

THE CHEVALLEY–WEIL FORMULA FOR FINITE GROUP ACTIONS ON HIGHER DIMENSIONAL COMPACT COMPLEX MANIFOLDS

WENFEI LIU AND RENJIE LYU

ABSTRACT. Building on the Atiyah–Singer holomorphic Lefschetz fixed-point theorem, we define ramification modules associated to the fixed loci of a finite group acting on a compact complex manifold. This allows us to generalize the Chevalley–Weil formula for compact Riemann surfaces to higher dimensions. More precisely, let G be a finite group acting on a compact complex manifold X , and let $\mathcal{E}$ be a G -equivariant locally free sheaf on X . Then, in the representation ring $R(G)_{\mathbb{Q}}$, we have

$$\chi_G(X, \mathcal{E}) := \sum_{i=0}^{\dim X} (-1)^i [H^i(X, \mathcal{E})] = \frac{1}{|G|} \chi(X, \mathcal{E})[\mathbb{C}[G]] + \sum_Z \Gamma(\mathcal{E})_Z$$

where Z runs over all connected components of the fixed-point sets X^g for $g \in G$, and each $\Gamma(\mathcal{E})_Z \in R(X)_{\mathbb{Q}}$, called the *ramification module* at Z , depends only on the restriction $\mathcal{E}|_Z$ and the normal bundle $N_{Z/X}$ as G_Z -equivariant bundles. We illustrate the computation of $\Gamma(\mathcal{E})_Z$ in several special cases and provide a detailed example for faithful actions of $G \cong (\mathbb{Z}/2\mathbb{Z})^n$ on a compact complex surface.

CONTENTS

1. Introduction	1
2. Preliminaries	3
2.1. Constructions around the representation rings	3
2.2. Recovering a representation from its restrictions to cyclic subgroups	4
2.3. G -equivariant K -theory and the holomorphic Lefschetz fixed-point theorem	5
3. The Chevalley–Weil formula for compact complex manifolds	8
4. Computation of the ramification module in special cases	12
5. The action of $(\mathbb{Z}/2\mathbb{Z})^n$ on a compact complex surface	14
References	16

1. INTRODUCTION

Let X be a compact Riemann surface, and $G \subset \text{Aut}(X)$ a finite subgroup of automorphisms. The classical Chevalley–Weil formula ([CW34, Wei35]) expresses the G -module $H^0(X, \omega_X^{\otimes n})$ of n -differentials as a rational multiple of the regular representation $\mathbb{C}[G]$, plus a correction term determined by the branched locus of the quotient map $X \rightarrow X/G$. This result was generalized in [EL80] to a formula for more general G -equivariant locally sheaves on smooth projective tame G -curves over an arbitrary algebraically closed field. Subsequent work has produced many further refinements with an arithmetic flavor; see [LL25] for a relatively complete list of references. Recently, the Chevalley–Weil formula for proper singular curves has been established in [Ton25, LL25].

As explicitly illustrated in [Koc05] and [Ara22], the Chevalley–Weil formula for G -equivariant locally free sheaves on a compact Riemann surface can be deduced from a more general fixed-point formulism ([AS68III]). In this paper, we extend this framework

Date: October 14, 2025.

2020 Mathematics Subject Classification. Primary: 14J50; Secondary: 20C15.

Key words and phrases. Compact complex manifold, finite Group action, Chevalley–Weil formula, holomorphic Lefschetz fixed-point formula.

to higher dimensions. Unlike the approach in [EL80], which relies on the quotient map $X \rightarrow X/G$, our method applies the Atiyah–Singer holomorphic Lefschetz fixed-point theorem directly to capture the contributions from the fixed loci via the so-called ramification modules; see Definition 3.9.

Theorem 1.1. *Let X be a compact complex manifold, possibly disconnected and non-equidimensional, and let G be a finite group acting on X . Let $\mathcal{E}$ be a G -equivariant locally free sheaf on X . Let $\chi_G(X, \mathcal{E}) := \sum_j (-1)^j [H^j(X, \mathcal{E})]$ be the G -Euler characteristic of $\mathcal{E}$, where $[H^j(X, \mathcal{E})]$ denotes the class of $H^j(X, \mathcal{E})$ in the Grothendieck group $R(G)$ of G -modules. Then we have an equality in $R(G)_{\mathbb{Q}}$:*

$$(1.1) \quad \chi_G(X, \mathcal{E}) := \sum_{i=0}^{\dim X} (-1)^i [H^i(X, \mathcal{E})] = \frac{1}{|G|} \chi(X, \mathcal{E}) [\mathbb{C}[G]] + \sum_Z \Gamma(\mathcal{E})_Z$$

where Z runs through the components of the fixed loci X^g with $g \in G$, and $\Gamma(\mathcal{E})_Z \in R(G)_{\mathbb{Q}}$ is the ramification module at Z , given in Definition 3.9.

One has $\chi_G(X, \mathcal{E}) = \frac{1}{|G|} \chi(X, \mathcal{E}) [\mathbb{C}[G]]$ if the action of G on X is free. Thus, the ramification modules $\Gamma(\mathcal{E})_Z$ can be viewed as correction terms arising from the deviation of the G -action being free.

The key part in proving Theorem 1.1 is the definition of the ramification modules $\Gamma(\mathcal{E})_Z$. For $g \in G$, let $H := \langle g \rangle$ be the cyclic group it generates, and let X^H be the fixed locus of H . Let $N^* := N_{X^H/X}^*$ be the conormal bundle of X^H , and let $\lambda_{-1} N^* = \sum_{i \geq 0} [\wedge^i N^*] \in K_H(X^H)$ the fundamental class of N^* in the H -equivariant K -group of X^H . Then Atiyah–Singer’s holomorphic Lefschetz fixed-point theorem (Theorem 2.5) computes the trace on the G -Euler characteristic:

$$(1.2) \quad \text{Tr}(g; \chi_G(X, \mathcal{E})) = \text{Tr} \left(g; \int_{X^H} \frac{\text{ch}_H(\mathcal{E}|_{X^H}) \text{td}(X^H)}{\text{ch}_H(\lambda_{-1} N^*)} \right).$$

Here, ch_H is the H -Chern character (Definition 3.3):

$$K_H(X^H) = K(X^H) \otimes R(H) \xrightarrow{\text{ch}_H} H^{\text{even}}(X^H, \mathbb{Z}) \otimes R(H).$$

Let $\mathfrak{p}_g := \ker(\text{Tr}_g: R(H) \rightarrow \mathbb{C})$ be the prime ideal of virtual H -modules with vanishing trace at g , and let $R(H)_g$ be the localization of $R(H)$ at $\mathfrak{p}_g$. The trace $\text{Tr}(g; \lambda_{-1} N^*)$ is non-zero, so $\lambda_{-1} N^*$ is a unit in $R(H)_g$. The trace formula (1.2) then implies that the integral expression $\int_{X^H} \frac{\text{ch}_H(\mathcal{E}|_{X^H}) \text{td}(X^H)}{\text{ch}_H(\lambda_{-1} N^*)}$ is the image of $\chi_G(X, \mathcal{E})$ in $R(H)_g$ under the homomorphism

$$R(G) \xrightarrow{\text{Res}_H^G} R(H) \xrightarrow{\text{loc}_g} R(H)_g,$$

where Res_H^G is the restriction map and loc_g is the localization at $\mathfrak{p}_g$. Thus, the problem reduces to recovering the global G -Euler characteristic $\chi_G(X, \mathcal{E})$ from its localized restrictions to cyclic subgroups.

Using Artin’s theorem in Section 2.2, one can theoretically recover the virtual G -module $\chi_G(X, \mathcal{E})$ from its restrictions to all cyclic subgroups of G (Proposition 2.3):

$$(1.3) \quad \chi_G(X, \mathcal{E}) = \sum_{H \subset G \text{ cyclic}} \frac{|H|}{|G|} \text{Ind}_H^G(\theta_H \otimes \text{Res}_H^G \chi_G(X, \mathcal{E})),$$

where θ_H is the characteristic module on the generators of H (see Definition 2.2).

Our goal is to reformulate the H -module $\theta_H \otimes \text{Res}_H^G \chi_G(X, \mathcal{E})$ in terms of ramification modules, which are characterized by the localized fixed-point contributions. For any cyclic subgroup $H \subset G$ and each component Z in X^H , we show in Lemma 3.6 the existence of an element $\tau_{Z,H} \in R(H)_{\mathbb{Q}}$ such that

$$\theta_H \otimes \text{Res}_H^G \chi_G(X, \mathcal{E}) = \sum_Z \int_Z \theta_H \tau_{Z,H} \text{ch}_H(\mathcal{E}|_Z) \text{td}(Z).$$

This leads to Definition 3.9 for the ramification module

$$\Gamma(\mathcal{E})_Z = \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G|} \text{Ind}_H^G \left(\theta_H \int_Z \text{ch}_H(\mathcal{E}|_Z) \tau_{Z,H} \text{td}(Z) \right),$$

where $\mathcal{H}_Z$ is the set of cyclic subgroups $H \subset G$ for which Z is a component of X^H . The equality in Theorem 1.1 is then a direct consequence of this definition.

The Chevalley–Weil formula (1.1) facilitates the computation of the multiplicity $\mu_M \chi_G(X, \mathcal{E})$ of an irreducible G -module M in the virtual G -module $\chi_G(X, \mathcal{E})$ (Corollary 3.14). Its practical utility depends on the computability of the ramification modules $\Gamma(\mathcal{E})_Z$. In Section 4, we present explicit forms of these ramification modules $\Gamma(\mathcal{E})_Z$ under one of the following conditions:

- (i) The stabilizer G_Z is cyclic (Lemma 4.1).
- (ii) $\text{codim}_X(Z) \leq 1$ or $\dim Z = 0$ ((Examples 4.2–4.4).

In Section 5, we determine $\chi_G(X, \mathcal{E})$ explicitly when $G \cong (\mathbb{Z}/2\mathbb{Z})^n$ and X is a compact complex surface. In this case, both conditions (i) and (ii) are easily verified.

Notation and Conventions. Let X be a set endowed with an action of a group G . We call X a G -set. For a subset $Z \subset X$, the subgroup

$$G_Z := \{g \in G \mid g(x) = x, \forall x \in Z\}$$

is called the (*pointwise*) *stabilizer* of Z . For a subset $K \subset G$, the *fixed locus* of K is

$$X^K := \{x \mid g(x) = x, \forall g \in K\}.$$

We denote the neutral element of G by id_G or id , and the order of G by $|G|$. For $g \in G$, we use $|g|$ to denote its order.

A G -manifold is a G -set X endowed with a manifold structure preserved by the G -action. Similarly, one defines a G -vector space, a complex G -manifold, and so on, by imposing a structure compatible with the G -action. A quasi-coherent G -sheaf (or G -equivariant sheaf) on a complex G -manifold X means a quasi-coherent sheaf $\mathcal{F}$ together with isomorphisms $\Phi_g: g^* \mathcal{F} \rightarrow \mathcal{F}$ for each $g \in G$ satisfying the following compatibility condition ([Bri25]):

$$\Phi_{gh} = \Phi_h \circ h^*(\Phi_g), \quad \text{for all } g, h \in G.$$

Let R be a commutative ring with unit. For two R -modules M and N , their tensor product $M \otimes_R N$ over R is simply denoted by $M \otimes N$ if R is clear from the context. Suppose that A and B are two commutative R -algebras with units and that one of them is free as a R -module. Then $\iota_A: A \rightarrow A \otimes_R B$, $a \mapsto a \otimes 1$ and $\iota_B: B \rightarrow A \otimes_R B$, $b \mapsto 1 \otimes b$ are injective ring homomorphisms, and we may view A and B as sub- R -algebras of $A \otimes_R B$ via ι_A and ι_B . For $a \in A$ and $b \in B$, we will often write ab or $a \cdot b$ for $a \otimes b$, which will result in no ambiguity. For a $\mathbb{Z}$ -module M and a $\mathbb{Z}$ -algebra K , We usually denote $M \otimes_{\mathbb{Z}} K$ by M_K .

Acknowledgment W.L. would like to thank Qing Liu for many helpful discussions on the Chevalley–Weil formula of curves during the cooperation on the paper [LL25]; the idea of defining the ramification module originates from there. He would also like to thank Lie Fu for his interest on the subject and for pointing out the notion of inertia stack, which inspires our definition of strata on a G -manifold. This research project has been supported by the NSFC (No. 12571046).

2. PRELIMINARIES

2.1. Constructions around the representation rings. The material of this subsection is based on [Ser77], [Seg68] and [AS68II, Section 1]; see also [LL25, Section 2.1].

Let G be a finite group and k a field. The *Grothendieck ring* $R_k(G)$ of k -representations of G , also called the *representation ring of G over k* , is the abelian group of formal finite $\mathbb{Z}$ -linear combinations $\sum_i n_i V_i$ of k -representations V_i of G , modulo the relations $V' + V'' - V$ for every G -equivariant exact sequence $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$. Since we primarily deal with complex representations, we denote $R_{\mathbb{C}}(G)$ by $R(G)$, and a representation without a specified base field means a complex representation.

The class of a representation V of G in $R(G)$ is denoted by $[V]_G$ (or $[V]$ if the group is clear). Elements of $R(G)$ are called *virtual G -modules*. The ring structure on $R(G)$ is induced by the tensor product, with $([V], [W]) \mapsto [V \otimes W]$, and the multiplicative unit is $[1_G]$, the class of the one-dimensional trivial representation of G . The classes of one-dimensional irreducible representations form a multiplicative subgroup of $R(G)$, and we can talk about the order of an element in this subgroup. To avoid confusion with other tensor products, we denote the multiplication of $\alpha, \beta \in R(G)$ by $\alpha \cdot \beta$ or $\alpha\beta$. For example, if $\alpha = [V]$ and $\beta = [W]$, then $\alpha \cdot \beta = [V \otimes W]$.

The underlying additive abelian group of $R(G)$ is free, with a basis given by the irreducible representations of G . Consequently, the natural homomorphisms $R(G) \rightarrow R(G)_{\mathbb{Q}} \rightarrow R(G)_{\mathbb{C}}$ are injective. A fundamental result is the ring isomorphism $R(G)_{\mathbb{C}} \rightarrow C(G)$, which sends $\alpha \in R(G)_{\mathbb{C}}$ to its character function $c(\alpha): g \mapsto \text{Tr}(g; \alpha)$, where $C(G)$ is the $\mathbb{C}$ -vector space of class functions on G . Since $C(G)$ has a trivial nilradical, so do $R(G)_{\mathbb{C}}$ and its subrings $R(G)$ and $R(G)_{\mathbb{Q}}$. The inner product of two elements $\alpha, \beta \in R(G)_{\mathbb{C}}$ is defined via their characters:

$$\langle \alpha, \beta \rangle_G := \langle c(\alpha), c(\beta) \rangle_G := \frac{1}{|G|} \sum_g \text{Tr}(g; \alpha) \cdot \overline{\text{Tr}(g; \beta)}.$$

The support of $\phi \in R(G)_{\mathbb{C}}$ is defined as

$$(2.1) \quad \text{Supp}(\phi) := \{g \in G \mid \text{Tr}(g; \phi) \neq 0\}.$$

We recall several standard maps associated with $R(G)$:

- For a subgroup $H \subset G$, we have the *restriction map* $\text{Res}_H^G: R(G) \rightarrow R(H)$ and the *induction map* $\text{Ind}_H^G: R(H) \rightarrow R(G)$.
- For $g \in G$, the trace defines a ring homomorphism

$$\text{Tr}_g: R(G) \rightarrow \mathbb{C}, \quad \alpha \mapsto \text{Tr}(g; \alpha)$$

Since $\mathbb{C}$ is an integral domain, the kernel $\mathfrak{p}_g := \ker \text{Tr}_g$ is a prime ideal of $R(G)$, and we denote by $R(G)_g := R(G)_{\mathfrak{p}_g}$ the localization at $\mathfrak{p}_g$. For $g = \text{id}$, the prime ideal $I_G := \mathfrak{p}_{\text{id}}$ is called the *augmentation ideal* of $R(G)$. Note that, $R(G)$ can have zero divisors,^{*} and the natural map $R(G) \rightarrow R(G)_g$ is in general neither injective nor surjective.

Lemma 2.1. *Suppose that $G = \langle g_0 \rangle$ is a nontrivial finite cyclic group, and let $\phi \in \widehat{G} \setminus \{1_G\}$ be a non-trivial irreducible representation. Let $K = \{g \in G \mid \phi(g) = 1\}$. Then, for $\psi \in R(G)$ such that $\text{Tr}(g; \psi) = 0$ for all $g \in K$, the element $\frac{|\phi|\psi}{1-\phi} \in R(G)_{g_0}$ is the image of $-\psi \cdot \sum_{d=0}^{|\phi|-1} d\phi^d \in R(G)$ under the localization map $R(G) \rightarrow R(G)_{g_0}$, where $|\phi|$ denotes the order of the character ϕ .*

Proof. Using [Koc05, Lemma 1.2] and the assumption that $\text{Tr}(g; \psi) = 0$ for $g \in K$, a direct calculation shows that

$$\text{Tr} \left(g; \psi \cdot \sum_{d=0}^{|\phi|-1} d\phi^d (1 - \phi) \right) = -|\phi| \text{Tr}(g; \psi), \quad \text{for all } g \in G.$$

The result follows from this identity. $\square$

2.2. Recovering a representation from its restrictions to cyclic subgroups. Let G be a finite group. By Artin's theorem [Ser77, 9.2], every element $\chi \in R(G)$ is a $\mathbb{Q}$ -linear combination of characters induced from cyclic subgroup of G . In this subsection, we provide a concrete expression of this fact in Proposition 2.3.

^{*}For example, $\chi_{\text{reg}} \cdot I_G = 0$, where χ_{reg} is the the class of the regular representation.

Definition 2.2. For a cyclic group H , we define an element $\theta_H \in \frac{1}{|H|}R(H)$ such that ([Ser77, Section 9.4])[†]:

$$(2.2) \quad \text{Tr}(h; \theta_H) = \begin{cases} 1 & \text{if } h \text{ generates } H, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\text{Tr}(\cdot; \theta_H)$ is the characteristic function on the set of generators of H . We call θ_H the *characteristic module* supported on the generators of H .

For any finite group G , it is known by [Ser77, Prop. 27] that

$$(2.3) \quad |G| \cdot [1_G] = \sum_{B \subset G} |B| \cdot \text{Ind}_B^G \theta_B$$

where B runs over all the cyclic subgroups of G . In particular, for any cyclic group H , we obtain

$$(2.4) \quad \theta_H = [1_H] - \sum_{B \subset H} \frac{|B|}{|H|} \cdot \text{Ind}_B^H \theta_B$$

where B runs over all the proper cyclic subgroups of H . This shows that θ_H belongs to $\frac{1}{|H|}R(H)$. If H has prime order, then

$$\theta_H = [1_H] - \text{Ind}_{\text{id}}^H \theta_{\text{id}} = [1_H] - \frac{1}{|H|}[\mathbb{C}[H]].$$

Note that, for a nontrivial cyclic group H , the trace $\text{Tr}(1; \theta_H) = 0$, meaning θ_H lies in the augmentation ideal $I(H)_{\mathbb{Q}}$.

Using θ_H , the following proposition recovers a virtual G -module from its restriction to the cyclic subgroups of G .

Proposition 2.3. *Let G be a finite group, and let $\chi \in R(G)_{\mathbb{Q}}$ be a virtual G -module with rational coefficients. Then*

$$(2.5) \quad \chi = \sum_{H \subset G} \frac{|H|}{|G|} \text{Ind}_H^G (\theta_H \cdot \text{Res}_H^G \chi) \in R(G)_{\mathbb{Q}}$$

where the sum runs over all the cyclic subgroups H of G .

Proof. Recall formula (2.3):

$$[1_G] = \frac{|H|}{|G|} \sum_{H \subset G} \text{Ind}_H^G \theta_H,$$

Multiplying this equation by χ yields

$$\chi = \frac{|H|}{|G|} \sum_{\substack{H \subset G \\ \text{cyclic}}} \text{Ind}_H^G \theta_H \cdot \chi = \sum_{\substack{H \subset G \\ \text{cyclic}}} \frac{|H|}{|G|} \text{Ind}_H^G (\theta_H \cdot \text{Res}_H^G \chi),$$

where the second equality follows from [Ser77, Section 3.3, Example 5]. $\square$

2.3. G -equivariant K -theory and the holomorphic Lefschetz fixed-point theorem.

Let G be a finite group and X be a compact complex G -manifold, probably disconnected and non-equidimensional. [‡] Let $K_G(X)$ be the Grothendieck ring of G -equivariant locally free sheaves (of finite rank) on X . If G is trivial, we drop the subscript and write $K(X)$.

Suppose we have a homomorphism of finite groups $\varphi: H \rightarrow G$, a compact complex H -manifold Z , and a map $f: Z \rightarrow X$ such that $f(h(z)) = \varphi(h)(f(z))$ for all $h \in H, z \in Z$. Then pulling back locally free sheaves induces a natural ring homomorphism:

$$\text{Res}_H^G: K_G(X) \rightarrow K_H(Z), \quad [\mathcal{E}] \mapsto [f^* \mathcal{E}]$$

This applies in particular to the inclusion of a subgroup $H \subset G$.

[†]Note that our definition of θ_H differs from that in [Ser77, Section 9.4] by a factor of $\frac{1}{|H|}$.

[‡]For example, the fixed locus M^σ for a finite-order automorphism σ of a connected compact complex manifold M can be disconnected and non-equidimensional.

The map $f : X \rightarrow \text{pt}$ to a point induces a pullback ring homomorphism

$$f^* : R(G) = K_G(\text{pt}) \rightarrow K_G(X),$$

making $K_G(X)$ into a $R(G)$ -algebra. For a locally free G -sheaf $\mathcal{E}$ on X , its image under the Gysin map $f_! : K_G(X) \rightarrow K_G(\text{pt}) \cong R(G)$ is the G -Euler characteristic of $\mathcal{E}$:

$$\chi_G(X, \mathcal{E}) := f_![\mathcal{E}] = \sum_{i=0}^{\dim X} (-1)^i [H^i(X, \mathcal{E})] \in R(G).$$

Now suppose the action of G on X is trivial. Then there is a ring isomorphism ([Seg68, Proposition 2.2])

$$(2.6) \quad \nu : K_G(X) \rightarrow K(X) \otimes R(G), \quad \nu(\mathcal{E}) = \sum_{M \in \widehat{G}} \text{Hom}_G(M, \mathcal{E}) \otimes [M],$$

where $\widehat{G}$ denotes the set of isomorphism classes of irreducible G -modules, and $\text{Hom}_G(M, \mathcal{E})$ denotes the sheaf of G -equivariant homomorphisms. Via this isomorphism, we have a ring homomorphism ([Ill68, 1.5.4])

$$(2.7) \quad \text{ch}_G : K_G(X) \cong K(X) \otimes R(G) \rightarrow H^{\text{even}}(X, \mathbb{Z}) \otimes R(G), \quad u \otimes \phi \mapsto \text{ch}(u)\phi$$

where $H^{\text{even}}(X, \mathbb{Z})$ denotes the even-degree part of cohomology ring $H^*(X, \mathbb{Z})$ and $\text{ch} : K(X) \rightarrow H^{\text{even}}(X, \mathbb{Z})$ is the usual Chern character. For a given $g \in G$, taking the trace of g yields a ring homomorphism

$$\text{Tr}_g : K(X) \otimes R(G) \rightarrow H^{\text{even}}(X, \mathbb{C}), \quad u \otimes \phi \mapsto \text{Tr}(g; \phi) \text{ch}(u)$$

There is also an integration map:

$$\int_X : H^{\text{even}}(X, \mathbb{Z}) \otimes R(G) \rightarrow R(G), \quad u \otimes \phi \mapsto \left(\int_X u \right) \phi.$$

Next we state the compatibility of the maps introduced above.

Lemma 2.4. *Let G be a finite group acting trivially on a compact complex manifold X , which may be disconnected and non-equidimensional, and let $H \subset G$ be a subgroup. Then the following diagrams commute:*

(i)

$$\begin{array}{ccc} K_G(X) & \xrightarrow{\nu} & K(X) \otimes R(G) \\ f_! \downarrow & & \downarrow f_! \otimes \text{id} \\ K_G(\text{pt}) & \xrightarrow{\nu} & K(\text{pt}) \otimes R(G). \end{array}$$

(ii)

$$\begin{array}{ccc} K_G(X) & \xrightarrow{\nu} & K(X) \otimes R(G) \\ \text{Res}_H^G \downarrow & & \downarrow \text{id} \otimes \text{Res}_H^G \\ K_H(X) & \xrightarrow{\nu} & K(X) \otimes R(H) \end{array}$$

(iii)

$$\begin{array}{ccc} K_G(X) & \xrightarrow{\text{ch}_G} & H^{\text{even}}(X, \mathbb{Z}) \otimes R(G) \\ \text{Res}_H^G \downarrow & & \downarrow \text{id} \otimes \text{Res}_H^G \\ K_H(X) & \xrightarrow{\text{ch}_H} & H^{\text{even}}(X, \mathbb{Z}) \otimes R(H) \end{array}$$

Proof. (i) By definition of $f_!$, we have

$$f_!(\mathcal{E}) = \chi_G(X, \mathcal{E}) = \sum_{[M] \in \widehat{G}} \chi(X, \text{Hom}_G(M, \mathcal{E})) \otimes [M] = \sum_{[M] \in \widehat{G}} f_!(\text{Hom}_G(M, \mathcal{E})) \otimes [M]$$

(ii) We need to prove the identity

$$(2.8) \quad \sum_{N \in \hat{H}} \text{Hom}_H(N, \text{Res}_H^G \mathcal{E}) \otimes N = \sum_{M \in \hat{G}} \text{Hom}_G(M, \mathcal{E}) \otimes \text{Res}_H^G M.$$

where M (resp. N) runs over irreducible G -modules (resp. H -modules).

Note that

$$\text{Res}_H^G M = \sum_{N \in \hat{H}} \mu_{MN} N, \quad \text{where } \mu_{MN} = \langle \text{Res}_H^G M, N \rangle_H.$$

Thus, the right-hand side of (2.8) equals

$$\sum_{N \in \hat{H}} \text{Hom}_G\left(\sum_{M \in \hat{G}} \mu_{MN} M, \mathcal{E}\right) \otimes N.$$

The Frobenius reciprocity

$$\langle \text{Res}_H^G M, N \rangle_H = \langle M, \text{Ind}_H^G N \rangle_G$$

implies that

$$\text{Ind}_H^G N = \sum_{M \in \hat{G}} \mu_{MN} M.$$

Hence

$$\sum_{N \in \hat{H}} \text{Hom}_G\left(\sum_{M \in \hat{G}} \mu_{MN} M, \mathcal{E}\right) \otimes N = \sum_{N \in \hat{H}} \text{Hom}_G(\text{Ind}_H^G N, \mathcal{E}) \otimes N.$$

This is equal to the left-hand side of (2.8) due to the adjunction isomorphism:

$$\text{Hom}_H(N, \text{Res}_H^G \mathcal{E}) = \text{Hom}_G(\text{Ind}_H^G N, \mathcal{E}).$$

(iii) For any $\epsilon \in K_G(X)$, write

$$\nu(\epsilon) = \sum_i u_i \otimes \phi_i, \quad u_i \in K(X), \phi_i \in R(G).$$

By part (ii), we have

$$(\text{id} \otimes \text{Res}_H^G) \circ \nu(\epsilon) = \nu(\text{Res}_H^G \epsilon).$$

It follows that

$$\begin{aligned} \text{ch}_H(\text{Res}_H^G \epsilon) &= \text{ch}_H((\text{id} \otimes \text{Res}_H^G) \circ \nu(\epsilon)) = \text{ch}_H\left(\sum_i u_i \otimes \text{Res}_H^G \phi_i\right) \\ &= \sum_i \text{ch}(u_i) \otimes \text{Res}_H^G \phi_i = \text{Res}_H^G \text{ch}_G(\epsilon). \end{aligned}$$

□

The main result of this paper is based on the following famous theorem of Atiyah and Singer.

Theorem 2.5 (Holomorphic Lefschetz fixed-point formula, [AS68III, Theorem 4.6]). *Let G be a finite group acting on a compact complex manifold X , probably disconnected and non-equidimensional, and let $\mathcal{E}$ be a locally free G -sheaf over X . Let X^g be the fixed locus of an element $g \in G$, and let $N^g = N_{X^g/X}$ the normal bundle of X^g in X . Then the following holds:*

$$(2.9) \quad \text{Tr}(g; \chi_G(X, \mathcal{E})) = \text{Tr}\left(g; \int_{X^g} \frac{\text{ch}_{\langle g \rangle}(\mathcal{E}|_{X^g}) \cdot \text{td}(X^g)}{\text{ch}_{\langle g \rangle}(\lambda_{-1}(N^g)^*)}\right),$$

where $(N^g)^*$ denotes the dual of N^g .

3. THE CHEVALLEY–WEIL FORMULA FOR COMPACT COMPLEX MANIFOLDS

Notation 3.1. Let X be a compact complex manifold, possibly disconnected and non-equidimensional, and let G be a finite group acting on X via a homomorphism $\rho: G \rightarrow \text{Aut}(X)$, which is not necessarily faithful. For brevity, we denote $g(x) := \rho(g)(x)$ for $g \in G$ and $x \in X$. Let $\mathcal{E}$ be a locally free G -sheaf on X . Note that $\ker \rho$ might act non-trivially on $\mathcal{E}$.

Definition 3.2. A component of X^g for some $g \in G$ is called a *stratum* of the pair (X, G) .

Let $\mathcal{Z}$ be the set of all strata. For two strata $Z_1, Z_2 \in \mathcal{Z}$, if $Z_1 \subsetneq Z_2$, their stabilizers satisfy $G_{Z_1} \supsetneq G_{Z_2}$. For each stratum $Z \in \mathcal{Z}$, define

$$\mathcal{H}_Z := \{H \subset G_Z \text{ cyclic} \mid Z \text{ is a component of } X^H\}$$

It is clear that $\mathcal{H}_Z$ is nonempty and its elements are subgroups of G_Z . Let $\mathcal{H}$ be the set of all cyclic subgroups of G . For each $H \in \mathcal{H}$, define

$$\mathcal{Z}_H := \{Z \in \mathcal{Z} \mid Z \text{ is a component of } X^H\}.$$

The set $\mathcal{Z}_H$ is empty if and only if X^H is empty.

Now for $Z \in \mathcal{Z}$ and a cyclic subgroup $H \subset G_Z$, the sheaf $\mathcal{E}|_Z$ is an H -equivariant sheaf on Z , and we may decompose $\mathcal{E}|_Z$ into eigensheaves with respect to the action of H :

$$\mathcal{E}|_Z = \bigoplus_{\phi \in \widehat{H}} \mathcal{E}_{Z,H,\phi}$$

where $\widehat{H}$ denotes the set of irreducible representations of H , and for $\phi \in \widehat{H}$, $\mathcal{E}_{Z,H,\phi}$ is the eigen-subsheaf of $\mathcal{E}|_Z$ with character ϕ .

Definition 3.3. The H -Chern character of $\mathcal{E}|_Z$ is defined as:

$$(3.1) \quad \text{ch}_H(\mathcal{E}|_Z) := \sum_{\phi \in \widehat{H}} \text{ch}(\mathcal{E}_{Z,H,\phi}) \otimes [\phi] \in H^{\text{even}}(Z, \mathbb{Z}) \otimes R(H).$$

Let us recall the description of $\text{ch}_H \lambda_{-1} N_{Z/X}^*$ given in [AS68III, Section 3]. Consider the eigen decomposition of the conormal bundle $N_{Z/X}^*$:

$$(3.2) \quad N_{Z/X}^* = \bigoplus_{\phi \in \widehat{H}} N_{Z,H,\phi}^*.$$

Let m_ϕ be the rank of the subbundle $N_{Z,H,\phi}^*$, and let $\{x_{k_j}, 1 \leq j \leq m_\phi\}$ be the Chern roots of $N_{Z,H,\phi}^*$. Then we have

$$(3.3) \quad \text{ch}_H \lambda_{-1}(N_{Z/X}^*) := \prod_{\phi \in \widehat{H}} \prod_{j=1}^{m_\phi} (1 - \phi e^{x_{k_j}}).$$

For notational convenience, we may write this as

$$\prod_{j=1}^m (1 - \phi_j e^{x_j})$$

where $\{x_j, 1 \leq j \leq m\}$ are the Chern roots of $N_{Z/X}^*$ with rank m , and $\{\phi_j, 1 \leq j \leq m\}$ is the set of (possibly repeated) eigen characters associated to $N_{Z/X}^*$. It is known that $\text{ch}_H \lambda_{-1}(N_{Z/X}^*)$ becomes a unit in $H^{\text{even}}(Z, \mathbb{Z}) \otimes R(H)_h$ by [AS68II, Lemma 2.7], where h is a generator of H . We show that there exists a "partial inverse" $\tau_{Z,H}$ of $\text{ch}_H \lambda_{-1}(N_{Z/X}^*)$ in $H^{\text{even}}(Z, \mathbb{Q}) \otimes R(H)_{\mathbb{Q}}$ in Lemma 3.5.

For a stratum Z and a cyclic group $H \in \mathcal{H}_Z$, define the subset $K_{Z,H} \subset H$ by:

$$(3.4) \quad K_{Z,H} = \begin{cases} \{h \in H \mid \text{Tr}(h; \phi_j) = 1 \text{ for some } 1 \leq j \leq m\} & \text{if } Z \text{ is not a component of } X \\ \{\text{id}\} & \text{if } Z \text{ is a component of } X \end{cases}$$

When Z is not a component of X , the set $K_{Z,H}$ consists exactly of those $h \in H$ for which Z is not a component of X^h . Define

$$(3.5) \quad \theta_{Z,H} = [1_H] - \sum_{B \subset K_{Z,H}} \frac{|B|}{|H|} \text{Ind}_B^H \theta_B \in R(H)_{\mathbb{Q}}$$

where the summation ranges over cyclic groups B contained in $K_{Z,H}$. Similar to θ_H in Definition 2.2, $\theta_{Z,H}$ is a characteristic module, whose support is the subset $H \setminus K_{Z,H}$. In fact, for any $h \in H \setminus K_{Z,H}$ and $B \subset K_{Z,H}$, we have

$$\theta_B(shs^{-1}) = \theta_B(h) = 0, \quad \forall s \in G.$$

For any $h \in K_{Z,H}$ with $\phi_j(h) = 1$ for some j , the cyclic subgroup $\langle h \rangle$ is contained in $K_{Z,H}$ since $\phi_j(h^k) = 1$ for all k . Then $\theta_B(h) = 1$ if and only if $\langle h \rangle = B$. Hence, we have

$$(3.6) \quad \text{Tr}(h; \theta_{Z,H}) = \begin{cases} 1 & \text{if } h \in H \setminus K_{Z,H}; \\ 0 & \text{if } h \in K_{Z,H}. \end{cases}$$

Remark 3.4. (i) For any $\phi \in R(H)$ with $\text{Supp}(\phi) \subset \text{Supp}(\theta_{Z,H})$, one has $\phi \cdot \theta_{Z,H} = \phi$. In particular, $\theta_{Z,H}^2 = \theta_{Z,H}$, meaning $\theta_{Z,H}$ is idempotent. Similarly, if H is nontrivial, then $\theta_H \cdot \theta_{Z,H} = \theta_H$.

(ii) $\theta_{Z,H}$ is contained in the augmentation ideal $I_{H,\mathbb{Q}}$.

Lemma 3.5. Let Z be a stratum of (X, G) and $H \in \mathcal{H}_Z$, with a generator h_0 . Then there exists a unique element $\tau_{Z,H} \in H^{\text{even}}(Z, \mathbb{Q}) \otimes R(H)$ that maps to $\theta_{Z,H} \text{ch}_H \lambda_{-1}(N_{Z/X}^*)^{-1}$ under the localization map $H^{\text{even}}(Z, \mathbb{Q}) \otimes R(H) \rightarrow H^{\text{even}}(Z, \mathbb{Q}) \otimes R(H)_{h_0}$, and that $\text{Tr}(h; \tau_{Z,H}) = 0$ for any $h \in K_{Z,H}$.

Proof. Using (3.3), we may write

$$(3.7) \quad \theta_{Z,H} \text{ch}_H \lambda_{-1}(N_{Z/X}^*)^{-1} = \theta_{Z,H} \prod_{j=1}^m (1 - \phi_j e^{x_j}) = \prod_{j=1}^m \frac{\theta_{Z,H}}{1 - \phi_j} \left(1 - \frac{\theta_{Z,H} \phi_j}{1 - \phi_j} \sum_{k \geq 1} \frac{x_j^k}{k!} \right)^{-1}$$

Expanding $\mathcal{U}(s, t) := \left(1 - s \sum_{k \geq 1} \frac{t^k}{k!} \right)^{-1}$ as a formal series in t , we obtain

$$\mathcal{U}(s, t) = \sum_{k \geq 0} A_k(s) t^k \in \mathbb{Q}[s][[t]]$$

where $A_k(s) \in \mathbb{Q}[s]$ is a polynomial with rational coefficients for each k . Then

$$\left(1 - \frac{\theta_{Z,H} \phi_j}{1 - \phi_j} \sum_{k \geq 1} \frac{x_j^k}{k!} \right)^{-1} = \sum_{k \geq 0} A_k \left(\frac{\theta_{Z,H} \phi_j}{1 - \phi_j} \right) x_j^k$$

and by (3.7), we have

$$\theta_{Z,H} \left(\text{ch}_H \lambda_{-1}(N_{Z/X}^*) \right)^{-1} = \prod_{j=1}^m \frac{\theta_{Z,H}}{1 - \phi_j} \sum_{k \geq 0} A_k \left(\frac{\theta_{Z,H} \phi_j}{1 - \phi_j} \right) x_j^k$$

By Lemma 2.1, the term $\frac{\theta_{Z,H}}{1 - \phi_j}$ are the image of some $\psi_j \in \frac{1}{|\phi_j|} R(H)$ satisfying $\psi_j(h) = 0$ for any $h \in K_{Z,H}$. Define

$$\tau_{Z,H} = \prod_{j=1}^m \psi_j \sum_{k \geq 0} A_k(\psi_j \phi_j) x_j^k \in H^{\text{even}}(Z, \mathbb{Q}) \otimes R(H).$$

Then $\tau_{Z,H}$ maps to $\theta_{Z,H} \text{ch}_H \lambda_{-1}(N_{Z/X}^*)^{-1}$ by the localization map, and $\text{Tr}(h; \tau_{Z,H}) = 0$ for any $h \in K_{Z,H}$. For $h \in H \setminus K_{Z,H}$, we have

$$\text{Tr}(h; \tau_{Z,H}) = \text{Tr}(h; \text{ch}_H \lambda_{-1}(N_{Z/X}^*)^{-1}) \in H^{\text{even}}(Z, \mathbb{Q}).$$

Therefore, $\text{Tr}(h; \tau_{Z,H})$ is uniquely determined for any $h \in H$ and it follows that $\tau_{Z,H}$ is uniquely determined as an element of $H^{\text{even}}(Z, \mathbb{Q}) \otimes R(H)$. $\square$

Remark 3.6. As the proof of Lemma 3.5 shows, the element $\left(\mathrm{ch}_H \lambda_{-1}(N_{Z/X}^*)\right)^{-1}$ in $H^{\mathrm{even}}(Z, \mathbb{Q}) \otimes R(H)_g$ does not have a preimage in $H^{\mathrm{even}}(Z, \mathbb{Q}) \otimes R(H)_{\mathbb{Q}}$ in general.

Definition 3.7. We define the *ramification Todd class* of $Z \subset X$ with respect to H as

$$\mathrm{td}_H(Z) = \mathrm{td}(Z) \cdot \tau_{Z,H} \in H^{\mathrm{even}}(Z, \mathbb{Q}) \otimes R(H)$$

Lemma 3.8. Let $Z \in \mathcal{Z}$ be a stratum.

- (i) If $\{\mathrm{id}\} \in \mathcal{H}_Z$, then Z is a component of X and $\mathrm{td}_{\{\mathrm{id}\}}(Z) = 0$.
- (ii) For $H \subset H'$ in $\mathcal{H}_Z$, we have

$$\mathrm{Res}_H^{H'} \mathrm{ch}_{H'}(\mathcal{E}|_Z) = \mathrm{ch}_H(\mathcal{E}|_Z), \quad \mathrm{Res}_H^{H'} \mathrm{td}_{H'}(Z) = \mathrm{td}_H(Z).$$

Proof. (i) Since $\{\mathrm{id}\} \in \mathcal{H}_Z$, Z is a component of $X^{\mathrm{id}} = X$. By definition, $\theta_{Z, \{\mathrm{id}\}} = 0$, and hence $\mathrm{td}_{\{\mathrm{id}\}}(Z) = 0$.

(ii) The first equality was proved in Lemma 2.4. For the second equality, we first show that $\mathrm{Res}_H^{H'} \theta_{Z,H'} = \theta_{Z,H}$. As both are characteristic modules, it suffices to note that they have the same support, which is clear by definition (3.4):

$$H \setminus K_{Z,H} = H \setminus K_{Z,H'}.$$

Now we prove that $\mathrm{Res}_H^{H'} \tau_{Z,H'} = \tau_{Z,H}$. The equality $\mathrm{Res}_H^{H'} \mathrm{td}_{H'}(Z) = \mathrm{td}_H(Z)$ then follows directly. The element $\tau_{Z,H}$, supported on $H \setminus K_{Z,H}$, is uniquely determined by the equation

$$\tau_{Z,H} \cdot \mathrm{ch}_H \lambda_{-1}(N_{Z/X}^*) = \theta_{Z,H}.$$

Through the above discussion and Lemma 2.4 we have

$$\begin{aligned} \theta_{Z,H} &= \mathrm{Res}_H^{H'} \theta_{Z,H'} \\ &= \mathrm{Res}_H^{H'} (\tau_{Z,H'}) \cdot \mathrm{Res}_H^{H'} (\mathrm{ch}_{H'} \lambda_{-1}(N_{Z/X}^*)) \\ &= \mathrm{Res}_H^{H'} (\tau_{Z,H'}) \cdot \mathrm{ch}_H \lambda_{-1}(\mathrm{Res}_H^{H'} N_{Z/X}^*). \end{aligned}$$

Since $\mathrm{Res}_H^{H'} (\tau_{Z,H'})$ is also supported on $H \setminus K_{Z,H}$, we conclude that $\mathrm{Res}_H^{H'} (\tau_{Z,H'}) = \tau_{Z,H}$. $\square$

The following definition of ramification module generalizes [LL25, Definition 3.2], originally conceived for smooth projective curves.

Definition 3.9. For each stratum $Z \in \mathcal{Z}$, we define the *ramification module* of the G -manifold X at Z as

$$(3.8) \quad \Gamma(\mathcal{E})_Z := \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G|} \mathrm{Ind}_H^G \left(\theta_H \int_Z \mathrm{ch}_H(\mathcal{E}|_Z) \mathrm{td}_H(Z) \right)$$

Lemma 3.10. Let $Z \in \mathcal{Z}$ be a stratum. Then the following holds for $\Gamma(X)_Z$:

- (i) $\Gamma(\mathcal{E})_Z$ is an element of the augmentation ideal $I_{G, \mathbb{Q}}$.
- (ii) If $G_Z = \{\mathrm{id}\}$, then Z is a component of X and $\Gamma(X)_Z = 0$.

Proof. (i) By Lemma 3.6, $\mathrm{Tr}(1; \theta_{Z,H}) = 0$. It follows that the traces of g on $\tau_{Z,H}$, $\mathrm{td}_H(Z)$, and $\Gamma(\mathcal{E})_Z$ are all zero.

(ii) Note that $\mathcal{H}_Z$ is always nonempty and its elements are subgroups of G_Z . If $G_Z = \{\mathrm{id}\}$, then $\mathcal{H}_Z = \{G_Z\}$. It follows that Z is a component of $X = X^{\mathrm{id}}$, and $\mathrm{td}_{G_Z}(Z) = 0$ by Lemma 3.8. Therefore, $\Gamma(X)_Z = 0$. $\square$

Theorem 3.11. Under Notation 3.1, the following equality holds in $R(G)_{\mathbb{Q}}$:

$$(3.9) \quad \chi_G(X, \mathcal{E}) = \frac{1}{|G|} \chi(X, \mathcal{E}) [\mathbb{C}[G]] + \sum_{Z \in \mathcal{Z}} \Gamma(\mathcal{E})_Z$$

Proof. By Proposition 2.3, we have

$$(3.10) \quad \chi_G(X, \mathcal{E}) = \sum_{H \subset G \text{ cyclic}} \frac{|H|}{|G|} \text{Ind}_H^G(\theta_H \cdot \text{Res}_H^G \chi_G(X, \mathcal{E}))$$

By the definition of θ_H , the class function $\theta_H \cdot \text{Res}_H^G \chi_G(X, \mathcal{E})$ is determined by the values on the generators h of the group H . By Theorem 2.5 and the definitions of ch_H and td_H , if H is nontrivial, then

$$(3.11) \quad \begin{aligned} \text{Tr}(h; \theta_H \cdot \text{Res}_H^G \chi_G(X, \mathcal{E})) &= \text{Tr} \left(h; \theta_H \sum_{Z \in \mathcal{Z}_H} \int_Z \frac{\text{ch}_H(\mathcal{E}|_Z) \cdot \text{td}(Z)}{\text{ch}_H \lambda_{-1}(N_{Z/X}^*)} \right) \\ &= \text{Tr} \left(h; \theta_H \theta_{Z,H} \sum_{Z \in \mathcal{Z}_H} \int_Z \frac{\text{ch}_H(\mathcal{E}|_Z) \cdot \text{td}(Z)}{\text{ch}_H \lambda_{-1}(N_{Z/X}^*)} \right) \\ &= \text{Tr} \left(h; \sum_{Z \in \mathcal{Z}_H} \int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \tau_{Z,H} \text{td}(Z) \right) \\ &= \text{Tr} \left(h; \sum_{Z \in \mathcal{Z}_H} \int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \text{td}_H(Z) \right). \end{aligned}$$

Hence, $\theta_H \text{Res}_H^G \chi_G(X, \mathcal{E})$ and $\int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \text{td}_H(Z)$ represent the same class in $R(H)_{\mathbb{Q}}$. Thus, the right hand side of (3.10) can be written as

$$(3.12) \quad \frac{1}{|G|} \text{Ind}_{\{\text{id}\}}^G(\theta_{\{\text{id}\}} \cdot \text{Res}_{\{\text{id}\}}^G \chi_G(X, \mathcal{E})) + \sum_{\{\text{id}\} \subsetneq H \subset G \text{ cyclic}} \frac{|H|}{|G|} \text{Ind}_H^G(\theta_H \cdot \text{Res}_H^G \chi_G(X, \mathcal{E})).$$

The first term simplifies to $\frac{1}{|G|} \chi(X, \mathcal{E}) [\mathbb{C}[G]]$. The second term becomes

$$\sum_{\{\text{id}\} \subsetneq H \subset G \text{ cyclic}} \frac{|H|}{|G|} \text{Ind}_H^G \sum_{Z \in \mathcal{Z}_H} \int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \text{td}_H(Z).$$

Reordering the summation gives

$$(3.13) \quad \sum_{Z \in \mathcal{Z}} \sum_{\{\text{id}\} \neq H \in \mathcal{H}_Z} \frac{|H|}{|G|} \text{Ind}_H^G \int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \text{td}_H(Z) = \sum_{Z \in \mathcal{Z}} \Gamma(\mathcal{E})_Z,$$

where we have used Definition 3.9 and Lemma 3.10. Substituting (3.13) into (3.12) yields the required equality (3.9). $\square$

Remark 3.12. Theorem 3.11 recovers [LL25, Theorem 3.6] in the case of (possibly disconnected) complex smooth projective curves.

Using [Don69, 5.5] instead of Theorem 2.5, one obtains an algebraic version of Theorem 3.11.

Theorem 3.13. *Let k be an algebraically closed field of characteristic $p \geq 0$ and X a smooth proper variety over k . Let G be a finite group such that $p \nmid |G|$, acting on X . Then we may define the set $\mathcal{Z}$ of strata as in Definition 3.2, and for each stratum $Z \in \mathcal{Z}$ and a locally free G -sheaf $\mathcal{E}$ on X , a ramification module $\Gamma(\mathcal{E})_Z$, depending only on the restriction $\mathcal{E}|_Z$ as a G_Z -sheaf, such that the following holds in $R_k(G)_{\mathbb{Q}}$:*

$$\chi_G(X, \mathcal{E}) = \frac{1}{|G|} \chi(X, \mathcal{E}) [\mathbb{C}[G]] + \sum_{Z \in \mathcal{Z}} \Gamma(\mathcal{E})_Z.$$

Using the formula (3.9), we can generalize the classical Chevalley–Weil formula on curves, which describes the multiplicity of any irreducible representation in the virtual G -module $\chi_G(X, \mathcal{E})$ in terms of the ramification data.

Corollary 3.14. *Under Notation 3.1, for each irreducible $M \in \widehat{G}$, its multiplicity in $\chi_G(X, \mathcal{E})$ is given by*

$$\mu_M \chi_G(X, \mathcal{E}) = \frac{\dim M}{|G|} \chi(X, \mathcal{E}) + \sum_{Z \in \mathcal{Z}} \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G|} \left\langle \text{Res}_H^G M, \int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \text{td}_H(Z) \right\rangle_H.$$

Proof. By Theorem 3.11, we have

$$\mu_M \chi_G(X, \mathcal{E}) = \langle [M], \chi_G(X, \mathcal{E}) \rangle_G = \langle [M], \frac{1}{|G|} \chi(X, \mathcal{E}) [\mathbb{C}[G]] \rangle_G + \sum_{Z \in \mathcal{Z}} \langle [M], \Gamma(\mathcal{E})_Z \rangle_G.$$

The first term equals $\frac{\dim M}{|G|} \chi(X, \mathcal{E})$ by a standard property of the regular representation $\mathbb{C}[G]$. For the second term, using the Frobenius reciprocity and the definition of $\Gamma(\mathcal{E})_Z$ yields

$$\langle [M], \Gamma(\mathcal{E})_Z \rangle_G = \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G|} \left\langle \text{Res}_H^G [M], \int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \text{td}_H(Z) \right\rangle_H$$

for any stratum $Z \in \mathcal{Z}$. □

4. COMPUTATION OF THE RAMIFICATION MODULE IN SPECIAL CASES

The practical utility of the Chevalley–Weil formula (3.9) often depends on the computability of the ramification modules $\Gamma(\mathcal{E})_Z$. In the following, we illustrate how $\Gamma(\mathcal{E})_Z$ can be simplified under restrictions on the stabilizer G_Z , or on the dimension and codimension of Z .

If the stabilizer G_Z is cyclic, then $\Gamma(\mathcal{E})_Z$ simplifies, as one does not need to sum over all of $\mathcal{H}_Z$.

Lemma 4.1. *Suppose that $Z \in \mathcal{Z}$ is a stratum with cyclic stabilizer G_Z . Then*

$$(4.1) \quad \Gamma(\mathcal{E})_Z = \frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \int_Z \text{ch}_{G_Z}(\mathcal{E}|_Z) \text{td}_{G_Z}(Z).$$

Proof. We first claim that $G_Z \in \mathcal{H}_Z$. This is clear if Z is a component of X . If Z is not a component in the fixed locus X^{G_Z} , G_Z acts identically along some normal direction $v \in N_{Z/X}$. Consequently, any cyclic subgroup of G_Z acts identically along v , which contradicts that Z , as a stratum, is a component in X^H for some subgroup $H \subset G_Z$.

Using Lemma 3.8, we have

$$(4.2) \quad \begin{aligned} \Gamma(\mathcal{E})_Z &= \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G|} \text{Ind}_H^G \int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \text{td}_H(Z) \\ &= \frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G_Z|} \text{Ind}_H^{G_Z} \theta_H \text{Res}_H^{G_Z} \int_Z \text{ch}_{G_Z}(\mathcal{E}|_Z) \text{td}_{G_Z}(Z). \end{aligned}$$

Let $K := \{g \in G_Z \mid Z \text{ is not a component of } X^g\}$. Then $\mathcal{H}_Z$ consists of subgroups of G_Z that are not contained in K . For a cyclic subgroup $H \subset G_Z$ contained in K , we have $\text{Res}_H^{G_Z} \text{td}_{G_Z}(Z) = 0$ and hence $\text{Res}_H^{G_Z} \int_Z \text{ch}_{G_Z}(\mathcal{E}|_Z) \text{td}_{G_Z}(Z) = 0$.

Thus, the last summation of (4.2) is the same as summing over all the subgroups $H \subset G_Z$:

$$\sum_{H \subset G_Z} \frac{|H|}{|G_Z|} \text{Ind}_H^{G_Z} \theta_H \text{Res}_H^{G_Z} \int_Z \text{ch}_{G_Z}(\mathcal{E}|_Z) \text{td}_{G_Z}(Z) = \int_Z \text{ch}_{G_Z}(\mathcal{E}|_Z) \text{td}_{G_Z}(Z)$$

where the equality follows from Proposition 2.3. Substituting this into (4.2) yields the required formula (4.1). □

Example 4.2. Suppose that $Z \in \mathcal{Z}$ is a connected component of X . Then

$$\mathcal{H}_Z = \{H \mid H \subset G_Z \text{ cyclic}\}.$$

The ramification module $\Gamma(\mathcal{E})_Z$ can be computed as follows:

$$\begin{aligned}\Gamma(\mathcal{E})_Z &= \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G|} \text{Ind}_H^G \int_Z \theta_H \text{ch}_H(\mathcal{E}|_Z) \text{td}_H(Z) \\ &= \frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G_Z|} \text{Ind}_H^{G_Z} \int_Z \theta_H \text{td}_H(Z) \cdot \text{Res}_H^{G_Z} \text{ch}_{G_Z}(\mathcal{E}|_Z) \\ &= \frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \sum_{H \in \mathcal{H}_Z} \frac{|H|}{|G_Z|} \int_Z \text{ch}_{G_Z}(\mathcal{E}|_Z) \cdot \text{Ind}_H^{G_Z} (\theta_H \text{td}_H(Z))\end{aligned}$$

Recall that $\text{td}_H(Z) = \text{td}(Z)$ if $H \neq \{\text{id}\}$, and $\text{td}_{\{\text{id}\}}(Z) = 0$. Then the last term equals

$$\frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \int_Z \text{ch}_{G_Z}(\mathcal{E}|_Z) \text{td}(Z) \left(\sum_{H \subset G_Z \text{ cyclic}} \frac{|H|}{|G_Z|} \text{Ind}_H^{G_Z} \theta_H - \frac{1}{|G_Z|} \text{Ind}_{\{\text{id}\}}^{G_Z} \theta_{\{\text{id}\}} \right)$$

By the Hirzebruch–Riemann–Roch theorem and Formula (2.5), this becomes

$$\frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \chi_{G_Z}(Z, \mathcal{E}|_Z) \left([1_{G_Z}] - \frac{1}{|G_Z|} \text{Ind}_{\{\text{id}\}}^{G_Z} \theta_{\{\text{id}\}} \right)$$

Simplifying yields

$$\begin{aligned}& \frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \chi_{G_Z}(Z, \mathcal{E}|_Z) - \frac{1}{|G|} \text{Ind}_{G_Z}^G \text{Ind}_{\{\text{id}\}}^{G_Z} (\theta_{\{\text{id}\}} \cdot \text{Res}_{\{\text{id}\}}^{G_Z} \chi_{G_Z}(Z, \mathcal{E}|_Z)) \\ &= \frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \chi_{G_Z}(Z, \mathcal{E}|_Z) - \frac{\chi(Z, \mathcal{E}|_Z)}{|G|} [\mathbb{C}[G]]\end{aligned}$$

where $\chi_{G_Z}(Z, \mathcal{E}|_Z)$ represents the G_Z -Euler characteristic of $\mathcal{E}|_Z$, and $\chi(Z, \mathcal{E}|_Z)$ is the usual Euler characteristic.

If $G_Z = \{\text{id}\}$ for each component Z of X , then $\mathcal{H}_Z$ consists of only the trivial group, and we have $\Gamma(\mathcal{E})_Z = 0$. In this case, (3.9) becomes

$$(4.3) \quad \chi_G(X, \mathcal{E}) = \frac{1}{|G|} \chi(X, \mathcal{E}) [\mathbb{C}[G]] + \sum_{Z' \in \mathcal{Z}'} \Gamma(\mathcal{E})_{Z'}.$$

where $\mathcal{Z}' \subset \mathcal{Z}$ consists of the strata with positive codimension in X .

Example 4.3. Suppose that P is an isolated point in X^g for some $g \in G$ and $\text{codim}_X \{P\} > 0$. Then $\{P\} \in \mathcal{Z}$ and $H := \langle g \rangle \in \mathcal{H}_P$. We have $\text{td}(\{P\}) = 1$. Decompose the conormal space $N_{Z/X}^* = T_P^* X$ into eigenspaces $T_P^* X = \bigoplus_{\phi \in \widehat{H}} V_\phi$, and let m_ϕ denote the dimension of V_ϕ . Then

$$\text{td}_H(Z) = \prod_{\phi \in \widehat{H}} \left(-\frac{1}{|\phi|} \sum_{d=0}^{|\phi|-1} d\phi^d \right)^{m_\phi}$$

where $|\phi|$ is the order of ϕ . Also,

$$\text{ch}_H(\mathcal{E}|_P) = [\mathcal{E}|_P]_H \in R(H).$$

Therefore,

$$\Gamma_G(\mathcal{E})_P = \sum_{H \in \mathcal{H}_P} \frac{|H|}{|G|} \text{Ind}_H^G \theta_H [\mathcal{E}|_P]_H \prod_{\phi \in \widehat{H}} \left(-\frac{1}{|\phi|} \sum_{d=0}^{|\phi|-1} d\phi^d \right)^{m_\phi}$$

Example 4.4. Suppose that X is connected and G -action on X is faithful. Let $Z \in \mathcal{Z}$ be a stratum with $\text{codim}_X Z = 1$. Then G_Z is cyclic. The conormal bundle $N_{Z/X}^*$ has rank 1, and G_Z acts on it by a character $\phi_Z: G_Z \rightarrow \mathbb{C}^*$. Since the codimension of Z is the smallest among all strata except X , $\mathcal{H}_Z$ consists of all the nontrivial subgroup of G_Z and the subset

K_{Z,G_Z} in (3.4) is the identity. Therefore, $\theta_{Z,G_Z} = [1_{G_Z}] - \frac{1}{|G_Z|}[\mathbb{C}[G_Z]]$ (see (3.5) for the definition of $\theta_{Z,H}$ with $H \in \mathcal{H}_Z$). By Lemma 4.1, we have

$$(4.4) \quad \Gamma(\mathcal{E})_Z = \frac{|G_Z|}{|G|} \text{Ind}_{G_Z}^G \sum_{\psi \in \hat{G}_Z} \psi \int_Z \text{ch}(\mathcal{E}_{Z,\psi}) \text{td}_{G_Z}(Z)$$

where $[\mathcal{E}|_Z]_{G_Z} = \sum_{\psi \in \hat{G}_Z} [\mathcal{E}_{Z,\psi}] \otimes [\psi] \in K(Z) \otimes R(G_Z)$.

5. THE ACTION OF $(\mathbb{Z}/2\mathbb{Z})^n$ ON A COMPACT COMPLEX SURFACE

We illustrate the general Chevalley–Weil formula of Theorem 3.11 by working out the case where $G \cong (\mathbb{Z}/2\mathbb{Z})^n$ acts on a connected compact complex surface.

First, we refine the expression of the ramification module from Example 4.4 at a fixed curve on a surface.

Lemma 5.1. *Let X be a connected compact complex surface, $G \subset \text{Aut}(X)$ a finite subgroup, and $\mathcal{E}$ a G -equivariant locally free sheaf on X . Suppose that C is a one-dimensional stratum of (X, G) . Denote by G_C the stabilizer group of C , and by $\phi_C: G_C \rightarrow \mathbb{C}^*$ the character of the rank 1 conormal bundle $N^* := N_{C/X}^*$ under the action of G_C . Then*

$$\Gamma(\mathcal{E})_C = -\frac{|G_C|}{|G|} \text{Ind}_{G_C}^G \theta_{C,G_C} \sum_{\psi \in \hat{G}_C} \psi \left(\frac{\chi(\mathcal{E}_{C,\psi})}{|G_C|} \sum_{d=0}^{|G_C|-1} d\phi_C^d + \frac{(\text{rk } \mathcal{E}_{C,\psi}) \cdot C^2}{|G_C|^2} \left(\sum_{d=0}^{|G_C|-1} d\phi_C^d \right)^2 \phi_C \right)$$

where $\theta_{C,G_C} = [1_{G_C}] - \frac{1}{|G_C|}[\mathbb{C}[G_C]]$.

Proof. Since $\dim X = 2$ and C is a curve, Example 4.4 gives $\theta_{C,G_C} = [1_{G_C}] - \frac{1}{|G_C|}[\mathbb{C}[G_C]]$, and

$$(5.1) \quad \begin{aligned} \tau_{C,G_C} &= \frac{\theta_{C,G_C}}{1 - \phi_C \cdot e^{c_1(N^*)}} = \frac{\theta_{C,G_C}}{1 - \phi_C} + \frac{\theta_{C,G_C} \phi_C}{(1 - \phi_C)^2} c_1(N^*) \\ &= -\frac{\theta_{C,G_C}}{|G_C|} \sum_{d=0}^{|G_C|-1} d\phi_C^d + \frac{\theta_{C,G_C} \phi_C}{|G_C|^2} \left(\sum_{d=0}^{|G_C|-1} d\phi_C^d \right)^2 c_1(N^*). \end{aligned}$$

We may write

$$\text{ch}_{G_C}(\mathcal{E}_{C,\psi}) = (\text{rk } \mathcal{E}_{C,\psi} + c_1(\mathcal{E}_{C,\psi})) \psi.$$

It follows that

$$\begin{aligned} & \int_C \text{ch}(\mathcal{E}_{C,\psi}) \text{td}_{G_C}(C) \\ &= \int_C (\text{rk } \mathcal{E}_{C,\psi} + c_1(\mathcal{E}_{C,\psi})) \left(\frac{\theta_{C,G_C}}{1 - \phi_C} + \frac{\theta_{C,G_C} \phi_C}{(1 - \phi_C)^2} c_1(N^*) \right) \left(1 + \frac{1}{2} c_1(T_C) \right) \\ &= \int_C \theta_{C,G_C} \left(\frac{\text{ch}(\mathcal{E}_{C,\psi}) \cdot \text{td}(C)}{1 - \phi_C} + \frac{(\text{rk } \mathcal{E}_{C,\psi}) \cdot \phi_C}{(1 - \phi_C)^2} c_1(N^*) \right). \end{aligned}$$

Note that $\int_C \text{ch}(\mathcal{E}_{C,\psi}) \cdot \text{td}(C) = \chi(\mathcal{E}_{C,\psi})$ and $c_1(N^*) = -C^2$. Hence, the above expression equals

$$(5.2) \quad \begin{aligned} & \theta_{C,G_C} \left(\frac{\chi(\mathcal{E}_{C,\psi})}{1 - \phi_C} - \frac{\text{rk } \mathcal{E}_{C,\psi} \cdot \phi_C}{(1 - \phi_C)^2} C^2 \right) \\ &= -\theta_{C,G_C} \left(\frac{\chi(\mathcal{E}_{C,\psi})}{|G_C|} \sum_{d=0}^{|G_C|-1} d\phi_C^d + \frac{\text{rk } \mathcal{E}_{C,\psi}}{|G_C|^2} \phi_C \left(\sum_{d=0}^{|G_C|-1} d\phi_C^d \right)^2 C^2 \right). \end{aligned}$$

Substituting (5.2) into (4.4) yields the desired equality. $\square$

Theorem 5.2. *Let X be a connected compact complex surface and $G \subset \text{Aut}(X)$ an automorphism subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Let $\mathcal{E}$ be a G -equivariant locally free sheaf of rank r on X . Suppose that there are N zero-dimensional strata $P_1, \dots, P_N$ and M one-dimensional strata $C_1, \dots, C_M$ for the G -action on X . Let $\mathcal{E}|_{C_k} = \mathcal{E}_{C_k}^+ \oplus \mathcal{E}_{C_k}^-$ be the decomposition of $\mathcal{E}|_{C_k}$ into eigensheaves with eigenvalues 1 and -1 under the action of G_{C_k} , and let r_k^+ and r_k^- be the ranks of $\mathcal{E}_{C_k}^+$ and $\mathcal{E}_{C_k}^-$ respectively. Then the following holds.*

- (i) *For each strata $Z \neq X$, $\mathcal{H}_Z$ has exactly one element, denoted by H_Z .*
- (ii) *We have*

$$\begin{aligned} \chi_G(X, \mathcal{E}) &= \frac{1}{2^n} \chi(X, \mathcal{E})[\mathbb{C}[G]] + \frac{1}{2^{n+1}} \sum_{i=1}^N \text{Ind}_{H_{P_i}}^G \left([\mathcal{E}|_{P_i}] - \frac{r}{2} [\mathbb{C}[H_{P_i}]] \right) \\ &+ \frac{1}{2^{n+1}} \sum_{k=1}^M \text{Ind}_{H_{C_k}}^G \left(-(K_X \cdot C_k)(r_k^+ - r_k^-) + 2(\deg \mathcal{E}_{C_k}^+ - \deg \mathcal{E}_{C_k}^-) \right) \left([1_{H_{C_k}}] - \frac{1}{2} [\mathbb{C}[H_{C_k}]] \right) \end{aligned}$$

Proof. (i) If $\dim Z = 1$, then G_Z is cyclic and hence has order 2. Clearly, $\mathcal{H}_Z = \{G_Z\}$. If $\dim Z = 0$, then $|G_Z| \leq 2^2$ by [CL24, Lemma 2.1]. By [CL24, Lemma 2.7], there is exactly one involution $\sigma \in G_Z$ having Z as an isolated fixed point, so $\mathcal{H}_Z = \{\langle \sigma \rangle\}$.

(ii) Based on (i), we have $\theta_{H_Z} = \theta_{Z, H_Z} = [1_{H_Z}] - \frac{1}{2} [\mathbb{C}[H_Z]]$. We now compute $\Gamma(\mathcal{E})_Z$ for each stratum of (X, G) . Denote by ϕ_Z the class of the nontrivial simple module of H_Z .

Since G -action on X is faithful, we have $\Gamma(\mathcal{E})_Z = 0$ if $Z = X$; see Example 4.2.

If $Z = C_k$ for some $1 \leq k \leq M$, then by Lemma 5.1, we have

$$\begin{aligned} (5.3) \quad \Gamma(\mathcal{E})_Z &= \frac{1}{2^{n-1}} \text{Ind}_{H_Z}^G ([1_{H_Z}] - \frac{1}{2} [\mathbb{C}[H_Z]]) \left(\left(-\frac{1}{2} \chi(\mathcal{E}_Z^+) - \frac{1}{4} r^+ Z^2 \right) \phi_Z + \left(-\frac{1}{2} \chi(\mathcal{E}_Z^-) - \frac{1}{4} r^- Z^2 \right) \right) \\ &= \frac{1}{2^{n+1}} \text{Ind}_{H_Z}^G ((r_k^+ - r_k^-) \cdot Z^2 + 2(\chi(\mathcal{E}_Z^+) - \chi(\mathcal{E}_Z^-))) \left([1_{H_Z}] - \frac{1}{2} [\mathbb{C}[H_Z]] \right) \\ &= \frac{1}{2^{n+1}} \text{Ind}_{H_Z}^G (-(K_X \cdot Z)(r_k^+ - r_k^-) + 2(\deg \mathcal{E}_Z^+ - \deg \mathcal{E}_Z^-)) \left([1_{H_Z}] - \frac{1}{2} [\mathbb{C}[H_Z]] \right) \end{aligned}$$

where we used the Riemann-Roch theorem $\chi(Z, \mathcal{F}) = (\text{rk } \mathcal{F})\chi(\mathcal{O}_Z) + \deg \mathcal{F}$ for a locally free sheaf $\mathcal{F}$ on Z , the adjunction formula $(K_X + Z) \cdot Z + 2\chi(Z, \mathcal{O}_Z) = 0$, and the identity

$$\left([1_{H_Z}] - \frac{1}{2} [\mathbb{C}[H_Z]] \right) \phi_Z = - \left([1_{H_Z}] - \frac{1}{2} [\mathbb{C}[H_Z]] \right).$$

If $Z = P_i$ for some $1 \leq i \leq N$, then by Example 4.3

$$\begin{aligned} (5.4) \quad \Gamma(\mathcal{E})_Z &= \frac{|H_Z|}{|G|} \text{Ind}_{H_Z}^G \theta_{H_Z} [\mathcal{E}|_Z] \left(\frac{1}{2} \phi_Z \right)^2 \\ &= \frac{1}{2^{n-1}} \text{Ind}_{H_Z}^G \frac{1}{4} \left([1_{H_Z}] - \frac{1}{2} [\mathbb{C}[H_Z]] \right) [\mathcal{E}|_Z] \\ &= \frac{1}{2^{n+1}} \left([\mathcal{E}|_Z] - \frac{r}{2} [\mathbb{C}[H_Z]] \right). \end{aligned}$$

Summing $\Gamma(\mathcal{E})_Z$ over all the strata Z and applying Theorem 3.11 yields the desired equality in (ii). $\square$

Applying Theorem 5.2 to the sheaves $\mathcal{E} = \mathcal{O}_X(nK_X)$ for $n \in \mathbb{Z}$, and Ω_X^1 , we obtain more explicit results.

Example 5.3. Take $\mathcal{E} = \Omega_X^1$ in Theorem 5.2. For a zero-dimensional stratum $Z = P_i$, we have

$$[\Omega_X^1|_Z] = [T_Z^* X] = 2[\phi_Z]$$

For a one-dimensional stratum $Z = C_k$, the short exact sequence $0 \rightarrow N_{Z/X}^* \rightarrow \Omega_X^1|_Z \rightarrow \Omega_Z^1 \rightarrow 0$ is H_Z -equivariant. Here, H_Z acts on $N_{Z/X}^*$ by the nontrivial character ϕ_Z and on

Ω_Z^1 by the trivial character 1_Z . Thus the sequence gives an eigen-subsheaf decomposition:

$$\Omega_X^1|_Z = N_{Z/X}^* \oplus \Omega_Z^1$$

In the notation of Theorem 5.2, for each $1 \leq k \leq M$,

$$r_k^+ = \text{rk } \Omega_Z^1 = 1, \quad r_k^- = \text{rk } N_{Z/X}^* = 1, \quad \deg N_{Z/X}^* = -Z^2, \quad \deg \Omega_Z^1 = 2g(Z) - 2$$

Using the exact sequence $0 \rightarrow \mathbb{C} \xrightarrow{\bar{\partial}} \mathcal{O}_X \xrightarrow{\bar{\partial}} \Omega_X^1 \xrightarrow{\bar{\partial}} \Omega_X^2 \rightarrow 0$, we find

$$\chi(X, \Omega_X^1) = -\chi(X, \mathbb{C}) + \chi(X, \mathcal{O}_X) + \chi(X, \Omega_X^2) = K_X^2 - 10\chi(X, \mathcal{O}_X)$$

where we use the Serre duality $\chi(X, \Omega_X^2) = \chi(X, \mathcal{O}_X)$ and the Noether formula $\chi(X, \mathbb{C}) = 12\chi(\mathcal{O}_X) - K_X^2$ for the second equality. Alternatively, one may apply the Hirzebruch–Riemann–Roch theorem directly.

By Theorem 5.2 (ii), we obtain,

$$\begin{aligned} \chi_G(X, \Omega_X^1) &= \frac{1}{2^n} (K_X^2 - 10\chi(\mathcal{O}_X)) [\mathbb{C}[G]] - \frac{1}{2^n} \sum_{i=1}^N \text{Ind}_{H_{P_i}}^G \left([1_{H_{P_i}}] - \frac{1}{2} [\mathbb{C}[H_{P_i}]] \right) \\ &\quad + \frac{1}{2^n} \sum_{k=1}^M \text{Ind}_{H_{C_k}}^G (2g(C_k) - 2 + C_k^2) \left([1_{H_{C_k}}] - \frac{1}{2} [\mathbb{C}[H_{C_k}]] \right) \end{aligned}$$

Example 5.4. Now take $\mathcal{E} = \mathcal{O}_X(nK_X)$, $n \in \mathbb{Z}$. This sheaf is invertible.

If $Z = P_i$ is a point, then the action near Z can be linearized analytically locally as $(x, y) \mapsto (-x, -y)$, which fixes the local basis $(dx \wedge dy)^{\otimes n}$ of $\mathcal{O}_X(nK_X)$. Thus,

$$[\mathcal{O}_X(nK_X)|_Z] = [1_{H_Z}]$$

If $Z = C_k$ is a curve, then there is analytically local coordinates x, y so that $Z = (x = 0)$ and the involution in H_Z acts as $(x, y) \mapsto (-x, y)$. Therefore, H_Z acts on $\mathcal{O}_X(nK_X)$ by the character ϕ_Z^n . Hence, we have

$$r_k^+ - r_k^- = (-1)^n, \quad \deg \mathcal{O}_X(nK_X)_Z^+ - \deg \mathcal{O}_X(nK_X)_Z^- = (-1)^n n(K_X \cdot Z)$$

Also, for $Z = X$, the Riemann–Roch theorem gives

$$\chi(X, \mathcal{O}_X(nK_X)) = \chi(X, \mathcal{O}_X) + \frac{1}{2} n(n-1) K_X^2$$

Substituting these computations into Theorem 5.2 (ii) with $\mathcal{E} = \mathcal{O}_X(nK_X)$ yields:

$$\begin{aligned} \chi_G(X, nK_X) &= \frac{1}{2^n} \left(\chi(X, \mathcal{O}_X) + \frac{1}{2} n(n-1) K_X^2 \right) [\mathbb{C}[G]] + \frac{1}{2^{n+1}} \sum_{1 \leq i \leq N} \text{Ind}_{H_{P_i}}^G ([1_{H_{P_i}}] - \frac{1}{2} [\mathbb{C}[H_{P_i}]])) \\ &\quad + (-1)^n \frac{2n-1}{2^{n+1}} \sum_{1 \leq k \leq M} \text{Ind}_{H_{C_k}}^G (K_X \cdot C_k) ([1_{H_{C_k}}] - \frac{1}{2} [\mathbb{C}[H_{C_k}]]). \end{aligned}$$

REFERENCES

- [AS68II] Michael Atiyah and Graeme Segal, The index of elliptic operators: II. Ann. Math. 87 (1968) no. 3, 531–545.
- [AS68III] Michael Atiyah and Isadore Singer, The index of elliptic operators: III. Ann. Math. 87 (1968), no. 3, 546–604.
- [Ara22] Donu Arapura, Residues of connections and the Chevalley–Weil formula for curves. L’Enseign. Math. 68 (2022), no. 3/4, 291–306.
- [Bri25] Michel Brion, Equivariantly normal varieties, <https://www-fourier.univ-grenoble-alpes.fr/~mbrion/env.pdf>, 2025.
- [CL24] Jin-Xing Cai and Wenfei Liu, On regular surfaces of general type with numerically trivial automorphism group of order 4, arXiv:2412.16501, 2024.
- [CW34] Claude Chevalley and André Weil, A, Über das verhalten der integrale 1. gattung bei automorphismen des funktionenkörpers. Abh. Math. Semin. Univ. Hambg. 10 (1934), 358–361.
- [Don69] Peter Donovan, The Lefschetz–Riemann–Roch formula, Bull. Soc. Math. France 97 (1969), 257–273.
- [EL80] Geir Ellingsrud and Knud Lønsted, An equivariant Lefschetz formula for finite reductive groups, Math. Ann. 251 (1980), 253–261.
- [Ill68] Luc Illusie, Nombres de Chern et groupes finis, Topology 7 (1968), 255–269.

- [Koc05] Bernhard Köck, Computing the equivariant Euler characteristic of Zariski and étale sheaves on curves, *Homology, homotopy and applications* 1 (2005), 1–13.
- [LL25] Qing Liu and Wenfei Liu, On the cohomological representations of finite automorphism groups of singular curves and compact complex spaces, *arXiv:2507.12290*, 2025.
- [Seg68] Graeme Segal, Equivariant K-theory, *Inst. Hautes Études Sci. Publ. Math.* No. 34 (1968), 129–151.
- [Ser77] Jean-Pierre Serre, Linear representations of finite groups, *Graduate texts in mathematics*, vol 42, 1977 (Translation of the French edition *Représentations linéaires des groupes finis*, Paris: Hermann 1971), Springer-Verlag, New York Inc.
- [Ton25] Yubo Tong, The Chevalley–Weil formula on nodal curves. *AIMS Math.* 10 (2025), no. 5, 12228–12253.
- [Wei35] André Weil, Über Matrizenringe auf Riemannschen Flächen und den Riemann-Rochschen Satz, *Abh. Math. Semin. Univ. Hambg.* 11, 110–115 (1935).

WENFEI LIU, SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN, FUJIAN 361005, P.R. CHINA

Email address: `wliu@xmu.edu.cn`

RENJIE LYU, SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN, FUJIAN 361005, P.R. CHINA

Email address: `r.lyu@xmu.edu.cn`