By the mean-value theorem, $\|w - \overline{w}\|_{L^\infty(\Omega_{\frac{3}{4}\eta(x_0')}(x_0))} \le C\eta(x_0')\|Dw\|_{L^\infty(\Omega_R)}$. Lemma 4.7 implies

$$\eta(x_0')^{-n/2} \|D(w-\overline{w})\|_{L^2(\Omega_{\eta(x_0')}(x_0))} \le C \sup_{x \in \Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)} \Big(|D\overline{w}(x)| + \frac{1}{\gamma}\overline{w}(x) \Big) + C.$$

From Proposition 4.6, we obtain

$$\sup_{x \in \Omega_{\frac{1}{16}\sqrt{\eta(x_0')}}(x_0)} \Big(|D\overline{w}(x)| + \frac{1}{\gamma}\overline{w}(x) \Big) \le C \left(\gamma^{\frac{\alpha}{2}} \eta(x_0')^{\frac{1-\alpha}{2}} + \big(\frac{1}{\gamma}\eta(x_0')\big)^{1/2} \right) \Big(\|Dw\|_{L^\infty(\Omega_R)} + 1 \Big).$$

Combining these estimates yields (4.32).

Let C_1 be the large universal constant appearing in (4.32). Define ε_0 and R_0 by $\varepsilon_0 = \frac{1}{2}\left(\frac{1}{16C_1^2}\right)^{\frac{1}{1-\alpha}}$ and $R_0 = \frac{1}{\sqrt{2}}\left(\frac{1}{4C_1}\right)^{\frac{1}{1-\alpha}}$. By choosing C_1 sufficiently large, we ensure that $\varepsilon_0 < 1/4$ and $R_0 < R/2$. Then, for any $\varepsilon \in (0, \varepsilon_0)$ and any $x_0 \in \Omega_{R_0}$, it follows from the definition $\eta(x') = \varepsilon + |x'|^2$ that $\eta(x_0')^{\frac{1-\alpha}{2}} \le \big(\varepsilon_0 + R_0^2\big)^{\frac{1-\alpha}{2}} \le \frac{1}{4C_1}$.

Now set $\mu = 16C_1^2$ and proceed by considering two cases:

Case 1: $\gamma \in (0, \mu\varepsilon]$. Since $\gamma \le \mu\varepsilon \le \mu\eta(x')$ for all $x \in \Omega_{\frac{R}{2}}$, Lemma 4.4 directly yields (4.3).

Case 2: $\gamma \in (\mu\varepsilon, \gamma_0)$. Taking the supremum in (4.32) over all $x_0 \in \Omega_{\frac{R_0}{2}}$ with $\mu\eta(x_0') < \gamma$, we obtain

$$\sup_{x \in \Omega_{R_0/2}, \mu\eta(x') < \gamma} |Dw(x)| \le \frac{1}{2}\|Dw\|_{L^\infty(\Omega_R)} + C.$$

Rearranging this inequality gives

$$\sup_{x \in \Omega_{R_0/2}, \mu\eta(x') < \gamma} |Dw(x)| \le \sup_{x \in \Omega_R, \mu\eta(x') \ge \gamma} |Dw(x)| + 2C.$$

By Lemma 4.4, $|Dw(x)| \leq C$ whenever $\mu\eta(x') \geq \gamma$, hence

$$\sup_{x\in\Omega_{R_0/2}} |Dw(x)| \leq C.$$

Moreover, by Theorem 1.4, we have $\sup_{x\in\Omega_R\setminus\Omega_{R_0/2}} |Dw(x)| \leq C$. Combining these estimates completes the proof of Theorem 4.1. $\square$

5 Proof of Theorem 1.1

We prove Theorem 1.1 in this section via a decomposition technique adapted from [6]. The strategy consists of separating the solution into singular and regular parts, determining the free constants multiplying the singular parts, and combining the resulting estimates to derive optimal gradient bounds. The estimates for the singular part rely on a specially constructed auxiliary function together with the results of Section 4, while those for the regular parts follow directly from the latter.

Let $u_{j,\gamma,\varepsilon} \in C^2(\overline{\widetilde{\Omega}^\varepsilon})$, for $j = 1, 2, 3$, be the solutions to the following boundary value problems:

$$\begin{cases} \Delta u_{1,\gamma,\varepsilon} = 0 & \text{in } \tilde{\Omega}^\varepsilon, \\ u_{1,\gamma,\varepsilon} + \gamma\partial_\nu u_{1,\gamma,\varepsilon} = 1 & \text{on } \partial D_1^\varepsilon, \\ u_{1,\gamma,\varepsilon} + \gamma\partial_\nu u_{1,\gamma,\varepsilon} = 0 & \text{on } \partial D_2^\varepsilon, \\ u_{1,\gamma,\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta u_{2,\gamma,\varepsilon} = 0 & \text{in } \tilde{\Omega}^\varepsilon, \\ u_{2,\gamma,\varepsilon} + \gamma\partial_\nu u_{2,\gamma,\varepsilon} = 0 & \text{on } \partial D_1^\varepsilon, \\ u_{2,\gamma,\varepsilon} + \gamma\partial_\nu u_{2,\gamma,\varepsilon} = 1 & \text{on } \partial D_2^\varepsilon, \\ u_{2,\gamma,\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \Delta u_{3,\gamma,\varepsilon} = 0 & \text{in } \tilde{\Omega}^\varepsilon, \\ u_{3,\gamma,\varepsilon} + \gamma\partial_\nu u_{3,\gamma,\varepsilon} = 0 & \text{on } \partial D_1^\varepsilon \cup \partial D_2^\varepsilon, \\ u_{3,\gamma,\varepsilon} = \phi & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

When no confusion arises, we abbreviate $u_{j,\gamma,\varepsilon}$ as u_j for $j = 1, 2, 3$, and drop the superscript ε by writing $\widetilde{\Omega} := \widetilde{\Omega}^\varepsilon$, $D_1 := D_1^\varepsilon$, and $D_2 := D_2^\varepsilon$.

By linearity, the solution may be expressed as

$$u = K_1u_1 + K_2u_2 + u_3 \quad \text{in } \tilde{\Omega}, \tag{5.3}$$

where $K_i = K_i(\gamma, \varepsilon)$ $(i = 1, 2)$ are free constants uniquely determined by the flux conditions in (1.4). This decomposition isolates specific physical effects: u_1 and u_2 are capacitary-type functions associated with each inclusion, whose gradients ∇u_i become singular in the narrow region between ∂D_1 and ∂D_2, while $|\nabla u_3|$ remains a regular term.

From (5.3), the gradient decomposes as

$$Du = (K_1 - K_2)Du_1 + K_2D(u_1 + u_2) + Du_3, \quad \text{in } \tilde{\Omega}. \tag{5.4}$$

The key step is to show that the singular behavior is captured entirely by $(K_1 - K_2)Du_1$, while $K_2D(u_1 + u_2)$ and Du_3 remain uniformly bounded. These estimates are established in Propositions 5.1 and 5.2, whose proofs are given in Subsections 5.2 and 5.3, respectively.

Without loss of generality, we assume $\|\varphi\|_{C^{2,\alpha}(\partial\Omega)} \leq 1$; the general case follows by scaling $u/\|\varphi\|_{C^{2,\alpha}(\partial\Omega)}$.

Proposition 5.1. *Let $u_1, u_2 \in H^1(\widetilde{\Omega})$ be the solutions to* (5.1)*, and let $u_3 \in H^1(\widetilde{\Omega})$ be the solution to* (5.2) *for $n \geq 2$, under assumptions* (1.5)–(1.6) *and the normalization*

$\|\varphi\|_{C^{2,\alpha}(\partial\Omega)} \leq 1$. *Let* γ_0 *be the constant specified by* (1.8). *Then, for all* $\varepsilon \in (0, 1/4)$ *and* $\gamma \in (0, \gamma_0)$, *the following estimates hold:*

$$\|Du_j\|_{L^\infty(\tilde{\Omega}\setminus\Omega_R)} \leq C, \quad j = 1, 2, 3, \tag{5.5}$$

$$\|D(u_1 + u_2)\|_{L^\infty(\tilde{\Omega})} \leq C, \quad \|Du_3\|_{L^\infty(\tilde{\Omega})} \leq C, \tag{5.6}$$

and

$$\frac{1}{C}\frac{1}{\gamma + \eta(x')} - C \leq |Du_j(x)| \leq \frac{C}{\gamma + \eta(x')} + C \quad \text{for } x \in \Omega_R,\ j = 1, 2, \tag{5.7}$$

where the constant C *depends only on* n, R, α *and* κ, *a lower bound of* $\gamma_0 - \gamma$ *and upper bounds of* $\|f_1\|_{C^{2,\alpha}}$ *and* $\|f_2\|_{C^{2,\alpha}}$.

Thus, the decomposition (5.4) isolates the singular contributions of Du_1 and Du_2 from the regular terms $D(u_1 + u_2)$ and Du_3.

The constants K_1 and K_2 are determined by the zero-flux conditions

$$\int_{\partial D_j} \partial_\nu u \, dS = 0 \quad j = 1, 2,$$

which lead to the 2×2 linear system:

$$\begin{cases} a_{11}K_1 + a_{12}K_2 + b_1 = 0, \\ a_{21}K_1 + a_{22}K_2 + b_2 = 0, \end{cases} \tag{5.8}$$

where

$$a_{ij} = \int_{\partial D_i} \frac{\partial u_j}{\partial \nu} dS, \qquad b_i = \int_{\partial D_i} \frac{\partial u_3}{\partial \nu} dS, \quad i, j = 1, 2. \tag{5.9}$$

Lemma 5.5 below shows that the diagonal entries a_{ii} are positive, the off-diagonals $a_{ij} (i \neq j)$ are negative, and $a_{1j} + a_{2j}$ is bounded away from zero, ensuring the system is well-posed. The magnitude of $|K_1 - K_2|$ is controlled by the size of a_{ii}. Proposition 5.6 provides upper and lower bounds for a_{ii} using the gradient estimates from Proposition 5.1.

The following estimates for $|K_1 - K_2|$ partially cancel the singularity of Du_1, yielding sharp bounds for Du.

Proposition 5.2. *Under the hypotheses of Proposition 5.1, we have* $|K_1| + |K_2| \leq C$. *Furthermore, for* $n \geq 2$, $\varepsilon \in (0, 1/4)$, *and* $\gamma \in (0, \gamma_0)$, *the following estimate holds:*

$$|K_1 - K_2| \leq C\rho_n(\varepsilon, \gamma), \text{ where } \rho_n(\varepsilon, \gamma) := \begin{cases} \sqrt{\varepsilon + \gamma} & n = 2, \\ \dfrac{1}{|\ln(\varepsilon + \gamma)|} & n = 3, \\ 1 & n \geq 4, \end{cases}$$

where the constant C *depends only on* n, R, α, κ, *a lower bound of* $\gamma_0 - \gamma$ *and upper bounds of* $\|f_1\|_{C^{2,\alpha}}$ *and* $\|f_2\|_{C^{2,\alpha}}$.

5.1 Proof of Theorem 1.1

Proof of Theorem 1.1. Substitute the gradient estimates from Proposition 5.1 and the constant estimates from Proposition 5.2 into the decomposition (5.4):

$$|Du| \leq |K_1 - K_2|\,|Du_1| + |K_2|\,|D(u_1 + u_2)| + |Du_3|.$$

In the narrow region $\tilde{\Omega}$, the term $|K_1 - K_2|\,|Du_1(x)|$ is bounded by

$$\frac{C\rho_n(\varepsilon,\gamma)}{\gamma+\varepsilon+|x'|^2},$$

yielding the dimension-dependent blow-up rate. The terms $|K_2|\,|D(u_1+u_2)|$ and $|Du_3|$ are uniformly bounded. Outside the narrow region, all terms are uniformly bounded. Combining these cases completes the proof of Theorem 1.1. □

5.2 Proof of Proposition 5.1

This subsection is devoted to establishing pointwise gradient estimates for the functions u_j $(j=1,2,3)$, distinguishing between the narrow region Ω_R and the bulk $\tilde{\Omega}\backslash\Omega_R$. We begin by proving uniform boundedness of the solutions by the maximum principle, taking careful account of the Robin boundary conditions.

Lemma 5.3. *Under the assumption of Proposition 5.1, we have*

$$\|u_j\|_{L^\infty(\tilde{\Omega})} \le 1, \quad \textit{for } j=1,2,3.$$

Proof. We first show $\|u_1\|_{L^\infty(\tilde{\Omega})} \le 1$; the case for u_2 follows by symmetry.

Suppose, for contradiction, that

$$u_1(x_0) = \min_{x\in\bar{\tilde{\Omega}}} u_1(x) < 0 \text{ for some } x_0 \in \bar{\tilde{\Omega}}.$$

By the strong maximum principle, $x_0 \in \partial D_1 \cup \partial D_2$. At this point, we would have $u_1(x_0) + \gamma\frac{\partial u_1}{\partial\nu}(x_0) < 0$, which contradicts the boundary condition $u_1 + \gamma\frac{\partial u_1}{\partial \nu} = \delta_{1j}$ on ∂D_j. Hence, $\min\limits_{x\in\bar{\tilde{\Omega}}} u_1(x) \ge 0$. A similar argument shows that $\max\limits_{x\in\bar{\tilde{\Omega}}} u_1(x) \le 1$.

To bound u_3, let $\tilde{u}_3 \in H^1(\tilde{\Omega})$ be the solution to

$$\begin{cases} \Delta\tilde{u}_3 = 0 & \text{in } \tilde{\Omega} \\ \tilde{u}_3 + \gamma\partial_\nu\tilde{u}_3 = 0 & \text{on } \partial D_1\cup\partial D_2, \\ \tilde{u}_3 = 1 & \text{on } \partial\Omega. \end{cases}$$

The same reasoning gives $0 \le \tilde{u}_3(x) \le 1$. Define

$$\hat{u}_3 = u_3 - \|\varphi\|_{C^{2,\alpha}(\partial\Omega)}\tilde{u}_3.$$

Then $\hat{u}_3$ satisfies

$$\begin{cases} \Delta\hat{u}_3 = 0 & \text{in } \tilde{\Omega}, \\ \hat{u}_3 + \gamma\partial_\nu\tilde{u}_3 = 0 & \text{on } \partial D_1\cup\partial D_2, \\ \hat{u}_3 \le 0 & \text{on } \partial\Omega. \end{cases}$$

The maximum principle implies $\hat{u}_3(x) \le 0$, so $u_3(x) \le \|\varphi\|_{C^{2,\alpha}(\partial\Omega)}\tilde{u}_3(x)$ for $x \in \bar{\tilde{\Omega}}$. Replacing $\hat{u}_3$ by $\hat{u}_3 = -u_3 - \|\varphi\|_{C^{2,\alpha}(\partial\Omega)}\tilde{u}_3$ yields the lower bound. Thus,

$$\|u_3\|_{L^\infty(\tilde{\Omega})} \le \|\varphi\|_{C^{2,\alpha}(\partial\Omega)} \le 1.$$

The lemma is proved. □

Proof of Proposition 5.1. Let ε_0 be as in Theorem 4.1.

For $\varepsilon \in [\varepsilon_0, 1/4)$, the estimates (5.5)–(5.7) follow directly from Theorem 1.4 and Lemma 5.3.

We now focus on the essential case $\varepsilon \in (0,\varepsilon_0)$. Here, the exterior estimate (5.5) again follows from Theorem 1.4 and Lemma 5.3. To establish the gradient estimates in the narrow region Ω_R, we apply Lemma 5.3 and Theorem 4.1 with $\mathfrak{R}=0$, $\Psi_1=\Psi_2=0$ to $1-u_1-u_2$ and u_3, which yields (5.6).

The core of the proof is to establish estimate (5.7) in Ω_R for small $\varepsilon\in(0,\varepsilon_0)$. We present the details for u_1 only, since the case of $1-u_2$ is identical. Define

$$\tilde{w}_1=\frac{x_n-f_2(x')+\gamma}{\delta(x')+2\gamma}. \tag{5.10}$$

To prove (5.7), it suffices to show

$$\|D(u_1-\tilde{w}_1)\|_{L^\infty(\Omega_R)}\leq C. \tag{5.11}$$

Let $w_1:=u_1-\tilde{w}_1$. We verify that $w_1$satisfies equation (4.1) with right-hand side and boundary data satisfying the assumption (4.2), so that Theorem 4.1 applies.

We begin by some basic computation. Under assumption (1.6), the following estimates hold for $i=1,2$:

$$\begin{aligned}
&|f_i(x')|\leq C|\eta(x')|,\quad |Df_i(x')|\leq C\eta(x')^{1/2},\quad |D^2f_i(x')|\leq C,\\
&[f_i]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}\leq C|Df_i|_{L^\infty(\Omega_{\eta(x_0')}(x_0))}\eta(x_0')^{1-\alpha}\leq C\eta(x_0')^{\frac{3}{2}-\alpha},\\
&[Df_i]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}\leq C\eta(x_0')^{1-\alpha}.
\end{aligned} \tag{5.12}$$

These imply, for $i=1,2$,

$$\begin{aligned}
&|(1+|Df_i(x')|^2)^{-1/2}-1|\leq C\eta(x'),\quad 1\leq(1+|Df_i(x')|^2)^{-1/2}\leq C,\\
&|D(1+|Df_i(x')|^2)^{-1/2}|\leq C\eta(x')^{1/2},\quad [(1+|Df_i(x')|^2)^{-1/2}]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}\leq\eta(x')^{\frac{3}{2}-\alpha},\\
&[D(1+|Df_i(x')|^2)^{-1/2}]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}\leq C.
\end{aligned} \tag{5.13}$$

We now verify that w_1 satisfies (4.1) with right-hand side and boundary data satisfying the assumptions (4.2). By virtue of (5.12), a direct computation yields, for $1\leq i\leq n-1$,

$$\begin{aligned}
&\partial_i\tilde{w}_1=\frac{-\partial_if_2(\delta(x')+2\gamma)-(x_n-f_2+\gamma)\partial_i(f_1-f_2)}{(\delta(x')+2\gamma)^2},\\
&\partial_n\tilde{w}_1=\frac{1}{\delta(x')+2\gamma},\quad \partial_{nn}\tilde{w}_1=0;\\
&|D_{x'}\tilde{w}_1(x)|\leq\frac{C\delta(x')^{1/2}}{\gamma+\delta(x')},\quad |\partial_n\tilde{w}_1(x)|\leq\frac{C}{\gamma+\delta(x')},\\
&|D^2_{x'}\tilde{w}_1(x)|\leq\frac{C}{\gamma+\delta(x')},\quad |D^2\tilde{w}_1(x)|\leq\frac{C\delta(x')^{1/2}}{(\gamma+\delta(x'))^2},\\
&[D\tilde{w}_1(x)]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}\leq\frac{C\delta(x')^{\frac{3}{2}-\alpha}}{(\gamma+\delta(x'))^2},\quad [D^2\tilde{w}_1(x)]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}\leq\frac{C\delta(x_0')^{1-\alpha}}{(\delta(x_0')+\gamma)^2}.
\end{aligned} \tag{5.14}$$

Since $\Delta u_1=0$ in Ω_R, by (5.14) we have

$$|\Delta w_1(x)|\leq\sum_{i=1}^{n-1}|\partial_{ii}\tilde{w}_1|\leq\frac{C}{\delta(x')+\gamma}. \tag{5.15}$$

Define the functions

$$\Psi_1(x) := -\gamma\Big(\big((1+|D_{x'}f_1|^2)^{-1/2}-1\big)\partial_n\tilde{w}_1-(1+|D_{x'}f_1|^2)^{-1/2}(D_{x'}\tilde{w}_1\cdot D_{x'}f_1)\Big),$$
$$\Psi_2(x) := \gamma\Big(\big((1+|D_{x'}f_2|^2)^{-1/2}-1\big)\partial_n\tilde{w}_1-(1+|D_{x'}f_2|^2)^{-1/2}(D_{x'}f_2\cdot D_{x'}\tilde{w}_1)\Big).$$

On $\Gamma_{1,2R}$, a direct calculation gives

$$\begin{aligned}
&w_1+\gamma\partial_\nu w_1 = (u_1+\gamma\partial_\nu u_1)-(\tilde{w}_1+\gamma\partial_\nu\tilde{w}_1)\\
&=1-\Big(\frac{\delta(x')+\gamma}{\delta(x')+2\gamma}+\gamma(1+|D_{x'}f_1|^2)^{-1/2}\big(\partial_n\tilde{w}_1-D_{x'}\tilde{w}_1\cdot D_{x'}f_1(x')\big)\Big)\\
&=1-\Big(1+\gamma\Big(\big((1+|D_{x'}f_1|^2)^{-1/2}-1\big)\partial_n\tilde{w}_1-(1+|D_{x'}f_1|^2)^{-1/2}(D_{x'}\tilde{w}_1\cdot D_{x'}f_1)\Big)\Big)\\
&=\Psi_1(x).
\end{aligned}$$

Similarly, $w_1+\gamma\partial_\nu w_1=\Psi_2(x)$ on $\Gamma_{2,2R}$.

Using (5.12), (5.13), and (5.14), we obtain, for $i=1,2$,

$$|\Psi_i(x)|\leq C\gamma\delta(x')|\partial_n\tilde{w}_1|+C\gamma|D_{x'}\tilde{w}_1||D_{x'}f_i|\leq C\frac{\gamma\delta(x')}{\gamma+\delta(x')},\tag{5.16}$$

and

$$\begin{aligned}
|D\Psi_i(x')|&\leq C\gamma\eta(x')^{1/2}|\partial_n\tilde{w}_1|+C\gamma\eta(x')|D^2\tilde{w}_1|+C\gamma\eta(x')^{1/2}|D_{x'}\tilde{w}_1||D_{x'}f_i|\\
&\quad+C\gamma|D^2\tilde{w}_1||D_{x'}f_i|+C\gamma|D_{x'}\tilde{w}_1||D^2f_i|\\
&\leq C\frac{\gamma}{\gamma+\delta(x')};\\
[D\Psi_i]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}&\leq C\gamma\eta(x_0')^{1/2}[D^2\tilde{w}_1]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}+C\gamma\eta(x_0')^{1-\alpha}\|D^2\tilde{w}_1\|_{L^\infty(\Omega_{\eta(x_0')}(x_0))}\\
&\quad+C\gamma[D\tilde{w}_1]_{C^\alpha(\Omega_{\eta(x_0')}(x_0))}+C\gamma\|D\tilde{w}_1\|_{L^\infty(\Omega_{\eta(x_0')}(x_0))}\\
&\leq C\Big(\frac{\gamma\delta(x_0')^{\frac{3}{2}-\alpha}}{(\gamma+\delta(x_0'))^2}+\frac{\gamma}{\gamma+\delta(x_0')}\Big).
\end{aligned}\tag{5.17}$$

From (5.15), (5.16), and (5.17), we conclude that

$$|\Delta w_1(x)|\leq\frac{C_1}{\gamma+\delta(x')}\ \text{ and }\ \|\Psi_i\|^*_{C^{1,\alpha}(\Omega_{\eta(x_0')}(x_0))}\leq C_1\frac{\gamma\delta(x_0')}{\gamma+\delta(x_0')}.$$

By Lemma 5.3, we have $\|w_1\|_{L^\infty(\Omega_{2R})}\leq 2$. Theorem 4.1 now applies to $\frac{w_1}{C_1+2}$, and (5.11) is proved. Combining with (5.14), we have (5.7). □

Remark 5.4. The proof of Proposition 5.1 actually yields the stronger estimate:

$$\Big|D(u_1(x)-\frac{x_n-f_2(x')+\gamma}{\delta(x')+2\gamma})\Big|\leq C\quad\forall x\in\Omega_{R/2}.$$

5.3 Proof of Proposition 5.2

To bound the free constants K_1 and K_2 and compute $|K_1-K_2|$, we require several basic properties of the coefficients a_{ij} and b_i defined in (5.9).

Lemma 5.5. *Let a_{ij} and b_i be defined by* (5.9). *Then*

(1) For any $\gamma\in(0,\infty)$, we have $a_{ii}>0$ and $a_{ij}<0$, for $i\neq j$,

(2) For any $M>0$ and $\gamma\in(0,M)$,

$$\frac{1}{C}<\sum_{i=1}^{2}a_{ij}<C \ \textit{ for } j=1,2,$$

where C depend only on n, R, α and κ, M, $\|f_1\|_{C^{2,\alpha}}$, and $\|f_2\|_{C^{2,\alpha}}$.

(3) For $\gamma\in(0,\infty)$,

$$\|\nabla u_3\|_{L^\infty(\tilde\Omega)}\le C\|\varphi\|_{C^{2,\alpha}(\tilde\Omega)},\quad |b_i|<C\|\varphi\|_{C^{2,\alpha}(\tilde\Omega)} \quad \textit{for } i=1,2.$$

Proof. (1) It suffices to prove the inequalities:

$$\frac{\partial u_1}{\partial\nu}\Big|_{\partial D_1}>0,\quad \frac{\partial u_1}{\partial\nu}\Big|_{\partial D_2}<0,\quad \frac{\partial u_1}{\partial\nu}\Big|_{\partial\Omega}<0. \tag{5.18}$$

Let $x_0\in\partial D_1$. We claim $u_1(x_0)<1$. Suppose, for contradiction, that $u_1(x_0)=1$. Since $0\le u_1\le 1$ by lemma 5.3, the function u_1 attains its maximum at x_0. By the strong maximum principle and the Hopf lemma, $\frac{\partial u_1}{\partial\nu}(x_0)>0$, which implies $u_1(x_0)+\frac{\partial u_1}{\partial\nu}(x_0)>1$, contradicting the boundary condition. Hence, $u_1(x_0)<1$, and the first inequality in (5.18) follows. The second inequality can be obtained by proving $u_1(x_0)>0$ for any $x_0\in\partial D_2$. Finally, since u_1 attains its minimum value on the boundary $\partial\Omega$, the Hopf lemma yields the last estimate.

A similar argument shows that $0<u_2<1$ on $\partial D_1\cup\partial D_2$ and

$$\frac{\partial u_2}{\partial\nu}\Big|_{\partial D_1}<0,\quad \frac{\partial u_2}{\partial\nu}\Big|_{\partial D_2}>0,\quad \frac{\partial u_2}{\partial\nu}\Big|_{\partial\Omega}<0.$$

(2) Since $0\le u_1\le 1$, Theorem 1.4 implies that $\|\nabla u_1\|_{L^\infty(\Omega\setminus\Omega_R)}\le C_0$. Choose a universal constant $\gamma_1\in(0,\gamma_0)$ such that $C_0\gamma_1<\frac{1}{2}$. We now consider two cases:

Case 1: $\gamma\in(0,\gamma_1)$. In this case, the boundary condition on ∂D_1 implies that $u_1|_{\partial D_1\setminus\Omega_R}>\frac{1}{2}$. Applying Theorem 1.4 to u_1 in $\tilde\Omega\setminus\Omega_R$, we find a point $\bar x\in\tilde\Omega\setminus\Omega_{2R}$ and a radius $r>0$ (independent of γ and ε) such that $u_1>\frac{1}{4}$ in $B(\bar x,2r)\subset\tilde\Omega\setminus\Omega_{2R}$. Let D be a fixed smooth domain containing both D_1 and D_2, with $\bar x\in\partial D$. Define a smooth function ϕ on ∂D such that $\phi=\frac{1}{4}$ on $B(\bar x,r)\cap\partial D$, $0\le\phi\le\frac{1}{4}$ on $\big(B(\bar x,2r)\setminus B(\bar x,r)\big)\cap\partial D$, and $\phi=0$ on $\partial D\setminus\overline{B(\bar x,2r)}$.

Let $\tilde u_1\in H^1(\Omega\setminus\overline{D})$ be the solution to

$$\begin{cases}\Delta\tilde u_1=0 & \text{in } \Omega\setminus\overline{D},\\ \tilde u_1=\phi & \text{on } \partial D,\\ \tilde u_1=0 & \text{on } \partial\Omega.\end{cases}$$

By the strong maximum principle, $0<\tilde u_1<\frac{1}{4}$ in $\Omega\setminus\overline{D}$. Moreover, the Hopf lemma implies $\frac{\partial\tilde u_1}{\partial\nu}\big|_{\partial\Omega}<-\frac{1}{C}$, where $C>0$ is a constant independent of γ and ε.

Since $u_1\ge\phi=\tilde u_1$ on ∂D and $u_1=\tilde u_1=0$ on $\partial\Omega$, the maximum principle implies $u_1-\tilde u_1\ge 0$ in $\Omega\setminus\overline{D}$. The harmonic function $u_1-\tilde u_1$ attains its minimum on $\partial\Omega$, so

$$\frac{\partial(u_1-\tilde u_1)}{\partial\nu}\bigg|_{\partial\Omega}\le 0.$$

Recalling that

$$a_{11}+a_{21}=-\int_{\partial\Omega}\frac{\partial u_1}{\partial\nu}dS,$$

we complete the proof of part (2) in this case.

Case 2: $\gamma \in [\gamma_1, M)$. Rewrite $u_1 = u_{1,\gamma,\varepsilon}$, which is also the unique minimizer of the strictly convex functional

$$I_{\gamma,\varepsilon}[w] := \int_{\tilde{\Omega}^\varepsilon} |\nabla w|^2 \, dx + \frac{1}{\gamma} \int_{\partial D_1^\varepsilon} (w-1)^2 \, dS + \frac{1}{\gamma} \int_{\partial D_2^\varepsilon} w^2 \, dS$$

over the space $\mathscr{A} := \{w \in H^1(\tilde{\Omega}^\varepsilon) : w = 0 \text{ on } \partial\Omega\}$. For any $\gamma \in [\gamma_1, M]$ and $\varepsilon \in [0, \varepsilon_0]$, we have

$$\|\nabla u_{1,\gamma,\varepsilon}\|^2_{L^2(\tilde{\Omega}^\varepsilon)} \le I_{\gamma,\varepsilon}(u_{1,\gamma,\varepsilon}) \le I_{\gamma,\varepsilon}[0] = \frac{1}{\gamma} \int_{\partial D_1^\varepsilon} 1 \, dS \le \frac{|\partial D_1^\varepsilon|}{\gamma_1}.$$

Together with the Dirichlet condition $u_{1,\gamma,\varepsilon} = 0$ on $\partial\Omega$, this yields

$$\|u_{1,\gamma,\varepsilon}\|^2_{H^1(\tilde{\Omega}^\varepsilon)} \le C.$$

We may extend $u_{1,\gamma,\varepsilon}$ to $H^1(\Omega)$ while maintaining $\|u_{1,\gamma,\varepsilon}\|_{H^1(\Omega)} \le C$.

We now claim that for any $\varepsilon \in [0, \varepsilon_0]$ and any $\gamma \in [\gamma_1, M]$,

$$\int_{\partial\Omega} \frac{\partial u_{1,\gamma,\varepsilon}}{\partial \nu} \, dS < -\frac{1}{C}, \tag{5.19}$$

where C is independent of ε and γ. Suppose, for contradiction, that this fails. Then there exist sequences $\{\varepsilon_j\} \subset [0, \varepsilon_0]$, $\{\gamma_j\} \subset [\gamma_1, M]$ with $\gamma_j \to \gamma^*$, $\varepsilon_j \to \varepsilon^*$, and a function $u^* \in H^1(\Omega)$ such that

$$\int_{\partial\Omega} \frac{\partial u_{1,\gamma_j,\varepsilon_j}}{\partial \nu} \, dS \to 0, \tag{5.20}$$

$$u_{1,\gamma_j,\varepsilon_j} \rightharpoonup u^* \text{ weakly in } H^1(\Omega). \tag{5.21}$$

Since $u_{1,\gamma_j,\varepsilon_j}$ minimizes $I_{\gamma_j,\varepsilon_j}$, for any $\phi \in H_0^1(\Omega)$ we have

$$\int_{\tilde{\Omega}^{\varepsilon_j}} \nabla u_{1,\gamma_j,\varepsilon_j} \cdot \nabla \phi \, dx + \frac{1}{\gamma_j} \int_{\partial D_1^{\varepsilon_j}} (u_{1,\gamma_j,\varepsilon_j} - 1)\phi \, dS + \frac{1}{\gamma_j} \int_{\partial D_2^{\varepsilon_j}} u_{1,\gamma_j,\varepsilon_j} \phi \, dS = 0.$$

Passing to the limit $j \to +\infty$ and using (5.21), we obtain

$$\int_{\tilde{\Omega}^{\varepsilon^*}} \nabla u^* \cdot \nabla \phi \, dx + \frac{1}{\gamma^*} \int_{\partial D_1^{\varepsilon^*}} (u^* - 1)\phi \, dS + \frac{1}{\gamma^*} \int_{\partial D_2^{\varepsilon^*}} u^* \phi \, dS = 0,$$

so that $u^* = u_{1,\gamma^*,\varepsilon^*}$ in $\tilde{\Omega}^{\varepsilon^*}$.

Now, by (5.21), the divergence theorem, and (5.20),

$$\begin{aligned}
&\int_{\partial D_1^{\varepsilon^*}} \frac{u_{1,\gamma^*,\varepsilon^*} - 1}{\gamma^*} \, dS + \int_{\partial D_2^{\varepsilon^*}} \frac{u_{1,\gamma^*,\varepsilon^*}}{\gamma^*} \, dS \\
&= \lim_{j \to +\infty} \left[\int_{\partial D_1^{\varepsilon_j}} \frac{u_{1,\gamma_j,\varepsilon_j} - 1}{\gamma_j} \, dS + \int_{\partial D_2^{\varepsilon_j}} \frac{u_{1,\gamma_j,\varepsilon_j}}{\gamma_j} \, dS \right] \\
&= -\lim_{j \to +\infty} \int_{\partial D_1^{\varepsilon_j} \cup \partial D_2^{\varepsilon_j}} \frac{\partial u_{1,\gamma_j,\varepsilon_j}}{\partial \nu} \, dS \\
&= -\lim_{j \to +\infty} \int_{\partial\Omega} \frac{\partial u_{1,\gamma_j,\varepsilon_j}}{\partial \nu} \, dS = 0.
\end{aligned}$$

However, from part (1) we know $u_{1,\gamma^*,\varepsilon^*} < 1$ on $\partial D_1^{\varepsilon^*}$ and $u_{1,\gamma^*,\varepsilon^*} > 0$ on $\partial D_2^{\varepsilon^*}$, contradicting the above. Thus, the claim (5.19) is proved.

(3) The estimate for b_i follows directly from its definition and Proposition 5.1. $\square$

Proposition 5.6. *For $i = 1, 2$, let a_{ii} be defined as in* (5.9)*, and let γ_0 be given by* (1.8)*. Then, for all $\varepsilon \in (0, 1/4)$ and $\gamma \in (0, \gamma_0)$,*

$$\frac{1}{C\,\rho_n(\varepsilon,\gamma)} \leq a_{ii} \leq \frac{C}{\rho_n(\varepsilon,\gamma)}, \tag{5.22}$$

where $\rho_n(\varepsilon,\gamma)$ is the function from Proposition 5.2. The constant C depends only on n, R, α, κ, a lower bound of $\gamma_0 - \gamma$ and upper bounds of $\|f_1\|_{C^{2,\alpha}}$ and $\|f_2\|_{C^{2,\alpha}}$.

Proof. We compute only a_{11}; the case of a_{22} is analogous.

We first establish the estimate

$$\left| a_{11} - \int_{|x'|\leq R/2} \frac{1}{\delta(x') + 2\gamma} dx' \right| \leq C. \tag{5.23}$$

The proof relies on an application of the divergence theorem. Introduce a test function $\psi_1(x) \in C^2(\bar{\tilde{\Omega}})$ satisfying

$$\psi_1(x) = 1 \text{ on } \partial D_1, \quad \psi_1(x) = 0 \text{ on } \partial D_2 \cup \partial\Omega,$$

with $\|\psi_1\|_{C^2(\tilde{\Omega}\setminus\Omega_{R/2})} \leq C$, and which varies linearly in x_n in the neck region:

$$\psi_1(x) := \frac{x_n - f_2(x')}{\delta(x')} = \frac{x_n - f_2(x')}{\varepsilon + f_1(x') - f_2(x')} \quad \text{in } \Omega_R.$$

By the divergence theorem, the flux a_{11} can be expressed as

$$a_{11} = \int_{\partial D_1} \frac{\partial u_1}{\partial \nu} dS = \int_{\partial\tilde{\Omega}} \frac{\partial u_1}{\partial \nu} \psi_1 dS = \int_{\tilde{\Omega}} \nabla\cdot(\psi_1 \nabla u_1) dx = \int_{\tilde{\Omega}} \nabla u_1 \cdot \nabla\psi_1 dx.$$

Since $\|Du_1\|_{L^\infty(\tilde{\Omega}\setminus\Omega_{R/2})} \leq C$, we obtain

$$a_{11} = \int_{\Omega_{R/2}} \nabla u_1 \cdot \nabla\psi_1 dx + \int_{\tilde{\Omega}\setminus\Omega_{R/2}} \nabla u_1 \cdot \nabla\psi_1 dx = \int_{\Omega_{R/2}} \nabla u_1 \cdot \nabla\psi_1 dx + O(1),$$

where $A = B + O(1)$ means $|A - B| \leq C$.

In the narrow region, we use the decomposition $u_1 = \tilde{w}_1 + w_1$, where $\tilde{w}_1$ is given by (5.10):

$$\int_{\Omega_{R/2}} \nabla u_1 \cdot \nabla\psi_1 dx = \int_{\Omega_{R/2}} \nabla w_1 \cdot \nabla\psi_1 dx + \int_{\Omega_{R/2}} \nabla\tilde{w}_1 \cdot \nabla\psi_1 dx.$$

The term involving ∇w_1 is bounded: since ∇w_1 is bounded (see (5.11)) and $|\nabla\psi_1(x)| \leq C/\delta(x')$, we have

$$\left| \int_{\Omega_{R/2}} \nabla w_1 \cdot \nabla\psi_1 dx \right| \leq C \int_{\Omega_{R/2}} \frac{1}{\delta(x')} dx \leq C.$$

For the term involving $\nabla\tilde{w}_1$, by using

$$|D_{x'}\tilde{w}_1(x)| + |D_{x'}\psi_1(x)| \leq \frac{C}{\delta(x')^{1/2}},$$

we can estimate the tangential part

$$\left| \int_{\Omega_{R/2}} D_{x'}\tilde{w}_1 \cdot D_{x'}\psi_1 dx \right| \leq C \int_{\Omega_{R/2}} \frac{1}{\delta(x')} dx \leq C.$$

A direct computation shows that

$$\int_{\Omega_{R/2}} \partial_n \tilde{w}_1 \cdot \partial_n \psi_1 dx = \int_{\Omega_{R/2}} \frac{1}{\delta(x')(\delta(x')+2\gamma)} dx = \int_{|x'|\le R/2} \frac{1}{\delta(x')+2\gamma} dx'.$$

Combining the above estimates yields (5.23).

Moreover, for the integral in (5.23), we have

$$\frac{1}{C\rho_n(\varepsilon,\gamma)} \le \int_{|x'|\le R/2} \frac{1}{\delta(x')+2\gamma} dx' \le \frac{C}{\rho_n(\varepsilon,\gamma)}. \tag{5.24}$$

By Lemma 5.5 (1)-(2), we also have

$$a_{11} > a_{11} + a_{21} > \frac{1}{C}. \tag{5.25}$$

For $n \ge 4$, the desired estimate follows directly from (5.23), (5.24), and (5.25).

For $n = 2,3$, note that $1/\rho_n(\varepsilon,\gamma) \to +\infty$ as $\varepsilon + \gamma \to 0$. Therefore, (5.22) holds when $\varepsilon + \gamma < 1/\tilde{M}$ for some sufficiently large universal constant $\tilde{M}$.

When $1/\tilde{M} \le \varepsilon + \gamma \le 1/4 + \gamma_0$ in dimensions $n = 2,3$, we have

$$\frac{1}{C} \le \frac{1}{\rho_n(\varepsilon,\gamma)} \le C.$$

In this case, (5.22) follows again from (5.23) and (5.25).

Combining all the above cases completes the proof. □

Proof of Proposition 5.2. Solving the system (5.8) via Cramer's rule gives

$$K_1 = \frac{-b_1 a_{22} + b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad K_2 = \frac{-b_2 a_{11} + b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}},$$

whence

$$K_1 - K_2 = \frac{b_2(a_{12}+a_{11}) - b_1(a_{21}+a_{22})}{a_{11}a_{22} - a_{12}a_{21}}. \tag{5.26}$$

Since

$$a_{11}a_{22} - a_{12}a_{21} = (a_{22}+a_{12})(a_{11} - \frac{a_{11}+a_{21}}{a_{22}+a_{12}} a_{12}),$$

Lemma 5.5 (1)–(2) implies $a_{22} > -a_{12} > 0$, which together with the bounds in Lemma 5.5 yields

$$0 < \frac{1}{C} a_{11} < a_{11}a_{22} - a_{12}a_{21} < C(a_{11}+a_{22}). \tag{5.27}$$

By symmetry, we also obtain

$$0 < \frac{1}{C} a_{22} < a_{11}a_{22} - a_{12}a_{21} < C(a_{11}+a_{22}).$$

Using the fact that $a_{22} > |a_{12}|$ and that $b_i (i = 1,2)$ are bounded (Lemma 5.5 (3)), we obtain

$$|K_1| \le \frac{C|b_1 a_{22}| + C|b_2 a_{12}|}{a_{22}} \le C,$$

and similarly, $|K_2| \le C$.

Finally, since the numerator in (5.26) is bounded (as both b_i and $\|D(u_1+u_2)\|_{L^\infty(\tilde{\Omega})}$ are bounded), it follows that

$$|K_1 - K_2| \le \frac{C}{a_{11}}.$$

The result now follows from Proposition 5.6. □

6 Proof of the lower bound

This section is devoted to the proof of Theorem 1.3, which establishes the optimality of the gradient estimates in Theorem 1.1. Since the decomposition (5.4) of Du contains exactly one singular term, namely Du_1, while all other contributions remain uniformly bounded, it suffices to obtain a positive lower bound for $|K_1 - K_2|$. This is achieved via the construction of a subsolution under a specific symmetry condition.

Proof of Theorem 1.3. From (5.27) and (5.26), we obtain

$$|K_1 - K_2| \geq \frac{1}{C(a_{11} + a_{22})} |b_2(a_{12} + a_{11}) - b_1(a_{21} + a_{22})|. \tag{6.1}$$

Under the symmetry assumptions of Theorem 1.3, the domain $\tilde{\Omega}$ is symmetry with respect to $\{x_n = \frac{\varepsilon}{2}\}$, which implies

$$a_{11} = a_{22}, \quad a_{12} = a_{21}.$$

Moreover, by Lemma 5.5 (2),

$$\frac{1}{C} < a_{11} + a_{12} = a_{22} + a_{21} = a_{11} + a_{12} < C.$$

Hence, inequality (6.1) simplifies to

$$|K_1 - K_2| \geq \frac{1}{Ca_{11}} |b_1 - b_2|.$$

It therefore suffices to prove that $|b_1 - b_2| > \frac{1}{C}$.

To this end, introduce the change of variables

$$y' = x', \quad y_n = x_n - \frac{\varepsilon}{2},$$

and define $w(y) = u_3(x)$. Then w satisfies

$$\begin{cases} \Delta w = 0 & \text{in } B_6 \backslash (D_1^\varepsilon \cup D_2^\varepsilon), \\ w + \gamma \partial_\nu w = 0 & \text{on } \partial D_1^\varepsilon \cup \partial D_2^\varepsilon, \\ w = y_n & \text{on } \partial\Omega, \end{cases}$$

where B_6 is the ball of radius 6 centered at the origin, and D_1^ε, D_2^ε are unit balls centered at $(0', 1 + \frac{\varepsilon}{2})$ and $(0', -1 - \frac{\varepsilon}{2})$, respectively. By symmetry, $w(y', y_n) = -w(y', -y_n)$, so $w(y', 0) = 0$, and

$$b_1 - b_2 = \int_{D_1^\varepsilon} \frac{\partial w}{\partial \nu} - \int_{D_2^\varepsilon} \frac{\partial w}{\partial \nu} = 2 \int_{D_1^\varepsilon} \frac{\partial w}{\partial \nu}.$$

We now claim that for any $\varepsilon, \gamma \in (0, \frac{1}{4})$, $-C < \int_{\partial D_1^\varepsilon} \frac{\partial w}{\partial \nu} \leq -\frac{1}{C}$ with $C > 0$ independent of ε and γ.

To prove this, observe that w solves the boundary value problem in the upper half-ball

$$\begin{cases} \Delta w = 0 & \text{in } B_6^+, \\ w + \gamma \partial_\nu w = 0 & \text{on } \partial D_1^\varepsilon, \\ w = y_n & \text{on } \partial B_6^+, \end{cases}$$

where $B_6^+ = \{B_6\} \cap y_n > 0$. The maximum principle implies $w \geq 0$ in B_6^+.

We now construct a barrier function. Define

$$\hat{w}(y) = c_1(y_n - 1 - \frac{\varepsilon}{2})(|y'|^2 + (y_n - 1 - \frac{\varepsilon}{2})^2 - 1) + c_2\gamma(y_n - 1 - \frac{\varepsilon}{2})^2$$

in the domain $A = B_6^+ \cap \{y_n > 1+\frac{\varepsilon}{2}\} \backslash \overline{D_1^\varepsilon}$, with $c_1 = \frac{1}{500}$ and $c_2 = \frac{1}{3000}$. A direct computation shows that

$$\begin{cases} \Delta \hat{w} \geq 0 & \text{in } A, \\ \hat{w} \leq y_n & \text{on } A_1 := \partial B_6^+ \cap \{y_n > 1 + \dfrac{\varepsilon}{2}\}, \\ \hat{w} = 0 & \text{on } A_2 := \{y_n = 1 + \dfrac{\varepsilon}{2}\} \cap \big(B_6^+ \backslash D_1^\varepsilon\big), \\ \hat{w} + \gamma \dfrac{\partial \hat{w}}{\partial \nu} \leq 0 & \text{on } A_3 := \{y_n > 1 + \dfrac{\varepsilon}{2}\} \cap \partial D_1^\varepsilon. \end{cases}$$

Since $w - \hat{w}$ is superharmonic in A, its minimum on $\bar{A}$ is attained on ∂A. Let $(w-\hat{w})(x_0) = \min\limits_{x \in \bar{A}} (w - \hat{w})(x)$ for some $x_0 \in \partial A$. Suppose $(w - \hat{w})(x_0) < 0$. Since $w - \hat{w} \geq 0$ on $A_1 \cup A_2$, it follows that $x_0 \in A_3$. At this point,

$$\frac{\partial(w - \hat{w})}{\partial \nu}(x_0) \leq 0, \quad \text{and } (w - \hat{w})(x_0) + \gamma \frac{\partial(w - \hat{w})}{\partial \nu}(x_0) < 0,$$

which contradicts the boundary condition $w + \gamma \partial_\nu w = 0$ on A_3. Hence, $w \geq \hat{w}$ throughout $\overline{A}$.

For $\varepsilon \in (0, \frac{1}{4})$, we find that $w(x) \geq \hat{w}(x) \geq \frac{c_2}{64}\gamma$ on the portion $\partial D_1^\varepsilon \cap \{y_n \in (\frac{5}{4}, 2)\}$. Since $w \geq 0$ in B_6^+, it follows that

$$\int_{\partial D_1^\varepsilon} \frac{\partial w}{\partial \nu}\, dS = \int_{\partial D_1^\varepsilon} -\frac{w}{\gamma}\, dS \leq \int_{\partial D_1^\varepsilon \cap \{y_n \in (\frac{5}{4}, 2)\}} -\frac{w}{\gamma}\, dS \leq \int_{\partial D_1^\varepsilon \cap \{y_n \in (\frac{5}{4}, 2)\}} -\frac{c_2}{64}\, dS \leq -\frac{1}{C}.$$

This, together with the upper bound from Lemma 5.5 (3), yields

$$-C \leq \int_{\partial D_1^\varepsilon} \frac{\partial w}{\partial \nu}\, dS \leq -\frac{1}{C},$$

which completes the proof of Theorem 1.3. □

A The perfect-conductivity limit

We consider the perfect-conductor model

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{(D_1 \cup D_2)}, \\ u = K_j & \text{on } \partial D_j, j = 1, 2, \\ \displaystyle\int_{\partial D_j} \partial_\nu u \, dS = 0 & j = 1, 2, \\ u = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where the free constant K_j is uniquely determined by the zero-flux conditions (third line). The existence and uniqueness of the solution to (A.1) follows from [6]. We now prove that solutions u_γ of (1.4) converge to the solution u_0 of (A.1) as $\gamma \to 0$.

Theorem A.1. *Let u_γ and u_0 be the solutions of* (1.4) *and* (A.1)*, respectively. Then*

$$u_\gamma \rightharpoonup u_0 \quad \textit{in } H^1(\tilde{\Omega}) \quad \textit{as } \gamma \to +0,$$

and

$$\lim_{\gamma \to 0} \|u_\gamma - u_0\|_{L^2(\Omega \setminus \overline{D_1 \cup D_2})} = 0.$$

Proof. By the uniqueness of u_0, it suffices to show that any subsequence $\{u_{\gamma_j}\}_{j=1}^{j=+\infty}$ with $\gamma_j \to 0^+$ as $j \to +\infty$ has a subsequence (still denoted by $\{u_{\gamma_j}\}$), such that

$$u_{\gamma_j} \rightharpoonup u_0 \quad \text{in } H^1(\tilde{\Omega}) \quad \text{as } \gamma_j \to +0,$$

and

$$\lim_{\gamma_j \to 0} \|u_{\gamma_j} - u_0\|_{L^2(\Omega \setminus \overline{D_1 \cup D_2})} = 0.$$

The proof is divided into three steps.

Step 1. Uniform bound. We first establish a uniform bound $|u_\gamma|_{H^1(\tilde{\Omega})} \le M$ for some constant $M > 0$ independent of γ.

Indeed, since u_γ is the minimizer of I_γ in $\mathscr{A} := \{v \in H^1(\widetilde{\Omega}) : v = \varphi \text{ on } \partial\Omega\}$, we have, for any fixed function $\eta \in H^1_\varphi$ with $\eta = 1$ on $\partial D_1 \cup \partial D_2$,

$$\frac{1}{2} \int_{\tilde{\Omega}} |Du_\gamma|^2 \, dx \le I_\gamma(u_\gamma) \le I_\gamma(\eta) = \frac{1}{2} \int_{\tilde{\Omega}} |D\eta|^2 \, dx.$$

Combining this with the boundary condition $u_\gamma = \varphi$ on $\partial\Omega$, we obtain the uniform estimate

$$\|u_\gamma\|_{H^1(\tilde{\Omega})} \le M.$$

Consequently, there exists a subsequence (still denoted by $\{u_{\gamma_j}\}$) that converges weakly in $H^1(\tilde{\Omega})$ to some limit u_0.

Step 2. The limit is harmonic. We show that u_0 is harmonic in $\tilde{\Omega}$.

For any test function $\eta \in C_c^\infty(\tilde{\Omega})$, the weak formulation gives

$$\int_{\tilde{\Omega}} Du_\gamma \cdot D\eta = 0.$$

By the weak convergence of u_{γ_j} to u_0, we may pass to the limit to obtain $\int_{\tilde{\Omega}} Du_0 \cdot D\eta = 0$, which implies $\Delta u_0 = 0$ in $\tilde{\Omega}$.

Step 3. Boundary conditions. We verify that u_0 satisfies the boundary conditions of (A.1).

The energy bound $I_\gamma(u_\gamma) \le M$ implies

$$\sum_{i=1}^{2} \int_{\partial D_i} |u_\gamma - (u_\gamma)_{\partial D_i}|^2 \, dS \le 2M\gamma,$$

where $(u_\gamma)_{\partial D_i}$ denotes the average of u_γ over ∂D_i. Since $u_{\gamma_j} \rightharpoonup u_0$ in $H^1(\tilde{\Omega})$, the trace operator implies convergence in $L^2(\partial D_1)$. Therefore, for any $\eta \in L^2(\partial D_1)$, we have

$$\int_{\partial D_1} (u_0 - (u_0)_{\partial_{D_1}})\eta \, dS = \lim_{j \to \infty} \int_{\partial D_1} (u_{\gamma_j} - (u_{\gamma_j})_{\partial_{D_1}})\eta \, dS = 0.$$

It follows that $u_0 - (u_0)_{\partial_{D_1}} = 0$ on ∂D_1, i.e., $u_0 = K_1$ on ∂D_1. A similar argument shows $u_0 = K_2$ on ∂D_2.

Finally, we prove the flux conditions. Let $\eta \in C^\infty(\bar{\tilde{\Omega}})$ be a test function such that $\eta = 1$ on ∂D_1 and $\eta = 0$ on $\partial\Omega \cup \partial D_2$. Consider the variation

$$i(t) = I_\gamma(u_\gamma + t\eta) \quad (t \in \mathbb{R}).$$

The first variation condition yields

$$i'(0) = \frac{di}{dt}\Big|_{t=0} = \int_{\tilde{\Omega}} Du_\gamma \cdot D\eta \, dx = 0.$$

Using the weak convergence $u_{\gamma_j} \rightharpoonup u_0$ and applying the divergence theorem, we find

$$\int_{\partial D_1} \partial_\nu u_0 \, dS = \int_{\tilde{\Omega}} Du_0 \cdot D\eta \, dx = \lim_{j\to\infty} \int_{\tilde{\Omega}} Du_{\gamma_j} \cdot D\eta \, dx = 0.$$

An analogous argument applied to ∂D_2 shows that the flux through ∂D_2 also vanishes. This completes the proof that u_0 is the solution to (A.1). □

Declarations

Data availability. This is a theoretical research, and no data was used or generated.
Conflict of interest. The authors declare that they have no conflict of interest.

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